

# Qutrit Dichromatic Calculus and Its Universality

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We introduce a dichromatic calculus (RG) for qutrit systems. We show that the decomposition of the qutrit Hadamard gate is non-unique and not derivable from the dichromatic calculus. As an application of the dichromatic calculus, we depict a quantum algorithm with a single qutrit. Since it is not easy to decompose an arbitrary  $d \times d$  unitary matrix into Z and X phase gates when  $d > 2$ , the proof of the universality of qudit ZX calculus for quantum mechanics is far from trivial. We construct a counterexample to Ranchin's universality proof, and give another proof by Lie theory that the qudit ZX calculus contains all single qudit unitary transformations, which implies that qudit ZX calculus, with qutrit dichromatic calculus as a special case, is universal for quantum mechanics.

## 1 Introduction

In [3], Coecke and Duncan developed dichromatic ZX-calculus for qubit systems. To extend the graphical calculus to higher dimensions, Ranchin considered qudit (d-dimensional quantum system) ZX-calculus [8]. At almost the same time, the authors of this paper investigated the theory and application of qutrit ZX-calculus [1]. Unlike in [8] and [1], we introduce two new rules P1 and P2 in this paper. The necessity of these two rules is demonstrated by depicting in dichromatic calculus the simplest quantum speed-up algorithm with a single qutrit [6].

In the qubit case, Duncan and Perdrix [5] proved that the Euler decomposition is not derivable from ZX calculus. In this paper, we prove similarly that the decomposition of the qutrit Hadamard gate is non-unique and not derivable from a dichromatic qutrit ZX-calculus.

For any d-dimensional quantum system ( $d \geq 2$ ), universality is a very important problem for ZX calculus. This means that the qudit ZX calculus can express any quantum state and gate. To the best of our knowledge, it is not easy to decompose an arbitrary  $d \times d$  unitary matrix into Z and X phase gates when  $d > 2$ . Thus the proof of the universality of qudit ZX calculus for quantum mechanics is far from trivial. Due to Brylinski [2], to prove the universality of qudit ZX calculus, it suffices to prove that the qudit ZX calculus contains all single qudit unitary transformations. Such a proof given in [8] is based on the fact [7] that the d-dimensional phase gates  $Z_d$  and  $X_d$  are sufficient to simulate all single qudit unitary transforms. For our understanding, only part of the whole family of  $Z_d$  phase gates can be represented by  $\Lambda_X$  phase gates (i.e., X phase gates) in [8]. Actually, we have a counterexample that some  $Z_d$  phase gates cannot be realized by  $\Lambda_X$  only. Thus another proof that the qudit ZX calculus contains all single qudit unitary transformations is requested. We solve this problem by the method of Lie algebra. Therefore the qudit ZX calculus, with qutrit dichromatic calculus as a special case, is universal for quantum mechanics.

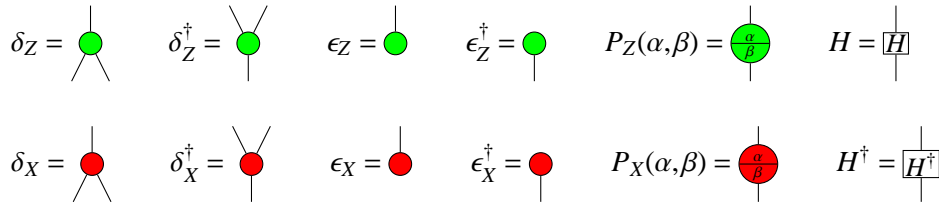
## 2 Red and Green Graphs

We fix some notations here. Let  $\mathbf{FdHilb}$  be the symmetric monoidal  $\dagger$ -category (SM $\dagger$ -category) of finite-dimensional complex Hilbert spaces and linear maps between them. Let  $\mathbf{FdHilb}_p$  be The SM $\dagger$ -category

of finite-dimensional complex Hilbert spaces and linear maps modulo the relation  $f \equiv g$  if  $\exists z \in \mathbb{C}, z \neq 0 : f = zg$ .  $\mathbf{FdHilb}_{\mathbb{Q}}$  is defined as the full subcategory of  $\mathbf{FdHilb}_{\mathbb{p}}$  generated by the objects  $\underbrace{\{Q \otimes \cdots \otimes Q \mid n \geq 0, \text{ where } Q := \mathbb{C}^3\}}_n$ . This is essentially the category of qutrits.

