Additive monotones for resource theories of parallel-combinable processes with discarding

Brendan Fong

Department of Computer Science
University of Oxford

brendan.fong@cs.ox.ac.uk

Hugo Nava-Kopp

Department of Computer Science
University of Oxford

hugo.nava.kopp@cs.ox.ac.uk

A partitioned process theory, as defined by Coecke, Fritz, and Spekkens, is a symmetric monoidal category together with an all-object-including symmetric monoidal subcategory. We think of the morphisms of this category as processes, and the morphisms of the subcategory as those processes that are freely executable. Via a construction we refer to as parallel-combinable processes with discarding, we obtain from this data a partially ordered monoid on the set of processes, with $f \succeq g$ if one can use the free processes to construct g from f. The structure of this partial order can then be probed using additive monotones: order-preserving monoid homomorphisms with values in the real numbers under addition. We first characterise these additive monotones in terms of the corresponding partitioned process theory.

Given enough monotones, we might hope to be able to reconstruct the order on the monoid. If so, we say that we have a complete family of monotones. In general, however, when we require our monotones to be additive monotones, such families do not exist or are hard to compute. We show the existence of complete families of additive monotones for various partitioned process theories based on the category of finite sets, in order to shed light on the way such families can be constructed.

1   Introduction

In [3], Coecke, Fritz, and Spekkens make a well-illustrated case for viewing symmetric monoidal categories as theories of resources: the objects of the category are interpreted as resources, the morphisms methods for converting one resource into another. In many examples, such as quantum entanglement, the resources themselves are processes, and the methods of converting one process into another involve composition with a set of 'freely executable' processes. This structure is formalised as a partitioned process theory: a pair comprising a symmetric monoidal category together with an all-object-including symmetric monoidal subcategory.

The question then arises: when can we build one resource from another? One technique for answering this question is to assign real numbers to each resource, according to their power to create other resources. We call such functions monotones. A collection of monotones that completely characterises this structure is known as a complete family of monotones. While complete families of monotones always exist, ones with nice properties are in general hard to come by, with even celebrated ones, such as entropy or Kolmogorov complexity, being difficult to compute.

In this article we explore the construction of families of so-called additive monotones—monotones that are also monoid homomorphisms into the real numbers under addition. Our main result fixes a particular method of building a resource theory from a partitioned process theory, and characterises the additive monotones on the resulting resource theory. We then explore two applications of this theorem, using it to construct complete families of additive monotones.

2   Resource theories of parallel-combinable processes with discarding

We formalise our ideas about resources and free processes using symmetric monoidal categories, following the work of Coecke, Fritz, and Spekkens [2, 3]. We say that a subcategory **D** of a category **C** is a *wide* subcategory if it includes all the objects of **C**.

Definition 1. *A* partitioned process theory $(\mathbf{C}, \mathbf{C}_{free})$ *consists of a symmetric monoidal category* **C** *together with a wide symmetric monoidal subcategory* $\mathbf{C}_{free}$.

With this definition we see that examples of partitioned process theories abound, not only describing structures such as entanglement and athermality arising in applied sciences [1, 6], but also simply natural ideas within mathematics itself. For example, any category **C** with products can be considered a symmetric monoidal category with monoidal product given by the categorical product, and in such a category each object is equipped with a commutative comonoid structure. We may generate a wide symmetric monoidal subcategory from these comonoid morphisms to serve as our category $\mathbf{C}_{free}$. A similar, topical [7, 4, 8], example arises from the analogous construction on so-called multigraph categories—categories in which every object is equipped with a special commutative Frobenius monoid.

We caution that a partitioned process theory was simply termed a resource theory in [2]; our terminology comes from [3], and following [3] we instead use *resource theory* to simply refer to a symmetric monoidal category in which we think of the objects as resources. In line with this viewpoint, we shall refer to the morphisms of **C** as *processes* and the morphisms of $\mathbf{C}_{free}$ as *free processes*. We then can construct a resource theory from a partitioned process theory by considering the processes as resources.

Indeed, given a partitioned process theory $(\mathbf{C}, \mathbf{C}_{free})$, we can construct a symmetric monoidal category in which the objects are the processes **C**, and the morphisms are methods of constructing one process from another using free processes. We term this new category the *resource theory of parallel-combinable processes with discarding* $\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})$ of the partitioned process theory $(\mathbf{C}, \mathbf{C}_{free})$.

