

One-dimensional Array Grammars and P Systems with Array Insertion and Deletion Rules

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We consider the (one-dimensional) array counterpart of contextual as well as insertion and deletion string grammars and consider the operations of array insertion and deletion in array grammars. First we show that the emptiness problem for P systems with (one-dimensional) insertion rules is undecidable. Then we show computational completeness of P systems using (one-dimensional) array insertion and deletion rules even of norm one only. The main result of the paper exhibits computational completeness of one-dimensional array grammars using array insertion and deletion rules of norm at most two.

1 Introduction

In the string case, the insertion operation was first considered in [14, 15, 16] and after that related insertion and deletion operations were investigated, e.g., in [17, 18]. Based on linguistic motivations, checking of insertion contexts was considered in [20] with introducing *contextual* grammars; these contextual grammars start from a set of strings (*axioms*), and new strings are obtained by using rules of the form (s, c) , where s and c are strings to be interpreted as inserting c in the context of s , either only at the ends of strings (*external* case, [20]) or in the *interior* of strings ([24]). The fundamental difference between contextual grammars and Chomsky grammars is that in contextual grammars we do not *rewrite* symbols, but we only *adjoin* symbols to the current string, i.e., contextual grammars are pure grammars. Hence, among the variants of these grammars as, for example, considered in [4, 5, 6, 25, 26, 22], the variant where we can retain only the set of strings produced by blocked derivations, i.e., derivations which cannot be continued, is of special importance. This corresponds to the maximal mode of derivation (called t-mode) in cooperating grammar systems (see [2]) as well as to the way results in P systems are obtained by halting computations; we refer the reader to [23, 27] and to the web page [30] for more details on P systems.

With the length of the contexts and/or of the inserted and deleted strings being big enough, the insertion-deletion closure of a finite language leads to computational completeness. There are numerous results establishing the descriptive complexity parameters sufficient to achieve this goal; for an overview of this area we refer to [32, 31]. In [13] it was shown that computational completeness can also be obtained with using only insertions and deletions of just one symbol at the ends of a string using the regulating framework of P systems, where the application of rules depends on the membrane region.

The contextual style of generating strings was extended to d -dimensional arrays in a natural way (see [12, 19]): a contextual array rule is a pair (s, c) of two arrays to be interpreted as inserting the new subarray c in the context of the array s provided that the positions where to put c are not yet occupied by a non-blank symbol. With retaining only the arrays produced in maximal derivations, interesting languages of two-dimensional arrays can be generated. In [7], it was shown that every recursively enumerable one-dimensional array language can be characterized as the projection of an array language generated by a two-dimensional contextual array grammar using rules of norm one only (the norm of a contextual array rule (s, c) is the maximal distance between two positions in the union of the two finite arrays s and c). A contextual array rule (s, c) can be interpreted as array insertion rule; by inverting the meaning of this operation, we get an array deletion rule (s, c) deleting the subarray c in the relative context of the subarray s . In [9], contextual array rules in P systems are considered. P systems using array insertion and deletion rules were investigated in [8], especially for the two-dimensional case, proving computational completeness with using array insertion and deletion rules even of norm one only.

In this paper, we focus on the one-dimensional case. First we show that the emptiness problem for P systems using one-dimensional contextual array rules is undecidable. We adapt the proof from [8] for proving the computational completeness of P systems using array insertion and deletion rules even of norm one only. The main result of the paper exhibits computational completeness of one-dimensional array grammars using array insertion and deletion rules of norm at most two.

2 Definitions and Examples

The set of integers is denoted by \mathbb{Z} , the set of non-negative integers by \mathbb{N} . An *alphabet* V is a finite non-empty set of abstract *symbols*. Given V , the free monoid generated by V under the operation of concatenation is denoted by V^* ; the elements of V^* are called strings, and the *empty string* is denoted by λ ; $V^* \setminus \{\lambda\}$ is denoted by V^+ . Each string $w \in T^+$ can be written as $w(1) \dots w(|w|)$, where $|w|$ denotes the length of w . The family of recursively enumerable string languages is denoted by RE . For more details of formal language theory the reader is referred to the monographs and handbooks in this area such as [3] and [29].

2.1 A General Model for Sequential Grammars

In order to be able to introduce the concept of membrane systems (P systems) for various types of objects, we first define a general model ([11]) of a grammar generating a set of terminal objects by derivations where in each derivation step exactly one rule is applied (sequential derivation mode) to exactly one object.

A (*sequential*) *grammar* G is a construct $(O, O_T, w, P, \Longrightarrow_G)$ where O is a set of *objects*, $O_T \subseteq O$ is a set of *terminal objects*, $w \in O$ is the *axiom (start object)*, P is a finite set of *rules*, and $\Longrightarrow_G \subseteq O \times O$ is the *derivation relation* of G . We assume that each of the rules $p \in P$ induces a relation $\Longrightarrow_p \subseteq O \times O$ with respect to \Longrightarrow_G fulfilling at least the following conditions: (i) for each object $x \in O$, $(x, y) \in \Longrightarrow_p$

for only finitely many objects $y \in O$; (ii) there exists a finitely described mechanism (as, for example, a Turing machine) which, given an object $x \in O$, computes all objects $y \in O$ such that $(x, y) \in \Longrightarrow_p$. A rule $p \in P$ is called *applicable* to an object $x \in O$ if and only if there exists at least one object $y \in O$ such that $(x, y) \in \Longrightarrow_p$; we also write $x \Longrightarrow_p y$. The derivation relation \Longrightarrow_G is the union of all \Longrightarrow_p , i.e., $\Longrightarrow_G = \cup_{p \in P} \Longrightarrow_p$. The reflexive and transitive closure of \Longrightarrow_G is denoted by \Longrightarrow_G^* .

In the following we shall consider different types of grammars depending on the components of G , especially on the rules in P ; these may define a special type X of grammars which then will be called *grammars of type X* .