### 2.1 RG category

We define a category  $\mathbf{RG}$  where the objects are  $n$ -fold monoidal products of an object  $*$ , denoted  $*^n (n \geq 0)$ . In  $\mathbf{RG}$ , a morphism from  $*^m$  to  $*^n$  is a finite undirected open graph from  $m$  wires to  $n$  wires, built from

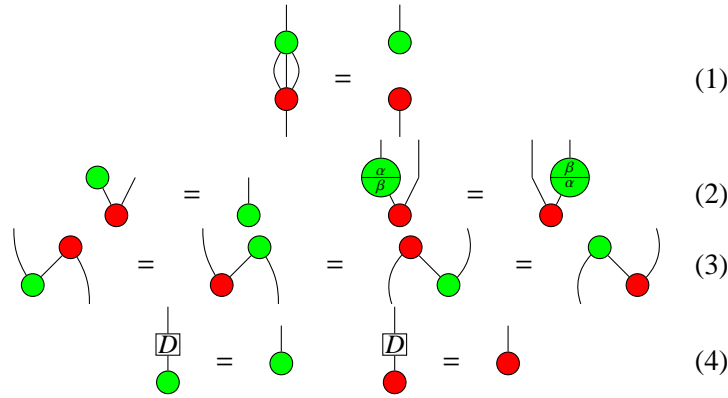


where  $\alpha, \beta \in [0, 2\pi)$ . For convenience, we denote the frequently used angles  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$  by 1 and 2 respectively. The generator  $H$  is called a Hadamard gate. Additionally, the identity morphism on  $*$  is represented as the straight wire. Composition is connecting up the edges, while tensor is simply putting two diagrams side by side. We also mention here that we ignore connected components of a graph which are connected to neither input nor output. This is in order to not have to deal with scalars.

$\mathbf{RG}$  morphisms are also subject to the equations depicted below.

1. Equations in Figure1.
2. All equations hold under flip of graphs, negation of angles, and exchange of  $H$  and  $H^\dagger$ .
3. All equations hold under flip of colours (except for rules  $K2$  and  $H2$ ).

The equations below can be derived from the rules of  $\mathbf{RG}$  given above. They are very useful when demonstrating some more complex equalities in describing quantum protocols [1] and algorithms[6].



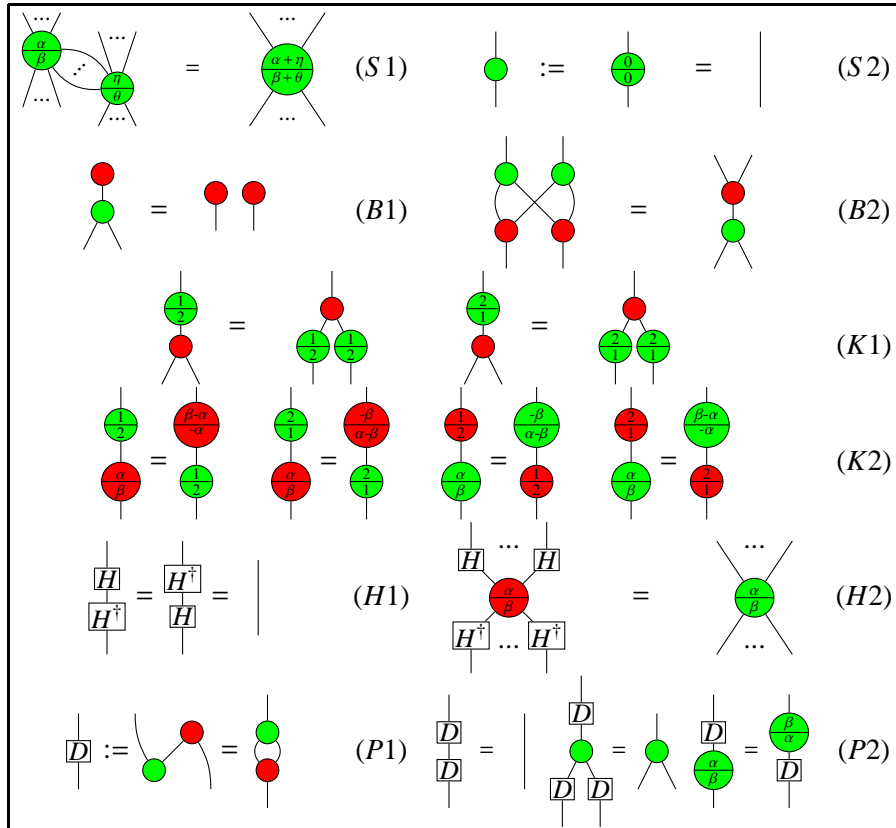
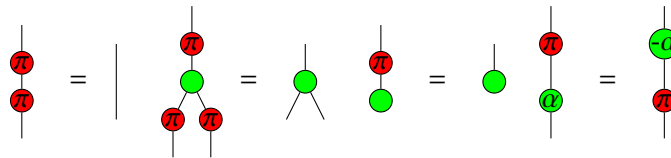


