

Categories of Quantum and Classical Channels

(extended abstract)

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We introduce the CP*-construction on a dagger compact closed category as a generalisation of Selinger’s CPM-construction. While the latter takes a dagger compact closed category and forms its category of “abstract matrix algebras” and completely positive maps, the CP*-construction forms its category of “abstract C*-algebras” and completely positive maps. This analogy is justified by the case of finite-dimensional Hilbert spaces, where the CP*-construction yields the category of finite-dimensional C*-algebras and completely positive maps.

The CP*-construction fully embeds Selinger’s CPM-construction in such a way that the objects in the image of the embedding can be thought of as “purely quantum” state spaces. It also embeds the category of classical stochastic maps, whose image consists of “purely classical” state spaces. By allowing classical and quantum data to coexist, this provides elegant abstract notions of preparation, measurement, and more general quantum channels.

1 Introduction

One of the motivations driving categorical treatments of quantum mechanics is to place classical and quantum systems on an equal footing in a single category, so that one can study their interactions. The main idea of categorical quantum mechanics [1] is to fix a category (usually dagger compact) whose objects are thought of as state spaces and whose morphisms are evolutions. There are two main variations.

- “Dirac style”: Objects form *pure* state spaces, and isometric morphisms form *pure* state evolutions.
- “von Neumann style”: Objects form spaces of *mixed* states, and morphisms form *mixed* quantum evolutions, also known as *quantum channels*.

The prototypical example of a “Dirac style” category is **FHilb**, the category of finite-dimensional Hilbert spaces and linear maps. One can pass to a “von Neumann style category” by considering the C*-algebras of operators on these Hilbert spaces, with completely positive maps between them. Selinger’s CPM-construction provides an abstract bridge from the former to the latter [18], turning morphisms in \mathbf{V} into quantum channels in $\text{CPM}[\mathbf{V}]$. However, this passage loses the connection between quantum and classical channels. For example, $\text{CPM}[\mathbf{FHilb}]$ only includes objects corresponding to the entire state space of some quantum system, whereas we would often like to focus in subspaces corresponding to particular classical contexts. There have been several proposals to remedy this situation, which typically involve augmenting $\text{CPM}[\mathbf{V}]$ with extra objects to carry this classical structure. Section 1.1 provides an overview.

This extended abstract¹ introduces a new solution to this problem, called the CP*-construction, which is closer in spirit to the study of quantum information using C*-algebras (see e.g. [15]). Rather than

*Supported by the John Templeton Foundation.

†Supported by EPSRC Fellowship EP/L002388/1.

¹A long version of this extended abstract is now available: [8].

freely augmenting a category of quantum data with classical objects, we define a new category whose objects are “abstract C^* -algebras”, and whose morphisms are the analogue of completely positive maps. It is then possible to construct a full embedding of the category $\mathbf{CPM}[\mathbf{V}]$, whose image yields the purely quantum channels. Furthermore, there exists another full embedding of the category $\mathbf{Stoch}[\mathbf{V}]$ of classical stochastic maps, whose image yields the purely classical channels. The remainder of the category $\mathbf{CP}^*[\mathbf{V}]$ can be interpreted as mixed classical/quantum state spaces, which carry partially coherent quantum states, such as degenerate quantum measurement outcomes.

$$\mathbf{V} \longrightarrow \mathbf{CPM}[\mathbf{V}] \xrightarrow{\text{full, faithful}} \mathbf{CP}^*[\mathbf{V}] \xleftarrow{\text{full, faithful}} \mathbf{Stoch}[\mathbf{V}]$$

The paper is structured as follows. Section 2 introduces normalisable dagger Frobenius algebras. These form a crucial ingredient to the \mathbf{CP}^* -construction, which is defined in Section 3. Sections 4 and 5 then show that the \mathbf{CP}^* -construction simultaneously generalises Selinger’s \mathbf{CPM} -construction, consisting of “quantum channels”, and the \mathbf{Stoch} -construction, consisting of “classical channels”. We then consider two examples: Section 6 shows that $\mathbf{CP}^*[\mathbf{FHilb}]$ is the category of finite-dimensional C^* -algebras and completely positive maps, and Section 7 shows that $\mathbf{CP}^*[\mathbf{Rel}]$ is the category of (small) groupoids and inverse-respecting relations. Section 8 compares $\mathbf{CP}^*[\mathbf{V}]$ with Selinger’s extension of the \mathbf{CPM} -construction using split idempotents. Finally, Section 9 discusses the many possibilities this opens up.

1.1 Related Work

Selinger introduced two approaches to add classical data to $\mathbf{CPM}[\mathbf{V}]$ by either freely adding biproducts to $\mathbf{CPM}[\mathbf{V}]$ or freely splitting the \dagger -idempotents of $\mathbf{CPM}[\mathbf{V}]$ [19]. These new categories are referred to as $\mathbf{CPM}[\mathbf{V}]^\oplus$ and $\mathbf{Split}^\dagger[\mathbf{CPM}[\mathbf{V}]]$, respectively.

When $\mathbf{V} = \mathbf{FHilb}$, both categories provide “enough space” to reason about classical and quantum data, as any finite-dimensional C^* -algebra can be defined as a sum of matrix algebras (as in $\mathbf{CPM}[\mathbf{FHilb}]^\oplus$) or as a certain orthogonal subspace of a larger matrix algebra (as in $\mathbf{Split}^\dagger[\mathbf{CPM}[\mathbf{FHilb}]]$). However, it is unclear whether the second construction captures too much: it may contain many more objects than simply mixtures of classical and quantum state spaces [19, Remark 4.9]. On the other hand, when $\mathbf{V} \neq \mathbf{FHilb}$, the category $\mathbf{CPM}[\mathbf{FHilb}]^\oplus$ may be too small. That is, there may be interesting objects that are not just sums of quantum objects.

