Paschke Dilations

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In 1973 Paschke defined a factorization for completely positive maps between C*-algebras. In this paper we show that for normal maps between von Neumann algebras, this factorization has a universal property, and coincides with Stinespring's dilation for normal maps into $B(\mathscr{H})$.

The Stinespring Dilation Theorem[17] entails that every normal completely positive linear map (*NCP-map*) $\varphi\colon \mathscr{A} \to B(\mathscr{H})$ is of the form $\mathscr{A} \xrightarrow{\pi} B(\mathscr{K}) \xrightarrow{V^*(\cdot)V} B(\mathscr{H})$ where $V\colon \mathscr{H} \to \mathscr{K}$ is a bounded operator and π a normal unital *-homomorphism (*NMIU-map*). Stinespring's theorem is fundamental in the study of quantum information and quantum computing: it is used to prove entropy inequalities (e.g. [10]), bounds on optimal cloners (e.g. [20]), full completeness of quantum programming languages (e.g. [16]), security of quantum key distribution (e.g. [8]), analyze quantum alternation (e.g. [1]), to categorify quantum processes (e.g. [14]) *and* as an axiom to single out quantum theory among information processing theories.[2] A fair overview of all uses of Stinespring's theorem and its consequences would warrant a separate article of its own.

One wonders: is the Stinespring dilation categorical in some way? Can the Stinespring dilation theorem be generalized to arbitrary NCP-maps $\varphi\colon \mathscr{A} \to \mathscr{B}$? In this paper we answer both questions in the affirmative. We use the dilation introduced by Paschke[11] for arbitrary NCP-maps, and we show that it coincides with Stinespring's dilation (a fact not shown before) by introducing a universal property for Paschke's dilation, which Stinespring's dilation also satisfies.

In the second part of this paper, we will study the class of maps that may appear on the right-hand side of a Paschke dilation, to prove the counter-intuitive fact that both maps in a Paschke dilation are extreme (among NCP maps with same value on 1).

Let us give the universal property and examples right off the bat; proofs are further down.

Theorem 1. *Every NCP-map* $\varphi\colon \mathscr{A} \to \mathscr{B}$ *has a **Paschke dilation**. A Paschke dilation of* φ *is a pair of maps* $\mathscr{A} \xrightarrow{\rho} \mathscr{P} \xrightarrow{f} \mathscr{B}$, *where* $\mathscr{P}$ *is a von Neumann algebra,* ρ *is an NMIU-map and* f *is an NCP-map with* $\varphi = f \circ \rho$ *such that for every other* $\mathscr{A} \xrightarrow{\rho'} \mathscr{P}' \xrightarrow{f'} \mathscr{B}$, *where* $\mathscr{P}'$ *is a von Neumann algebra,* ρ' *is an NMIU-map, and* f' *is an NCP-map with* $\varphi = f' \circ \rho'$, *there is a unique NCP-map* $\sigma\colon \mathscr{P}' \to \mathscr{P}$ *such that the diagram on the right commutes.*
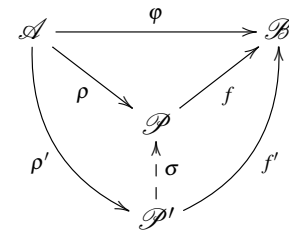
Example 2. A *minimal* Stinespring dilation $\mathscr{A} \xrightarrow{\pi} B(\mathscr{K}) \xrightarrow{V(\cdot)V^*} B(\mathscr{H})$ of an NCP-map is a Paschke dilation, see Theorem 14.

Example 3. As a special case of the previous example, we see the GNS construction for a normal state φ on a von Neumann algebra $\mathscr{A}$, gives a Paschke dilation $\mathscr{A} \xrightarrow{\pi} B(\mathscr{H}) \xrightarrow{\langle \xi, (\cdot)\xi \rangle} \mathbb{C}$ of φ. In particular, the Paschke dilation of $(\lambda, \mu) = \frac{1}{2}(\lambda + \mu)$, $\mathbb{C}^2 \to \mathbb{C}$ is

$$\mathbb{C}^2 \xrightarrow{(\lambda,\mu) \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}} M_2 \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{2}(a+b+c+d)} \mathbb{C} .$$

This gives a universal property to the von Neumann algebra M_2 of 2×2 complex matrices, (which is a model of the qubit.)

The following examples can be proven using only the universal property of a Paschke dilation.

Example 4. The Paschke dilation of an NMIU-map $\rho \colon \mathscr{A} \to \mathscr{B}$ is $\mathscr{A} \xrightarrow{\ \rho\ } \mathscr{B} \xrightarrow{\ \mathrm{id}\ } \mathscr{B}$.

Example 5. If $\mathscr{A} \xrightarrow{\ \rho\ } \mathscr{P} \xrightarrow{\ f\ } \mathscr{B}$ is a Paschke dilation, then $\mathscr{P} \xrightarrow{\ \mathrm{id}\ } \mathscr{P} \xrightarrow{\ f\ } \mathscr{B}$ is a Paschke dilation of f.

Example 6. Let $\varphi \colon \mathscr{A} \to \mathscr{B}_1 \oplus \mathscr{B}_2$ be any NCP-map. $\mathscr{A} \xrightarrow{\langle \rho_1, \rho_2 \rangle} \mathscr{P}_1 \oplus \mathscr{P}_2 \xrightarrow{f_1 \oplus f_2} \mathscr{B}_1 \oplus \mathscr{B}_2$ is a Paschke dilation of φ if $\mathscr{A} \xrightarrow{\ \rho_i\ } \mathscr{P}_i \xrightarrow{\ f_i\ } \mathscr{B}_i$ is a Paschke dilation of $\pi_i \circ \varphi$ for $i = 1, 2$.

Example 7. Let $\varphi \colon \mathscr{A} \to \mathscr{B}$ be any NCP-map with Paschke dilation $\mathscr{A} \xrightarrow{\ \rho\ } \mathscr{P} \xrightarrow{\ f\ } \mathscr{B}$ and $\lambda > 0$. Then $\mathscr{A} \xrightarrow{\ \rho\ } \mathscr{P} \xrightarrow{\ \lambda f\ } \mathscr{B}$ is a Paschke dilation of $\lambda \varphi$.

Example 8. Let $\mathscr{A} \xrightarrow{\ \rho\ } \mathscr{P} \xrightarrow{\ f\ } \mathscr{B}$ be a Paschke dilation for a map $\varphi \colon \mathscr{A} \to \mathscr{B}$. If $\vartheta \colon \mathscr{P} \to \mathscr{P}'$ is any isomorhism, then $\mathscr{A} \xrightarrow{\vartheta \circ \rho} \mathscr{P}' \xrightarrow{f \circ \vartheta^{-1}} \mathscr{B}$ is also a Paschke dilation of φ.

There is a converse to the last example:

Lemma 9. *If $\mathscr{A} \xrightarrow{\ \rho_i\ } \mathscr{P}_i \xrightarrow{\ f_i\ } \mathscr{B}$ $(i = 1, 2)$ are Paschke dilations for the same map $\varphi \colon \mathscr{A} \to \mathscr{B}$, then there is a unique (NMIU) isomorphism $\vartheta \colon \mathscr{P}_1 \to \mathscr{P}_2$ such that $\vartheta \circ \rho_1 = \rho_2$ and $f_2 \circ \vartheta = f_1$.*

Proof. There are unique mediating maps $\sigma_1 \colon \mathscr{P}_1 \to \mathscr{P}_2$ and $\sigma_2 \colon \mathscr{P}_2 \to \mathscr{P}_1$. It is easy to see $\sigma_1 \circ \sigma_2$ satisfies the same property as the unique mediating map $\mathrm{id} \colon \mathscr{P}_1 \to \mathscr{P}_1$ and so $\sigma_1 \circ \sigma_2 = \mathrm{id}$. Similarly $\sigma_2 \circ \sigma_1 = \mathrm{id}$. Define $\vartheta = \sigma_1$. We just saw ϑ is an NCP-isomorphism. Note $\vartheta(1) = \vartheta(\rho_1(1)) = \rho_2(1) = 1$ and so ϑ is unital. But then by [22, Corollary 47] ϑ is an NMIU isomorphism. $\square$

1   Two universal properties for Stinespring's dilation

Let $\varphi \colon \mathscr{A} \to B(\mathscr{H})$ be a NCP-map where $\mathscr{A}$ is a von Neumann algebra and $\mathscr{H}$ is a Hilbert space. In this section, we prove that any minimal normal Stinespring dilation of φ gives a Paschke dilation of φ. Let us first recall the relevant definitions.

Definition 10. A **normal Stinespring dilation** of φ, is a triple $(\mathscr{K}, \pi, V)$, where $\mathscr{K}$ is a Hilbert space, $\pi \colon \mathscr{A} \to B(\mathscr{K})$ is an NMIU-map, and $V \colon \mathscr{H} \to \mathscr{K}$ a bounded operator such that $\varphi = \mathrm{Ad}_V \circ \pi$, where $\mathrm{Ad}_v \colon B(\mathscr{K}) \to B(\mathscr{H})$ is the NCP-map given by $\mathrm{Ad}_V(A) = V^* A V$ for all $A \in B(\mathscr{K})$.[1] If the linear span of $\{ \pi(a) V x \colon a \in \mathscr{A} \ x \in \mathscr{H} \}$ is dense in $\mathscr{K}$, then $(\mathscr{K}, \pi, V)$ is called **minimal**.

It is a well-known fact that all minimal normal Stinespring dilations of φ are unitarily equivalent (see e.g. [12, Prop. 4.2]). We will adapt its proof to show that a minimal Stinespring dilation admits a universal property (Prop. 13), which we will need later on. The adaptation is mostly straight-forward, except for the following lemma.

Lemma 11. *Let $\pi \colon \mathscr{A} \to \mathscr{B}$, $\pi' \colon \mathscr{A} \to \mathscr{C}$ be NMIU-maps between von Neumann algebras, and let $\sigma \colon \mathscr{C} \to \mathscr{B}$ be an NCP-map such that $\sigma \circ \pi' = \pi$. Then $\sigma(\pi'(a_1) c \pi'(a_2)) = \pi(a_1) \sigma(c) \pi(a_2)$ for any $a_1, a_2 \in \mathscr{A}$ and $c \in \mathscr{C}$.*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ \pi\ } & \mathscr{B} \\ & \searrow{\scriptstyle \pi'} & \uparrow{\scriptstyle \sigma} \\ & & \mathscr{C} \end{array}$$

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[1] Be warned: many authors prefer to define Ad_V by $\mathrm{Ad}_V(A) = VAV^*$ instead.