Figure 1: RG rules

It is worth noting that there are some remarkable differences between qutrit rules and qubit rules. First, in qubit case we have  $\begin{matrix} \bullet \\ \diagup \\ \bullet \end{matrix} = \begin{matrix} | \\ | \end{matrix}$ , while in qutrit case we have  $\begin{matrix} \bullet \\ \diagup \\ \bullet \end{matrix} = \begin{matrix} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{matrix}$ . Second, the dualizer of the two observables  $Z$  and  $X$  is an even permutation, i.e., the identical permutation. And there is only one odd permutation  $\pi$  in qubit case such that



While in qutrit case, the dualizer of  $Z$  and  $X$  is an odd permutation which satisfies rule  $P2$ . Third, in qubit case the  $K2$  rule still holds when flipping the colours, while it doesn't hold under flip of colours in qutrit case.

Now  $\mathbf{RG}$  is a symmetric monoidal category, which can further be made into a  $\dagger$ -SMC by having  $\dagger$  act on the generators as follows:

$$\begin{array}{cccccc}
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} \alpha \\ \beta \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} -\alpha \\ -\beta \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \boxed{H} \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \boxed{H^\dagger} \end{array} \\
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \bullet \end{array} &
\left( \begin{array}{c} \alpha \\ \beta \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} -\alpha \\ -\beta \\ \bullet \end{array} &
\left( \begin{array}{c} | \\ | \\ \boxed{H^\dagger} \end{array} \right)^\dagger = \begin{array}{c} | \\ | \\ \boxed{H} \end{array}
\end{array}$$

where functoriality of  $(\cdot)^\dagger$  is guaranteed by Rule 2.

## 2.2 RG interpretation

Here, we give an interpretation for these graphs by describing a monoidal functor  $[\cdot]_{RG} : \mathbf{RG} \rightarrow \mathbf{FdHilb}_{\mathbb{Q}}$ , mapping the morphisms as follows (expressed in Dirac notation):

