

# On the Generative Capacity of Contextual Grammars with Strictly Locally Testable Selection Languages

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We continue the research on the generative capacity of contextual grammars where contexts are adjoined around whole words (externally) or around subwords (internally) which belong to special regular selection languages. All languages generated by contextual grammars where all selection languages are elements of a certain subregular language family form again a language family. We investigate contextual grammars with strictly locally testable selection languages and compare those families to families which are based on finite, monoidal, nilpotent, combinational, definite, suffix-closed, ordered, commutative, circular, non-counting, power-separating, or union-free languages.

## 1 Introduction

Contextual grammars were introduced by Solomon Marcus in [17] as a formal model that might be used for the generation of natural languages. The derivation steps consist of adjoining contexts to given sentences starting from a finite set. A context is given by a pair  $(u, v)$  of words. The external adjoining to a word  $x$  gives the word  $uxv$  and the internal adjoining gives all words  $x_1ux_2vx_3$  with  $x_1x_2x_3 = x$ . Following the linguistic motivation, conditions are given for each context which have to be met by the word in order to allow the context to be adjoined. Contextual grammars where the contexts are adjoined ex- or internally are called external or internal contextual grammars, respectively. If conditions are given to the subword where a context is to be adjoined, we speak about external or internal contextual grammars with selection. Contextual grammars with ex- or internal derivation and selection in a certain family  $F$  of languages were defined where it is required that the word where a context is wrapped around belongs to a language of the family  $F$ . Contextual grammars have been studied where the family  $F$  is taken from the Chomsky hierarchy (see [15, 20, 22] and references therein).

The study of external contextual grammars with selection in special regular sets was started by Jürgen Dassow in [4]. The research was continued by Jürgen Dassow, Florin Manea, and Bianca Truthe (see [8, 9, 10, 16]) where further subregular families of selection languages were considered and the effect of subregular selection languages on the generative power of external and internal contextual grammars was investigated. A recent survey can be found in [28]. The internal case is different from the case of external contextual grammars, as there are two main differences between the ways in which words are derived. In the case of internal contextual grammars, it is possible that the insertion of a context into a sentential form can be done at more than one place, such that the derivation becomes in some sense non-deterministic; in the case of external grammars, once a context was selected, there is at most one way to insert it: wrapped around the sentential form, when this word is in the selection language of the context.

If a context can be added internally, then it can be added arbitrarily often (because the subword where the context is wrapped around does not change) which does not necessarily hold for external grammars.

In the present paper, we investigate the impact of strictly locally testable selection languages in contextual grammars on the generative capacity and compare it to those of the families which are based on finite, monoidal, nilpotent, combinational, definite, suffix-closed, ordered, commutative, circular, non-counting, power-separating, or union-free languages. External contextual grammars with such selection languages have been investigated in [5]. We give here some further results. Internal contextual grammars with strictly locally testable selection languages have not been investigated so far. So, this paper gives first results in this area. In the end, we mention some open problems.

## 2 Preliminaries

After giving some notations used in this paper, we first recall the subregular families of languages under investigation and then recall the contextual grammars with external or internal language generating modes.

We assume that the reader is familiar with the basic concepts of the theory of automata and formal languages. For details, we refer to [22].

Given an alphabet  $V$ , we denote by  $V^*$  and  $V^+$  the set of all words and the set of all non-empty words over  $V$ , respectively. The empty word is denoted by  $\lambda$ . By  $V^k$  and  $V^{\leq k}$  for some natural number  $k$ , we denote the set of all words of the alphabet  $V$  with exactly  $k$  letters and the set of all words over  $V$  with at most  $k$  letters, respectively. For a word  $w$ , we denote the length of  $w$  by  $|w|$ .

### 2.1 Subregular Language Families

#### 2.1.1 Definitions of Subregular Language Families

We consider the following restrictions for regular languages. Let  $L$  be a language over an alphabet  $V$ . We say that the language  $L$  – with respect to the alphabet  $V$  – is

- *monoidal* if and only if  $L = V^*$ ,
- *nilpotent* if and only if it is finite or its complement  $V^* \setminus L$  is finite,
- *combinational* if and only if it has the form  $L = V^*X$  for some subset  $X \subseteq V$ ,
- *definite* if and only if it can be represented in the form  $L = A \cup V^*B$  where  $A$  and  $B$  are finite subsets of  $V^*$ ,
- *suffix-closed* (or *fully initial* or *multiple-entry* language) if and only if, for any two words  $x \in V^*$  and  $y \in V^*$ , the relation  $xy \in L$  implies the relation  $y \in L$ ,
- *ordered* if and only if the language is accepted by some deterministic finite automaton

$$\mathcal{A} = (V, Z, z_0, F, \delta)$$

with an input alphabet  $V$ , a finite set  $Z$  of states, a start state  $z_0 \in Z$ , a set  $F \subseteq Z$  of accepting states and a transition mapping  $\delta$  where  $(Z, \preceq)$  is a totally ordered set and, for any input symbol  $a \in V$ , the relation  $z \preceq z'$  implies  $\delta(z, a) \preceq \delta(z', a)$ ,

- *commutative* if and only if it contains with each word also all permutations of this word,
- *circular* if and only if it contains with each word also all circular shifts of this word,

- *non-counting* (or *star-free*) if and only if there is a natural number  $k \geq 1$  such that, for any three words  $x \in V^*$ ,  $y \in V^*$ , and  $z \in V^*$ , it holds  $xy^kz \in L$  if and only if  $xy^{k+1}z \in L$ ,
- *power-separating* if and only if, there is a natural number  $m \geq 1$  such that for any word  $x \in V^*$ , either  $J_x^m \cap L = \emptyset$  or  $J_x^m \subseteq L$  where  $J_x^m = \{x^n \mid n \geq m\}$ ,
- *union-free* if and only if  $L$  can be described by a regular expression which is only built by product and star,
- *strictly locally  $k$ -testable* if and only if there are three subsets  $B, I$ , and  $E$  of  $V^k$  such that any word  $a_1a_2 \dots a_n$  with  $n \geq k$  and  $a_i \in V$  for  $1 \leq i \leq n$  belongs to the language  $L$  if and only if  $a_1a_2 \dots a_k \in B$ ,  $a_{j+1}a_{j+2} \dots a_{j+k} \in I$  for  $1 \leq j \leq n - k - 1$ , and  $a_{n-k+1}a_{n-k+2} \dots a_n \in E$ ,
- *strictly locally testable* if and only if it is strictly locally  $k$ -testable for some natural number  $k$ .

We remark that monoidal, nilpotent, combinational, definite, ordered, union-free, and strictly locally ( $k$ -)testable languages are regular, whereas non-regular languages of the other types mentioned above exist. Here, we consider among the commutative, circular, suffix-closed, non-counting, and power-separating languages only those which are also regular.

Some properties of the languages of the classes mentioned above can be found in [23] (monoids), [11] (nilpotent languages), [13] (combinational and commutative languages), [19] (definite languages), [12] and [2] (suffix-closed languages), [24] (ordered languages), [3] (circular languages), [18] (non-counting and strictly locally testable languages), [25] (power-separating languages), [1] (union-free languages).

By *FIN*, *MON*, *NIL*, *COMB*, *DEF*, *SUF*, *ORD*, *COMM*, *CIRC*, *NC*, *PS*, *UF*, *SLT<sub>k</sub>* (for any natural number  $k \geq 1$ ), *SLT*, and *REG*, we denote the families of all finite, monoidal, nilpotent, combinational, definite, regular suffix-closed, ordered, regular commutative, regular circular, regular non-counting, regular power-separating, union-free, strictly locally  $k$ -testable, strictly locally testable, and regular languages, respectively.