Usually, the *language generated by G* (in the $*$ -mode) is the set of all terminal objects (we also assume $v \in O_T$ to be decidable for every $v \in O$) derivable from the axiom, i.e., $L_*(G) = \{v \in O_T \mid w \xrightarrow{*}_G v\}$. The *language generated by G in the t -mode* is the set of all terminal objects derivable from the axiom in a halting computation, i.e., $L_t(G) = \{v \in O_T \mid (w \xrightarrow{*}_G v) \wedge \nexists z (v \Longrightarrow_G z)\}$. The family of languages generated by grammars of type X in the derivation mode δ , $\delta \in \{*, t\}$, is denoted by $\mathcal{L}_\delta(X)$. If for every G of type X , $G = (O, O_T, w, P, \Longrightarrow_G)$, we have $O_T = O$, then X is called a *pure type*, otherwise it is called *extended*.

2.2 String grammars

In the general notion as defined above, a *string grammar* G_S is represented as $((N \cup T)^*, T^*, w, P, \Longrightarrow_P)$ where N is the alphabet of *non-terminal symbols*, T is the alphabet of *terminal symbols*, $N \cap T = \emptyset$, $w \in (N \cup T)^+$ is the *axiom*, P is a finite set of *string rewriting rules*, and the derivation relation \Longrightarrow_{G_S} is the classic one for string grammars defined over $V^* \times V^*$, with $V = N \cup T$. As classic types of string grammars we consider string grammars with arbitrary rules of the form $u \rightarrow v$ with $u \in V^+$ and $v \in V^*$ as well as context-free rules of the form $A \rightarrow v$ with $A \in N$ and $v \in V^*$. The corresponding types of grammars are denoted by *ARB* and *CF*, thus yielding the families of languages $\mathcal{L}(ARB)$ and $\mathcal{L}(CF)$, i.e., the family of recursively enumerable languages *RE* and the family of context-free languages, respectively.

In [13], left and right insertions and deletions of strings were considered; the corresponding types of grammars using rules inserting strings of length at most k and deleting strings of length at most m are denoted by $D^m I^k$.

2.3 Array grammars

We now introduce the basic notions for d -dimensional arrays and array grammars in a similar way as in [10, 12]. Let $d \in \mathbb{N}$; then a *d -dimensional array* \mathcal{A} over an alphabet V is a function $\mathcal{A} : \mathbb{Z}^d \rightarrow V \cup \{\#\}$, where $shape(\mathcal{A}) = \{v \in \mathbb{Z}^d \mid \mathcal{A}(v) \neq \#\}$ is finite and $\# \notin V$ is called the *background* or *blank symbol*. We usually write $\mathcal{A} = \{(v, \mathcal{A}(v)) \mid v \in shape(\mathcal{A})\}$. The set of all d -dimensional arrays over V is denoted by V^{*d} . The *empty array* in V^{*d} with empty shape is denoted by Λ_d . Moreover, we define $V^{+d} = V^{*d} \setminus \{\Lambda_d\}$. Let $v \in \mathbb{Z}^d$, $v = (v_1, \dots, v_d)$; the norm of v is defined as $\|v\| = \max\{|v_i| \mid 1 \leq i \leq d\}$. The *translation* $\tau_v : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ is defined by $\tau_v(w) = w + v$ for all $w \in \mathbb{Z}^d$. For any array $\mathcal{A} \in V^{*d}$ we define $\tau_v(\mathcal{A})$, the corresponding d -dimensional array translated by v , by $(\tau_v(\mathcal{A}))(w) = \mathcal{A}(w - v)$ for all $w \in \mathbb{Z}^d$. For a (non-empty) finite set $W \subset \mathbb{Z}^d$ the norm of W is defined as $\|W\| = \max\{\|v - w\| \mid v, w \in W\}$. The vector $(0, \dots, 0) \in \mathbb{Z}^d$ is denoted by Ω_d .

Usually (e.g., see [1, 28, 33]) arrays are regarded as equivalence classes of arrays with respect to linear translations, i. e., only the relative positions of the symbols different from $\#$ in the plane are taken

into account: the equivalence class $[\mathcal{A}]$ of an array $\mathcal{A} \in V^{*d}$ is defined by

$$[\mathcal{A}] = \left\{ \mathcal{B} \in V^{*d} \mid \mathcal{B} = \tau_v(\mathcal{A}) \text{ for some } v \in \mathbb{Z}^d \right\}.$$

The set of all equivalence classes of d -dimensional arrays over V with respect to linear translations is denoted by $[V^{*d}]$ etc.

Let $d_1, d_2 \in \mathbb{N}$ with $d_1 < d_2$. The natural embedding $i_{d_1, d_2} : \mathbb{Z}^{d_1} \rightarrow \mathbb{Z}^{d_2}$ is defined by $i_{d_1, d_2}(v) = (v, \Omega_{d_2-d_1})$ for all $v \in \mathbb{Z}^{d_1}$. To a d_1 -dimensional array $\mathcal{A} \in V^{+d_1}$ with $\mathcal{A} = \{(v, \mathcal{A}(v)) \mid v \in \text{shape}(\mathcal{A})\}$, we assign the d_2 -dimensional array $i_{d_1, d_2}(\mathcal{A}) = \{(i_{d_1, d_2}(v), \mathcal{A}(v)) \mid v \in \text{shape}(\mathcal{A})\}$; moreover, we have $i_{d_1, d_2}(\Lambda_{d_1}) = \Lambda_{d_2}$.

Any one-dimensional array $\mathcal{A} = \{(v, \mathcal{A}(v)) \mid v \in \text{shape}(\mathcal{A})\}$ with $\text{shape}(\mathcal{A}) = \{(m_i) \mid 1 \leq i \leq n\}$, $m_1 < \dots < m_n$, can also be represented as the sequence $(m_1) \mathcal{A}((m_1)) \dots (m_n) \mathcal{A}((m_n))$ and simply as a string $\mathcal{A}((m_1)) \#^{m_2-m_1-1} \dots \mathcal{A}((m_n))$; for example, we may write $[(-2)a(-1)a(3)a(5)a]$ and $a^2 \#^3 a \# a$ for the array $\{((-2), a), ((-1), a), ((3), a), ((5), a)\}$.