$$\begin{array}{l}
\left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = |+\rangle \quad \left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = \langle +| \quad \left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = |00\rangle\langle 0| + |11\rangle\langle 1| + |22\rangle\langle 2| \\
\left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = |0\rangle\langle 00| + |1\rangle\langle 11| + |2\rangle\langle 22| \quad \left[ \begin{array}{c} \alpha \\ \beta \\ \bullet \end{array} \right]_{RG} = |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1| + e^{i\beta}|2\rangle\langle 2| \\
\left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = |0\rangle \quad \left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = \langle 0| \quad \left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = |++\rangle\langle +| + |\omega\omega\rangle\langle \omega| + |\bar{\omega}\bar{\omega}\rangle\langle \bar{\omega}| \\
\left[ \begin{array}{c} | \\ | \\ \bullet \end{array} \right]_{RG} = |+\rangle\langle ++| + |\omega\rangle\langle \omega\omega| + |\bar{\omega}\rangle\langle \bar{\omega}\bar{\omega}| \quad \left[ \begin{array}{c} \alpha \\ \beta \\ \bullet \end{array} \right]_{RG} = |+\rangle\langle +| + e^{i\alpha}|\omega\rangle\langle \omega| + e^{i\beta}|\bar{\omega}\rangle\langle \bar{\omega}| \\
\left[ \begin{array}{c} | \\ | \\ \boxed{H} \end{array} \right]_{RG} = |+\rangle\langle 0| + |\omega\rangle\langle 1| + |\bar{\omega}\rangle\langle 2| \quad \left[ \begin{array}{c} | \\ | \\ \boxed{H^\dagger} \end{array} \right]_{RG} = |0\rangle\langle +| + |1\rangle\langle \omega| + |2\rangle\langle \bar{\omega}|
\end{array}$$

where  $\omega = e^{\frac{2}{3}\pi i}$ ,  $\bar{\omega} = e^{\frac{4}{3}\pi i}$ , and

$$\begin{cases} |+\rangle &= |0\rangle + |1\rangle + |2\rangle \\ |\omega\rangle &= |0\rangle + \omega|1\rangle + \bar{\omega}|2\rangle \\ |\bar{\omega}\rangle &= |0\rangle + \bar{\omega}|1\rangle + \omega|2\rangle \end{cases}$$

**Proposition 2.1**  $[\cdot]_{RG}$  is a symmetric monoidal  $\dagger$ -functor.

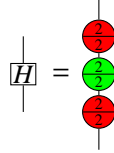
**Proof:** This involves checking for each rule  $f = g$  in  $\mathbf{RG}$  that  $[f]_{RG} = [g]_{RG}$ , that  $[\cdot]_{RG}$  respects the symmetric monoidal structure on the generators, and for each generator  $f$ , we have  $[f]_{RG}^\dagger = [f^\dagger]_{RG}$ .  $\square$

### 3 Decomposition of the Hadamard Gate

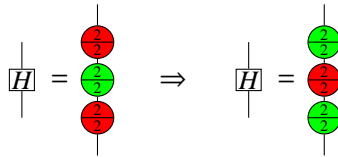
It can be directly checked that in  $\mathbf{FdHilb}_Q$ , we have

$$[H]_{RG} = [P_X(\frac{4\pi}{3}, \frac{4\pi}{3})]_{RG} \circ [P_Z(\frac{4\pi}{3}, \frac{4\pi}{3})]_{RG} \circ [P_X(\frac{4\pi}{3}, \frac{4\pi}{3})]_{RG}$$

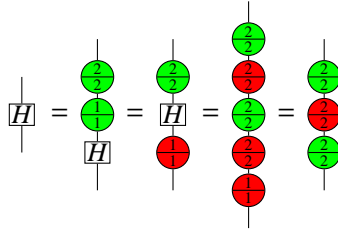
We call the following graph an Euler decomposition of the Hadamard gate:



**Proposition 3.1** *The Euler decomposition is not unique:*



**Proof:**



□

In the qubit case, Duncan and Perdrix [5] proved that the Euler decomposition is not derivable from ZX calculus. Similarly, we have

**Proposition 3.2** *The Euler decomposition is not derivable from RG.*

**Proof:** We define an alternative interpretation functor  $[\cdot]_0 : \mathbf{RG} \rightarrow \mathbf{FdHilb}_Q$  exactly as  $[\cdot]_{RG}$  with the following change:

$$[P_X(\alpha, \beta)]_0 = [P_X(0, 0)]_{RG} \quad [P_Z(\alpha, \beta)]_0 = [P_Z(0, 0)]_{RG}$$

This functor preserves all the rules introduced in Figure 1, so its image is indeed a valid model of the theory. However we have the following inequality

$$[H]_0 \neq [P_X(\frac{4\pi}{3}, \frac{4\pi}{3})]_0 \circ [P_Z(\frac{4\pi}{3}, \frac{4\pi}{3})]_0 \circ [P_X(\frac{4\pi}{3}, \frac{4\pi}{3})]_0$$

hence the Euler decomposition is not derivable from  $\mathbf{RG}$ .

