

Symmetry constraints on temporal order in measurement-based quantum computation

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We discuss the interdependence of resource state, measurement setting and temporal order in measurement-based quantum computation. The possible temporal orders of measurement events are constrained by the principle that the randomness inherent in quantum measurement should not affect the outcome of the computation. We provide a classification for all temporal relations among measurement events compatible with a given initial stabilizer state and measurement setting, in terms of a matroid. Conversely, we show that classical processing relations necessary for turning the local measurement outcomes into computational output determine the resource state and measurement setting up to local equivalence. Further, we find a symmetry transformation related to local complementation that leaves the temporal relations invariant.

1 Introduction

Quantum states can change over time in two fundamentally different ways, unitary evolution and measurement. The former is deterministic and reversible whereas the latter is probabilistic and irreversible. Quantum computation [1] can be built on either. Employing unitary evolution leads to the quantum circuit model [2, 3] (the standard model of quantum computation), and is also present in adiabatic quantum computation [4]. Using measurement as the central tool to drive the computation leads to various measurement and teleportation-based schemes [5, 6, 7].

Here we consider the one-way quantum computer (MBQC) [6], a measurement based scheme of universal quantum computation. Therein, the process of computation is driven by local measurements on an entangled state of many qubits. Since the entanglement monotonically decreases over the course of computation, the initial state can be viewed as an entanglement resource. Known examples of universal resources include certain graph states [8] and ground states of two-body Hamiltonians [9], among them two-dimensional AKLT states of spin $3/2$ [10, 11]. A classification of universal resource states is to date not available, but it is known that only a tiny fraction of quantum states can possibly be universal [12, 13].

The object of study in this paper is the temporal ordering of measurements in MBQC, and what determines it. We point out explicitly that here we do *not* consider a scenario where a specific quantum algorithm is given in the circuit model and translated into MBQC. In that case, the temporal order of measurements can be straightforwardly obtained from the quantum circuit by a known mapping. Rather, we look at the initial quantum state and the planes of the Bloch sphere in which the locally measured observables reside, with no further information provided. We ask which temporal orders of measurements are compatible with this information. It may at first sight appear surprising that there is any constraint at all. However, the resource state and temporal order in MBQC become interrelated through *the principle*

that randomness of measurement outcomes must be prevented from affecting the logical processing. If the resource is a stabilizer state then all constraints on temporal order follow from the symmetries inferred by its stabilizer group. The purpose of this paper is to work out the consequences of this connection.

In addition to the theory of MBQC itself, we may investigate MBQC temporal order towards a different goal: toy models for generating temporal order from none. Consider the case where the resource is a graph state $|G\rangle$. How much of the temporal order follows from the interaction graph G ? The graph G is an undirected object whereas the partial order is directed. Thus, if G constrained the temporal orders severely, MBQC would provide a mechanism for generating temporal order. A quantum theory of gravity must achieve this in a far more complicated setting.

Previous work on MBQC temporal order has shown that for resource graph states $|G\rangle$ the temporal order of measurements is fully specified if the sets I and O of first- and last-measurable qubits are known [14], [15]. But not every pair I, O leads to a possible temporal order. A condition on admissible pairs I, O in terms of the adjacency matrix of G has been given [16].

In this paper, we present the following results. (i) We show that all transitive temporal relations which prevent measurement randomness from affecting the logical processing correspond to bases of a matroid derived from a resource stabilizer state and the local measurement bases. (ii) It is known that the adaption of measurement bases according to previously obtained measurement outcomes and the extraction of the computational result are governed by linear processing relations. Further, in all known schemes, the Bloch vectors specifying the measured local observables lie in the (X, Y) , (Y, Z) or (Z, X) equators of the Bloch sphere which are called “measurement planes”. Here we show that the linear processing relations determine the resource stabilizer state and the measurement planes up to equivalence. For this result to hold we need to slightly extend the previously known processing relations by introducing gauge variable that affect the processing but not the probability distribution of the computational output (See Section 3). (iii) We identify a transformation that leaves the temporal order of an MBQC unchanged and which is a slight generalization of local complementation. Local complementation has previously been found useful for the discussion of local equivalence among graph states [17]-[19].

The remainder of this paper is organized as follows. In Section 2 we provide the necessary background and notation, and briefly review prior work on temporal order in MBQC. In Section 3 we introduce the aforementioned gauge degrees of freedom and a resulting normal form of the resource state stabilizer. Based on this normal form, in Section 4 we derive our two main results on the interdependence of resource state and temporal order, (i) and (ii). In Section 5 we introduce a symmetry transformation that leaves the temporal relation in any given MBQC invariant and which is related to local complementation. In Section 6 we show that MBQC mimics certain aspects of General Relativity [20, 21]. We present an MBQC analogue of Malament’s theorem [22] and discuss the emergence of an event horizon. In Section 7 we conclude and point out open questions.

2 Background

In this section we review some basic facts about measurement-based quantum computation, essential definitions for the discussion of MBQC temporal order, as well as previous work in this area. The scheme of MBQC itself, with a proof of its computational universality, is not reviewed here since various articles on this subject exist in the literature [6], [23] - [27] Also, we require familiarity with the stabilizer formalism [28].

2.1 MBQC and cluster states

In MBQC, the process of computation is driven by local (=1-qubit) measurements on an initial highly entangled state, generally taken to be a so-called cluster state. The local measurements can only reduce entanglement, and therefore all entanglement needed for the computation must come from the initial state. For this reason, the initial state is often called the ‘resource state’¹.

The computational power of a given MBQC strongly depends on the choice of the initial resource state. For example, a local resource state has obviously no computational power. Other states may be used for a restricted class of computations. Two-or-higher dimensional cluster states of unbounded size have the property that they enable universal quantum computation. That is, *any* quantum computation can be realized on such a state, by suitable choice of the local measurement bases.

We now define graph states, to be used later on, and cluster states as a subclass thereof.

Definition 1 (Graph states and cluster states.). *Be G a graph with vertex set $V(G)$ and edge set $E(G)$, such that there is one qubit for each vertex $a \in V(G)$. Then, the graph state $|G\rangle$ is the unique (up to global phase) joint eigenstate, $|G\rangle = K_a |G\rangle$, for all $a \in V(G)$, of the operators*

$$K_a = \sigma_x^{(a)} \bigotimes_{b|(a,b) \in E(G)} \sigma_z^{(b)}. \quad (1)$$

The cluster state $|\Phi_{\mathcal{L}}\rangle$ is a graph state where the corresponding graph is a d -dimensional lattice \mathcal{L} .

In the standard scheme [6], the measured observables are of the form

$$O_a[q_a] = \cos \varphi_a \sigma_x^{(a)} + (-1)^{q_a} \sin \varphi_a \sigma_y^{(a)}, \quad (2)$$

with a the qubit to which the measurement is applied, and $q_a \in \mathbb{Z}_2$ depending on outcomes of (earlier) measurements on other qubit locations.

2.2 Temporal order in MBQC

As noted above, the temporal order of measurement in MBQC is a consequence of the randomness of measurement outcomes. By adjusting measurement bases according to measurement outcomes obtained on other qubits, this randomness inherent in quantum mechanical measurement can be kept from creeping into the logical processing [6]. If the measurement outcome of qubit a influences the choice of measurement basis for qubit b , clearly, qubit a must be measured before qubit b can. This is how a temporal order among the measurement events arises.

We now generalize the above scenario of measuring observables of form Eq. (2) on cluster states. Namely, we now consider general stabilizer states $|\Psi\rangle$ as resources, with $\text{supp}(|\Psi\rangle) = \Omega$ and stabilizer group $\mathcal{S}(|\Psi\rangle)$. Furthermore, we generalize the local observables whose measurement drives the computation from Eq. (2) to

$$O_a[q_a] = \cos \varphi_a \sigma_\phi^{(a)} + (-1)^{q_a} \sin \varphi_a \sigma_{s\phi}^{(a)}, \quad \forall a \in \Omega, \quad (3)$$

with $\sigma_\phi, \sigma_{s\phi} \neq \sigma_\phi \in \{X, Y, Z\}$. Therein, the measurement angles φ_a are in the range $-\pi/2 \leq \varphi_a < \pi/2$, and $q_a \in \mathbb{Z}_2$ may depend on measurement outcomes from several (other) qubits in Ω .

¹Large entanglement is a necessary [29, 30] but not sufficient condition for the usefulness of a quantum state as resource in MBQC. Somewhat paradoxically, quantum states exist that are too entangled to be useful [12].

Definition 2 (Measurement plane). *For every qubit $a \in \text{supp}(|\Psi\rangle)$, the measurement plane at a is the ordered pair $[\sigma_\phi^{(a)}, \sigma_s^{(a)}]$.*

We define a third Pauli operator, $\sigma_s = i\sigma_{s\phi}\sigma_\phi$. As we will see shortly, the Pauli operators σ_ϕ and σ_s are useful because of the relations

$$\sigma_\phi O[q]\sigma_\phi^\dagger = O[q \oplus 1], \quad \sigma_s O[q]\sigma_s^\dagger = -O[q]. \quad (4)$$

The basic mechanism of accounting for an ‘‘undesired’’ measurement outcome is the following. Suppose on some qubit $a \in \Omega$, instead of the ‘‘desired’’ post-measurement state $|\varphi_a\rangle_a$ the ‘‘undesired’’ post-measurement state $|\varphi_a^\perp\rangle_a$ has been obtained. The goal is to get the computation back on track by only adjusting the subsequent measurements. To do that, we require a stabilizer operator $\tilde{K}(a) \in \mathcal{S}(|\Psi\rangle)$ with the following properties [14]: (1) $\tilde{K}(a)$ has support only on a and the yet unmeasured qubits, and (2) $\tilde{K}(a)|_a = \sigma_s^{(a)}$. Recall that $\sigma_s|\varphi_a\rangle = |\varphi_a^\perp\rangle$ for the eigenstates $|\varphi_a(q_a, s_a)\rangle$ of the local measured observable $O[q_a]$, c.f. Eq. (4). Denote by $\mathcal{P}(a)$ and $\mathcal{F}(a)$ the past and future of a , respectively. Then,

$$\begin{aligned} (\mathcal{P}(a)\langle\varphi_{\text{loc}}| \otimes_a \langle\varphi_a^\perp|) |\Psi\rangle &= (\mathcal{P}(a)\langle\varphi_{\text{loc}}| \otimes_a \langle\varphi_a^\perp|) \tilde{K}(a) |\Psi\rangle \\ &= (\mathcal{P}(a)\langle\varphi_{\text{loc}}| \otimes_a \langle\varphi_a|) \tilde{K}(a) |_{\mathcal{F}(a)} |\Psi\rangle \end{aligned} \quad (5)$$

Therein, the first equality follows from $\tilde{K}(a) \in \mathcal{S}(|\Psi\rangle)$, and the second from the above properties (1) and (2).

Since the overlaps between local states (representing the local measurements) with the resource state $|\Psi\rangle$ contain all information about the computation, we thus find that we can correct for ‘‘undesired’’ outcomes by (a) adjusting measurement bases of future measurements (caused by tensor factors σ_ϕ in $\tilde{K}(a)|_{\mathcal{F}(a)}$) and (b) re-interpretation of measurement outcomes (caused by tensor factors σ_s in $\tilde{K}(a)|_{\mathcal{P}(a)}$).

Example: Consider MBQC on a cluster state $|\Phi_3\rangle$ of three qubits on a line, each measured in the $[\sigma_x, \sigma_y]$ -plane. That is, for all three qubits $\sigma_s = Z$ and $\sigma_\phi = X$. (Here and from now on, we use the shorthand $X \equiv \sigma_x$, $Y \equiv \sigma_y$ and $Z \equiv \sigma_z$.) The stabilizer generators of $|\Phi_3\rangle$ are

$$\begin{aligned} K_1 &= X_1 \otimes Z_2 \otimes I_3 = \sigma_\phi^{(1)} \otimes \sigma_s^{(2)} \otimes I^{(3)}, \\ K_2 &= Z_1 \otimes X_2 \otimes Z_3 = \sigma_s^{(1)} \otimes \sigma_\phi^{(2)} \otimes \sigma_s^{(3)}, \\ K_3 &= I_1 \otimes Z_2 \otimes X_3 = I^{(1)} \otimes \sigma_s^{(2)} \otimes \sigma_\phi^{(3)}. \end{aligned} \quad (6)$$

When the three cluster qubits are measured in the order $1 \prec 2 \prec 3$, the corresponding quantum circuit is [24]

$$|+\rangle \text{---} \boxed{U_z} \text{---} \boxed{U_x} \text{---} \boxed{U_z} \text{---} \| X \quad (7)$$

Using Eq. (5), here we show that if qubits 1, 2, 3 are measured in the order $1 \prec 2 \prec 3$, then the randomness of the measurement outcomes on qubits 1 and 2 can be corrected for. First we consider the stabilizer operator $K_2 = \sigma_s^{(1)} \otimes \sigma_\phi^{(2)} \otimes \sigma_s^{(3)} =: K(1)$. If inserted in the state overlap of Eq. (5), the measurement outcome of qubits 1 and 3 is flipped, as well as the measurement basis at qubit 2. Since qubits 2 and 3 are yet unmeasured when qubit 1 is measured, this is a valid correction operation for qubit 1; hence the notation $K(1)$. Similarly, $K_3 = I^{(1)} \otimes \sigma_s^{(2)} \otimes \sigma_\phi^{(3)} =: K(2)$ can be used as correction operation for qubit 2. $K(2)$ flips the measurement outcome of qubit 2 and the measurement basis of qubit 3. Since qubit 3 is yet unmeasured when qubit 2 is measured, this corresponds to a valid correction operation.

The above argument also works in reverse. If the correction operations $K(1)$ and $K(2)$ are used, then $1 \prec 2 \prec 3$ follows. $K(1)$ implies that the measurement basis of qubit 2 depends on the measurement outcome at qubit 1, hence $1 \prec 2$. Note that $1 \prec 3$ does not yet follow! $K(1)$ does *not* affect the measurement basis at qubit 3. Only the meaning of the eigenstates is interchanged, which by itself does not require qubit 3 to be measured after qubit 1. The interpretation of the measurement outcome may take place long after the measurement itself has taken place.

$K_3 = K(2)$ implies that the measurement basis of qubit 3 depends on the measurement outcome of qubit 2, and hence $2 \prec 3$. Both relations taken together yield $1 \prec 2 \prec 3$.

From the equivalence with the circuit of Eq. (7) one would expect one bit of classical output. Indeed, if no correction operations need to be used, the eigenvalue measured at the output of the circuit corresponds to the eigenvalue λ_3 measured on qubit 3 of the cluster. Now recall that $K(1)$, applied conditioned upon $\lambda_1 = -1$, flips λ_3 . Therefore, with or without corrections, the eigenvalue measured in the circuit Eq. (7) equals $\lambda_1 \lambda_3$. Or, in binary notation $\lambda_1 \equiv (-1)^{s_1}$, $\lambda_3 \equiv (-1)^{s_3}$, the single bit of classical output takes the value $s_1 + s_3 \pmod 2$.

Note that we have not made any use of $K_1 = \sigma_\phi^{(1)} \otimes \sigma_s^{(2)}$ in the above argument. Still, K_1 has a role to play, as we discuss in Section 3.