We shall assume all categories here are small, and write $|\mathbf{C}|$ and $\mathrm{Mor}(\mathbf{C})$ for the sets of objects and morphisms of a category **C** respectively.

Proposition 2. *Let* $(\mathbf{C}, \mathbf{C}_{free})$ *be a partitioned process theory. Then we may define a symmetric monoidal category* $\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})$ *with objects* $f \in \mathrm{Mor}(\mathbf{C})$, *and morphisms* $f \to g$ *triples* (Z, ξ_1, ξ_2), *where* $Z \in |\mathbf{C}|$, $\xi_1, \xi_2 \in \mathrm{Mor}(\mathbf{C}_{free})$, *such that there exists* $j \in \mathrm{Mor}(\mathbf{C})$ *with*

$$\xi_2 \circ (f \otimes 1_Z) \circ \xi_1 = g \otimes j. \tag{1}$$

In string diagrams equation (1) becomes:
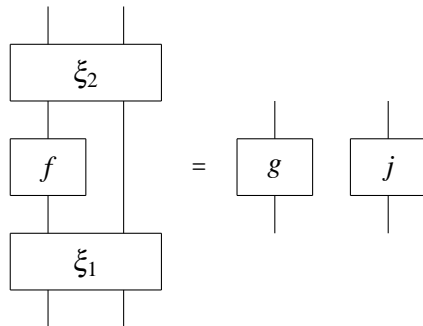
Proof. A proof of this proposition may be found in [2, Theorem 11].[1]   □

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[1]While it makes little difference for our purposes here, to do away with this assumption that the categories must be small, we may define a notion of equivalence on the triples (Z, ξ_1, ξ_2) so that we consider two triples to be equivalent if they have, roughly speaking, the same 'operational behaivour'. For similar constructions this approach is favoured in [3, Section 3].

Remark 3. Note that there are many ways to construct interesting resource theories from a partitioned process theory, depending on the methods we allow for turning one process into another using free processes. Other examples include the resource theory of parallel-combinable processes $PC(\mathbf{C}, \mathbf{C}_{\text{free}})$ [3, Subsection 3.3] and the resource theory of universally-combinable processes $UC(\mathbf{C}, \mathbf{C}_{\text{free}})$ [3, Subsection 3.4]. While we shall not define these constructions here, for the reader familiar with them, we note that we have the inclusion of symmetric monoidal categories

$$PC(\mathbf{C}, \mathbf{C}_{\text{free}}) \longrightarrow PCD(\mathbf{C}, \mathbf{C}_{\text{free}}).$$

Moreover, when the partitioned process theory obeys certain conditions, it can be shown that these definitions coincide, although in general they do not. We also note that it is possible to interpret these different constructions as different methods of constructing an operad of 'wiring diagrams' [9] from the set of free processes.

In this paper we are interested in understanding the ordered monoid corresponding to this resource theory. Recall that an *ordered monoid* $(X, \geq, \cdot)$ is a set X together with a partial order $\geq$ and a monoid multiplication $\cdot$ such that for all $x, y, z, w \in X$,

$$\text{if} \quad x \geq y \text{ and } z \geq w, \quad \text{then} \quad x \cdot z \geq y \cdot w. \tag{2}$$

We may partially decategorify a resource theory to obtain an ordered monoid in the following way.

Theorem 4. *Let $\mathbf{R}$ be a symmetric monoidal category, and call objects f, g in $\mathbf{R}$ equivalent if there exists morphisms $f \to g$ and $g \to f$. This defines an equivalence relation. Write $[f]$ for the equivalence class of the object f; we shall frequently abuse notation to simply write f for the equivalence class of $[f]$ and $|\mathbf{R}|$ for the set of equivalence classes of objects in $\mathbf{R}$.*

Then there exists an ordered monoid $(|\mathbf{R}|, \geq, \otimes)$ on the set of these equivalence classes, with $[f] \geq [g]$ if there exists a morphism $f \to g$ in $\mathbf{R}$, and using the monoidal product in $\mathbf{R}$ to define $[f] \otimes [g] = [f \otimes g]$. Moreover, this monoid is commutative.