□

### 4 Quantum Algorithm with a Single Qutrit

Recently, Gedik[6] introduces a simple algorithm using only a single qutrit to determine the parity of permutations of a set of three objects. As in the case of Deutsch’s algorithm, a speed-up relative to corresponding classical algorithms is obtained.

















Consider the six permutations of the set  $\{0, 1, 2\}$ . Each permutation can be treated as a function  $f(x)$  defined on the set  $x \in \{0, 1, 2\}$ . Then the task is to determine its parity. The problem could be solved by evaluating  $f(x)$  for two different values of  $x$ .

The function  $f$  has a domain and range of three values. These three values correspond to the three states of a qutrit  $|m\rangle$  where  $m = 0, 1, 2$ . The unitary  $U_f$  corresponding to the function  $f$  is a simple transposition of orthonormal states  $|m\rangle$ . Applying  $U_f$  to the eigenstate  $|\omega\rangle$  of the  $X$  observable we obtain

$$\begin{cases} U_f|\omega\rangle = |\omega\rangle(\text{up to a phase}) & \text{if } f \text{ is an even permutation;} \\ U_f|\omega\rangle = |\bar{\omega}\rangle(\text{up to a phase}) & \text{if } f \text{ is an odd permutation.} \end{cases}$$

Thus, a single evaluation of the function is enough to determine its parity.

The above algorithm can be depicted by the dichromatic calculus as follows:

$f$	(0)	(1 2)(0 1)	(1 2)(0 2)	(1 2)	(0 1)	(0 2)
$U_f$						
$U_f w\rangle$		 = 	 = 		 = 	 = 
Parity	Even			Odd		

### 5 The Qudit ZX Calculus Is Universal

It is important to prove that the qudit ZX calculus is universal for quantum mechanics for any  $d$ . Since it is not easy to decompose an arbitrary  $d \times d$  unitary matrix into Z and X phase gates (i.e.,  $\Lambda_Z$  and  $\Lambda_X$  gates) when  $d > 2$ , the proof of universality is far from trivial. Due to Brylinski [2], to prove the universality of qudit ZX calculus for quantum mechanics, it suffices to prove that the qudit ZX calculus contains all single qudit unitary transformations. Such a proof given in [8] is based on the fact [7] that the  $d$ -dimensional phase gates  $Z_d, X_d$  are sufficient to simulate all single qudit unitary transforms, where

$$Z_d(b_0, b_1, \dots, b_{d-1}) : b_0|0\rangle + b_1|1\rangle + \dots + b_{d-1}|d-1\rangle \mapsto |d-1\rangle$$

(the  $d$  complex coefficients,  $b_0, b_1, \dots, b_{d-1}$  are normalized to unity)

$$X_d(\phi) : \begin{cases} |d-1\rangle \mapsto e^{i\phi}|d-1\rangle \\ |p\rangle \mapsto |p\rangle \text{ for } p \neq d-1 \end{cases}$$

It was checked in [8] that each  $X_d$  can be encoded to a phase gate  $\Lambda_Z$  of the qudit ZX calculus, where

$$\Lambda_Z(\alpha_1, \alpha_2, \dots, \alpha_{d-1}) := \begin{pmatrix} 1 & & & \\ & e^{i\alpha_1} & & \\ & & \ddots & \\ & & & e^{i\alpha_{d-1}} \end{pmatrix}$$

Meanwhile, some  $Z_d$  phase gates were shown to be realized by  $\Lambda_X$  phase gate in the qudit ZX calculus, where

$$\Lambda_X(\alpha_1, \alpha_2, \dots, \alpha_{d-1}) := \frac{1}{d} \begin{pmatrix} c_0 & c_{d-1} & c_{d-2} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{d-1} & \dots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \dots & c_4 & c_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{d-1} & c_{d-2} & c_{d-3} & \dots & c_1 & c_0 \end{pmatrix}$$

$c_k = 1 + \sum_{l=1}^{d-1} \eta^{r_k(l)} e^{i\alpha_l}$ ,  $r_k$  permutes the entries 1 (there is one  $r_k$  for each  $k$ ).