2.3 Influence matrix, forward and backward cones

To counteract the randomness of measurement outcomes, two measurement settings (i.e., bases) per qubit suffice, which may be labeled by $q_a = 0$ and $q_a = 1$, respectively, for each qubit a . The measurement settings may collectively be described by a binary vector \mathbf{q} , $[\mathbf{q}]_a = q_a$ for all $a \in \Omega$, and the measurement outcomes by a binary vector \mathbf{s} , $[\mathbf{s}]_a = s_a$, for all $a \in \Omega$. It turns out that the relation between measurement bases \mathbf{q} and measurement outcomes \mathbf{s} is *linear* [31],

$$\mathbf{q} = T\mathbf{s} \pmod 2, \tag{8}$$

with T a binary matrix. We call T the influence matrix.

The set of all qubits b whose measurement basis must be adjusted according to the measurement outcome on a is denoted as the *forward cone* of a qubit a . Similarly, the backward cone of a qubit b is the set of all those qubits a whose measurement outcome influence the measurement basis at b . More formally, with Eq. (8),

Definition 3 (Forward and backward cones). *For any $a \in \Omega$ the forward cone $fc(a)$ is given by*

$$fc(a) := \{b \in \Omega \mid \partial q_b / \partial s_a = 1\}. \tag{9}$$

For any $b \in \Omega$ the backward cone $bc(b)$ is given by

$$bc(b) := \{a \in \Omega \mid \partial q_b / \partial s_a = 1\}. \tag{10}$$

We denote the characteristic vectors of $fc(a)$ and $bc(b)$ by $\mathbf{fc}(a)$ and $\mathbf{bc}(b)$, respectively. Then, the influence matrix T takes the form

$$T = \begin{pmatrix} (\mathbf{bc}(1)) \\ (\mathbf{bc}(2)) \\ \cdot \\ \cdot \\ (\mathbf{bc}(n)) \end{pmatrix} = \left(\begin{pmatrix} \mathbf{fc}(1) \end{pmatrix} \begin{pmatrix} \mathbf{fc}(2) \end{pmatrix} \dots \begin{pmatrix} \mathbf{fc}(n) \end{pmatrix} \right). \tag{11}$$

The influence matrix T generates a temporal relation among the measurement events under transitivity. We say $a \prec b$ (a precedes b) if $b \in fc(a)$. A priori, it is not forbidden that for two qubits $a, b \in \Omega$, $a \prec b$ and $b \prec a$. However, such a computation could not be run deterministically in a world like ours where time progresses linearly. An MBQC is *deterministically runnable* if the temporal relation “ \prec ” between the measurement events is a strict partial order.

Definition 4 (Strict partial order). *A strict partial order is a relation among the elements $a \in \Omega$ with the following properties*

$$\begin{aligned} a \not\prec a, & \quad \forall a \in \Omega, & \quad (\text{irreflexivity}) \\ a \prec b \implies b \not\prec a, & \quad \forall a, b \in \Omega & \quad (\text{antisymmetry}) \\ a \prec b, b \prec c \implies a \prec c, & \quad \forall a, b, c \in \Omega. & \quad (\text{transitivity}) \end{aligned} \quad (12)$$

Definition 5 (Input and output sets.). *For a given MBQC, the input set $I \subseteq \Omega$ is the set of qubits whose backward cones are empty, $I = \{a \in \Omega \mid bc(a) = \emptyset\}$. The output set $O \subseteq \Omega$ is the set of qubits whose forward cones are empty, $O = \{b \in \Omega, fc(b) = \emptyset\}$.*

That is, with respect to a given temporal relation among the measurement events, I is the maximal set of qubits which can be measured first, and O is the maximal set of qubits which can be measured last.

Regarding the computational output, for the purpose of this paper we are exclusively interested in MBQCs for which the computational result is a classical bit string. That is, every qubit in Ω is measured. Then, as a consequence of the randomness of individual measurement outcomes, the classical output \mathbf{o} of an MBQC is given by *correlations* among measurement outcomes. Again, the relation between classical output and measurement outcomes is linear,

$$\mathbf{o} = Z\mathbf{s} \pmod{2}, \quad (13)$$

for a suitable binary matrix Z .

Example: To illustrate the above notions, we briefly return to the three-qubit cluster state example of Section 2.2. From the previous discussion we find that

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \pmod{2}. \quad (14)$$

Therefore, $fc(1) = \{2\}$, $fc(2) = \{3\}$, $fc(3) = \emptyset$ and $bc(1) = \emptyset$, $bc(2) = \{1\}$, $bc(3) = \{2\}$. Hence, $I = \{1\}$ and $O = \{3\}$. Also, $1 \prec 2$ and $2 \prec 3$. The latter two relations generate a third under transitivity, namely $1 \prec 3$. Asymmetry and irreflexivity are obeyed in this example. Regarding the single bit of output in this computation, the matrix Z of Eq. (13) is $Z = (101)$.

2.4 Brief review of prior work on MBQC temporal order

MBQC temporal order as an (almost) emergent phenomenon. In [14] the following question is asked: “Given graph G and the set Σ of measurement planes for all vertices, can the temporal order of measurements in MBQC with a graph state $|G\rangle$ be uniquely reconstructed from this information?” The graph G and the measurement planes Σ are undirected objects. Thus, if the answer to this question was yes, then temporal order in MBQC were truly emergent.

However, it turns out that the pair G, Σ does not specify the temporal order of measurements in MBQC uniquely; there are in general a number of consistent temporal orders respecting the requirement

that the randomness of measurement outcomes should not affect the logical processing. One may then ask how constraining on temporal order this requirement actually is. To this question, the following answer is provided by [14]: If in addition to G and Σ the set I of first-measurable and the set O of last-measurable qubits is known, then the complete temporal order (if existing) can be uniquely reconstructed from this information. Thus, MBQC temporal order is not emergent in the strict sense; a seed I, O must be provided in addition to G and Σ , and the complete temporal order then follows.

But not every pair I, O will lead to a consistent temporal order. The question that now arises is which pairs I, O do. For the case where the stabilizer resource state is a graph state and all qubits are measured in the $[X, Y]$ -plane² then the answer to this question is given in [16]. Denote by A_G the adjacency matrix of the graph G describing the resource state $|G\rangle$, and by $A_G|_{O^c \times I^c}$ the submatrix of A_G where the rows are restricted to $O^c := \Omega \setminus O$ and the columns are restricted to I^c . Then, the pair I, O leads to a partial order of measurement events in MBQC iff there exists a matrix T such that $A_G|_{O^c \times I^c} T = I$, and T is free of cycles (that is $T_{aa} = 0, \forall a, T_{ab}T_{ba} = 0, \forall a, b, T_{ab}T_{bc}T_{ca} = 0, \forall a, b, c$, etc). The resulting temporal order is generated by T under transitivity.

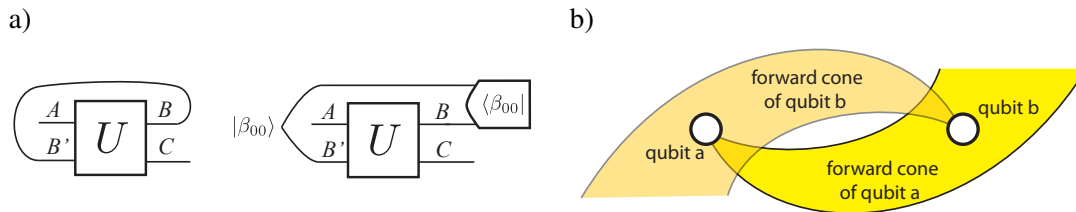


Figure 1: Closed time-like curves in MBQC. a) Bennett, Schumacher and Svetlichny’s post-selection model [33, 34] of CTCs (left: circuit with wires ‘going backwards in time’, right: implementation thereof using teleportation and post-selection). b) Nested forward cones in the MBQC equivalent of the teleportation circuit in (a).

MBQC and closed time-like curves. In [32] it is shown that MBQC encompasses the post-selection model of closed time-like curves (CTC’s) proposed by Bennett, Schumacher [33] and Svetlichny [34]. The CTCs arise from circuits such as the one displayed in Fig. 1a, translated into MBQC. The result are forward cones with the property $b \in fc(a) \wedge a \in fc(b)$, for two suitably chosen qubits $a, b \in \Omega$; see Fig. 1b. Such nested forward cones are an obstruction to deterministic runnability of MBQC, but mimic closed time-like curves of General Relativity in the MBQC setting.

3 Gauge degrees of freedom

Here we introduce the notion of “gauge transformations” acting on a given quantum computation. These transformations exist for both the circuit model and MBQC.

3.1 Gauge transformations in the circuit model

To obtain an intuition for the gauge transformations introduced here, it is instructive to first inspect them in the circuit model. Specifically, we consider a quantum circuit which consists of (1) the preparation of

²The former condition alone is not a restriction, since all stabilizer states are local Clifford equivalent to graph states [35], but both conditions jointly are.

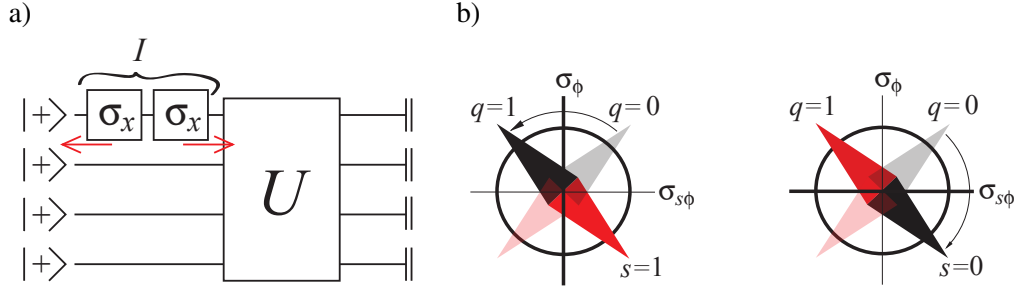


Figure 2: Two symmetry transformations. a) Gauge transformation in the circuit model. For any logical qubit l , an identity $I = \sigma_x^{(l)} \sigma_x^{(l)}$ is inserted into the circuit next to the input. The left σ_x is propagated backwards in time, and absorbed by the input state $|+\rangle$. The right σ_x is propagated forward in time, flipping rotation angles and, potentially, measurement outcomes in the passing. b) Flipping a measurement plane in MBQC. In the measurement plane $[\sigma_\phi, \sigma_{s\phi}]$, the Pauli operator σ_ϕ is distinguished over $\sigma_{s\phi}$ because the rule for adjusting a local observable O for measurement is $O[q=1] = \sigma_\phi O[q=0] \sigma_\phi^\dagger$. If, for a qubit a , $T_{aa} = 0$ (with T the influence matrix) then the exchange $\sigma_\phi^{(a)} \longleftrightarrow \sigma_{s\phi}^{(a)}$ is a symmetry transformation for the given MBQC.

the quantum register in the initial state $\bigotimes_{i=1}^n |+\rangle_i$, (2) unitary evolution composed of, say, CNOT gates and one-qubit rotations about the X - and Z -axes, and (3) local measurements for readout. Such a circuit is displayed in Fig. 2 above. Then, into every qubit line individually, we may insert an identity $I = \sigma_x \sigma_x$ next to the input; See Fig. 2. The left σ_x is propagated backwards in time until absorbed by the input state $|+\rangle$. The right σ_x is propagated forward in time, flipping rotation angles and readout-measurement outcomes in the passing.

This transformation is an *equivalence transformation*, since it is caused by the insertion of an identity gate into the circuit of Fig. 2. It changes the sign for certain rotation angles, i.e. when angles are counted positive or negative. Specifically, for the z -rotation gates next to each input qubit we can individually choose our convention for which rotation angles are called positive or negative, respectively. Once those signs are fixed on the input side, they are fixed throughout the circuit. Changing this reference affects the procedure of computation, but leaves the distribution of computational results unchanged. We therefore call it a gauge transformation.

3.2 The gauge transformations in the measurement-based model

Translating the above discussed gauge transformations from the circuit model into MBQC it is easily seen that the above relations Eq. (8) and (13) are incomplete. We find the more general relations

$$\mathbf{q} = T\mathbf{s} + H\mathbf{g} \pmod{2}, \quad (15a)$$

$$\mathbf{o} = Z\mathbf{s} + R\mathbf{g} \pmod{2}, \quad (15b)$$

with \mathbf{g} a choice of gauge.

Under a change of \mathbf{g} , the measurement bases in a particular MBQC change, but the probability distribution for the classical output values remains unchanged. Of course, knowledge of the \mathbf{g} -dependent extra parts in Eq. (15a,15b) is not necessary to run any given MBQC, since $\mathbf{g} = \mathbf{0}$ is always a valid choice.

However, the presence of the extra terms in the processing relations strengthens their interdependence with the resource state, which is the reason why we discuss them here.

We may now want to derive the generalized processing relations Eq. (15a,15b) directly in MBQC, without reference to the circuit model. Before discussing the general case, we return to the specific example of the three-qubit cluster state in Section 2.2.

Example: Consider the product $K_1 K_3 = \sigma_\phi^{(1)} \otimes \sigma_\phi^{(3)}$ of K_1, K_3 in Eq. (6). When used in Eq. (5), the effect of this stabilizer element is to flip the measurement bases of qubits 1 and 3. Hence, the relation Eq. (14) generalizes to

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} g_1 \pmod{2}. \quad (16)$$

This is precisely what one would expect from the insertion of $\sigma_x \sigma_x$ next to the input of the equivalent quantum circuit in Eq. (7).

We now turn to the general case. Via Eq. (5), the stabilizer group $\mathcal{S}(|\Psi\rangle)$ acts on \mathbf{s} and \mathbf{q} . Denote the post-measurement state of qubit $a \in \Omega$ by $|s_a, q_a\rangle_a$, with s_a the measurement outcome and q_a specifying the measurement basis. Then, as in Eq. (5),

$$\left(\bigotimes_{a \in \Omega} a \langle s_a, q_a | \right) |\Psi\rangle = \left(\bigotimes_{a \in \Omega} a \langle s_a, q_a | \right) K |\Psi\rangle = \left(\bigotimes_{a \in \Omega} a \langle s_a, q_a | K \right) |\Psi\rangle, \quad \forall K \in \mathcal{S}(|\Psi\rangle). \quad (17)$$

Since $\sigma_s |s, q\rangle = |s \oplus 1, q\rangle$ and $\sigma_\phi |s, q\rangle = |s, q \oplus 1\rangle$, for a stabilizer element $K = \bigotimes_{a \in \Omega} (\sigma_s^{(a)})^{v_a} (\sigma_\phi^{(a)})^{w_a}$ the action of G_K on \mathbf{s}, \mathbf{q} is

$$G_K : \begin{array}{l} \mathbf{s} \longrightarrow \mathbf{s} + \mathbf{v} \pmod{2}, \\ \mathbf{q} \longrightarrow \mathbf{q} + \mathbf{w} \pmod{2}. \end{array} \quad (18)$$

Again, nothing changes by the insertion of a stabilizer (identity) operator into the state overlap of Eq. (5), and transformations G_K are therefore equivalence transformations. They can be used to constrain the possible temporal orders in MBQC, as we discuss explicitly in Appendix A.