For this reason, it is interesting to study $\mathbf{CP}^*[\mathbf{V}]$, as it lies between these two constructions:

$$\mathbf{CPM}[\mathbf{V}]^\oplus \xrightarrow{\text{full, faithful}} \mathbf{CP}^*[\mathbf{V}] \xrightarrow{\text{full, faithful}} \mathbf{Split}^\dagger[\mathbf{CPM}[\mathbf{V}]]$$

The first embedding is well-defined whenever \mathbf{V} has biproducts, and the second when \mathbf{V} satisfies a technical axiom about square roots (see Definition 8.2). In the former case, $\mathbf{CP}^*[\mathbf{V}]$ inherits biproducts from \mathbf{V} , so it is possible to lift the embedding of $\mathbf{CPM}[\mathbf{V}]$ by the universal property of the free biproduct completion; this will be proved in detail in a subsequent paper. In the latter case, one can always construct the associated dagger-idempotent of an object in $\mathbf{CP}^*[\mathbf{V}]$, and (with the assumption from Definition 8.2), the notions of complete positivity in $\mathbf{CP}^*[\mathbf{V}]$ and $\mathbf{Split}^\dagger[\mathbf{CPM}[\mathbf{V}]]$ coincide. We provide the details of this construction in Section 8.

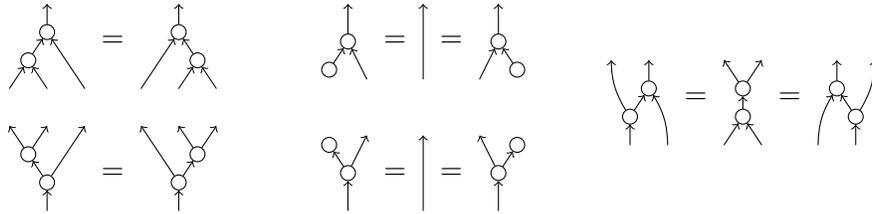
A third approach by Coecke, Paquette, and Pavlovic is similar to ours in that it makes use of commutative Frobenius algebras to represent classical data [9]. As in the previous two approaches, it takes $\mathbf{CPM}[\mathbf{V}]$ and freely adds classical structure, this time by forming the comonad associated with a particular commutative Frobenius algebra and taking the Kleisli category. All such categories are then glued

together by Grothendieck construction. The CP*-construction was originally conceived as a way to simplify this approach and overcome its limitations.

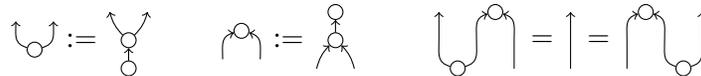
2 Abstract C*-algebras

This section defines so-called normalisable dagger Frobenius algebras, which will play a central role in the CP*-construction of the next section. We start by recalling the notion of a Frobenius algebra. For an introduction to dagger (compact) categories and their graphical calculus, we refer to [1, 20].

Definition 2.1. A *Frobenius algebra* is an object A in a monoidal category together with morphisms depicted as $\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}$, $\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array}$, $\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}$ and $\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array}$ on it satisfying the following diagrammatic equations.



Any Frobenius algebra defines a cap and a cup that satisfy the snake equation.



Definition 2.2. A Frobenius algebra is *symmetric* when $\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \begin{array}{c} \cup \\ \cup \end{array}$ and $\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} = \begin{array}{c} \cap \\ \cap \end{array}$.

In a dagger compact category \mathbf{V} , a *dagger Frobenius algebra* is a Frobenius algebra that satisfies $\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = (\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array})^\dagger$ and $\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} = (\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array})^\dagger$. Their import for us starts with the following theorem.

Theorem 2.3 ([21]). *Given a dagger Frobenius algebra $(A, \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array})$ in \mathbf{FHilb} , the following operation gives A the structure of a C*-algebra.*

$$\left(\begin{array}{c} \uparrow \\ \boxed{v} \end{array} \right)^* := \begin{array}{c} \boxed{v^\dagger} \\ \circlearrowright \end{array}$$

Furthermore, all finite-dimensional C*-algebras arise this way. □

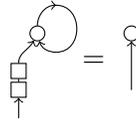
In light of this theorem, one might be tempted to consider dagger Frobenius algebras to be the “correct” way to define the abstract analogue of finite-dimensional C*-algebras. However, there is one more condition, called *normalisability*, that is satisfied by *all* dagger Frobenius algebras in \mathbf{FHilb} , yet not by dagger Frobenius algebras in general. Before we come to that, we introduce the notion of a central map for a monoid.

Definition 2.4. A map $z : A \rightarrow A$ is *central* for a monoid when $\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \circ (z \otimes 1_A) = z \circ \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \circ (1_A \otimes z)$.

We call such a map central, because it corresponds uniquely to a point $p_z : I \rightarrow A$ in the centre of the monoid via left (or equivalently right) multiplication by p_z .

Recall that a map $g : A \rightarrow A$ in a dagger category is *positive* when $g = h^\dagger \circ h$ for some map h . It is called *positive definite* when it is a positive isomorphism. Using these conditions, we give the definition of a normalisability as a well-behavedness property of the “loop” map $\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \circ \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array}$.

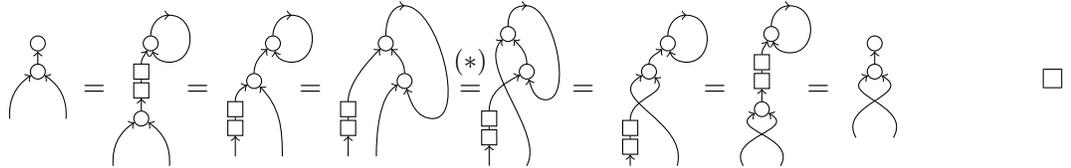
Definition 2.5. A dagger Frobenius algebra is *normalisable* when there is a central, positive-definite map z , called the *normaliser* and depicted as $\begin{array}{c} \uparrow \\ \boxed{z} \end{array}$, satisfying the following diagrammatic equation.



Special dagger Frobenius algebras are normalisable dagger Frobenius algebra where $z^2 = 1_A$. Normalisable dagger Frobenius algebras are always symmetric.

Theorem 2.6. *Normalisable dagger Frobenius algebras are symmetric.*

Proof. The proof follows from expanding the counit and applying cyclicity of the trace (*).



Definition 2.7. An object X in a dagger compact category is *positive-dimensional* if there is a positive definite scalar z satisfying

$$X \left| \begin{array}{c} z \\ z \end{array} \right. \bigcirc_X = \left| \begin{array}{c} z \\ z \end{array} \right. X$$

A dagger compact closed category is called positive-dimensional if all its objects are.

