

Directed Containers as Categories

Danel Ahman

LFCS, School of Informatics, University of Edinburgh, United Kingdom

d.ahman@ed.ac.uk

Tarmo Uustalu

Institute of Cybernetics at Tallinn University of Technology, Estonia

tarmo@cs.ioc.ee

Directed containers make explicit the additional structure of those containers whose set functor interpretation carries a comonad structure. The data and laws of a directed container resemble those of a monoid, while the data and laws of a directed container morphism resemble those of a monoid morphism in the reverse direction. With some reorganization, a directed container is the same as a small category, but a directed container morphism is opcleavage-like. We draw some conclusions for comonads from this observation, considering in particular basic constructions and concepts like the opposite category and a groupoid.

1 Introduction

Abbott, Altenkirch, Ghani, Hancock, McBride and Morris's containers [1, 5] are a representation of a wide class of set functors (datatypes) in terms of shapes and positions. We could say that they are a form of syntax for datatypes as semantic entities. Polynomials à la Gambino and Hyland [6] are a close variation of containers. They are nice in particular because they can be made to work in a very general setting (categories with pullbacks) [8], but the definitions and proofs get very involved very early on.

Ahman, Chapman and Uustalu [2] introduced directed containers to make explicit the additional structure of those containers whose set functor interpretation carries a comonad structure. Typical examples of such functors with a comonad structure are node-labelled tree datatypes. The counit extracts the root label of a tree while the comultiplication relabels every node with the whole subtree rooted by that node. In a directed container every position in a shape determines another shape, the subshape corresponding to this position. Ahman, Chapman and Uustalu [2] also spelled out the polynomial version of directed containers (directed polynomials), but did not put it to any use.

In this paper, we revisit directed polynomials and take advantage of them to make some observations about the structure of the category of directed containers. In particular, we develop some new constructions on directed containers and some specializations of directed containers. Specifically, we see that the category of directed containers is isomorphic to the category of small categories and (what we here call) relative split pre-opcleavages of a certain type. We consider the opposite category construction (for a datatype of node-(i.e., subtree-)labelled trees, the corresponding datatype of context-labelled trees) and the concept of a groupoid (of which zipper datatypes are typical examples).

The significance of containers for functional programming consists primarily in providing a tool for generic programming with both datatypes in general as well as with datatypes with special structure. But containers are also particularly well-suited for analyzing special classes of datatypes from a combinatorial perspective. For instance, the concepts of directed containers and distributive laws of directed

containers enable us to recognize and enumerate the comonad structures that are available on different functors, the distributive laws that exist between these comonads etc.

This paper is organized as follows. First, in Section 2, we briefly review containers and directed containers. Then, we go to polynomials and directed polynomials in Section 3, recognizing that a directed polynomial is just a small category while a directed polynomial morphism is not a functor, but something different and considerably more involved. Next, in Section 4, we consider the coproduct of two directed containers and a certain tensor of two directed containers as constructions of categories. Finally, we analyze opposite categories and groupoids as directed containers in Sections 5 and 6, respectively.

Throughout the paper, we use syntax similar to that of the dependently-typed functional programming language Agda. In particular, when declaring the type of a function, we mark the arguments that are derivable in most contexts as implicit by enclosing them/their types in braces. In applications of the function, we either omit these arguments or provide them in braces. In infix applications of functions, we place the implicit arguments (if we want to provide them) after the function symbol.

2 Directed Containers

We begin with a brief recap of containers and directed containers.

2.1 Containers

A *container* is given by a set S (of shapes) and a S -indexed family of sets P (of positions). Any container (S, P) interprets into a set functor $\llbracket S, P \rrbracket^c$, defined on objects as $\llbracket S, P \rrbracket^c X =_{\text{df}} \Sigma s : S. P s \rightarrow X$ and on morphisms as $\llbracket S, P \rrbracket^c f(s, v) =_{\text{df}} (s, \lambda p. f(v p))$.

A *container morphism* between (S, P) and (S', P') is given by maps $t : S \rightarrow S'$ and $q : \Pi s : S. P'(t s) \rightarrow P s$, correspondingly interpreting into a natural transformation $\llbracket t, q \rrbracket^c$ between the functors $\llbracket S, P \rrbracket^c$ and $\llbracket S', P' \rrbracket^c$, defined as $\llbracket t, q \rrbracket^c(s, v) =_{\text{df}} (t s, \lambda p. v(q s p))$.