3.3 Closed time-like curves – good or bad?

Partial temporal orders for an MBQC with a given resource stabilizer state and fixed measurement planes are the solution to two constraints, namely

1. Every qubit in the resource state must *have* a forward cone³.
2. The forward cones generate an irreflexive and antisymmetric relation under transitivity.

In this paper we allow closed time like curves (CTC) in the computation; we are interested in classifying all transitive *temporal relations* consistent with a given resource stabilizer state and set of measurement planes. We do not restrict to partial orders per se, and therefore *drop the above Condition 2*. The remaining Condition 1 may, at first sight, hardly seem to pose any constraints at all. However, it does. It imposes a self-consistency condition on each forward cone. As we make explicit in Section 6.2, this self-consistency condition in certain cases takes the form of a wave equation.

A temporal relation among measurements which is not a partial order contains closed time-like curves, and can be implemented in familiar linear time only by employing postselection. On the other

³Forward cones are allowed to be empty.

hand, temporal relations which contain closed time-like curves have recently been found of independent interest [32]. They are the translation of quantum circuits with post-selection CTCs [33], [34] into MBQC. We may compare with the theory of General Relativity, where certain solutions of the Einstein equations contain closed time-like curves. Such solutions give rise to a host of paradoxes, and whether they are physical is under debate [36]. But Einstein's field equations are not abandoned because they allow for CTCs.

3.4 Correction and gauge operations in the stabilizer formalism

To state and prove our results on MBQC temporal order, we need to make a few more definitions. It turns out that the possible temporal relations, and indeed the classical processing relations Eq. (15a), (15b), can be parametrized by two subsets of Ω , namely the computational output set O_{comp} and the gauge input set I_{gauge} . We define these sets next.

We say that the measurement outcome s_a of qubit a is *corrected* in a given MBQC, if by insertion of a suitable stabilizer operator K in Eq. (17) equivalence of $s_a = 1$ with the reference outcome $s_a = 0$ is established, at the cost of adjustment of measurement bases and/or re-interpretation of measurement outcomes on other qubits.

We observe that if for a given MBQC *all* measurement outcomes s_a , $a \in \Omega$, can be corrected then these measurement outcomes contain no information and no linear combination of them is worth outputting. Thus, in general there will be a set of qubits whose measurement outcomes are not corrected.

Definition 6 (Computational output set). *For a given MBQC, the computational output set $O_{\text{comp}} \subseteq \Omega$ is the set of qubits whose measurement outcomes are not corrected.*

The correction operations for the qubits in $a \in (O_{\text{comp}})^c$, c.f. Eq. (5), are each implemented by correction operators $K(a) \in \mathcal{S}(|\Psi\rangle)$. For each $a \in (O_{\text{comp}})^c$, $K(a)$ has the property that it flips the measurement outcome s_a at qubit a , but does not flip the measurement outcome on any other qubit in $(O_{\text{comp}})^c$. In this way, it is ensured that the correction operation is for qubit a individually.

Definition 7 (Correction operator). *For an MBQC with a given stabilizer resource state $|\Psi\rangle$, fixed set Σ of measurement planes and computational output set O_{comp} , for each $a \in (O_{\text{comp}})^c$ the corresponding correction operator $K(a) \in \mathcal{S}(|\Psi\rangle)$ is a Pauli operator satisfying the conditions*

$$\begin{aligned} K(a)|_a &\in \{\sigma_s^{(a)}, \sigma_{s\phi}^{(a)}\}, \\ K(a)|_b &\in \{I^{(b)}, \sigma_\phi^{(b)}\}, \quad \forall b \in (O_{\text{comp}})^c \setminus a. \end{aligned} \quad (19)$$

The correction operator $K(a)$ can be used to correct an ‘‘undesired’’ measurement outcome at qubit a , c.f. Eq. (5). If, for a given operator $K(a)$, $K(a)|_b \in \{\sigma_\phi^{(b)}, \sigma_{s\phi}^{(b)}\}$ then, by Eq. (5), the measurement basis at qubit b depends on the measurement outcome for qubit a . In terms of the influence matrix T ,

$$K(a)|_b \in \{\sigma_\phi^{(b)}, \sigma_{s\phi}^{(b)}\} \iff [T]_{ba} = 1, \quad \forall a \in (O_{\text{comp}})^c, b \in \Omega. \quad (20)$$

Note that in the first line of Eq. (19) we allow $K(a)|_a = \sigma_{s\phi}^{(a)}$ only because we are admitting closed time-like curves in the present discussion. $K(a)|_a = \sigma_s^{(a)}$ means that the measurement basis for qubit a depends on the outcome s_a of the measurement of qubit a . This amounts to a closed time-like curve only involving qubit a (self-loop) and is an obstacle to deterministic runnability.

Because the qubits $a \in O_{\text{comp}}$ have no correction operations, $T_{ba} = 0$ for all $b \in \Omega$, and thus

$$O_{\text{comp}} \subseteq O. \quad (21)$$

Thus, O_{comp} and the correction operators $\{K(a), a \in (O_{\text{comp}})^c\}$ completely determine the influence matrix T and hence the temporal relation among the measurements.

Conversely, if O_{comp} and $\{K(a), a \in (O_{\text{comp}})^c\}$ are unknown, then the constraints Eq. (19) pose self-consistency conditions on them. We discuss these conditions further below.

As we have seen in the concrete three-qubit example above, for a given MBQC the relation between the choice \mathbf{q} of measurement bases and the measurement outcomes \mathbf{s} in general allows for an offset term $H\mathbf{g}$, c.f. Eq. (15a). Thus, the measurement bases for a certain set of qubits can be freely chosen until the initially arbitrary \mathbf{g} becomes fixed. This observation leads to

Definition 8 (Gauge input set). *For a given MBQC with input set I , the gauge input set $I_{\text{gauge}} \subseteq I$ is a set of qubits such that for each $i \in I_{\text{gauge}}$ the parameter q_i specifying the locally measured observable $O_i[q_i]$ can be freely chosen.*

Analogously to the correction operations, there will be gauge operators which implement the ‘corrections’ of measurement bases for the qubits in I_{gauge} . The definition of the gauge operators ensures that for all $i \in I_{\text{gauge}}$ the corresponding q_i can be changed individually without changing the others.

Definition 9 (Gauge operators). *For an MBQC with a given stabilizer resource state $|\Psi\rangle$, fixed set Σ of measurement planes, computational output set O_{comp} and gauge input set I_{gauge} , for each $i \in I_{\text{gauge}}$, the corresponding gauge operator $\bar{K}(i) \in \mathcal{S}(|\Psi\rangle)$ is a Pauli operator satisfying the conditions*

$$\begin{aligned} \bar{K}(i)|_i &= \sigma_\phi^{(i)}, \\ \bar{K}(i)|_j &= I^{(j)}, \quad \forall j \in (I_{\text{gauge}} \cap (O_{\text{comp}})^c) \setminus i, \\ \bar{K}(i)|_k &\in \{I^{(k)}, \sigma_\phi^{(k)}\}, \quad \forall k \in ((I_{\text{gauge}})^c \cap (O_{\text{comp}})^c) \setminus i, \\ \bar{K}(i)|_l &\in \{I^{(l)}, \sigma_s^{(l)}\}, \quad \forall l \in (I_{\text{gauge}} \cap O_{\text{comp}}) \setminus i. \end{aligned} \tag{22}$$

Like previously for the correction operations, Eq. (22) poses self-consistency condition on the possible sets I_{gauge} and $\{\bar{K}(i), i \in I_{\text{gauge}}\}$.

Lemma 1. *The correction operators $K(a)$ of Eq. (19), $a \in (O_{\text{comp}})^c$, and the gauge operators $\bar{K}(i)$ of Eq. (22), $i \in I_{\text{gauge}}$, are independent. That is, for all sets $J \subset I_{\text{gauge}}$, $L \subset (O_{\text{comp}})^c$ with $J \neq \emptyset \vee L \neq \emptyset$, $\tilde{K}(J, L) := \prod_{i \in J} \bar{K}(i) \prod_{a \in L} K(a) \neq I^{(\Omega)}$.*

Proof of Lemma 1. (indirect) Assume that there exists a pair J, L , with $J \neq \emptyset \vee L \neq \emptyset$, such that $\tilde{K}(J, L) = I^{(\Omega)}$. Then, for all $b \in (O_{\text{comp}})^c$, $\tilde{K}(J, L)|_b = 1$. Now, with Eq. (19), $K(b)|_b \in \{\sigma_s^{(b)}, \sigma_{s\phi}^{(b)}\}$, and $K(c)|_b \in \{\sigma_\phi^{(b)}, I^{(b)}\}$ for all $c \in (O_{\text{comp}})^c \setminus b$. With Eq. (22), $\bar{K}(i)|_b \in \{\sigma_\phi^{(b)}, I^{(b)}\}$ for all $i \in I_{\text{gauge}}$. Therefore, no other $K(\cdot)$, $\bar{K}(\cdot)$ can cancel a $\sigma_s^{(b)}$ -contribution from $K(b)$ to $\tilde{K}(J, L)$. Hence, $b \notin L$ for all $b \in (O_{\text{comp}})^c$, and thus $L = \emptyset$. By an analogous argument, no $K(\cdot)$ can cancel the $\sigma_\phi^{(i)}$ -contribution of $\bar{K}(i)$ to $\tilde{K}(J, L)$, hence $i \notin J$ for all $i \in I_{\text{gauge}}$, and $J = \emptyset$. Thus, $\tilde{K}(J, L) = I^{(\Omega)} \implies J, L = \emptyset$. Contradiction. Hence, the $K(a)$, $\bar{K}(i)$ are independent. \square

Remark: For any given MBQC with fixed temporal relation, the computational output set O_{comp} is a subset of the output set O , see Eq. (21). But O_{comp} and O are not necessarily equal. Likewise, $I_{\text{gauge}} \subseteq I$ by definition, but I_{gauge} and I may not be equal. To illustrate this point, we consider the following two examples.

Example 1. Consider the 3-qubit cluster state of Section 2.2, with measurement planes $[X/Y]$ for all three qubits. We consider the correction operators $K(1) = K_2$, $K(2) = K_3$ for qubits 1 and 2, and gauge operator $\bar{K}(1) = K_1 K_3$. $O_{\text{comp}} = \{3\}$ and $I_{\text{gauge}} = \{1\}$ are then permitted by Eqs. (19) and (22),

respectively. As was discussed previously for the above choice of correction operations, $I = \{1\}$ and $O = \{3\}$. Thus, in the present example $I_{\text{gauge}} = I$ and $O_{\text{comp}} = O$.

Example 2. Consider a Greenberger-Horne-Zeilinger state $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ as resource state, with all three qubits measured in a basis in the $[X, Y]$ -plane. We use the stabilizer elements $K(1) := Z_1Z_3 = \sigma_s^{(1)}\sigma_s^{(3)}$ and $K(2) := Z_2Z_3 = \sigma_s^{(2)}\sigma_s^{(3)}$ as correction operations for qubits 1 and 2, and $\bar{K}(1) := X_1X_2X_3 = \sigma_\phi^{(1)}\sigma_\phi^{(2)}\sigma_\phi^{(3)}$ as gauge operator. Then, the choice $I_{\text{gauge}} = \{1\}$ and $O_{\text{comp}} = \{3\}$ is admitted by Eqs. (19) and (22), respectively. On the other hand, this is an example of a temporarily flat MQC, $T = 0$. Therefore, $I = O = \{1, 2, 3\}$, and $O_{\text{comp}} \neq O$, $I_{\text{gauge}} \neq I$.

Lemma 2. *For any MBQC on a stabilizer state, $|I_{\text{gauge}}| \leq |O_{\text{comp}}|$.*

We prove Lemma 2 in Section 3.5.

Definition 10. *A pair $I_{\text{gauge}}, O_{\text{comp}}$ is called extremal iff $|I_{\text{gauge}}| = |O_{\text{comp}}|$.*

As will become clear in the next section, extremal pairs $I_{\text{gauge}}, O_{\text{comp}}$ are easier to handle than general pairs, and are not very restrictive (c.f. Theorem 2).

We still need to relate the classical output vector \mathbf{o} appearing in Eq. (15b) to the set O_{comp} . To this end, we make the following

Definition 11 (Optimal classical output). *A classical output vector \mathbf{o} , with processing relations $\mathbf{o} = Z\mathbf{s} + R\mathbf{g}$, is optimal iff the following conditions hold*

1. Maximality: *Upon left-multiplication by an invertible matrix, Z can be brought into a unique normal form*

$$Z \sim (Z|I), \quad (23)$$

where the column split is between $(O_{\text{comp}})^c|O_{\text{comp}}$, and

2. Determinism: *For the matrix Z in Eq. (23),*

$$[Z]_{ij} = 1 \iff K(j)|_i \in \{\sigma_s^{(i)}, \sigma_{s\phi}^{(i)}\}. \quad (24)$$

In other words Eq. (24) informs us that the j th column of Z is simply the restriction of the support of $K(j)$ to O_{comp} .

The reason for defining an ‘optimal classical output’ besides a ‘classical output’ is the following: One could, in principle, run an MBQC perfectly deterministically and then choose \mathbf{o} such that nothing is outputted at all, or all outputted bits, independent of the measurement angles chosen, are zero guaranteed or perfectly random guaranteed. The above definition of an optimal classical output eliminates such choices. Maximality says that there is one bit of optimal output per qubit of O_{comp} . The determinism condition can be understood from the correction procedure explained in Eq. (5). For a qubit $j \in (O_{\text{comp}})^c$, we account for the undesired outcome $s_j = 1$ by inserting $K(j)$ into the state overlap $\langle \varphi_{\text{loc}} | \Psi \rangle = \langle \varphi_{\text{loc}} | K(j) | \Psi \rangle$. Now consider a qubit $i \in O_{\text{comp}}$. By Eq. (23), s_i contributes to the output bit o_i , the i -th bit of \mathbf{o} . If $K(j)|_i \in \{\sigma_s^{(i)}, \sigma_{s\phi}^{(i)}\}$, then the insertion of $K(j)$ into the overlap flips s_i , i.e., $s_i \rightarrow s_i \oplus 1$. This needs to be taken into account when reading out s_i . The linear combination $s_j \oplus s_i$ remains unaffected by the correction for s_j .

We have now made the definitions needed to state and prove our results on temporal order in MBQC. In Section 3.5 we establish a normal form for the stabilizer generator matrix of the resource state $|\Psi\rangle$. This normal form is the basis for our results on MBQC temporal order, which are stated in Sections 4 and A.

3.5 Normal form of the resource state stabilizer

Let us briefly review which pieces of information specify an MBQC. A priori, there are four: the set of measurement angles, the set Σ of measurement planes, the resource state $|\Psi\rangle$ and the classical processing relations Eq. (15a,15b). The measurement angles entirely drop out of all our considerations about temporal order. Next, we observe that when specifying the measurement planes and the resource state separately, we really specify too much. Starting from a given pair $|\Psi\rangle, \Sigma$ of resource state and set of measurement planes, for any local Clifford unitary U , the pair $U|\Psi\rangle, U(\Sigma)$ obtained by applying U to both the measurement planes Σ and the stabilizer state $|\Psi\rangle$ is again a valid pair, i.e., it consists of a set of a stabilizer state and a set of measurement planes. Furthermore, it amounts to exactly the same computation as the original pair. The pair $|\Psi\rangle, \Sigma$ is thus redundant. To remove this redundancy, we combine the measurement planes and the stabilizer state $|\Psi\rangle$ into the stabilizer generator matrix $\mathcal{G}(|\Psi\rangle)$ in the σ_ϕ/σ_s -stabilizer basis,

$$\mathcal{G}(|\Psi\rangle) = (\Phi||S). \quad (25)$$

Therein, the columns to the left (right) form the σ_ϕ (σ_s -) part of the stabilizer generator matrix. $\mathcal{G}(|\Psi\rangle)$ comprises all information from the resource state $|\Psi\rangle$ and the set of measurement planes Σ relevant for the discussion of MBQC. We do not need to know the state and the measurement planes separately.