However, not every  $Z_d$  phase gate can be represented by  $\Lambda_X(\alpha_1, \alpha_2, \dots, \alpha_{d-1})$ . In fact, to realize any  $Z_d(b_0, b_1, \dots, b_{d-1})$  in this way, we need to find  $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$  such that

$$c_{d-1}b_0 + c_{d-2}b_1 + \dots + c_1b_{d-2} + c_0b_{d-1} = d \quad (1)$$

$$c_k b_0 + c_{k-1} b_1 + \dots + c_0 b_k + c_{d-1} b_{k+1} + \dots + c_{k+1} b_{d-1} = 0, \quad \forall k \neq d-1 \quad (2)$$

Since  $\sum_{k=0}^{d-1} c_k = d$ , summing up all the equations in (1) and (2), we have  $\sum_{k=0}^{d-1} b_k = 1$ . Clearly, not every unit complex vector  $(b_0, b_1, \dots, b_{d-1})$  satisfies  $\sum_{k=0}^{d-1} b_k = 1$  or  $\sum_{k=0}^{d-1} b_k = e^{i\alpha}$  up to a global phase.

For example,  $(b_0, b_1, \dots, b_{d-1}) = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$ ,  $d > 2$ , is such a counterexample.

The above argument means that we need to find another proof that the qudit ZX calculus contains all single qudit unitary transformations. Next we solve this problem using the theory of Lie algebra.

Let

$$H = \left\{ \left( \begin{array}{cccc} e^{i\alpha_0} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & e^{i\alpha_{d-1}} \end{array} \right) \mid \alpha_0, \dots, \alpha_{d-1} \in \mathbb{R} \right\}, \quad V = \frac{1}{\sqrt{d}} \sum_{j,k=0}^{d-1} \omega^{jk} |j\rangle \langle k|, \quad \omega = e^{i\frac{2\pi}{d}}, \quad H' = VHV^{-1}$$

**Proposition 5.1** *Both  $H$  and  $H'$  are closed connected subgroups of the compact Lie group of unitaries  $G = U(d)$ .*

**Proof:** Let  $S^1 = \{e^{i\alpha} \mid \alpha \in \mathbb{R}\}$ . Clearly, the circle  $S^1$  is closed and connected. Since  $H \cong S^1 \times \dots \times S^1$ ,  $H$  is also a closed connected group. It is obvious that  $H'$  is topologically isomorphic to  $H$ . Thus  $H'$  is a closed connected group.  $\square$

We need two lemmas as follows.

**Lemma 5.2** [2] *Let  $G$  be a compact Lie group. If  $H_1, \dots, H_k$  are closed connected subgroups and they generate a dense group of  $G$ , then in fact they generate  $G$ .*

**Lemma 5.3** [2] *Let  $\mathfrak{h} = \text{Lie } H$ ,  $\mathfrak{h}' = \text{Lie } H'$ ,  $\mathfrak{g} = \text{Lie } U(d)$ . If  $\mathfrak{h}$  and  $\mathfrak{h}'$  generate  $\mathfrak{g}$  as a Lie algebra, and  $H$  and  $H'$  are closed connected groups, then  $H$  and  $H'$  generate a dense subgroup of  $U(d)$ .*

We choose the following matrices [4] as a basis of the Lie algebra  $\mathfrak{g}$ :

$$\sigma_x^{(jk)} (0 \leq j < k \leq d-1), \sigma_y^{(jk)} (0 \leq j < k \leq d-1), \sigma_z^{(jk)} (j=0, 1 \leq k \leq d-1), iI_d$$

where

$$\sigma_x^{(jk)} = i|j\rangle\langle k| + i|k\rangle\langle j|, \sigma_y^{(jk)} = |j\rangle\langle k| - |k\rangle\langle j|, \sigma_z^{(jk)} = i|j\rangle\langle j| - i|k\rangle\langle k|$$