A d -dimensional array grammar G_A is represented as

$$\left(\left[(N \cup T)^{*d} \right], \left[T^{*d} \right], [\mathcal{A}_0], P, \Longrightarrow_{G_A} \right)$$

where N is the alphabet of *non-terminal symbols*, T is the alphabet of *terminal symbols*, $N \cap T = \emptyset$, $\mathcal{A}_0 \in (N \cup T)^{*d}$ is the *start array*, P is a finite set of d -dimensional array rules over V , $V = N \cup T$, and $\Longrightarrow_{G_A} \subseteq \left[(N \cup T)^{*d} \right] \times \left[(N \cup T)^{*d} \right]$ is the derivation relation induced by the array rules in P .

A “classical” d -dimensional array rule p over V is a triple $(W, \mathcal{A}_1, \mathcal{A}_2)$ where $W \subseteq \mathbb{Z}^d$ is a finite set and \mathcal{A}_1 and \mathcal{A}_2 are mappings from W to $V \cup \{\#\}$. In the following, we will also write $\mathcal{A}_1 \rightarrow \mathcal{A}_2$, as W is implicitly given by the finite arrays $\mathcal{A}_1, \mathcal{A}_2$. The norm of the d -dimensional array production $(W, \mathcal{A}_1, \mathcal{A}_2)$ is defined by $\|(W, \mathcal{A}_1, \mathcal{A}_2)\| = \|W\|$. We say that the array $\mathcal{C}_2 \in V^{*d}$ is *directly derivable* from the array $\mathcal{C}_1 \in V^{+d}$ by $(W, \mathcal{A}_1, \mathcal{A}_2)$ if and only if there exists a vector $v \in \mathbb{Z}^d$ such that $\mathcal{C}_1(w) = \mathcal{C}_2(w)$ for all $w \in \mathbb{Z}^d - \tau_v(W)$ as well as $\mathcal{C}_1(w) = \mathcal{A}_1(\tau_{-v}(w))$ and $\mathcal{C}_2(w) = \mathcal{A}_2(\tau_{-v}(w))$ for all $w \in \tau_v(W)$, i.e., the subarray of \mathcal{C}_1 corresponding to \mathcal{A}_1 is replaced by \mathcal{A}_2 , thus yielding \mathcal{C}_2 ; we also write $\mathcal{C}_1 \Longrightarrow_p \mathcal{C}_2$. Moreover we say that the array $\mathcal{B}_2 \in [V^{*d}]$ is *directly derivable* from the array $\mathcal{B}_1 \in [V^{+d}]$ by the d -dimensional array production $(W, \mathcal{A}_1, \mathcal{A}_2)$ if and only if there exist $\mathcal{C}_1 \in \mathcal{B}_1$ and $\mathcal{C}_2 \in \mathcal{B}_2$ such that $\mathcal{C}_1 \Longrightarrow_p \mathcal{C}_2$; we also write $\mathcal{B}_1 \Longrightarrow_p \mathcal{B}_2$.

A d -dimensional array rule $p = (W, \mathcal{A}_1, \mathcal{A}_2)$ in P is called *monotonic* if $\text{shape}(\mathcal{A}_1) \subseteq \text{shape}(\mathcal{A}_2)$ and *#-context-free* if $\text{shape}(\mathcal{A}_1) = \{\Omega_d\}$; if it is *#-context-free* and, moreover, $\text{shape}(\mathcal{A}_2) = W$, then p is called *context-free*. A d -dimensional array grammar is said to be of type X , $X \in \{d\text{-ARBA}, d\text{-MONA}, d\text{-}\#\text{-CFA}, d\text{-CFA}\}$ if every array rule in P is of the corresponding type, the corresponding families of array languages of equivalence classes of d -dimensional arrays by d -dimensional array grammars are denoted by $\mathcal{L}_*(X)$. These families form a Chomsky-like hierarchy, i.e., $\mathcal{L}_*(d\text{-CFA}) \subsetneq \mathcal{L}_*(d\text{-MONA}) \subsetneq \mathcal{L}_*(d\text{-ARBA})$ and $\mathcal{L}_*(d\text{-CFA}) \subsetneq \mathcal{L}_*(d\text{-}\#\text{-CFA}) \subsetneq \mathcal{L}_*(d\text{-ARBA})$.

Two d -dimensional arrays \mathcal{A} and \mathcal{B} in $[V^{*d}]$ are called *shape-equivalent* if and only if $\text{shape}(\mathcal{A}) = \text{shape}(\mathcal{B})$. Two d -dimensional array languages L_1 and L_2 from $[V^{*d}]$ are called *shape-equivalent* if and only if $\{\text{shape}(\mathcal{A}) \mid \mathcal{A} \in L_1\} = \{\text{shape}(\mathcal{B}) \mid \mathcal{B} \in L_2\}$.

2.4 Contextual, Insertion and Deletion Array Rules

A d -dimensional contextual array rule (see [12]) over the alphabet V is a pair of finite d -dimensional arrays $((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2))$ where $W_1 \cap W_2 = \emptyset$ and $\text{shape}(\mathcal{A}_1) \cup \text{shape}(\mathcal{A}_2) \neq \emptyset$. The effect of this

contextual rule is the same as of the array rewriting rule $(W_1 \cup W_2, \mathcal{A}_1, \mathcal{A}_1 \cup \mathcal{A}_2)$, i.e., in the context of \mathcal{A}_1 we insert \mathcal{A}_2 . Hence, such an array rule $((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2))$ can also be called an *array insertion rule*, and then we write $I((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2))$; if $\text{shape}(\mathcal{A}_i) = W_i, i \in \{1, 2\}$, we simply write $I(\mathcal{A}_1, \mathcal{A}_2)$. Yet we may also interpret the pair $((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2))$ as having the effect of the array rewriting rule $\mathcal{A}_1 \cup \mathcal{A}_2 \rightarrow \mathcal{A}_1$, i.e., in the context of \mathcal{A}_1 we delete \mathcal{A}_2 ; in this case, we speak of an *array deletion rule* and write $D((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2))$ or, if $\text{shape}(\mathcal{A}_i) = W_i, i \in \{1, 2\}$, then even only $D(\mathcal{A}_1, \mathcal{A}_2)$. For any (contextual, insertion, deletion) array rule $((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2))$ we define its norm by

$$\|((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2))\| = \|W_1 \cup W_2\|.$$

