On the Lambek Calculus with an Exchange Modality

Jiaming Jiang
Computer Science
North Carolina State University
Raleigh, North Carolina, USA
jjiang13@ncsu.edu

Harley Eades III
Computer Science
Augusta University
Augusta, Georgia, USA
harley.eades@gmail.com

Valeria de Paiva
Nuance Communications
Sunnyvale, California, USA
valeria.depaiva@gmail.com

In this paper we introduce Commutative/Non-Commutative Logic (CNC logic) and two categorical models for CNC logic. This work abstracts Benton's Linear/Non-Linear Logic [4] by removing the existence of the exchange structural rule. One should view this logic as composed of two logics; one sitting to the left of the other. On the left, there is intuitionistic linear logic, and on the right is a mixed commutative/non-commutative formalization of the Lambek calculus. Then both of these logics are connected via a pair of monoidal adjoint functors. An exchange modality is then derivable within the logic using the adjunction between both sides. Thus, the adjoint functors allow one to pull the exchange structural rule from the left side to the right side. We then give a categorical model in terms of a monoidal adjunction, and then a concrete model in terms of dialectica Lambek spaces.

1 Introduction

Joachim Lambek first introduced the Syntactic Calculus, now known as the Lambek Calculus, in 1958 [14]. Since then the Lambek Calculus has largely been motivated by providing an explanation of the mathematics of sentence structure, and can be found at the core of Categorical Grammar; a term first used in the title of Bar-Hillel, Gaifman and Shamir (1960), but categorical grammar began with Ajdukiewicz (1935) quite a few years earlier. At the end of the eighties the Lambek calculus and other systems of categorical grammars were taken up by computational linguists as exemplified by [18, 17, 2, 10].

It was computational linguists who posed the question of whether it is possible to isolate exchange using a modality in the same way that the of-course modality of linear logic, $!A$, isolates weakening and contraction. de Paiva and Eades [8] propose one solution to this problem by extending the Lambek calculus with the modality characterized by the following sequent calculus inference rules:

$$\frac{\kappa\Gamma \vdash B}{\kappa\Gamma \vdash \kappa B}\,\text{E}_\text{R} \qquad \frac{\Gamma_1, A, \Gamma_2 \vdash B}{\Gamma_1, \kappa A, \Gamma_2 \vdash B}\,\text{E}_\text{L} \qquad \frac{\Gamma_1, \kappa A, B, \Gamma_2 \vdash C}{\Gamma_1, B, \kappa A, \Gamma_2 \vdash C}\,\text{E1} \qquad \frac{\Gamma_1, A, \kappa B, \Gamma_2 \vdash C}{\Gamma_1, \kappa B, A, \Gamma_2 \vdash C}\,\text{E2}$$

The thing to note is that the modality κA appears on only one of the operands being exchanged. That is, these rules along with those for the tensor product allow one to prove that $\kappa A \otimes B \multimap B \otimes \kappa A$ holds. This is somewhat at odds with algebraic intuition, and it is unclear how this modality could be decomposed into adjoint functors in a linear/non-linear (LNL) formalization of the Lambek calculus.

In this paper we show how to add an exchange modality, eA, where the modality now occurs on both operands being exchanged. That is, one can show that $eA \otimes eB \multimap eB \otimes eA$ holds. We give a sequent calculus and a LNL natural deduction formalization for the Lambek calculus with this new modality, and two categorical models: a LNL model and a concrete model in dialectica spaces. Thus giving a second solution to the problem proposed above.

The Lambek Calculus also has the potential for many applications in other areas of computer science, such as, modeling processes. Linear Logic has been at the forefront of the study of process calculi for

T. Ehrhard, M. Fernández, V. de Paiva, L. Tortora de Falco (Eds.):
Linearity-TLLA 2018
EPTCS 292, 2019, pp. 43–89, doi:10.4204/EPTCS.292.4

many years [11, 20, 1]. We can think of the commutative tensor product of linear logic as a parallel operator. For example, given a process A and a process B, then we can form the process $A \otimes B$ which runs both processes in parallel. If we remove commutativity from the tensor product we obtain a sequential composition instead of parallel composition. That is, the process $A \triangleright B$ first runs process A and then process B in that order. Vaughan Pratt has stated that , "sequential composition has no evident counterpart in type theory" see page 11 of [20]. We believe that the Lambek Calculus will lead to filling this hole.

Acknowledgments. The first two authors were supported by NSF award #1565557. We thank the anonymous reviewers for their helpful feedback that made this a better paper.

2 A Sequent Calculus Formalization of CNC Logic

We now introduce Commutative/Non-commutative (CNC) logic in the form of a sequent calculus. One should view this logic as composed of two logics; one sitting to the left of the other. On the left, there is intuitionistic linear logic, denoted by $\mathcal{C}$ and on the right is the Lambek calculus denoted by $\mathcal{L}$. Then we connect these two systems by a pair of monoidal adjoint functors $\mathcal{C} : \mathsf{F} \dashv \mathsf{G} : \mathcal{L}$. Keeping this intuition in mind we now define the syntax for CNC logic.

Definition 1. *The following grammar describes the syntax of the sequent calculus of CNC logic:*

($\mathcal{C}$-Types)	$W, X, Y, Z ::= \mathsf{Unit} \mid X \otimes Y \mid X \multimap Y \mid \mathsf{G}A$
($\mathcal{L}$-Types)	$A, B, C, D ::= \mathsf{Unit} \mid A \triangleright B \mid A \rightharpoonup B \mid B \leftharpoonup A \mid \mathsf{F}X$
($\mathcal{C}$-Contexts)	$\Phi, \Psi ::= \cdot \mid X \mid \Phi, \Psi$
($\mathcal{L}$-Contexts)	$\Gamma, \Delta ::= \cdot \mid A \mid X \mid \Gamma; \Delta$

The syntax for $\mathcal{C}$-types are the standard types for intuitionistic linear logic. We have a constant Unit, tensor product $X \otimes Y$, and linear implication $X \multimap Y$, but just as in LNL logic we also have a type $\mathsf{G}A$ where A is an $\mathcal{L}$-type; that is, a type from the non-commutative side corresponding to the right-adjoint functor between $\mathcal{L}$ and $\mathcal{C}$. This functor can be used to import types from the non-commutative side into the commutative side. Now a sequent in the the commutative side is denoted by $\Phi \vdash_{\mathcal{C}} X$ where Φ is a $\mathcal{C}$-context, which is a sequence of types X.

The non-commutative side is a bit more interesting than the commutative side just introduced. Sequents in the non-commutative side are denoted by $\Gamma \vdash_{\mathcal{L}} A$ where Γ is now a $\mathcal{L}$-context. These contexts are ordered sequences of types from *both* sides denoted by B and X respectively. Given two contexts Γ and Δ we denote their concatenation by $\Gamma; \Delta$; we use a semicolon here to emphasize the fact that the contexts are ordered.

The context consisting of hypotheses from both sides goes back to Benton [4] and is a property unique to adjoint logics such as Benton's LNL logic and CNC logic. This is also a very useful property because it allows one to make use of both sides within the Lambek calculus without the need to annotate every formula with a modality.

The reader familiar with LNL logic will notice that our sequent, $\Gamma \vdash_{\mathcal{L}} A$, differs from Benton's. His is of the form $\Gamma; \Delta \vdash_{\mathcal{L}} A$, where Γ contains non-linear types, and Δ contains linear formulas. Just as Benton remarks, the splitting of his contexts was a presentational device. One should view his contexts as merged, and hence, linear formulas were fully mixed with non-linear formulas. Now why did we not use this presentational device? Because, when contexts from LNL logic become out of order Benton could use the exchange rule to put them back in order again, but we no longer have general exchange. Thus, we are not able to keep the context organized in this way.

The syntax for $\mathcal{L}$-types are of the typical form for the Lambek Calculus. We have two unit types Unit (one for each side), a non-commutative tensor product $A \triangleright B$, right implication $A \rightharpoonup B$, and left implication

$B \leftharpoonup A$. In standard Lambek Calculus [19], $A \rightharpoonup B$ is written as B/A and $B \leftharpoonup A$ as $A \backslash B$. We use $\rightharpoonup$ and $\leftharpoonup$ here instead to indicate they are two directions of the linear implication $\multimap$.

The sequent calculus for CNC logic can be found in Figure 1. We split the figure in two: the top of the figure are the rules of intuitionistic linear logic whose sequents are the $\mathcal{C}$-sequents denoted by $\Psi \vdash_{\mathcal{C}} X$, and the bottom of the figure are the rules for the mixed commutative/non-commutative Lambek calculus whose sequents are the $\mathcal{L}$-sequents denoted by $\Gamma \vdash_{\mathcal{L}} A$, but the two halves are connected via the functor rules $\mathcal{C}$-G_R, $\mathcal{L}$-G_L, $\mathcal{L}$-F_L, and $\mathcal{L}$-F_R, and the rules $\mathcal{LC}$-Unit_L, $\mathcal{L}$-ex, $\mathcal{LC}$-$\otimes_L$, $\mathcal{L}$-$\multimap_L$, $\mathcal{LC}$-Cut.

$$\frac{}{X \vdash_{\mathcal{C}} X}\ \mathcal{C}\text{-ax} \qquad \frac{\Phi,\Psi \vdash_{\mathcal{C}} X}{\Phi,\mathsf{Unit},\Psi \vdash_{\mathcal{C}} X}\ \mathcal{C}\text{-Unit}_L \qquad \frac{}{\cdot \vdash_{\mathcal{C}} \mathsf{Unit}}\ \mathcal{C}\text{-Unit}_L \qquad \frac{\Phi,X,Y,\Psi \vdash_{\mathcal{C}} Z}{\Phi,X\otimes Y,\Psi \vdash_{\mathcal{C}} Z}\ \mathcal{C}\text{-}\otimes_L$$

$$\frac{\Phi \vdash_{\mathcal{C}} X \quad \Psi \vdash_{\mathcal{C}} Y}{\Phi,\Psi \vdash_{\mathcal{C}} X\otimes Y}\ \mathcal{C}\text{-}\otimes_R \qquad \frac{\Phi \vdash_{\mathcal{C}} X \quad \Psi_1,Y,\Psi_2 \vdash_{\mathcal{C}} Z}{\Psi_1,X\multimap Y,\Phi,\Psi_2 \vdash_{\mathcal{C}} Z}\ \mathcal{C}\text{-}\multimap_L \qquad \frac{\Phi,X,\Psi \vdash_{\mathcal{C}} Y}{\Phi,\Psi \vdash_{\mathcal{C}} X\multimap Y}\ \mathcal{C}\text{-}\multimap_R \qquad \frac{\Phi \vdash_{\mathcal{L}} A}{\Phi \vdash_{\mathcal{C}} \mathsf{G}A}\ \mathcal{C}\text{-G}_R$$

$$\frac{\Phi,X,Y,\Psi \vdash_{\mathcal{C}} Z}{\Phi,Y,X,\Psi \vdash_{\mathcal{C}} Z}\ \mathcal{C}\text{-ex} \qquad \frac{\Phi \vdash_{\mathcal{C}} X \quad \Psi_1,X,\Psi_2 \vdash_{\mathcal{C}} Y}{\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{C}} Y}\ \mathcal{C}\text{-Cut}$$

$$\frac{}{A \vdash_{\mathcal{L}} A}\ \mathcal{L}\text{-ax} \qquad \frac{\Gamma;\Delta \vdash_{\mathcal{L}} A}{\Gamma;\mathsf{Unit};\Delta \vdash_{\mathcal{L}} A}\ \mathcal{LC}\text{-Unit}_L \qquad \frac{\Gamma;\Delta \vdash_{\mathcal{L}} A}{\Gamma;\mathsf{Unit};\Delta \vdash_{\mathcal{L}} A}\ \mathcal{L}\text{-Unit}_L \qquad \frac{}{\cdot \vdash_{\mathcal{L}} \mathsf{Unit}}\ \mathcal{L}\text{-Unit}_R$$

$$\frac{\Gamma;X;Y;\Delta \vdash_{\mathcal{L}} A}{\Gamma;Y;X;\Delta \vdash_{\mathcal{L}} A}\ \mathcal{L}\text{-ex} \qquad \frac{\Gamma;X;Y;\Delta \vdash_{\mathcal{L}} A}{\Gamma;X\otimes Y;\Delta \vdash_{\mathcal{L}} A}\ \mathcal{LC}\text{-}\otimes_L \qquad \frac{\Gamma;A;B;\Delta \vdash_{\mathcal{L}} C}{\Gamma;A\triangleright B;\Delta \vdash_{\mathcal{L}} C}\ \mathcal{L}\text{-}\otimes_L \qquad \frac{\Gamma \vdash_{\mathcal{L}} A \quad \Delta \vdash_{\mathcal{L}} B}{\Gamma;\Delta \vdash_{\mathcal{L}} A\triangleright B}\ \mathcal{L}\text{-}\otimes_R$$

$$\frac{\Phi \vdash_{\mathcal{C}} X \quad \Gamma;Y;\Delta \vdash_{\mathcal{L}} A}{\Gamma;X\multimap Y;\Phi;\Delta \vdash_{\mathcal{L}} A}\ \mathcal{L}\text{-}\multimap_L \qquad \frac{\Gamma \vdash_{\mathcal{L}} A \quad \Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C}{\Delta_1;A\rightharpoonup B;\Gamma;\Delta_2 \vdash_{\mathcal{L}} C}\ \mathcal{L}\text{-}\rightharpoonup_L \qquad \frac{\Gamma;A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} A\rightharpoonup B}\ \mathcal{L}\text{-}\rightharpoonup_R$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad \Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C}{\Delta_1;\Gamma;B\leftharpoonup A;\Delta_2 \vdash_{\mathcal{L}} C}\ \mathcal{L}\text{-}\leftharpoonup_L \qquad \frac{A;\Gamma \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B\leftharpoonup A}\ \mathcal{L}\text{-}\leftharpoonup_R \qquad \frac{\Gamma;X;\Delta \vdash_{\mathcal{L}} A}{\Gamma;\mathsf{F}X;\Delta \vdash_{\mathcal{L}} A}\ \mathcal{L}\text{-F}_L \qquad \frac{\Phi \vdash_{\mathcal{C}} X}{\Phi \vdash_{\mathcal{L}} \mathsf{F}X}\ \mathcal{L}\text{-F}_R$$

$$\frac{\Gamma;A;\Delta \vdash_{\mathcal{L}} B}{\Gamma;\mathsf{G}A;\Delta \vdash_{\mathcal{L}} B}\ \mathcal{L}\text{-G}_L \qquad \frac{\Phi \vdash_{\mathcal{C}} X \quad \Delta_1;X;\Delta_2 \vdash_{\mathcal{L}} A}{\Delta_1;\Phi;\Delta_1 \vdash_{\mathcal{L}} A}\ \mathcal{LC}\text{-Cut} \qquad \frac{\Gamma \vdash_{\mathcal{L}} A \quad \Delta_1;A;\Delta_2 \vdash_{\mathcal{L}} B}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} B}\ \mathcal{L}\text{-Cut}$$

Figure 1: Sequent Calculus for CNC Logic

We prove cut elimination for the sequent calculus. We define the *rank* $|X|$ (resp. $|A|$) of a commutative (resp. non-commutative) formula to be the number of logical connectives in the proposition. For instance, $|X\otimes Y| = |X| + |Y| + 1$. The *cut rank* $c(\Pi)$ of a proof Π is one more than the maximum of the ranks of all the cut formulae in Π, and 0 if Π is cut-free. Then the *depth* $d(\Pi)$ of a proof Π is the length of the longest path in the proof tree (so the depth of an axiom is 0). The key to the proof of cut elimination is the following lemma, which shows how to transform a single cut, either by removing it or by replacing it with one or more simpler cuts.