It is easily checked that

$$\mathfrak{h} = \left\{ \sum_{j=0}^{d-1} i\alpha_j |j\rangle\langle j| \mid \alpha_j \in \mathbb{R} \right\} = \text{span} \{ \sigma_z^{(0j)} (1 \leq j \leq d-1), iI_d \}$$

$$\mathfrak{h}' = \left\{ \sum_{j=0}^{d-1} i\alpha_j V|j\rangle\langle j|V^{-1} \mid \alpha_j \in \mathbb{R} \right\} = \text{span} \{ V\sigma_z^{(0j)}V^{-1} (1 \leq j \leq d-1), iI_d \}$$

**Theorem 5.4** *Let  $\mathfrak{m}$  be the Lie subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{h}$  and  $\mathfrak{h}'$ . Then all the  $\sigma_x^{(jk)}$  ( $0 \leq j < k \leq d-1$ ) and  $\sigma_y^{(jk)}$  ( $0 \leq j < k \leq d-1$ ) are included in  $\mathfrak{m}$ .*

**Proof:**  $\forall t \in \{1, \dots, d-1\}$ ,  $V\sigma_z^{(0t)}V^{-1} \in \mathfrak{m}$ .

$$V\sigma_z^{(0t)}V^{-1} = \left( \frac{1}{\sqrt{d}} \sum_{j,k=0}^{d-1} \omega^{jk} |j\rangle\langle k| (i|0\rangle\langle 0| - i|t\rangle\langle t|) \left( \frac{1}{\sqrt{d}} \sum_{j_1, k_1=0}^{d-1} \bar{\omega}^{j_1 k_1} |k_1\rangle\langle j_1| \right) \right) = \frac{i}{d} \sum_{j, j_1=0}^{d-1} (1 - \omega^{(j-j_1)t}) |j\rangle\langle j_1|$$

Thus  $\sum_{t=1}^{d-1} V\sigma_z^{(0t)}V^{-1} \in \mathfrak{m}$ . By direct calculation,

$$\mathcal{X} := \sum_{t=1}^{d-1} V\sigma_z^{(0t)}V^{-1} = \sum_{t=1}^{d-1} \frac{i}{d} \left( \sum_{j, j_1=0}^{d-1} (1 - \omega^{(j-j_1)t}) |j\rangle\langle j_1| \right) = \sum_{j, j_1=0, j \neq j_1}^{d-1} i|j\rangle\langle j_1| = \sum_{0 \leq j < k \leq d-1} \sigma_x^{(jk)} \in \mathfrak{m}$$

We give the Lie products between  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  as follows.

$$\left\{ \begin{array}{l} [\sigma_x^{(0t)}, \sigma_z^{(0t)}] = 2\sigma_y^{(0t)} \\ [\sigma_x^{(0k)}, \sigma_z^{(0t)}] = \sigma_y^{(0k)}, k \neq t \\ [\sigma_x^{(tk)}, \sigma_z^{(0t)}] = -\sigma_y^{(tk)}, 0 < t \neq k \\ [\sigma_x^{(jt)}, \sigma_z^{(0t)}] = \sigma_y^{(jk)}, 0 < j \neq t \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} [\sigma_y^{(0k)}, \sigma_z^{(0k)}] = -2\sigma_x^{(0k)} \\ [\sigma_y^{(0k)}, \sigma_z^{(0t)}] = -\sigma_x^{(kt)}, k \neq t \\ [\sigma_y^{(jk)}, \sigma_z^{(0j)}] = -\sigma_x^{(jk)}, 0 < j \neq k \\ [\sigma_y^{(jk)}, \sigma_z^{(0k)}] = \sigma_x^{(jk)}, 0 < j \neq k \end{array} \right. \quad (2)$$