Proof. By Theorem 3.1 of [5], we know that, for all $c, d \in \mathscr{C}$,

$$\sigma(d^*d) = \sigma(d)^*\sigma(d) \quad \Longrightarrow \quad \sigma(cd) = \sigma(c)\sigma(d). \tag{1}$$

Let $a \in \mathscr{A}$. We have $\sigma(\pi'(a)^*\pi'(a)) = \sigma(\pi'(a^*a)) = \pi(a^*a) = \pi(a)^*\pi(a) = \sigma(\pi'(a))^*\sigma(\pi'(a))$. By (1), we have $\sigma(c\pi'(a)) = \sigma(c)\sigma(\pi'(a)) \equiv \sigma(c)\pi(a)$ for all $c \in \mathscr{C}$. Then also $\sigma(\pi'(a)c) = \pi(a)\sigma(c)$ for all $c \in \mathscr{C}$ (by taking adjoints). Thus $\sigma(\pi'(a_1)c\pi'(a_2)) = \pi(a_1)\sigma(c\pi'(a_2)) = \pi(a_1)\sigma(c)\pi(a_2)$ for all $a_1, a_2 \in \mathscr{A}$ and $c \in \mathscr{C}$. $\qquad\square$

Lemma 12. *Let $\mathscr{K}$ be a Hilbert space. If $\mathrm{Ad}_S = \mathrm{Ad}_T$ for $S, T \in B(\mathscr{K})$, then $S = \lambda T$ for some $\lambda \in \mathbb{C}$.*

Proof. Let $x \in \mathscr{K}$ be given, and let P be the projection onto $\{\lambda x \colon \lambda \in \mathbb{C}\}$. Then

$$\{\lambda S^*x \colon \lambda \in \mathbb{C}\} = \mathrm{Ran}(S^*PS) = \mathrm{Ran}(T^*PT) = \{\lambda T^*x \colon \lambda \in \mathbb{C}\}.$$

It follows that $S^*x = \alpha T^*x$ for some $\alpha \in \mathbb{C}$ with $\alpha \neq 0$. While α might depend on x, there is $\alpha_0 \in \mathbb{C}$ with $\alpha_0 \neq 0$ and $S^* = \alpha_0 T^*$ by Lemma 9 of [22]. Then $S = \alpha_0^* T$. $\qquad\square$

Proposition 13. *Let $(\mathscr{K}, \pi, V)$ and $(\mathscr{K}', \pi', V')$ be normal Stinespring dilations of φ. If $(\mathscr{K}, \pi, V)$ is minimal, then there is a unique isometry $S \colon \mathscr{K} \to \mathscr{K}'$ such that $SV = V'$ and $\pi = \mathrm{Ad}_S \circ \pi'$.*

Proof. Let us deal with a pathological case. If $V = 0$, then $\varphi = 0$, $V' = 0$, $\mathscr{K} = \{0\}$ and $\pi = 0$, and so the unique linear map $S \colon \{0\} \to \mathscr{K}'$ satisfies the requirements. Assume $V \neq 0$.
(Uniqueness) Let $S_1, S_2 \colon \mathscr{K} \to \mathscr{K}'$ be isometries with $S_k V = V'$ and $\mathrm{Ad}_{S_k} \circ \pi' = \pi$. We must show that $S_1 = S_2$. We have, for all, $a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n \in \mathscr{A}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathscr{H}$, and $c \in \mathscr{C}$,

$$
\begin{aligned}
\Big\langle \mathrm{Ad}_{S_k}(c)\sum_i \pi(a_i)Vx_i, \ \sum_j \pi(\alpha_j)Vy_j \Big\rangle &= \sum_{i,j} \big\langle V^*\pi(\alpha_j^*)\,\mathrm{Ad}_{S_k}(c)\,\pi(a_i)Vx_i, \ y_j \big\rangle && \text{by rearranging} \\
&= \sum_{i,j} \big\langle V^*\mathrm{Ad}_{S_k}(\,\pi'(\alpha_j^*)c\pi'(a_i)\,)Vx_i, \ y_j \big\rangle && \text{by Lemma 11} \\
&= \sum_{i,j} \big\langle (V')^*\pi'(\alpha_j^*)c\pi'(a_i)V'x_i, \ y_j \big\rangle && \text{as } S_k V = V'
\end{aligned}
$$

Since the linear span of $\pi(\mathscr{A})V\mathscr{H}$ is dense in $\mathscr{K}$, we get $\mathrm{Ad}_{S_1} = \mathrm{Ad}_{S_2}$. Thus $\lambda S_1 = S_2$ for some $\lambda \in \mathbb{C}$ by Lemma 12. Since $V \neq 0$, there is $x \in \mathscr{H}$ with $Vx \neq 0$. Then $S_1 Vx = V'x = S_2 Vx = \lambda S_1 Vx$, and so $\lambda = 1$. Thus $S_1 = S_2$, as desired.
(Existence) Note that for all $a_1, \ldots, a_n \in \mathscr{A}$ and $x_1, \ldots, x_n \in \mathscr{H}$, we have

$$\Big\| \sum_i \pi(a_i)Vx_i \Big\|^2 = \sum_{i,j} \big\langle V^*\pi(a_j^*a_i)Vx_i, x_j \big\rangle = \sum_{i,j} \big\langle \varphi(a_j^*a_i)x_i, x_j \big\rangle = \Big\| \sum_i \pi'(a_i)V'x_i \Big\|^2.$$

Hence there is a unique isometry $S \colon \mathscr{K} \to \mathscr{K}'$ such that $S\pi(a)Vx = \pi'(a)V'x$ for all $a \in \mathscr{A}$ and $x \in \mathscr{H}$. Since $SVx = S\pi(1)Vx = \pi'(1)V'x = V'x$ for all $x \in \mathscr{H}$, we have $SV = V'$. Further, for all $a, a_1, \ldots, a_n \in \mathscr{A}$, and $x_1, \ldots, x_n \in \mathscr{H}$, we have

$$S\pi(a)\sum_i \pi(a_i)Vx_i = \sum_i S\pi(aa_i)Vx_i = \sum_i \pi'(aa_i)V'x_i = \pi'(a)\sum_i \pi'(a_i)V'x_i = \pi'(a)S\sum_i \pi(a_i)Vx_i.$$

Since the linear span of $\pi(\mathscr{A})V\mathscr{H}$ is dense in $\mathscr{K}$, we get $S\pi(a) = \pi'(a)S$. Note that $S^*S = 1$, because S is an isometry. Thus $S^*\pi'(a)S = S^*S\pi(a) = \pi(a)$, and so $\mathrm{Ad}_S \circ \pi' = \pi$. $\qquad\square$

Theorem 14. *Let $(\mathscr{K}, \pi, V)$ be a minimal normal Stinespring dilation of an NCP-map $\varphi \colon \mathscr{A} \to B(\mathscr{H})$. Then $\mathscr{A} \xrightarrow{\pi} B(\mathscr{K}) \xrightarrow{V(\cdot)V^*} B(\mathscr{H})$ is a Paschke dilation of φ.*

Proof. Let $\mathscr{P}'$ be a von Neumann algebra. Let $\rho'\colon \mathscr{A} \to \mathscr{P}'$ be a NMIU-map, and $f'\colon \mathscr{P}' \to B(\mathscr{K})$ an NCP-map with $f' \circ \rho' = \varphi$. We must show that there is a unique NCP-map $\sigma\colon \mathscr{P}' \to B(\mathscr{K})$ with $\sigma \circ \rho' = \pi$ and $\mathrm{Ad}_V \circ \sigma = f'$. The uniqueness of σ follows by the same reasoning we used to show that $\mathrm{Ad}_{S_1} = \mathrm{Ad}_{S_2}$ in Proposition 13. To show such σ exists, let $(\mathscr{K}', \pi', V')$ be a minimal normal Stinespring dilation of f'. Note that $(\mathscr{K}', \pi' \circ \rho', V')$ is a normal Stinespring dilation of φ. Thus, by Proposition 13, there is a (unique) isometry $S\colon \mathscr{K} \to \mathscr{K}'$ such that $SV = V'$ and $\mathrm{Ad}_S \circ \pi' \circ \rho' = \pi$. Define $\sigma \equiv \mathrm{Ad}_S \circ \pi'$. Clearly $\sigma \circ \rho' = \mathrm{Ad}_s \circ \pi' \circ \rho' = \pi$ and $\mathrm{Ad}_V \circ \sigma = \mathrm{Ad}_V \circ \mathrm{Ad}_S \circ \pi' = \mathrm{Ad}_{V'} \circ \pi' = f'$, as desired. $\square$
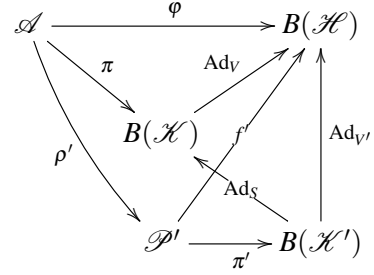
If we combine Theorem 14 with Lemma 9 we get the following.

Corollary 15. *Let $(\mathscr{K}, \pi, V)$ be a minimal normal Stinespring dilation of an NCP-map $\varphi\colon \mathscr{A} \to B(\mathscr{H})$, and let $\mathscr{A} \xrightarrow{\rho} \mathscr{P} \xrightarrow{f} B(\mathscr{H})$ be a Paschke dilation of φ. Then there is a unique NMIU-isomorphism $\vartheta\colon B(\mathscr{K}) \to \mathscr{P}$ with $\rho = \vartheta \circ \pi$ and $f \circ \vartheta = \mathrm{Ad}_V$.*

2   Existence of the Paschke Dilation

We will show that every *NCP*-map φ between von Neumann algebras has a Paschke dilation, see Theorem 18. For this we employ the theory of self-dual Hilbert $\mathscr{B}$-modules — developed by Paschke — which are, roughly speaking, Hilbert spaces in which the field of complex numbers has been replaced by a von Neumann algebra $\mathscr{B}$. Nowadays, the more general (not necessarily self-dual) Hilbert $\mathscr{B}$-modules, where $\mathscr{B}$ is a C^*-algebra, have become more prominent, and so it seems appropriate to point out from the get–go that both self-duality and the fact that $\mathscr{B}$ is a von Neumann algebra (also a type of self-duality, by the way) seem to be essential in the proof of Theorem 18.

We review the definitions and results we need from the theory of self-dual Hilbert $\mathscr{B}$-modules.

Overview 16. Let $\mathscr{B}$ be a von Neumann algebra.

1. A **pre-Hilbert $\mathscr{B}$-module** X (see Def. 2.1 of [11][2]) is a right $\mathscr{B}$-module equipped with a $\mathscr{B}$**-valued inner product**, that is, a map $\langle \cdot, \cdot \rangle\colon X \times X \to \mathscr{B}$ such that, for all $x, y, y' \in X$ and $b \in \mathscr{B}$,

   (a) $\langle x, (y+y')b \rangle = \langle x, y \rangle b + \langle x, y' \rangle b$;   (c) $\langle x, y \rangle^* = \langle y, x \rangle$.
   (b) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$;

2. A **Hilbert $\mathscr{B}$-module** (see Def. 2.4 of [11]) is a pre-Hilbert $\mathscr{B}$-module X which is complete with respect to the norm $\| \cdot \|$ on X given by $\|x\| = \sqrt{\| \langle x, x \rangle \|}$.