Containers and container morphisms form a monoidal category **Cont**. The interpretation $\llbracket - \rrbracket^c$ is a fully faithful monoidal functor from **Cont** to $[\mathbf{Set}, \mathbf{Set}]$, with the monoidal structures given by the composition of containers and the composition of set functors, respectively.

We emphasize the fully-faithfulness of $\llbracket - \rrbracket^c$: it means that all natural transformations between functors representable by containers are representable as container morphisms and uniquely so.

2.2 Directed Containers

Directed containers are containers with additional structure.

A *directed container* is a container (S, P) together with maps $\downarrow : \Pi s : S. P s \rightarrow S$ (the subshape corresponding to a position), $\circ : \Pi \{s : S\}. P s$ (the root position in a shape), $\oplus : \Pi \{s : S\}. \Pi p : P s. P(s \downarrow p) \rightarrow P s$ (translation of subshape positions into the global shape) such that

$$\begin{aligned} s \downarrow \circ &= s \\ s \downarrow (p \oplus p') &= (s \downarrow p) \downarrow p' \\ p \oplus \circ &= p \\ \circ \oplus p &= p \\ (p \oplus p') \oplus p'' &= p \oplus (p' \oplus p'') \end{aligned}$$

where the 4th displayed equation is welltyped thanks to the 1st (to type the left-hand side we need $p : P(s \downarrow \circ)$ while to type the right-hand side we need $p : P s$) and the 5th thanks to the 2nd (to type the

left-hand side we need $p'' : P(s \downarrow (p \oplus p'))$ while to type the right-hand side we need $p'' : P((s \downarrow p) \downarrow p)$. In the special case $S = 1$, the equations simply require that $(P*, \circ\{*\}, \oplus\{*\})$ is a monoid.

Any directed container $(S, P, \downarrow, \circ, \oplus)$ interprets into a comonad $\llbracket S, P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}} =_{\text{df}} (D, \varepsilon, \delta)$ on **Set**, with $D =_{\text{df}} \llbracket S, P \rrbracket^{\text{c}}$, $\varepsilon(s, v) =_{\text{df}} v(\circ\{s\})$, $\delta(s, v) =_{\text{df}} (s, \lambda p. (s \downarrow p, \lambda p'. v(p \oplus \{s\} p')))$.

A *directed container morphism* between $(S, P, \downarrow, \circ, \oplus)$ and $(S', P', \downarrow', \circ', \oplus')$ is a container morphism (t, q) between (S, P) and (S', P') such that

$$\begin{aligned} t(s \downarrow qsp) &= ts \downarrow' p \\ \circ\{s\} &= qs(\circ'\{ts\}) \\ qsp \oplus \{s\} q(s \downarrow qsp) p' &= qs(p \oplus' \{ts\} p') \end{aligned}$$

where the 3rd displayed equation is welltyped as the 1st holds (to type the left-hand side we need $p' : P'(t(s \downarrow qsp))$ while to type the right-hand side we need $p' : P'(ts \downarrow' p)$). In the special case $S = S' = 1$, which trivializes t , the map $q*$ is a monoid morphism between $(P'*, \circ'\{*\}, \oplus'\{*\})$ and $(P*, \circ\{*\}, \oplus\{*\})$ —in the opposite direction compared to the directed container morphism.

A directed container morphism (t, q) between $(S, P, \downarrow, \circ, \oplus)$ and $(S', P', \downarrow', \circ', \oplus')$ interprets into a comonad morphism between $\llbracket S, P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}}$ and $\llbracket S', P', \downarrow', \circ', \oplus' \rrbracket^{\text{dc}}$, given by $\llbracket t, q \rrbracket^{\text{dc}} =_{\text{df}} \llbracket t, q \rrbracket^{\text{c}}$.

Directed containers and directed container morphisms form a category **DCont**. The interpretation $\llbracket - \rrbracket^{\text{dc}}$ is a fully faithful functor from **DCont** to **Comonads(Set)**. In fact, $\llbracket - \rrbracket^{\text{dc}}$ is a pullback of $\llbracket - \rrbracket^{\text{c}}$ along $U : \mathbf{Comonads}(\mathbf{Set}) \rightarrow [\mathbf{Set}, \mathbf{Set}]$, meaning that directed containers are precisely those containers whose set functor interpretation carries a comonad structure.

Here are some simple examples of directed containers.

Taking $S =_{\text{df}} 1$, $P* =_{\text{df}} \text{Nat}$, $* \downarrow p =_{\text{df}} *$, $\circ =_{\text{df}} 0$, $p \oplus p' =_{\text{df}} p + p'$, we get a directed container interpreting into the stream comonad: $DX =_{\text{df}} \Sigma* : 1. \text{Nat} \rightarrow X \cong \text{Str}X$. The counit extracts the head element from a stream while the comultiplication turns a stream into a stream of its suffixes.