Lemma 2 (Cut Reduction)**.** *The cut-reduction steps are as follows:*

1. *If Π_1 is a proof of $\Phi \vdash_{\mathcal{C}} X$ and Π_2 is a proof of $\Psi_1,X,\Psi_2 \vdash_{\mathcal{C}} Y$ with $c(\Pi_1)$, $c(\Pi_2) \leq |X|$, then there exists a proof Π of $\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{C}} Y$ with $c(\Pi) \leq |X|$.*

2. *If Π_1 is a proof of $\Phi \vdash_{\mathcal{C}} X$ and Π_2 is a proof of $\Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} A$ with $c(\Pi_1)$, $c(\Pi_2) \leq |X|$, then there exists a proof Π of $\Gamma_1;\Phi;\Gamma_2 \vdash_{\mathcal{L}} A$ with $c(\Pi) \leq |X|$.*

3. *If* Π_1 *is a proof of* $\Gamma \vdash_{\mathcal{L}} A$ *and* Π_2 *is a proof of* $\Delta_1;A;\Delta_2 \vdash_{\mathcal{L}} B$ *with* $c(\Pi_1)$*,* $c(\Pi_2) \leq |A|$*, then there exists a proof* Π *of* $\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} B$ *with* $c(\Pi) \leq |A|$*.*

Proof. This proof is done case by case on the last step of Π_1 and Π_2 and by induction on $d(\Pi_1)$ and $d(\Pi_2)$, following [16]. For instance, suppose Π_1 is a proof of $\Phi_1, X_2, X_1, \Phi_2 \vdash_{\mathcal{C}} Y$ and Π_2 is a proof of $\Psi_1, Y, \Psi_2 \vdash_{\mathcal{C}} Z$. Consider the case where the last step in Π_1 uses the rule $\mathcal{C}$-ex. Π_1 can be depicted as follows, where the previous steps are denoted by π:

$$\Pi_1: \quad \dfrac{\begin{array}{c}\pi \\ \Phi_1, X_1, X_2, \Phi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1, X_2, X_1, \Phi_2 \vdash_{\mathcal{C}} Y}\ \mathcal{C}\text{-ex}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. By induction on π and Π_2, there is a proof Π' for the sequent $\Psi_1, \Phi_1, X_1, X_2, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} Z$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |Y|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Psi_1, \Phi_1, X_1, X_2, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} Z\end{array}}{\Psi_1, \Phi_1, X_2, X_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} Z}\ \mathcal{C}\text{-ex}$$

The full proof can be found in Appendix A. □

Then we have the following lemma.

Lemma 3. *Let* Π *be a proof of a sequent* $\Phi \vdash_{\mathcal{C}} X$ *or* $\Gamma \vdash_{\mathcal{L}} A$ *s.t.* $c(\Pi) > 0$*. Then there is a proof* Π' *of the same sequent with* $c(\Pi') < c(\Pi)$*.*

Proof. We prove the lemma by induction on $d(\Pi)$. We denote the proof Π by $\pi + r$, where r is the last inference of Π and π denotes the rest of the proof. If r is not a cut, then by induction hypothesis on π, there is a proof π' s.t. $c(\pi') < c(\pi)$ and $\Pi' = \pi' + r$. Otherwise, we assume r is a cut on a formula Y. If $c(\Pi) > |X| + 1$, then there is a cut on $|Y|$ in π with $|Y| > |X|$. So we can apply the induction hypothesis on π to get Π' with $c(\Pi') < c(\Pi)$. The last case to consider is when $c(\Pi) = |X| + 1$ (note that $c(\Pi)$ cannot be less than $|X| + 1$). In this case, Π is in the form of

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ \Phi \vdash_{\mathcal{C}} X & \Psi_1, X, \Psi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} Y}\ \mathcal{C}\text{-Cut} \qquad \text{or,} \qquad \dfrac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ \Phi \vdash_{\mathcal{C}} X & \Gamma_1; X; \Gamma_2 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; \Phi; \Gamma_2 \vdash_{\mathcal{L}} A}\ \mathcal{LC}\text{-Cut}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X| + 1$. By induction, we can construct $c(\Pi'_1)$ proving $\Phi \vdash_{\mathcal{C}} X$ and $c(\Pi'_2)$ proving $\Psi_1, X, \Psi_2 \vdash_{\mathcal{C}} Y$ (or $< \Gamma_1; X; \Gamma_2 \vdash_{\mathcal{L}} A$) with $c(\Pi'_1), c(\Pi'_2) \leq |X|$. Then by Lemma 2, we can construct Π' proving $\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} Y$ (or $\Gamma_1; \Phi; \Gamma_2 \vdash_{\mathcal{L}} A$) with $c(\Pi') \leq |X|$.
The case where the last inference is a cut on a formula A is similar as when it is a cut on X. □

By induction on $c(\Pi)$ and Lemma 3, the cut elimination theorem follows immediately.

Theorem 4 (Cut Elimination)**.** *Let* Π *be a proof of a sequent* $\Phi \vdash_{\mathcal{C}} X$ *or* $\Gamma \vdash_{\mathcal{L}} A$ *s.t.* $c(\Pi) > 0$*. Then there is an algorithm which yields a cut-free proof* Π' *of the same sequent.*

3 A Type Theoretic Formalization of CNC Logic

Similar as the sequent calculus, the term assignment for CNC logic is also composed of two logics; intuitionistic linear logic on the left, denoted by $\mathcal{C}$, and the Lambek calculus on the right, denoted by $\mathcal{L}$. The syntax for types and contexts we use in the term assignment is the same as in the sequent calculus. The rest of the syntax for the term assignment is defined as follows.

Definition 5. *The following grammar describes the syntax of the term assignment of the CNC logic:*

($\mathcal{C}$-Terms)	$t ::= x \mid \mathsf{triv} \mid t_1 \otimes t_2 \mid \mathsf{let}\, t_1 : X\, \mathsf{be}\, q\, \mathsf{in}\, t_2 \mid \lambda x : X.t \mid t_1 t_2 \mid \mathsf{ex}\, t_1, t_2\, \mathsf{with}\, x_1, x_2\, \mathsf{in}\, t_3 \mid \mathsf{G}s$
($\mathcal{L}$-Terms)	$s ::= x \mid \mathsf{triv} \mid s_1 \triangleright s_2 \mid \mathsf{let}\, s_1 : A\, \mathsf{be}\, p\, \mathsf{in}\, s_2 \mid \mathsf{let}\, t : X\, \mathsf{be}\, q\, \mathsf{in}\, s \mid \lambda_l x : A.s \mid \lambda_r x : A.s$ $\mid \mathsf{app}_l\, s_1\, s_2 \mid \mathsf{app}_r\, s_1\, s_2 \mid \mathsf{F}t$
($\mathcal{C}$-Patterns)	$q ::= \mathsf{triv} \mid x \mid q_1 \otimes q_2 \mid \mathsf{G}p$
($\mathcal{L}$-Patterns)	$p ::= \mathsf{triv} \mid x \mid p_1 \triangleright p_2 \mid \mathsf{F}q$
($\mathcal{C}$-Typing Judgment)	$\Phi \vdash_{\mathcal{C}} t : X$
($\mathcal{L}$-Typing Judgment)	$\Gamma \vdash_{\mathcal{L}} s : A$

Now $\mathcal{C}$-typing judgments are denoted by $\Psi \vdash_{\mathcal{C}} t : X$ where Ψ is a sequence of pairs of variables and their types, denoted by $x : X$, t is a $\mathcal{C}$-term, and X is a $\mathcal{C}$-type. The $\mathcal{C}$-terms are all standard, but $\mathsf{G}s$ corresponds to the morphism part of the right-adjoint of the adjunction between both logics, and $\mathsf{ex}\, t_1, t_2\, \mathsf{with}\, x_1, x_2\, \mathsf{in}\, t_3$ is the introduction form for the structural rule exchange.

The $\mathcal{L}$-typing judgment has the form $\Gamma \vdash_{\mathcal{L}} s : A$ where Γ is now a $\mathcal{L}$-context, denoted by Γ or Δ. These contexts are ordered sequences of pairs of free variables with their types from *both* sides denoted by $x : B$ and $x : X$ respectively. Finally, the term s is a $\mathcal{L}$-term, and A is a $\mathcal{L}$-type. Given two typing contexts Γ and Δ we denote their concatenation by $\Gamma;\Delta$; we use a semicolon here to emphasize the fact that the contexts are ordered. $\mathcal{L}$-terms correspond to introduction and elimination forms for each of the previous types. For example, $s_1 \triangleright s_2$ introduces a tensor, and $\mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2$ eliminates a tensor.

The typing rules for CNC logic can be found in Figure 2. We split the figure in two: the top of the figure are the rules of intuitionistic linear logic whose judgment is the $\mathcal{C}$-typing judgment denoted by $\Psi \vdash_{\mathcal{C}} t : X$, and the bottom of the figure are the rules for the mixed commutative/non-commutative Lambek calculus whose judgment is the $\mathcal{L}$-judgment denoted by $\Gamma \vdash_{\mathcal{L}} s : A$, and the two halves are connected via the rules rules $\mathcal{C}$-G_I, $\mathcal{L}$-G_E, $\mathcal{L}$-F_I, and $\mathcal{L}$-F_E, $\mathcal{LC}$-Unit_E, $\mathcal{LC}$-$\otimes_E$, and $\mathcal{LC}$-Cut.

The one step β-reduction rules are listed in Figure 3. Similarly to the typing rules, the figure is split in two: the top lists the rules of the intuitionistic linear logic, and the bottom are those of the mixed commutative/non-commutative Lambek calculus.

The commuting conversions can be found in Figures 4-6. We divide the rules into three parts due to the length. The first part, Figure 4, includes the rules for the intuitionistic linear logic. The second, Figure 5, includes the rules for the commutative/non-commutative Lambek calculus. The third, Figure 6, includes the mixed rules $\mathcal{LC}$-Unit_E and $\mathcal{LC}$-$\otimes_E$.

We also proved that the sequent calculus formalization given in Figure 1 is equivalent to the typing rules (or else called the natural deduction formalization) given in Figure 2 are equivalent, as stated in the following theorem.

Theorem 6. *The sequent calculus (SC) and natural deduction (ND) formalizations for CNC logic are equivalent in the sense that there are two mappings $N : SC \to ND$ and $S : ND \to SC$ that map each rule in SC to a proof in ND, and each rule in ND to a proof in SC, respectively.*

Proof. The proof is done case by case on each rule in the sequence calculus and natural deduction formalizations. It is obvious that the axioms in one formalization can be mapped to the axioms in the other. The introduction rules in *ND* are mapped to the right rules in *SC*, and vice versa. The elimination

$$\frac{}{x:X \vdash_{\mathcal{C}} x:X}\ \mathcal{C}\text{-ax} \qquad \frac{}{\cdot \vdash_{\mathcal{C}} \mathsf{triv}:\mathsf{Unit}}\ \mathcal{C}\text{-Unit}_I \qquad \frac{\Phi \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi,\Psi \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t_2 : Y}\ \mathcal{C}\text{-Unit}_E$$

$$\frac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi,\Psi \vdash_{\mathcal{C}} t_1 \otimes t_2 : X \otimes Y}\ \mathcal{C}\text{-}\otimes_I \qquad \frac{\Phi \vdash_{\mathcal{C}} t_1 : X\otimes Y \quad \Psi_1, x:X, y:Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : X\otimes Y\, \mathsf{be}\, x\otimes y\, \mathsf{in}\, t_2 : Z}\ \mathcal{C}\text{-}\otimes_E \qquad \frac{\Phi, x:X \vdash_{\mathcal{C}} t : Y}{\Phi \vdash_{\mathcal{C}} \lambda x:X.t : X \multimap Y}\ \mathcal{C}\text{-}\multimap_I$$

$$\frac{\Phi \vdash_{\mathcal{C}} t_1 : X \multimap Y \quad \Psi \vdash_{\mathcal{C}} t_2 : X}{\Phi,\Psi \vdash_{\mathcal{C}} t_1 t_2 : Y}\ \mathcal{C}\text{-}\multimap_E \qquad \frac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_{\mathcal{C}} \mathsf{G}s : \mathsf{G}A}\ \mathcal{C}\text{-G}_I \qquad \frac{\Phi, x:X, y:Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z:Y, w:X, \Psi \vdash_{\mathcal{C}} \mathsf{ex}\, w,z\, \mathsf{with}\, x,y\, \mathsf{in}\, t : Z}\ \mathcal{C}\text{-ex}$$

$$\frac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi_1, x:X, \Psi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{C}} [t_1/x]t_2 : Y}\ \mathcal{C}\text{-Cut}$$

$$\frac{}{x:A \vdash_{\mathcal{L}} x:A}\ \mathcal{L}\text{-ax} \qquad \frac{}{\cdot \vdash_{\mathcal{L}} \mathsf{triv}:\mathsf{Unit}}\ \mathcal{L}\text{-Unit}_I \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma;\Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2 : A}\ \mathcal{L}\text{-Unit}_E$$

$$\frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{Unit} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi;\Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \mathcal{LC}\text{-Unit}_E \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma;\Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B}\ \mathcal{L}\text{-}\otimes_I$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1; x:A; y:B; \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\ \mathcal{L}\text{-}\otimes_E \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X\otimes Y \quad \Gamma_1; x:X; y:Y; \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1;\Phi;\Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, t : X\otimes Y\, \mathsf{be}\, x\otimes y\, \mathsf{in}\, s : A}\ \mathcal{LC}\text{-}\otimes_E$$

$$\frac{\Gamma; x:A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x:A.s : A \rightharpoonup B}\ \mathcal{L}\text{-}\rightharpoonup_I \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma;\Delta \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B}\ \mathcal{L}\text{-}\rightharpoonup_E \qquad \frac{x:A;\Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x:A.s : B \leftharpoonup A}\ \mathcal{L}\text{-}\leftharpoonup_I$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta;\Gamma \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B}\ \mathcal{L}\text{-}\leftharpoonup_E \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X}{\Phi \vdash_{\mathcal{L}} \mathsf{F}t : \mathsf{F}X}\ \mathcal{L}\text{-F}_I \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{F}X \quad \Delta_1; x:X; \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}x\, \mathsf{in}\, s_2 : A}\ \mathcal{L}\text{-F}_E$$

$$\frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\, t : A}\ \mathcal{L}\text{-G}_E \qquad \frac{\Gamma; x:X; y:Y; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; z:Y; w:X; \Delta \vdash_{\mathcal{L}} \mathsf{ex}\, w,z\, \mathsf{with}\, x,y\, \mathsf{in}\, s : A}\ \mathcal{L}\text{-ex}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta_1; x:A; \Delta_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_2 : B}\ \mathcal{L}\text{-Cut} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X \quad \Gamma_1; x:X; \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1;\Phi;\Gamma_1 \vdash_{\mathcal{L}} [t/x]s : A}\ \mathcal{LC}\text{-Cut}$$

Figure 2: Typing Rules for CNC Logic

$$\frac{}{\mathsf{let}\,\mathsf{triv} : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t \rightsquigarrow_\beta t} \qquad \frac{}{\mathsf{let}\,t_1 \otimes t_2 : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_3 \rightsquigarrow_\beta [t_1/x][t_2/y]t_3} \qquad \frac{}{(\lambda x : X.t_1)t_2 \rightsquigarrow_\beta [t_2/x]t_1}$$

$$\frac{}{\mathsf{let}\,\mathsf{triv} : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s \rightsquigarrow_\beta s} \qquad \frac{}{\mathsf{let}\,t_1 \otimes t_2 : X \otimes Y\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s \rightsquigarrow_\beta [t_1/x][t_2/y]s}$$

$$\frac{}{\mathsf{let}\,s_1 \triangleright s_2 : A \triangleright B\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_3 \rightsquigarrow_\beta [s_1/x][s_2/y]s_3} \qquad \frac{}{\mathsf{let}\,\mathsf{F}t : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s \rightsquigarrow_\beta [t/x]s} \qquad \frac{}{\mathsf{app}_l\,(\lambda_l x : A.s_1)\,s_2 \rightsquigarrow_\beta [s_2/x]s_1}$$

$$\frac{}{\mathsf{app}_r\,(\lambda_r x : A.s_1)\,s_2 \rightsquigarrow_\beta [s_2/x]s_1} \qquad \frac{}{\mathsf{derelict}\,(\mathsf{G}\,s) \rightsquigarrow_\beta s}$$

Figure 3: β-reductions for CNC Logic

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,t_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{let}\,t_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,t_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_1) : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{let}\,t_1 : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_3)}$$

$$\frac{}{(\mathsf{let}\,t_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_1)t_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(t_1 t_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,t_2 : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t_2 : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,(\mathsf{let}\,t_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_1) : X_1 \otimes Y_1\,\mathsf{be}\,w \otimes z\,\mathsf{in}\,t_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,(\mathsf{let}\,t_1 : X_1 \otimes Y_1\,\mathsf{be}\,w \otimes z\,\mathsf{in}\,t_3)}$$

$$\frac{}{(\mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_1)t_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,(t_1 t_3)} \qquad \frac{}{\mathsf{let}\,(t_1 t_2) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3 \rightsquigarrow_{\mathsf{c}} t_1(\mathsf{let}\,t_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3)}$$

Figure 4: Commuting Conversions: Intuitionistic Linear Logic

rules and lefts rules are mapped to each other with some fiddling. For instance, the elimination rule for the non-commutative tensor is mapped to the following proof in SC:

$$\dfrac{\Phi \vdash_C t_1 : X \otimes Y \qquad \dfrac{\Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, z : X \otimes Y, \Psi_2 \vdash_C \mathsf{let}\,z : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_2 : Z}\ C\text{-}\otimes_L}{\Psi_1, \Phi, \Psi_2 \vdash_C [t_1/z](\mathsf{let}\,z : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_2) : Z}\ C\text{-}\mathsf{Cut}$$

The full proof is in Appendix D. □

4 An Adjoint Model

In this section we introduce Lambek Adjoint Models (LAMs). Benton's LNL model consists of a symmetric monoidal adjunction $F : \mathcal{C} \dashv \mathcal{L} : G$ between a Cartesian closed category $\mathcal{C}$ and a symmetric monoidal closed category $\mathcal{L}$. LAM consists of a monoidal adjunction between a symmetric monoidal closed category and a Lambek category.