The norm of the set of contextual array productions in G , $\|P\|$, is defined by

$$\|P\| = \max \{ \|W_1 \cup W_2\| \mid ((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2)) \in P \}.$$

Let G_A be a d -dimensional array grammar $([V^{*d}], [T^{*d}], [\mathcal{A}_0], P, \Longrightarrow_{G_A})$ with P containing array insertion and deletion rules. The norm of the set of array insertion and deletion rules in G , $\|P\|$, is defined as for a set of contextual array productions, i.e., we again define

$$\|P\| = \max \{ \|W_1 \cup W_2\| \mid ((W_1, \mathcal{A}_1), (W_2, \mathcal{A}_2)) \in P \}.$$

For G_A we consider the array languages $L_*(G_A)$ and $L_t(G_A)$ generated by G_A in the modes $*$ and t , respectively; the corresponding families of array languages are denoted by $\mathcal{L}_\delta(d\text{-DIA})$, $\delta \in \{*, t\}$; if only array insertion (i.e., contextual) rules are used, we have the case of pure grammars, and we also write $\mathcal{L}_\delta(d\text{-CA})$. For interesting relations between the families of array languages $\mathcal{L}_*(d\text{-CA})$ and $\mathcal{L}_t(d\text{-CA})$ as well as $\mathcal{L}_*(d\text{-\#CFA})$ and $\mathcal{L}_*(d\text{-CFA})$ we refer the reader to [12].

In the following, instead of using the notation $([V^{*d}], [T^{*d}], [\mathcal{A}_0], P, \Longrightarrow_{G_A})$ for a d -dimensional array grammar of a specific type, we may also simply write $(V, T, [\mathcal{A}_0], P)$. For contextual array grammars or for array grammars only containing array insertion rules we may even write $(V, [\mathcal{A}_0], P)$.

Our first example shows how we can generate one-dimensional arrays of the form $LE^n \bar{S} E^m R$, $n, m \geq 1$, with a contextual array grammar containing only rules of norm 1:

Example 1 Consider the contextual array grammar

$$G_{line} = (\{\bar{S}, E, L, R\}, E\bar{S}E, P) \text{ with } P = \left\{ \boxed{E}E, E\boxed{E}, \boxed{E}R, L\boxed{E} \right\};$$

in order to represent the contextual array (or array insertion and deletion) rules in a depictive way, the symbols of the selector are enclosed in boxes). Obviously, $\|P\| = 1$. Starting from the axiom $E\bar{S}E$, the sequence of symbols E is prolonged to the right by the contextual array rule $\boxed{E}E$ and prolonged to the left by the contextual array rule $E\boxed{E}$. The derivation only halts as soon as we have used both the rules $\boxed{E}R$ and $L\boxed{E}$ to introduce the right and left endmarkers R and L , respectively. In sum, we obtain

$$[L_t(G_{line})] = \{LE^n \bar{S} E^m R \mid n, m \geq 1\},$$

whereas

$$[L_*(G_{line})] = \{LE^n \bar{S} E^m R, E^n \bar{S} E^m R, LE^n \bar{S} E^m, E^n \bar{S} E^m \mid n, m \geq 1\}.$$

3 (Sequential) P Systems

For controlling the derivations in an array grammar with array insertion and deletion rules, in [8] the model of sequential P systems using array insertion and deletion rules was considered. In general, for arbitrary types of underlying grammars, P systems are defined as follows:

A (sequential) P system of type X with tree height n is a construct $\Pi = (G, \mu, R, i_0)$ where

- $G = (O, O_T, A, P, \Longrightarrow_G)$ is a sequential grammar of type X ;
- μ is the membrane (tree) structure of the system with the height of the tree being n (μ usually is represented by a string containing correctly nested marked parentheses); we assume the membranes to be the nodes of the tree representing μ and to be uniquely labelled by labels from a set Lab ;
- R is a set of rules of the form (h, r, tar) where $h \in Lab$, $r \in P$, and tar , called the *target indicator*, is taken from the set $\{here, in, out\} \cup \{in_h \mid h \in Lab\}$; R can also be represented by the vector $(R_h)_{h \in Lab}$, where $R_h = \{(r, tar) \mid (h, r, tar) \in R\}$ is the set of rules assigned to membrane h ;
- i_0 is the initial membrane containing the axiom A .

As we only have to follow the trace of a single object during a computation of the P system, a configuration of Π can be described by a pair (w, h) where w is the current object (e.g., string or array) and h is the label of the membrane currently containing the object w . For two configurations (w_1, h_1) and (w_2, h_2) of Π we write $(w_1, h_1) \Longrightarrow_{\Pi} (w_2, h_2)$ if we can pass from (w_1, h_1) to (w_2, h_2) by applying a rule $(h_1, r, tar) \in R$, i.e., $w_1 \Longrightarrow_r w_2$ and w_2 is sent from membrane h_1 to membrane h_2 according to the target indicator tar . More specifically, if $tar = here$, then $h_2 = h_1$; if $tar = out$, then the object w_2 is sent to the region h_2 immediately outside membrane h_1 ; if $tar = in_{h_2}$, then the object is moved from region h_1 to the region h_2 immediately inside region h_1 ; if $tar = in$, then the object w_2 is sent to one of the regions immediately inside region h_1 .

A sequence of transitions between configurations of Π , starting from the initial configuration (A, i_0) , is called a *computation* of Π . A *halting computation* is a computation ending with a configuration (w, h) such that no rule from R_h can be applied to w anymore; w is called the *result* of this halting computation if $w \in O_T$. As the language generated by Π we consider $L_t(\Pi)$ which consists of all terminal objects from O_T being results of a halting computation in Π .

By $\mathcal{L}_t(X-LP)$ ($\mathcal{L}_t(X-LP^{(n)})$) we denote the family of languages generated by P systems (of tree height at most n) using grammars of type X . If only the targets *here*, *in*, and *out* are used, then the P system is called *simple*, and the corresponding families of languages are denoted by $\mathcal{L}_t(X-LsP)$ ($\mathcal{L}_t(X-LsP^{(n)})$).