Taking $S =_{\text{df}} \text{Nat}$, $P_s =_{\text{df}} [0..s]$, $s \downarrow p =_{\text{df}} s - p$, $\circ =_{\text{df}} 0$, $p \oplus p' =_{\text{df}} p + p'$, we get a directed container interpreting into the nonempty list comonad: $DX =_{\text{df}} \Sigma_s : \text{Nat}. [0..s] \rightarrow X \cong \text{NEList}X$. The counit extracts the head element from a nonempty list while the comultiplication turns a nonempty list into a nonempty list of its suffixes.

If we instead take $S =_{\text{df}} \text{Nat}$, $P_s =_{\text{df}} [0..s]$, $s \downarrow p =_{\text{df}} s$, $\circ =_{\text{df}} 0$, $p \oplus \{s\} p' =_{\text{df}} (p + p') \bmod (s + 1)$, we get a directed container interpreting into a different comonad on the nonempty list functor. The counit extracts the head element from a nonempty list while the comultiplication turns a nonempty list into a nonempty list of its cyclic shifts.

Finally, taking S to be any given set and $P_s =_{\text{df}} 1$, we get a directed container interpreting into the reader comonad $DX =_{\text{df}} \Sigma_s : S. 1 \rightarrow X \cong S \times X$. Taking S to be any given set, $P_s =_{\text{df}} S$, $s \downarrow s' =_{\text{df}} s'$, $\circ\{s\} =_{\text{df}} s$ and $s' \oplus \{s\} s'' =_{\text{df}} s''$, the directed container obtained interprets into the array comonad $DX =_{\text{df}} \Sigma_s : S. S \rightarrow X \cong S \times (S \rightarrow X)$.

3 Directed Containers as Categories

We now proceed to polynomials and directed polynomials.

3.1 Containers as Polynomials

Polynomials are an alternative view of containers where the S -indexed family P of positions is replaced with a single set \bar{P} that collects all positions across all shapes. This set must be fibred over S in the sense of coming together with a map $s : \bar{P} \rightarrow S$ that tells which shape each position belongs to.

A *polynomial* is hence given by two sets S and \bar{P} and a map $s : \bar{P} \rightarrow S$.

Containers can be converted into polynomials and vice versa by $\bar{P} =_{\text{df}} \Sigma s : S.Ps$, $s(s, p) =_{\text{df}} s$ resp. $Ps =_{\text{df}} \Sigma p : \bar{P}. \{s p = s\}$.¹ This gives us a bijection up to isomorphism between the collections of containers and polynomials.

In a morphism between polynomials, instead of an S -indexed family of maps $q_s : P'(t s) \rightarrow Ps$, we have a single map \bar{q} sending elements of \bar{P}' to \bar{P} which is defined only for those $p : \bar{P}'$ whose shape is in the image of $t : S \rightarrow S'$, i.e., $t s = s' p$ for some $s : S$. This s must be explicitly supplied, as it is not uniquely determined by p , unless t is injective. Moreover, the shape for the position returned by \bar{q} on (s, p) must be s , i.e., $s(\bar{q}(s, p)) = s$.²

A *polynomial morphism* between (S, \bar{P}, s) and (S', \bar{P}', s') is therefore given by maps $t : S \rightarrow S'$ and $\bar{q} : (\Sigma s : S. \Sigma p : \bar{P}'. \{t s = s' p\}) \rightarrow \bar{P}$ such that $s(\bar{q}(s, p)) = s$. Polynomials and polynomial morphisms form a category **Poly**.

Container morphisms and polynomial morphisms are interconverted by $\bar{q}(s, p) =_{\text{df}} q s p$ resp. $q s p =_{\text{df}} \bar{q}(s, p)$. These conversions extend the bijection up to isomorphism between the collections of containers and polynomials to an equivalence between the categories **Cont** and **Poly** of containers resp. polynomials.

3.2 Directed Containers as Directed Polynomials as Small Categories

Let us apply a similar change of view (from “indexed” to “fibred”) to directed containers. We arrive at the following result.