Now note that the correction operators $K(a)$, $a \in (O_{\text{comp}})^c$, and the gauge operators $\bar{K}(i)$, $i \in I_{\text{gauge}}$, are all elements of the stabilizer $\mathcal{S}(|\Psi\rangle)$, and, by Lemma 1, are independent. Thus, they either form or can be completed to a set of generators for $\mathcal{S}(|\Psi\rangle)$. This observation leads us to the following

Lemma 3. *For any MBQC on a stabilizer state $|\Psi\rangle$ with extremal I_{gauge} , O_{comp} , the generator matrix \mathcal{G} of $\mathcal{S}(|\Psi\rangle)$ can be written in the normal form*

$$\mathcal{G} \cong \begin{array}{c} \begin{array}{cc} \sigma_\phi & \sigma_s \\ I_{\text{gauge}} & (I_{\text{gauge}})^c \end{array} \begin{array}{cc} (O_{\text{comp}})^c & O_{\text{comp}} \end{array} \\ \left(\begin{array}{c|c||c|c} 0 & T^T & I & Z^T \\ \hline I & H^T & 0 & R^T \end{array} \right). \end{array} \quad (26)$$

The matrices H, R, T, Z are related to the matrices H, R, T, Z governing the classical processing of measurement outcomes in MBQC, $\mathbf{q} = T\mathbf{s} + H\mathbf{g} \pmod 2$ and $\mathbf{o} = Z\mathbf{s} + R\mathbf{g} \pmod 2$, via

$$T = \left(\begin{array}{c|c} 0 & 0 \\ \hline T & 0 \end{array} \right), \quad H = \left(\begin{array}{c} I \\ \hline H \end{array} \right), \quad Z = (Z|I), \quad R = R. \quad (27)$$

Proof. By Definition 7, a correction operator $K(a)$ exists for every $a \in (O_{\text{comp}})^c$. By Eq. (19) these operators have no support in $I \supseteq I_{\text{gauge}}$ and are $|(O_{\text{comp}})^c|$ in number and must take the following form: $(0|A||I|B)$, where the σ_ϕ part is split as $I_{\text{gauge}}, I_{\text{gauge}}^c$ while the σ_s -part is split as $O_{\text{comp}}, (O_{\text{comp}})^c$. The gauge operators are, by definition, of the form: $(I|C||0|D)$, where the column splits are as for the correction operators. Since there are $|\Omega|$ generators for the stabilizer $\mathcal{S}(|\Psi\rangle)$, of which $|(O_{\text{comp}})^c|$ are already accounted for, there can be at most $|O_{\text{comp}}|$ independent gauge operators, i.e., $|I_{\text{gauge}}| \leq |O_{\text{comp}}|$ which proves Lemma 2. In the present setting, the pair $I_{\text{gauge}}, O_{\text{comp}}$ is extremal by assumption, thus

these two sets of generators exhaust the stabilizer generators and we can write the stabilizer as

$$\mathcal{G}(|\Psi\rangle) = \begin{matrix} I_{\text{gauge}} & (I_{\text{gauge}})^c & (O_{\text{comp}})^c & O_{\text{comp}} \\ \left(\begin{array}{c|c|c|c} 0 & A & I & B \\ \hline I & C & 0 & D \end{array} \right), \end{matrix} \quad (28)$$

for suitable matrices A , B , C and D . We now need to identify these matrices. By definition, $I_{\text{gauge}} \subseteq I$. Measurement outcomes on qubits $a \in O_{\text{comp}}$ are not corrected, hence $fc(a) = \emptyset$ for all $a \in O_{\text{comp}}$, and $O_{\text{comp}} \subseteq O$ follows from the definition of the output set O . Then, the influence matrix T takes the form

$$T = \begin{pmatrix} 0 & 0 \\ \mathbf{T} & 0 \end{pmatrix}, \quad (29)$$

where the column split is $(O_{\text{comp}})^c | O_{\text{comp}}$ and the row split is $I_{\text{gauge}} | (I_{\text{gauge}})^c$. Now consider the correction operator $K(a)$ for $a \in (O_{\text{comp}})^c$ in the upper part of $\mathcal{G}(|\Psi\rangle)$ in Eq. (28). We already know from Eq. (20), that the σ_ϕ part of $K(a)$ is the forward cone of a . Therefore we must have $A = \mathbf{T}^T$. Further comparing, with Eq. (24) which states that the restriction of the σ_s part of the correction operator $K(a)$ to O_{comp} is the a th column of Z . But this is precisely the a th row of B , thus we infer that $B = \mathbf{Z}^T$.

Next, consider row i of the lower part of $\mathcal{G}(|\Psi\rangle)$ in Eq. (28). Row i is $(0, \dots, 0, 1, 0, \dots, 0 | \mathbf{c}^T | | \mathbf{0} | \mathbf{d}^T)$. The corresponding stabilizer operator $\bar{K}(i)$, when inserted into the overlap $\langle \phi_{\text{loc}} | \Psi \rangle$ as in Eq. (5), flips the measurement basis at qubit i and of qubits $l \in (I_{\text{gauge}})^c$ with $[\mathbf{c}]_l = 1$. It further flips the measurement outcomes at qubits $m \in O_{\text{comp}}$ with $[\mathbf{d}]_m = 1$. Therefore,

$$H = \begin{pmatrix} I \\ \mathbf{H} \end{pmatrix}, \quad \text{with } \mathbf{H} = C^T, \text{ and} \\ R = \mathbf{R}, \quad \text{with } \mathbf{R} = D^T.$$

We thus arrive at the normal form Eq. (26). □

4 Interdependence of resource state and temporal order

Let us return to our discussion from the beginning of Section 3.5, on which pieces of information are needed to describe an MBQC. At this stage, apart from the set of measurement angles which do not enter our discussion, we remain with two pieces of data specifying a given MBQC, namely $\mathcal{G}(|\Psi\rangle)$ and the processing relations Eq. (15a), (15b). But it doesn't stop there. As the results of [14], [15], [16] show, the resource state $|\Psi\rangle$, the measurement planes Σ and the influence matrix T —being part of the classical processing relations Eq. (15a), (15b)—are not independent. Specifically, given the sets I of first-measurable and O of last-measurable qubits in addition to $|\Psi\rangle$ and Σ , the temporal order (generated by T) can be worked out completely.

Here we prove a statement about the interdependence of $\mathcal{G}(|\Psi\rangle)$ and the MBQC classical processing relations which goes the opposite direction. Namely we show that the classical processing relations Eq. (15a), (15b) uniquely specify the stabilizer generator matrix $\mathcal{G}(|\Psi\rangle)$, i.e., the pair $|\Psi\rangle$, Σ up to equivalence; See Theorem 4 below. Thus, only two pieces of data are needed to specify an MBQC that satisfies the determinism constraints, namely the measurement angles and the classical processing relations for the measurement outcomes.

A further question is whether the temporal relations compatible with a resource state $|\Psi\rangle$ and set of measurement planes Σ fit into a common framework. In this regard, we show that the classical processing relations (containing the temporal order) for MBQC with a fixed resource state and set of measurement planes, for extremal pairs $I_{\text{gauge}}, O_{\text{comp}}$, are in one-to-one correspondence with the bases of $\mathcal{G}(|\Psi\rangle)$, c.f. Theorem 3.

4.1 Results

We now present four theorems on the mutual dependence of the resource state and the classical processing relations.

Theorem 1. *Consider an MBQC on a stabilizer state $|\Psi\rangle$, with fixed measurement planes and an extremal pair of gauge input set I_{gauge} and computational output set O_{comp} . Then, the relations $\mathbf{q} = T\mathbf{s} + H\mathbf{g} \pmod 2$, and $\mathbf{o} = Z\mathbf{s} + R\mathbf{g} \pmod 2$ for an optimal output \mathbf{o} are unique.*

That is, once the resource state $|\Psi\rangle$, the measurement planes and $I_{\text{gauge}}, O_{\text{comp}}$ are fixed, there is no freedom left to choose the classical processing relations. They are uniquely determined by the former. In particular, for fixed stabilizer state $|\Psi\rangle$ and measurement planes, $T = T(I_{\text{gauge}}, O_{\text{comp}})$, $H = H(I_{\text{gauge}}, O_{\text{comp}})$ etc.

A corollary of Theorem 1 is that given the measurement planes and an extremal pair $I_{\text{gauge}}, O_{\text{comp}}$, the resource state $|\Psi\rangle$ uniquely determines the influence matrix T . One may ask how restrictive a condition the extremality of the pair $I_{\text{gauge}}, O_{\text{comp}}$ is. In this regard, note

Theorem 2. *Consider an MBQC on a fixed resource stabilizer state for fixed measurement planes, with an influence matrix T and input and output sets $I(T), O(T)$, such that no qubit $a \in I^c$ can be individually gauged with respect to $I(T), O(T)$. Then, there exists an extremal pair $I_{\text{gauge}} \subseteq I, O_{\text{comp}} \subseteq O$ such that $T = T(I_{\text{gauge}}, O_{\text{comp}})$.*

The input set $I(T)$ and the output set $O(T)$ which appear in Theorem 2 are uniquely specified by T through Definition 5. $T(I_{\text{gauge}}, O_{\text{comp}})$ is uniquely specified by the pair $I_{\text{gauge}}, O_{\text{comp}}$ through Theorem 1.

Theorem 2 states that all temporal relations for an MBQC, subject to the extra condition on the qubits which can be individually gauged, arise from extremal pairs $I_{\text{gauge}}, O_{\text{comp}}$. By establishing Theorem 2 we trade the condition of the pairs $I_{\text{gauge}}, O_{\text{comp}}$ being extremal for the condition that no qubit in I^c can be individually gauged. The latter is a more meaningful condition. Suppose a qubit a in I^c could be individually gauged wrt I, O . Then $\bar{K}(a)$ exists. For any b with $K(b)|_a = \sigma_\phi^{(a)}$, $\tilde{K}(b) := K(b)\bar{K}(a)$ is a valid correction operator for qubit b , and $\tilde{K}(b)|_a = I^{(a)}$. Hence, a could be removed from all forward cones and thereby be made a qubit in I . By imposing the extra condition in Theorem 2, we exclude temporal relations where certain qubits could be in the input set I but aren't.

Theorem 1 is mute on the question of which extremal pairs $I_{\text{gauge}}, O_{\text{comp}}$ are admissible. Theorem 3 below describes how much freedom remains for the choice of the classical processing relations, given $\mathcal{G}(|\Psi\rangle)$.

Theorem 3. *For MBQC with a fixed resource stabilizer state $|\Psi\rangle$ and fixed measurement planes, the classical processing relations for extremal $I_{\text{gauge}}, O_{\text{comp}}$, as specified by the matrices H, R, T, Z and the sets $I_{\text{gauge}}, O_{\text{comp}}$, are in one-to-one correspondence with the bases of the matroid $\mathcal{G}(|\Psi\rangle)$.*

After we have justified our restriction to extremal pairs of gauge input and computational output sets in Theorem 2 and have characterized the set of temporal relations compatible with a given resource state and set of measurement planes in Theorem 3, we now return to Theorem 1, and show that a converse also holds.

Theorem 4. Consider an MBQC on a stabilizer state $|\Psi\rangle$, with classical processing relations $\mathbf{q} = Ts + Hg \pmod{2}$, $\mathbf{o} = Zs + Rg \pmod{2}$ for an optimal classical output \mathbf{o} , such that $\text{rk}H = \text{rk}Z$. Then the classical processing relations uniquely specify the stabilizer generator matrix $\mathcal{G}(|\Psi\rangle)$ in the σ_ϕ/σ_s -basis, i.e. the resource stabilizer state $|\Psi\rangle$ and set Σ of measurement planes up to equivalence.

Remark: For Theorem 4 it does not matter whether or not the classical processing relations codify a temporal relation which is a partial order.

Remark: There is a constructive procedure for obtaining $\mathcal{G}(|\Psi\rangle)$ (in the σ_ϕ/σ_s -basis) from the linear processing relations. If the processing relation did not stem from an actual computation but rather was “made up”, the resulting $\mathcal{G}(|\Psi\rangle)$ may not be a valid stabilizer generator matrix. I.e., the rows of $\mathcal{G}(|\Psi\rangle)$ may correspond to Pauli operators which do not pairwise commute.

4.2 Proofs of Theorems 1-4

Theorem 1 is an immediate consequence of Lemma 3.

Proof of Theorem 2. Assume that I is valid input set and O is a valid output set for a given MBQC. Then, the stabilizer generator matrix of the resource state can be written in the σ_ϕ/σ_s -basis as

$$\mathcal{G}(|\Psi\rangle) = \left(\begin{array}{c|c|c|c} I & I^c & O^c & O \\ \hline 0 & \tilde{T}^T & I & A \\ \hline B & C & 0 & D \end{array} \right), \quad (30)$$

for some matrices A, B, C and D . The influence matrix T can be obtained as

$$T^T = \left(\begin{array}{c|c} 0 & \tilde{T}^T \\ \hline 0 & 0 \end{array} \right), \quad (31)$$

with the column split between I and I^c , and the row split between O^c and O . The matrices B and $(B|C)$ do not necessarily have maximal row-rank. By row transformations of $(B|C||0|D)$ we extract the dependent rows, and obtain

$$\mathcal{G}(|\Psi\rangle) = \left(\begin{array}{c|c|c|c} I & I^c & O^c & O \\ \hline 0 & \tilde{T}^T & I & A \\ \hline 0 & 0 & 0 & D' \\ \hline 0 & C'' & 0 & D'' \\ \hline B''' & C''' & 0 & D''' \end{array} \right), \quad (32)$$

However, note that each row in the third set of rows in the above matrix can either be interpreted as a correction operator $K(a)$, with non-empty forward cone $fc(a)$, for some $a \in O$, or a gauge operator $\bar{K}(i)$ for some $i \in I^c$. The former is ruled out because every $a \in O$ must have an empty forward cone. The latter is ruled out by the assumption that no qubit outside the set I can be individually gauged. Therefore that set of rows must identically vanish.