Proposition 2.8. *For a positive-dimensional dagger compact closed category \mathbf{V} , every object of the form $A^* \otimes A$ carries a canonical normalisable dagger Frobenius algebra, given as follows.*

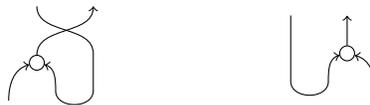


Proof. The Frobenius axioms follow immediately from compact closure. By positive-dimensionality, there exists a positive-definite scalar z such that $(z^2 \circ \text{Tr}_A(1_A)) \otimes 1_A = 1_A$. It is then possible to show that $1_{A^* \otimes A} \otimes z$ is the normaliser. \square

From now on, we will always take \mathbf{V} to be positive-dimensional.

3 The CP*-construction

This section defines the CP*-construction. In fact, defining it is easy; most work goes into proving that it yields a dagger compact category. For the definition we need some (graphical) notation, generalising the left and right regular actions $A \mapsto \text{End}(A)$ given by $x \mapsto x \cdot (-)$ and $x \mapsto (-) \cdot x$ for a finite-dimensional algebra A . More generally, for a monoid $(A, \uparrow, \downarrow)$ in a compact category, define its left and right action maps $A \rightarrow A^* \otimes A$ as follows.



Similarly, we can define for any comonoid left and right coaction maps. By convention, we work primarily with right action and coaction maps, and express them more succinctly as follows.



Definition 3.1. Let \mathbf{V} be a dagger compact category. Objects of $\text{CP}^*[\mathbf{V}]$ are normalizable dagger Frobenius algebras in \mathbf{V} . Morphisms from $(A, \hat{\circlearrowleft})$ to $(B, \hat{\circlearrowright})$ in $\text{CP}^*[\mathbf{V}]$ are morphisms $f: A \rightarrow B$ in \mathbf{V} such that there exists an object X and $g: A \rightarrow X \otimes B$ in \mathbf{V} satisfying the following equation.

$$\begin{array}{c} \text{---} \\ \circlearrowleft \\ \text{---} \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \downarrow \\ \boxed{g_*} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{g} \\ \uparrow \end{array} \quad (1)$$

Composition and identities are inherited from \mathbf{V} .

Equation (1) is called the *CP*-condition*. We first establish that $\text{CP}^*[\mathbf{V}]$ is a well-defined category.

Lemma 3.2. Any symmetric Frobenius algebra satisfies $\hat{\circlearrowleft} = \hat{\circlearrowright}$.

Proof. We can use symmetry to prove the result.

$$\begin{array}{c} \text{---} \\ \circlearrowleft \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowleft \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} \quad \square$$

Lemma 3.3. Any normalisable dagger Frobenius algebra satisfies $\hat{\circlearrowleft} \hat{\circlearrowright} = \uparrow$.

Proof. Follows from Frobenius equations and a trace identity (*) for symmetric Frobenius algebras.

$$\begin{array}{c} \text{---} \\ \circlearrowleft \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowleft \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} \stackrel{(*)}{=} \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \circlearrowright \\ \text{---} \end{array} \quad \square$$

These lemmas can express the *CP*-condition* in the sometimes more convenient “convolution form”.

Proposition 3.4. Let \mathbf{V} be a dagger compact category, $(A, \hat{\circlearrowleft})$ and $(B, \hat{\circlearrowright})$ be normalisable dagger Frobenius algebras, and $f: A \rightarrow B$ a morphism.

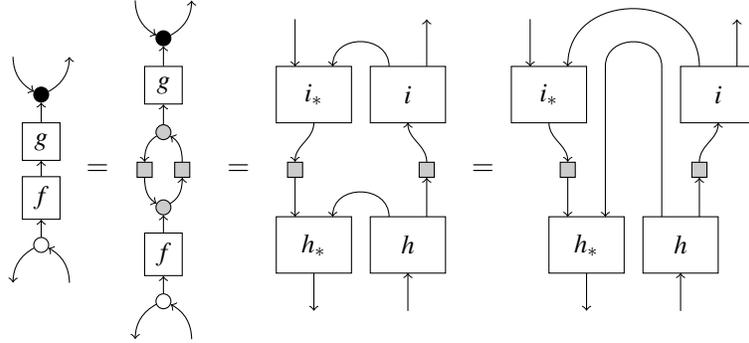
$$\exists g. \begin{array}{c} \text{---} \\ \circlearrowleft \\ \text{---} \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \downarrow \\ \boxed{g_*} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{g} \\ \uparrow \end{array} \Leftrightarrow \exists h. \begin{array}{c} \text{---} \\ \circlearrowleft \\ \text{---} \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \circlearrowright \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \text{---} \\ \downarrow \\ \boxed{h_*} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{h} \\ \uparrow \end{array}$$

Proof. For (\Rightarrow) , apply $(\hat{\circlearrowleft} \circ - \circ \hat{\circlearrowright})$ to both sides and use Lemma 3.3 and properties of normalisers. For (\Leftarrow) , apply $(\hat{\circlearrowright} \circ - \circ \hat{\circlearrowleft})$ to both sides and apply Lemma 3.2. □

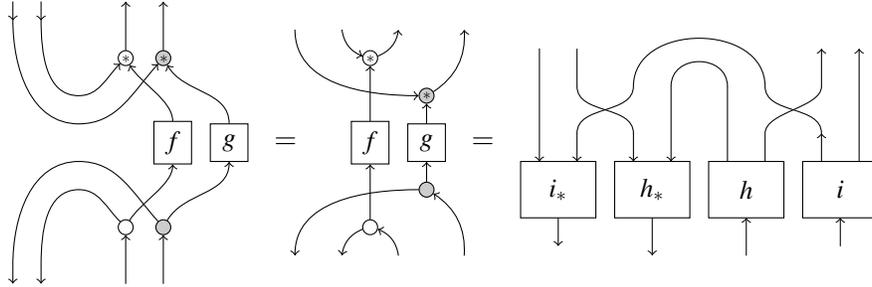
Theorem 3.5. If \mathbf{V} is a dagger compact category \mathbf{V} , so is $\text{CP}^*[\mathbf{V}]$.

Proof. Identity maps $1_A: (A, \hat{\rho}_A) \rightarrow (A, \hat{\rho}_A)$ satisfy the CP*-condition: letting $g = \hat{\rho}_A$ in (1) does the job by Lemma 3.2, whose left-hand side is $\hat{\rho}_A \circ 1_A \circ \hat{\rho}_A$.

Next, suppose $f: (A, \hat{\rho}_A) \rightarrow (B, \hat{\rho}_B)$ and $g: (B, \hat{\rho}_B) \rightarrow (C, \hat{\rho}_C)$ satisfy the CP*-condition. It then follows from Lemma 3.3 that their composition does, too.