A *directed polynomial* is given by sets S , \bar{P} and maps $s, t : \bar{P} \rightarrow S$, $\text{id} : \{S\} \rightarrow P$, $;$: $(\Sigma p : \bar{P}. \Sigma p' : \bar{P}. \{t p = s p'\}) \rightarrow \bar{P}$ such that $s(\text{id}\{s\}) = s$, $s(p; p') = s p$ and

$$\begin{aligned} t(\text{id}\{s\}) &= s \\ t(p; p') &= t p' \\ p; \text{id}\{s\} &= p \\ \text{id}\{s\}; p &= p \\ (p; p'); p'' &= p; (p'; p'') \end{aligned}$$

Similarly to the situation with directed containers, here the 4th displayed equation is welltyped because the 1st equation holds and the 5th equation is welltyped because the 2nd equation holds. (But the more basic nondisplayed equations need also to be used.)

A directed container is converted into a directed polynomial by $\bar{P} =_{\text{df}} \Sigma s : S.Ps$, $s(s, p) =_{\text{df}} s$, $t(s, p) =_{\text{df}} s \downarrow p$, $\text{id}\{s\} =_{\text{df}} (s, \circ\{s\})$, $(s, p); (s \downarrow p, p') =_{\text{df}} (s, p \oplus \{s\} p')$.

A conversion in the opposite direction is given by $Ps =_{\text{df}} \Sigma p : \bar{P}. \{s p = s\}$, $s \downarrow p =_{\text{df}} t p$, $\circ\{s\} =_{\text{df}} \text{id}\{s\}$, $p \oplus \{s\} p' =_{\text{df}} p; p'$. These conversions form a bijection up to isomorphism between the collections of directed containers and directed polynomials.

It is easy to notice that the data and laws of a directed polynomial are just those of a small category: shapes are objects, positions are morphisms, s and t give the source and target of a morphism, id and $;$ are the identities and composition. We learn that the reason why a directed container is like a monoid,

¹Elements of Ps are dependent pairs with an equality proof as the second component. We mark this component as implicit by enclosing its type in braces in the definition of Ps and omit it in actual pairs of this type. We apply the same practice also to other similar dependent tuple types.

²These considerations concern general polynomial morphisms, which is what we need. In works on polynomials, it is commonplace to consider only linear polynomial morphisms. They are much easier to handle, but too restrictive for our purposes.

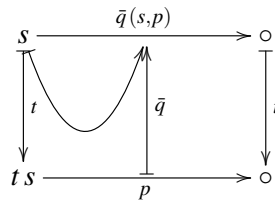
albeit not quite, is that it is a proper generalization of a monoid, and a well-known proper generalization at that, a small category!

What are the implications of this? Can it be that the category of directed containers is nothing but the category **Cat** of small categories and functors? That would be a hasty conclusion. We should diligently spell out what a directed container morphism amounts to. It is this.

A *directed polynomial morphism* between $E = (S, \bar{P}, s, t, \text{id}, ;)$ and $E' = (S', \bar{P}', s', t', \text{id}', ;')$ is given by maps $t : S \rightarrow S'$ and $\bar{q} : (\Sigma s : S. \Sigma p : \bar{P}'. \{t s = s' p\}) \rightarrow \bar{P}$ such that $s(\bar{q}(s, p)) = s$ and

$$\begin{aligned} t(t(\bar{q}(s, p))) &= t' p \\ \text{id} \{s\} &= \bar{q}(s, \text{id}' \{t s\}) \\ \bar{q}(s, p); \bar{q}(t(\bar{q}(s, p)), p') &= \bar{q}(s, p; ' p') \end{aligned}$$

where the 3rd displayed equation is welltyped because the 1st holds. Pictorially,



Directed polynomials and directed polynomial morphisms form a category **DPoly**.³

Directed container morphisms and directed polynomial morphisms are interconvertible by $\bar{q}(s, p) =_{\text{df}} q s p$ resp. $q s p =_{\text{df}} \bar{q}(s, p)$. This extends the bijection up to isomorphism between the collections of directed containers and directed polynomials into an equivalence of the categories **DCont** and **DPoly**.

We see that a directed polynomial morphism (t, \bar{q}) between directed polynomials E and E' is very far from anything like a functor between E and E' as small categories. Instead, it is a bit like \bar{q} being an opcleavage for t , but with two big reservations. First, t is not a functor, but only an object mapping, and second, the requirements on \bar{q} are somewhat weak.

We could reasonably say that \bar{q} is a split pre-opcleavage for $t^\dagger : E \rightarrow S'^\dagger$ relative to $! : E' \rightarrow S'^\dagger$ where S'^\dagger is the cofree category on S' , i.e., the small category with S' as the set of objects and a single morphism between any two objects. This is according to the following terminology that we have invented for the occasion.