Since $\mathcal{G}(|\Psi\rangle)$ has full row rank, so does the matrix D' appearing in Eq. (32). We may then choose a set $\Delta O \subseteq O$ such that the columns of D' indexed by ΔO form a maximal independent set. We set $O_{\text{comp}} := O \setminus \Delta O$. Then, by further row transformations which do not affect \tilde{T} , the matrix in Eq. (32) can be converted to

$$\mathcal{G}(|\Psi\rangle) = \left(\begin{array}{c|c|c|c|c} I & I^c & O^c & \Delta O & O_{\text{comp}} \\ \hline 0 & \tilde{T}^T & I & 0 & A' \\ \hline 0 & 0 & 0 & I & A'' \\ \hline B''' & C''' & 0 & 0 & D''' \end{array} \right). \quad (33)$$

B''' has full row rank by construction. We can therefore find a set $I_{\text{gauge}} \subseteq I$ such that the columns of B''' indexed by I_{gauge} form a maximal independent set. For any such set I_{gauge} we can convert the matrix in Eq. (33) fully into the normal form Eq. (26) without affecting T .

For any of the above choices for $I_{\text{gauge}} \subseteq I$ and $O_{\text{comp}} \subseteq O$, the resulting influence matrix $T(I_{\text{gauge}}, O_{\text{comp}})$ can be extracted as

$$T(I_{\text{gauge}}, O_{\text{comp}})^T = \left(\begin{array}{c|c} 0 & \tilde{T}^T \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array} \right), \quad (34)$$

with the column split between I and I^c , and the row split between O^c , ΔO and $O_{\text{comp}} = O \setminus \Delta O$. By comparison of Eqs. (31) and (34) we verify $T = T(I_{\text{gauge}}, O_{\text{comp}})$. \square

Remark: Comparing Eq.(33) with the normal form in Eq. (26), we can write the stabilizer matrix in a slightly varied form that can be useful later.

$$\mathcal{G}(|\Psi\rangle) = \left(\begin{array}{c|c|c|c|c|c} I_{\text{gauge}} & \Delta I & I^c & O^c & \Delta O & O_{\text{comp}} \\ \hline 0 & 0 & \tilde{T}^T & I & 0 & Z_1^T \\ \hline 0 & 0 & 0 & 0 & I & Z_2^T \\ \hline I & H_1^T & H_2^T & 0 & 0 & R^T \end{array} \right), \quad (35)$$

where $Z = (Z_1 | Z_2)$ and where $H^T = (H_1^T | H_2^T)$.

Proof of Theorem 3. Denote by \mathcal{B} the set of bases of \mathcal{G} , and by \mathcal{T} the set of extremal classical processing relations of form Eq. (15a), (15b), specified by the triple $(I_{\text{gauge}}, O_{\text{comp}}, \{H, R, T, Z\})$. Then, the mapping $h : \mathcal{B} \rightarrow \mathcal{T}$ exists and is a bijection. (1) Existence of h : By the normal form Eq. (26) of $\mathcal{G} = (\Phi | S)$, for a given basis $B(\mathcal{G})$ the sets I_{gauge} and O_{comp} are extracted as follows. A qubit i is in I_{gauge} if and only if the corresponding column of Φ appears in $B(\mathcal{G})$. A qubit a is in O_{comp} if and only if the corresponding column of S does not appear in $B(\mathcal{G})$. Knowing I_{gauge} and O_{comp} , $\{H, R, T, Z\}$ is uniquely determined via Theorem 1. (2) Surjectivity of h : By definition of ‘‘extremal’’. (3) Injectivity of h : given I_{gauge} and O_{comp} , $B(G) = (\Phi|_{I_{\text{gauge}}} | S_{O_{\text{comp}}})$ is unique. \square

Proof of Theorem 4. We divide the proof into the following steps: i) First, with $\text{rk}H = \text{rk}Z$, the processing relations stem from an extremal pair $I_{\text{gauge}}, O_{\text{comp}}$. We show that given an extremal pair $I_{\text{gauge}}, O_{\text{comp}}$, the matrices H, R, T and Z are uniquely determined by the classical processing relations Eq. (15a,15b). ii)

From these matrices we can derive the corresponding normal form of the resource state uniquely. Let us denote this by \mathcal{N} . In iii) and iv) we show that the following diagram commutes, which establishes the equivalence of normal forms of all extremal pairs.

$$\begin{array}{ccc}
 (I_{\text{gauge}}, O_{\text{comp}}) & \xrightarrow{\Lambda} & (I'_{\text{gauge}}, O'_{\text{comp}}) \\
 \downarrow T, H, Z, R & & \downarrow T', H', Z', R' \\
 \mathcal{N} & \xrightarrow{M(\Lambda)} & \mathcal{N}'
 \end{array} \tag{36}$$

In Eq. (36), $M(\Lambda)$ is a transformation on the normal form \mathcal{N} , dependent on Λ . Because of their independence, we consider the transformations $I_{\text{gauge}} \rightarrow I'_{\text{gauge}}$ and $O_{\text{comp}} \rightarrow O'_{\text{comp}}$ separately in iii) and iv) respectively.

- i) Extracting H , R , T and Z : Given any set of inputs I and outputs O by Theorem 2, we know that there always exists an extremal pair $(I_{\text{gauge}}, O_{\text{comp}})$, where $I_{\text{gauge}} \subseteq I$, $O_{\text{comp}} \subseteq O$. By Eq. (29), T is uniquely specified by T . By the assumption of the classical output being optimal, Z can be brought into the unique normal form of Eq. (23) by left multiplication with an invertible matrix and (permuting the columns if necessary), Z can be extracted. The matrix H may not appear in Eq. (15a) in its normal form Eq. (27), nonetheless for an invertible matrix Λ , the vector \mathbf{q} specifying the measurement bases is invariant under $H \rightarrow H\Lambda$, $\mathbf{g} \rightarrow \Lambda^{-1}\mathbf{g}$. By definition of I_{gauge} , every qubit in I_{gauge} can be individually gauged with respect to $I_{\text{gauge}}, O_{\text{comp}}$. Therefore, (up to row permutations), we can choose Λ such that

$$H\Lambda = \begin{pmatrix} I \\ H \end{pmatrix}, \tag{37}$$

where the row split is between I_{gauge} (upper) and $(I_{\text{gauge}})^c$ (lower). Λ and H are unique. The classical output \mathbf{o} is invariant under the transformation $R \rightarrow R\Lambda'$, $\mathbf{g} \rightarrow (\Lambda')^{-1}\mathbf{g}$. However, since Eqs. (15a) and (15b) refer to \mathbf{g} in the same basis, Λ' is now fixed: $\Lambda' = \Lambda$. Then, $R = R\Lambda$, with the Λ of Eq. (37).

- ii) Assembling the normal form: Using the unique matrices H , R , T , Z , by Lemma 3, we can now assemble the normal form Eq. (26) of the stabilizer generator matrix for the resource state $|\Psi\rangle$. By assumption of Theorem 4, the matrices H, R, T, Z describe a valid computation, and the normal form Eq. (26) derived from them must thus yield a valid description of a quantum state. In particular, all Pauli operators specified by the rows of $\mathcal{G}(\{H, R, T, Z\})$ in Eq. (26) must commute. (The rows are independent by design of the normal form.) We have thus constructed a description of $|\Psi\rangle$. Since H, R, T, Z are unique, so is $|\Psi\rangle$.

We now proceed to construct the stabilizer $\mathcal{S}(|\Psi\rangle)$ from the processing relations when I_{gauge} and O_{comp} are not specified. From the classical processing relation Eq. (15a) we can still extract the input set I and the output set O , by testing which rows and columns of T identically vanish. Then, the possible choices for I_{gauge} and O_{comp} are limited to $I_{\text{gauge}} \subseteq I$ and $O_{\text{comp}} \subseteq O$ by definition.

- iii) Equivalence under $\Lambda : I_{\text{gauge}} \rightarrow I'_{\text{gauge}}$. In order to prove this we rely on the slightly variant version of the normal form of $\mathcal{G}(|\Psi\rangle)$ as shown in Eq. (35) which also includes additional detail about the correction operators. As noted above, the different normal forms for H can be interconverted by right-multiplication with an invertible matrix Λ (change of basis for \mathbf{g}), i.e. $H, R \rightarrow H\Lambda, R\Lambda$, where $H^T = (I|H^T)$. Under such a transformation, the upper part of the normal form Eq. (26) for $\mathcal{G}(|\Psi\rangle)$ remains unchanged, and the lower part is transformed $(H^T||0|R^T) \rightarrow \Lambda^T(H^T||0|R^T)$. Invertible row transformations on $\mathcal{G}(|\Psi\rangle)$ leave $|\Psi\rangle$ unchanged, and $|\Psi\rangle$ is thus independent on the precise choice of $I_{\text{gauge}} \subseteq I$.

iv) Equivalence under $\Lambda : O_{\text{comp}} \rightarrow O'_{\text{comp}}$. Proving that a different choice of O_{comp} does not change the stabilizer state is a little more complicated. We proceed in the following manner. From Theorem 3, we know that $I_{\text{gauge}} \cup O_{\text{comp}}^c$ are the bases of a matroid. Therefore for two distinct computational output sets O_{comp} and O''_{comp} , there exists another computational output set $O'_{\text{comp}} = O_{\text{comp}} \setminus \{i\} \cup \{j\}$, where $i \in O_{\text{comp}} \setminus O''_{\text{comp}}$ and $j \in O''_{\text{comp}} \setminus O_{\text{comp}}$. Therefore, it suffices if we show that the stabilizer state does not change if we change the computational output set from O_{comp} to O'_{comp} . Assume that the classical relation for the computational output set O_{comp} is given as

$$\mathbf{o} = \mathbf{Zs} + \mathbf{Rg} = (\mathbf{Z}_1 | \mathbf{Z}_2 | I) \mathbf{s} + \mathbf{Rg}. \quad (38)$$

where the column split of Z is between O_{comp}^c , ΔO , and O_{comp} . The corresponding normal form (with the column split in σ_ϕ -part $I_{\text{gauge}} | I_{\text{gauge}}^c$) is

$$\left(\begin{array}{c|c|c|c|c} 0 & \tilde{\mathbf{T}}^T & I & 0 & \mathbf{Z}_1^T \\ \hline 0 & 0 & 0 & I & \mathbf{Z}_2^T \\ \hline I & \mathbf{H}^T & 0 & 0 & \mathbf{R}^T \end{array} \right), \quad (39)$$

where $\mathbf{T}^T = \begin{pmatrix} \tilde{\mathbf{T}}^T \\ 0 \end{pmatrix}$. Suppose that we transform O_{comp} to $O_{\text{comp}} \setminus \{i\} \cup \{j\}$, where $i \in O_{\text{comp}}$ and $j \in \Delta O$. Without loss of generality assume that i is the last column of Z_2 . (It cannot be an all zero column because, then it would not be possible for it to be in O_{comp} .) Let $\mathbf{Z}_1 = \begin{pmatrix} x^T \\ \mathbf{Z}_A \end{pmatrix}$ and $\mathbf{Z}_2 = \begin{pmatrix} a^T & 1 \\ \mathbf{Z}_B & b \end{pmatrix}$. Then $\Lambda = \begin{pmatrix} 1 & 0 \\ b & I \end{pmatrix}$ acting on Z achieves the transformation O_{comp} to O'_{comp} .

$$\Lambda(\mathbf{Z}_1 | \mathbf{Z}_2 | I) = \left(\begin{array}{c|c|c|c|c} x^T & a^T & 1 & 1 & 0 \\ \hline \mathbf{Z}_A + bx^T & \mathbf{Z}_B + ba^T & 0 & b & I \end{array} \right) \quad (40)$$

$$\sim \left(\begin{array}{c|c|c|c|c} x^T & a^T & 1 & 1 & 0 \\ \hline \mathbf{Z}_A + bx^T & \mathbf{Z}_B + ba^T & b & 0 & I \end{array} \right) = (\mathbf{Z}'_1 | | \mathbf{Z}'_2 | | I) \quad (41)$$

We claim that this same transformation can be effected by row transformations of $\mathcal{G}(|\Psi\rangle)$. First let us focus on the middle set of rows in $\mathcal{G}(|\Psi\rangle)$, namely the correction operators for ΔO . Then acting by $M(\Lambda) = \begin{pmatrix} I & a \\ 0 & 1 \end{pmatrix}$ gives us

$$M(\Lambda)(0|0|I|\mathbf{Z}_2^T) = \left(\begin{array}{c|c|c|c|c} 0 & 0 & I & a & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ \hline & & & & \mathbf{Z}_B^T + ab^T \\ & & & & b^T \end{array} \right) \quad (42)$$

$$\sim \left(\begin{array}{c|c|c|c|c} 0 & 0 & I & 0 & a \\ \hline 0 & 0 & 0 & 1 & 1 \\ \hline & & & & \mathbf{Z}_B^T + ab^T \\ & & & & b^T \end{array} \right) = (0|0|I|\mathbf{Z}_2'^T). \quad (43)$$

Now if take the last row in Eq. (43), namely $(0|0|0|1|b^T) = c$ and add xc to the top set of rows in Eq. (39) we obtain

$$\left(0 \mid \tilde{\mathbf{T}}^T \mid I \mid 0 \mid x \mid 0 \mid \mathbf{Z}_A^T + xb^T \right) \sim \left(0 \mid \tilde{\mathbf{T}}^T \mid I \mid 0 \mid \mathbf{Z}_1'^T \right) \quad (44)$$

showing the equivalence of Z and Z' . The equivalence of R and R' under Λ can be shown in exactly the same fashion as for Z_1 and Z'_1 .

This concludes the proof that the extremal classical relations completely determine the stabilizer state. \square

5 Flipping measurement planes and temporal invariance

In this section, we introduce a second group of symmetry transformations on MBQCs which is related to local complementation [17] - [19]. These transformations leave the temporal relation of the measurements in a any given MBQC unchanged. An addition, if the temporal relation is a partial order, the distribution of the computational output is left unchanged.

5.1 Flipping measurement planes

According to Eq. (3), for any qubit a in a given resource state, the local observable measured to drive the computation is

$$O_a[q_a] = \cos \varphi_a \sigma_\phi^{(a)} + (-1)^{q_a} \sin \varphi_a \sigma_{s\phi}^{(a)}.$$

Therein, q_a is a linear function of the measurement outcomes $\{s_b, b \in \Omega\}$, c.f. Eq. (15a).

Let's see what happens if we use a different rule for the adjustment of measurement bases, namely

$$O'_a[q_a] = (-1)^{q_a} \cos \varphi_a \sigma_\phi^{(a)} + \sin \varphi_a \sigma_{s\phi}^{(a)}, \quad (45)$$

That is, if $q_a = 1$, to obtain $O'_a[1]$ we are flipping the observable $O_a[0]$ about the $\sigma_{s\phi}$ -axis rather than the σ_ϕ -axis. Comparing Eqs. (3) and (45), we find that

$$O'_a[q_a] \equiv (-1)^{q_a} O_a[q_a], \quad (46)$$

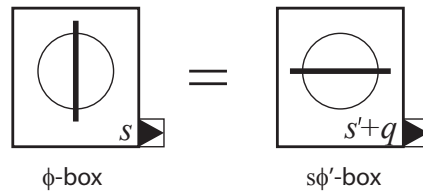
independent of the measurement angle φ_a . The measurements of $O'_a[q_a]$ and $O_a[q_a]$ are always in the same basis for the same q_a , and the measured eigenvalues differ by a factor of $(-1)^{q_a}$.