For the monoidal structure, take $(A, \hat{\rho}_A) \otimes (B, \hat{\rho}_B) := (A \otimes B, \hat{\rho}_{A \otimes B})$. For maps $f: (A, \hat{\rho}_A) \rightarrow (C, \hat{\rho}_C)$ and $g: (B, \hat{\rho}_B) \rightarrow (D, \hat{\rho}_D)$ satisfying (1), also $f \otimes g: (A \otimes B, \hat{\rho}_{A \otimes B}) \rightarrow (C \otimes D, \hat{\rho}_{C \otimes D})$ satisfies the CP*-condition. This can be seen by applying the coaction of $\hat{\rho}_{A \otimes B}$ and the action of $\hat{\rho}_{C \otimes D}$, then decomposing f into h_*, h and g into i_*, i , as follows.



As for the tensor unit, note that $I := (I, \rho_I)$ is a normalisable dagger Frobenius algebra by monoidal coherence in \mathbf{V} . Using this definition of \otimes and I , the monoidal structure maps α , λ , and ρ from \mathbf{V} trivially satisfy the CP*-condition. Thus $\text{CP}^*[\mathbf{V}]$ is a monoidal category. $\text{CP}^*[\mathbf{V}]$ inherits the dagger from \mathbf{V} . Symmetry and dual maps in \mathbf{V} lift to the following morphisms in $\text{CP}^*[\mathbf{V}]$.

$$\sigma_{A,B}: (A \otimes B, \hat{\rho}_{A \otimes B}) \rightarrow (B \otimes A, \hat{\rho}_{B \otimes A}) \quad e_{A^*}: (A, \hat{\rho}_A) \otimes (A^*, \hat{\rho}_{A^*}) \rightarrow I$$

The Frobenius identities and Lemma 3.2 establish the CP*-condition for these maps. \square

We refer to a morphism $I \rightarrow (A, \hat{\rho}_A)$ of $\text{CP}^*[\mathbf{V}]$ as a *positive element* of $(A, \hat{\rho}_A)$. Another way to express the CP*-condition for a \mathbf{V} -morphism is to say that it preserves the property of being a positive element when applied to a some subsystem, as in the following theorem. This will be useful to connect to the traditional notion of complete positivity of linear maps between C*-algebras.

Theorem 3.6. *Let $(A, \hat{\rho}_A)$ and $(B, \hat{\rho}_B)$ be normalisable dagger Frobenius algebras and $f: A \rightarrow B$ a morphism in a dagger compact category \mathbf{V} . The following are equivalent:*

- (a) f satisfies the CP*-condition;
- (b) postcomposing with $f \otimes 1_C$ sends positive elements of $(A, \hat{\rho}_A) \otimes (C, \hat{\rho}_C)$ to positive elements of $(B, \hat{\rho}_B) \otimes (C, \hat{\rho}_C)$ for all dagger normalisable Frobenius algebras $(C, \hat{\rho}_C)$;

(c) *postcomposing with $f \otimes 1_{X^* \otimes X}$ sends positive elements of $(A, \hat{\rho}_A) \otimes (X^* \otimes X, \hat{\rho}_{X^* \otimes X})$ to positive elements of $(B, \hat{\rho}_B) \otimes (X^* \otimes X, \hat{\rho}_{X^* \otimes X})$ for all objects X in \mathbf{V} .*

Proof. For (a) \Rightarrow (b): if ρ is a positive element of $(A, \hat{\rho}_A) \otimes (C, \hat{\rho}_C)$ and f satisfies the CP*-condition, then so does $(f \otimes 1_C) \circ \rho$, by Theorem 3.5. The implication (b) \Rightarrow (c) is trivial. Finally, for (c) \Rightarrow (a), take $(C, \hat{\rho}_C) = (A^*, \hat{\rho}_{A^*})$. The action map $\hat{\rho}_A : (C, \hat{\rho}_C) \rightarrow (C^* \otimes C, \hat{\rho}_{C^* \otimes C})$ is a morphism in $\text{CP}^*[\mathbf{V}]$. As a consequence of this fact and Theorem 3.5, the following is a positive element of $(A, \hat{\rho}_A) \otimes (X^* \otimes X, \hat{\rho}_{X^* \otimes X})$.

So, by assumption, $(f \otimes 1_{A^*}) \circ \rho$ is also a positive element. Applying white caps to both sides finishes the proof.

See also [8, Proposition 3.4]. □

4 Embedding Selinger’s CPM–construction

This section will concentrate on the “purely quantum” objects in $\text{CP}^*[\mathbf{V}]$, by proving that the latter embeds $\text{CPM}[\mathbf{V}]$ in a full, faithful, and strongly dagger symmetric monoidal way. First, we recall Selinger’s CPM–construction [18].

Definition 4.1. For a dagger compact category \mathbf{V} , form the dagger compact category $\text{CPM}[\mathbf{V}]$ as follows. Objects are the same as those in \mathbf{V} , and morphisms $f \in \text{CPM}[\mathbf{V}](A, B)$ are \mathbf{V} -morphisms $f: A^* \otimes A \rightarrow B^* \otimes B$ for which there is $g: A \rightarrow X \otimes B$ satisfying the following condition.

Composition, identity maps, and \otimes on objects are defined as in \mathbf{V} . On morphisms, \otimes is defined as:

A strongly dagger symmetric monoidal functor is a functor F along with a unitary natural isomorphism $\varphi_{A,B}: F(A \otimes B) \rightarrow F(A) \otimes F(B)$ satisfying several coherence properties that we have no space to go in to. Our next theorem shows that $\text{CPM}[\mathbf{V}]$ is equivalent to the full subcategory of $\text{CP}^*[\mathbf{V}]$ consisting of objects of the form $(A^* \otimes A, \hat{\rho}_A)$. Its proof uses $*$ -homomorphisms, which we first define.

Definition 4.2. If $(A, \hat{\circlearrowleft})$ and $(B, \hat{\circlearrowright})$ are normalisable dagger Frobenius algebras in a dagger compact category \mathbf{V} , a morphism $f: A \rightarrow B$ is called a **-homomorphism* when it satisfies the following equations.

Lemma 4.3. Let $(A, \hat{\circlearrowleft})$ and $(B, \hat{\circlearrowright})$ be normalisable dagger Frobenius algebras in a dagger compact category \mathbf{V} . If $f: A \rightarrow B$ is a *-homomorphism, then it is a well-defined morphism in $\text{CP}^*[\mathbf{V}]$.