Proof. The relation on the objects of $\mathbf{R}$ specified by the existence of a morphism $f \to g$ is reflexive due to identity morphisms, and the transitive due to composition. Thus we obtain a partial order on the set of equivalence classes of $|\mathbf{R}|$.

The unit for the monoid $|\mathbf{R}|$ is given by the monoidal unit in $\mathbf{R}$; the unitors show that it is indeed a unit for the monoid multiplication. Similarly, the associativity of the monoid multiplication follows from the existence of the associators for the monoidal product, and the commutativity from the braiding. Moreover, the compatibility condition (2) follows from the functoriality of the monoidal product.   □

Under this equivalence relation, we consider two resources the same if we may convert one into the other, and vice versa. We then think of this ordered monoid as a *theory of resource convertibility*, with the monoid structure describing how we can combine two resources to make another, and the partial order describing when we can turn one resource (more precisely, equivalence class of resources) into another.

Note that in the theory of resource convertibility for a resource theory of parallel-combinable processes with discarding, the free morphisms themselves form an equivalence class, and that this equivalence class acts as the identity element for the monoid multiplication. Note also that this equivalence class contains the identity morphisms 1_X for all X, so there is no confusion to be had by writing this equivalence class as 1.

By 'discarding' in 'resource theory of parallel-combinable processes with discarding' we mean that no cost is incurred by replacing some resource $g \otimes j$ with just some subpart g of it. In terms of theories

of resource convertibility, this means that the corresponding monoid is non-negative. Recall that we call an ordered monoid $(X, \geq, \cdot)$ *non-negative* if the identity element 1 of the monoid is the bottom element for the partial order; that is, if we have

$$\text{for all } x \in X, \;\; x \geq 1.$$

This is also equivalent to the property that for all $x, y \in X$ we have $x \cdot y \geq y$. The ordered monoid corresponding to a resource theory of parallel-combinable processes with discarding is always non-negative.

Lemma 5. *Let* $(\mathbf{C}, \mathbf{C}_{free})$ *be a partitioned process theory. Then the ordered monoid* $(|\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})|, \geq, \otimes)$ *is non-negative.*

Proof. Given $\xi : A \to B \in \mathrm{Mor}(\mathbf{C}_{\mathrm{free}})$ and $f : X \to Y \in \mathrm{Mor}(\mathbf{C})$, we see that $\sigma_{Y,B}^{-1} \circ (f \otimes 1_B) \circ (1_X \otimes \xi) \circ \sigma_{A,X} = \xi \otimes f$, where $\sigma_{A,X} : A \otimes X \to X \otimes A$ is the braiding. Thus $[f] \geq [\xi] = [1]$. $\qquad\square$

Remark 6. Although [2] demonstrates that the resource theory of parallel-combinable processes with discarding is a highly applicable structure, there are some instances in which the discarding means it does not yield an interesting ordered monoid.

For example, consider the partitioned process theory $(\mathbf{Rel}_\times, \mathbf{Set}_\times)$, where $\mathbf{Rel}_\times$ is the symmetric monoidal category with objects finite sets, morphisms relations, and monoidal product cartesian product, and $\mathbf{Set}_\times$ is the wide symmetric monoidal subcategory with morphisms restricted to the functions. We might give the partitioned process theory $(\mathbf{Rel}_\times, \mathbf{Set}_\times)$ the following interpretation. Each relation $f \subseteq X \times Y$ may be viewed as a noisy possibilistic communication channel, with $x \in X$ possibly mapping to any of the $y \in Y$ such that $(x, y) \in f$. The free morphisms, in this case functions, represent encoding and decoding functions, while the cartesian product models the fact that any channel can be used arbitrarily many times. The resource theory $\mathrm{PCD}(\mathbf{Rel}_\times, \mathbf{Set}_\times)$ might thus be an attempt at a model for simulatability of one possibilistic channel by another.

The ordered monoid corresponding to this resource theory, however, is trivial. Write the empty set $\varnothing$ and let 1 be a singleton set. Given relations $f : X \to Y$ and $g : A \to B$, we may choose $Z = \varnothing$, $\xi_1 : \varnothing \to X$, $\xi_2 : Y \to B$, and $j : \varnothing \to 1$. Then both $\xi_2 \circ (f \times 1_Z) \circ \xi_1$ and $g \times j$ are the unique relation $\varnothing \to B$. This implies that for all $f, g \in \mathrm{Mor}(\mathbf{Rel}_\times)$, we have a morphism $f \to g \in \mathrm{PCD}(\mathbf{Rel}_\times, \mathbf{Set}_\times)$, thus yielding a trivial monoid.