Definition 7. *A **Lambek category** is a monoidal category $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ with two functors $- \rightharpoonup - : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ and $- \leftharpoonup - : \mathcal{L} \times \mathcal{L}^{\mathsf{op}} \longrightarrow \mathcal{L}$ such that the following two natural bijections hold:*

$$\mathsf{Hom}_{\mathcal{L}}(A \triangleright B, C) \cong \mathsf{Hom}_{\mathcal{L}}(A, B \rightharpoonup C) \qquad \mathsf{Hom}_{\mathcal{L}}(A \triangleright B, C) \cong \mathsf{Hom}_{\mathcal{L}}(B, C \leftharpoonup A)$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_3)}$$

$$\frac{}{\mathsf{app}_r\,(\mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1)\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{app}_r\,s_1\,s_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1) : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : A \triangleright B\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : A \triangleright B\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : A_2 \triangleright B_2\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_1) : A_1 \triangleright B_1\,\mathsf{be}\,w \triangleright z\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : A_2 \triangleright B_2\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,(\mathsf{let}\,s_1 : A_1 \triangleright B_1\,\mathsf{be}\,w \triangleright z\,\mathsf{in}\,s_3)}$$

$$\frac{}{\mathsf{app}_r\,(\mathsf{let}\,s_2 : A_2 \triangleright B_2\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_1)\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : A_2 \triangleright B_2\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,(\mathsf{app}_r\,s_1\,s_3)}$$

$$\frac{}{\mathsf{app}_l\,(\mathsf{let}\,s_2 : A_2 \triangleright B_2\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_1)\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : A_2 \triangleright B_2\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,(\mathsf{app}_l\,s_1\,s_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : A \triangleright B\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_1) : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}z\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : A \triangleright B\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}z\,\mathsf{in}\,s_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_2)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_1) : A \triangleright B\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,(\mathsf{let}\,s_1 : A \triangleright B\,\mathsf{be}\,x \triangleright y\,\mathsf{in}\,s_3)}$$

$$\frac{}{\mathsf{app}_r\,(\mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_1)\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,(\mathsf{app}_r\,s_1\,s_3)}$$

$$\frac{}{\mathsf{app}_l\,(\mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_1)\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,(\mathsf{app}_l\,s_1\,s_3)}$$

$$\frac{}{\mathsf{let}\,(\mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_1) : \mathsf{F}Y\,\mathsf{be}\,\mathsf{F}y\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,s_2 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{F}Y\,\mathsf{be}\,\mathsf{F}y\,\mathsf{in}\,s_3)}$$

Figure 5: Commuting Conversions: Commutative/Non-commutative Lambek Calculus

Lambek categories are also known as monoidal bi-closed categories.

Definition 8. *A* ***Lambek Adjoint Model (LAM)****,* $(\mathcal{C},\mathcal{L},F,G,\eta,\varepsilon)$*, consists of*

- *a symmetric monoidal closed category* $(\mathcal{C},\otimes,I,\alpha,\lambda,\rho)$*;*
- *a Lambek category* $(\mathcal{L},\triangleright,I',\alpha',\lambda',\rho')$*;*
- *a monoidal adjunction* $F:\mathcal{C}\dashv\mathcal{L}:G$ *with unit* $\eta:\mathsf{Id}_{\mathcal{C}}\to GF$ *and counit* $\varepsilon:FG\to\mathsf{Id}_{\mathcal{L}}$*, where* $(F:\mathcal{C}\to\mathcal{L},m)$ *and* $(G:\mathcal{L}\to\mathcal{C},n)$ *are monoidal functors.*

Following the tradition, we use letters X, Y, Z for objects in $\mathcal{C}$ and A, B, C for objects in $\mathcal{L}$. The rest of this section proves essential properties of any LAM.

An isomorphism. Let $(\mathcal{C},\mathcal{L},F,G,\eta,\varepsilon)$ be a LAM, where (F,m) and (G,n) are monoidal functors. Similarly as in Benton's LNL model, $\mathsf{m}_{X,Y}: FX\triangleright FY \longrightarrow F(X\otimes Y)$ are components of a natural isomorphism, and $\mathsf{m}_I : I' \longrightarrow FI$ is an isomorphism. This is essential for modeling certain rules of CNC logic, such as tensor elimination in natural deduction. We define the inverses of $\mathsf{m}_{X,Y}: FX\triangleright FY\to F(X\otimes Y)$ and $\mathsf{m}_I : I'\to FI$ as:

$$\mathsf{p}_{X,Y} : F(X\otimes Y) \xrightarrow{F(\eta_X\otimes\eta_Y)} F(GFX\otimes GFY) \xrightarrow{F\mathsf{n}_{FX,FY}} FG(FX\triangleright FY) \xrightarrow{\varepsilon_{FX\triangleright FX}} FX\triangleright FY$$

$$\mathsf{p}_I : FI \xrightarrow{F\mathsf{n}_{I'}} FGI' \xrightarrow{\varepsilon_{I'}} I'$$

$$\overline{\mathsf{let}\,(\mathsf{let}\,t : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_2 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_2)}$$

$$\overline{\mathsf{app}_r\,(\mathsf{let}\,t : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1)\,s_2 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{app}_r\,s_1\,s_2)}$$

$$\overline{\mathsf{let}\,(\mathsf{let}\,t : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1) : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_2 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}x\,\mathsf{in}\,s_2)}$$

$$\overline{\mathsf{let}\,(\mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,s_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_2 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_2)}$$

$$\overline{\mathsf{let}\,(\mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,s_1) : A_1 \triangleright B_1\,\mathsf{be}\,w \triangleright z\,\mathsf{in}\,s_2 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,(\mathsf{let}\,s_1 : A_1 \triangleright B_1\,\mathsf{be}\,w \triangleright z\,\mathsf{in}\,s_2)}$$

$$\overline{\mathsf{app}_r\,(\mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,s_1)\,s_2 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,(\mathsf{app}_r\,s_1\,s_2)}$$

$$\overline{\mathsf{app}_l\,(\mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,s_1)\,s_2 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,(\mathsf{app}_l\,s_1\,s_2)}$$

$$\overline{\mathsf{let}\,(\mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,s_1) : \mathsf{F}Z\,\mathsf{be}\,\mathsf{F}z\,\mathsf{in}\,s_3 \rightsquigarrow_{\mathsf{c}} \mathsf{let}\,t : X\otimes Y\,\mathsf{be}\,x\otimes y\,\mathsf{in}\,(\mathsf{let}\,s_1 : \mathsf{F}Z\,\mathsf{be}\,\mathsf{F}z\,\mathsf{in}\,s_3)}$$

Figure 6: Commuting Conversions: Mixed Rules

Due to [12], it can be easily shown that m_I is an isomorphism with inverse, and that $\mathsf{m}_{X,Y}$ are components of a natural isomorphism with inverses $\mathsf{p}_{X,Y}$.

Strong non-commutative monad. Next we show that the monad on $\mathcal{C}$ in LAM is strong but non-commutative. In Benton's LNL model, the monad on the Cartesian closed category is commutative, but later Benton and Wadler [3] wonder, is it possible to model non-commutative monads using adjoint models similar to LNL models? The following shows that LAMs correspond to strong non-commutative monoidal monads.

Lemma 9. *The monad induced by any LAM, $GF : \mathcal{C} \longrightarrow \mathcal{C}$, is monoidal.*

Proof. The proof is done by checking the conditions for a functor being monoidal. The detail of the proof is in Appendix B. □

However, the monad is not symmetric because the following diagram does not commute.

$$\begin{array}{ccccc}
GFX \otimes GFY & \xrightarrow{\mathsf{ex}_{GFX,GFY}} & GFY \otimes GFX & \xrightarrow{\mathsf{n}_{FY,FX}} & G(FY \triangleright FX) \\
\downarrow{\scriptstyle \mathsf{n}_{FX,FY}} & & & & \downarrow{\scriptstyle G\mathsf{m}_{Y,X}} \\
G(FX \triangleright FY) & \xrightarrow[G\mathsf{m}_{X,Y}]{} & GF(X\otimes Y) & \xrightarrow[GF\mathsf{ex}_{X,Y}]{} & GF(Y \otimes X)
\end{array}$$

Commutativity fails, because the functors defining the monad are not symmetric monoidal, but only monoidal. This means that the diagram

$$\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{\mathsf{ex}_{FA,FB}} & FB \otimes' FA \\
\downarrow{\scriptstyle \mathsf{m}_{A,B}} & & \downarrow{\scriptstyle \mathsf{m}_{B,A}} \\
F(A \otimes B) & \xrightarrow[F\mathsf{ex}_{A,B}]{} & F(B \otimes A)
\end{array}$$

does not hold for G nor F. However, we can prove the monad is strong.

Lemma 10. *The monad, $GF : C \longrightarrow C$, on the symmetric monoidal closed category in LAM is strong.*

Proof. The proof is done by first defining a natural transformation τ, called the **tensorial strength**, with components $\tau_{A,B} : A \triangleright TB \to T(A \triangleright B)$, and then proving the commutativity of several diagrams through diagram chasing. The formal definition for a strong monad and the full proof are in Appendix C. □

Finally, we obtain the non-commutativity of the monad induced by some LAM as follows.

Lemma 11 (Due to Kock [13])**.** *Let $\mathcal{M}$ be a symmetric monoidal category and T be a strong monad on $\mathcal{M}$. Then T is commutative iff it is symmetric monoidal.*

Theorem 12. *There exists a LAM whose monad, $GF : C \longrightarrow C$, on the SMCC in the LAM is strong but non-commutative.*

Proof. This proof follows from Lemma 10 and Lemma 11. □

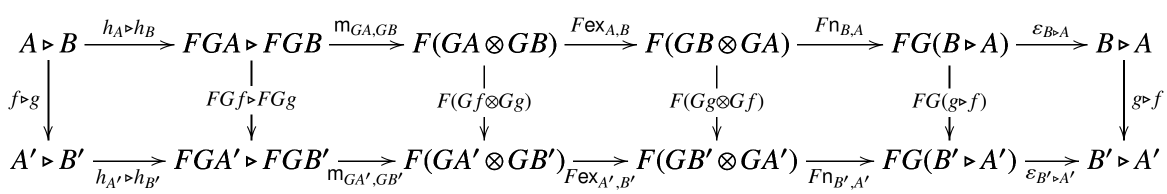
Comonad for exchange. We conclude this section by showing that the comonad induced by some LAM is monoidal and extends $\mathcal{L}$ with exchange. The former is proved in [12]. The latter is shown by proving that its corresponding co-Eilenberg-Moore category is symmetric monoidal.

Theorem 13. *Given a LAM $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ and the comonad $FG : \mathcal{L} \longrightarrow \mathcal{L}$, the co-Eilenberg-Moore category $\mathcal{L}^{FG}$ has an exchange natural transformation $\mathsf{ex}^{FG}_{A,B} : A \triangleright B \to B \triangleright A$, and $\mathsf{ex}^{FG}_{A,B}$ is a symmetry, i.e., $\mathsf{ex}^{FG}_{A,B} \circ \mathsf{ex}^{FG}_{A,B} = id_A$.*

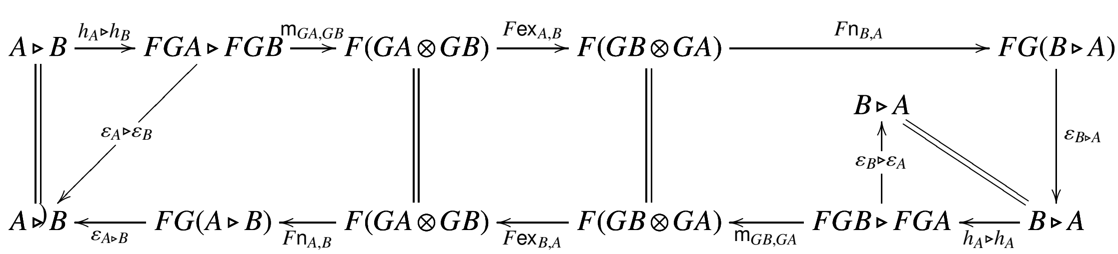
Proof. The natural transformation $\mathsf{ex}^{FG}_{A,B} : A \triangleright B \to B \triangleright A$ is defied as follows:

$$A \triangleright B \xrightarrow{h_A \triangleright h_B} FGA \triangleright FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{ex}_{GA,GB}} F(GB \otimes GA) \xrightarrow{F\mathsf{n}_{B,A}} FG(B \triangleright A) \xrightarrow{\varepsilon_{B \triangleright A}} B \triangleright A$$

in which ex is the exchange for C. Then ex^{FG} is a natural transformation because the following diagrams commute for morphisms $f : A \to A'$ and $g : B \to B'$:



$\mathsf{ex}^{FG}_{A,B}$ is a symmetry because the following diagrams commute:



□

5 A Model in Dialectica Spaces

In this section we give a different categorical model in terms of dialectica categories; which are a sound and complete categorical model of the Lambek Calculus as was shown by de Paiva and Eades [8]. This section is largely the same as the corresponding section de Paiva and Eades give, but with some modifications to their definition of biclosed posets with exchange (see Definition 18). However, we try to make this section as self contained as possible.

Dialectica categories were first introduced by de Paiva as a categorification of Gödel's Dialectica interpretation [7]. Dialectica categories were one of the first sound categorical models of intuitionistic linear logic with linear modalities. We show in this section that they can be adapted to become a sound and complete model for CNC logic, with both the exchange and of-course modalities. Due to the complexities of working with dialectica categories we have formally verified[1] this section in the proof assistant Agda [6].

First, we define the notion of a biclosed poset. These are used to control the definition of morphisms in the dialectica model.

Definition 14. *Suppose $(M,\leq,\circ,e)$ is an ordered non-commutative monoid. If there exists a largest $x \in M$ such that $a \circ x \leq b$ for any $a,b \in M$, then we denote x by $a \rightharpoonup b$ and called it the **left-pseudocomplement** of a w.r.t b. Additionally, if there exists a largest $x \in M$ such that $x \circ a \leq b$ for any $a,b \in M$, then we denote x by $b \leftharpoonup a$ and called it the **right-pseudocomplement** of a w.r.t b.*

*A **biclosed poset**, $(M,\leq,\circ,e,\rightharpoonup,\leftharpoonup)$, is an ordered non-commutative monoid, $(M,\leq,\circ,e)$, such that $a \rightharpoonup b$ and $b \leftharpoonup a$ exist for any $a,b \in M$.*

Now using the previous definition we define dialectica Lambek spaces.

Definition 15. *Suppose $(M,\leq,\circ,e,\rightharpoonup,\leftharpoonup)$ is a biclosed poset. Then we define the category of **dialectica Lambek spaces**, $\mathsf{Dial}_M(\mathsf{Set})$, as follows:*

- *objects, or dialectica Lambek spaces, are triples (U,X,α) where U and X are sets, and $\alpha : U \times X \longrightarrow M$ is a generalized relation over M, and*
- *maps that are pairs $(f,F) : (U,X,\alpha) \longrightarrow (V,Y,\beta)$ where $f : U \longrightarrow V$, and $F : Y \longrightarrow X$ are functions such that the weak adjointness condition $\forall u \in U.\forall y \in Y.\alpha(u,F(y)) \leq \beta(f(u),y)$ holds.*

Notice that the biclosed poset is used here as the target of the relations in objects, but also as providing the order relation in the weak adjoint condition on morphisms. This will allow the structure of the biclosed poset to lift up into $\mathsf{Dial}_M(\mathsf{Set})$.