Proof. Using the definition: $\hat{\circlearrowleft} \circ f \circ \hat{\circlearrowright} = (\hat{\circlearrowleft} \circ \hat{\circlearrowright}) \circ (f^* \otimes f)$. Applying Lemma 3.2 completes the proof. \square

Theorem 4.4. Let \mathbf{V} be a positive-dimensional dagger compact category. Define $L: \text{CPM}[\mathbf{V}] \rightarrow \text{CP}^*[\mathbf{V}]$ by setting $L(A) := (A^* \otimes A, \hat{\circlearrowleft})$ on objects and $L(f) = f$ on morphisms. Then L is a well-defined functor that is full, faithful, and strongly dagger symmetric monoidal.

Proof. For well-definedness, we show that a \mathbf{V} -morphism $f: A^* \otimes A \rightarrow B^* \otimes B$ is a $\text{CPM}[\mathbf{V}]$ -morphism from A to B if and only if it is a $\text{CP}^*[\mathbf{V}]$ -morphism from $(A^* \otimes A, \hat{\circlearrowleft})$ to $(B^* \otimes B, \hat{\circlearrowright})$. First, assume $f \in \text{CPM}[\mathbf{V}]$ and compose with the action and coaction of the respective algebras to see that f satisfies the CP^* -condition, as follows.

Conversely, if f is in $\text{CP}^*[\mathbf{V}]$, then it is also in $\text{CPM}[\mathbf{V}]$, as follows.

Composition is the same in $\text{CPM}[\mathbf{V}]$ and $\text{CP}^*[\mathbf{V}]$, so L is a functor that is furthermore full and faithful. Similarly, daggers are the same in $\text{CPM}[\mathbf{V}]$ and $\text{CP}^*[\mathbf{V}]$, so L trivially preserves daggers. It now suffices to show that L is strongly monoidal. Define the isomorphism $\varphi_{A,B}: L(A \otimes B) \rightarrow L(A) \otimes L(B)$ as the reshuffling map (321)(4): $B^* \otimes A^* \otimes A \otimes B \rightarrow A^* \otimes A \otimes B^* \otimes B$. One can verify that this map is a *-homomorphism from $L(A \otimes B)$ to $L(A) \otimes L(B)$, so it must satisfy the CP^* -condition. This map is also unitary, and it is a routine calculation to show that it satisfies the coherence equations for a strong symmetric monoidal functor. \square

5 Generalised stochastic maps and measurement of quantum states

Whereas the previous section focused on “purely quantum” objects in $\mathbf{CP}^*[\mathbf{V}]$, this section looks at the “purely classical” ones. We will define a “purely probabilistic” category $\mathbf{Stoch}[\mathbf{V}]$, that by construction embeds into $\mathbf{CP}^*[\mathbf{V}]$. Thus objects in $\mathbf{CP}^*[\mathbf{V}]$ can be interpreted as being “combined classical and quantum”. The category $\mathbf{Stoch}[\mathbf{V}]$ was first considered in [9]. It was defined in a slightly different form there, but one can prove that this coincides with the following definition.

Definition 5.1. For a dagger compact category \mathbf{V} , define $\mathbf{Stoch}[\mathbf{V}]$ to be the full subcategory of $\mathbf{CP}^*[\mathbf{V}]$ consisting of all commutative normalisable dagger Frobenius algebras.

The next proposition justifies why this category is that of “classical channels”. We call a morphism $f: (A, \hat{\bullet}) \rightarrow (B, \hat{\bullet})$ in $\mathbf{CP}^*[\mathbf{V}]$ *normalised* if it preserves counits: $\hat{\bullet} \circ f = \hat{\bullet}$. Because commutative finite-dimensional C^* -algebras correspond to finite-dimensional Hilbert spaces with a choice of orthonormal basis, we can think of the latter as objects of $\mathbf{Stoch}[\mathbf{FHilb}]$ [10, 2, 13]. Recall that a stochastic map between finite-dimensional Hilbert spaces is a matrix with positive real entries whose every column sums to one.

Proposition 5.2. *Normalised morphisms in $\mathbf{Stoch}[\mathbf{FHilb}]$ correspond to stochastic maps between finite-dimensional Hilbert spaces.*

Proof. See [15, 3.2.3 and 2.1.3]. □

For any Frobenius algebra $(A, \hat{\bullet})$ we can consider its *copyable points*: the morphisms $p: I \rightarrow A$ that are “copied” by the comultiplication, in the sense that $\hat{\bullet} \circ p = p \otimes p$. This is especially interesting for commutative normalisable dagger Frobenius algebras, because in \mathbf{FHilb} , copyable points form an orthonormal basis for A . Writing vectors in the basis of classical points, one can show that the normalised positive elements are precisely those vectors with positive coefficients that sum to 1. Thus, normalised positive elements of a commutative normalisable dagger Frobenius algebra can be regarded as probability distributions over its copyable points.

So far we have mostly looked at classical and quantum systems in isolation. But as they live together in a category $\mathbf{CP}^*[\mathbf{V}]$, we can also consider maps between them. Consider a normalised morphism $P: L(H) \rightarrow (A, \hat{\bullet})$ from a quantum to a classical system. Then $P^\dagger \circ x_i$ is a positive element of $\hat{\bullet}$ for each copyable point x_i . Furthermore, any commutative normalisable dagger Frobenius algebra in \mathbf{FHilb} satisfies $\hat{\bullet} = \sum_i x_i$. Since P is normalised, $\sum P^\dagger \circ x_i = P^\dagger \circ \sum x_i = P^\dagger \circ \hat{\bullet} = e_H: I \rightarrow H^* \otimes H$ is the cup from the compact structure on H . Positive elements in $H^* \otimes H$ represent positive operators from H to itself, and e_H represents the identity operator. Thus the set $\{P^\dagger \circ x_i\}$ corresponds to a positive operator-valued measure (POVM). Furthermore, for any quantum state ρ (i.e. normalised positive element in $L(H)$), it is straightforward to show that $P \circ \rho$ yields the probability distribution whose i th element is the probability of getting outcome x_i , computed via the Born rule.

The dual notion of a morphism $E: (A, \hat{\bullet}) \rightarrow L(H)$ from a classical system to a quantum system can be thought of as a preparation. Or, more precisely, as a map carrying a classical probability distribution over some fixed set of states to a single mixed state. Choosing a particular decomposition $E \circ \rho$ for a quantum state ρ gives us a way to represent quantum ensembles. See also [16, 3.2.4].