3. A Hilbert $\mathscr{B}$-module X is **self-dual** (see §3 of [11]) if every bounded module map $\tau\colon X \to \mathscr{B}$ is of the form $\tau = \langle x, \cdot \rangle$ for some $x \in X$.

4. Self-duality is essential for the following result. Let $T\colon X \to Y$ be a bounded module map between Hilbert $\mathscr{B}$-modules. If X is self-dual, then it is 'adjointable'; that is: there is a unique bounded module map $T^*\colon Y \to X$, called the **adjoint** of T, with $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in X$ and $y \in Y$ (see Prop. 3.4 of [11]).

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[2]Although in [11] $\langle x, y \rangle$ is linear in x and anti-linear in y, we have chosen to adopt the now dominant convention that $\langle x, y \rangle$ is anti-linear in x and linear in y.

5. Let X be a pre-Hilbert $\mathscr{B}$-module. One can extend X to a self-dual Hilbert $\mathscr{B}$-module as follows. The set X' of bounded module maps from $\tau\colon X \to \mathscr{B}$ is called the **dual** of X, and is a $\mathscr{B}$-module via $(\tau \cdot b)(x) = b^* \cdot \tau(x)$ (see line 4 of p. 450 of [11]). Note that X sits inside X' via the injective module map $x \mapsto \hat{x} \equiv \langle x, \cdot \rangle$. In fact, X' can be equipped with an $\mathscr{B}$-valued inner product that makes X' into a self-dual Hilbert $\mathscr{B}$-module with $\langle \tau, \hat{x} \rangle = \tau(x)$ for all $\tau \in X'$ and $x \in X$ (see Thm. 3.2 of [11]).

6. Any bounded module map $T\colon X \to Y$ between pre-Hilbert $\mathscr{B}$-modules has a unique extension to a bounded module map $\tilde{T}\colon X' \to Y'$ (see Prop. 3.6 of [11]).

   It follows that any bounded module map $T\colon X \to Y$ from a pre-Hilbert $\mathscr{B}$-module into a self-dual Hilbert $\mathscr{B}$-module has a unique extension $\overline{T}\colon X' \to Y$.

7. Let X be a self-dual Hilbert $\mathscr{B}$-module. The set $B^a(X)$ of bounded module maps on X forms a von Neumann algebra (see Prop. 3.10 of [11]).[3] Addition and scalar multiplication are computed coordinate-wise in $B^a(X)$; multiplication is given by composition, and involution is the adjoint.

   An element t of $B^a(X)$ is positive iff $\langle x, x \rangle \geq 0$ for all $x \in X$ (see Lem 4.1 of [9]). If X happens to be the dual of a pre-Hilbert $\mathscr{B}$-module X_0, then we even have $t \geq 0$ iff $\langle x, tx \rangle \geq 0$ for all $x \in X_0$.

   It follows that T^*T is positive in $B^a(X)$ for any bounded module map $T\colon X \to Y$.

   We will also need the fact that $\langle x, (\cdot)x \rangle \colon B^a(X) \to \mathscr{B}$ is normal for every $x \in X$, which follows from the observation that $f(\langle x, (\cdot)x \rangle) \colon B^a(X) \to \mathbb{C}$ is normal for every positive normal map $f\colon \mathscr{B} \to \mathbb{C}$, which in turn follows form the description of the predual of $B^a(X)$ in Proposition 3.10 of [11].

Definition 17. Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be an NCP-map between von Neumann algebras. A complex bilinear map of the form $B\colon \mathscr{A} \times \mathscr{B} \to X$, where X is a self-dual Hilbert $\mathscr{B}$-module, is called φ**-compatible** if, there is $r > 0$ such that, for all $a_1, \ldots, a_n \in \mathscr{A}$ and $b_1, \ldots, b_n \in \mathscr{B}$,

$$\Big\| \sum_i B(a_i, b_i) \Big\|^2 \;\leq\; r \cdot \Big\| \sum_{i,j} b_i^* \varphi(a_i^* a_j) b_j \Big\|, \tag{2}$$

and $B(a, b_1)b_2 = B(a, b_1 b_2)$ for all $a \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$.

Theorem 18. *Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be an NCP-map between von Neumann algebras.*

1. *There is a self-dual Hilbert $\mathscr{B}$-module $\mathscr{A} \otimes_\varphi \mathscr{B}$ and a φ-compatible bilinear map*

$$\otimes\colon \mathscr{A} \times \mathscr{B} \to \mathscr{A} \otimes_\varphi \mathscr{B}$$

   such that for every φ-compatible bilinear map $B\colon \mathscr{A} \times \mathscr{B} \to Y$ there is a unique bounded module map $T\colon \mathscr{A} \otimes_\varphi \mathscr{B} \longrightarrow Y$ such that $T(a \otimes b) = B(a, b)$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.

2. *For every $a_0 \in \mathscr{A}$ there is a unique bounded module map $\rho(a_0)$ on $\mathscr{A} \otimes_\varphi \mathscr{B}$ given by*

$$\rho(a_0)(a \otimes b) \;=\; (a_0 a) \otimes b,$$

   and the assignment $a \mapsto \rho(a)$ yields an NMIU-map $\rho\colon \mathscr{A} \to B^a(\mathscr{A} \otimes_\varphi \mathscr{B})$.

3. *The assignment $T \mapsto \langle 1 \otimes 1, T(1 \otimes 1) \rangle$ gives an NCP-map $f\colon B^a(\mathscr{A} \otimes_\varphi \mathscr{B}) \longrightarrow \mathscr{B}$.*

4. *$\mathscr{A} \xrightarrow{\ \rho\ } B^a(\mathscr{A} \otimes_\varphi \mathscr{B}) \xrightarrow{\ f\ } \mathscr{B}$ is a Paschke dilation of φ.*

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[3]The superscript a in $B^a(X)$ stands for adjointable, which is automatic for bounded module-maps on a self-dual X.

Proof. *(1)* To construct $\mathscr{A} \otimes_\varphi \mathscr{B}$, we follow the lines of Theorem 5.2 of [11], and start with the algebraic tensor product, $\mathscr{A} \odot \mathscr{B}$, whose elements are finite sums of the form $\sum_i a_i \otimes b_i$, and which is a right $\mathscr{B}$-module via $(\sum_i a_i \otimes b_i)\beta = \sum_i a_i \otimes (b_i\beta)$. If we define $[\,\cdot\,,\,\cdot\,]$ on $\mathscr{A} \odot \mathscr{B}$ by

$$\Big[\sum_i a_i \otimes b_i, \ \sum_j \alpha_j \otimes \beta_j \Big] \ = \ \sum_{i,j} b_i^* \varphi(a_i^* \alpha_j)\beta_j,$$

we get a $\mathscr{B}$-valued semi-inner product on $\mathscr{A} \odot \mathscr{B}$, and a (proper) $\mathscr{B}$-valued inner product on the quotient $X_0 = (\mathscr{A} \odot \mathscr{B})/N$, where $N = \{x \in \mathscr{A} \odot \mathscr{B} : [x,x] = 0\}$, so that X_0 is a pre-Hilbert $\mathscr{B}$-module. (That N is a submodule of $\mathscr{A} \odot \mathscr{B}$ is not entirely obvious, see Remark 2.2 of [11].) Now, define $\mathscr{A} \otimes_\varphi \mathscr{B} := X_0'$ (where X_0' is the dual of X_0 from Overview 16(5)), and let $\otimes \colon \mathscr{A} \times \mathscr{B} \to \mathscr{A} \otimes_\varphi \mathscr{B}$ be given by $a \otimes b = a \widehat{\otimes b + N}$. Then $\mathscr{A} \otimes_\varphi \mathscr{B}$ is a self-dual Hilbert $\mathscr{B}$-module, and $\otimes$ is a φ-compatible bilinear map.

Let $B \colon \mathscr{A} \times \mathscr{B} \to Y$ be a φ-compatible bilinear map to some self-dual Hilbert $\mathscr{B}$-module Y. We must show that there is a unique bounded module map $T \colon \mathscr{A} \otimes_\varphi \mathscr{B} \to Y$ such that $T(a \otimes b) = B(a,b)$. By Overview 16(6), it suffices to show that there is a unique bounded module map $T \colon X_0 \to Y$ with $T(a \otimes b) = B(a,b)$. Since $\mathscr{A} \odot \mathscr{B}$ is generated by elements of the form $a \otimes b$, uniqueness is obvious. Concerning existence, there is a (unique) linear map $S \colon \mathscr{A} \odot \mathscr{B} \to X$ with $S(a \otimes b) = B(a,b)$ by the universal property of the algebraic tensor product. Note that the kernel of S contains N, because if $x = \sum_i a_i \otimes b_i$ is from N, then $[x,x] = \sum_{i,j} b_i^* \varphi(a_i^* a_j) b_j = 0$, and so $\|S(x)\| \equiv \|\sum_i B(a_i, b_i)\| = 0$ by Equation (2). Thus there is a unique linear map $T \colon X_0 \to Y$ with $T(a \otimes b) = B(a,b)$. By Equation (2), T is bounded. Finally, since $B(a, b\beta) = B(a,b)\beta$, it is easy to see that S and T are module maps.

(2) Let $a_0 \in \mathscr{A}$ be given. To obtain the bounded module map $\rho(a_0)$ on $\mathscr{A} \otimes_\varphi \mathscr{B}$, it suffices to show the bilinear map $B \colon \mathscr{A} \times \mathscr{B} \to \mathscr{A} \otimes_\varphi \mathscr{B}$ given by $B(a,b) = (a_0 a) \otimes b$ is φ-compatible. It is easy to see that $B(a,b)\beta = B(a,b\beta)$. Concerning Equation (2), let $a_1, \ldots, a_n \in \mathscr{A}$ and $b_1, \ldots, b_n \in \mathscr{B}$ be given. Then we have, writing a for the row vector $(a_1 \cdots a_n)$, b for the column vector $(b_1 \cdots b_n)$,

$$\Big\| \sum_i B(a_i, b_i) \Big\|^2 \ = \ \Big\| \sum_i (a_0 a_i) \otimes b_i \Big\|^2 \ = \ \Big\| \sum_{i,j} b_i^* \varphi(a_i^* a_0^* a_0 a_j) b_j \Big\| \ = \ \|b^*(M_n\varphi)(a^* a_0^* a_0 a)b\|$$

$$\leq \ \|a_0^* a_0\| \cdot \|b^*(M_n\varphi)(a^* a)b\| \ = \ \|a_0\|^2 \cdot \Big\| \sum_{i,j} b_i^* \varphi(a_i^* a_j) b_j \Big\|.$$

Thus B is φ-compatible, and so there is a unique bounded module map $\rho(a_0) \colon \mathscr{A} \otimes_\varphi \mathscr{B} \longrightarrow \mathscr{A} \otimes_\varphi \mathscr{B}$ with $\rho(a_0)(a \otimes b) = (a_0 a) \otimes b$. Since it is easy to see that $a_0 \mapsto \rho(a_0)$ gives a multiplicative involutive unital linear map $\rho \colon \mathscr{A} \to B^a(\mathscr{A} \otimes_\varphi \mathscr{B})$, the only thing left to prove is that ρ is normal.