In the string case (see [13]), the operations of left and right insertion (I) of strings of length m and left and right deletion (D) of strings of length k were investigated; the corresponding types are abbreviated by $D^k I^m$. Every language $L \subseteq T^*$ in $\mathcal{L}_*(D^1 I^1)$ can be written in the form $T_l^* S T_r^*$ where $T_l, T_r \subseteq T$ and S is a finite subset of T^* . Using the regulating mechanism of P systems, we get $\{a^{2^n} \mid n \geq 0\} \in \mathcal{L}_t(D^1 I^2-LP^{(1)})$ and even obtain computational completeness:

Theorem 1 (see [13]) $\mathcal{L}_t(D^1 I^1-LsP^{(8)}) = RE$.

One-dimensional arrays can also be interpreted as strings; left/right insertion of a symbol a corresponds to taking the set containing all rules $I\left(a \begin{array}{|c|} \hline b \\ \hline \end{array}\right) / I\left(\begin{array}{|c|} \hline b \\ \hline \end{array} a\right)$ for any b ; left/right deletion of

a symbol a corresponds to taking the rule $D\left(\boxed{\#} \ a\right)/D\left(a \ \boxed{\#}\right)$; these array insertion and deletion rules have norm one, but the array deletion rules also sense for the blank symbol $\#$ in the selector. Hence, from Theorem 1, we immediately infer the following result, with $D^k I^m A$ denoting the type of array grammars using array deletion and insertion rules of norms at most k and m , respectively:

Corollary 2 $\mathcal{L}_t(1-D^1 I^1 A-LsP^{(8)}) = \mathcal{L}_*(1-ARBA)$.

With respect to the tree height of the simple P systems, this result will be improved considerably in Section 4. One-dimensional array grammars with only using array insertion and deletion rules of norm at most two will be shown to be computationally complete in Section 5.

3.1 Encoding the Post Correspondence Problem With Array Insertion P Systems

An instance of the *Post Correspondence Problem* is a pair of sequences of non-empty strings (u_1, \dots, u_n) and (v_1, \dots, v_n) over an alphabet T . A solution of this instance is a sequence of indices i_1, \dots, i_k such that $u_{i_1} \dots u_{i_k} = v_{i_1} \dots v_{i_k}$; we call $u_{i_1} \dots u_{i_k}$ the result of this solution. Let

$$\{u_{i_1} \dots u_{i_k} \mid i_1, \dots, i_k \text{ is a solution of } ((u_1, \dots, u_n), (v_1, \dots, v_n))\}$$

be the set of results of all solutions of the instance $((u_1, \dots, u_n), (v_1, \dots, v_n))$ of the Post Correspondence Problem, denoted by $L((u_1, \dots, u_n), (v_1, \dots, v_n))$.

We now show how $L((u_1, \dots, u_n), (v_1, \dots, v_n))$ can be represented in a very specific way as the language generated by an array insertion P system. Consider the homomorphism h_Σ defined by $h_\Sigma : \Sigma \rightarrow \Sigma\Sigma'$ with $h_\Sigma(a) = aa'$ for all $a \in \Sigma$.

Lemma 3 *Let $I = ((u_1, \dots, u_n), (v_1, \dots, v_n))$ be an instance of the Post Correspondence Problem over T . Then we can effectively construct a one-dimensional array insertion P system Π such that*

$$[L(\Pi)] = \{LL'h_T(w)RR' \mid w \in L((u_1, \dots, u_n), (v_1, \dots, v_n))\}.$$

Proof. The main idea for constructing the one-dimensional array insertion P system

$$\Pi = (G, [{}_0 [{}_1]_1 \dots [{}_n]_n \dots [{}_{n+1}]_{n+1}]_0, R, 0)$$

with

$$G = \left(\{F, L, R, L', R'\} \cup T \cup T', LL', P \cup \left\{ \boxed{a} \boxed{a'} RR' \mid a \in T \right\} \right)$$

is to generate sequences $u_{i_1} \dots u_{i_k}$ and $(v_{i_1})' \dots (v_{i_k})'$ for sequences of indices i_1, \dots, i_k in an interleaving way, i.e., each symbol $a \in T$ from the first sequence is followed by the corresponding primed symbol a' in the second sequence; as soon as both sequences have reached the same length, i.e., if we have got the encoding of a solution for this instance of the Post Correspondence Problem, we may use a fitting contextual array rule $\boxed{a} \boxed{a'} RR'$ for some $a \in T$ to stop the derivation in Π .

For generating these interleaving sequences of symbols a and a' we take the following rules into P :

For prolonging the first sequence by u_i , $u_i = u_i(1) \dots u_i(|u_i|)$, we add all rules of the form

$$\boxed{u_i(0)} \boxed{u_i(0)''} u_i(1) \boxed{u_i(1)''} \dots u_i(|u_i|) \boxed{u_i(|u_i|)''}$$

with $u_i(0) \in T \cup \{L\}$ and $u_i(j)'' \in \{u_i(j)', \#\}$, $0 \leq j \leq |u_i|$, $1 \leq i \leq n$, fulfilling the following constraints:

- if $u_i(0) = L$ then $u_i(0)'' = L'$;
- if $u_i(k)'' = \#$ for some $k \geq 0$, then $u_i(j)'' = \#$ for all $k \leq j \leq |u_i|$;
- if $u_i(k)'' = u_i(k)'$ for some $k \geq 0$, then $u_i(j)'' = u_i(j)'$ for all $j \leq k$.

In the same way, for prolonging the second sequence by v_i , $v_i = v_i(1) \dots v_i(|v_i|)$, represented with primed symbols, we add all rules of the form

$$\boxed{v_i(0)''} \boxed{v_i(0)'} \boxed{v_i(1)''} v_i(1)' \dots \boxed{v_i(|v_i|)''} v_i(|v_i|)'$$

with $v_i(0) \in T \cup \{L\}$ and $v_i(j)'' \in \{v_i(j), \#\}$, $0 \leq j \leq |v_i|$, $1 \leq i \leq n$, fulfilling the following constraints:

- if $v_i(0)' = L'$ then $v_i(0)'' = L$;
- if $v_i(k)'' = \#$ for some $k \geq 0$, then $v_i(j)'' = \#$ for all $k \leq j \leq |v_i|$;
- if $v_i(k)'' = v_i(k)$ for some $k \geq 0$, then $v_i(j)'' = v_i(j)$ for all $j \leq k$.