We say that a *split pre-opcleavage* for a functor $F : \mathbb{C} \rightarrow \mathbb{D}$ is an assignment, to any object X of \mathbb{C} and any morphism f of \mathbb{D} with $F X$ as the source, of a morphism $f_* X$ of \mathbb{C} that has X as the source and satisfies $F(f_* X) = f$, in such a way that $\text{id} \{X\} = (\text{id} \{F X\})_* X$ and $f_* X; g_* Y = (f; g)_* X$. This is weaker than an opcleavage in that $f_* X$ does not have to be opCartesian. But at the same time, this is also stronger in that the equations of a split opcleavage are already required. As a variation, a *split pre-opcleavage* for a functor $F : \mathbb{C} \rightarrow \mathbb{D}$ relative to a functor $J : \mathbb{J} \rightarrow \mathbb{D}$ is an assignment, to any object X of \mathbb{C} and any morphism f of \mathbb{J} whose source Z satisfies $F X = J Z$, of a morphism $f_* X$ of \mathbb{C} that has X as the source and satisfies $F(f_* X) = J f$, in such a way that $\text{id} \{X\} = (\text{id} \{F X\})_* X$ and $f_* X; g_* Y = (f; g)_* X$. Here the meaning of ‘relative’ is the same as in relative adjunctions [7] and relative monads [4].

We are not sure at this stage that this is the optimal analysis of directed polynomial morphisms in terms of standard or close to standard concepts, but it is the best we currently have.

³Linear directed polynomial morphisms are much simpler than general directed polynomial morphisms: they are nothing but fully-faithful functors between small categories.

Let us revisit our examples. The small category for the stream comonad has $S =_{\text{df}} 1$, $\bar{P} =_{\text{df}} \text{Nat}$, $s p =_{\text{df}} *$, $t p =_{\text{df}} *$, $\text{id} =_{\text{df}} 0$, $p; p' =_{\text{df}} p + p'$. (Of course this is nothing else than the monoid $(\text{Nat}, 0, +)$ seen as one-object category.)

The small category for the nonempty lists and suffixes comonad is given by $S =_{\text{df}} \text{Nat}$, $\bar{P} =_{\text{df}} \Sigma s : \text{Nat}. [0..s]$, $s(s, p) =_{\text{df}} s$, $t(s, p) =_{\text{df}} s - p$, $\text{id} \{s\} =_{\text{df}} (s, 0)$, $(s, p); (s - p, p') =_{\text{df}} (s, p + p')$.

The small category for the nonempty lists and cyclic shifts comonad has $S =_{\text{df}} \text{Nat}$, $\bar{P} =_{\text{df}} \Sigma s : \text{Nat}. [0..s]$, $s(s, p) =_{\text{df}} s$, $t(s, p) =_{\text{df}} s$, $\text{id} \{s\} =_{\text{df}} (s, 0)$, $(s, p); (s, p') =_{\text{df}} (s, (p + p') \bmod (s + 1))$.

The small category for the reader comonad has as S any given set, $\bar{P} =_{\text{df}} \Sigma s : S. 1 \cong S$, $s s =_{\text{df}} s$, $t s =_{\text{df}} s$, $\text{id} \{s\} =_{\text{df}} s$, $s; s =_{\text{df}} s$. This is the discrete category on the set of objects S —the free category on S . The small category for the array comonad has as S any given set, $\bar{P} =_{\text{df}} \Sigma s : S. S \cong S \times S$, $s(s, s') =_{\text{df}} s$, $t(s, s') =_{\text{df}} s'$, $\text{id} \{s\} =_{\text{df}} (s, s)$, $(s, s'); (s', s'') =_{\text{df}} (s, s'')$. This category has a unique morphism between any two objects and is the cofree category on S .

4 The Coproduct and a Tensor of Directed Containers

We can now look at some basic constructions of directed containers as constructions of categories.