Let D be a bounded directed set of self-adjoint elements of $\mathscr{A}$. To show that ρ is normal, we must prove that $\rho(\sup D) = \sup_{d \in D} \rho(d)$. It suffices to show that $\langle x, \rho(\sup D)x \rangle = \langle x, \sup_{d \in D} \rho(d)x \rangle$ for all $x \in X_0$. Let $x \in X_0$ be given and write $x = \sum_i a_i \otimes b_i$, where $a_1, \ldots, a_n \in \mathscr{A}$ and $b_1, \ldots, b_n \in \mathscr{A}$. Then, if a stands for the row vector $(a_1 \cdots a_n)$ and b is the column vector $(b_1, \ldots, b_n)$, we have

$$\langle x, \rho(\sup D)x \rangle \ = \ b^*(M_n\varphi)(a^* \sup D\, a)b \ = \ \sup_{d \in D} b^*(M_n\varphi)(a^* da)b \ = \ \sup_{d \in D} \langle x, \rho(d)x \rangle \ = \ \langle x, \sup_{d \in D} \rho(d)x \rangle,$$

where we used that $b^*(M_n\varphi)(a^*(\,\cdot\,)a)b$ and $\langle x, (\,\cdot\,)x \rangle$ are normal. Thus ρ is normal.

(3) Write $e = 1 \otimes 1$. We already know that $\langle e, (\,\cdot\,)e \rangle$ is normal, and since for $t_1, \ldots, t_n \in B^a(\mathscr{A} \otimes_\varphi \mathscr{B})$ and $b_1, \ldots, b_n \in \mathscr{B}$, we have $\sum_{i,j} b_i^* \langle e, t_i^* t_j e \rangle b_j = \langle \sum_i t_i e b_i, \sum_j t_j e b_j \rangle \geq 0$, we see by Remark 5.1 of [11], that $\langle e, (\,\cdot\,)e \rangle$ is completely positive.

(4) To begin, note that $(f \circ \rho)(a) = \langle 1 \otimes 1, a \otimes 1 \rangle = \varphi(a)$ for all $a \in \mathscr{A}$, and so $\varphi = f \circ \rho$.

Suppose that φ factors as $\mathscr{A} \xrightarrow{\rho'} \mathscr{P}' \xrightarrow{f'} \mathscr{B}$, where $\mathscr{P}'$ is a von Neumann algebra, ρ' is an NMIU-map, and f' is an NCP-map. We must show that there is a unique NCP-map $\sigma\colon \mathscr{P}' \to B^a(\mathscr{A} \otimes_\varphi \mathscr{B})$ with $f \circ \sigma = f'$ and $\sigma \circ \rho' = \rho$.

(Uniqueness) Let $\sigma_1, \sigma_2\colon \mathscr{P} \to B^a(\mathscr{A} \otimes_\varphi \mathscr{B})$ be NCP-maps with $f \circ \sigma_k = f'$ and $\sigma_k \circ \rho' = \rho$. We must show that $\sigma_1 = \sigma_2$. Let $c \in \mathscr{P}'$ and $x \in X_0$ be given. It suffices to prove that $\langle x, \sigma_1(c)x \rangle = \langle x, \sigma_2(c)x \rangle$ (see Overview 16(7)). Write $x = \sum_i a_i \otimes b_i$ where $a_i \in \mathscr{A}$, $b_i \in \mathscr{B}$. Then, $a_i \otimes b_i = \rho(a_i)(1 \otimes 1)b_i$, and so

$$
\begin{aligned}
\langle x, \sigma_i(c)x \rangle &= \sum_{i,j} b_i^* f(\rho(a_i^*)\sigma_i(c)\rho(a_j))b_j && \text{by an easy computation} \\
&= \sum_{i,j} b_i^* f(\sigma_i(\rho'(a_i^*)c\rho'(a_j)))b_j && \text{by Lemma 11} \\
&= \sum_{i,j} b_i^* f'(\rho'(a_i^*)c\rho'(a_j))b_j && \text{since } f' = f \circ \sigma_i.
\end{aligned}
\tag{3}
$$

Hence $\langle x, \sigma_1(x) \rangle = \langle x, \sigma_2(x) \rangle$, and so $\sigma_1 = \sigma_2$.

(Existence) Recall that each self-adjoint bounded operator A on Hilbert space $\mathscr{H}$ gives a bounded quadratic form $x \mapsto \langle x, Ax \rangle$, that every quadratic form arises in this way, and that the operator A can be reconstructed from its quadratic form. One can develop a similar correspondence in the case of Hilbert $\mathscr{B}$-modules, which can be used to define σ from Equation (3). We will, however, give a shorter proof of the existence of σ, which was suggested to us by Michael Skeide.

The trick is to see that the construction that gave us $\mathscr{A} \otimes_\varphi \mathscr{B}$ may also be applied to $f'\colon \mathscr{P}' \to \mathscr{B}$ yielding maps $\mathscr{P}' \xrightarrow{\rho''} B^a(\mathscr{P}' \otimes_{f'} \mathscr{B}) \xrightarrow{f''} \mathscr{B}$. It suffices to find an NCP-map $\sigma'\colon B^a(\mathscr{P}' \otimes_{f'} \mathscr{B}) \longrightarrow B^a(\mathscr{A} \otimes_\varphi \mathscr{B})$ with $f'' = f \circ \sigma'$ and $\rho = \sigma' \circ \rho'' \circ \rho'$ for then $\sigma = \sigma' \circ \rho''$ will have the desired properties.

Let $S\colon \mathscr{A} \otimes_\varphi \mathscr{B} \longrightarrow \mathscr{P}' \otimes_{f'} \mathscr{B}$ be the bounded module map given by $S(a \otimes b) = \rho'(a) \otimes b$, which exists by part 1, because a straightforward computation shows that $(a,b) \mapsto \rho'(a) \otimes b$ gives a φ-compatible bilinear map $\mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{P}' \otimes_{f'} \mathscr{B}$. We claim that $\sigma' = S^*(\,\cdot\,)S$ fits the bill.

Let us begin by proving that $\sigma'(1) \equiv S^*S = 1$. Let $x \in X_0$. It suffices to show that $\langle x, S^*Sx \rangle = \langle x, x \rangle$. Writing $x \equiv \sum_i a_i \otimes b_i$, we have $\langle x, S^*Sx \rangle = \langle Sx, Sx \rangle = \sum_{i,j} b_i^* f'(\rho'(a_i^*)\rho'(a_j))b_j = \sum_{i,j} b_i^* \varphi(a_i^*a_j)b_j = \langle x, x \rangle$, because ρ' is multiplicative and $f' \circ \rho' = \varphi$. Thus $S^*S = 1$.

Note that σ' is completely positive, because for all $s_1, \ldots, s_n \in B^a(\mathscr{P}' \otimes_{f'} \mathscr{B})$ and $t_1, \ldots, t_n \in B^a(\mathscr{A} \otimes_\varphi \mathscr{B})$, we have $\sum_{i,j} t_i^* \sigma'(s_i^* s_j)t_j = (\sum_i s_i St_i)^*(\sum_j s_j St_j) \geq 0$ (see Remark 5.1 of [11]).

Let $x \in \mathscr{A} \otimes_\varphi \mathscr{B}$ be given. Note that $\langle x, \sigma'(\,\cdot\,)x \rangle = \langle Sx, (\,\cdot\,)Sx \rangle$ is normal. From this it follows that σ' is normal (in the same way we proved that ρ is normal in (2)).

Since from $S(1 \otimes 1) = \rho'(1) \otimes 1 = 1 \otimes 1$ it swiftly follows that $f \circ \sigma' = f''$, the only thing left to show is that $\rho = \sigma' \circ \rho'' \circ \rho'$. Let $a, a_0 \in \mathscr{A}$ and $b \in \mathscr{B}$ be given. By point (1), it suffices to show that $\rho(a_0)(a \otimes b) = \sigma'(\rho''(\rho'(a_0)))(a \otimes b)$. Unfolding gives $\sigma'(\rho''(\rho'(a_0)))(a \otimes b) = S^*(\rho'(a_0a) \otimes b) = S^*S(\rho(a_0)(a \otimes b))$, but we already saw that $S^*S = 1$, and so we are done. $\qquad\square$

3 Pure maps

Schrödinger's equation is invariant under the reversal of time, and so any isolated purely quantum mechanical process is invertible. However, NCP-maps include non-invertible processes such as measurement and discarding. However, not all is lost, for a broad class of processes is pure enough to be 'reversed', e.g. $(\mathrm{Ad}_V)^\dagger = \mathrm{Ad}_{V^*}$. In fact, the Stinespring dilation theorem states that every NCP-map into $B(\mathscr{H})$ factors as a reversible Ad_V, after a (possibly) non-reversible NMIU-map.

In this section we study two seemingly unrelated definitions of pure (i.e. reversible) for arbitrary NCP-maps. The first is a direct generalization of Ad_V, and the second uses the Paschke dilation. Both definitions turn out to be equivalent.

Before we continue, let's rule out two alternative definitions of pure. 1. Recall a state $\varphi \colon \mathscr{A} \to \mathbb{C}$ is called pure, if is an extreme point among all states. It does not make sense to define an NCP-map to be pure if it is extreme, because every NMIU-map is extreme (among the unital NCP-maps). 2. Inspired by the GNS-correspondence between pure states and irreducible representations, Størmer defines a map $\varphi \colon \mathscr{A} \to B(\mathscr{H})$ to be pure if the only maps below φ in the completely positive order are scalar multiples of φ. One can show that the Størmer pure NCP-maps between $B(\mathscr{H}) \to B(\mathscr{K})$ are exactly those of the form Ad_V, see Prop. 33. In a non-factor every central element gives a different completely positive map below the identity and so if one generalizes Størmer's definition to arbitrary NCP-maps, the identity need not be pure.

Now, let us sketch our first definition of pure. Consider $V \colon \mathscr{K} \to \mathscr{H}$. By polar decomposition we may factor $V = UA$, where $A \colon \mathscr{K} \to r(A)\mathscr{H}$ is a positive map and $U \colon r(A)\mathscr{H} \to \mathscr{K}$ is an isometry and so $\text{Ad}_V = \text{Ad}_A \circ \text{Ad}_U$. The pure maps Ad_A and Ad_U are of a particularly simple form. We will see they admit a dual universal property, the first is (up to scaling) a *compression* and the second a *corner* (Def. 19). This allows us to generalize the notion of pure to arbitrary NCP-maps (Def. 21). We show maps on the right-hand side of a Paschke dilation are pure. Then we will show the main result of the section: an NCP-map is pure if and only if the map on the left-hand side of its Paschke dilation is surjective. As a corollary, we show both the left- and right-hand side of a Paschke dilation are extreme among the maps with the same value on 1.