The set R consists of the following rules:

Starting from the axiom LL' , in membrane region 0 we have all rules

$$\left(0, I \left(\boxed{u_i(0)} \boxed{u_i(0)''} u_i(1) \boxed{u_i(1)''} \dots u_i(|u_i|) \boxed{u_i(|u_i|)''} \right), in_i \right),$$

i.e., when adding the sequence corresponding to the string u_i , the resulting array is sent into membrane i , where the sequence of primed strings corresponding to the string v_i is added and the resulting array is sent out again into the skin region 0 using any of the rules

$$\left(i, I \left(\boxed{v_i(0)''} \boxed{v_i(0)'} \boxed{v_i(1)''} v_i(1)' \dots \boxed{v_i(|v_i|)''} v_i(|v_i|)' \right), out \right).$$

For the cases when no fitting rules for prolonging the array exist, we take the rules

$$\left(0, I \left(\boxed{X} \boxed{\#} F \right), here \right)$$

for any $X \in T \cup \{F\}$ and

$$\left(i, I \left(\boxed{X} \boxed{\#} F \right), here \right)$$

for any $X \in T' \cup \{L', F\}$ and $1 \leq i \leq n$.

The observation that (only) the application of an array insertion rule

$$\left(0, I \left(\boxed{a} \boxed{a'} RR' \right), in_{n+1} \right)$$

for some $a \in T$ stops the derivation, with sending the terminal array into membrane $n+1$, completes the proof. ■

As is well known (see [21]), the Post Correspondence Problem is undecidable, hence, the emptiness problem for $\mathcal{L}_t(DI-LP^{(1)})$ is undecidable:

Corollary 4 *For any $k \geq 1$, the emptiness problem for $\mathcal{L}_t(DI-LP^{(k)})$ is undecidable.*

For $d \geq 2$, even the emptiness problem for $\mathcal{L}_t(d-CA)$ is undecidable, which follows from the result obtained in [7], where it was shown that every recursively enumerable one-dimensional array language can be characterized as the projection of an array language generated by a two-dimensional contextual array grammar using rules of norm one only.

4 Computational Completeness of Array Insertion and Deletion P Systems Using Rules with Norm at Most One

We now show the first of our main results: any recursively enumerable one-dimensional array language can be generated by an array insertion and deletion P system which only uses rules of norm at most one and the targets *here*, *in*, and *out* and whose membrane structure has only tree height 2; for two-dimensional array languages, the corresponding result was established in [8].

Theorem 5 $\mathcal{L}_t(1-D^1I^1A-LsP^{(2)}) = \mathcal{L}_*(1-ARBA)$.

Proof. The main idea of the proof is to construct the simple P system Π of type 1-DIA with a membrane structure of height two generating a recursively enumerable one-dimensional array language L_A given by a grammar G_A of type 1-ARBA in such a way that we first generate the “workspace”, i.e., the lines as described in Example 1 and then simulate the rules of the one-dimensional array grammar G_A inside this “workspace”; finally, the superfluous symbols E and L, R have to be erased to obtain the terminal array.

Now let $G_A = \left(\left[(N \cup T)^{*1} \right], [T^{*1}], [\mathcal{A}_0], P, \Longrightarrow_{G_A} \right)$ be an array grammar of type 1-ARBA generating L_A . In order to make the simulation in Π easier, without loss of generality, we may make some assumptions about the forms of the array rules in P : First of all, we may assume that the array rules are in a kind of Chomsky normal form (e.g., compare [10]), i.e., only of the following forms: $A \rightarrow B$ for $A \in N$ and $B \in N \cup T \cup \{\#\}$ as well as $AvD \rightarrow BvC$ with $\|v\| = 1$ (i.e., $v \in \{(1), (-1)\}$), $A, B, C \in N \cup T$, and $D \in N \cup T \cup \{\#\}$ (we would like to emphasize that usually A, B, C, D in the array rule $AvD \rightarrow BvC$ are not allowed to be terminal symbols); in a more formal way, the rule $AvD \rightarrow BvC$ represents the rule $(W, \mathcal{A}_1, \mathcal{A}_2)$ with $W = \{(0), v\}$, $\mathcal{A}_1 = \{((0), A), (v, D)\}$, and $\mathcal{A}_2 = \{((0), B), (v, C)\}$. As these rules in fact are simulated in Π with the symbol E representing the blank symbol $\#$, a rule $Av\# \rightarrow BvC$ now corresponds to a rule $AvE \rightarrow BvC$. Moreover, a rule $A \rightarrow B$ for $A \in N$ and $B \in N \cup T$ can be replaced by the set of all rules $AvD \rightarrow BvD$ for all $D \in N \cup T \cup \{E\}$ and $v \in \{(1), (-1)\}$, and $A \rightarrow \#$ can be replaced by the set of all rules $AvD \rightarrow EvD$ for all $D \in N \cup T \cup \{E\}$ and $v \in \{(1), (-1)\}$.

After these replacements described above, in the P system Π we now only have to simulate rules of the form $AvD \rightarrow BvC$ with $v \in \{(1), (-1)\}$ as well as $A, B, C, D \in N \cup T \cup \{E\}$. Yet in order to obtain a P system Π with the required features, we make another assumption for the rules to be simulated: any intermediate array obtained during a derivation contains exactly one symbol marked with a bar; as we only have to deal with sequential systems where at each moment exactly one rule is going to be applied, this does not restrict the generative power of the system as long as we can guarantee that the marking can be moved to any place within the current array. Instead of a rule $AvD \rightarrow BvC$ we therefore take the corresponding rule $\bar{A}vD \rightarrow Bv\bar{C}$; moreover, to move the bar from one position in the current array to another position, we add all rules $\bar{A}vC \rightarrow Av\bar{C}$ for all $A, C \in N \cup T \cup \{E\}$ and $v \in \{(1), (-1)\}$. We collect all these rules obtained in the way described so far in a set of array rules P' and assume them to be uniquely labelled by labels from a set of labels Lab' , i.e., $P' = \{l : \bar{A}_lvD_l \rightarrow B_lv\bar{C}_l \mid l \in Lab'\}$.