6 Hilbert spaces

It is high time we looked at some examples. This section proves that $\mathbf{CP}^*[\mathbf{FHilb}]$ is the category of all finite-dimensional C^* -algebras and completely positive maps. The proof also illuminates Theorem 4.4.

Namely, $\text{CP}^*[\mathbf{FHilb}]$ has finite-dimensional C^* -factors for objects. Recall that a C^* -algebra is a factor when its centre is 1-dimensional. Finite-dimensional C^* -algebras in fact enjoy an easy structure theorem: they are finite direct sums of factors, and factors are precisely matrix algebras [12, Theorem III.1.1]. The following lemma recalls the structure of these factors, and the subsequent proposition determines the objects of $\text{CP}^*[\mathbf{FHilb}]$.

Lemma 6.1. *If H is an n -dimensional Hilbert space, then there is an isomorphism of algebras between $L(H)$ and $\mathbb{M}_n(\mathbb{C})$. Therefore, $H^* \otimes H$ carries C^* -algebra structure; the involution is as in Theorem 2.3.*

Proof. First of all, $\mathbb{M}_n(\mathbb{C})$ is a Hilbert space under the Hilbert–Schmidt inner product $\langle a | b \rangle = \text{Tr}(a^\dagger b)$. It has a canonical orthonormal basis $\{e_{ij} \mid i, j = 1, \dots, n\}$, where e_{ij} is the matrix all of whose entries are 0 except the (i, j) -entry, which is 1. Pick an orthonormal basis $\{|1\rangle, \dots, |n\rangle\}$ for H , so that $\{\langle i | \otimes | j \rangle \mid i, j = 1, \dots, n\}$ forms an orthonormal basis for $H^* \otimes H$. Then the assignment $\langle i | \otimes | j \rangle \mapsto e_{ij}$ implements a unitary isomorphism between $H^* \otimes H$ and $\mathbb{M}_n(\mathbb{C})$. Direct computation shows that matrix multiplication translates across this isomorphism to $\hat{\curvearrowright}$ on $H^* \otimes H$. Similarly, taking the conjugate transpose of a matrix corresponds to the involution on $H^* \otimes H$ given in Theorem 2.3. \square

Proposition 6.2. *All dagger Frobenius algebras in \mathbf{FHilb} are normalisable.*

Proof. Let $(A, \hat{\curvearrowright}, \hat{\circ})$ be a dagger Frobenius algebra in \mathbf{FHilb} . By Theorem 2.3, it must be isomorphic to a C^* -algebra of the form $\bigoplus_k \mathbb{M}_{n_k}(\mathbb{C})$, giving a unitary isomorphism of the associated dagger Frobenius algebras. Let $\{e_{ij}^{(k)} : 0 \leq i, j < n_k\}$ form an orthonormal basis for A . We can define $\hat{\curvearrowright}$ in terms of this basis as $\hat{\curvearrowright}(e_{ij}^{(k)} \otimes e_{i'j'}^{(k)}) = \delta_k^{k'} \delta_j^{j'} e_{ij}^{(k)}$. From this, we can compute $\text{Tr}_A(\hat{\curvearrowright})$ directly.

$$\text{Tr}_A(\hat{\curvearrowright})(e_{ij}^{(k)}) = \sum_{i'j'k'} \left(e_{i'j'}^{(k')}\right)^\dagger \hat{\curvearrowright}(e_{ij}^{(k)} \otimes e_{i'j'}^{(k)}) = \sum_{i'j'k'} \left(e_{i'j'}^{(k')}\right)^\dagger \delta_k^{k'} \delta_j^{j'} e_{ij}^{(k)} = \sum_{j'} \left(e_{jj'}^{(k)}\right)^\dagger e_{ij}^{(k)} = \sum_{j'} \delta_i^j = n_k \delta_i^j.$$

Note that $\hat{\circ}(e_{ij}^{(k)}) = \delta_i^j$. We can now define the normaliser $\hat{\square}$ as $e_{ij}^{(k)} \mapsto \frac{1}{\sqrt{n_k}} e_{ij}^{(k)}$: this map is invertible, satisfies $\text{Tr}_A(\hat{\curvearrowright}) \circ (\hat{\square})^2 = \hat{\circ}$, and acts by a constant scalar on each summand of A and so is central. \square

Combining the previous proposition with Theorem 2.3, we see that the objects of $\text{CP}^*[\mathbf{FHilb}]$ are (in 1-to-1 correspondence with) finite-dimensional C^* -algebras. We now turn to the morphisms of $\text{CP}^*[\mathbf{FHilb}]$. First, let us review what (concrete) completely positive maps between C^* -algebras are. Good references are [3, 17]. The main notion is that of a matrix algebra $\mathbb{M}_n(A)$ over a C^* -algebra A . Its elements are n -by- n matrices with entries in A , given C^* -structure by $(a_{ij}) \cdot (b_{ij}) = \sum_{k=1}^n a_{ik} b_{kj}$, $(a_{ij})^* = (a_{ji}^*)$, and $\|(a_{ij})\| = \sup\{\|(a_{ij})x\| \mid x \in A^n, \|x\| = 1\}$, where $\|(x_1, \dots, x_n)\|^2 = \sum_{i=1}^n \|x_i\|^2$. The following well-known lemma then follows directly.

Lemma 6.3. *If A is a finite-dimensional C^* -algebra, then so is $\mathbb{M}_n(A)$. If $f: A \rightarrow B$ is a linear map, then so is the function $\mathbb{M}_n(f): \mathbb{M}_n(A) \rightarrow \mathbb{M}_n(B)$ that sends (a_{ij}) to $(f(a_{ij}))$. If f is a $*$ -homomorphism, then so is $\mathbb{M}_n(f)$. \square*

Definition 6.4. A linear map $f: A \rightarrow B$ between C^* -algebras is *positive* when for every $a \in A$ there exists $b \in B$ satisfying $f(a^*a) = b^*b$. It is *completely positive* when $\mathbb{M}_n(f)$ is positive for every $n \in \mathbb{N}$.

For us it will be convenient to take another, well-known, viewpoint. If A and B are finite-dimensional C^* -algebras, then so is the algebraic tensor product $A \otimes B$.