A strictly locally testable language characterized by three finite sets  $B, I$ , and  $E$  as above which includes additionally a finite set  $F$  of words which are shorter than those of the sets  $B, I$ , and  $E$  is denoted by  $[B, I, E, F]$ .

As the set of all families under consideration, we set

$$\mathcal{F} = \{FIN, MON, NIL, COMB, DEF, SUF, ORD, COMM, CIRC, NC, PS, UF, SLT\} \cup \{SLT_k \mid k \geq 1\}.$$

### 2.1.2 Hierarchy of Subregular Language Families

Many inclusion relations and incomparabilities between these families have been proved in the past, see [28] for a survey. We now insert the families of the strictly locally ( $k$ -)testable languages into the existing hierarchy.

The families of strictly locally  $k$ -testable languages form an infinite hierarchy of proper inclusions. This is shown in [21] with the witness languages

$$L_h = \{ab^h\}^+ \in SLT_{h+1} \setminus SLT_h \text{ for } h \geq 1.$$

From [18], we know the proper inclusion  $SLT \subset NC$ . In [5], the proper inclusions  $COMB \subset SLT_1$  and  $DEF \subset SLT$  as well as the incomparability of each family  $SLT_k$  for  $k \geq 1$  with the families *FIN*, *NIL*, and *DEF* were mentioned but not proved. This will be done in the sequel. We first give a witness language which will be useful in all these proofs.

**Lemma 2.1** Let  $L_{SLT_1, \neg DEF} = \{a\} \cup \{ab^n a \mid n \geq 0\}$ . Then  $L_{SLT_1, \neg DEF} \in SLT_1 \setminus DEF$ .

*Proof.* The language  $L_{SLT_1, \neg DEF}$  can be represented as  $[\{a\}, \{b\}, \{a\}, \emptyset]$ , hence  $L_{SLT_1, \neg DEF} \in SLT_1$ .

Suppose, this language is definite. Then there are two finite subsets  $D_s \subset \{a, b\}^*$  and  $D_e \subset \{a, b\}^*$  such that

$$L_{SLT_1, \neg DEF} = D_s \cup \{a, b\}^* D_e.$$

Let  $k = \max\{|w| \mid w \in D_s \cup D_e\} + 1$ . The word  $ab^k a$  belongs to the language  $L_{SLT_1, \neg DEF}$  but not to the subset  $D_s$  due to its length. Hence,  $ab^k a \in \{a, b\}^* D_e$  and also

$$ab^k a \in \{a, b\}^+ D_e$$

due to the length of the word. Then we have also  $b^{k+1} a \in \{a, b\}^+ D_e$  and, therefore,

$$b^{k+1} a \in L_{SLT_1, \neg DEF}$$

which is a contradiction. Thus,  $L_{SLT_1, \neg DEF} \notin DEF$ .  $\square$

The language  $L_{SLT_1, \neg DEF}$  is a witness language for the properness of the three inclusions stated in the following lemmas.

**Lemma 2.2** The proper inclusion  $MON \subset SLT_1$  holds.

*Proof.* We first prove that  $MON$  is included in  $SLT_1$ . Let  $L$  be a monoidal language over an alphabet  $V$ . Then  $L = V^*$ . With  $[V, V, V, \lambda]$ , we have a representation of the language  $L$  as a strictly locally 1-testable language. Hence,  $MON \subseteq SLT_1$ .

A witness language for the properness is the language  $L_{SLT_1, \neg DEF}$  which, according to Lemma 2.1, belongs to the class  $SLT_1$  but not to  $DEF$  and not to  $MON$  because  $MON \subseteq DEF$ .  $\square$

**Lemma 2.3** The proper inclusion  $COMB \subset SLT_1$  holds.

*Proof.* We first prove that  $COMB$  is included in  $SLT_1$ . Let  $L$  be a combinational language over an alphabet  $V$ . Then  $L = V^* X$  for some subset  $X \subseteq V$ . With  $[V, V, X, \emptyset]$ , we have a representation of the language  $L$  as a strictly locally 1-testable language. Hence,  $COMB \subseteq SLT_1$ .

A witness language for the properness is the language  $L_{SLT_1, \neg DEF}$  which, according to Lemma 2.1, belongs to the class  $SLT_1$  but not to  $DEF$  and not to  $COMB$  because  $COMB \subseteq DEF$ .  $\square$

**Lemma 2.4** The proper inclusion  $DEF \subset SLT$  holds.

*Proof.* We first prove  $DEF \subseteq SLT$ . Let  $L$  be a definite language over an alphabet  $V$ . Then  $L = D_s \cup V^* D_e$  for some finite subsets  $D_s \subset V^*$  and  $D_e \subset V^*$ . Let  $k = \max\{|w| \mid w \in D_s \cup D_e\} + 1$ . Further, let

- $F = \{w \mid w \in L \cap V^{\leq k-1}\}$  be the set of all words of  $L$  with a length smaller than  $k$ ,
- $B = V^k$  and  $I = V^k$  the set of all words over the alphabet  $V$  with a length of  $k$ ,
- $E = V^* D_e \cap V^k$  the set of all words of the set  $V^* D_e$  with length  $k$ ,

and  $L'$  be the strictly locally  $k$ -testable language represented by  $[B, I, E, F]$ . We now prove that  $L = L'$  holds.

We first show  $L \subseteq L'$ . Let  $w \in L$ . If  $|w| < k$ , then  $w \in F$  and, hence,  $w \in L'$ . Otherwise,  $w \in V^* D_e$  and there are words  $w_0$  and  $w_1$  such that  $w = w_1 w_0$  and  $w_0 \in V^* D_e \cap V^k$  (the word  $w_0$  is the suffix of  $w$  of length  $k$ ). Every subword of  $w$  of length  $k$  belongs to the set  $V^k$ . Hence, the prefix of  $w$  of length  $k$

belongs to the set  $B$ , every proper infix of  $w$  of length  $k$  belongs to the set  $I$ , and the suffix  $w_0$  belongs to the set  $E$ . Therefore, we have  $w \in L'$  also in this case.

We now show  $L' \subseteq L$ . Let  $w \in L'$ . If  $w \in F$ , then  $w \in L \cap V^{\leq k-1}$  and, hence,  $w \in L$ . Otherwise, the length  $m = |w|$  of  $w$  is at least  $k$  and the word  $w$  is composed of  $m$  letters  $x_i \in V$  for  $1 \leq i \leq m$  such that  $w = x_1x_2 \dots x_m$ . Then we have for the prefix  $x_1x_2 \dots x_k \in B$ , for each infix  $x_{j+1}x_{j+2} \dots x_{j+k} \in I$  for  $1 \leq j \leq m-1-k$ , and for the suffix  $x_{m-k+1}x_{m-k+2} \dots x_m \in E$ . Therefore,  $x_{m-k+1}x_{m-k+2} \dots x_m \in V^*D_e$  and  $x_1x_2 \dots x_{m-k} \in V^*$ . Hence,  $w \in V^*D_e$  and  $w \in L$ .

Since  $L = L'$  and  $L' \in SLT_k$  by construction, we also have that  $L \in SLT_k$  and, thus, also  $L \in SLT$ .

A witness language for the properness of the inclusion  $DEF \subseteq SLT$  is the language  $L_{SLT_1, \neg DEF}$  which, according to Lemma 2.1, belongs to the class  $SLT_1$  and therefore also to  $SLT$  but not to  $DEF$ .  $\square$

The language  $L_{SLT_1, \neg DEF}$  from Lemma 2.1 serves also partially for proving the incomparability of the families of the strictly locally  $k$ -testable languages with the families of the finite languages, of the nilpotent languages, and of the definite languages.