After all these preparatory steps we now are able to construct the simple P system Π with array insertion and deletion rules:

$$\Pi = \left(G, \left[\begin{array}{c} 0 \\ l_1 \quad l_2 \\ l_2 \end{array} \right]_{l_1} \cdots \left[\begin{array}{c} l_1 \quad l_2 \\ l_2 \end{array} \right]_{l_1} \cdots \left[\begin{array}{c} F_1 \quad F_2 \\ F_2 \\ F_1 \end{array} \right]_{F_1} \right]_0, R, I_2$$

with I_1 and I_2 being the membranes for generating the initial lines, F_1 and F_2 are the membranes to extract the final terminal arrays in halting computations, and l_1 and l_2 for all $l \in Lab'$ are the membranes to simulate the corresponding array rule from P' labelled by l . The components of the underlying array grammar G can easily be collected from the description of the rules in R as described below.

We start with the initial array $\mathcal{A}_0 = E\bar{S}E$ from Example 1 and take all rules

$(I_2, I(r), \text{here})$

with all rules $r \in \{\boxed{E}E, E\boxed{E}\}$, taken as array insertion rules; using the insertion rule

$(I_2, I(L\boxed{E}), \text{out})$

we get out of membrane I_2 into membrane region I_1 , and by using

$(I_1, I(\boxed{E}R), \text{out})$

we move the initial line $LE^n\bar{S}E^mR$ for some $n, m \geq 1$ out into the skin membrane.

To be able to simulate a derivation from G_A for a specific terminal array, the workspace in this initial line has to be large enough, but as we can generate such lines with arbitrary size, such an initial array can be generated for any terminal array in $L_*(G_A)$.

An array rule from $P' = \{l : \bar{A}_l \nu D_l \rightarrow B_l \nu \bar{C}_l \mid l \in Lab'\}$ is simulated by applying the following sequence of array insertion and deletion rules in the membranes l_1 and l_2 , which send the array twice the path from the skin membrane to membrane l_2 via membrane l_1 and back to the skin membrane:

$$\begin{aligned} & (0, I(\boxed{R}K_l), \text{in}), (l_1, D(\boxed{\bar{A}}_l \nu D_l), \text{in}), \\ & (l_2, I(\boxed{\bar{A}}_l \nu \bar{D}_l^{(l)}), \text{out}), (l_1, D(\boxed{\bar{D}}_l^{(l)} (-\nu) \bar{A}_l), \text{out}), \\ & (0, I(\boxed{\bar{D}}_l^{(l)} (-\nu) B_l), \text{in}), (l_1, D(\boxed{B}_l \nu \bar{D}_l^{(l)}), \text{in}), \\ & (l_2, D(\boxed{R}K_l), \text{out}), (l_1, I(\boxed{B}_l \nu \bar{C}_l), \text{out}). \end{aligned}$$

Whenever reaching the skin membrane, the current array contains exactly one barred symbol. If we reach any of the membranes l_1 and/or l_2 with the wrong symbols (which implies that none of the rules listed above is applicable), we introduce the trap symbol F by the rules

$$(m, I(F\boxed{L}), \text{out}) \text{ and } (m, I(F\boxed{F}), \text{out})$$

for $m \in \{l_1, l_2 \mid l \in Lab' \cup \{I\}\}$; as soon as F has been introduced once, with

$$(0, I(F\boxed{F}), \text{in})$$

we can guarantee that the computation in Π will never stop.

As soon as we have obtained an array representing a terminal array, the corresponding array computed in Π is moved into membrane F_1 by the rule $(0, D(R), \text{in})$ (for any X , $D(X) / I(K)$ just means deleting/inserting X without taking care of the context). In membrane F_1 , the left endmarker L and all superfluous symbols E as well as the marked blank symbol \bar{E} (without loss of generality we may assume that at the end of the simulation of a derivation from G_A in Π the marked symbol is \bar{E}) are erased by using the rules $(F_1, D(X), \text{here})$ with $X \in \{E, \bar{E}, L\}$. The computation in Π halts with yielding a terminal array in membrane F_1 if and only if no other non-terminal symbols have occurred in the array we have moved into F_1 ; in the case that non-terminal symbols occur, we start an infinite loop between membrane F_1 and membrane F_2 by introducing the trap symbol F :

$$(F_1, D(X), \text{in}) \text{ for } X \notin T \cup \{E, \bar{E}, L\} \text{ and } (F_2, I(F), \text{out}).$$

As can be seen from the description of the rules in Π , we can simulate all terminal derivations in G_A by suitable computations in Π , and a terminal array \mathcal{A} is obtained as the result of a halting computation (always in membrane F_1) if and only if $\mathcal{A} \in L_*(G_A)$; hence, we conclude $L_t(\Pi) = L_*(G_A)$. ■

5 Computational Completeness of Array Grammars with Array Insertion and Deletion Rules with Norms of at Most Two

When allowing array insertion and deletion rules with norms of at most two, computational completeness can even be obtained without any additional control mechanism (as for example, using P systems as considered in Section 4).

Theorem 6 $\mathcal{L}_l(1-D^2I^2A) = \mathcal{L}_*(1-ARBA)$.

Proof. The main idea of the proof is to construct a one-dimensional array grammar with array insertion and deletion rules simulating the actions of a Turing machine M_A with a bi-infinite tape which generates a given one-dimensional array language (we identify each position of the tape with the corresponding position in \mathbb{Z} ; in that sense, M_A can be seen as a machine generating arrays). Let $M_A = (Q, V, T, \delta, q_0, q_f)$ where Q is a finite set of states, V is the tape alphabet, $T \subseteq V$ is the input alphabet, δ is the transition function, q_0 is the initial state, and q_f is the final state. The Turing machine starts on the empty tape, which means that on each position there is the blank symbol represented by the special symbol $E \in V$.