From the Lie products listed above, we have

$$[\mathcal{X}, \sigma_z^{(0t)}] = \sum_{k=1}^{d-1} \sigma_y^{(0k)} + \sum_{k=0}^{d-1} \sigma_y^{(kt)} \in \mathfrak{m}, \forall t \in \{1, \dots, d-1\}.$$



$$\forall u \in \{1, \dots, d-1\}, \left[ \sum_{k=1}^{d-1} \sigma_y^{(0k)} + \sum_{k=0}^{d-1} \sigma_y^{(kt)}, \sigma_z^{(0u)} \right] = \begin{cases} -\sigma_x^{0u} - \sum_{k=0, k \neq u}^{d-1} \sigma_x^{(ku)}, & t \neq u \\ -2\sigma_x^{(0u)} - 2 \sum_{k=0, k \neq u}^{d-1} \sigma_x^{(ku)}, & t = u \end{cases}$$

Therefore,  $\sigma_x^{(0u)} + \sum_{k=0, k \neq u}^{d-1} \sigma_x^{(ku)} \in \mathfrak{m}, \forall u \in \{1, \dots, d-1\}$ .

Furthermore, for  $u, v \in \{1, \dots, d-1\}$ ,

$$[\sigma_x^{(0u)} + \sum_{k=0, k \neq u}^{d-1} \sigma_x^{(ku)}, \sigma_z^{(0v)}] = \begin{cases} 2\sigma_y^{(0u)} - \sigma_y^{(uv)} \in \mathfrak{m}, & v \neq u \\ 4\sigma_y^{(0u)} + \sum_{k=1, k \neq u}^{d-1} \sigma_y^{(ku)} \in \mathfrak{m}, & v = u \end{cases}$$

Thus

$$\sum_{v=1, v \neq u}^{d-1} (2\sigma_y^{(0u)} - \sigma_y^{(vu)}) = 2(d-2)\sigma_y^{(0u)} - \sum_{v=1, v \neq u}^{d-1} \sigma_y^{(vu)} \in \mathfrak{m}$$

$$\left( 2(d-2)\sigma_y^{(0u)} - \sum_{v=1, v \neq u}^{d-1} \sigma_y^{(vu)} \right) + 4\sigma_y^{(0u)} + \sum_{k=1, k \neq u}^{d-1} \sigma_y^{(ku)} = 2d\sigma_y^{(0u)} \in \mathfrak{m}$$

i.e.,  $\sigma_y^{(0u)} \in \mathfrak{m}, \forall u \in \{1, \dots, d-1\}$ .

Immediately, we get  $\sigma_y^{(vu)} \in \mathfrak{m}, \forall u \neq v, u, v \in \{1, \dots, d-1\}$ .

Up to now, all the  $\sigma_y^{(jk)} (0 \leq j < k \leq d-1)$  are included in  $\mathfrak{m}$ . Still from the Lie products listed in (2), we know that all the  $\sigma_x^{(jk)} (0 \leq j < k \leq d-1)$  are included in  $\mathfrak{m}$ .  $\square$

Theorem (5.4) means that  $\mathfrak{h}$  and  $\mathfrak{h}'$  generate the Lie algebra  $\mathfrak{g}$ . It follows from proposition(5.1), lemma (5.2) and lemma (5.3) that  $H$  and  $H'$  generate  $U(d)$ , which means qudit ZX Calculus contains all single qudit unitary transformations. Therefore the qudit ZX calculus is universal for quantum mechanics.

## 6 Conclusion and Future Work

In this paper, we introduce a dichromatic calculus (RG) for qutrit systems. We show that the decomposition of the qutrit Hadamard gate is non-unique and not derivable from the dichromatic calculus. As an application of the dichromatic calculus, we depict a quantum algorithm with a single qutrit. Furthermore, for any  $d$ , we prove that the qudit ZX calculus contains all single qudit unitary transformations. It follows that qudit ZX calculus, with qutrit dichromatic calculus as a special case, is universal for quantum mechanics.

There are many issues requiring further exploration. Here we just list a few of them as follows. First, does there exist a formula in which each unitary is decomposed into X and Z phase gates? Second, is the dichromatic ZX calculus complete for qutrit stabilizer quantum mechanics? Finally, is the dichromatic ZX calculus incomplete for qutrit quantum mechanics?

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