We call the transformation $\tau_s[i] : O_i[q_i] \rightarrow O'_i[q_i]$ *flipping of the measurement plane at qubit i* . On the elementary degrees of freedom, namely the resource state $|\Psi\rangle$, the Pauli observables $\sigma_s^{(i)}$, $\sigma_{s\phi}^{(i)}$ (action on $\sigma_s^{(i)}$ is implied), and the measurement angle φ_i , the flipping $\tau_s[i]$ acts as

$$\begin{aligned} \tau_s[i] : \quad & \sigma_\phi^{(i)} \longleftrightarrow \sigma_{s\phi}^{(i)}, \\ & \varphi_i \longrightarrow (-1)^{q_i} \frac{\pi}{2} - \varphi_i, \\ & |\Psi\rangle \longrightarrow |\Psi\rangle, \end{aligned} \quad (47)$$

The action of $\tau_s[i]$ on the Pauli operators $\sigma^{(j)}$ and the measurement angles φ_j , for $j \neq i$ is trivial. All our considerations are independent of the values of the measurement angles. In particular, the second part of the transformation Eq. (47) does not affect temporal order.

We now discuss the effect of the flipping $\tau_s[a]$, $a \in \Omega$ on a given MBQC. Denote the measurement outcome of a measurement of $O'_a(q_a)$ by s'_a . By Eq. (46), the following two measurement procedures are always equivalent. (I) Measuring $O_a[q_a]$ and outputting s_a , and (II) Measuring $O'_a[q_a]$ and outputting $s'_a + q_a \pmod 2$. We may call the device that performs Procedure I a ϕ -box, and the device that performs Procedure II a $s\phi'$ -box. Then,



The prime in the $s\phi'$ -box accounts for the fact that not the measurement outcome s'_a itself is outputted, but rather the locally post-processed value $s'_a + q_a \pmod 2$. Now, instead of outputting $s'_a + q_a$, the device at a may only output s'_a (that is an $s\phi$ -box), and the classical post-processing relations for the adaption of measurement bases are modified accordingly, i.e. $s_a \longrightarrow s'_a + q_a \pmod 2$. Can the resulting relations again be written in a form $\mathbf{q} = T'\mathbf{s}' + H'\mathbf{g}$?

We now attempt transforming a ϕ -box into an $s\phi$ -box at qubit a . The vectors of measurement outcomes \mathbf{s} and \mathbf{s}' are related via $\mathbf{s} = \mathbf{s}' + e_a e_a^T \mathbf{q}$. Inserting this relation into Eq. (15a), we obtain

$$(I + T e_a e_a^T) \mathbf{q} = T \mathbf{s}' + H \mathbf{g} \pmod 2. \tag{48}$$

Case I: $T_{aa} = 0$. Physically, this means that the measurement basis at the flipped qubit a does not depend on the measurement outcome at a , before the transformation. Multiplying Eq. (48) with e_a^T from the left yields $q_a = e_a^T T \mathbf{s}' + e_a^T H \mathbf{g} \pmod 2$. Inserting back into Eq. (48), we obtain

$$T' = T + T e_a e_a^T T \pmod 2. \tag{49}$$

Likewise,

$$H' = H \oplus T e_a e_a^T H, Z' = Z \oplus Z e_a e_a^T T, R' = R \oplus Z e_a e_a^T H. \tag{50}$$

Eqs. (49) and (50) completely describe the effect of the flipping $\tau_s[a]$ of the measurement plane at qubit a on the classical processing relations Eq. (15a), (15b).

Remark: If the matrices H, R, T, Z are given their normal form Eq. (27) wrt the pair $I_{\text{gauge}}, O_{\text{comp}}$ then the flipping of the measurement plane at any vertex a with $T_{aa} = 0$ leaves this normal form intact. We can therefore state a transformation rule equivalent to Eqs. (49), (50) for the matrices H, R, T, Z . This rule is, in fact, simpler. We assemble the composite matrix

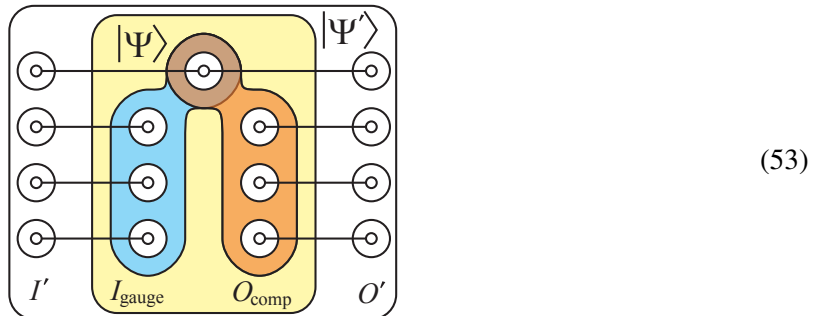
$$T_{\text{ext}} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline H & T & 0 & 0 \\ \hline R & Z & I & 0 \end{array} \right), \tag{51}$$

which is a square matrix of size $(|\Omega| + |I_{\text{gauge}}| + |O_{\text{comp}}|) \times (|\Omega| + |I_{\text{gauge}}| + |O_{\text{comp}}|)$. The effect of flipping of the measurement plane at a then is

$$T_{\text{ext}} \longrightarrow T'_{\text{ext}} = T_{\text{ext}} + T_{\text{ext}} e_a e_a^T T_{\text{ext}} \pmod 2. \tag{52}$$

Thus, the rule is just the same as Eq. (49) for the original influence matrix T .

Furthermore, T_{ext} is the influence matrix for MBQC on a bigger resource state $|\Psi'\rangle$ constructed from $|\Psi\rangle$. For the support Ω' of $|\Psi'\rangle$ we require two additional sets of qubits, I' and O' , with $|I'| = |O'| = |I_{\text{gauge}}| = |O_{\text{comp}}|$, such that $\Omega' = \Omega \cup I' \cup O'$. $|\Psi'\rangle$ is obtained from $|\Psi\rangle$ by the following construction:



$$\tag{53}$$

Therein, the gates $\bigcirc\text{---}\bigcirc = \Lambda_s$ are σ_s -controlled σ_s -gates, i.e., $\Lambda_s \sigma_s^{(1)} \Lambda_s^\dagger = \sigma_s^{(1)}$, $\Lambda_s \sigma_\phi^{(1)} \Lambda_s^\dagger = \sigma_\phi^{(1)} \otimes \sigma_s^{(2)}$, etc, and the extra qubits in I' and O' are initially prepared in the eigenstate of σ_ϕ with eigenvalue 1. With the definition Eq. (53) of $|\Psi'\rangle$, the labelling of the blocks of rows and columns for the matrix on the r.h.s. of Eq. (52) is $I'|(O_{\text{comp}})^c|O_{\text{comp}}|O'$ for the columns and $I'|I_{\text{gauge}}|(I_{\text{gauge}})^c|O'$ for the rows.

We thus find that all information in the classical MBQC processing relations is temporal information for the computation on a slightly extended resource state.

Case 2: $T_{aa} = 1$. In this case, the measurement basis at a does depend on the measurement outcome at a . This is an example for a closed time-like curve (only involving the measurement device at a), and an obstacle to deterministic runnability. Now, the matrix $I \oplus T \mathbf{e}_a \mathbf{e}_a^T$ on the left side in Eq. (48) is not invertible, $\mathbf{e}_a^T (I \oplus T \mathbf{e}_a \mathbf{e}_a^T) = 0$. Hence, the relation (48) can not be solved for \mathbf{q} in this case. There is no relation Eq. (15a) with the same sets I_{gauge} , O_{comp} before and after flipping.

We now discuss the consequences of flipping measurement planes for the above two cases.

5.2 Flipping measurement planes and local complementation

We now return to the above Case 1, namely when flipping of a measurement plane yields a computation with a new relation Eq. (15a). Note that the computation before and after the flip generate the same output distribution. Flipping a ϕ -box into an $s\phi'$ -box is an equivalence transformation, only based on the operator identity Eq. (46). Changing an $s\phi'$ -box into an $s\phi$ -box is again an equivalence transformation, provided it can be carried out.

The influence matrices T and T' before and after the flipping, respectively, are in general not the same, c.f. Eq. (49). However, T and T' still generate the same temporal order, as we now show.

Lemma 4. *Be T an influence matrix with $T_{ii} = 0$. Then, T and $T' = T \oplus T \mathbf{e}_i \mathbf{e}_i^T T$ generate the same temporal relation under transitivity.*

Proof of Lemma 4. Let's introduce a shorthand $a \rightarrow c$ for $c \in fc(a)$ (meaning that the measurement outcome at a influences the measurement basis at c). Now, we have to show that $e \prec_T f \iff e \prec_{T'} f$, for any T' generated from T by the transformation Eq. (49).

(I) " \implies ": Assume that $e \prec_T f$. Then there exists a sequence of measurement events $e \rightarrow m_1 \rightarrow m_2 \rightarrow \dots \rightarrow a \rightarrow c \rightarrow \dots \rightarrow f$. Can we break the arrow $a \rightarrow c$, say? To investigate this, let us rewrite the transformation rule Eq. (49) for the flip $\tau_s[i]$ as

$$\tau_s[i]: \begin{array}{ll} \mathbf{fc}(a) & \longrightarrow \mathbf{fc}(a) \oplus \mathbf{fc}(i), & \text{if } i \in fc(a), \\ \mathbf{fc}(a) & \longrightarrow \mathbf{fc}(a), & \text{if } i \notin fc(a), \end{array} \quad (54)$$

Case 1: $a \rightarrow i$ before the transformation $\tau_s[i]$. Then, $\mathbf{fc}'(a) = \mathbf{fc}(a) \oplus \mathbf{fc}(i)$. Since $T_{ii} = 0$ by assumption, $a \rightarrow i$ after the transformation $\tau_s[i]$. Sub-case 1a: $i \rightarrow c$ before the transformation $\tau_s[i]$. Since $T_{ii} = 0$ ($i \notin fc(i)$), $i \rightarrow c$ after the transformation $\tau_s[i]$. Thus $a \rightarrow i \rightarrow c$ after the transformation, and hence $a \prec_{T'} c$. Sub-case 1b: $i \not\rightarrow c$ before $\tau_s[i]$. Then, $a \rightarrow c$ remains after the transformation. Case 2: $a \not\rightarrow i$ before the transformation $\tau_s[i]$. Then $a \rightarrow c$ after $\tau_s[i]$. Thus, in all cases $a \prec_{T'} c$, and therefore $e \prec_{T'} f$.

(II) " \impliedby ": From Eq. (54), $\tau_s[i]^2 = I$. \square

Apply a series of transformations Eq. (49) on an initial influence matrix T with vanishing diagonal part may produce an influence matrix with a non-vanishing diagonal part. Thus, the application of the transformation Eq. (49) is restricted. To circumvent this problem, we introduce a modified transformation

$$\tilde{\tau}[i]: T \longrightarrow T' = T + T \mathbf{e}_i \mathbf{e}_i^T T + \mathcal{D}(T \mathbf{e}_i \mathbf{e}_i^T T) \pmod{2}. \quad (55)$$

Clearly, this transformation takes influence matrices with vanishing diagonal part to influence matrices with vanishing diagonal part, and thereby avoids the problem of restricted applicability of transformation Eq. (49). Note that the transformation Eq. (55) has the form of local complementation, albeit the influence matrix T that it acts on will in general not be symmetric.

But what is the physical significance of transformation Eq. (55)? The only additional effect of the transformation $\tilde{\tau}[i]$ over $\tau_s[i]$ is the cancelling of the diagonal part of the influence matrix after the transformation, c.f. the last term in Eq. (55). This can be achieved by a local unitary that exchanges $\sigma_{s\phi} \leftrightarrow \sigma_s$ on a respective qubit. The action of $\tilde{\tau}[i]$ on the elementary degrees of freedom therefore is

$$\tilde{\tau}[i] : \begin{array}{l} \sigma_{\phi}^{(i)} \quad \longleftrightarrow \quad \sigma_{s\phi}^{(i)}, \\ \varphi_i \quad \longrightarrow \quad (-1)^{q_i} \frac{\pi}{2} - \varphi_i, \\ \sigma_s^{(j)} \quad \longleftrightarrow \quad \sigma_{s\phi}^{(j)}, \\ |\Psi\rangle \quad \longrightarrow \quad |\Psi\rangle. \end{array} \quad \forall j \in fc(i) \cap bc(i), \quad (56)$$

We find that the local measured operators for all qubits $j \in bc(i) \cap fc(i)$ change in a way that cannot be accommodated by a change of the respective measurement angle. For those qubits, the new measured observables lie in a different equatorial plane of the Bloch sphere. Therefore, the transformation Eq. (56), unlike the transformation Eq. (49), does *not* necessarily map a given computation onto itself. What it does, however, is mapping a given computation to a computation with the same temporal relation.

Lemma 5. *Be T an influence matrix with $T_{ii} = 0$. Then, T and $T' = T \oplus T\mathbf{e}_i\mathbf{e}_i^T T \oplus \mathcal{D}(T\mathbf{e}_i\mathbf{e}_i^T T)$ generate the same temporal relation under transitivity.*

Proof of Lemma 5. Assuming the initial influence matrix has vanishing diagonal part, we split the transformation $T \longrightarrow T \oplus T\mathbf{e}_i\mathbf{e}_i^T T \oplus \mathcal{D}(T\mathbf{e}_i\mathbf{e}_i^T T)$ into two steps, namely $T \longrightarrow T' = T \oplus T\mathbf{e}_i\mathbf{e}_i^T T$ and $T' \longrightarrow T'' = T' \oplus \mathcal{D}(T')$. By Lemma 4, T and T' generate the same temporal order. Now assume that $T'_{kk} = 1$ for some $k \in \Omega$, $k \neq i$. This requires that $T_{ik} = T_{ki} = 1$. Then, we also have $T'_{ik} = T'_{ki} = 1$ and $T''_{ik} = T''_{ki} = 1$. Thus, $k \prec_{T'} k$ and $k \prec_{T''} k$. The closed time-like curve involving k is not changed by setting $T''_{kk} = 0$. All other relations trivially remain unaffected by the transformation $T' \longrightarrow T''$. \square

6 MBQC – a toy model for quantum space time?

In attempts to unify the theory of general relativity with quantum mechanics, often the viewpoint is taken that spacetime is not an independent construct, but rather a consequence of the laws of quantum mechanics. Once this assertion is spelled out, the natural next step is to identify the key quantum property which yields a mechanism for generating temporal order, and to illustrate this mechanism in a toy model.

We do not solve any puzzle of quantum gravity here, but argue that measurement-based quantum computation possesses certain properties that one expects to find in a toy model generating spacetime from none. Namely, (i) Despite its origin in non-relativistic quantum mechanics, time in MBQC is a binary relation among spacetime events (here quantum-mechanical measurements). In computations without closed time-like curves—which are the ones of practical interest—this relation is a partial ordering. There is no external time parameter. (ii) MBQC invokes a quantum mechanical principle that strongly constrains the possible temporal orders, namely that the logical processing is not affected by the randomness inherent in quantum mechanical measurement. This principle can be formulated in terms of an underlying symmetry group.

Below we discuss aspects of MBQC that we find are of interest for toy models generating spacetime. These are: an analogue of Malement’s theorem, forward cones arising as solutions of a wave equation,

and event horizons. We emphasize that these analogies arise at a formal level—MBQC is about bits, not matter fields.