3 Complete families of monotones

In information theory, one is often interested in the trying to define an entropy of a source. These entropies give a real number quantifying, loosely speaking, the randomness of the source. Such functions provide insight into whether one source might be simulated by another [10]. In terms of partitioned process theories $(\mathbf{C}, \mathbf{C}_{\mathrm{free}})$, this suggests we might look at order-preserving functions from the ordered monoid derived from $\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{\mathrm{free}})$ to the real numbers. We call such functions monotones.

Definition 7. *Let* $(X, \geq)$ *be a partially ordered set. A* monotone *is an order-preserving function* $M : (X, \geq) \to (\mathbb{R}, \geq)$. *It is further called* complete *if it is also order-reflecting: that is, if for all* $x, y \in X$ *we have*

$$x \geq y \quad \text{if and only if} \quad M(x) \geq M(y).$$

A complete monotone exists for an partially ordered set if and only if it embeds into the reals. This is rarely the case. It is always possible, however, to find a *complete family* of monotones [3, Proposition 5.2].

Definition 8. *Given a partially ordered set $(X, \geq)$, we call a collection $\{M_i\}_{i \in I}$ of monotones on $(X, \geq)$ a complete family of monotones if for all $x, y \in X$ we have*

$$x \geq y \quad \textit{if and only if} \quad M_i(x) \geq M_i(y) \textit{ for all } i \in I.$$

Some families, however, are better than others. To provide additional insight into the structure of the ordered monoid, we frequently require some extra properties, such as preservation of some sort of monoid structure.

Definition 9. *We say that a monotone on an ordered set $(X, \geq, \cdot)$ is*

- additive *if it is also a monoid homomorphism into $(\mathbb{R}, \geq, +)$.*

- non-negative *if its image in $\mathbb{R}$ forms a non-negative ordered monoid.*

Additive monotones are also known as *functionals* [5, Section 3]. One advantage of working with additive monotones is that we can use them to recover the structure of the commutative monoid. Indeed, a complete family of monotones $\{M_i\}_{i \in I}$ on $(X, \geq, \cdot)$ induces an injective order-preserving monoid homomorphism into $(\mathbb{R}^I, \geq, +)$, where the partial order $\geq$ and monoid multiplication $+$ are given by the pointwise construction, and so locates $(X, \geq, \cdot)$ as a sub-ordered monoid of $\mathbb{R}^I$.

We shall work towards providing some examples of complete families of additive monotones for resource theories of parallel-combinable processes with discarding. Our next step is to characterise the non-negative additive monotones for such resource theories.

Theorem 10. *Let $(\mathbf{C}, \mathbf{C}_{free})$ be a partitioned process theory and let $(X, \geq, \cdot)$ be a non-negative ordered monoid. A function $\mu : \mathrm{Mor}(\mathbf{C}) \to X$ induces an order-preserving monoid homomorphism*

$$M : (|\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})|, \geq, \otimes) \longrightarrow (X, \geq, \cdot)$$
$$[f] \longmapsto \mu(f)$$

if and only if for all $Z \in |\mathbf{C}|$, $f, g \in \mathrm{Mor}(\mathbf{C})$, and $\xi \in \mathrm{Mor}(\mathbf{C}_{free})$ we have

(i) $\mu(f \otimes g) = \mu(f) \cdot \mu(g)$*;*

(ii) $\mu(1_Z) = 1$*; and*

(iii) $\mu(f) \geq \mu(\xi \circ f)$ *and* $\mu(f) \geq \mu(f \circ \xi)$ *whenever such composites are well-defined.*

Moreover, this gives a one-to-one correspondence: every order-preserving monoid homomorphism on $(|\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})|, \geq, \otimes)$ arises from a unique such function μ.

Conditions *(i)* and *(ii)* are unsurprising; they simply ask that the function respect the monoid structure. Condition *(iii)*, however, is a bit more illuminating: it tells us that as long as composition with free morphisms reduces the value of the function μ, the function induces a monotone.