Definition 19. Let a be an element of a von Neumann algebra $\mathscr{A}$ with $0 \leq a \leq 1$.

1. The least projection above a, we call the **support projection** of a, and we denote it as $\lceil a \rceil$. Its de Morgan dual $\lfloor a \rfloor \equiv 1 - \lceil 1 - a \rceil$ is the greatest projection below a. For any projection p, write C_p for the **central carrier**, that is: the least central projection above p (see e.g. [6, Def. 5.5.1]).

2. For an NCP-map $\varphi \colon \mathscr{A} \to \mathscr{B}$, we write $\text{car}\,\varphi$ for the **carrier** of φ, the least projection of $\mathscr{A}$ such that $\varphi(\text{car}\,\varphi) = \varphi(1)$. The map φ is said to be **faithful** if $\text{car}\,\varphi = 1$. Equivalently, φ is faithful if $\varphi(a^*a) = 0$ implies $a^*a = 0$ for all $a \in \mathscr{A}$.

3. We call the map $h_a \colon \mathscr{A} \to \lfloor a \rfloor \mathscr{A} \lfloor a \rfloor$ given by $b \mapsto \lfloor a \rfloor b \lfloor a \rfloor$, the **standard corner** of a.

4. We call the map $c_a \colon \lceil a \rceil \mathscr{A} \lceil a \rceil \to \mathscr{A}$ given by $b \mapsto \sqrt{a}b\sqrt{a}$, the **standard compression** of a.

5. A contractive NCP-map $h \colon \mathscr{A} \to \mathscr{B}$ is said to be a **corner** for an $a \in [0,1]_{\mathscr{A}}$ if $h(a) = h(1)$ and for every (other) contractive NCP-map $f \colon \mathscr{A} \to \mathscr{C}$ with $f(a) = f(1)$, there is a unique $f' \colon \mathscr{B} \to \mathscr{C}$ with $f = f' \circ h$.

6. A contractive NCP-map $c \colon \mathscr{B} \to \mathscr{A}$ is said to be a **compression** for an $a \in [0,1]_{\mathscr{A}}$ if $c(1) = a$ and for every (other) contractive NCP-map $g \colon \mathscr{C} \to \mathscr{A}$ with $g(1) \leq a$, there is a unique $g' \colon \mathscr{C} \to \mathscr{B}$ with $g = c \circ g'$.

Proposition 20. *Let $\mathscr{A}$ be any von Neumann algebra with effect $a \in [0,1]_{\mathscr{A}}$.*

1. *The standard corner h_a is a corner for a and the standard compression c_a is a compression for a.*

2. *Corners are surjective. Compressions are injective. Restricted to self-adjoint elements, compressions are order-embeddings.*

3. *Every corner h for a is of the form $h = \vartheta \circ h_a$ for some isomorphism ϑ. Every compression c for a is of the form $c = c_a \circ \vartheta$ for some isomorphism ϑ.*

4. Assume $\varphi \colon \mathscr{A} \to \mathscr{B}$ is a contractive NCP-map. Let φ' denote the unique contractive NCP-map such that $c_{\varphi(1)} \circ \varphi' = \varphi$ and φ'' the unique contractive NCP-map with $\varphi'' \circ h_{\mathrm{car}\,\varphi} = \varphi$. Then φ' is unital and φ'' is faithful.

5. With φ as above, there is a unique NCP-map $\varphi_{\measuredangle} \colon (\mathrm{car}\,\varphi)\mathscr{A}(\mathrm{car}\,\varphi) \to \lceil \varphi(1) \rceil \mathscr{B} \lceil \varphi(1) \rceil$ such that $c_{\varphi(1)} \circ \varphi_{\measuredangle} \circ h_{\mathrm{car}\,\varphi} = \varphi$. The map $\varphi_{\measuredangle}$ is faithful and unital.

Proof of Proposition 20. The proposition is true in greater generality.[4] We give a direct proof.

1. Proven in [22, Prop. 5 & 6].

2. Let $c \colon \mathscr{A} \to \mathscr{B}$ be any compression. Assume $a, a' \in \mathscr{A}$ are self-adjoint elements with $c(a) \leq c(a')$. Without loss of generality, we may assume $a, a' \geq 0$ as $\|a\| + a, \|a\| + a' \geq 0$. Define $p_{\alpha} \colon \mathscr{C} \to \mathscr{B}$ by $p_{\alpha}(1) = \alpha$ for $\alpha \in \mathscr{A}$. Note $0 \leq c(a' - a) \leq c(1)$ and so by the universal property of c, there is a unique map $f' \colon \mathbb{C} \to \mathscr{A}$ with $c \circ f' = p_{c(a'-a)}$. We compute

$$c(p_{\frac{1}{2}a'}(1)) = \frac{1}{2}c(a') = \frac{c(a) + p_{c(a'-a)}(1)}{2} = \frac{c(a) + c(f'(1))}{2} = c(p_{\frac{1}{2}(a+f'(1))}(1))$$

and so by the universal property of c, we have $p_{\frac{1}{2}a'} = p_{\frac{1}{2}(a+f'(1))}$ and so $a \leq a'$, as desired. As a corollary, c is injective on self-adjoint elements. It follows c is injective. (See e.g. [3, Proof Lemma 4.2]). Clearly isomorphisms and a standard corner are surjective. Thus by point 3, every corner is surjective.

3. A standard argument gives us that there are mediating NCP-isomorphisms. They are actual NMIU-isomorphisms by e.g. [22, Corollary 47].

4. Write $u_{\mathscr{A}} \colon \mathbb{C} \to \mathscr{A}$ for the NCP-map $u_{\mathscr{A}}(\lambda) = \lambda \cdot 1$. Then $\varphi \circ u_{\mathscr{A}} = c_{\varphi(1)} \circ \varphi' \circ u_{\mathscr{A}} = c_{\varphi(1)} \circ u_{\lceil \varphi(1) \rceil \mathscr{B} \lceil \varphi(1) \rceil}$ and so by the universal property of $c_{\varphi(1)}$, we get $\varphi' \circ u_{\mathscr{A}} = u_{\lceil \varphi(1) \rceil \mathscr{B} \lceil \varphi(1) \rceil}$ and so $1 = \varphi'(1)$, as desired. Write $p \equiv \mathrm{car}\,\varphi$. Now, to show φ'' is faithful, assume $\varphi''(pap) = 0$ for some $pap \in [0,1]_{p\mathscr{A}p}$. Then $0 = \varphi''(pap) = \varphi''(h_p(pap)) = \varphi(pap)$ and so $pap \leq 1 - \mathrm{car}\,\varphi = 1 - p$. Hence $pap = 0$, as desired.

5. By point 2, we know $\varphi_{\measuredangle}$ is unique. Note $c_{\varphi(1)} \circ \varphi_{\measuredangle}(1) = \varphi(1)$ and $\mathrm{car}(\varphi_{\measuredangle} \circ h_{\mathrm{car}\,\varphi}) = \mathrm{car}\,\varphi$ and so $\varphi_{\measuredangle}$ is unital and faithful by the previous point. $\qquad\square$

In the previous Definition and Proposition we chose to restrict ourselves to contractive maps as a 'non-contractive compression' might sound confusing. For a non-contractive φ, define $\varphi_{\measuredangle} := (\frac{1}{\|\varphi\|} \cdot \varphi)_{\measuredangle}$.

Definition 21. A NCP-map $\varphi \colon \mathscr{A} \to \mathscr{B}$ is said to be **pure** whenever $\varphi_{\measuredangle}$ is an isomorphism.

Example 22. The pure NCP-maps between $B(\mathscr{H})$ and $B(\mathscr{K})$ are exactly those of the form Ad_V for some $V \colon \mathscr{K} \to \mathscr{H}$. The contractive pure maps are exactly those with $V^*V \leq 1$.

Example 23. If $\varphi \colon \mathscr{A} \to \mathscr{B}$ is unital and faithful, then $\varphi_{\measuredangle} = \varphi$. Thus a state is pure if and only if it is faithful. Also isomorphisms are pure.

Proposition 24. *Corners, compressions and pure maps are closed under composition.*

Proof. First we show corners are closed under composition. Let $h \colon \mathscr{A} \to \mathscr{B}$ be a corner for a and $h' \colon \mathscr{B} \to \mathscr{C}$ be a corner for b. There is a unique isomorphism $\vartheta \colon \lfloor a \rfloor \mathscr{A} \lfloor a \rfloor \to \mathscr{B}$ such that $h = \vartheta \circ h_{\lfloor a \rfloor}$. Note $h_{\lfloor a \rfloor} \circ c_{\lfloor a \rfloor} = \mathrm{id}$ and so $h \circ c_{\lfloor a \rfloor} \circ \vartheta^{-1} = \mathrm{id}$. We will show $h' \circ h$ is a corner for $c_{\lfloor a \rfloor}(\vartheta^{-1}(\lfloor b \rfloor))$. To this end, assume $g \colon \mathscr{A} \to \mathscr{D}$ is any contractive NCP-map for which it holds $g(c_{\lfloor a \rfloor}(\vartheta^{-1}(\lfloor b \rfloor))) = g(1)$.

Clearly $g(1) = g(c_{\lfloor a \rfloor}(\vartheta^{-1}(\lfloor b \rfloor))) = g(1) \leq g(\lfloor a \rfloor) \leq g(1)$ and so by the universal property of h, there is a unique contractive NCP-map $g' \colon \mathscr{B} \to \mathscr{D}$ such that $g' \circ h = g$. Now

$$g'(\lfloor b \rfloor) = g'(h(c_{\lfloor a \rfloor}(\vartheta^{-1}(\lfloor b \rfloor)))) = g(c_{\lfloor a \rfloor}(\vartheta^{-1}(\lfloor b \rfloor))) = g(1) = g'(h(1)) = g'(1)$$

and so by the universal property of h' there is a unique contractive NCP-map $g'' \colon \mathscr{C} \to \mathscr{D}$ with $g'' \circ h' = g'$. Clearly $g'' \circ h' \circ h = g$. It is easy to see g'' is unique.