We will show that $\mathsf{Dial}_M(\mathsf{Set})$ is a model of the Lambek Calculus with modalities. First, we must show that $\mathsf{Dial}_M(\mathsf{Set})$ is monoidal biclosed.

Definition 16. *Suppose (U,X,α) and (V,Y,β) are two objects of $\mathsf{Dial}_M(\mathsf{Set})$. Then their tensor product is defined as follows:*

$$(U,X,\alpha) \triangleright (V,Y,\beta) = (U \times V,(V \to X) \times (U \to Y),\alpha \triangleright \beta)$$

where $- \to -$ is the function space from Set, and $(\alpha \triangleright \beta)((u,v),(f,g)) = \alpha(u,f(v)) \circ \beta(g(u),v)$.

The unit of the above tensor product is defined as follows:

$$I = (\top,\top,\iota)$$

where $\top$ is the initial object in Set, and $\iota(*,*) = e$.

[1] The complete formalization can be found online at `https://bit.ly/2TpoyWU`.

It follows from de Paiva and Eades [8] that this does indeed define a monoidal tensor product, but take note of the fact that this tensor product is indeed non-commutative, because the non-commutative multiplication of the biclosed poset is used to define the relation of the tensor product.

The tensor product has two right adjoints making $\mathsf{Dial}_M(\mathsf{Set})$ biclosed.

Definition 17. *Suppose* (U,X,α) *and* (V,Y,β) *are two objects of* $\mathsf{Dial}_M(\mathsf{Set})$*. Then two internal-homs can be defined as follows:*

$$(U,X,\alpha) \rightharpoonup (V,Y,\beta) = ((U \to V)\times(Y \to X), U \times Y, \alpha \rightharpoonup \beta)$$
$$(V,Y,\beta) \leftharpoondown (U,X,\alpha) = ((U \to V)\times(Y \to X), U \times Y, \alpha \leftharpoondown \beta)$$

It is straightforward to show that the typical bijections defining the corresponding adjunctions hold; see de Paiva and Eades for the details [8].

We now extend $\mathsf{Dial}_M(\mathsf{Set})$ with two modalities: the usual modality, of-course, denoted $!A$, and the exchange modality denoted ξA. However, we must first extended biclosed posets to include an exchange operation.

Definition 18. *A* ***biclosed poset with exchange*** *is a biclosed poset* $(M,\leq,\circ,e,\rightharpoonup,\leftharpoondown)$ *equipped with an unary operation* $\xi : M \to M$ *satisfying the following:*

(Compatibility)	$a \leq b$ *implies* $\xi a \leq \xi b$ *for all* $a,b,c \in M$	*(Duplication)*	$\xi a \leq \xi\xi a$ *for all* $a \in M$
(Minimality)	$\xi a \leq a$ *for all* $a \in M$	*(Exchange)*	$(\xi a \circ \xi b) \leq (\xi b \circ \xi a)$ *for all* $a,b \in M$

This definition is where the construction given here departs from the definition of biclosed posets with exchange given by de Paiva and Eades [8].

We can now define the two modalities in $\mathsf{Dial}_M(\mathsf{Set})$ where M is a biclosed poset with exchange.

Definition 19. *Suppose* (U,X,α) *is an object of* $\mathsf{Dial}_M(\mathsf{Set})$ *where* M *is a biclosed poset with exchange. Then the* ***of-course*** *and* ***exchange*** *modalities can be defined as* $!(U,X,\alpha) = (U, U \to X^*, !\alpha)$ *and* $\xi(U,X,\alpha) = (U,X,\xi\alpha)$ *where* X^* *is the free commutative monoid on* X*,* $(!\alpha)(u,f) = \alpha(u,x_1)\circ\cdots\circ\alpha(u,x_i)$ *for* $f(u) = (x_1,\ldots,x_i)$*, and* $(\xi\alpha)(u,x) = \xi(\alpha(u,x))$*.*

This definition highlights a fundamental difference between the two modalities. The definition of the exchange modality relies on an extension of biclosed posets with essentially the exchange modality in the category of posets. However, the of-course modality is defined by the structure already present in $\mathsf{Dial}_M(\mathsf{Set})$, specifically, the structure of Set.

Both of the modalities have the structure of a comonad. That is, there are monoidal natural transformations $\varepsilon_! : !A \longrightarrow A$, $\varepsilon_\xi : \xi A \longrightarrow A$, $\delta_! : !A \longrightarrow !!A$, and $\delta_\xi : \xi A \longrightarrow \xi\xi A$ which satisfy the appropriate diagrams; see the formalization for the full proofs. Furthermore, these comonads come equipped with arrows $w : !A \longrightarrow I$, $d : !A \longrightarrow !A \otimes !A$, $\mathsf{ex}_{A,B} : \xi A \otimes \xi B \longrightarrow \xi B \otimes \xi A$.

Finally, using the fact that $\mathsf{Dial}_M(\mathsf{Set})$ for any biclosed poset is essentially a non-commutative formalization of Bierman's linear categories [5] we can use Benton's construction of an LNL model from a linear category to obtain a LAM model, and hence, obtain the following.

Theorem 20. *Suppose* M *is a biclosed poset with exchange. Then* $\mathsf{Dial}_M(\mathsf{Set})$ *is a sound model for CNC logic.*

6 Future Work

We introduce the idea above of having a modality for exchange, but what about individual modalities for weakening and contraction? Indeed it is possible to give modalities for these structural rules as well using adjoint models. Now that we have each structural rule isolated into their own modality is it possible to put them together to form new modalities that combine structural rules? The answer to this question has already been shown to be positive, at least for weakening and contraction, by Melliés [15], but we plan to extend this line of work to include exchange.

The monads induced by the adjunction in CNC logic is non-commutative, but Benton and Wadler show that the monads induced by the adjunction in LNL logic [3] are commutative. Using the extension of Melliés' work we mention above would allow us to combine both CNC logic with LNL logic, and then be able to support both commutative monads as well as non-commutative monads. We plan on exploring this in the future.

Hasegawa [9] studies the linear of-course modality, $!A$, as a comonad induced by an adjunction between a Cartesian closed category a (non-symmetric) monoidal category. The results here generalizes his by generalizing the Cartesian closed category to a symmetric monoidal closed category. However, his approach focuses on the comonad rather than the adjunctions. It would be interesting to do the same for LAM as well.

References

[1] Samson Abramsky. Proofs as processes. *Theoretical Computer Science*, 135(1):5 – 9, 1994. URL: http://www.sciencedirect.com/science/article/pii/0304397594001030, doi:http://dx.doi.org/10.1016/0304-3975(94)00103-0.

[2] Guy Barry, Mark Hepple, Neil Leslie, and Glyn Morrill. Proof figures and structural operators for categorial grammar. In *Proceedings of the Fifth Conference on European Chapter of the Association for Computational Linguistics*, EACL '91, pages 198–203, Stroudsburg, PA, USA, 1991. Association for Computational Linguistics. URL: https://doi.org/10.3115/977180.977215, doi:10.3115/977180.977215.

[3] Nick Benton and Philip Wadler. Linear logic, monads and the lambda calculus. In *Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science*, LICS '96, pages 420–, Washington, DC, USA, 1996. IEEE Computer Society.

[4] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994.

[5] G. M. Bierman. *On Intuitionistic Linear Logic*. PhD thesis, Wolfson College, Cambridge, December 1993.

[6] Ana Bove, Peter Dybjer, and Ulf Norell. A brief overview of Agda-a functional language with dependent types. *TPHOLs*, 5674:73–78, 2009.

[7] Valeria de Paiva. *The dialectica categories*. PhD thesis, Computer Laboratory, University of Cambridge, PhD Thesis, 1990. Computer Laboratory, University of Cambridge, PhD Thesis.

[8] Valeria de Paiva and Harley Eades. *Dialectica Categories for the Lambek Calculus*, pages 256–272. Springer International Publishing, Cham, 2018. URL: https://doi.org/10.1007/978-3-319-72056-2_16, doi:10.1007/978-3-319-72056-2_16.

[9] Masahito Hasegawa. Linear exponential comonads without symmetry. In Iliano Cervesato and Maribel Fernández, editors, Proceedings Fourth International Workshop on *Linearity,* Porto, Portugal, 25 June 2016, volume 238 of *Electronic Proceedings in Theoretical Computer Science*, pages 54–63. Open Publishing Association, 2017. doi:10.4204/EPTCS.238.6.

[10] Mark Hepple. *The grammar and processing of order and dependency: A categorial approach*. PhD thesis.

[11] Kohei Honda and Olivier Laurent. An exact correspondence between a typed pi-calculus and polarised proof-nets. *Theoretical Computer Science*, 411(22):2223 – 2238, 2010. URL: http://www.sciencedirect.com/science/article/pii/S0304397510000538, doi:http://dx.doi.org/10.1016/j.tcs.2010.01.028.

[12] G. M. Kelly. Doctrinal adjunction. In *Category Seminar*, volume 420 of *Lecture Notes in Mathematics*, pages 257–280. Springer, 1974.

[13] Anders Kock. Strong functors and monoidal monads. *Archiv der Mathematik*, 23(1):113–120, 1972.

[14] Joachim Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, pages 154–170, 1958.

[15] Paul-André Melliés. Comparing hierarchies of types in models of linear logic. *Information and Computation*, 189(2):202 – 234, 2004.

[16] Paul-André Melliés. Categorical semantics of linear logic. In *Interactive Models of Computation and Program Behaviour*, volume 27 of *Panoramas et Synthses*, pages 1–196. Socit Mathmatique de France, 2009.

[17] Michael Moortgat. Categorial investigations: Logical and linguistics aspects of the lambek calculus. 1988.

[18] Richard T Oehrle, Emmon Bach, and Deirdre Wheeler. *Categorial grammars and natural language structures*, volume 32. Springer Science & Business Media, 2012.

[19] Mati Pentus. Models for the Lambek calculus. *Annals of Pure and Applied Logic*, 75(1):179 – 213, 1995. URL: `http://www.sciencedirect.com/science/article/pii/0168007294000639`, `doi:http://dx.doi.org/10.1016/0168-0072(94)00063-9`.

[20] Vaughan R. Pratt. Types as processes, via Chu spaces. *Electronic Notes in Theoretical Computer Science*, 7:227 – 247, 1997.

A Proof For Lemma 2

A.1 Commuting Conversion Cut vs. Cut

A.1.1 $\mathcal{C}$-Cut vs. $\mathcal{C}$-Cut

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_C X\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi_1\\ \Psi_2, X, \Psi_3 \vdash_C Y\end{array} \quad \begin{array}{c}\pi_2\\ \Psi_1, Y, \Psi_4 \vdash_C Z\end{array}}{\Psi_1, \Psi_2, X, \Psi_3, \Psi_4 \vdash_C Z}\ \text{CUT}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. Therefore, $c(\pi_1)$, $c(\pi_2) \leq |X|$. Since Y is the cut formula on π_1 and π_2, we have $|Y|+1 \leq |X|$. By induction on Π_1 and π_1 there exists a proof Π' for sequent $\Psi_2, \Phi, \Psi_3 \vdash_C Y$ s.t. $c(\Pi') \leq |X|$. So Π can be constructed as follows, with $c(\Pi) \leq max\{c(\Pi'), c(\pi_2), |Y|+1\} \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Psi_2, \Phi, \Psi_3 \vdash_C Y\end{array} \quad \begin{array}{c}\pi_2\\ \Psi_1, Y, \Psi_4 \vdash_C Z\end{array}}{\Psi_1, \Psi_2, \Phi, \Psi_3, \Psi_4 \vdash_C Z}\ \text{CUT}$$

- Case 2:

$$\Pi_1 : \dfrac{\begin{array}{c}\pi_1\\ \Phi \vdash_C X\end{array} \quad \begin{array}{c}\pi_2\\ \Psi_2, X, \Psi_3 \vdash_C Y\end{array}}{\Psi_2, \Phi, \Psi_3 \vdash_C Y}\ \text{CUT} \qquad \begin{array}{c}\Pi_2\\ \Psi_1, Y, \Psi_4 \vdash_C Z\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. Since the cut rank of the last cut in Π_1 is $|X|+1$, then $|X|+1 \leq |Y|$. By induction on Π_1 and Π_2, there is a proof Π' for sequent $\Psi_1, \Psi_2, X, \Psi_3, \Psi_4 \vdash_C Z$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\Pi'), |X|+1\} \leq |Y|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Phi \vdash_C X\end{array} \quad \begin{array}{c}\Pi'\\ \Psi_1, \Psi_2, X, \Psi_3, \Psi_4 \vdash_C Z\end{array}}{\Psi_1, \Psi_2, \Phi, \Psi_3, \Psi_4 \vdash_C Z}\ \text{CUT}$$

A.1.2 $\mathcal{C}$-Cut vs. $\mathcal{LC}$-Cut

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi_2\\ \Psi_1, X, \Psi_2 \vdash_{\mathcal{C}} Y\end{array} \quad \begin{array}{c}\pi_3\\ \Gamma_1; Y; \Gamma_2 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; \Psi_1; X; \Psi_2; \Gamma_2 \vdash_{\mathcal{L}} A}\ \text{CUT1}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. Therefore, $c(\pi_1)$, $c(\pi_2) \leq |X|$. Since Y is the cut formula on π_1 and π_2, we have $|Y| + 1 \leq |X|$. By induction on Π_1 and π_1, there exists a proof Π' for sequent $\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} Y$ s.t. $c(\Pi') \leq |X|$. So Π can be constructed as follows, with $c(\Pi) \leq max\{c(\Pi'), c(\pi_2), |Y| + 1\} \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} Y\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1; Y; \Gamma_2 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; \Psi_1; \Phi; \Psi_2; \Gamma_2 \vdash_{\mathcal{L}} A}\ \text{CUT1}$$

- Case 2:

$$\Pi_1 : \dfrac{\begin{array}{c}\pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \quad \begin{array}{c}\pi_2\\ \Psi_1, X, \Psi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} Y}\ \text{CUT} \qquad \begin{array}{c}\Pi_2\\ \Gamma_1; Y; \Gamma_2 \vdash_{\mathcal{L}} A\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. Similar as above, $|X| + 1 \leq |Y|$ and there is a proof Π' constructed from π_2 and Π_2 for sequent $\Gamma_1; \Psi_1; X; \Psi_2; \Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\Pi'), |X| + 1\} \leq |Y|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \quad \begin{array}{c}\Pi'\\ \Gamma_1; \Psi_1; X; \Psi_2; \Gamma_2 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; \Psi_1; \Phi; \Psi_2; \Gamma_2 \vdash_{\mathcal{L}} A}\ \text{CUT}$$

A.1.3 $\mathcal{LC}$-Cut vs. $\mathcal{L}$-Cut

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi_1\\ \Gamma_2; X; \Gamma_3 \vdash_{\mathcal{L}} A\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1; A; \Gamma_4 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1; \Gamma_2; X; \Gamma_3; \Gamma_4 \vdash_{\mathcal{L}} B}\ \text{CUT2}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. Therefore, $c(\pi_1)$, $c(\pi_2) \leq |X|$. Since A is the cut formula on π_1 and π_2, we have $|A| + 1 \leq |X|$. By induction on Π_1 and π_1, there exists a proof Π' for sequent $\Gamma_2; \Phi; \Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. So Π can be constructed as follows, with $c(\Pi) \leq max\{c(\Pi'), c(\pi_2), |A| + 1\} \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_2; \Phi; \Gamma_3 \vdash_{\mathcal{L}} A\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1; A; \Gamma_4 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1; \Gamma_2; \Phi; \Gamma_3; \Gamma_4 \vdash_{\mathcal{L}} B}\ \text{CUT2}$$