**Lemma 2.5** *The classes  $SLT_k$  for  $k \geq 1$  are incomparable to the classes  $FIN$ ,  $NIL$ , and  $DEF$ .*

*Proof.* Due to the inclusion relations, it suffices to show that there is a language in the class  $SLT_1$  (which belongs also to each other family  $SLT_k$  for  $k > 1$ ) but which is not definite (and hence neither nilpotent nor finite) and that there are languages  $L_k$  (for  $k \geq 1$ ) which are finite (and, hence, nilpotent and definite) but not strictly locally  $k$ -testable.

A language for the first case is  $L_{SLT_1, \neg DEF}$  from Lemma 2.1 since  $L_{SLT_1, \neg DEF} \in SLT_1 \setminus DEF$ .

Languages for the other incomparabilities are  $L_k = \{a\}^{k+1}$  for  $k \geq 1$ . Every such language  $L_k$  is finite. Let  $k$  be a natural number. Suppose that the language  $L_k$  is also strictly locally  $k$ -testable. Then it is represented by  $[\{a\}^k, \emptyset, \{a\}^k, \emptyset]$ . But then, we also have  $a^k \in L_k$  which is a contradiction. Hence,  $L_k \in FIN \setminus SLT_k$  for  $k \geq 1$ .  $\square$

The incomparabilities of the families of the strictly locally ( $k$ -)testable languages to the families  $UF$  of the union-free languages,  $SUF$  of the suffix-closed languages,  $COMM$  of the commutative languages, and  $CIRC$  of the circular languages follow, due to the inclusion relations, from the incomparabilities of the classes  $UF$ ,  $SUF$ ,  $COMM$ , and  $CIRC$  to the classes  $COMB$  of the combinational languages and  $NC$  of the non-counting languages which were proved in [14].

Regarding the class  $ORD$  of the ordered languages, we give the following relations without proofs<sup>1</sup>.

**Lemma 2.6** *The proper inclusion  $SLT_1 \subset ORD$  holds. The classes  $SLT_k$  for  $k \geq 2$  are incomparable to the class  $ORD$ .*

If we combine these results with those mentioned in [28], we obtain the following statement.

**Theorem 2.7** *The inclusion relations presented in Figure 1 hold. An arrow from an entry  $X$  to an entry  $Y$  depicts the proper inclusion  $X \subset Y$ ; if two families are not connected by a directed path, then they are incomparable.*

An edge label in Figure 1 refers to a paper or a lemma in the present paper where the respective inclusion is proved (it is not necessarily the first paper where the inclusion is already mentioned). The incomparabilities which are not related to strictly locally testable languages are proved in [27].

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<sup>1</sup>Proofs have been found by be the authors after the acceptance of this paper.



for  $\alpha \in \{\text{ex}, \text{in}\}$ . If the derivation mode is known from the context, we omit the index  $\alpha$ . For a family  $\mathfrak{L}$  of languages, we denote by  $\mathcal{EC}(\mathfrak{L})$  and  $\mathcal{IC}(\mathfrak{L})$  the family of all languages generated externally and internally, respectively, by contextual grammars with selection in  $\mathfrak{L}$  (where all selection languages belong to the family  $\mathfrak{L}$ ).

From the definition follows that the subset relation is preserved under the use of contextual grammars: if we allow more, we do not obtain less.

**Lemma 2.8** *For any two language classes  $X$  and  $Y$  with  $X \subseteq Y$ , we have the inclusions*

$$\mathcal{EC}(X) \subseteq \mathcal{EC}(Y) \quad \text{and} \quad \mathcal{IC}(X) \subseteq \mathcal{IC}(Y).$$

### 3 Results

#### 3.1 External Contextual Grammars

When we speak about contextual grammars in this subsection, we mean contextual grammars with external derivation (also called external contextual grammars). A language of an external contextual grammar is a language which is externally generated.

In [4], contextual grammars were investigated where the selection languages are finite, monoidal, combinational, definite, nilpotent, commutative, or suffix-closed and a hierarchy of the language families generated was presented. In the papers [8, 9, 26], results on the power of external contextual grammars with circular, ordered, union-free, or definite selection languages are given. The language families generated by such systems were inserted into the hierarchy from [4]. Furthermore, subregular language families  $\mathcal{F}_n$  were considered and integrated which are obtained by restricting to  $n$  states, non-terminal symbols, or production rules to accept or to generate regular languages ([28]). We consider here only subregular families defined by structural properties (not resources).

We now present a witness language to prove a proper inclusion and incomparabilities regarding ordered languages as selection languages.

**Lemma 3.1** *Let  $L_{ORD, \neg SLT} = \{a\}^* \cup \{a\}^* \{b\} \{a\}^* \cup \{c\} \{a\}^* \{b\} \{a\}^* \{c\}$ . Then*

$$L_{ORD, \neg SLT} \in \mathcal{EC}(ORD) \setminus \mathcal{EC}(SLT).$$

*Proof.* The language  $L_{ORD, \neg SLT}$  can be generated by the contextual grammar

$$(\{a, b, c\}, \{(\{a, b\}^*, \{(\lambda, a), (a, \lambda)\}), (\{a\}^* \{b\} \{a, b\}^*, \{(c, c)\})\}, \{\lambda, b\})$$

where the selection languages are ordered: For  $\{a, b\}^*$ , only one state is needed; the other selection language is accepted by a deterministic finite automaton where the transition function is given by  $\delta(z_0, a) = z_0$  and  $\delta(z_0, b) = z_1 = \delta(z_1, a) = \delta(z_1, b)$ .

Assume that the language  $L_{ORD, \neg SLT}$  can be generated by a contextual grammar with strictly locally testable selection languages. The subset  $\{c\} \{a\}^* \{b\} \{a\}^* \{c\}$  of  $L_{ORD, \neg SLT}$  is infinite. Therefore, there is an infinite selection language  $S \subseteq \{a\}^* \{b\} \{a\}^*$  which is used to obtain words of the set  $\{c\} \{a\}^* \{b\} \{a\}^* \{c\}$ . Hence, to  $S$  belongs some context  $(u, v)$  with  $u \in \{c\} \{a\}^*$  and  $v \in \{a\}^* \{c\}$ . If  $S$  is a strictly locally  $k$ -testable language, then  $S = [B, I, E, F]$  and  $a^k \in B \cap I \cap E$ . Then, we have also  $a^k \in S$ . Therefore, a word from the set  $\{c\} \{a\}^* \{c\}$  is generated which does not belong to the language  $L_{ORD, \neg SLT}$ . This contradiction implies that  $L_{ORD, \neg SLT} \notin \mathcal{EC}(SLT)$ .  $\square$

We now prove the mentioned proper inclusion.

**Lemma 3.2** *The proper inclusion  $\mathcal{E}\mathcal{C}(SLT_1) \subset \mathcal{E}\mathcal{C}(ORD)$  holds.*

*Proof.* The inclusion  $\mathcal{E}\mathcal{C}(SLT_1) \subseteq \mathcal{E}\mathcal{C}(ORD)$  follows from Lemma 2.6 and Lemma 2.8. A witness language for the properness is  $L_{ORD, \neg SLT} \in \mathcal{E}\mathcal{C}(ORD) \setminus \mathcal{E}\mathcal{C}(SLT)$  from Lemma 3.1.  $\square$

Many incomparability results have been published in [4, 5, 28]. The only open questions are whether the class  $\mathcal{E}\mathcal{C}(ORD)$  is incomparable to the classes  $\mathcal{E}\mathcal{C}(SLT)$  and  $\mathcal{E}\mathcal{C}(SLT_k)$  for  $k \geq 2$ . We have the following conjecture. If this proves to be true, then we have the incomparabilities together with Lemma 3.1.

**Conjecture 3.3** *There is a language  $L_{SLT_2, \neg ORD} \in \mathcal{E}\mathcal{C}(SLT_2) \setminus \mathcal{E}\mathcal{C}(ORD)$ .*

Summarizing, we have the following result.