Lemma 6.5. *Any finite-dimensional C^* -algebra A has a canonical $*$ -isomorphism $\mathbb{M}_n(A) \cong A \otimes \mathbb{M}_n(\mathbb{C})$. Under this correspondence, a linear map $f: A \rightarrow B$ between C^* -algebras is completely positive if and only if $f \otimes 1_{\mathbb{M}_n(\mathbb{C})}$ is positive for every $n \in \mathbb{N}$.*

Proof. Borrowing notation from Lemma 6.1, the $*$ -isomorphism $\mathbb{M}_n(A) \rightarrow A \otimes \mathbb{M}_n(\mathbb{C})$ is given by $(a_{ij}) \mapsto \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$. One easily verifies that this preserves the multiplication and involution. The second statement follows directly from Definition 6.4 by unfolding this isomorphism. \square

Now we have phrased the concrete notion of complete positivity in terms of tensor products with matrix algebras (cf. Theorem 3.6), and given that matrix algebras in **FHilb** are precisely the algebras of the form $L(H)$ for some Hilbert space H , we can determine $\text{CP}^*[\mathbf{FHilb}]$.

Theorem 6.6. $\text{CP}^*[\mathbf{FHilb}]$ is equivalent to the category of finite-dimensional C^* -algebras and completely positive maps.

Proof. Define a functor E from $\text{CP}^*[\mathbf{FHilb}]$ to the category of finite-dimensional C^* -algebras and completely positive maps, acting on objects as in Theorem 2.3 and as identity on morphisms.

Proposition 6.2 and Theorem 2.3 show that E is surjective on objects. Suppose $f: (A, \hat{\otimes}_{\mathcal{R}}) \rightarrow (B, \hat{\otimes}_{\mathcal{R}})$ is a morphism in $\text{CP}^*[\mathbf{V}]$. Then Theorem 3.6 shows that $E(f)$ must be completely positive, as characterised by Lemma 6.5. Therefore E is well-defined. Conversely, any completely positive map g between C^* -algebras satisfies the CP^* -condition because of Lemma 6.5, so E is full and hence an equivalence of categories. \square

Remark 6.7. It follows from the previous theorem that the embedding L does not extend to an equivalence of categories because it is not essentially surjective. That is, there are finite-dimensional C^* -algebras, such as $A = \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$, that are not isomorphic to a factor: $\dim(A) = 1^2 + 2^2 = 5 \neq n^2 = \dim(\mathbb{M}_n(\mathbb{C}))$.

7 Sets and relations

The previous section justified regarding normalisable dagger Frobenius algebras as generalised (finite-dimensional) C^* -algebras. This section considers $\text{CP}^*[\mathbf{V}]$ for $\mathbf{V} = \mathbf{Rel}$, the category of sets and relations. Starting with objects, we immediately see that these ‘generalised C^* -algebras’ are quite different.

Proposition 7.1. All normalisable dagger Frobenius algebras in **Rel** are special. Therefore they are (in 1-to-1 correspondence with small) groupoids.

Proof. We have to prove that normalisability implies speciality in **Rel**; for this it suffices to show that $z^2 = 1$. Now, the normaliser z is an isomorphism. In **Rel**, this means it is (the graph of) a bijection. But z is also positive, and hence self-adjoint. This means it is equal to its own inverse. Therefore $z^2 = 1$. The second statement now follows directly from [14, Theorem 7]. \square

Next, we turn to determining the morphisms of $\text{CP}^*[\mathbf{Rel}]$.

Definition 7.2. A relation $R \subseteq \text{Mor}(\mathbf{G}) \times \text{Mor}(\mathbf{H})$ between groupoids \mathbf{G}, \mathbf{H} respects inverses when

$$gRh \iff g^{-1}Rh^{-1}, \quad gRh \implies 1_{\text{dom}(g)}R1_{\text{dom}(h)}. \quad (2)$$

Proposition 7.3. $\text{CP}^*[\mathbf{Rel}]$ is (isomorphic to) the category of groupoids and relations respecting inverses.

Proof. In general, a morphism $R \subseteq (X \times X) \times (Y \times Y)$ in **Rel** is completely positive if and only if

$$(x', x)R(y', y) \iff (x, x')R(y, y'), \quad (x', x)R(y', y) \implies (x, x)R(y, y). \quad (3)$$

If \mathbf{G} and \mathbf{H} are groupoids, corresponding to Frobenius algebras $(G, \hat{\circlearrowleft})$ and $(H, \hat{\circlearrowleft})$, and $R \subseteq G \times H$,

$$\begin{aligned} \hat{\circlearrowleft} &= \{((g, g'), g^{-1} \circ g') \in G^3 \mid \text{cod}(g) = \text{cod}(g')\}, \\ \hat{\circlearrowleft} \circ R \circ \hat{\circlearrowleft} &= \{((g, g'), (h, h')) \in G^2 \times H^2 \mid \text{cod}(g) = \text{cod}(g'), \text{cod}(h) = \text{cod}(h'), (g^{-1} \circ g')R(h^{-1} \circ h')\}. \end{aligned}$$

Substituting this into (3) translates precisely into (2). \square

We close this section by investigating the embedding $L: \text{CPM}[\mathbf{Rel}] \rightarrow \text{CP}^*[\mathbf{Rel}]$. Recall that a category is *indiscrete* when there is precisely one morphism between each two objects. Indiscrete categories are automatically groupoids.

Lemma 7.4. *The essential image of the embedding $L: \text{CPM}[\mathbf{Rel}] \rightarrow \text{CP}^*[\mathbf{Rel}]$ is the full subcategory of $\text{CP}^*[\mathbf{Rel}]$ consisting of indiscrete (small) groupoids.*

Proof. Let X be an object in $\text{CPM}[\mathbf{Rel}]$, that is, a set. By definition, $L(X)$ corresponds to a groupoid with set of morphisms $X \times X$, and composition

$$(y_1, y_2) \circ (x_1, x_2) = \begin{cases} (y_1, x_2) & \text{if } y_2 = x_1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We deduce that the objects of $L(X)$ correspond to identities, i.e. pairs (x_1, x_2) with $x_1 = x_2$. So objects of $L(X)$ just correspond to elements of X . Similarly, we find that $\text{dom}(x_1, x_2) = x_2$ and $\text{cod}(x_1, x_2) = x_1$. Hence (x_1, x_2) is a morphism $x_2 \rightarrow x_1$ in $L(X)$, and it is the unique such. \square

8 Splitting idempotents

This section compares the CP^* -construction to Selinger's second solution to the problem of classical channels. First, recall this construction of splitting dagger idempotents [19].