- Case 2:

$$\Pi_1 : \dfrac{\begin{array}{c}\pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_2; X; \Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_2; \Phi; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{CUT} \qquad \begin{array}{c}\Pi_2\\ \Gamma_1; A; \Gamma_4 \vdash_{\mathcal{L}} B\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A|$. Similar as above, $|X| + 1 \leq |A|$ and there is a proof Π' constructed from' π_2 and Π_2 for sequent $\Gamma_1; \Gamma_2; X; \Gamma_3; \Gamma_4 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |A|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\Pi'), |X| + 1\} \leq |A|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \quad \begin{array}{c}\Pi'\\ \Gamma_1; \Gamma_2; X; \Gamma_3; \Gamma_4 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1; \Gamma_2; \Phi; \Gamma_3; \Gamma_4 \vdash_{\mathcal{L}} B}\ \text{CUT}$$

A.1.4 $\mathcal{L}$-Cut vs. $\mathcal{L}$-Cut

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Gamma \vdash_{\mathcal{L}} A\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi_1\\ \Delta_2;A;\Delta_3 \vdash_{\mathcal{L}} B\end{array} \quad \begin{array}{c}\pi_2\\ \Delta_1;B;\Delta_4 \vdash_{\mathcal{L}} C\end{array}}{\Delta_1;\Delta_2;A;\Delta_3;\Delta_4 \vdash_{\mathcal{L}} C}\ \textsc{cut2}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A|$. Therefore, $c(\pi_1)$, $c(\pi_2) \leq |A|$. Since B is the cut formula on π_1 and π_3, we have $|B|+1 \leq |A|$. By induction on Π_1 and π_1, there exists a proof Π' for sequent $\Delta_2;\Gamma;\Delta_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |A|$. So Π can be constructed as follows, with $c(\Pi) \leq max\{c(\Pi'), c(\pi_2), |B|+1\} \leq |A|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Delta_2;\Gamma;\Delta_3 \vdash_{\mathcal{L}} B\end{array} \quad \begin{array}{c}\pi_2\\ \Delta_1;B;\Delta_4 \vdash_{\mathcal{L}} C\end{array}}{\Delta_1;\Delta_2;\Gamma;\Delta_3;\Delta_4 \vdash_{\mathcal{L}} C}\ \textsc{cut}$$

- Case 2:

$$\Pi_1 : \dfrac{\begin{array}{c}\pi_1\\ \Delta \vdash_{\mathcal{L}} A\end{array} \quad \begin{array}{c}\pi_2\\ \Delta_2;A;\Delta_3 \vdash_{\mathcal{L}} B\end{array}}{\Delta_2;\Delta;\Delta_3 \vdash_{\mathcal{L}} A}\ \textsc{cut} \qquad \begin{array}{c}\Pi_2\\ \Delta_1;B;\Delta_4 \vdash_{\mathcal{L}} C\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. Similar as above, $|A|+1 \leq |B|$ and there is a proof Π' constructed from π_2 and Π_2 for sequent $\Delta_1;\Delta_2;A;\Delta_3;\Delta_4 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |A|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\Pi'), |A|+1\} \leq |B|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash_{\mathcal{L}} A\end{array} \quad \begin{array}{c}\Pi'\\ \Delta_1;\Delta_2;A;\Delta_3;\Delta_4 \vdash_{\mathcal{L}} C\end{array}}{\Delta_1;\Delta_2;\Gamma;\Delta_3;\Delta_4 \vdash_{\mathcal{L}} C}\ \textsc{cut}$$

A.2 The Axiom Steps

A.2.1 $\mathcal{C}$-ax

- Case 1:

$$\Pi_1 : \dfrac{}{X \vdash_{\mathcal{C}} X}\ \textsc{ax} \qquad \begin{array}{c}\Pi_2\\ \Phi_1, X, \Phi_2 \vdash_{\mathcal{C}} Y\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. The proof Π is the same as Π_2.

- Case 2:

$$\Pi_1 : \begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2 : \dfrac{}{X \vdash_{\mathcal{C}} X}\ \textsc{ax}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. The proof Π is the same as Π_1.

- Case 3:

$$\Pi_1 : \dfrac{}{X \vdash_{\mathcal{C}} X}\ \textsc{ax} \qquad \begin{array}{c}\Pi_2\\ \Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} A\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. The proof Π is the same as Π_2.

A.2.2 $\mathcal{C}$-ax

- Case 1:

$$\Pi_1 : \frac{}{A \vdash_{\mathcal{L}} A}\text{AX} \qquad \begin{array}{c}\Pi_2 \\ \Gamma_1; A; \Gamma_2 \vdash_{\mathcal{L}} B\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A|$. The proof Π is the same as Π_2.

- Case 2:

$$\Pi_1 : \begin{array}{c}\Pi_1 \\ \Delta \vdash_{\mathcal{L}} A\end{array} \qquad \Pi_2 : \frac{}{A \vdash_{\mathcal{L}} A}\text{AX}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. The proof Π is the same as Π_1.

A.3 The Exchange Steps

A.3.1 $\mathcal{C}$-ex

- Case 1:

$$\begin{array}{c}\Pi_1 \\ \Psi \vdash_{\mathcal{C}} X_1\end{array} \qquad \Pi_2 : \frac{\begin{array}{c}\pi \\ \Phi_1, X_1, X_2, \Phi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1, X_2, X_1, \Phi_2 \vdash_{\mathcal{C}} Y}\text{EX}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X_1|$. By induction on π and Π_1, there is a proof Π' for sequent $\Phi_1, \Psi, X_2, \Phi_2 \vdash_{\mathcal{C}} Y$ s.t. $c(\Pi') \leq |X_1|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X_1|$.

$$\frac{\begin{array}{c}\Pi' \\ \Phi_1, \Psi, X_2, \Phi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1, X_2, \Psi, \Phi_2 \vdash_{\mathcal{C}} Y}\text{SERIES OF EX}$$

- Case 2:

$$\begin{array}{c}\Pi_1 \\ \Psi \vdash_{\mathcal{C}} X_2\end{array} \qquad \Pi_2 : \frac{\begin{array}{c}\pi \\ \Phi_1, X_1, X_2, \Phi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1, X_2, X_1, \Phi_2 \vdash_{\mathcal{C}} Y}\text{EX}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X_2|$. By induction on π and Π_1, there is a proof Π' for sequent $\Phi_1, X_1, \Psi, \Phi_2 \vdash_{\mathcal{C}} Y$ s.t. $c(\Pi') \leq |X_2|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X_2|$.

$$\frac{\begin{array}{c}\Pi' \\ \Phi_1, X_1, \Psi, \Phi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1, \Psi, X_1, \Phi_2 \vdash_{\mathcal{C}} Y}\text{SERIES OF EX}$$

A.3.2 $\mathcal{L}$-ex

- Case 1:

$$\begin{array}{c}\Pi_1 \\ \Psi \vdash_{\mathcal{C}} X_1\end{array} \qquad \Pi_2 : \frac{\begin{array}{c}\pi \\ \Delta_1; X_1; X_2; \Delta_2 \vdash_{\mathcal{L}} A\end{array}}{\Delta_1; X_2; X_1; \Delta_2 \vdash_{\mathcal{L}} A}\text{EX}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X_1|$. By induction on π and Π_1, there is a proof Π' for sequent $\Delta_1; \Psi; X_2; \Delta_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X_1|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X_1|$.

$$\dfrac{\overset{\Pi'}{\Delta_1;\Psi;X_2;\Delta_2 \vdash_{\mathcal{L}} A}}{\Delta_1;X_2;\Psi;\Delta_2 \vdash_{\mathcal{L}} A}\ \textsc{series of ex}$$

- Case 2:

$$\overset{\Pi_1}{\Psi \vdash_C X_2} \qquad \Pi_2:\ \dfrac{\overset{\pi}{\Delta_1;X_1;X_2;\Delta_2 \vdash_{\mathcal{L}} A}}{\Delta_1;X_2;X_1;\Delta_2 \vdash_{\mathcal{L}} A}\ \textsc{ex}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X_2|$. By induction on π and Π_1, there is a proof Π' for sequent $\Delta_1;X_1;\Psi;\Delta_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X_2|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X_2|$.

$$\dfrac{\overset{\Pi'}{\Delta_1;X_1;\Psi;\Phi_2 \vdash_{\mathcal{L}} A}}{\Phi_1;\Psi;X_1;\Phi_2 \vdash_{\mathcal{L}} A}\ \textsc{series of ex}$$

A.4 Principal Formula vs. Principal Formula

A.4.1 The Commutative Tensor Product $\otimes$

$$\Pi_1:\ \dfrac{\overset{\pi_1}{\Phi_1 \vdash_C X} \quad \overset{\pi_2}{\Phi_2 \vdash_C Y}}{\Phi_1,\Phi_2 \vdash_C X \otimes Y}\ \textsc{tenR} \qquad \Pi_2:\ \dfrac{\overset{\pi_3}{\Psi_1, X, Y, \Psi_2 \vdash_C Z}}{\Psi_1, X \otimes Y, \Psi_2 \vdash_C Z}\ \textsc{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X \otimes Y| = |X| + |Y| + 1$. The proof Π can be constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\pi_2), c(\pi_3), |X|+1, |Y|+1\} \leq |X| + |Y| + 1 = |X \otimes Y|$.

$$\dfrac{\overset{\pi_1}{\Phi_1 \vdash_C X} \quad \dfrac{\overset{\pi_2}{\Phi_2 \vdash_C Y} \quad \overset{\pi_3}{\Psi_1, X, Y, \Psi_2 \vdash_C Z}}{\Psi_1, X, \Phi_2, \Psi_2 \vdash_C Z}\ \textsc{cut}}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C Z}\ \textsc{cut}$$

A.4.2 The Non-commutative Tensor Product $\triangleright$

$$\Pi_1:\ \dfrac{\overset{\pi_1}{\Gamma_1 \vdash_{\mathcal{L}} A} \quad \overset{\pi_2}{\Gamma_2 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Gamma_2 \vdash_{\mathcal{L}} A \triangleright B}\ \textsc{tenR} \qquad \Pi_2:\ \dfrac{\overset{\pi_3}{\Delta_1;A;B;\Delta_2 \vdash_{\mathcal{L}} C}}{\Delta_1;A \triangleright B;\Delta_2 \vdash_{\mathcal{L}} C}\ \textsc{tenL1}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A \triangleright B| = |X| + |Y| + 1$. The proof Π can be constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\pi_2), c(\pi_3), |A|+1, |B|+1\} \leq |A| + |B| + 1 = |A \triangleright B|$.

$$\dfrac{\overset{\pi_1}{\Gamma_1 \vdash_{\mathcal{L}} A} \quad \dfrac{\overset{\pi_2}{\Gamma_2 \vdash_{\mathcal{L}} B} \quad \overset{\pi_3}{\Delta_1;A;B;\Delta_2 \vdash_{\mathcal{L}} C}}{\Delta_1;A;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} C}\ \textsc{cut2}}{\Delta_1;\Gamma_1;\Gamma_2;\Psi_2 \vdash_{\mathcal{L}} C}\ \textsc{cut2}$$

A.4.3 The Commutative Implication $\multimap$

$$\Pi_1:\ \dfrac{\overset{\pi_1}{\Phi_1, X \vdash_C Y}}{\Phi_1 \vdash_C X \multimap Y}\ \textsc{tenR} \qquad \Pi_2:\ \dfrac{\overset{\pi_2}{\Phi_2 \vdash_C X} \quad \overset{\pi_3}{\Psi_1, Y, \Psi_2 \vdash_C Z}}{\Psi_1, X \multimap Y, \Phi, \Psi_2 \vdash_C Z}\ \textsc{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X \multimap Y| = |X| + |Y| + 1$. The proof Π is constructed as follows $c(\Pi) \leq max\{c(\pi_1), c(\pi_2), c(\pi_3), |X|+1, |Y|+1\} \leq |X| + |Y| + 1 = |X \multimap Y|$.

$$\dfrac{\dfrac{\overset{\pi_1}{\Phi_1, X \vdash_C Y} \quad \overset{\pi_2}{\Phi_2 \vdash_C X}}{\Phi_1, \Phi_2 \vdash_C Y}\ \textsc{tenR} \quad \overset{\pi_3}{\Psi_1, Y, \Psi_2 \vdash_C Z}}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C Z}\ \textsc{tenR}$$

A.4.4 The Non-commutative Right Implication $\rightharpoonup$

$$\Pi_1 : \dfrac{\begin{array}{c}\pi_1\\ \Gamma;A \vdash_{\mathcal{L}} B\end{array}}{\Gamma \vdash_{\mathcal{L}} A \rightharpoonup B}\ \textsc{imprR} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi_2\\ \Delta_1 \vdash_{\mathcal{L}} A\end{array} \quad \begin{array}{c}\pi_3\\ \Delta_2;B \vdash_{\mathcal{L}} C\end{array}}{\Delta_2;A \rightharpoonup B;\Delta_1 \vdash_{\mathcal{L}} C}\ \textsc{imprL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A \rightharpoonup B| = |A| + |B| + 1$. The proof Π is constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\pi_2), c(\pi_3), |A| + 1, |B| + 1\} \leq |A| + |B| + 1 = |A \rightharpoonup B|$.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma;A \vdash_{\mathcal{L}} B\end{array} \quad \begin{array}{c}\pi_2\\ \Delta_1 \vdash_{\mathcal{L}} A\end{array}}{\Gamma;\Delta_1 \vdash_{\mathcal{L}} B}\ \textsc{cut2} \quad \begin{array}{c}\pi_3\\ \Delta_2;B \vdash_{\mathcal{L}} C\end{array}}{\Delta_2;\Gamma;\Delta_1 \vdash_{\mathcal{L}} C}\ \textsc{cut2}$$

A.4.5 The Non-commutative Left Implication $\leftharpoonup$

$$\Pi_1 : \dfrac{\begin{array}{c}\pi_1\\ A;\Gamma \vdash_{\mathcal{L}} B\end{array}}{\Gamma \vdash_{\mathcal{L}} B \leftharpoonup A}\ \textsc{implR} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi_2\\ \Delta_1 \vdash_{\mathcal{L}} A\end{array} \quad \begin{array}{c}\pi_3\\ B;\Delta_2 \vdash_{\mathcal{L}} C\end{array}}{\Delta_1;B \leftharpoonup A;\Delta_2 \vdash_{\mathcal{L}} C}\ \textsc{implL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B \leftharpoonup A| = |A| + |B| + 1$. The proof Π is constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\pi_2), c(\pi_3), |A| + 1, |B| + 1\} \leq |A| + |B| + 1 = |B \leftharpoonup A|$.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ A;\Gamma \vdash_{\mathcal{L}} B\end{array} \quad \begin{array}{c}\pi_2\\ \Delta_1 \vdash_{\mathcal{L}} A\end{array}}{\Delta_1;\Gamma \vdash_{\mathcal{L}} B}\ \textsc{cut2} \quad \begin{array}{c}\pi_3\\ B;\Delta_2 \vdash_{\mathcal{L}} C\end{array}}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} C}\ \textsc{cut1}$$

A.4.6 The Commutative Unit Unit

- Case 1:

$$\Pi_1 : \dfrac{}{\cdot \vdash_{\mathcal{C}} \mathsf{Unit}}\ \textsc{unitR} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi\\ \Phi,\Psi \vdash_{\mathcal{C}} X\end{array}}{\Phi,\mathsf{Unit},\Psi \vdash_{\mathcal{C}} X}\ \textsc{unitL}$$

  By assumption, $c(\Pi_1), c(\Pi_2) \leq |\mathsf{Unit}|$. The proof Π is the subproof π in Π_2 for sequent $\Phi \vdash_{\mathcal{C}} X$. So $c(\Pi) = c(\Pi_2) \leq |\mathsf{Unit}|$.

- Case 2:

$$\Pi_1 : \dfrac{}{\cdot \vdash_{\mathcal{C}} \mathsf{Unit}}\ \textsc{unitR} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi\\ \Gamma;\Delta \vdash_{\mathcal{L}} A\end{array}}{\Gamma;\mathsf{Unit};\Delta \vdash_{\mathcal{L}} A}\ \textsc{unitL1}$$

  Similar as above, Π is π.

A.4.7 The Non-commutative Unit Unit

$$\Pi_1 : \dfrac{}{\cdot \vdash_{\mathcal{L}} \mathsf{Unit}}\ \textsc{unitR} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi\\ \Gamma;\Delta \vdash_{\mathcal{L}} A\end{array}}{\Gamma;\mathsf{Unit};\Delta \vdash_{\mathcal{L}} A}\ \textsc{unitL2}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |\mathsf{Unit}|$. The proof Π is the subproof π in Π_2 for sequent $\Delta \vdash_{\mathcal{L}} A$. So $c(\Pi) = c(\Pi_2) \leq |\mathsf{Unit}|$.