6.1 An MBQC counterpart of Malament's theorem?

Malament's theorem [22] states that in any spacetime manifold the light cones determine the metric up to a conformal factor. Here we argue that Theorem 4 is an MBQC counterpart of that. To begin, we must identify an MBQC counterpart of spacetime. We say that the MBQC equivalent of a spacetime is the entire measurement-based quantum computation, with its resource state and temporally ordered measurement events. The reason for this identification is that in Malament's setting of General Relativity, spacetime is all there is to reason about while in our setting it is the process of MBQC. We further identify the spacetime points with the location of measurement events.

Recall that an MBQC is fully specified by (i) the stabilizer generator matrix \mathcal{G} of the resource state, (ii) the set of measurement angles, and (iii) the linear processing relations Eq. (15a), (15b) for the adaptation of measurement bases and generating the output. Theorem 4 states that (iii) determines (ii), i.e. all of the MBQC except the measurement angles.

We can thus read Theorem 4 as an MBQC counterpart of Malament's theorem if we make two further identifications: (a) the light cone structure in GR corresponds to the linear processing relations in MBQC, and (b) the conformal factor in the spacetime metric at every spacetime point corresponds to the measurement angle at every measurement location.

Regarding (a), sure, the linear processing relations contain the forward cones in MBQC through the matrix T , but don't they also contain the additional matrices H , R and Z that have nothing to do with MBQC temporal order? In this regard, note that by doubling the qubits on the input and output boundaries I_{gauge} and O_{comp} in the support of the resource state, the influence matrix T_{ext} of the extended resource state $|\Psi''\rangle$ comprises all information about H , R , T and Z (c.f. Section 5.1). In this sense, all information in the processing relations is temporal.

Regarding (b), we presently do not know of a physical reason for identifying a scale factor for the metric at any spacetime point with a measurement angle at every MBQC spacetime point. However, we note that both Malamat's theorem and its MBQC analogy Theorem 4, in their respective settings, leave one real-valued parameter per spacetime point unfixed.

6.2 Forward cones satisfy a wave equation

We consider an MBQC on a resource stabilizer state in graph normal form $|G\rangle$. We show that if the measurement plane is (X/Y) for every qubit then the forward cones satisfy a wave equation. Here, (X/Y) is an unordered pair such that we are left with both the possibilities of $\sigma_\phi = X$ and $\sigma_\phi = Y$. The choices may differ for different qubits.

Following [37], we define the discretization of the Laplacian $\Delta = \sum_{i=1}^d \frac{d^2}{dx_i^2}$ as $\Delta := DD^T$, where D is the incidence matrix of the graph G . In our case, all addition is mod 2. An equivalent formulation then is

$$\Delta := \Gamma + \mathcal{D} \pmod{2}, \quad (57)$$

with Γ the adjacency matrix of G and \mathcal{D} a diagonal matrix such that $[\mathcal{D}]_{vv} = \deg(v)$, for all $v \in V(G)$. The discretized Laplacian can act on the characteristic vector \mathbf{f}_a of $fc(a)$, yielding $\Delta \mathbf{f}_a$. For further use we define the function $f_a : V(G) \rightarrow \mathbb{Z}_2$, by $f_a(v) = 1(0)$ if $v \in fc(a)$ ($v \notin fc(a)$). Then, the action of Δ on f_a is defined through $\Delta f_a(v) := [\Delta \mathbf{f}_a]_v$.

The definition Eq. (57) of the Laplacian is in accordance with intuition, as the following example for two-dimensional lattice graphs (embedded on a torus) shows. There,

$$\begin{aligned} (\Delta f_a)(x, y) &= [(f_a(x+1, y) - f_a(x, y)) - (f_a(x, y) - f_a(x-1, y))] + \\ &\quad + [(f_a(x, y+1) - f_a(x, y)) - (f_a(x, y) - f_a(x, y-1))] \pmod 2 \\ &= f_a(x+1, y) + f_a(x-1, y) + f_a(x, y+1) + f_a(x, y-1) \pmod 2 \\ &= [(\Gamma + \mathcal{D})\mathbf{f}_a]_{(x,y)} \pmod 2. \end{aligned}$$

In the last line, $\mathcal{D} = 0$ because all vertices have degree 4. Back to general graphs G , the offset \mathcal{D} in Eq. (58) is necessary such that $\Delta \mathbf{f} = \mathbf{0}$ whenever $f = \text{const}$. We now have the following

Lemma 6. *Consider MBQC on a graph state $|G\rangle$ where every qubit is measured in the (X, Y) -plane. Then, the forward cones satisfy a wave equation with position-dependent mass m ,*

$$m(v)f_a(v) + \Delta f_a(v) = \delta_{a,v}, \quad \forall v \in O_{\text{comp}}^c, \quad (58)$$

Therein, addition is mod 2, and $m(v) = \deg(v) + b(v) \pmod 2$ with $b(v) = 0(1)$ if the measurement plane at v is $[X, Y]$ (is $[Y, X]$).

Proof. For a given qubit b , irrespective of whether the measurement plane at b is $[X, Y]$ or $[Y, X]$, with Eq. (19) for the correction operations $K(a)$ we find that $b \in fc(a)$ iff $K(a)|_b \in X_b, Y_b$. In words, $K(a)$ has Pauli operators $\sim X$ exactly in those places that are in the forward cone of qubit a . Then, using the Definition 1 of graph states, we find

$$K(a) \equiv X(\mathbf{f}_a)Z(\Gamma \mathbf{f}_a), \quad (59)$$

where $X(\mathbf{g}) := \bigotimes_{v \in V(G) | |g|_v=1} X_v$.

Special case: The measurement plane is $[X, Y]$ for all qubits ($\sigma_\phi^{(a)} = X_a, \forall a \in V(G)$). Now, for all $b \neq a$ we require $K(a)|_b \in \{I_b, X_b\}$, and $K(a)|_a \in \{Z_a, Y_a\}$. Using Eq. (59), we thus find the constraint

$$\Gamma \mathbf{f}_a|_{O_{\text{comp}}^c} = \mathbf{e}_a. \quad (60)$$

Therein, \mathbf{e}_a is a $|O_{\text{comp}}^c|$ -component vector which has an entry 1 in the a th component, and zeros everywhere else. With Eq. (57), this reproduces Eq. (58) for $m(v) = \deg(v)$, in accordance with Lemma 6 for $b(v) \equiv 0$.

General case. For $b \neq a$ we require that $K(a)|_b \in \{I_b, X_b\}$ if the measurement basis at b is $[X, Y]$ and $K(a)|_b \in \{I_b, Y_b\}$ if the basis at b is $[Y, X]$. Furthermore, we require that $K(a)|_a \in \{Z_a, Y_a\}$ if the measurement basis at a is $[X, Y]$ and $K(a)|_a \in \{Z_a, X_a\}$ if the measurement basis at a is $[Y, X]$. Using Eq. (59) again, we find Eq. (58), with $m(v) = \deg(v) + b(v) \pmod 2$ as required. \square

6.3 CTCs of length 1 and event horizons

In this section we consider an MBQC that has a closed time-like curve of length 1 at a given qubit i . That is, the measurement basis at i depends on the measurement outcome at i . We argue that if we flip the measurement plane at qubit i we obtain an MBQC from which the closed time-like curve is removed, but instead qubit i vanishes behind the MBQC counterpart of an event horizon.

Be i a qubit such that $T_{ii} = 1$ before flipping the measurement plane at qubit i . Such a qubit i cannot be in I_{gauge} , since $I_{\text{gauge}} \subseteq I$ by definition. If $T_{ii} = 1$ then $i \in bc(i)$. The backward cones of all qubits in i

are empty by definition of I , however. Likewise, $i \notin O_{\text{comp}}$. If $T_{ii} = 1$ then $i \in fc(i)$. However, $fc(a) = \emptyset$ for all $a \in O_{\text{comp}}$. Thus, there is only one case to consider, namely $i \in (I_{\text{gauge}})^c \cap (O_{\text{comp}})^c$.

In this case, there exists a correction operator $K(i)$ for qubit i before the flipping, $K(i) = \sigma_{s\phi} \otimes K(i)|_{\Omega \setminus i}$. After flipping at i , this operator turns into

$$\tau_s[i](K(i)) = \sigma_{\phi}^{(i)} \otimes K(i)|_{\Omega \setminus i} =: \bar{K}'(i). \quad (61)$$

That is, the operator $\tau_s[i](K(i))$ resulting from flipping at i is a gauge type operator, c.f. Eq. (22). Thus, the flipping transformation $\tau_s[i]$ (when $T_{ii} = 1$) enlarges I_{gauge} by one qubit,

$$\tau_s[i] : I_{\text{gauge}} \longrightarrow I_{\text{gauge}} \cup \{i\}, \text{ if } T_{ii} = 1.$$

Furthermore, after the flipping at i there no longer is a correction operation for qubit i , hence

$$\tau_s[i] : O_{\text{comp}} \longrightarrow O_{\text{comp}} \cup \{i\}, \text{ if } T_{ii} = 1.$$

This has two consequences. First, the forward cone of i becomes empty. In particular $T_{ii} = 0$ after the flipping. Thus, the closed time-like curve consisting of qubit i has been removed. Second, an additional bit of optimal classical output is being created by the flipping at i . What does that output bit signify?

Recall that before the flipping at i , the rule for adjusting the measurement basis at i is

$$q_i \stackrel{!}{=} s_i + \sum_{j \in J \setminus i} s_j \pmod{2},$$

for some set $J \subseteq \Omega$. Here, we have dropped a constant offset $\mathbf{h}^T \mathbf{g}$ on the r.h.s. The symbol “!” above the equality means that equality is a requirement for the correctness of the computation, but it cannot be deterministically implemented. As follows from Eq. (46), the measurement outcomes before and after the flip, s_i and s'_i are related via $s_i = s'_i \oplus q_i$. For all the other qubits, $s'_j = s_j$. Substituting this into the above relation, we obtain

$$s'_i + \sum_{j \in J \setminus i} s'_j \pmod{2} \stackrel{!}{=} 0, \quad \forall q'_a \in \mathbb{Z}_2. \quad (62)$$

Thus, the additional output bit $o_i = s'_i + \sum_{j \in J \setminus i} s'_j \pmod{2}$ is a flag. If $o_i = 0$ then the computation succeeded, and if $o_i = 1$ then it did not.

Now suppose that the problem solved by the given MBQC is in NP. Then, this flag bit is not necessary. The remaining output may be efficiently checked for correctness anyway. Thus, one may safely discard the extra bit o_i of output. Not post-selecting on $o_i = 0$ can, if anything, only increase the success probability of the computation. We thus arrive at

Lemma 7. *Be \mathcal{M}_1 an MBQC with a classical output \mathbf{o} and influence matrix T such that $T_{ii} = 1$, i.e., \mathcal{M}_1 has a closed time-like curve involving a single qubit $i \in \Omega$. Be \mathcal{M}_2 the MBQC with the same classical output \mathbf{o} , obtained from \mathcal{M}_1 by flipping the measurement plane at i . Then, the closed time-like curve of i in \mathcal{M}_1 is removed in \mathcal{M}_2 . Furthermore, if \mathcal{M}_1 solves a problem in the complexity class NP with probability p , then \mathcal{M}_2 solves the same problem with probability $\geq p$.*

Remark: Lemma 7 does not guard against the inefficiencies of post-selection, in particular if multiple CTCs of length 1 are being removed. While the success probability after removing the CTCs is guaranteed not to be smaller than for the original computation with the CTCs (which can only be executed using post-selection), neither it is provably significantly larger.

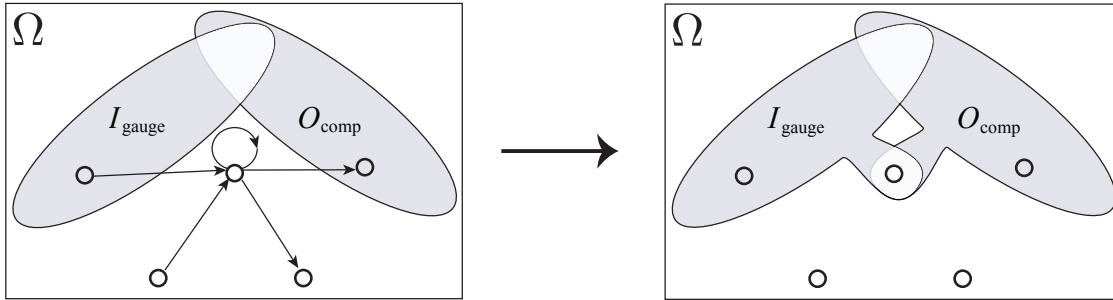


Figure 3: Breaking a closed time-like curve of length 1. The looped qubit becomes an element of $I_{\text{gauge}} \cap O_{\text{comp}}$ after flipping the measurement plane. As such both its forward cone ($O_{\text{comp}} \subseteq O$), and backward cone ($I_{\text{gauge}} \subseteq I$) must be empty.

Event horizons. Let us consider the flow of information between qubit i whose measurement plane has been flipped and the other qubits. Before the flip (MBQC \mathcal{M}_1 of Lemma 7), $i \in (O_{\text{comp}})^c \cap (I_{\text{gauge}})^c$. After the flip (MBQC \mathcal{M}_2 of Lemma 7), $i \in O_{\text{comp}} \cap I_{\text{gauge}}$. In \mathcal{M}_2 , since $i \in I_{\text{gauge}}$, no information for the adaption of measurement basis is flowing into site i from the other sites. Likewise, since $i \in O_{\text{comp}}$, no information for the adaption of measurement bases is flowing out of site i . Finally, because of the normal form Eq. (27), the measurement outcome s_i appears in only one readout bit. This readout bit is o_i as given in l.h.s. of Eq. (62), which is precisely the bit of classical output that can be discarded if the problem solved by the quantum computation is in NP. If o_i is discarded, then no information is flowing out of the site i at all. Thus, in summary, from the viewpoint of classical processing, qubit i in \mathcal{M}_2 becomes entirely disconnected from the computation. It vanishes behind the MBQC counterpart of an event horizon.

7 Conclusions and outlook

In this paper we have studied the constraints on temporal order in measurement-based quantum computation which arise from the principle that the randomness inherent in quantum measurement should not affect the logical processing. We have established a classification of temporal relations consistent with a given resource stabilizer state and set of measurement planes. Conversely, we have shown that the linear processing relations in measurement based quantum computation, subject to the above principle, specify the resource state and set of measurement planes up to equivalence. We identified gauge degrees of freedom which need to be included in order to establish the above results. Furthermore, we found a transformation that leaves the temporal order in every MBQC invariant and is related to local complementation. Finally, we pointed out formal MBQC analogues of a result and a piece of phenomenology in the theory of General Relativity, namely of Malament’s theorem and event horizons.

At this point, we are led to ask the following questions:

1. We introduced a group of gauge transformations Eq. (18), and a group symmetry transformations Eq. (55), generated by flipping measurement planes. Both transformations preserve MBQC temporal orders. Can the two groups be unified?
2. Some of the temporal relations admitted by the matroid $\mathcal{G}(|\Psi\rangle)$ contain closed time-like curves. Given a stabilizer state $|\Psi\rangle$ and set of measurement planes Σ , can we find an algebraic (or other)

structure which comprises only the partial orders? Can the partial order of measurements with the smallest set O_{comp} be efficiently computed?