Proof. Suppose first that μ induces an ordered monoid homomorphism M. Note in particular this means that M is well-defined on the equivalence classes of objects in $\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})$, and also that M preserves the order and monoid multiplication. Given that M preserves the monoid multiplication, we have

$$\mu(f \otimes g) = M([f \otimes g]) = M([f]) \cdot M([g]) = \mu(f) \cdot \mu(g),$$

and as M preserves identities, we have $\mu(1_Z) = M([1_Z]) = 0$. Recall now that $f \geq g$ if and only if there exists free processes ξ_1, ξ_2, an object Z, and a morphism j in $\mathbf{C}$ such that $\xi_1 \circ (f \otimes 1_Z) \circ \xi_2 = g \otimes j$. Then, whenever they are defined, we thus have $f \geq f \circ \xi$ and $f \geq \xi \circ f$, and so since M is monotone we have

$$\mu(f) = M([f]) \geq M([f \circ \xi]) = \mu(f \circ \xi)$$

and similarly $\mu(f) \geq \mu(\xi \circ f)$. Thus μ obeys *(i)–(iii)*.

Conversely, suppose that μ has the properties *(i)–(iii)*. Now suppose that we have processes f and g such that $\xi_1 \circ (f \otimes 1_Z) \circ \xi_2 = g \otimes j$. We thus have

$$\mu(f) \overset{(i,ii)}{=} \mu(f \otimes 1_Z) \overset{(iii)}{\geq} \mu(\xi_1 \circ (f \otimes 1_Z) \circ \xi_2) = \mu(g \otimes j) \overset{(i)}{\geq} \mu(g).$$

This implies that if there exists a morphism $f \to g$ in $\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{\text{free}})$ then $\mu(f) \geq \mu(g)$, and hence that $M([f]) = \mu(f)$ is well-defined and monotone. Properties *(i)* and *(ii)* then immediately imply M is a monoid homomorphism.

Finally, given a homomorphism of ordered monoids $M : (|\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{\text{free}})|, \geq, \otimes) \to (X, \geq, \cdot)$, we may define $\mu(f) = M([f])$ to obtain the unique function $\mu : \mathrm{Mor}(\mathbf{C}) \to X$ that induces it. $\qquad\square$

This theorem allows us to construct non-negative additive monotones by working from the partitioned process theory $(\mathbf{C}, \mathbf{C}_{\text{free}})$. We take advantage of this fact in the next section to produce some families of complete monotones.

4   Example: encoding functions as resources

In this section we construct two examples of complete families of non-negative additive monotones for resource theories of parallel-composable processes with discarding. Both partitioned process theories at hand have as processes the functions between finite sets. They may hence be understood as modelling encoding schemes, where a resource is a method for assigning a code symbol to each element of some finite input set. Our ordered monoids thus answer the question of when we may use the free morphisms and the construction (1) to turn one encoding scheme into another.

We emphasise here the concreteness of these results: given two functions between finite sets, one can quickly use the complete families of additive monotones we construct to evaluate whether one resource is convertible into the other.

4.1   The partitioned process theory of functions and bijections

Let $\mathbf{Set}$ the symmetric monoidal category with objects finite sets, morphisms functions, and monoidal product disjoint union, and let $\mathbf{Bij}$ be the wide symmetric monoidal subcategory with morphisms restricted to the bijective functions. Write $\#X$ for the cardinality of a set X.

Proposition 11. *For $i \in \mathbb{N}$, define functions:*

$$\varphi_i : \mathrm{Mor}(\mathbf{Set}) \longrightarrow \mathbb{N};$$
$$(f : X \to Y) \longmapsto \#\{y \in Y \mid \#f^{-1}(y) = i\}.$$

The family of monotones $\{F_i\}_{i \in \mathbb{N} \setminus \{1\}}$ induced by the family of functions $\{\varphi_i\}_{i \in \mathbb{N} \setminus \{1\}}$ is a complete family of additive monotones for the resource theory $\mathrm{PCD}(\mathbf{Set}, \mathbf{Bij})$.

Observe that each φ_i takes a function $f : X \to Y$ and returns the number of elements of Y that have i elements of X map to it. Also note that for every list of $\ell : \mathbb{N} \to \mathbb{N}$ of natural numbers with only finitely many entries nonzero, there exists a function $j_\ell : C \to D \in \mathrm{Mor}(\mathbf{Set})$ such that $\varphi_i(j_\ell) = \ell(i)$—indeed, simply choose C_i of cardinality $i \cdot \ell(i)$, D_i of cardinality $\ell(i)$, let $j_{\ell,i}$ map i elements of C_i to each element of D, and then define $j_\ell = \bigsqcup_{i=0}^{\infty} j_{\ell,i}$.