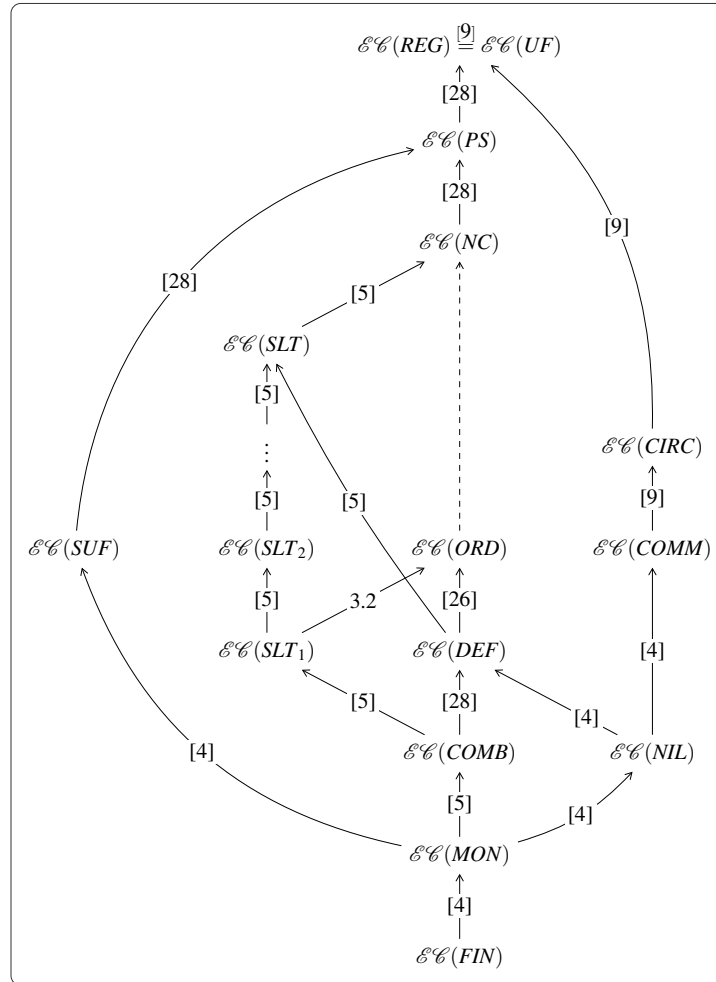


Figure 2: Hierarchy of language families by external contextual grammars with selection languages defined by structural properties. An edge label refers to the paper or lemma where the respective inclusion is proved.

**Theorem 3.4** *The inclusion relations presented in Figure 2 hold. An arrow from an entry  $X$  to an entry  $Y$  depicts the proper inclusion  $X \subset Y$ ; the dashed arrow from  $\mathcal{E}\mathcal{C}(ORD)$  to  $\mathcal{E}\mathcal{C}(NC)$  indicates*



that it is not known so far whether the inclusion is proper or whether equality holds. With exception of the pairs  $(\mathcal{E}\mathcal{C}(\text{ORD}), \mathcal{E}\mathcal{C}(\text{SLT}))$  and  $(\mathcal{E}\mathcal{C}(\text{ORD}), \mathcal{E}\mathcal{C}(\text{SLT}_k))$  for  $k \geq 2$ , if two families  $X$  and  $Y$  are not connected by a directed path, then  $X$  and  $Y$  are incomparable; in the exceptional cases,  $\mathcal{E}\mathcal{C}(\text{ORD}) \not\subseteq \mathcal{E}\mathcal{C}(\text{SLT})$  and  $\mathcal{E}\mathcal{C}(\text{ORD}) \not\subseteq \mathcal{E}\mathcal{C}(\text{SLT}_k)$  for  $k \geq 2$  hold.

### 3.2 Internal Contextual Grammars

When we speak about contextual grammars in this subsection, we mean contextual grammars with internal derivation (also called internal contextual grammars). A language of an internal contextual grammar is a language which is internally generated.

In [16], such contextual grammars were investigated where the selection languages belong to families  $\mathcal{F}_n$  which are obtained by restriction to  $n$  states or  $n$  non-terminal symbols, productions, or symbols to accept or to generate regular languages. In [10], the research was continued and extended to the investigation of the effect of finite, monoidal, nilpotent, combinational, definite, ordered, regular commutative, regular circular, regular suffix-closed, and union-free selection languages on the generative capacity of internal contextual grammars. We consider here only subregular families defined by structural properties (not resources).

In contrast to the external derivation mode, contextual grammars can internally apply a context infinitely often if it can be applied once. If a word contains a subword which belongs to a selection language, also the word after inserting the context contains a subword (namely the same as before) which belongs to this selection language. This difference has as a consequence that finite selection languages not only yield finitely many words as in the case of contextual grammars working in the external mode. Another consequence is that ‘outer’ parts of a word do not have to be added at the end of the derivation process but can be produced at some time whereas ‘inner’ parts can be ‘blown up’ later. For this reason, the results obtained for external contextual grammars are not of much help here.

According to Lemma 2.8, we have the inclusion  $\mathcal{I}\mathcal{C}(X) \subseteq \mathcal{I}\mathcal{C}(Y)$  whenever we have the proper inclusion  $X \subset Y$  for two families of languages  $X$  and  $Y$ .

We now present witness languages for proving the properness of the inclusions

$$\mathcal{I}\mathcal{C}(\text{COMB}) \subset \mathcal{I}\mathcal{C}(\text{SLT}_1) \subset \mathcal{I}\mathcal{C}(\text{SLT}_2) \subset \cdots \subset \mathcal{I}\mathcal{C}(\text{SLT}_k) \subset \cdots \subset \mathcal{I}\mathcal{C}(\text{SLT}) \subset \mathcal{I}\mathcal{C}(\text{NC})$$

and  $\mathcal{I}\mathcal{C}(\text{DEF}) \subset \mathcal{I}\mathcal{C}(\text{SLT})$ .

**Lemma 3.5** *Let  $L = \{ac^nbd^n \mid n \geq 0\}$ . Then  $L \in \mathcal{I}\mathcal{C}(\text{SLT}_1) \setminus \mathcal{I}\mathcal{C}(\text{COMB})$ .*

*Proof.* The internal contextual grammar  $(\{a, b, c, d\}, \{(\{b\}^+, \{(c, d)\})\}, \{ab\})$  with the strictly locally 1-testable selection language  $\{b\}^+$  (which has a representation as  $[\{b\}, \{b\}, \{b\}, \emptyset]$ ) generates the language  $L$ . Thus,  $L \in \mathcal{I}\mathcal{C}(\text{SLT}_1)$ .

Assume that  $L = L(G)$  for some internal contextual grammar  $G$  with combinational selection languages. Then, for sufficiently large  $n$  (which is larger than the sum of the longest length of axioms in  $G$  and the maximum of  $|uv|$  for contexts  $(u, v)$  of  $G$ ), we have a derivation  $x \Longrightarrow ac^nbd^n$ . Because  $x \in L$  holds, the used context  $(\alpha, \beta)$  contains no letter  $a$  and no letter  $b$  (otherwise, we can produce a word with more than two occurrences of  $a$  or  $b$ ), we have  $x = ac^mbd^m$ ,  $\alpha = c^{n-m}$ ,  $\beta = d^{n-m}$ , and the context is wrapped around a subword  $c^tbd^s$  for some numbers  $t$  and  $s$  with  $m \geq t \geq 0$ ,  $m \geq s \geq 0$ . Since the selection language  $C$  is combinational, we get  $ac^mbd^s \in C$  by  $c^tbd^s \in C$ . Therefore, we have the derivation  $x = ac^mbd^s d^{m-s} \Longrightarrow \alpha ac^mbd^s \beta d^{m-s} = c^{n-m} ac^mbd^n$ , i. e., we can derive a word not in  $L$ . Thus,  $L \notin \mathcal{I}\mathcal{C}(\text{COMB})$ .  $\square$