A.4.8 The Functor F

$$\Pi_1 : \dfrac{\begin{matrix}\pi_1 \\ \Phi \vdash_C X\end{matrix}}{\Phi \vdash_{\mathcal{L}} \mathsf{F}X}\,\text{FR} \qquad \Pi_2 : \dfrac{\begin{matrix}\pi_2 \\ \Gamma;X;\Delta \vdash_{\mathcal{L}} A\end{matrix}}{\Gamma;\mathsf{F}X;\Delta \vdash_{\mathcal{L}} A}\,\text{FL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |\mathsf{F}X| = |X| + 1$. The proof Π is constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\pi_2), |X| + 1\} \leq |\mathsf{F}X|$.

$$\dfrac{\begin{matrix}\pi_1 \\ \Phi \vdash_C X\end{matrix} \quad \begin{matrix}\pi_2 \\ \Gamma;X;\Delta \vdash_{\mathcal{L}} A\end{matrix}}{\Gamma;\Phi;\Delta \vdash_{\mathcal{L}} A}\,\text{cut2}$$

A.4.9 The Functor G

$$\Pi_1 : \dfrac{\begin{matrix}\pi_1 \\ \Phi \vdash_{\mathcal{L}} A\end{matrix}}{\Phi \vdash_C \mathsf{G}A}\,\text{GR} \qquad \Pi_2 : \dfrac{\begin{matrix}\pi_2 \\ \Gamma;A;\Delta \vdash_{\mathcal{L}} B\end{matrix}}{\Gamma;\mathsf{G}A;\Delta \vdash_{\mathcal{L}} B}\,\text{GL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |\mathsf{G}A| = |A| + 1$. The proof Π is constructed as follows, and $c(\Pi) \leq max\{c(\pi_1), c(\pi_2), |A| + 1\} \leq |\mathsf{G}A|$.

$$\dfrac{\begin{matrix}\pi_1 \\ \Phi \vdash_{\mathcal{L}} A\end{matrix} \quad \begin{matrix}\pi_2 \\ \Gamma;A;\Delta \vdash_{\mathcal{L}} B\end{matrix}}{\Gamma;\Phi;\Delta \vdash_{\mathcal{L}} B}\,\text{GL}$$

A.5 Secondary Conclusion

A.5.1 Left introduction of the commutative implication $\multimap$

- Case 1:

$$\Pi_1 : \dfrac{\begin{matrix}\pi_1 \\ \Phi_1 \vdash_C X_1\end{matrix} \quad \begin{matrix}\pi_2 \\ \Phi_2, X_2, \Phi_3 \vdash_C Y\end{matrix}}{\Phi_2, X_1 \multimap X_2, \Phi_1, \Phi_3 \vdash_C Y}\,\text{impL} \qquad \begin{matrix}\Pi_2 \\ \Psi_1, Y, \Psi_2 \vdash_C Z\end{matrix}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. By induction, there is a proof Π' from π_2 and Π_2 for sequent $\Psi_1, \Phi_2, X_2, \Phi_3, \Psi_2 \vdash_C Z$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |Y|$.

$$\dfrac{\begin{matrix}\pi_1 \\ \Phi_1 \vdash_C X_1\end{matrix} \quad \dfrac{\begin{matrix}\pi_2 \\ \Phi_2, X_2, \Phi_3 \vdash_C Y\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Psi_1, Y, \Psi_2 \vdash_C Z\end{matrix}}{\Psi_1, \Phi_2, X_2, \Phi_3, \Psi_2 \vdash_C Z}\,\text{cut}}{\Psi_1, \Phi_2, X_1 \multimap X_2, \Phi_1, \Phi_3, \Psi_2 \vdash_C Z}\,\text{impL}$$

- Case 2:

$$\Pi_1 : \dfrac{\begin{matrix}\pi_1 \\ \Phi_1 \vdash_C X_1\end{matrix} \quad \begin{matrix}\pi_2 \\ \Phi_2, X_2, \Phi_3 \vdash_C Y\end{matrix}}{\Phi_2, X_1 \multimap X_2, \Phi_1, \Phi_3 \vdash_C Y}\,\text{impL} \qquad \begin{matrix}\Pi_2 \\ \Gamma_1; Y; \Gamma_2 \vdash_{\mathcal{L}} A\end{matrix}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. By induction, there is a proof Π' from π_2 and Π_2 for sequent $\Gamma_1; \Phi_2; X_2; \Phi_3; \Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |Y|$.

$$\dfrac{\begin{matrix}\pi_1 \\ \Phi_1 \vdash_C X_1\end{matrix} \quad \dfrac{\begin{matrix}\pi_2 \\ \Phi_2, X_2, \Phi_3 \vdash_C Y\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma_1; Y; \Gamma_2 \vdash_{\mathcal{L}} A\end{matrix}}{\Gamma_1; \Phi_2; X_2; \Phi_3; \Gamma_2 \vdash_{\mathcal{L}} A}\,\text{cut}}{\Gamma_1; \Phi_2; X_1 \multimap X_2; \Phi_1; \Phi_3; \Gamma_2 \vdash_{\mathcal{L}} A}\,\text{impL}$$

A.5.2 Left introduction of the non-commutative left implication $\rightharpoonup$

$$\Pi_1 : \quad \dfrac{\begin{array}{c}\pi_1 \\ \Gamma_1 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2 \\ \Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_2;A_1 \rightharpoonup A_2;\Gamma_1;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{impL} \qquad\qquad \begin{array}{c}\Pi_2 \\ \Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction, there is a proof Π' from π_2 and Π_2 for sequent $\Delta_1;\Gamma_2;A_2;\Gamma_3;\Delta_2 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |B|$.

$$\dfrac{\begin{array}{c}\pi_1 \\ \Gamma_1 \vdash_{\mathcal{L}} A_1\end{array} \quad \dfrac{\begin{array}{c}\pi_2 \\ \Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array} \quad \begin{array}{c}\Pi_2 \\ \Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C\end{array}}{\Delta_1;\Gamma_2;A_2;\Gamma_3;\Delta_2 \vdash_{\mathcal{L}} C}\ \text{cut}}{\Delta_1;\Gamma_2;A_1 \rightharpoonup A_2;\Gamma_1;\Gamma_3;\Delta_2 \vdash_{\mathcal{L}} C}\ \text{impL}$$

A.5.3 Left introduction of the non-commutative right implication $\leftharpoonup$

$$\Pi_1 : \quad \dfrac{\begin{array}{c}\pi_1 \\ \Gamma_1 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2 \\ \Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_2;\Gamma_1;A_2 \leftharpoonup A_1;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{impL} \qquad\qquad \begin{array}{c}\Pi_2 \\ \Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction, there is a proof Π' from π_2 and Π_2 for sequent $\Delta_1;\Gamma_2;A_2;\Gamma_3;\Delta_2 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |B|$.

$$\dfrac{\begin{array}{c}\pi_1 \\ \Gamma_1 \vdash_{\mathcal{L}} A_1\end{array} \quad \dfrac{\begin{array}{c}\pi_2 \\ \Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array} \quad \begin{array}{c}\Pi_2 \\ \Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C\end{array}}{\Delta_1;\Gamma_2;A_2;\Gamma_3;\Delta_2 \vdash_{\mathcal{L}} C}\ \text{cut}}{\Delta_1;\Gamma_2;\Gamma_1;A_2 \leftharpoonup A_1;\Gamma_3;\Delta_2 \vdash_{\mathcal{L}} C}\ \text{impL}$$

A.5.4 $\mathcal{C}$-ex

- Case 1:

$$\Pi_1 : \quad \dfrac{\begin{array}{c}\pi \\ \Phi_1,X_1,X_2,\Phi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1,X_2,X_1,\Phi_2 \vdash_{\mathcal{C}} Y}\ \text{ex} \qquad\qquad \begin{array}{c}\Pi_2 \\ \Psi_1,Y,\Psi_2 \vdash_{\mathcal{C}} Z\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. By induction on π and Π_2, there is a proof Π' for sequent $\Psi_1,\Phi_1,X_1,X_2,\Phi_2,\Psi_2 \vdash_{\mathcal{C}} Z$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |Y|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Psi_1,\Phi_1,X_1,X_2,\Phi_2,\Psi_2 \vdash_{\mathcal{C}} Z\end{array}}{\Psi_1,\Phi_1,X_2,X_1,\Phi_2,\Psi_2 \vdash_{\mathcal{C}} Z}\ \text{ex}$$

- Case 2:

$$\Pi_1 : \quad \dfrac{\begin{array}{c}\pi \\ \Phi_1,X,Y,\Phi_2 \vdash_{\mathcal{C}} Z\end{array}}{\Phi_1,Y,X,\Phi_2 \vdash_{\mathcal{C}} Z}\ \text{beta} \qquad\qquad \begin{array}{c}\Pi_2 \\ \Gamma_1;Z;\Gamma_2 \vdash_{\mathcal{L}} A\end{array}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Z|$. Similar as above, there is a proof Π' constructed from π and Π_2 for $\Gamma_1;\Phi_1;X;Y;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |Z|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |Z|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;\Phi_1;X;Y;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\Phi_1;Y;X;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A}\ \text{beta}$$

A.5.5 $\mathcal{L}$-ex

$$\Pi_1:\ \dfrac{\overset{\pi}{\Gamma_1;X;Y;\Gamma_2 \vdash_{\mathcal{L}} A}}{\Gamma_1;Y;X;\Gamma_2 \vdash_{\mathcal{L}} A}\ \text{BETA} \qquad \overset{\Pi_2}{\Delta_1;A;\Delta_2 \vdash_{\mathcal{L}} B}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A|$. Similar as above, there is a proof Π' constructed from π and Π_2 for sequent $\Delta_1;\Gamma_1;X;Y;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |A|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |A|$.

$$\dfrac{\overset{\Pi'}{\Delta_1;\Gamma_1;X;Y;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B}}{\Delta_1;\Gamma_1;Y;X;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B}\ \text{BETA}$$

A.5.6 Left introduction of the commutative tensor product $\otimes$

- Case 1:

$$\Pi_1:\ \dfrac{\overset{\pi}{\Phi_1,X_1,X_2,\Phi_2 \vdash_{C} Y}}{\Phi_1,X_1 \otimes X_2,\Phi_2 \vdash_{C} Y}\ \text{TENL} \qquad \overset{\Pi_2}{\Psi_1,Y,\Psi_2 \vdash_{C} Z}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. By induction, there is a proof Π' from π and Π_2 for sequent $\Psi_1,\Phi_1,X_1,X_2,\Phi_2,\Psi_2 \vdash_{C} Z$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |Y|$.

$$\dfrac{\dfrac{\overset{\pi}{\Phi_1,X_1,X_2,\Phi_2 \vdash_{C} Y} \quad \overset{\Pi_2}{\Psi_1,Y,\Psi_2 \vdash_{C} Z}}{\Psi_1,\Phi_1,X_1,X_2,\Phi_2,\Psi_2 \vdash_{C} Z}\ \text{CUT}}{\Psi_1,\Phi_1,X_1 \otimes X_2,\Phi_2,\Psi_2 \vdash_{C} Z}\ \text{TENL}$$

- Case 2:

$$\Pi_1:\ \dfrac{\overset{\pi}{\Phi_1,X_1,X_2,\Phi_2 \vdash_{C} Y}}{\Phi_1,X_1 \otimes X_2,\Phi_2 \vdash_{C} Y}\ \text{TENL} \qquad \overset{\Pi_2}{\Gamma_1;Y;\Gamma_2 \vdash_{\mathcal{L}} A}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |Y|$. By induction, there is a proof Π' from π and Π_2 for sequent $\Gamma_1;\Phi_1;X_1;X_2;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |Y|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |Y|$.

$$\dfrac{\dfrac{\overset{\pi}{\Phi_1,X_1,X_2,\Phi_2 \vdash_{C} Y} \quad \overset{\Pi_2}{\Gamma_1;Y;\Gamma_2 \vdash_{\mathcal{L}} A}}{\Gamma_1;\Phi_1;X_1;X_2;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A}\ \text{CUT1}}{\Gamma_1;\Phi_1;X_1 \otimes X_2;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A}\ \text{TENL1}$$

- Case 3:

$$\Pi_1:\ \dfrac{\overset{\pi}{\Gamma_1;X;Y;\Gamma_2 \vdash_{\mathcal{L}} A}}{\Gamma_1;X \otimes Y;\Gamma_2 \vdash_{\mathcal{L}} A}\ \text{TENL} \qquad \overset{\Pi_2}{\Delta_1;A;\Delta_2 \vdash_{\mathcal{L}} B}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A|$. By induction, there is a proof Π' from π and Π_2 for sequent $\Delta_1;X;Y;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |A|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |A|$.

$$\dfrac{\dfrac{\overset{\pi}{\Gamma_1;X;Y;\Gamma_2 \vdash_{\mathcal{L}} A} \quad \overset{\Pi_2}{\Delta_1;A;\Delta_2 \vdash_{\mathcal{L}} B}}{\Delta_1;\Gamma_1;X;Y;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B}\ \text{CUT2}}{\Delta_1;\Gamma_1;X \otimes Y;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B}\ \text{TENL1}$$

A.5.7 Left introduction of the non-commutative tensor products ▷

$$\Pi_1: \dfrac{\overset{\pi}{\Gamma_1;A_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B}}{\Gamma_1;A_1 \triangleright A_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{tenL2} \qquad \overset{\Pi_2}{\Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction, there is a proof Π' from π and Π_2 for sequent $\Delta_1;\Gamma_1;A_1;A_2;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |B|$.

$$\dfrac{\dfrac{\overset{\pi}{\Gamma_1;A_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B} \quad \overset{\Pi_2}{\Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C}}{\Delta_1;\Gamma_1;A_1;A_2;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} C}\ \text{cut2}}{\Delta_1;\Gamma_1;A_1 \triangleright A_2;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} C}\ \text{tenL2}$$

A.5.8 Left introduction of the commutative unit Unit

- Case 1:

$$\Pi_1: \dfrac{\overset{\pi}{\Phi_1,\Phi_2 \vdash_{\mathcal{C}} X}}{\Phi_1,\mathsf{Unit},\Phi_2 \vdash_{\mathcal{C}} X}\ \text{unitL} \qquad \overset{\Pi_2}{\Psi_1,X,\Psi_2 \vdash_{\mathcal{C}} Y}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction, there is a proof Π' from π and Π_2 for sequent $\Psi_1,\Phi_1,\Phi_2,\Psi_2 \vdash_{\mathcal{C}} Y$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\Pi'}{\Psi_1,\Phi_1,\Phi_2,\Psi_2 \vdash_{\mathcal{C}} Y}}{\Psi_1,\Phi_1,\mathsf{Unit},\Phi_2,\Psi_2 \vdash_{\mathcal{C}} Y}\ \text{unitL}$$

- Case 2:

$$\Pi_1: \dfrac{\overset{\pi}{\Phi_1,\Phi_2 \vdash_{\mathcal{C}} X}}{\Phi_1,\mathsf{Unit},\Phi_2 \vdash_{\mathcal{C}} X}\ \text{unitL} \qquad \overset{\Pi_2}{\Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} A}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction, there is a proof Π' from π and Π_2 for sequent $\Gamma_1;\Phi_1;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\Pi'}{\Gamma_1;\Phi_1;\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A}}{\Gamma_1;\Phi_1;\mathsf{Unit};\Phi_2;\Gamma_2 \vdash_{\mathcal{L}} A}\ \text{unitL}$$

- Case 3:

$$\Pi_1: \dfrac{\overset{\pi}{\Delta_1;\Delta_2 \vdash_{\mathcal{L}} A}}{\Delta_1;\mathsf{Unit};\Delta_2 \vdash_{\mathcal{L}} A}\ \text{unitL} \qquad \overset{\Pi_2}{\Gamma_1;A;\Gamma_2 \vdash_{\mathcal{L}} B}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction, there is a proof Π' from π and Π_2 for sequent $\Gamma_1;\Delta_1;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\Pi'}{\Gamma_1;\Delta_1;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Delta_1;\mathsf{Unit};\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{unitL}$$