4.1 Coproduct

Given two small categories $(S_0, \bar{P}_0, s_0, t_0, \text{id}_0, ;_0)$ and $(S_1, \bar{P}_1, s_1, t_1, \text{id}_1, ;_1)$, we can construct a small category $(S, \bar{P}, s, t, \text{id}, ;)$ by

$$\begin{aligned}
 S &=_{\text{df}} S_0 + S_1 \\
 P &=_{\text{df}} P_0 + P_1 \\
 s(\text{inl } p) &=_{\text{df}} \text{inl}(s_0 p) \\
 s(\text{inr } p) &=_{\text{df}} \text{inr}(s_1 p) \\
 t(\text{inl } p) &=_{\text{df}} \text{inl}(t_0 p) \\
 t(\text{inr } p) &=_{\text{df}} \text{inr}(t_1 p) \\
 \text{id} \{\text{inl } s\} &=_{\text{df}} \text{inl}(\text{id}_0 \{s\}) \\
 \text{id} \{\text{inr } s\} &=_{\text{df}} \text{inr}(\text{id}_1 \{s\}) \\
 \text{inl } p; \text{inl } p' &=_{\text{df}} \text{inl}(p;_0 p') \\
 \text{inr } p; \text{inr } p' &=_{\text{df}} \text{inr}(p;_1 p')
 \end{aligned}$$

This small category $(S, \bar{P}, s, t, \text{id}, ;)$ is a coproduct of $(S_0, \bar{P}_0, s_0, t_0, \text{id}_0, ;_0)$ and $(S_1, \bar{P}_1, s_1, t_1, \text{id}_1, ;_1)$ in the category of small categories and functors, but it is likewise their coproduct in the category of small categories and pre-opcleavages.

Via the isomorphism of the categories of directed polynomials and directed containers, we have cheaply got a construction for the coproduct of two directed containers. Given two directed containers

$(S_0, P_0, \downarrow_0, \circ_0, \oplus_0)$ and $(S_1, P_1, \downarrow_1, \circ_1, \oplus_1)$, their coproduct is $(S, P, \downarrow, \circ, \oplus)$ where

$$\begin{aligned}
S &=_{\text{df}} S_0 + S_1 \\
P(\text{inl } s) &=_{\text{df}} P_0 s \\
P(\text{inr } s) &=_{\text{df}} P_1 s \\
\text{inl } s \downarrow p &=_{\text{df}} \text{inl } (s \downarrow_0 p) \\
\text{inr } s \downarrow p &=_{\text{df}} \text{inr } (s \downarrow_1 p) \\
\circ \{\text{inl } s\} &=_{\text{df}} \circ_0 s \\
\circ \{\text{inr } s\} &=_{\text{df}} \circ_1 s \\
p \oplus \{\text{inl } s\} p' &=_{\text{df}} p \oplus_0 \{s\} p' \\
p \oplus \{\text{inr } s\} p' &=_{\text{df}} p \oplus_1 \{s\} p'
\end{aligned}$$

The interpretation $\llbracket - \rrbracket^{\text{dc}}$ takes the coproduct of two directed containers into the coproduct of the corresponding comonads, i.e., it preserves coproducts. The category of set comonads inherits its coproducts from the category of set functors.

4.2 A Tensor

Given two small categories $(S_0, \bar{P}_0, s_0, t_0, \text{id}_0, ;_0)$ and $(S_1, \bar{P}_1, s_1, t_1, \text{id}_1, ;_1)$, we can also build a small category $(S, \bar{P}, s, t, \text{id}, ;)$ by

$$\begin{aligned}
S &=_{\text{df}} S_0 \times S_1 \\
P &=_{\text{df}} P_0 \times P_1 \\
s(p_0, p_1) &=_{\text{df}} (s_0 p_0, s_1 p_1) \\
t(p_0, p_1) &=_{\text{df}} (t_0 p_0, t_1 p_1) \\
\text{id } \{s_0, s_1\} &=_{\text{df}} (\text{id } \{s_0\}, \text{id } \{s_1\}) \\
(p_0, p_1) ; (p'_0, p'_1) &=_{\text{df}} (p_0 ;_0 p'_0, p_1 ;_1 p'_1)
\end{aligned}$$

In the category of small categories and functors, $(S, \bar{P}, s, t, \text{id}, ;)$ is the Cartesian product of $(S_0, \bar{P}_0, s_0, t_0, \text{id}_0, ;_0)$ and $(S_1, \bar{P}_1, s_1, t_1, \text{id}_1, ;_1)$. If we replace functors with relative split pre-opcleavages, this ceases to be the case; the problem is that we cannot define pairing. So we only get a binary operation on small categories that is associative and functorial in both arguments (a semimonoidal structure).