We continue with compressions. Assume $c \colon \mathscr{C} \to \mathscr{B}$ is a compression for b and $c' \colon \mathscr{B} \to \mathscr{A}$ is a compression for a. We will show $c' \circ c$ is a compression for $c'(b)$. To this end, let $g \colon \mathscr{D} \to \mathscr{A}$ be any contractive NCP-map such that $g(1) \leq c'(b)$. As $g(1) \leq c'(b) \leq a$, there is a unique $g' \colon \mathscr{D} \to \mathscr{B}$ with $c' \circ g' = g$. Clearly $c'(g'(1)) = g(1) \leq c'(b)$. Thus $g'(1) \leq b$ and so there is a unique contractive NCP-map $g'' \colon \mathscr{D} \to \mathscr{C}$ such that $c \circ g'' = g'$. Now $c' \circ c \circ g'' = g$ It is easy to see g'' is unique.

To show pure maps are closed under composition, it is sufficient to show that $h_p \circ c_a$ is pure for any von Neumann algebra $\mathscr{A}$, projection $p \in \mathscr{A}$ and effect $a \in [0,1]_{\mathscr{A}}$. To this end, we will define a u such that the diagram on the right makes sense, commutes and Ad_u is an isomorphism. First some facts.

$$
\begin{array}{ccc}
 & \mathscr{A} & \\
{}^{c_a}\nearrow & & \searrow^{h_p} \\
\lceil a \rceil \mathscr{A} \lceil a \rceil & & p\mathscr{A}p \\
{}^{h_{\lceil \sqrt{a}p\sqrt{a} \rceil}}\downarrow & & \uparrow^{c_{pap}} \\
\lceil \sqrt{a}p\sqrt{a} \rceil \mathscr{A} \lceil \sqrt{a}p\sqrt{a} \rceil & \xrightarrow[\mathrm{Ad}_u]{} & \lceil pap \rceil \mathscr{A} \lceil pap \rceil
\end{array}
$$

1. By polar decomposition (see e.g. [19, p.15]), there is a partial isometry $u \in \mathscr{A}$ such that $\sqrt{a}p = u\sqrt{pap}$ with domain $u^*u = \lceil pap \rceil$ and range $uu^* = r(\sqrt{a}p) = r(\sqrt{a}p(\sqrt{a}p)^*) = \lceil \sqrt{a}p\sqrt{a} \rceil$, where with $r(b)$ we denote the projection onto the closed range of b. Note that adjoining by u restricts to an isomorphism $\mathrm{Ad}_u \colon \lceil \sqrt{a}p\sqrt{a} \rceil \mathscr{A} \lceil \sqrt{a}p\sqrt{a} \rceil \to \lceil pap \rceil \mathscr{A} \lceil pap \rceil$.

2. We have $\sqrt{a}p\sqrt{a} \leq a \leq \lceil a \rceil$ and so $\sqrt{a}p\sqrt{a} \in \lceil a \rceil \mathscr{A} \lceil a \rceil$. Clearly $pap \in p\mathscr{A}p$. Hence

$$\lceil \sqrt{a}p\sqrt{a} \rceil (\lceil a \rceil \mathscr{A} \lceil a \rceil) \lceil \sqrt{a}p\sqrt{a} \rceil = \lceil \sqrt{a}p\sqrt{a} \rceil \mathscr{A} \lceil \sqrt{a}p\sqrt{a} \rceil$$
$$\lceil pap \rceil (p\mathscr{A}p) \lceil pap \rceil = \lceil pap \rceil \mathscr{A} \lceil pap \rceil.$$

3. We have $ur(u)u = r(uu^*)u = uu^*u = \lceil \sqrt{a}p\sqrt{a} \rceil u$.

Now we see the diagram makes sense and commutes:

$$
\begin{aligned}
h_p(c_a(\lceil a \rceil x \lceil a \rceil)) &= p\sqrt{a}\lceil a \rceil x \lceil a \rceil \sqrt{a}p \\
&= \sqrt{pap}u^*\lceil a \rceil x \lceil a \rceil u\sqrt{pap} \\
&= \sqrt{pap}u^*\lceil \sqrt{a}p\sqrt{a} \rceil \lceil a \rceil x \lceil a \rceil \lceil \sqrt{a}p\sqrt{a} \rceil u\sqrt{pap} \\
&= c_{pap}(\mathrm{Ad}_u(h_{\lceil \sqrt{a}p\sqrt{a} \rceil}(\lceil a \rceil x \lceil a \rceil))).
\end{aligned}
$$

Consequently $h_p \circ c_a$ is pure. $\qquad\square$

Remark 25. On the category of von Neumann algebra with pure maps, one may define a dagger which turns it into a dagger category. It is not directly clear it is unique, but with additional assumptions is can be shown to be unique. This is beyond the scope of this paper and will appear elsewhere.

Proposition 26. *Let* $\varphi \colon \mathscr{A} \to \mathscr{B}'$ *be an NCP-map with Paschke dilation* $\mathscr{A} \xrightarrow{\rho} \mathscr{P} \xrightarrow{f} \mathscr{B}'$. *Let* $b \in [0,1]_{\mathscr{B}}$ *together with a compression* $c \colon \mathscr{B}' \to \mathscr{B}$ *for* b. *Then* $\mathscr{A} \xrightarrow{\rho} \mathscr{P} \xrightarrow{c \circ f} \mathscr{B}$ *is a Paschke dilation of* $c \circ \varphi$.

Proof. Assume $\mathscr{P}'$ is any von Neumann algebra together with NMIU-map $\rho' \colon \mathscr{A} \to \mathscr{P}'$ and NCP-map $f' \colon \mathscr{P}' \to \mathscr{B}$ such that $f' \circ \rho' = c \circ \varphi$. Note $f'(1) = f'(\rho'(1)) = c(\varphi(1)) \leq c(1) \leq b$. Hence there is a unique NCP-map $f'' \colon \mathscr{P}' \to \mathscr{B}'$ with $c \circ f'' = f'$. Observe $c \circ f'' \circ \rho' = f' \circ \rho' = c \circ \varphi$ and so $f'' \circ \rho' = \varphi$ as c is injective. There is a unique $\sigma \colon \mathscr{P}' \to \mathscr{P}$ with $\sigma \circ \rho' = \rho$ and $f \circ \sigma = f''$. But then $c \circ f \circ \sigma = c \circ f'' = f'$ and so we have shown existence of a mediating map. To show uniqueness, assume $\sigma' \colon \mathscr{P}' \to \mathscr{P}$ is any NCP-map such that $c \circ f \circ \sigma' = f'$ and $\sigma \circ \rho' = \rho$. Clearly $c \circ f \circ \sigma' = f' = c \circ f''$ and so $f \circ \sigma' = f''$. Thus $\sigma = \sigma'$ by definition of σ. $\square$

Corollary 27. *Let* $\varphi \colon \mathscr{A} \to \mathscr{B}$ *be an NCP-map with Paschke dilation* $\mathscr{A} \overset{\rho}{\twoheadrightarrow} \mathscr{P} \overset{f}{\twoheadrightarrow} \mathscr{B}$ *. Then f is pure. Furthermore, if φ is contractive, then so is f and if φ is unital, then f is a corner.*

Proof. First, we will prove that if φ is unital, then f is a corner. Thus, assume φ is unital. Let $\mathscr{A} \overset{\rho_s}{\twoheadrightarrow} \mathscr{P}_s \overset{f_s}{\twoheadrightarrow} \mathscr{B}$ be the standard Paschke dilation of φ. By [11, Corollary 5.3], f_s is a corner. By Lemma 9, we have $f = f_s \circ \vartheta$ for some isomorphism ϑ. But then f is also a corner.

Now, we will prove that for arbitrary contractive φ, the map f is pure and contractive. The non-contractive case follows by scaling. Write $\varphi' \colon \mathscr{A} \to \lceil \varphi(1) \rceil \mathscr{B} \lceil \varphi(1) \rceil$ for the unique unital NCP-map such that $\varphi = c_{\varphi(1)} \circ \varphi'$. Let $\mathscr{A} \overset{\rho'}{\twoheadrightarrow} \mathscr{P}' \overset{f'}{\twoheadrightarrow} \lceil \varphi(1) \rceil \mathscr{B} \lceil \varphi(1) \rceil$ denote the Paschke dilation of φ'. By Proposition 26, $\mathscr{A} \overset{\rho'}{\twoheadrightarrow} \mathscr{P}' \overset{c_{\varphi(1)} \circ f'}{\twoheadrightarrow} \mathscr{B}$ is a Paschke dilation of φ. Thus by Lemma 9, we know $f = c_{\varphi(1)} \circ f' \circ \vartheta$ for some isomorphism ϑ. As these are all pure, f is pure as well. $\square$

Theorem 28. *Let $\mathscr{A}$ be a von Neumann algebra together with a projection $p \in \mathscr{A}$. Then a Paschke dilation of the standard corner h_p is given by* $\mathscr{A} \overset{hc_p}{\twoheadrightarrow} C_p \mathscr{A} \overset{h_p}{\twoheadrightarrow} p \mathscr{A} p$ *.*

Proof. Let $\mathscr{A} \overset{\rho}{\twoheadrightarrow} A(\mathscr{A} \otimes_{h_p} p\mathscr{A}p) \overset{f}{\twoheadrightarrow} p\mathscr{A}p$ be the Paschke dilation of φ from Theorem 18. The plan is to first prove that $\mathscr{A} \otimes_{h_p} p\mathscr{A}p$ can be identified with $\mathscr{A}p$, and then to show that $B^a(\mathscr{A}p) = C_p\mathscr{A}$.

Note that $\mathscr{A}p$ is a right $p\mathscr{A}p$-module by $(ap) \cdot (pbp) = apbp$, and a pre-Hilbert $p\mathscr{A}p$-module via $\langle ap, \alpha p \rangle = pa^*\alpha p$. Since by the C^*-identity the norm on $\mathscr{A}p$ as a pre-Hilbert $\mathscr{B}$-module coincides with the norm of $\mathscr{A}p$ as subset of the von C^*-algebra $\mathscr{A}$, and $\mathscr{A}p$ is norm closed in $\mathscr{A}$, and $\mathscr{A}$ is norm complete, we see that $\mathscr{A}p$ is complete, and thus a Hilbert $p\mathscr{A}p$-module.

The next step is to show that $\mathscr{A}p$ is self-dual. Let $\tau \colon \mathscr{A}p \to p\mathscr{A}p$ be a bounded $p\mathscr{A}p$-module map. We must find $\alpha p \in \mathscr{A}p$ with $\tau(ap) = p\alpha^*ap$ for all $a \in \mathscr{A}$. This requires some effort.

1. We claim that $C_p = \sup\{r \colon r \lesssim p\}$, where $r \lesssim p$ denotes that r is a projection which is *von Neumann-Murray below* p, i.e. $r = vv^*$ and $v^*v \leq p$ for some $v \in \mathscr{A}$, see [6, Def. 6.2.1]. To begin, writing $q = \sup\{r \colon r \lesssim p\}$, we have $C_q = C_p$. Indeed, since $p \lesssim p$, we have $p \leq q$, and so $C_p \leq C_q$. For the other direction, $C_q \leq C_p$, note that if r is a projection with $r \lesssim p$, then $r \leq C_r \leq C_p$ by [6, Prop. 6.2.8]. Thus $q \leq C_p$, and so $C_q \leq C_p$.