**Lemma 3.6** *Let  $n$  be a natural number with  $n \geq 2$  and*

$$L_n = \{a^m b^{2n} c^m \mid m \geq n\} \cup \{a^{n-1} b^n c^{n-1}\}.$$

*Then  $L_n \in (\mathcal{SC}(SLT_n) \cap \mathcal{SC}(FIN)) \setminus \mathcal{SC}(SLT_{n-1})$ .*

*Proof.* Let  $n$  be a natural number with  $n \geq 2$  and  $L_n$  the language mentioned in the claim. The language  $L_n$  is generated by the contextual grammar

$$(\{a, b, c\}, \{([\{a^n\}, \{a, b, c\}^n, \{c^n\}], \{(a, c)\})\}, \{a^n b^{2n} c^n, a^{n-1} b^n c^{n-1}\})$$

with a selection language from the family  $SLT_n$  and by

$$(\{a, b, c\}, \{(\{b^{2n}\}, \{(a, c)\})\}, \{a^n b^{2n} c^n, a^{n-1} b^n c^{n-1}\})$$

with a finite selection language.

The language  $L_n$  is not generated by a contextual grammar where all selection languages belong to the family  $SLT_{n-1}$ . Assume the contrary. Since the subset  $\{a^m b^{2n} c^m \mid m \geq n\}$  of  $L_n$  is infinite, there is a selection language  $S = [B, I, E, F]$  used with a word  $a^p b^{2n} c^q$  for two natural numbers  $p \geq 0$  and  $q \geq 0$ . As  $S \in SLT_{n-1}$ , we have  $b^{n-1} \in I$ . Then also the word  $a^p b^n c^q$  belongs to the selection language which is a subword of the word  $a^{n-1} b^n c^{n-1} \in L_n$ . Hence, another word with exactly  $n$  letters  $b$  would be generated which is a contradiction to the form of the words in the language  $L_n$ .  $\square$

**Lemma 3.7** *Let  $L = \{a^n b^m c^n d^m \mid m \geq 1, n \geq 1\}$ . Then  $L \in \mathcal{SC}(SLT_1) \setminus \mathcal{SC}(DEF)$ .*

*Proof.* The language  $L$  can be generated by the contextual grammar

$$G = (\{a, b, c, d\}, \{(S_{ac}, \{(a, c)\}), (S_{bd}, \{(b, d)\})\}, \{abcd\})$$

with the strictly locally 1-testable selection languages  $S_{ac} = \{a\}\{b\}^*\{c\}$  and  $S_{bd} = \{b\}\{c\}^*\{d\}$ :

$$S_{ac} = [\{a\}, \{b\}, \{c\}, \emptyset] \quad \text{and} \quad S_{bd} = [\{b\}, \{c\}, \{d\}, \emptyset].$$

Assume that the language  $L$  can be generated by a contextual grammar  $G'$  with definite selection languages. Let  $S_i = A_i \cup V^* B_i$  for  $1 \leq i \leq q$  be the selection languages of  $G'$ . Further, let

$$p = \max \left\{ |w| \mid w \in \bigcup_{i=1}^q (A_i \cup B_i) \right\}.$$

Since the language  $L$  is infinite and the number of the letters  $a$  and  $b$  are unbounded in its words, there is a word  $a^r b^s c^r d^s \in L$  with  $r \geq p$  and  $s \geq p$  such that from this word another one is generated. Hence, there is a selection language  $S_i$  with  $1 \leq i \leq q$  which contains a word which is a subword of  $a^r b^s c^r d^s$ . This word is  $a^{r'} b^{s'} c^{r''}$  with  $1 \leq r' \leq r$  and  $1 \leq r'' \leq r$  or  $b^{s'} c^{r'} d^{s''}$  with  $1 \leq s' \leq s$  and  $1 \leq s'' \leq s$  in order to maintain the form of the words of the language. Since  $S_i$  is definite and  $s - 1 + r'' \geq p$  or  $r - 1 + s'' \geq p$ , the word  $b^{s-1} c^{r''}$  or  $c^{r-1} d^{s''}$  also belongs to the selection language  $S_i$ . But then a letter  $a$  would be inserted inside the  $b$ -block or a letter  $b$  would be inserted inside the  $c$ -block. In both cases, a word would be generated which does not belong to the language  $L$ . Therefore, the language  $L$  cannot be generated by a contextual grammar with definite selection languages.  $\square$

**Lemma 3.8** *Let  $L = \{a^{p_1} b a^{p_2} b a^{p_3+p_1} b a^{p_2} b a^{p_3} \mid p_i \geq 1, 1 \leq i \leq 3\}$ . Then  $L \in \mathcal{SC}(ORD) \setminus \mathcal{SC}(SLT)$ .*

*Proof.* The language  $L$  can be generated by the contextual grammar

$$(\{a, b\}, \{(\{a\}^* \{b\} \{a\}^* \{b\} \{a\}^*, \{(a, a)\})\}, \{ababaababa\})$$

where the selection language is ordered since it is accepted by a deterministic finite automaton where the transition function is given by the following table (the order is  $z_0 \preceq z_1 \preceq z_2 \preceq z_3$ , start state is  $z_0$ , accepting state is  $z_2$ ):

$\delta$	$z_0$	$z_1$	$z_2$	$z_3$
$a$	$z_0$	$z_1$	$z_2$	$z_3$
$b$	$z_1$	$z_2$	$z_3$	$z_3$

Assume that the language  $L$  can be generated by a contextual grammar with strictly locally testable selection languages. The length of each  $a$ -block is unbounded. Therefore, there is an infinite selection language  $S \subseteq \{a\}^* \{b\} \{a\}^* \{b\} \{a\}^*$  used where the length of the  $a$ -block between the two letters  $b$  is unbounded and which has a context  $(a^l, a^l)$  associated to it (otherwise, a word would be generated which has not the required form of the words of the language  $L$ ). If  $S$  is a strictly locally  $k$ -testable language, then it contains with a word  $a^q b a^r b a^s$  with  $q \geq 0$ ,  $r \geq k$ , and  $s \geq 0$  also the word  $a^q b a^r b a^s$ . Adding the context  $(a^l, a^l)$  around such a subword of a word of  $L$  would yield a word which does not belong to the language  $L$  (a word with a wrong format). This contradiction implies that  $L \notin \mathcal{I}\mathcal{C}(SLT)$ .  $\square$

We now prove the proper inclusions mentioned above.

**Theorem 3.9** *The relations*

$$\mathcal{I}\mathcal{C}(COMB) \subset \mathcal{I}\mathcal{C}(SLT_1) \subset \mathcal{I}\mathcal{C}(SLT_2) \subset \dots \subset \mathcal{I}\mathcal{C}(SLT_k) \subset \dots \subset \mathcal{I}\mathcal{C}(SLT) \subset \mathcal{I}\mathcal{C}(NC)$$

and  $\mathcal{I}\mathcal{C}(DEF) \subset \mathcal{I}\mathcal{C}(SLT)$  hold.