The corresponding construction on directed containers is the following. Given two directed containers $(S_0, P_0, \downarrow_0, \circ_0, \oplus_0)$ and $(S_1, P_1, \downarrow_1, \circ_1, \oplus_1)$, we build a new directed container $(S, P, \downarrow, \circ, \oplus)$ by

$$\begin{aligned}
S &=_{\text{df}} S_0 \times S_1 \\
P(s_0, s_1) &=_{\text{df}} P s_0 \times P s_1 \\
(s_0, s_1) \downarrow (p_0, p_1) &=_{\text{df}} (s_0 \downarrow_0 p_0, s_1 \downarrow_1 p_1) \\
\circ \{s_0, s_1\} &=_{\text{df}} (\circ_0 \{s_0\}, \circ_1 \{s_1\}) \\
(p_0, p_1) \oplus (p'_0, p'_1) &=_{\text{df}} (p_0 \oplus_0 p'_0, p_1 \oplus_1 p'_1)
\end{aligned}$$

In Glasgow, this construction on the underlying containers has been named Hancock's tensor.

The sum and tensor of containers provide a semiring category structure on the category of containers. The category of small categories and split pre-opcleavages (or directed containers) inherits this structure.

5 The Opposite Directed Container

Given a small category $(S, \bar{P}, s, t, \text{id}, ;)$, an obvious related small category to look at is the *opposite* category $(S^{\text{op}}, \bar{P}^{\text{op}}, s^{\text{op}}, t^{\text{op}}, \text{id}^{\text{op}}, ;^{\text{op}})$ where

$$\begin{aligned} S^{\text{op}} &=_{\text{df}} S \\ \bar{P}^{\text{op}} &=_{\text{df}} \bar{P} \\ s^{\text{op}} p &=_{\text{df}} t p \\ t^{\text{op}} p &=_{\text{df}} s p \\ \text{id}^{\text{op}} \{s\} &=_{\text{df}} \text{id} \{s\} \\ f ;^{\text{op}} g &=_{\text{df}} g ; f \end{aligned}$$

Translated to directed containers, we get the following construction. Given a directed container $(S, P, \downarrow, \circ, \oplus)$, the “opposite” directed container is $(S^{\text{op}}, P^{\text{op}}, \downarrow^{\text{op}}, \circ^{\text{op}}, \oplus^{\text{op}})$ where

$$\begin{aligned} S^{\text{op}} &=_{\text{df}} S \\ P^{\text{op}} s &=_{\text{df}} \Sigma s' : S. \Sigma p : P s'. \{s = s' \downarrow p\} \\ s \downarrow^{\text{op}} (s', p) &=_{\text{df}} s' \\ \circ^{\text{op}} \{s\} &=_{\text{df}} (s, \circ \{s\}) \\ (s', p) \oplus^{\text{op}} \{s\} (s'', p') &=_{\text{df}} (s'', p' \oplus \{s''\} p) \end{aligned}$$

For a datatype of node-labelled trees of some branching type, this construction delivers the datatype of context-labelled trees of the same branching type. Let us work out what happens to the directed container for the non-empty list and suffixes comonad. The opposite category is given by

$$\begin{aligned} S^{\text{op}} &=_{\text{df}} \text{Nat} \\ \bar{P}^{\text{op}} &=_{\text{df}} \Sigma s : \text{Nat}. [0..s] \\ s^{\text{op}} (s, p) &=_{\text{df}} s - p \\ t^{\text{op}} (s, p) &=_{\text{df}} s \\ \text{id}^{\text{op}} \{s\} &=_{\text{df}} (s, 0) \\ (s - p, p') ;^{\text{op}} (s, p) &=_{\text{df}} (s, p + p') \end{aligned}$$

Accordingly, the opposite directed container is given by

$$\begin{aligned} S^{\text{op}} &=_{\text{df}} \text{Nat} \\ P^{\text{op}} s &=_{\text{df}} \Sigma s' : \text{Nat}. \Sigma p : [0..s']. \{s = s' - p\} \\ s \downarrow^{\text{op}} (s', p) &=_{\text{df}} s' \\ \circ^{\text{op}} \{s\} &=_{\text{df}} (s, 0) \\ (s', p) \oplus^{\text{op}} \{s\} (s'', p') &=_{\text{df}} (s'', p' + p) \end{aligned}$$

It is easy to see that the second component of a position determines the first: s' must be $s + p$, so we can leave s' out, removing the bound on p and letting it range over all of Nat . Hence we can simplify:

$$\begin{aligned} S^{\text{op}} &=_{\text{df}} \text{Nat} \\ P^{\text{op}} s &=_{\text{df}} \text{Nat} \\ s \downarrow^{\text{op}} p &=_{\text{df}} s + p \\ \circ^{\text{op}} &=_{\text{df}} 0 \\ p \oplus^{\text{op}} p' &=_{\text{df}} p' + p \end{aligned}$$