3. The generators of the resource state stabilizer commute by definition. As pointed out in the second remark below Theorem 4, this commutativity condition constrains the MBQC linear processing relations, including temporal order. What is the physical meaning of this constraint to the processing relations?
4. In MBQC, the link between the randomness in quantum mechanical measurement and temporal order is the principle that the randomness of measurement outcomes should not affect the logical processing. In a context more general than quantum computation, what could this principle be replaced by?

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A Invariance under the gauge transformations

A.1 Gauge transformations and temporal order

In this section we provide a different angle at Theorem 1, namely we show that the gauge transformations Eq. (18) impose severe constraints on the possible temporal orders for a given stabilizer state and set of measurement planes.

The gauge transformations Eq. (18) act on \mathbf{q} and \mathbf{s} , and can therefore have a non-trivial effect on the classical processing relation Eq. (15a), $\mathbf{q} = T\mathbf{s} + H\mathbf{g}$. We now study this effect. Since the transformations Eq. (18) are caused by the insertion of stabilizer operators into the state overlap in Eq. (5), they do not change the physical situation. Therefore, the temporal relations before and after any such transformation must be equally valid, although not necessarily identical. By insertion of the stabilizer operator into the overlap $\langle \Phi_{\text{loc}} | \Psi \rangle$, the stabilizer of $|\Psi\rangle$ and the sets I_{gauge} , O_{comp} do not change. Therefore, by Theorem 1, the matrices T and H do not change. Thus, besides \mathbf{q} and \mathbf{s} , all that can change in the relation Eq. (15a) under a transformation Eq. (18) is \mathbf{g} . The following two viewpoints are always equivalent: (A) The relation $\mathbf{q} = f_{\mathbf{g}}(\mathbf{s})$, under the action Eq. (18) of a G_K on (\mathbf{s}, \mathbf{q}) is changed into an equivalent such relation $\mathbf{q} = f_{\mathbf{g}'}(\mathbf{s})$, with $G_K : \mathbf{g} \rightarrow \mathbf{g}'$. (B) The relation $\mathbf{q} = \tilde{f}(\mathbf{s}, \mathbf{g})$ remains invariant under all transformations G_K , acting on the triple $(\mathbf{s}, \mathbf{q}, \mathbf{g})$. We choose the latter viewpoint.

We now infer the action of the transformations G_K on \mathbf{g} . Without loss of generality we assume that the relations Eq. (15a) are given with H in its normal form Eq. (27),

$$\mathbf{q} = T\mathbf{s} + \left(\frac{I}{H} \right) \mathbf{g} \pmod{2}.$$

Furthermore, we assume that the pair $I_{\text{gauge}}, O_{\text{comp}}$ is extremal. Since $I_{\text{gauge}} \subseteq I$ by definition, for each $i \in I_{\text{gauge}}$, q_i only depends on \mathbf{g} but not on the measurement outcomes \mathbf{s} , $q_i = g_i$. Now, the correction operators $K(a)$, $a \in (O_{\text{comp}})^c$ derived from the normal form Eq. (26) of $\mathcal{G}(|\Psi\rangle)$, $K(a)|_{I_{\text{gauge}}}$ has no σ_ϕ -part. Therefore, the corresponding transformations G_K do not flip q_i , for all $i \in I_{\text{gauge}}$. In order to preserve the relation $q_i = g_i$, they thus leave \mathbf{g} unchanged. Now consider the other stabilizer generators, $\bar{K}(i)$, $i \in I_{\text{gauge}}$, obeying the conditions Eq. (22). By construction, $G_{\bar{K}(i)}$ flips q_i but no other q_j , for $i \neq j \in I_{\text{gauge}}$. Hence, to preserve the relations $q_i = g_i$, it must also flip g_i , but no other g_j , $i \neq j \in I_{\text{gauge}}$. Thus, for a stabilizer element $K = \bigotimes_{a \in \Omega} (\sigma_s^{(a)})^{v_a} (\sigma_\phi^{(a)})^{w_a}$,

$$G_K : \mathbf{g} \rightarrow \mathbf{g} \oplus \mathbf{w}|_{I_{\text{gauge}}}. \quad (63)$$

Therein, we have assumed that the basis choice for \mathbf{g} is such that the matrix H appearing in Eq. (15a) is of normal form Eq. (27).

We have now fully specified the action of G_K on the triple $(\mathbf{q}, \mathbf{s}, \mathbf{g})$, c.f. Eq. (18), (63). MBQCs satisfy the invariance condition

$$\mathbf{q} = T\mathbf{s} + H\mathbf{g} \pmod{2} \iff G_K(\mathbf{q}) = T G_K(\mathbf{s}) + H G_K(\mathbf{g}) \pmod{2}, \forall K \in \mathcal{S}(|\Psi\rangle). \quad (64)$$

It is evident that the requirement (64) of invariance of the processing relations (15a) under the gauge transformations poses constraints on the possible matrices T and H . In fact, as we show below, given O_{comp} the matrices T and H are uniquely specified uniquely by the above invariance condition.

To check the invariance condition Eq. (64) in a specific case, we return to our 3-qubit cluster state example of Section 2.2. We consider the effect of the transformations induced by generators $K_1 = \sigma_\phi^{(1)} \sigma_s^{(2)}$, $K_2 = \sigma_s^{(1)} \sigma_\phi^{(2)} \sigma_s^{(3)}$ and $K_3 = \sigma_s^{(2)} \sigma_\phi^{(3)}$ on the processing relations Eq. (16). As noted earlier, $I_{\text{gauge}} = \{1\}$. Then, with Eqs. (18) and (63),

$$\begin{aligned} G_{K_1} : \mathbf{q} &\rightarrow \mathbf{q} \oplus (1, 0, 0)^T, & \mathbf{s} &\rightarrow \mathbf{s} \oplus (0, 1, 0)^T, & g_1 &\rightarrow g_1 \oplus 1, \\ G_{K_2} : \mathbf{q} &\rightarrow \mathbf{q} \oplus (0, 1, 0)^T, & \mathbf{s} &\rightarrow \mathbf{s} \oplus (1, 0, 1)^T, & g_1 &\rightarrow g_1, \\ G_{K_3} : \mathbf{q} &\rightarrow \mathbf{q} \oplus (0, 0, 1)^T, & \mathbf{s} &\rightarrow \mathbf{s} \oplus (0, 1, 0)^T, & g_1 &\rightarrow g_1. \end{aligned} \quad (65)$$

It is easily checked that the relation Eq. (16) is invariant under the transformations G_{K_1} , G_{K_2} and G_{K_3} of Eq. (65). However, if the transformations are restricted to \mathbf{q} , \mathbf{s} , the relation Eq. (16) is no longer invariant under the transformation induced by K_1 .

We now return to the general case and show that, given the set O_{comp} and the action Eq. (18), (63) of the gauge transformations on the triple $(\mathbf{q}, \mathbf{s}, \mathbf{g})$, the invariance condition Eq. (64) uniquely specifies the classical processing relations Eq. (15a) for the adaption of measurement bases.

Recall that we write the stabilizer generator matrix for $|\Psi\rangle$ in the σ_ϕ/σ_s -basis as $\mathcal{G}(|\Psi\rangle) = (\Phi|S)$. Then, for the stabilizer generator $K_a \in \mathcal{S}(|\Psi\rangle)$ corresponding to the a -th row of $\mathcal{G}(|\Psi\rangle)$, with Eq. (18) the action of the gauge transformation G_{K_a} on \mathbf{s} , \mathbf{q} is

$$G_{K_a} : \mathbf{s} \longrightarrow \mathbf{s} \oplus \text{row}_a(S), \quad \mathbf{q} \longrightarrow \mathbf{q} \oplus \text{row}_a(\Phi). \quad (66)$$

With Eq. (63), the action of G_{K_a} on \mathbf{g} is

$$G_{K_a} : \longrightarrow \mathbf{g} \oplus \text{row}_a(\Phi)|_{I_{\text{gauge}}}. \quad (67)$$

Here, $\text{row}_a(\Phi)|_{I_{\text{gauge}}}$ denotes $\text{row}_a(\Phi)$ restricted to I_{gauge} . Then, the condition Eq. (64) for invariance of $\mathbf{q} = T\mathbf{s} + H\mathbf{g}$ under G_{K_a} becomes

$$\text{row}_a(\Phi) = T\text{row}_a(S) + H\text{row}_a(\Phi)|_{I_{\text{gauge}}} \pmod{2}.$$

This condition must hold for all stabilizer generators K_a simultaneously, hence

$$\Phi^T = TS^T + H\Phi^T|_{I_{\text{gauge}} \times \Omega} \pmod{2}. \quad (68)$$

By definition, the qubits in I_{gauge} have empty backward cones, and the qubits in O_{comp} have empty forward cones, hence T is of the form

$$T = \left(\begin{array}{c|c} 0 & 0 \\ \hline \mathbb{T} & 0 \end{array} \right),$$

where the column split is $(O_{\text{comp}})^c|O_{\text{comp}}$ and the row split is $I_{\text{gauge}}|(I_{\text{gauge}})^c$, c.f. Eq. (29). By right-multiplication of relation Eq. (68) with a suitable matrix, we transform $\Phi^T|_{I_{\text{gauge}} \times \Omega}$ into a matrix of form $(I|0)$ where the column split is between I_{gauge} and I_{gauge}^c . By definition of I_{gauge} , such a transformation is always possible. Under the same transformation,

$$\Phi^T \longrightarrow \left(\begin{array}{c|c} I & 0 \\ \hline \Phi_1 & \Phi_2 \end{array} \right), \quad S^T|_{(O_{\text{comp}})^c \times \Omega} \longrightarrow (S_1|S_2). \quad (69)$$

Inserting the above into Eq. (68), we find that H must be of normal form Eq. (27), $H = \left(\begin{array}{c} I \\ \hline H \end{array} \right)$, and

$$\begin{aligned} \Phi_1 &= \mathbb{T}S_1 + H \pmod{2}, \\ \Phi_2 &= \mathbb{T}S_2 \pmod{2}. \end{aligned} \quad (70)$$

Now, S_2 must be an invertible matrix. This is the condition that, by definition of O_{comp} , every measurement outcome in $(O_{\text{comp}})^c$ is correctable. Then,

$$\mathbb{T} = \Phi_2 S_2^{-1} \pmod{2}, \quad H = \Phi_1 + \Phi_2 S_2^{-1} S_1 \pmod{2}. \quad (71)$$

Hence the relation $\mathbf{q} = T\mathbf{s} + H\mathbf{g}$ is uniquely specified.

A.2 Gauge transformations and computational output

In addition to Eq. 64, we also require invariance of the classical output under the transformations Eq. (18), (63),

$$\mathbf{o} = \mathbf{Zs} + \mathbf{Rg} \pmod{2} = \mathbf{Z}G_K(\mathbf{s}) + \mathbf{R}G_K(\mathbf{g}) \pmod{2}, \forall K \in \mathcal{S}(|\Psi\rangle). \quad (72)$$

Like Eq. (64), Eq. (72) is a determinism constraint. If for a single output bit o the relation $o = \mathbf{z}^T \mathbf{s} + \mathbf{r}^T \mathbf{g}$ is not invariant under all gauge transformations Eq. (18), (63), then the value of o is guaranteed to be random, and thus useless as readout bit of a computation. Specifically,

Lemma 8. *Assume an MBQC where the relation $\mathbf{q} = \mathbf{T}\mathbf{s} + \mathbf{H}\mathbf{g}$ is invariant under the gauge transformations Eq. (18), (63), but an output bit o exists whose defining relation $o = \mathbf{z}^T \mathbf{s} + \mathbf{r}^T \mathbf{g}$ is not invariant under the action of G_K for some $K \in \mathcal{S}(|\Psi\rangle)$. Then, the value of o is completely random, independent of the choice of measurement angles.*

Proof of Lemma 8. For simplicity, consider first the special case where $K \in \mathcal{S}(|\Psi\rangle)$ acts trivially on \mathbf{g} , $G_K(\mathbf{g}) = \mathbf{g}$, $\forall \mathbf{g}$. We may then write $o = \sum_{i \in J} s_i + c$ for an offset $c = \mathbf{r}^T \mathbf{g}$. We call the string $\mathbf{s}|_J$ of measurement outcomes on J even (odd) if it has even (odd) weight. We denote the local post-measurement state on qubit a by $|\varphi_a, s_a, q_a(\mathbf{s}, \mathbf{g})\rangle$, where φ_a is the measurement angle, s_a the measurement outcome and q_a specifies the chosen measurement basis.

Under the transformation G_K , $\mathbf{s} \rightarrow \mathbf{s} \oplus \Delta \mathbf{s}_K$, where, by assumption, $\Delta \mathbf{s}_K|_J$ is odd. Now, the probability of outputting $o = c$ is

$$\begin{aligned} p(o = c) &= \sum_{\mathbf{s}|_J = \text{even}} \left| \left(\bigotimes_{a \in \Omega} \langle \varphi_a, s_a, q_a(\mathbf{s}, \mathbf{g}) | \right) |\Psi\rangle \right|^2 \\ &= \sum_{\mathbf{s}|_J = \text{even}} \left| \left(\bigotimes_{a \in \Omega} \langle \varphi_a, s_a, q_a(\mathbf{s}, \mathbf{g}) | \right) K |\Psi\rangle \right|^2 \\ &= \sum_{\mathbf{s}|_J = \text{even}} \left| \left(\bigotimes_{a \in \Omega} \langle \varphi_a, s_a \oplus \Delta s_{K,a}, q_a \oplus \Delta q_{K,a} | \right) |\Psi\rangle \right|^2 \\ &= \sum_{\mathbf{s}|_J = \text{even}} \left| \left(\bigotimes_{a \in \Omega} \langle \varphi_a, s_a \oplus \Delta s_{K,a}, q_a(\mathbf{s} \oplus \Delta \mathbf{s}_K, \mathbf{g}) | \right) |\Psi\rangle \right|^2 \\ &= \sum_{\mathbf{s}|_J = \text{odd}} \left| \left(\bigotimes_{a \in \Omega} \langle \varphi_a, s_a, q_a(\mathbf{s}, \mathbf{g}) | \right) |\Psi\rangle \right|^2 \\ &= p(o = \bar{c}). \end{aligned} \quad (73)$$

Thus, $p(o = c) = p(o = \bar{c}) = 1/2$. Note that in transitioning from the third to the fourth line of Eq. (73) we have used the invariance property Eq. (64), i.e., the assumption that the adaption of measurement bases is deterministic.

In the general case, $G_K : \mathbf{s} \rightarrow \mathbf{s} \oplus \Delta \mathbf{s}_K$, $\mathbf{g} \rightarrow \mathbf{g} \oplus \Delta \mathbf{g}_K$. We note that we can choose any gauge fixing \mathbf{g} , and thus $p(o = c) = \frac{1}{2^{|\text{gauge}|}} \sum_{\mathbf{g}} \sum_{\mathbf{s}|_J = \text{even}} \left| \left(\bigotimes_{a \in \Omega} \langle \varphi_a, s_a, q_a(\mathbf{s}, \mathbf{g}) | \right) |\Psi\rangle \right|^2$. By an argument analogous to the above we then find $p(o = 0) = p(o = 1) = 1/2$. \square