*Proof.* The inclusions follow from the inclusions of the underlying language families (see [18, 21]) and Lemma 2.8. The properness is shown by the witness languages in the previous lemmas:

$$\begin{aligned} \mathcal{I}\mathcal{C}(COMB) &\subset \mathcal{I}\mathcal{C}(SLT_1) \text{ according to Lemma 3.5,} \\ \mathcal{I}\mathcal{C}(SLT_1) &\subset \mathcal{I}\mathcal{C}(SLT_2) \subset \dots \subset \mathcal{I}\mathcal{C}(SLT_k) \subset \dots \subset \mathcal{I}\mathcal{C}(SLT) \text{ due to Lemma 3.6,} \\ \mathcal{I}\mathcal{C}(SLT) &\subset \mathcal{I}\mathcal{C}(NC) \text{ according to Lemma 3.8, since } \mathcal{I}\mathcal{C}(ORD) \subseteq \mathcal{I}\mathcal{C}(NC) \text{ ([28]),} \\ \mathcal{I}\mathcal{C}(DEF) &\subset \mathcal{I}\mathcal{C}(SLT) \text{ due to Lemma 3.7, since } \mathcal{I}\mathcal{C}(SLT_1) \subset \mathcal{I}\mathcal{C}(SLT). \end{aligned}$$

$\square$

**Lemma 3.10** *The proper inclusion  $\mathcal{I}\mathcal{C}(SLT_1) \subset \mathcal{I}\mathcal{C}(ORD)$  holds.*

*Proof.* The inclusion  $\mathcal{I}\mathcal{C}(SLT_1) \subseteq \mathcal{I}\mathcal{C}(ORD)$  follows from Lemma 2.6 and Lemma 2.8. A witness language for the properness is the language  $L \in \mathcal{I}\mathcal{C}(ORD) \setminus \mathcal{I}\mathcal{C}(SLT)$  from Lemma 3.8.  $\square$

The incomparabilities of the families  $\mathcal{I}\mathcal{C}(COMM)$  and  $\mathcal{I}\mathcal{C}(CIRC)$  with the families  $\mathcal{I}\mathcal{C}(SLT_k)$  for  $k \geq 1$  and  $\mathcal{I}\mathcal{C}(SLT)$  follow from the incomparabilities of the sets  $\mathcal{I}\mathcal{C}(COMM)$  and  $\mathcal{I}\mathcal{C}(CIRC)$  with the sets  $\mathcal{I}\mathcal{C}(COMB)$  and  $\mathcal{I}\mathcal{C}(NC)$  shown in [28], since

$$\mathcal{I}\mathcal{C}(COMB) \subseteq \mathcal{I}\mathcal{C}(SLT_1) \subseteq \mathcal{I}\mathcal{C}(SLT_2) \subseteq \dots \subseteq \mathcal{I}\mathcal{C}(SLT) \subseteq \mathcal{I}\mathcal{C}(NC).$$

Regarding  $\mathcal{I}\mathcal{C}(SUF)$ , we know that there is a language in the set  $\mathcal{I}\mathcal{C}(COMB) \setminus \mathcal{I}\mathcal{C}(SUF)$  ([10]) which also belongs to each set  $\mathcal{I}\mathcal{C}(SLT_k)$  for  $k \geq 1$  and  $\mathcal{I}\mathcal{C}(SLT)$  due to the inclusion relations.

However, it is still open whether there is a language in the set  $\mathcal{S}\mathcal{C}(SUF) \setminus \mathcal{S}\mathcal{C}(NC)$  (which would not belong to subsets of  $\mathcal{S}\mathcal{C}(NC)$  either). So, we cannot use the method as for the classes  $\mathcal{S}\mathcal{C}(COMM)$  and  $\mathcal{S}\mathcal{C}(CIRC)$ .

In the sequel, we show that, for every number  $k \geq 1$ , there is a language which belongs to the set  $\mathcal{S}\mathcal{C}(SUF)$  but not to  $\mathcal{S}\mathcal{C}(SLT_k)$ . We first note that the internal contextual grammar

$$(\{c, d\}, \{(\{c, d\}^*, \{(c, d)\})\}, \{\lambda\})$$

generates the Dyck language  $D$  over  $\{c, d\}$ . For  $k \geq 1$ , we set

$$\begin{aligned} K'_k &= \{c^{m_0} a c^{m_1} a \dots c^{m_k} a c^{m_{k+1}} b d^{m_0+m_1+\dots+m_{k+1}} \mid m_i \geq 0, 0 \leq i \leq k+1\}, \\ K''_k &= K'_k \cup \{c^k a^{2k-1}\} K'_k \{d^k\}, \end{aligned}$$

and define  $K_k$  as the language obtained from  $K''_k$  by inserting a word of  $D$  at any position.

**Lemma 3.11** *For all  $k \geq 1$ , we have  $K_k \in \mathcal{S}\mathcal{C}(SUF) \setminus \mathcal{S}\mathcal{C}(SLT_k)$ .*

*Proof.* The internal contextual grammar

$$(\{a, b, c, d\}, \{(\{a^r b \mid 0 \leq r \leq k+1\} \cup \{\lambda\}, \{(c, d)\})\}, \{a^{k+1} b, c^k a^{3k} b d^k\})$$

with a suffix closed selection language generates the language  $K_k$ . Thus,  $K_k \in \mathcal{S}\mathcal{C}(SUF)$ .

Assume that  $K_k = L(G)$  for some internal contextual grammar  $G$  where all selection languages are strictly locally  $k$ -testable. Let  $n$  be sufficiently large. Then there is a derivation  $x \Rightarrow c^n a^{k+1} b d^n \in K_k$ . Since  $x \in K_k$  holds, the used context  $(\alpha, \beta)$  contains no letter  $a$  and no letter  $b$ . We have  $x = c^m a^{k+1} b d^m$ ,  $\alpha = c^{n-m}$ ,  $\beta = d^{n-m}$ , and the context is wrapped around a subword  $c^t a^{k+1} b d^s$  for some numbers  $t$  and  $s$  with  $m \geq t \geq 0$  and  $m \geq s \geq 0$ .

Let  $t \geq k$ . Since the strictly locally  $k$ -testable selection language  $C$  which is used contains the word  $c^t a^{k+1} b d^s$ , it also contains the word  $c^k a^{3k} b d^s$ . Analogously, if  $s \geq k$ , the selection language  $C$  also contains the word  $c^t a^{3k} b d^k$ . Let  $k' = \min\{k, t\}$  and  $k'' = \min\{k, s\}$ . Then we have  $c^{k'} a^{3k} b d^{k''} \in C$ . Hence, we have the derivation

$$c^k a^{3k} b d^k \Rightarrow c^{k-k'} \alpha c^{k'} a^{3k} b d^{k''} \beta d^{k-k''} = c^{k+n-m} a^{3k} b d^{k+n-m}$$

which produces a word not in  $K_k$ . Therefore,  $K_k \notin \mathcal{S}\mathcal{C}(SLT_k)$ .  $\square$

Together with the result  $\mathcal{S}\mathcal{C}(COMB) \setminus \mathcal{S}\mathcal{C}(SUF) \neq \emptyset$  (recalled from [10]) which also implies that  $\mathcal{S}\mathcal{C}(SLT_k) \setminus \mathcal{S}\mathcal{C}(SUF) \neq \emptyset$  for all  $k \geq 1$ , we obtain the following incomparability result.

**Lemma 3.12** *The families  $\mathcal{S}\mathcal{C}(SLT_k)$  for  $k \geq 1$  are incomparable to the family  $\mathcal{S}\mathcal{C}(SUF)$ .*

It is left open, whether the family  $\mathcal{S}\mathcal{C}(SUF)$  is also incomparable to the family  $\mathcal{S}\mathcal{C}(SLT)$  or whether it is a proper subset (since we know already that  $\mathcal{S}\mathcal{C}(SLT) \setminus \mathcal{S}\mathcal{C}(SUF) \neq \emptyset$ ).

Now we investigate the relations of the families  $\mathcal{S}\mathcal{C}(FIN)$ ,  $\mathcal{S}\mathcal{C}(NIL)$ , and  $\mathcal{S}\mathcal{C}(DEF)$  to the families  $\mathcal{S}\mathcal{C}(SLT_k)$  for  $k \geq 1$  as well as the relation of the family  $\mathcal{S}\mathcal{C}(SLT)$  to the family  $\mathcal{S}\mathcal{C}(ORD)$ .