We get that the underlying functor of the comonad is defined by $D^{\text{op}}X =_{\text{df}} \Sigma s : \text{Nat. Nat} \rightarrow X \cong \text{Nat} \times \text{Str}X$. This is exactly the datatype of context-labelled trees of our chosen branching type; i.e., a datatype whose every element is a tree together with a label for every possible one-hole context it can fill. In our example, a tree is a nonempty list over 1, identified with a natural number. Its contexts are longer nonempty lists. The counit extracts the label of the empty context in a context-labelled tree. The comultiplication replaces the label of every context with the corresponding context-labelled tree. Formally, $\varepsilon(s, xs) =_{\text{df}} \text{hd } xs$, $\delta(s, xs) =_{\text{df}} (s, \delta_0(s, xs))$ where $\delta_0(s, xs) =_{\text{df}} (s, xs) :: \delta_0(s + 1, \text{tl } xs)$.

6 Bidirected Containers as Groupoids

A groupoid is a category where every morphism is iso. In algebraicized form, a *groupoid* is a category $(S, \bar{P}, s, t, \text{id}, ;)$ together with a map $(-)^{-1} : \bar{P} \rightarrow \bar{P}$ such that $s(p^{-1}) = t p$ and

$$\begin{aligned} t(p^{-1}) &= s p \\ p ; (p^{-1}) &= \text{id } \{s p\} \\ (p^{-1}) ; p &= \text{id } \{t p\} \end{aligned}$$

Here the 3rd displayed equation is welltyped because the 1st holds. The 1st equation is in fact redundant, since it follows from the 2nd equation together with the 1st and 2nd equations of a category.

Translating this axiomatization to directed containers, we get what we call bidirected containers.

A *bidirected container* is a directed container $(S, P, \downarrow, \circ, \oplus)$ together with a map $\ominus : \Pi\{s : S\}. \Pi p : P s. P(s \downarrow p)$ such that

$$\begin{aligned} (s \downarrow p) \downarrow (\ominus \{s\} p) &= s \\ p \oplus \{s\} (\ominus \{s\} p) &= \circ \{s\} \\ (\ominus \{s\} p) \oplus \{s \downarrow p\} p &= \circ \{s \downarrow p\} \end{aligned}$$

Again the 3rd displayed equation is welltyped because the 1st holds, and again the 1st equation is redundant as derivable from the 2nd equation together with the 1st and 2nd equations of a directed container.

The conversions between the two are given by $\ominus \{s\} p =_{\text{df}} p^{-1}$ and $(s, p)^{-1} =_{\text{df}} (s \downarrow p, \ominus \{s\} p)$.

Intuitively, in a bidirected container, not only can positions of a shape's subshape be translated to it, but also the other way around. Indeed, $\ominus \{s\} p$ should be seen as the translation of the root position of the shape s into the subshape determined by the position p .

We recall that, if a category is a groupoid, then it is isomorphic to its dual. But the converse is not generally true. For example, the small category for the streams comonad is isomorphic to its own dual (as is any category with one object), but it is not a groupoid.

The small category for the nonempty lists and cyclic shifts comonad is a groupoid. In particular, we have $(s, p)^{-1} =_{\text{df}} (s, -p \bmod (s + 1))$. In the corresponding bidirected container, we have $\ominus \{s\} p =_{\text{df}} -p \bmod (s + 1)$.

The small category for the reader comonad is also a groupoid (as a discrete category), via $s^{-1} =_{\text{df}} s$. In the corresponding bidirected container, we have $\ominus \{s\} s =_{\text{df}} s$. The small category for the array comonad is a groupoid via $(s, s')^{-1} =_{\text{df}} (s', s)$. The corresponding bidirected container has $\ominus \{s\} s' =_{\text{df}} s$.

7 Conclusion

We have witnessed that the polynomial view of containers reveals a symmetry between shapes/subshapes of positions in the concept of a directed container—the symmetry between sources/targets of morphisms

in the concept of a category. This makes specific constructions and specializations available for those comonads whose underlying functors are containers. The concept of a directed container morphism however breaks this symmetry; it is also quite intricate. This situation seems to be quite special for containers that interpret into comonads. For instance, containers that interpret into monads do not admit a comparably simple explicit description.

As a continuation of this work, we would like to analyze our previous results on compatible compositions and distributive laws of directed containers [3] from the polynomial viewpoint. We expect that this will lead to a generalization of the concepts of a Zappa-Szép product and a matching pair of actions from monoids to small categories. We would also like to see if some analogs of the construction of the opposite directed container and the concept of a bidirected container are available for comonads directly.

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