   Thus we must prove that $C_q - q = 0$. It suffices to show that $C_{C_q-q} = 0$. Note that for every projection r with $r \leq C_q - q$ and $r \lesssim p$ we have $r = 0$, because $r \lesssim p$ implies $r \leq q$ and so $2r \leq C_q$. Thus, by [6, Prop. 6.1.8], we get $C_{C_q-q}C_p = 0$. But since $C_q - q \leq C_q = C_p$, we have $C_{C_q-q} \leq C_p$, and so $C_{C_q-q} = C_{C_q-q}C_p = 0$.

2. Using Zorn's lemma, we can find a family $(q_i)_{i \in I}$ of pairwise orthogonal projections in $\mathscr{A}$ with $q_i \lesssim p$ and $C_p \equiv q = \sum_{i \in I} q_i$. (Here, and in the remainder of this proof, infinite sums in von Neumann algebras are taken with respect to the ultraweak topology.) For each $i \in I$, pick $v_i \in \mathscr{A}$ with $v_i v_i^* = q_i$ and $v_i^* v_i \leq p$. Since $p^\perp v_i^* v_i p^\perp \leq p^\perp p p^\perp = 0$, we have $v_i p^\perp = 0$ by the C^*-identity, and so $v_i \in \mathscr{A}p$ for all $i \in I$.

3. Our plan is to prove that $\tau(ap) = \langle(\sum_{i\in I}\tau(v_i)v_i^*)^*, ap\rangle$ for all $a \in \mathscr{A}$, but first we must show that $\sum_{i\in I}\tau(v_i)v_i^*$ converges ultraweakly. This requires a slight detour. Let $J \subseteq I$ be any finite subset. Note that $(\tau(v_i)^*\tau(v_j))_{ij}$ is a matrix over $p\mathscr{A}p$ where i and j range over J. By [11, Thm. 2.8 (ii)], we have for all $(b_i)_{i\in J}$ from $p\mathscr{A}p$,

$$\sum_{i,j\in J} b_i^*\tau(v_i)^*\tau(v_j)b_j \;=\; \tau(\sum_{i\in J}v_ib_i)^*\tau(\sum_{i\in J}v_ib_i) \;\leq\; \|\tau\|\langle\sum_{i\in J}v_ib_i,\sum_{i\in J}v_ib_i\rangle \;=\; \sum_{i,j\in J}b_i^*\langle v_i,v_j\rangle b_j,$$

and so $(\tau(v_i)^*\tau(v_j))_{ij} \leq \|\tau\|(\langle v_i,v_j\rangle)_{ij}$ as matrices over $p\mathscr{A}p$ by [11, Prop. 6.1]. Since the projections $(q_i)_{i\in I}$ are pairwise orthogonal, and $\langle v_i,v_j\rangle = v_i^*v_j = v_i^*q_iq_jv_j$, we see that $(\langle v_i,v_j\rangle)_{ij}$ is a diagonal matrix below p, and since $\langle v_i,v_i\rangle = v_i^*v_i \leq p$ we get

$$\sum_{i,j\in J}v_i\tau(v_i)^*\tau(v_j)v_j^* \;\leq\; \|\tau\|\sum_{i\in J}v_ipv_i^* \;=\; \|\tau\|\sum_{i\in J}q_i \;\leq\; \|\tau\|1.$$

From this it follows that the net of partial sums of $\tau(v_i)v_i^*$ is norm bounded (by the C^*-identity), and ultraweakly Cauchy (by Cauchy–Schwarz and the fact that $\sum_{j\in J}q_j$ converges ultraweakly as J increases), and thus ultraweakly convergent[22, Prop. 40]. Define $\alpha \equiv (\sum_i\tau(v_i)v_i^*)^*$.

4. Note that $p\alpha^* = \sum_i p\tau(v_i)v_i^* = \alpha^*$ (because $\tau(v_i) \in p\mathscr{A}p$) and so $\alpha \in \mathscr{A}p$.

5. The linear map $\tau((\cdot)p)\colon \mathscr{A}\to\mathscr{A}$ is ultrastrongly continuous, because if $a_ip \to 0$ ultrastrongly, then for any normal state ω on $p\mathscr{A}p$ we have

$$\omega(\tau(a_ip)^*\tau(a_ip)) \;\leq\; \|\tau\|\omega(\langle a_ip,a_ip\rangle) \;=\; \|\tau\|\omega(pa_i^*a_ip) \;\to\; 0.$$

6. Pick any $ap \in \mathscr{A}p$. As $q = \sup_i q_i$, we have $q = \sum_i q_i$ ultrastrongly (combine [6, Lemma 5.1.4] with [13, Prop. 1.15.2]), and thus $\tau(qap) = \sum_i \tau(q_iap)$ ultraweakly. Hence

$$\tau(ap) \;=\; \tau(qap) \;=\; \sum_i \tau(q_iap) \;=\; \sum_i \tau(v_ipv_i^*ap) \;=\; \sum_i \tau(v_i)pv_i^*ap \;=\; \langle\alpha p,ap\rangle,$$

where we use that $qap = ap$ (since $q = C_p$), and $q_i = q_i^2 = v_iv_i^*v_iv_i^* = v_ipv_i^*$. Thus $\mathscr{A}p$ is self-dual.

Now we will show $\mathscr{A}p$ is isomorphic to $\mathscr{A}\otimes_{h_p}p\mathscr{A}p$. Note $[ap\otimes p, ap\otimes p] = pa^*ap$ and so $ap\otimes p \in N$ if and only if $ap = 0$. A straight-forward computation shows $a\otimes p\alpha p - ap\alpha p\otimes p \in N$ and so every $x \in \mathscr{A}\odot p\mathscr{A}p$ is N-equivalent to exactly one $ap\otimes p$ for some $a \in \mathscr{A}$. Thus $ap\otimes p + N \mapsto ap$ fixes an isomorphism $X_0 \to \mathscr{A}p$. As $\mathscr{A}p$ is already complete and self-dual, so is X_0. Hence via $ap \mapsto \widehat{ap\otimes p}$ we have $\mathscr{A}p \cong X_0 \cong X \cong X' \equiv \mathscr{A}\otimes_{h_p}p\mathscr{A}p$ and so $\mathscr{A}\xrightarrow{\;\rho\;}B^a(\mathscr{A}p)\xrightarrow{\;f\;}p\mathscr{A}p$ is a Paschke dilation for h_p, where $\rho(\alpha)ap = \alpha ap$ and $f(t) = pt(p)$.

If in the proof above — that $\tau \equiv \alpha^*(\cdot)p$ for some $\alpha \in \mathscr{A}p$ — we replace τ by a bounded $p\mathscr{A}p$-module map $t\colon \mathscr{A}p \to \mathscr{A}p$, then the reasoning is still valid (except for point (4)), and so we see that the map $\rho\colon \mathscr{A}\to B^a(\mathscr{A}p)$ given by $\rho(\alpha_0)(ap) = \alpha_0 ap$ is surjective.

Now we show $\operatorname{car}\rho = C_p$. It is sufficient to show that for each $\alpha \geq 0$ with $\alpha \in \mathscr{A}$, we have that $\alpha ap = 0$ for all $a \in \mathscr{A}$ if and only if $\alpha C_p = 0$. The reverse direction follows from $\alpha ap = \alpha(1 - C_p)ap = 0$ whenever $\alpha C_p = 0$. Thus assume $\alpha ap = 0$ for all $a \in \mathscr{A}$. In particular $0 = \alpha v_ipv_i^* = \alpha q_i$ and so $\alpha C_p = \alpha q = \sum_i \alpha q_i = 0$, as desired. We now know $\rho = \vartheta \circ h_{C_p}$ for some isomorphism $\vartheta\colon B^a(\mathscr{A}p) \to C_p\mathscr{A}$. It is easy to see $f \circ \vartheta^{-1} = h_p$ and so we have proven our Theorem. $\qquad\square$

Corollary 29. *Let $\varphi\colon \mathscr{A}\to\mathscr{B}$ be an NCP-map with Paschke dilation $\mathscr{A}\xrightarrow{\;\rho\;}\mathscr{P}\xrightarrow{\;f\;}\mathscr{B}$. Then the map φ is pure if and only if ρ is surjective.*

Proof of Corollary 29. Assume ρ is surjective. The kernel of ρ is $z\mathscr{A}$ for some central projection z. (See e.g. [13, 1.10.5].) Note the quotient-map for the kernel of ρ is a corner for z, hence by the isomorphism theorem ρ is a corner. By Corollary 27 f is pure. Thus φ is the composition of pure maps and hence pure.

Now assume φ is pure. By scaling, we may assume φ is contractive. Write $p \equiv \operatorname{car}\varphi$. Note $\varphi = c \circ h_p$ for some compression $c\colon p\mathscr{A}p \to \mathscr{B}$ and standard corner h_p for p. $\mathscr{A} \overset{h_{C_p}}{\longrightarrow} C_p\mathscr{A} \overset{h_p}{\longrightarrow} \mathscr{A}$ is a Paschke dilation for h_p and so by Proposition 26, $\mathscr{A} \overset{h_{C_p}}{\longrightarrow} \mathscr{A} \overset{c \circ h_p}{\longrightarrow} \mathscr{B}$ is a Paschke dilation for φ. By Lemma 9, we know $\rho = \vartheta \circ h_{C_p}$ for some isomorphism ϑ and so ρ is surjective. $\qquad\square$

Now we have a better grip on when a Paschke embedding is surjective. The following is a characterization of when a Paschke embedding is injective. This is a generalization of our answer[21] to the same question for the Stinespring embedding.

Theorem 30. *Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be an NCP-map with Paschke dilation $\mathscr{A} \overset{\rho}{\longrightarrow} \mathscr{P} \overset{f}{\longrightarrow} \mathscr{B}$. The map ρ is injective if and only φ maps no non-zero central projection to zero. (Equivalently: $C_{\operatorname{car}\varphi} = 1$.)*

Proof. Let $\alpha \in \mathscr{A}$. As a first step, we claim $\rho(\alpha) = 0$ if and only if $\varphi(a^*\alpha^*\alpha a) = 0$ for every $a \in \mathscr{A}$. From left to right is easy (expand $\rho(\alpha)a \otimes 1$). To show the converse, assume $\varphi(a^*\alpha^*\alpha a) = 0$ for all $a \in \mathscr{A}$. Then for every sequence $a_1, \ldots, a_n \in \mathscr{A}$ we may use $a := \sum_i a_i$ and so $\sum_{i,j} \varphi(a_j^*\alpha^*\alpha a_i) = 0$. From this and [11, Prop. 6.1] it follows that for every sequence $b_1, \ldots, b_n \in \mathscr{B}$, we have $\sum_{i,j} b_j^*\varphi(a_j^*\alpha^*\alpha a_i)b_i = 0$. Hence $\rho(\alpha)\sum_i a_i \otimes b_i = 0$ for any $\sum_i a_i \otimes b_i$. This is sufficient to conclude $\rho(\alpha) = 0$, as desired.