A.5.9 Left introduction of the non-commutative unit Unit

$$\Pi_1: \dfrac{\overset{\pi}{\Delta_1;\Delta_2 \vdash_{\mathcal{L}} A}}{\Delta_1;\mathsf{Unit};\Delta_2 \vdash_{\mathcal{L}} A}\ \textsc{unitL} \qquad \overset{\Pi_2}{\Gamma_1;A;\Gamma_2 \vdash_{\mathcal{L}} B}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction, there is a proof Π' from π and Π_2 for sequent

$$\Gamma_1;\Delta_1;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B$$

s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows, and $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\Pi'}{\Gamma_1;\Delta_1;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Delta_1;\mathsf{Unit};\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \textsc{unitL}$$

A.5.10 Left introduction of the functor F

$$\Pi_1: \dfrac{\overset{\pi_1}{\Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} A}}{\Gamma_1;\mathsf{F}X;\Gamma_2 \vdash_{\mathcal{L}} A}\ \text{FL} \qquad \overset{\Pi_2}{\Delta_1;A;\Delta_2 \vdash_{\mathcal{L}} B}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |A|$. By induction, there is a proof Π' from π_2 and Π_2 for sequent $\Delta_1;\Gamma_1;X;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |A|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |A|$.

$$\dfrac{\dfrac{\overset{\pi_2}{\Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} A} \quad \overset{\Pi_2}{\Delta_1;A;\Delta_2 \vdash_{\mathcal{L}} B}}{\Delta_1;\Gamma_1;X;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B}\ \textsc{cut2}}{\Delta_1;\Gamma_1;\mathsf{F}X;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} B}\ \text{FL}$$

A.5.11 Left introduction of the functor G

$$\Pi_1: \dfrac{\overset{\pi_1}{\Gamma_1;A;\Gamma_2 \vdash_{\mathcal{L}} B}}{\Gamma_1;\mathsf{G}A;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{GL} \qquad \overset{\Pi_2}{\Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction, there is a proof Π' from π_2 and Π_2 for sequent $\Delta_1;\Gamma_1;A;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) \leq |B|$.

$$\dfrac{\dfrac{\overset{\pi_2}{\Gamma_1;A;\Gamma_2 \vdash_{\mathcal{L}} B} \quad \overset{\Pi_2}{\Delta_1;B;\Delta_2 \vdash_{\mathcal{L}} C}}{\Delta_1;\Gamma_1;A;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} C}\ \textsc{cut2}}{\Delta_1;\Gamma_1;\mathsf{G}A;\Gamma_2;\Delta_2 \vdash_{\mathcal{L}} C}\ \text{GL}$$

A.6 Secondary Hypothesis

A.6.1 Right introduction of the commutative tensor product $\otimes$

- Case 1:

$$\overset{\Pi_1}{\Phi_2 \vdash_{\mathcal{C}} X} \qquad \Pi_2: \dfrac{\overset{\pi_1}{\Psi_1,X,\Psi_2 \vdash_{\mathcal{C}} Y_1} \quad \overset{\pi_2}{\Phi_1 \vdash_{\mathcal{C}} Y_2}}{\Psi_1,X,\Psi_2,\Phi_1 \vdash_{\mathcal{C}} Y_1 \otimes Y_2}\ \textsc{tenR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Psi_1,\Phi_2,\Psi_2 \vdash_{\mathcal{C}} Y_1$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{matrix}\Pi' \\ \Psi_1, \Phi_2, \Psi_2 \vdash_C Y_1\end{matrix} \quad \begin{matrix}\pi_1 \\ \Phi_1 \vdash_C Y_2\end{matrix}}{\Psi_1, \Phi_2, \Psi_2, \Phi_1 \vdash_C Y_1 \otimes Y_2}\ \text{tenR}$$

- Case 2:

$$\begin{matrix}\Pi_1 \\ \Phi_2 \vdash_C X\end{matrix} \qquad \Pi_2: \dfrac{\begin{matrix}\pi_1 \\ \Phi_1 \vdash_C Y_1\end{matrix} \quad \begin{matrix}\pi_2 \\ \Psi_1, X, \Psi_2 \vdash_C Y_2\end{matrix}}{\Phi_1, \Psi_1, X, \Psi_2 \vdash_C Y_1 \otimes Y_2}\ \text{tenR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Psi_1, \Phi_2, \Psi_2 \vdash_C Y_2$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{matrix}\pi_1 \\ \Phi_1 \vdash_C Y_1\end{matrix} \quad \begin{matrix}\Pi' \\ \Psi_1, \Phi_2, \Psi_2 \vdash_C Y_2\end{matrix}}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_C Y_1 \otimes Y_2}\ \text{tenR}$$

A.6.2 Right introduction of the non-commutative tensor product $\triangleright$

- Case 1:

$$\begin{matrix}\Pi_1 \\ \Phi \vdash_C X\end{matrix} \qquad \Pi_2: \dfrac{\begin{matrix}\pi_1 \\ \Gamma_1; X; \Gamma_2 \vdash_{\mathcal{L}} A\end{matrix} \quad \begin{matrix}\pi_2 \\ \Gamma_3 \vdash_{\mathcal{L}} B\end{matrix}}{\Gamma_1; X; \Gamma_2; \Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\ \text{tenR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Gamma_1; \Phi; \Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{matrix}\Pi' \\ \Gamma_1; \Phi; \Gamma_2 \vdash_{\mathcal{L}} A\end{matrix} \quad \begin{matrix}\pi_1 \\ \Gamma_3 \vdash_{\mathcal{L}} B\end{matrix}}{\Gamma_1; \Phi; \Gamma_2; \Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\ \text{tenR}$$

- Case 2:

$$\begin{matrix}\Pi_1 \\ \Delta \vdash_{\mathcal{L}} C\end{matrix} \qquad \Pi_2: \dfrac{\begin{matrix}\pi_1 \\ \Gamma_1; C; \Gamma_2 \vdash_{\mathcal{L}} A\end{matrix} \quad \begin{matrix}\pi_2 \\ \Gamma_3 \vdash_{\mathcal{L}} B\end{matrix}}{\Gamma_1; C; \Gamma_2; \Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\ \text{tenR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |C|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Gamma_1; \Delta; \Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |C|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |C|$.

$$\dfrac{\begin{matrix}\Pi' \\ \Gamma_1; \Delta; \Gamma_2 \vdash_{\mathcal{L}} A\end{matrix} \quad \begin{matrix}\pi_1 \\ \Gamma_3 \vdash_{\mathcal{L}} B\end{matrix}}{\Gamma_1; \Delta; \Gamma_2; \Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\ \text{tenR}$$

- Case 3:

$$\begin{matrix}\Pi_1 \\ \Phi \vdash_C X\end{matrix} \qquad \Pi_2: \dfrac{\begin{matrix}\pi_1 \\ \Gamma_1 \vdash_{\mathcal{L}} A\end{matrix} \quad \begin{matrix}\pi_2 \\ \Gamma_2; X; \Gamma_3 \vdash_{\mathcal{L}} B\end{matrix}}{\Gamma_1; \Gamma_2; X; \Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\ \text{tenR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_2; \Phi; \Gamma_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\pi_1}{\Gamma_1 \vdash_{\mathcal{L}} A} \qquad \overset{\Pi'}{\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\,\textsc{tenR}$$

- Case 4:

$$\overset{\Pi_1}{\Delta \vdash_{\mathcal{L}} C} \qquad \Pi_2: \dfrac{\overset{\pi_1}{\Gamma_1 \vdash_{\mathcal{L}} A} \qquad \overset{\pi_2}{\Gamma_2;C;\Gamma_3 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Gamma_2;C;\Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\,\textsc{tenR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |C|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |C|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |C|$.

$$\dfrac{\overset{\pi_1}{\Gamma_1 \vdash_{\mathcal{L}} A} \qquad \overset{\Pi'}{\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A \triangleright B}\,\textsc{tenR}$$

A.6.3 Left introduction of the commutative implication $\multimap$

- Case 1:

$$\overset{\Pi_1}{\Phi \vdash_{\mathcal{C}} X} \qquad \Pi_2: \dfrac{\overset{\pi_1}{\Psi_2, X, \Psi_3 \vdash_{\mathcal{C}} Y_1} \qquad \overset{\pi_2}{\Psi_1, Y_2, \Psi_4 \vdash_{\mathcal{C}} Z}}{\Psi_1, Y_1 \multimap Y_2, \Psi_2, X, \Psi_3, \Psi_4 \vdash_{\mathcal{C}} Z}\,\textsc{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Psi_2, \Phi, \Psi_3 \vdash_{\mathcal{C}} Y_1$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\Pi'}{\Psi_2, \Phi, \Psi_3 \vdash_{\mathcal{C}} Y_1} \qquad \overset{\pi_2}{\Psi_1, Y_2, \Psi_4 \vdash_{\mathcal{C}} Z}}{\Psi_1, Y_1 \multimap Y_2, \Psi_2, \Phi, \Psi_3, \Psi_4 \vdash_{\mathcal{C}} Z}\,\textsc{impL}$$

- Case 2:

$$\overset{\Pi_1}{\Phi \vdash_{\mathcal{C}} X} \qquad \Pi_2: \dfrac{\overset{\pi_1}{\Psi_3 \vdash_{\mathcal{C}} Y_1} \qquad \overset{\pi_2}{\Psi_1, X, \Psi_2, Y_2, \Psi_4 \vdash_{\mathcal{C}} Z}}{\Psi_1, X, \Psi_2, Y_1 \multimap Y_2, \Psi_3, \Psi_4 \vdash_{\mathcal{C}} Z}\,\textsc{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Psi_1, \Phi, \Psi_2, Y_2, \Psi_4 \vdash_{\mathcal{C}} Z$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\pi_1}{\Psi_3 \vdash_{\mathcal{C}} Y_1} \qquad \overset{\Pi'}{\Psi_1, \Phi, \Psi_2, Y_2, \Psi_4 \vdash_{\mathcal{C}} Z}}{\Psi_1, \Phi_1, \Psi_2, Y_1 \multimap Y_2, \Psi_3, \Psi_4 \vdash_{\mathcal{C}} Z}\,\textsc{impL}$$

- Case 3:

$$\overset{\Pi_1}{\Phi \vdash_{\mathcal{C}} X} \qquad \Pi_2: \dfrac{\overset{\pi_1}{\Psi_2 \vdash_{\mathcal{C}} Y_1} \qquad \overset{\pi_2}{\Psi_1, Y_2, \Psi_3, X, \Psi_4 \vdash_{\mathcal{C}} Z}}{\Psi_1, Y_1 \multimap Y_2, \Psi_2, \Psi_3, X, \Psi_4 \vdash_{\mathcal{C}} Z}\,\textsc{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Psi_1, \Phi, \Psi_2, Y_2, \Psi_4 \vdash_{\mathcal{C}} Z$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\dfrac{\pi_1}{\Psi_2 \vdash_C Y_1} \quad \dfrac{\Pi'}{\Psi_1, Y_2, \Psi_3, \Phi, \Psi_4 \vdash_C Z}}{\Psi_1, Y_1 \multimap Y_2, \Psi_2, \Psi_3, \Phi, \Psi_4 \vdash_C Z}\ \text{impL}$$

- Case 4:

$$\dfrac{\Pi_1}{\Phi \vdash_C X} \qquad \Pi_2: \dfrac{\dfrac{\pi_1}{\Psi_1, X, \Psi_2 \vdash_C Y_1} \quad \dfrac{\pi_2}{\Gamma_1; Y_2; \Gamma_2 \vdash_{\mathcal{L}} A}}{\Gamma_1; Y_1 \multimap Y_2; \Psi_1; X; \Psi_2; \Gamma_2 \vdash_{\mathcal{L}} A}\ \text{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Psi_1, \Phi, \Psi_2 \vdash_C Y_1$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\dfrac{\Pi'}{\Psi_1, \Phi, \Psi_2 \vdash_C Y_1} \quad \dfrac{\pi_2}{\Gamma_1; Y_2; \Gamma_2 \vdash_{\mathcal{L}} A}}{\Gamma_1; Y_1 \multimap Y_2; \Psi_1; \Phi; \Psi_2; \Gamma_2 \vdash_{\mathcal{L}} A}\ \text{impL}$$

- Case 5:

$$\dfrac{\Pi_1}{\Phi \vdash_C X} \qquad \Pi_2: \dfrac{\dfrac{\pi_1}{\Psi \vdash_C Y_1} \quad \dfrac{\pi_2}{\Gamma_1; X; \Gamma_2; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1; X; \Gamma_2; Y_1 \multimap Y_2; \Psi; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1; \Phi; \Gamma_2; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\dfrac{\pi_1}{\Psi \vdash_C Y_1} \quad \dfrac{\Pi'}{\Gamma_1; \Phi; \Gamma_2; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1; \Phi; \Gamma_2; Y_1 \multimap Y_2; \Psi; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{impL}$$

- Case 6:

$$\dfrac{\Pi_1}{\Delta \vdash_{\mathcal{L}} B} \qquad \Pi_2: \dfrac{\dfrac{\pi_1}{\Psi \vdash_C Y_1} \quad \dfrac{\pi_2}{\Gamma_1; B; \Gamma_2; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1; B; \Gamma_2; Y_1 \multimap Y_2; \Psi; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1; \Delta; \Gamma_2; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\dfrac{\pi_1}{\Psi \vdash_C Y_1} \quad \dfrac{\Pi'}{\Gamma_1; \Delta; \Gamma_2; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1; \Delta; \Gamma_2; Y_1 \multimap Y_2; \Psi; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{impL}$$

- Case 7:

$$\dfrac{\Pi_1}{\Phi \vdash_C X} \qquad \Pi_2: \dfrac{\dfrac{\pi_1}{\Psi \vdash_C Y_1} \quad \dfrac{\pi_2}{\Gamma_1; Y_2; \Gamma_2; X; \Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1; Y_1 \multimap Y_2; \Psi; \Gamma_2; X; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1; Y_2; \Gamma_2; \Phi; \Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\dfrac{\pi_1}{\Psi \vdash_C Y_1} \quad \dfrac{\Pi'}{\Gamma_1; Y_2; \Gamma_2; \Phi; \Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1; Y_1 \multimap Y_2; \Psi; \Gamma_2; \Phi; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{impL}$$

- Case 8:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2:\ \dfrac{\begin{array}{c}\pi_1\\ \Psi \vdash_{C} Y_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;Y_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_1 \multimap Y_2;\Psi;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{IMPL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;Y_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Psi \vdash_{C} Y_1\end{array} \quad \begin{array}{c}\Pi'\\ \Gamma_1;Y_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_1 \multimap Y_2;\Psi;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{IMPL}$$

A.6.4 Left introduction of the non-commutative left implication $\rightharpoonup$

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{C} X\end{array} \qquad \Pi_2:\ \dfrac{\begin{array}{c}\pi_1\\ \Delta_1;X;\Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta_1;X;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{IMPRL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Delta_1;\Phi;\Delta_2 \vdash_{\mathcal{L}} A_1$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Delta_1;\Phi;\Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta_1;\Phi;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{IMPL}$$

- Case 2:

$$\begin{array}{c}\Pi_1\\ \Gamma \vdash_{\mathcal{L}} C\end{array} \qquad \Pi_2:\ \dfrac{\begin{array}{c}\pi_1\\ \Delta_1;C;\Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta_1;C;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{IMPRL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |C|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} A_1$ s.t. $c(\Pi') \leq |C|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |C|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta_1;\Gamma;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{IMPRL}$$

- Case 3:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{C} X\end{array} \qquad \Pi_2:\ \dfrac{\begin{array}{c}\pi_1\\ \Delta \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;X;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;X;\Gamma_2;A_1 \rightharpoonup A_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{IMPRL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;\Phi;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Delta \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\Pi'\\ \Gamma_1;\Phi;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;\Phi;\Gamma_2;A_1 \rightharpoonup A_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{IMPRL}$$

- Case 4:

$$\begin{array}{c}\Pi_1\\ \Delta_1 \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi_1\\ \Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;B;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;B;\Gamma_2;A_1 \rightharpoonup A_2;\Delta_2;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{imprL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;\Delta_1;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\Pi'\\ \Gamma_1;\Delta_1;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;\Delta_1;\Gamma_2;A_1 \rightharpoonup A_2;\Delta_2;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{imprL}$$

- Case 5:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi_1\\ \Delta \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{imprL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;A_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Delta \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\Pi'\\ \Gamma_1;A_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{imprL}$$

- Case 6:

$$\begin{array}{c}\Pi_1\\ \Delta_1 \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi_1\\ \Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{imprL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;A_2;\Gamma_2;\Delta_1;\Gamma_3 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\pi_1\\ \Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\Pi'\\ \Gamma_1;A_2;\Gamma_2;\Delta_1;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;A_1 \rightharpoonup A_2;\Delta_2;\Gamma_2;\Delta_1;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{imprL}$$