Proof of Proposition 11. That we have defined a family of additive monotones is an immediate consequence of Theorem 10. Remembering that $i \neq 1$ and that the free morphisms in this case are the bijections, it is clear that each φ_i has the properties *(i)–(iii)* required.

We turn our attention to completeness. Fix functions $f : X \to Y$ and $g : A \to B$. For completeness we need to show that if for all $i \in \mathbb{N} \smallsetminus \{1\}$ we have $F_i(f) \geq F_i(g)$, then there exists a finite set Z, bijections ξ_1, ξ_2, and a function j such that

$$\xi_2 \circ (f \sqcup 1_Z) \circ \xi_1 = g \sqcup j.$$

Let us construct such data as follows. Consider the list of numbers $\varphi_i(f) - \varphi_i(g)$ for all $i \in \mathbb{N}$ (*including* $i = 1$). Then we have two cases:

1. if $\varphi_1(f) - \varphi_1(g) \geq 0$, choose j such that $\varphi_i(j) = \varphi_i(f) - \varphi_i(g)$ for all $i \in \mathbb{N}$, and choose Z to be the empty set $\varnothing$.

2. if $\varphi_1(f) - \varphi_1(g) < 0$, choose j such that $\varphi_1(j) = 0$ and $\varphi_i(j) = \varphi_i(f) - \varphi_i(g)$ for all $i \in \mathbb{N} \smallsetminus \{1\}$, and Z to be a set of cardinality $\varphi_1(g) - \varphi_1(f)$.

We now have obtained $j \in \mathrm{Mor}(\mathbf{Set})$ and $Z \in |\mathbf{Set}|$ such that

$$\varphi_i(f \sqcup 1_Z) = \varphi_i(g \sqcup j)$$

for all $i \in \mathbb{N}$. Write C and D respectively for the domain and codomain of j. As $f \sqcup 1_Z$ and $g \sqcup j$ both have the same list of cardinalities of preimages of elements of their codomains, we may now choose bijections $\xi_1 : A \sqcup C \to X \sqcup Z$ and $\xi_2 : Y \sqcup Z \to B \sqcup D$ such that $\xi_2 \circ (f \sqcup 1_Z) \circ \xi_1 = g \sqcup j$, as required. $\square$

Write $\mathrm{FinSupp}_{\mathbb{N}}(X)$ for the set of finitely supported functions on X; that is, the set of functions $f : X \to \mathbb{N}$ for which $f(x) \neq 0$ for at most finitely many $x \in X$. Such functions are partially ordered by setting $f \geq g$ if for all $x \in X$ we have $f(x) \geq g(x)$, and may be given a commutative monoid structure by setting $(f + g)(x) = f(x) + g(x)$.

The above discussion thus characterises the theory of resource convertibility for $\mathrm{PCD}(\mathbf{Set}, \mathbf{Bij})$:

Corollary 12. *We have an isomorphism of ordered monoids*

$$F : \big(|\mathrm{PCD}(\mathbf{Set}, \mathbf{Bij})|, \geq, \sqcup\big) \longrightarrow \big(\mathrm{FinSupp}_{\mathbb{N}}(\mathbb{N} \smallsetminus \{1\}), \geq, +\big);$$
$$f \longmapsto \big(i \mapsto F_i(f)\big).$$

4.2   The partitioned process theory of functions and injections

Write $\mathbf{Inj}$ for the wide symmetric monoidal subcategory of $\mathbf{Set}$ with morphisms injective functions. We consider the partitioned process theory $(\mathbf{Set}, \mathbf{Inj})$.

Proposition 13. *For $i \in \mathbb{N}$, define functions:*

$$\gamma_i : \mathrm{Mor}(\mathbf{Set}) \longrightarrow \mathbb{N};$$
$$(f : X \to Y) \longmapsto \#\big\{y \in Y \,\big|\, \#f^{-1}(y) \geq i\big\}.$$

The family of monotones $\{G_i\}_{i \in \mathbb{N} \smallsetminus \{0,1\}}$ induced by the family of functions $\{\gamma_i\}_{i \in \mathbb{N} \smallsetminus \{0,1\}}$ is a complete family of additive monotones for the resource theory $\mathrm{PCD}(\mathbf{Set}, \mathbf{Inj})$.