Assume ρ is injective. For brevity, write $p := \operatorname{car}\phi$. Let $a \in \mathscr{A}$ be given. Note

$$a^*(1 - C_p)a = (1 - C_p)a^*a(1 - C_p) \le \|a\|^2(1 - C_p) \le \|a\|^2(1 - p)$$

and so $\varphi(a^*(1 - C_p)a) \le \|a\|^2\varphi(1 - p) = 0$. By the initial claim, we see $\rho(1 - C_p) = 0$ and so $C_p = 1$, as desired.

For the converse, assume $C_p = 1$ and $\rho(\alpha) = 0$. By Zorn's lemma, find a maximal family of orthogonal projections $(q_i)_{i \in I}$ from $\mathscr{A}$ with $q_i \lesssim p$. Then $\sup_{i \in I} q_i = C_p$; see point 2 from the proof of Thm. 28. For each q_i pick a v_i such that $v_i v_i^* = q_i$ and $v_i^* v_i \le p$. From $\rho(\alpha) = 0$ we saw it follows that $\varphi(a^*\alpha^*\alpha a) = 0$ for all $a \in \mathscr{A}$. In particular $\varphi(v_i^*\alpha^*\alpha v_i) = 0$. Without loss of generality we may assume $a^*a \le 1$ and then $v_i^*\alpha^*\alpha v_i \le 1 - p$. Hence

$$q_i\alpha^*\alpha q_i = v_i v_i^*\alpha^*\alpha v_i v_i^* \le v_i(1 - p)v_i^* = v_i v_i^* - v_i v_i^* = 0.$$

Consequently $\alpha^*\alpha \le 1 - q_i$ for every $i \in I$. Thus $\alpha^*\alpha \le 1 - \sup_{i \in I} q_i = 1 - C_p = 0$. By the C*-identity, $\alpha = 0$ and so ρ is indeed injective. $\qquad\square$

To continue our study of pure maps, we need some preparation.

Definition 31. Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be any NCP-map.

1. Write $[0, \varphi]_{\text{NCP}} \equiv \{\psi\colon \mathscr{A} \to \mathscr{B} \text{ NCP-map}; \varphi - \psi \text{ is completely positive}\}$.

2. We say φ is **NCP-extreme**, if it is an extreme point among the NCP-maps with same value on 1; that is: $\lambda\varphi_1 + (1 - \lambda)\varphi_2 = \varphi$ for $\varphi_1, \varphi_2\colon \mathscr{A} \to \mathscr{B}$ NCP-maps and $0 < \lambda < 1$ implies $\varphi_1 = \varphi_2 = \varphi$

3. If $\mathscr{A} \overset{\rho}{\longrightarrow} \mathscr{P} \overset{f}{\longrightarrow} \mathscr{B}$ is a Paschke dilation of φ and $t \in \rho(\mathscr{A})'$ with $t \ge 0$, define $\varphi_t\colon \mathscr{A} \to \mathscr{B}$ by $\varphi_t(a) \equiv f(\sqrt{t}\rho(a)\sqrt{t})$.

Theorem 32. *Assume $\varphi\colon \mathscr{A} \to \mathscr{B}$ is an NCP-map with Paschke dilation $\mathscr{A} \overset{\rho}{\longrightarrow} \mathscr{P} \overset{f}{\longrightarrow} \mathscr{B}$.*

1. *The map $t \mapsto \varphi_t$ is an affine order isomorphism $[0,1]_{\rho(\mathscr{A})'} \to [0, \varphi]_{\text{NCP}}$.*

2. *φ is NCP-extreme if and only if $t \mapsto \varphi_t(1)$ is injective on $[0,1]_{\rho(\mathscr{A})'}$.*

Proof. The Paschke dilation constructed in Theorem 18 is called the standard Paschke dilation, for which point 1 is shown in [11, Prop. 5.4] and point 2 in [11, Thm. 5.4]. We show the result carries to an arbitrary Paschke dilation. Write $\mathscr{A} \xrightarrow{\rho_s} \mathscr{P}_s \xrightarrow{f_s} \mathscr{B}$ for the standard Paschke dilation of φ. Write $\varphi_t^{\mathscr{P}}$ and $\varphi_t^{\mathscr{P}_s}$ to distinguish between φ_t relative to the given and standard Paschke dilation. Let $\vartheta \colon \mathscr{P} \to \mathscr{P}_s$ be the mediating isomorphism from Lemma 9.

It is easy to see ϑ restricts to an affine order isomorphism $[0,1]_{\rho(\mathscr{A})'} \to [0,1]_{\rho_s(\mathscr{A})'}$. Note

$$\varphi_t^{\mathscr{P}}(a) = f(\sqrt{t}\rho(a)\sqrt{t}) = f_s(\vartheta(\sqrt{t}\rho(a)\sqrt{t})) = f_s(\sqrt{\vartheta(t)}\rho_s(a)\sqrt{\vartheta(t)}) = \varphi_{\vartheta(t)}^{\mathscr{P}_s}(a)$$

and so $t \mapsto \varphi_t^{\mathscr{P}}$ is the composition of affine order isomorphisms ϑ^{-1} and $t \mapsto \varphi_t^{\mathscr{P}_s}$, which proves 1. Finally, for 2, note $t \mapsto \varphi_t^{\mathscr{P}_s}(1)$ is injective if and only if $t \mapsto \varphi_t^{\mathscr{P}}(1)$ is as ϑ^{-1} is injective. $\qquad\square$

Proposition 33. *Let $\varphi \colon \mathscr{A} \to B(\mathscr{H})$ be any NCP-map. Then φ is pure in the definition of Størmer [18, Def. 3.5.4] if and only if φ is pure as in Def. 21.*

Proof. Let $\varphi \colon \mathscr{A} \to B(\mathscr{H})$ be any NCP-map with standard Paschke dilation $\mathscr{A} \xrightarrow{\rho} \mathscr{P} \xrightarrow{f} B(\mathscr{H})$. Note that by Theorem 14, we know $\mathscr{P} \cong B(\mathscr{K})$ for some Hilbert space $\mathscr{K}$ and so it is a factor.

Assume $\varphi \colon \mathscr{A} \to B(\mathscr{H})$ is pure as in Def. 21. Let $\psi \colon \mathscr{A} \to B(\mathscr{H})$ with $\varphi - \psi$ completely positive. To show φ is pure in the sense of Størmer, we have to show $\psi = \lambda \varphi$ for some $\lambda \in [0,1]$. By [11, Prop. 5.4], $\psi = \varphi_t$ for some $t \in \rho(\mathscr{A})'$ with $0 \leq t \leq 1$. In particular ψ is normal. As φ is pure we know by Corollary 29 that ρ is surjective. Thus $\rho(\mathscr{A})' = Z(\mathscr{P}) = \mathbb{C}1$. Thus $t = \lambda 1$ for some $\lambda \in [0,1]$. We conclude $\psi = \varphi_t = \varphi_{\lambda 1} = \lambda \varphi_1 = \lambda \varphi$ as desired.

For the converse, assume φ is Størmer pure. By Corollary 29, it is sufficient to show ρ is surjective. As $\rho(\mathscr{A})$ is a von Neumann subalgebra of $\mathscr{P}$, we may conclude that ρ is surjective if we can show $\rho(\mathscr{A})' \subseteq Z(\mathscr{P})$ as $\mathscr{P} = Z(\mathscr{P})' \subseteq \rho(\mathscr{A})'' = \rho(\mathscr{A}) \subseteq \mathscr{P}$ by the double commutant theorem. To this end, let $t \in \rho(\mathscr{A})'$. Without loss of generality, we may assume $0 \leq t \leq 1$. Then $\varphi_t \in [0,1]_{\text{NCP}}$ and so $\varphi_t = \lambda \varphi$ for some $\lambda \in [0,1]$. Hence $\varphi_t = \lambda \varphi = \varphi_{\lambda 1}$ and so $t = \lambda 1 \in Z(\mathscr{P})$, which completes the proof. $\qquad\square$

Theorem 34. *NMIU-maps and pure NCP-maps are NCP-extreme.*

Proof. Let $\rho \colon \mathscr{A} \to \mathscr{B}$ be any NMIU-map. $\mathscr{A} \xrightarrow{\rho} \mathscr{B} \xrightarrow{\text{id}} \mathscr{B}$ is a Paschke dilation of ρ. If $t \in \rho(\mathscr{A})'$, then $\rho_t(1) = t$ and so, clearly, $t \mapsto \rho_t(1)$ is injective on $\rho(\mathscr{A})'$. Hence by Theorem 32, we see ρ is NCP-extreme.

Assume $\varphi \colon \mathscr{A} \to \mathscr{B}$ is pure. By scaling, we may assume φ is contractive. Write $p \equiv \operatorname{car} \varphi$. Then $\varphi = c \circ h_p$ for some compression $p\mathscr{A}p \to \mathscr{B}$. By Proposition 26 and Theorem 28 we know $\mathscr{A} \xrightarrow{h_{C_p}} C_p\mathscr{A} \xrightarrow{c \circ h_p} \mathscr{B}$ is a Paschke dilation of φ. We have to show $t \mapsto \varphi_t(1)$ is injective on $[0,1]_{h_{C_p}(\mathscr{A})'}$. As h_{C_p} is surjective, we have $h_{C_p}(\mathscr{A})' = Z(C_p\mathscr{A})$ hence $\varphi_t(1) = c(h_p(\sqrt{t}h_{C_p}(1)\sqrt{t})) = c(ptp)$ for $t \in Z(C_p\mathscr{A})$. As compressions are injective, it is sufficient to show h_p is injective on $Z(C_p\mathscr{A})$. Assume $t \in Z(C_p\mathscr{A})$ such that $ptp = 0$. Then $0 = r(pt) = pr(t)$ and so $p \leq 1 - r(t)$. As $1 - r(t)$ is a central projection, we must have $1 - r(t) \leq C_p = 1_{C_p\mathscr{A}}$. Hence $r(t) = 0$ and so $t = 0$. We are done. $\qquad\square$

Problem 35. *Is there an NCP-extreme NCP-map into a factor which is neither pure nor a compression after an NMIU-map?*

Remarks

The construction of the standard Paschke dilation is a generalization of the GNS-construction to Hilbert C*-modules. For this reason, for those studying C*-modules, the construction is known as Paschke's GNS (e.g. [15, Remark 8.4]). There is also a generalization of Stinespring to C*-modules due to Kasparov.[7] This Theorem, however, is like Stinespring only applicable to NCP-maps of which the codomain has a certain form and hence as far as it applies to arbitrary NCP-maps, it reduces to Paschke.

Acknowledgments

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