We now construct the one-dimensional array grammar $G_A = (V', T, P, A)$ with

$$\begin{aligned} V' &= V \cup \{L, R, L', R', E', F\} \\ &\cup \{[AqXD] \mid A \in V \cup \{L\}, q \in Q \cup \{q_f\}, X \in V, D \in V \cup \{R\}\}, \\ A &= LE[Eq_0EE]ER \end{aligned}$$

and with the set of array insertion and deletion rules P constructed according to the following ‘‘program’’:

The simulation of a computation of M_A starts with the axiom $LE[Eq_0EE]ER$; L and R are the left and the right endmarker, respectively. Throughout the whole simulation, the position of the (read/write-)head of the Turing machine M_A is marked by the special symbol $[AqXD]$ indicating that the head currently is on a symbol X with an A to its left and a D to its right. Whenever new ‘‘workspace’’ is needed, L or R are moved one position to the left or right, respectively, at the same time inserting another E :

$$\begin{aligned} &I\left(\boxed{[AqXE]}\boxed{R}\boxed{R}\right), D\left(\boxed{[AqXE]}\boxed{R}\boxed{R}\right), I\left(\boxed{[AqXE]}\boxed{E}\boxed{R}\right); \\ &I\left(\boxed{L}\boxed{L}\boxed{[EqXD]}\right), D\left(\boxed{L}\boxed{L}\boxed{[EqXD]}\right), I\left(\boxed{L}\boxed{E}\boxed{[EqXD]}\right). \end{aligned}$$

Any transition $(p, Y, R) \in \delta(q, X)$ (reading X in state q , M_A enters state p , rewrites X by Y and moves its head one position to the right) is simulated by the rules

$$\begin{aligned} &D\left(\boxed{A}\boxed{[AqXD]}\boxed{D}\right), I\left(\boxed{[AqXD]}\boxed{[YpDC]}\boxed{C}\right), \\ &D\left(\boxed{[AqXD]}\boxed{[YpDC]}\boxed{C}\right), I\left(\boxed{Y}\boxed{[YpDC]}\boxed{C}\right); \end{aligned}$$

any transition $(p, Y, L) \in \delta(q, X)$ (reading X in state q , M_A enters state p , rewrites X by Y and moves its head one position to the left) is simulated by the rules

$$\begin{aligned} &D\left(\boxed{A}\boxed{[AqXD]}\boxed{D}\right), I\left(\boxed{C}\boxed{[CpAY]}\boxed{[AqXD]}\right), \\ &D\left(\boxed{C}\boxed{[CpAY]}\boxed{[AqXD]}\right), I\left(\boxed{C}\boxed{[CpAY]}\boxed{Y}\right). \end{aligned}$$

As soon as M_A has reached the final state q_f (without loss of generality, we may assume that this state is the only one where M_A may halt), we start the final procedure in G_A to obtain the terminal array: First, we go to the left until q_f reaches the left border L of the workspace, delete the left endmarker L and go into the state q'_f :

$$\begin{aligned} & I\left(L \boxed{L} \boxed{[Lq_fXD]}\right), D\left(\boxed{L'} \boxed{L} \boxed{[Lq_fXD]}\right), \\ & I\left(\boxed{L'} \boxed{[Eq'_fEX]} \boxed{[Lq_fXD]}\right), D\left(\boxed{L'} \boxed{[Eq'_fEX]} \boxed{[Lq_fXD]}\right), \\ & I\left(\boxed{L'} \boxed{[Eq'_fEX]} \boxed{X}\right), D\left(\boxed{L'} \boxed{[Eq'_fEX]} \boxed{X}\right), I\left(\boxed{E'} \boxed{[Eq'_fEX]} \boxed{X}\right). \end{aligned}$$

With state q'_f , G_A now goes to the right, keeping each $a \in T$, but replacing E by E' (these rules are the same as if simulating the corresponding transitions in a Turing machine, hence, we do not specify them here) until the right endmarker R is reached:

$$\begin{aligned} & I\left(\boxed{[Aq'_fER]} \boxed{R} \boxed{R'}\right), D\left(\boxed{[Aq'_fER]} \boxed{R} \boxed{R'}\right), \\ & I\left(\boxed{[Aq'_fER]} \boxed{E'} \boxed{R'}\right), D\left(\boxed{[Aq'_fER]} \boxed{E'} \boxed{R'}\right), \\ & I\left(\boxed{E'} \boxed{E'} \boxed{R'}\right), D\left(\boxed{E'} \boxed{E'} \boxed{R'}\right). \end{aligned}$$

After the deletion of R' , only the symbols E' remain to be deleted by the array deletion rule $D(E)$.

Whenever something goes wrong in the process of simulating the transitions of M_A in G_A , the application of a trap rule will be enforced, yielding an unbounded sequence of trap symbols F to the right:

$$I\left(\boxed{R} \boxed{F}\right), I\left(\boxed{R'} \boxed{F}\right), I\left(\boxed{F} \boxed{F}\right).$$

As can be seen from the description of the rules in G_A , we can simulate all terminal computations in M_A by suitable derivations in G_A , and a terminal array \mathcal{A} is obtained as the result of a halting computation if and only if $\mathcal{A} \in L_t(G_A)$; hence, we conclude $L_t(G_A) = L(M_A)$. \blacksquare

6 Conclusion

Array insertion grammars have already been considered as contextual array grammars in [12], whereas the inverse interpretation of a contextual array rule as a deletion rule has newly been introduced in [9], which continued the research on P systems with left and right insertion and deletion of strings, see [13].

In the main part of our paper, we have restricted ourselves to exhibit examples of one-dimensional array languages that can be generated by array insertion (contextual array) grammars as well as to show that array insertion and deletion P systems using rules with norm at most one and even array grammars only using array insertion and deletion rules with norm at most two are computationally complete.

In [9], the corresponding computational completeness result has been shown for two-dimensional array insertion and deletion P systems using rules with norm at most one. It remains as an interesting question for future research whether the result for array grammars only using array insertion and deletion rules with norm at most two can also be achieved in higher dimensions, but at least for dimension two.

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