The function γ_i maps a function $f : X \to Y$ to the number of elements of Y that have at least i elements of X map to it; it is a sum of the functions φ_k for $k \geq i$.

Proof. Theorem 10 again easily gives us that the G_i are additive monotones for $i \in \mathbb{N}$, $i \geq 2$. In particular, note for condition *(iii)* that pre-composing a function $f : X \to Y$ with an injection never increases the cardinality of the preimage of a point in Y, and that f followed by an injection has preimages of points in the codomain that are either empty or equal to the preimage some point in Y, with no two points of the codomain sharing the same point in Y.

It thus remains to prove the completeness of this family. Fix functions $f : X \to Y$ and $g : A \to B$, and suppose that for all $i \geq 2$ we have $G_i(f) \geq G_i(g)$. Again we wish to construct witnesses $Z \in |\mathbf{Set}|$, $\xi_1, \xi_2 \in \mathrm{Mor}(\mathbf{Inj})$, and $j \in \mathrm{Mor}(\mathbf{Set})$ such that $\xi_2 \circ (f \sqcup 1_Z) \circ \xi_1 = g \sqcup j$. There are many ways to construct such witnesses. We offer the following algorithm.

Choose Z to be a set of cardinality $\max\{0, \#B - \#Y\}$, D to be a set of cardinality $\max\{0, \#Y - \#B\}$, and j to be the unique function $j : \varnothing \to D$. This ensures that for all $i \in \mathbb{N}$, including 0 and 1, we have $\gamma_i(f \sqcup 1_Z) \geq \gamma_i(g \sqcup j)$. By definition, this means that for all $i \in \mathbb{N}$ we have

$$\#\{y \in Y \sqcup Z \,|\, \#(f \sqcup 1_Z)^{-1}(y) \geq i\} \geq \#\{b \in B \sqcup D \,|\, \#(g \sqcup j)^{-1}(b) \geq i\}.$$

This allows us to define an injection (in fact a bijection) $\xi_2 : Y \sqcup Z \to B \sqcup D$ mapping each element $y \in Y \sqcup Z$ to an element $b \in B$ such that $\#(f \sqcup 1_Z)^{-1}(y) \geq \#(g \sqcup j)^{-1}(b)$. We then may choose an injection $\xi_1 : A \to X \sqcup Z$ such that for all $a \in A$ we have $\xi_1(a) \in (\xi_2 \circ (f \sqcup 1_Z))^{-1}(g(a))$. This proves the proposition. $\qquad\square$

Analogous to the previous case, we reach the following characterisation of the theory of resource convertibility for $\mathrm{PCD}(\mathbf{Set}, \mathbf{Inj})$.

Corollary 14. *We have an isomorphism of ordered monoids*

$$G : \big(|\mathrm{PCD}(\mathbf{Set}, \mathbf{Inj})|, \geq, \sqcup\big) \longrightarrow \big(\mathrm{FinSupp}_{\mathbb{N}}(\mathbb{N} \setminus \{0, 1\}), \geq, +\big);$$
$$f \longmapsto \big(i \mapsto G_i(f)\big).$$

5   Some remarks on further directions

While we have indicated how to construct additive monotones on resource theories of parallel-combinable processes with discarding, there is work to be done to understand their complete families better. In particular, an existence theorem or otherwise for complete families of additive monotones would be of interest, as well as a notion of minimally complete family of monotones. Some first steps towards such results can be found in [5, §6-7].

Observe also that the two examples of Section 4 have an interesting property: they in fact form a triple of inclusions of symmetric monoidal categories

$$\mathbf{Bij} \longhookrightarrow \mathbf{Inj} \longhookrightarrow \mathbf{Set}.$$

We might call this a doubly-partitioned process theory. This nested structure seems to have been reflected in the construction of complete families of additive monotones: we built one complete family from the other. We wonder whether this could be done more generally.

A third salient question is the relationship between different methods of constructing resource theories from partitioned process theories. A place to start is perhaps to examine whether Theorem 10 has analogues for related constructions.

Acknowledgements

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