**Lemma 3.13** *The families  $\mathcal{S}\mathcal{C}(SLT_k)$  for  $k \geq 1$  are incomparable to the families  $\mathcal{S}\mathcal{C}(FIN)$ ,  $\mathcal{S}\mathcal{C}(NIL)$ , and  $\mathcal{S}\mathcal{C}(DEF)$ .*

*Proof.* Due to the inclusion relations, it suffices to show that there are languages

$$L_0 \in \mathcal{I}\mathcal{C}(SLT_1) \setminus \mathcal{I}\mathcal{C}(DEF) \quad \text{and} \quad L_n \in \mathcal{I}\mathcal{C}(FIN) \setminus \mathcal{I}\mathcal{C}(SLT_n) \text{ for } n \geq 1.$$

From Lemma 3.7, we know for  $L_0 = \{a^n b^m c^n d^m \mid m \geq 1, n \geq 1\}$  that  $L_0 \in \mathcal{I}\mathcal{C}(SLT_1) \setminus \mathcal{I}\mathcal{C}(DEF)$ . From Lemma 3.6, we know  $L_n = \{a^m b^{2^n} c^m \mid m \geq n\} \cup \{a^{n-1} b^n c^{n-1}\} \in \mathcal{I}\mathcal{C}(FIN) \setminus \mathcal{I}\mathcal{C}(SLT_{n-1})$  for  $n \geq 2$ .  $\square$

From [18] and by Lemma 2.8, we know the inclusion  $\mathcal{I}\mathcal{C}(SLT) \subseteq \mathcal{I}\mathcal{C}(NC)$ ; from [28], we have the relation  $\mathcal{I}\mathcal{C}(ORD) \subseteq \mathcal{I}\mathcal{C}(NC)$ . Here, we have shown with Lemma 3.8 that there is a language in the family  $\mathcal{I}\mathcal{C}(ORD)$  which does not belong to the family  $\mathcal{I}\mathcal{C}(SLT)$ . The question whether the family  $\mathcal{I}\mathcal{C}(SLT)$  is a proper subset of the family  $\mathcal{I}\mathcal{C}(ORD)$  or whether these two families are incomparable is left open.

Summarizing, we have the following result.

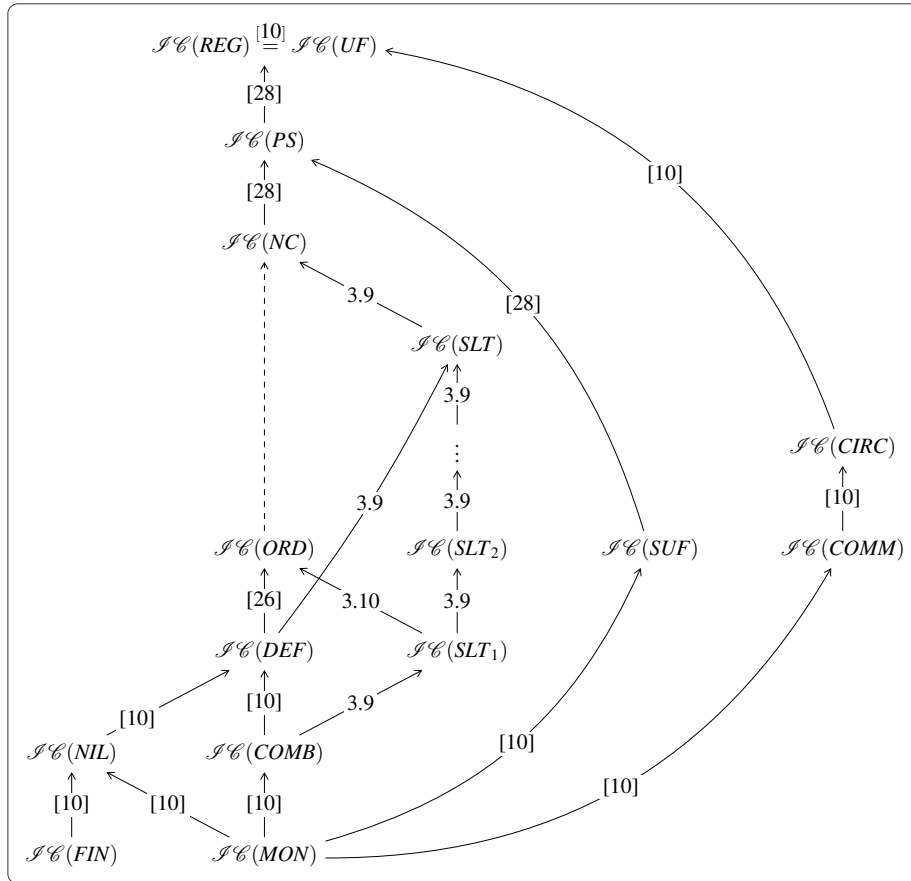


Figure 3: Hierarchy of language families by internal contextual grammars with selection languages defined by structural properties. An edge label refers to the paper where the respective inclusion is proved.

**Theorem 3.14** *The inclusion relations presented in Figure 3 hold. An arrow from an entry X to an entry Y depicts the proper inclusion  $X \subset Y$ ; the dashed arrow from  $\mathcal{I}\mathcal{C}(ORD)$  to  $\mathcal{I}\mathcal{C}(NC)$  indicates*

that it is not known so far whether the inclusion is proper or whether equality holds. If two families are not connected by a directed path, then they are incomparable with the exception of the family  $\mathcal{I}\mathcal{C}(SUF)$  and the families  $\mathcal{I}\mathcal{C}(ORD)$ ,  $\mathcal{I}\mathcal{C}(NC)$ , and  $\mathcal{I}\mathcal{C}(SLT)$  where  $\mathcal{I}\mathcal{C}(ORD) \not\subseteq \mathcal{I}\mathcal{C}(SUF)$ ,  $\mathcal{I}\mathcal{C}(NC) \not\subseteq \mathcal{I}\mathcal{C}(SUF)$ , and  $\mathcal{I}\mathcal{C}(SLT) \not\subseteq \mathcal{I}\mathcal{C}(SUF)$  hold, and with exception of the family  $\mathcal{I}\mathcal{C}(ORD)$  and the families  $\mathcal{I}\mathcal{C}(SLT_k)$  for  $k \geq 2$  and  $\mathcal{I}\mathcal{C}(SLT)$ , where  $\mathcal{I}\mathcal{C}(ORD) \not\subseteq \mathcal{I}\mathcal{C}(SLT_k)$  for  $k \geq 2$  and  $\mathcal{I}\mathcal{C}(ORD) \not\subseteq \mathcal{I}\mathcal{C}(SLT)$  hold.

## 4 Conclusions

The inclusion relations obtained for the families of languages generated by external or internal contextual grammars are in most cases the same as for the families where the selection languages are taken from.

For further research, the open questions already mentioned should be considered: What is the relation between the families  $\mathcal{E}\mathcal{C}(SLT)$  and  $\mathcal{E}\mathcal{C}(ORD)$  as well as between the families  $\mathcal{I}\mathcal{C}(SLT)$  and  $\mathcal{I}\mathcal{C}(ORD)$ , especially, is there a language in the set  $\mathcal{E}\mathcal{C}(SLT_2) \setminus \mathcal{E}\mathcal{C}(ORD)$  or a language in the set  $\mathcal{I}\mathcal{C}(SLT_2) \setminus \mathcal{I}\mathcal{C}(ORD)$ ? Is the family  $\mathcal{I}\mathcal{C}(SUF)$  incomparable to the family  $\mathcal{I}\mathcal{C}(SLT)$  or is it a proper subset? Additionally, it remains to investigate the relations of the family  $\mathcal{I}\mathcal{C}(SUF)$  to the families  $\mathcal{I}\mathcal{C}(ORD)$  and  $\mathcal{I}\mathcal{C}(NC)$ .