Definition 8.1. Let \mathbf{V} be a dagger category. The category $\text{Split}^\dagger[\mathbf{V}]$ has as objects (A, p) where $p: A \rightarrow A$ is a morphism in \mathbf{V} satisfying $p \circ p = p = p^\dagger$; its morphisms $(A, p) \rightarrow (B, q)$ are morphisms $f: A \rightarrow B$ in \mathbf{V} satisfying $f = q \circ f \circ p$.

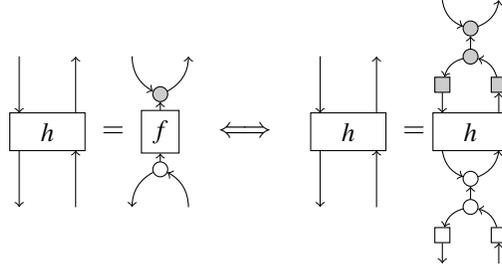
If \mathbf{V} is a dagger compact category, then so is $\text{Split}^\dagger[\mathbf{V}]$ (see [19, Proposition 3.16]). We will need the following assumption, that is satisfied in both \mathbf{Rel} and \mathbf{FHilb} .

Definition 8.2. A dagger compact category \mathbf{V} is said to *have algebraic square roots* when, given any normalisable Frobenius algebra on A and any central positive definite morphism $f: A \rightarrow A$, there exists a central morphism $g: A \rightarrow A$ such that $f = g \circ g$.

We thank the anonymous referee for pointing us towards the following proposition.

Proposition 8.3. *Let \mathbf{V} be a dagger compact category that has algebraic square roots. There is a functor $F: \text{CP}^*[\mathbf{V}] \rightarrow \text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$, acting as $F(A, \hat{\circlearrowleft}, \hat{\circlearrowright}) = \hat{\circlearrowleft} \circ \hat{\circlearrowright} \circ (\hat{\circlearrowleft} \otimes \hat{\circlearrowright})$ on objects, and as $F(f) = \hat{\circlearrowleft} \circ \hat{\circlearrowright} \circ f \circ \hat{\circlearrowleft} \circ \hat{\circlearrowright}$ on morphisms. It is full, faithful, and strongly dagger symmetric monoidal.*

Proof. First, notice that $p = F(A, \hat{\circlearrowleft}, \hat{\circlearrowright})$ is indeed a well-defined object of $\text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$: clearly $p = \hat{\circlearrowleft} \circ \hat{\circlearrowright} \circ \hat{\circlearrowleft} \circ \hat{\circlearrowright} = p^\dagger$ by centrality of the normaliser, $p \circ p = p$ follows from Lemma 3.3, and p is completely positive by Lemma 3.2. The assumption of algebraic square roots guarantees that $F(f)$ is indeed a well-defined morphism in $\text{CPM}[\mathbf{V}]$; by Lemma 3.3, it is in fact a well-defined morphism in $\text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$. Moreover, it is easy to see that an arbitrary \mathbf{V} -morphism $h: A^* \otimes A \rightarrow B^* \otimes B$ is a well-defined morphism in $\text{CP}^*[\mathbf{V}]$ if and only if it is a well-defined morphism in $\text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$.



Both $\text{CP}^*[\mathbf{V}]$ and $\text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$ inherit composition, identities, and daggers from \mathbf{V} , so F is a full and faithful functor preserving daggers. The symmetric monoidal structure in both $\text{CP}^*[\mathbf{V}]$ and $\text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$ is similarly defined in terms of that of \mathbf{V} , making F strongly symmetric monoidal. \square

Thus, when \mathbf{V} has algebraic square roots, $\text{CP}^*[\mathbf{V}]$ is equivalent to a full subcategory of $\text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$: the one obtained by splitting only the idempotents of the form $F(A, \begin{smallmatrix} \uparrow \\ \circ \\ \downarrow \end{smallmatrix}, \begin{smallmatrix} \uparrow \\ \square \\ \downarrow \end{smallmatrix})$. A variation of [19, Proposition 3.16] shows that splitting any family of dagger idempotents that is closed under tensor gives a dagger compact category, and Theorem 3.5 follows. The proof that we have written out does not need the assumption of algebraic square roots; moreover, it more explicitly exhibits the structure of $\text{CP}^*[\mathbf{V}]$ as a category of algebras.

In summary, in sufficiently nice cases, the CP^* -construction fits between the CPM -construction and its idempotent splitting.

$$\text{CPM}[\mathbf{V}] \xrightarrow{L} \text{CP}^*[\mathbf{V}] \xrightarrow{F} \text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$$

Both functors are strongly dagger symmetric monoidal, as well as full and faithful. Moreover, their composition is naturally isomorphic to the canonical inclusion $\text{CPM}[\mathbf{V}] \rightarrow \text{Split}^\dagger[\text{CPM}[\mathbf{V}]]$. However, the image of the left functor does not include classical channels. Similarly, a priori there is no reason why the right functor should be an equivalence. In particular, the middle category seems to capture the right amount of objects, and provides a constructive way to access them.

9 Future work

Having an abstract notion of C^* -algebra, (and, by extension, an abstract categorical construction placing classical and quantum information on equal footing) opens up many avenues for exploration.

- Quantum mechanics can be characterised in information-theoretic terms [4], but this argument is often criticised because it assumes a C^* -algebraic framework from the start. The CP^* -construction can investigate to what extent this criticism is valid and improve on those foundations.
- We can now abstractly study all sorts of notions from the C^* -algebraic formulation of quantum information theory [15, 16]. For example, notions of complementarity can be translated between abstract and concrete C^* -algebras [6, 13].
- The category **Stab** of stabiliser quantum mechanics embeds into $\text{CP}^*[\mathbf{Rel}]$, opening the door to abstract considerations of classical simulable circuits.
- It would be good to see whether algebraic square roots are necessary for Proposition 8.3. We also expect to find an example showing that F is not an equivalence.
- One could characterise categories of the form $\text{CP}^*[\mathbf{V}]$, perhaps using environments [5, 11, 7].
- It is worth investigating to what extent our construction generalises to infinite dimension [2, 7].

- On the theoretical side, the CP^* -construction might be (lax) monadic.
- Our construction seems related in spirit to [22]; it would be good to make connections precise.

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