In [28], two independent hierarchies have been obtained for each type of contextual grammars, one based on selection languages defined by structural properties (as considered in this present paper), the other based on resources (number of non-terminal symbols, production rules, or states). These hierarchies should be merged.

The families of languages which are locally ( $k$ -)testable (not necessarily in the strict sense) are the Boolean closure of the families in the strict sense. For contextual grammars where the selection languages are intersections or unions of strictly locally ( $k$ -)testable languages, nothing has to be done since the classes  $SLT_k$  for  $k \geq 1$  and  $SLT$  are closed under intersection and, for union in a selection pair  $(S_1 \cup S_2, C)$ , one can take several selection pairs  $(S_1, C)$ ,  $(S_2, C)$  instead. It remains to investigate the impact of locally ( $k$ -)testable selection languages which are the complement of a strictly locally ( $k$ -)testable language.

Additionally, other subfamilies of regular languages could be taken into consideration. Recently, by Jürgen Dassow, external contextual grammars have been investigated where the selection languages are ideals or codes ([6, 7]). This research could be extended to internal contextual grammars with ideals or codes as selection languages.

## References

- [1] J. A. Brzozowski (1962): *Regular Expression Techniques for Sequential Circuits*. Ph.D. thesis, Princeton University, Princeton, NJ, USA.
- [2] J. A. Brzozowski, G. Jirásková & C. Zou (2014): *Quotient complexity of closed languages*. *Theory of Computing Systems* 54, pp. 277–292, doi:10.1007/s00224-013-9515-7.
- [3] Jürgen Dassow (1979): *On the Circular Closure of Languages*. *Elektronische Informationsverarbeitung und Kybernetik/Journal of Information Processing and Cybernetics* 15(1–2), pp. 87–94.
- [4] Jürgen Dassow (2005): *Contextual grammars with subregular choice*. *Fundamenta Informaticae* 64(1–4), pp. 109–118.
- [5] Jürgen Dassow (2015): *Contextual languages with strictly locally testable and star free selection languages*. *Analele Universității București* 62, pp. 25–36.

- [6] Jürgen Dassow (2018): *Grammars with control by ideals and codes*. *Journal of Automata, Languages and Combinatorics* 23(1–3), pp. 143–164, doi:10.25596/jal-c-2018-143.
- [7] Jürgen Dassow (2021): *Remarks on external contextual grammars with selection*. *Theoretical Computer Science* 862, pp. 119–129, doi:10.1016/j.tcs.2020.07.028.
- [8] Jürgen Dassow, Florin Manea & Bianca Truthe (2011): *On Contextual Grammars with Subregular Selection Languages*. In Markus Holzer, Martin Kutrib & Giovanni Pighizzini, editors: *Descriptive Complexity of Formal Systems – 13th International Workshop, DCFS 2011, Gießen/Limburg, Germany, July 25–27, 2011. Proceedings*, LNCS 6808, Springer-Verlag, pp. 135–146, doi:10.1007/978-3-642-22600-7\_11.
- [9] Jürgen Dassow, Florin Manea & Bianca Truthe (2012): *On External Contextual Grammars with Subregular Selection Languages*. *Theoretical Computer Science* 449(1), pp. 64–73, doi:10.1016/j.tcs.2012.04.008.
- [10] Jürgen Dassow, Florin Manea & Bianca Truthe (2012): *On Subregular Selection Languages in Internal Contextual Grammars*. *Journal of Automata, Languages, and Combinatorics* 17(2–4), pp. 145–164, doi:10.25596/jal-c-2012-145.
- [11] F. Gécseg & I. Peak (1972): *Algebraic Theory of Automata*. Akadémiai Kiadó, Budapest.
- [12] A. Gill & L. T. Kou (1974): *Multiple-entry finite automata*. *Journal of Computer and System Sciences* 9(1), pp. 1–19, doi:10.1016/S0022-0000(74)80034-6.
- [13] Ivan M. Havel (1969): *The theory of regular events II*. *Kybernetika* 5(6), pp. 520–544.
- [14] Markus Holzer & Bianca Truthe (2015): *On Relations Between Some Subregular Language Families*. In Rudolf Freund, Markus Holzer, Nelma Moreira & Rogério Reis, editors: *Seventh Workshop on Non-Classical Models of Automata and Applications (NCMA), Porto, Portugal, August 31 – September 1, 2015, Proceedings*, books@ocg.at 318, Österreichische Computer Gesellschaft, pp. 109–124.
- [15] Sorin Istrail (1978): *Gramatici contextuale cu selectiva regulata*. *Stud. Cerc. Mat.* 30, pp. 287–294.
- [16] Florin Manea & Bianca Truthe (2012): *On Internal Contextual Grammars with Subregular Selection Languages*. In Martin Kutrib, Nelma Moreira & Rogério Reis, editors: *Descriptive Complexity of Formal Systems – 14th International Workshop, DCFS 2012, Braga, Portugal, July 23–25, 2012. Proceedings*, LNCS 7386, Springer-Verlag, pp. 222–235, doi:10.1007/978-3-642-31623-4\_17.
- [17] Solomon Marcus (1969): *Contextual grammars*. *Revue Roum. Math. Pures Appl.* 14, pp. 1525–1534.
- [18] Robert McNaughton & Seymour Papert (1971): *Counter-free Automata*. MIT Press, Cambridge, USA.
- [19] M. Perles, M. M. Rabin & E. Shamir (1963): *The theory of definite automata*. *IEEE Trans. Electronic Computers* 12, pp. 233–243, doi:10.1109/PGEC.1963.263534.
- [20] Gheorghe Păun (1998): *Marcus Contextual Grammars*. Kluwer Publ. House, Dordrecht, doi:10.1007/978-94-015-8969-7.
- [21] Stefano Crespi Reghizzi & Pierluigi San Pietro (2011): *From Regular to Strictly Locally Testable Languages*. In P. Ambrož, Š. Holub & Z. Masáková, editors: *8th International Conference WORDS 2011, EPTCS* 63, pp. 103–111, doi:10.4204/EPTCS.63.14.
- [22] Grzegorz Rozenberg & Arto Salomaa, editors (1997): *Handbook of Formal Languages*. Springer-Verlag, Berlin, doi:10.1007/978-3-642-59136-5.
- [23] H. J. Shyr (1991): *Free Monoids and Languages*. Hon Min Book Co., Taichung, Taiwan.
- [24] H. J. Shyr & G. Thierrin (1974): *Ordered Automata and Associated Languages*. *Tankang Journal of Mathematics* 5(1), pp. 9–20.
- [25] H. J. Shyr & G. Thierrin (1974): *Power-Separating Regular Languages*. *Mathematical Systems Theory* 8(1), pp. 90–95, doi:10.1007/BF01761710.
- [26] Bianca Truthe (2014): *A Relation Between Definite and Ordered Finite Automata*. In Suna Bensch, Rudolf Freund & Friedrich Otto, editors: *Sixth Workshop on Non-Classical Models of Automata and Applications (NCMA), Kassel, Germany, July 28–29, 2014, Proceedings*, books@ocg.at 304, Österreichische Computer Gesellschaft, pp. 235–247.

- [27] Bianca Truthe (2018): *Hierarchy of Subregular Language Families*. Technical Report, Justus-Liebig-Universität Giessen, Institut für Informatik, IFIG Research Report 1801.
- [28] Bianca Truthe (2021): *Generative Capacity of Contextual Grammars with Subregular Selection Languages*. *Fundamenta Informaticae* 180, pp. 1–28, doi:10.3233/FI-2021-2037.
- [29] Barbara Wiedemann (1978): *Vergleich der Leistungsfähigkeit endlicher determinierter Automaten*. Diplomarbeit, Universität Rostock.