A.6.5 Left introduction of the non-commutative right implication $\leftharpoonup$

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi_1\\ \Delta_1;X;\Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;\Delta_1;A_2 \leftharpoonup A_1;X;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{implL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Delta_1;\Phi;\Delta_2 \vdash_{\mathcal{L}} A_1$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Delta_1;\Phi;\Delta_2 \vdash_{\mathcal{L}} A_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;\Delta_1;A_2 \leftharpoonup A_1;\Phi;\Delta_2;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{implL}$$

- Case 2:

$$\begin{array}{c}\Pi_1\\ \Gamma \vdash_{\mathcal{L}} C\end{array} \qquad \Pi_2: \ \frac{\overset{\pi_1}{\Delta_1;C;\Delta_2 \vdash_{\mathcal{L}} A_1} \quad \overset{\pi_2}{\Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Delta_1;C;\Delta_2;A_2 \leftharpoonup A_1;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{IMPLL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |C|$. By induction on Π_1 and π_1, there is a proof Π' for sequent $\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} A_1$ s.t. $c(\Pi') \leq |C|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |C|$.

$$\frac{\overset{\Pi'}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} A_1} \quad \overset{\pi_2}{\Gamma_1;A_2;\Gamma_2 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Delta_1;\Gamma;\Delta_2;A_2 \leftharpoonup A_1;\Gamma_2 \vdash_{\mathcal{L}} B}\ \text{IMPLL}$$

- Case 3:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \ \frac{\overset{\pi_1}{\Delta \vdash_{\mathcal{L}} A_1} \quad \overset{\pi_2}{\Gamma_1;X;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B}}{\Gamma_1;X;\Gamma_2;\Delta;A_2 \leftharpoonup A_1;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{IMPLL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;\Phi;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\frac{\overset{\pi_1}{\Delta \vdash_{\mathcal{L}} A_1} \quad \overset{\Pi'}{\Gamma_1;\Phi;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Phi;\Gamma_2;\Delta;A_2 \leftharpoonup A_1;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{IMPLL}$$

- Case 4:

$$\begin{array}{c}\Pi_1\\ \Delta_1 \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \ \frac{\overset{\pi_1}{\Delta_2 \vdash_{\mathcal{L}} A_1} \quad \overset{\pi_2}{\Gamma_1;B;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} C}}{\Gamma_1;B;\Gamma_2;\Delta_2;A_2 \leftharpoonup A_1;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{IMPLL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;\Delta_1;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\frac{\overset{\pi_1}{\Delta_2 \vdash_{\mathcal{L}} A_1} \quad \overset{\Pi'}{\Gamma_1;\Delta_1;\Gamma_2;A_2;\Gamma_3 \vdash_{\mathcal{L}} C}}{\Gamma_1;\Delta_1;\Gamma_2;\Delta_2;A_2 \leftharpoonup A_1;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{IMPLL}$$

- Case 5:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \ \frac{\overset{\pi_1}{\Delta \vdash_{\mathcal{L}} A_1} \quad \overset{\pi_2}{\Gamma_1;A_2;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Delta;A_2 \leftharpoonup A_1;\Delta;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{IMPLL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;A_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\frac{\overset{\pi_1}{\Delta \vdash_{\mathcal{L}} A_1} \quad \overset{\Pi'}{\Gamma_1;A_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B}}{\Gamma_1;\Delta;A_2 \leftharpoonup A_1;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{IMPLL}$$

- Case 6:

$$\begin{array}{c}\Pi_1\\ \Delta_1 \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \ \frac{\overset{\pi_1}{\Delta_2 \vdash_{\mathcal{L}} A_1} \quad \overset{\pi_2}{\Gamma_1;A_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} C}}{\Gamma_1;\Delta_2;A_2 \leftharpoonup A_1;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{IMPLL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π_2, there is a proof Π' for sequent $\Gamma_1;A_2;\Gamma_2;\Delta_1;\Gamma_3 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\frac{\overset{\pi_1}{\Delta_2 \vdash_{\mathcal{L}} A_1} \quad \overset{\Pi'}{\Gamma_1;A_2;\Gamma_2;\Delta_1;\Gamma_3 \vdash_{\mathcal{L}} C}}{\Gamma_1;\Delta_2;A_2 \leftharpoonup A_1;\Gamma_2;\Delta_1;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{IMPLL}$$

A.6.6 Left introduction of the commutative tensor ⊗ (with low priority)

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_C X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Psi_1, X, \Psi_2, Y_1, Y_2, \Psi_3 \vdash_C Z\end{array}}{\Psi_1, X, \Psi_2, Y_1 \otimes Y_2, \Psi_3 \vdash_C Z}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Psi_1, \Phi, \Psi_2, Y_1, Y_2, \Psi_3 \vdash_C Z$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Psi_1, \Phi, \Psi_2, Y_1, Y_2, \Psi_3 \vdash_C Z\end{array}}{\Psi_1, \Phi, \Psi_2, Y_1 \otimes Y_2, \Psi_3 \vdash_C Z}\ \text{tenL}$$

- Case 2:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_C X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Psi_1, Y_1, Y_2, \Psi_2, X, \Psi_3 \vdash_C Z\end{array}}{\Psi_1, Y_1 \otimes Y_2, \Psi_2, X, \Psi_3 \vdash_C Z}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Psi_1, Y_1, Y_2, \Psi_2, \Phi, \Psi_3 \vdash_C Z$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Psi_1, Y_1, Y_2, \Psi_2, \Phi, \Psi_3 \vdash_C Z\end{array}}{\Psi_1, Y_1 \otimes Y_2, \Psi_2, \Phi, \Psi_3 \vdash_C Z}\ \text{tenL}$$

- Case 3:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_C X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1; X; \Gamma_2; Y_1; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; X; \Gamma_2; Y_1 \otimes Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1; \Phi; \Gamma_2; Y_1; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1; \Phi; \Gamma_2; Y_1; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; \Phi; \Gamma_2; Y_1 \otimes Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

- Case 4:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1; B; \Gamma_2; Y_1; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; B; \Gamma_2; Y_1 \otimes Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1; B; \Gamma_2; Y_1; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1; \Delta; \Gamma_2; Y_1; Y_2; \Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1; \Delta; \Gamma_2; Y_1 \otimes Y_2; \Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

- Case 5:

$$\begin{array}{c}\Pi_1 \\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;Y_1;Y_2;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_1\otimes Y_2;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;Y_1;Y_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;Y_1;Y_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_1\otimes Y_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

- Case 6:

$$\begin{array}{c}\Pi_1 \\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;Y_1;Y_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_1\otimes Y_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;Y_1;Y_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;Y_1;Y_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_1\otimes Y_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{tenL}$$

A.6.7 Left introduction of the non-commutative tensor $\triangleright$ (with low priority)

:

- Case 1:

$$\begin{array}{c}\Pi_1 \\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;X;\Gamma_2;A_1;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;X;\Gamma_2;A_1\triangleright A_2;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Phi;\Gamma_2;A_1;A_2;\Gamma_3 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;\Phi;\Gamma_2;A_1;A_2;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;\Phi;\Gamma_2;A_1\triangleright A_2;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{tenL}$$

- Case 2:

$$\begin{array}{c}\Pi_1 \\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;B;\Gamma_2;A_1;A_2;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;B;\Gamma_2;A_1\triangleright A_2;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Delta;\Gamma_2;A_1;A_2;\Gamma_3 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;\Delta;\Gamma_2;A_1;A_2;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;\Delta;\Gamma_2;A_1\triangleright A_2;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{tenL}$$

- Case 3:

$$\begin{array}{c}\Pi_1 \\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;A_1;A_2;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \triangleright A_2;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;A_1;A_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;A_1;A_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;A_1 \triangleright A_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} B}\ \text{tenL}$$

- Case 4:

$$\begin{array}{c}\Pi_1 \\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;A_1;A_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;A_1 \triangleright A_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{tenL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;A_1;A_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} C$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;A_1;A_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} C\end{array}}{\Gamma_1;A_1 \triangleright A_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} C}\ \text{tenL}$$

A.6.8 $\mathcal{C}$-ex

- Case 1:

$$\begin{array}{c}\Pi_1 \\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Psi_1,X,\Psi_2,Y_1,Y_2,\Psi_3 \vdash_{\mathcal{C}} Z\end{array}}{\Psi_1,X,\Psi_2,Y_2,Y_1,\Psi_3 \vdash_{\mathcal{C}} Z}\ \text{beta}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Psi_1,\Phi,\Psi_2,Y_1,Y_2,\Psi_3 \vdash_{\mathcal{C}} Z$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Psi_1,\Phi,\Psi_2,Y_1,Y_2,\Psi_3 \vdash_{\mathcal{C}} Z\end{array}}{\Psi_1,\Phi,\Psi_2,Y_2,Y_1,\Psi_3 \vdash_{\mathcal{C}} Z}\ \text{cut}$$

- Case 2:

$$\begin{array}{c}\Pi_1 \\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Psi_1,Y_1,Y_2,\Psi_2,X,\Psi_3 \vdash_{\mathcal{C}} Z\end{array}}{\Psi_1,X,\Psi_2,Y_2,Y_1,\Psi_3 \vdash_{\mathcal{C}} Z}\ \text{beta}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Psi_1,Y_1,Y_2,\Psi_2,\Phi,\Psi_3 \vdash_{\mathcal{C}} Z$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Psi_1,Y_1,Y_2,\Psi_2,\Phi,\Psi_3 \vdash_{\mathcal{C}} Z\end{array}}{\Psi_1,Y_2,Y_1,\Psi_2,\Phi,\Psi_3 \vdash_{\mathcal{C}} Z}\ \text{cut}$$

A.6.9 $\mathcal{L}$-ex

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;X;\Gamma_2;Y_1;Y_2;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;X;\Gamma_2;Y_2;Y_1;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{beta}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Phi;\Gamma_2;Y_1;Y_2;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;\Phi;\Gamma_2;Y_1;Y_2;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\Phi;\Gamma_2;Y_2;Y_1;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{cut}$$

- Case 2:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;B;\Gamma_2;Y_1;Y_2;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;B;\Gamma_2;Y_2;Y_1;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{beta}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Delta;\Gamma_2;Y_1;Y_2;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;\Delta;\Gamma_2;Y_1;Y_2;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\Delta;\Gamma_2;Y_2;Y_1;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{cut}$$

- Case 3:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;Y_1;Y_2;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;X;\Gamma_2;Y_2;Y_1;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{beta}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;Y_1;Y_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;Y_1;Y_2;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_2;Y_1;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{cut}$$

- Case 4:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;Y_1;Y_2;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_2;Y_1;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{beta}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;Y_1;Y_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;Y_1;Y_2;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;Y_2;Y_1;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\ \textsc{cut}$$

A.6.10 Left introduction of the commutative unit Unit (with low priority)

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Psi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi\\ \Phi_1,\Phi_2,X,\Phi_3 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1,\mathsf{Unit},\Phi_2,X,\Phi_3 \vdash_{\mathcal{C}} Y}\ \text{UNITL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Phi_1,\Phi_2,\Psi,\Phi_3 \vdash_{\mathcal{C}} Y$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Phi_1,\Phi_2,\Psi,\Phi_3 \vdash_{\mathcal{C}} Y\end{array}}{\Phi_1,\mathsf{Unit},\Phi_2,\Psi,\Phi_3 \vdash_{\mathcal{C}} Y}\ \text{UNITL}$$

- Case 2:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi\\ \Gamma_1;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{Unit};\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{UNITL1}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Gamma_2;\Phi;\Gamma_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{Unit};\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{UNITL1}$$

- Case 3:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi\\ \Gamma_1;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{Unit};\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{UNITL1}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{Unit};\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{UNITL1}$$

A.6.11 Left introduction of the non-commutative unit Unit (with low priority)

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2 : \dfrac{\begin{array}{c}\pi\\ \Gamma_1;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{Unit};\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{UNITL2}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{Unit};\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{UNITL2}$$

- Case 2:

$$\Pi_1 \quad \Delta \vdash_{\mathcal{L}} B \qquad \Pi_2: \dfrac{\overset{\pi}{\Gamma_1;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1;\mathsf{Unit};\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{unitL2}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\overset{\Pi'}{\Gamma_1;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}}{\Gamma_1;\mathsf{Unit};\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{unitL2}$$

A.6.12 Right introduction of the commutative implication ⊸ (with low priority)

$$\Pi_1 \quad \Phi \vdash_{\mathcal{C}} X \qquad \Pi_2: \dfrac{\overset{\pi}{\Psi_1, X, \Psi_2, Y_1 \vdash_{\mathcal{C}} Y_2}}{\Psi_1, X, \Psi_2 \vdash_{\mathcal{C}} Y_1 \multimap Y_2}\ \text{impR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent

$$\Psi_1, \Phi, \Psi_2, Y_1 \vdash_{\mathcal{C}} Y_2 \text{ s.t. } c(\Pi') \leq |X|$$

Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\Pi'}{\Psi_1, \Phi, \Psi_2, Y_1 \vdash_{\mathcal{C}} Y_2}}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} Y_1 \multimap Y_2}\ \text{impR}$$

A.6.13 Right introduction of the non-commutative left implication ⇀ (with low priority)

- Case 1:

$$\Pi_1 \quad \Phi \vdash_{\mathcal{C}} X \qquad \Pi_2: \dfrac{\overset{\pi}{\Gamma_1;X;\Gamma_2;A \vdash_{\mathcal{L}} B}}{\Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} A \rightharpoonup B}\ \text{impR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Phi;\Gamma_2;A \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\overset{\Pi'}{\Gamma_1;\Phi;\Gamma_2;A \vdash_{\mathcal{L}} B}}{\Gamma_1;\Phi;\Gamma_2 \vdash_{\mathcal{L}} A \rightharpoonup B}\ \text{implR}$$

- Case 2:

$$\Pi_1 \quad \Delta \vdash_{\mathcal{L}} C \qquad \Pi_2: \dfrac{\overset{\pi}{\Gamma_1;C;\Gamma_2;A \vdash_{\mathcal{L}} B}}{\Gamma_1;C;\Gamma_2 \vdash_{\mathcal{L}} A \rightharpoonup B}\ \text{impR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |C|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Delta;\Gamma_2;A \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |C|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |C|$.

$$\dfrac{\overset{\Pi'}{\Gamma_1;\Delta;\Gamma_2;A \vdash_{\mathcal{L}} B}}{\Gamma_1;\Delta;\Gamma_2 \vdash_{\mathcal{L}} A \rightharpoonup B}\ \text{implR}$$

A.6.14 Right introduction of the non-commutative right implication $\leftharpoonup$ (with low priority)

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ A;\Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;X;\Gamma_2 \vdash_{\mathcal{L}} B \leftharpoonup A}\ \text{impL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $A;\Gamma_1;\Phi;\Gamma_2 \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ A;\Gamma_1;\Phi;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;\Phi;\Gamma_2 \vdash_{\mathcal{L}} B \leftharpoonup A}\ \text{impR}$$

- Case 2:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} C\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ A;\Gamma_1;C;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;C;\Gamma_2 \vdash_{\mathcal{L}} B \leftharpoonup A}\ \text{impR}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |C|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Delta;\Gamma_2;A \vdash_{\mathcal{L}} B$ s.t. $c(\Pi') \leq |C|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |C|$.

$$\dfrac{\begin{array}{c}\Pi'\\ A;\Gamma_1;\Delta;\Gamma_2 \vdash_{\mathcal{L}} B\end{array}}{\Gamma_1;\Delta;\Gamma_2 \vdash_{\mathcal{L}} B \leftharpoonup A}\ \text{impR}$$

A.6.15 Right introduction of the functor F

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Psi_1,X,\Psi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Psi_1,X,\Psi_2 \vdash_{\mathcal{L}} \mathsf{F}Y}\ \text{Fr}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{C}} Y$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{C}} Y\end{array}}{\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{L}} \mathsf{F}Y}\ \text{Fr}$$

A.6.16 Left introduction of the functor F (with low priority)

- Case 1:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;X;\Gamma_2;Y;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;X;\Gamma_2;\mathsf{F}Y;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{Fl}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Phi;\Gamma_2;Y;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;\Phi;\Gamma_2;Y;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\Phi;\Gamma_2;\mathsf{F}Y;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{Fl}$$

- Case 2:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;B;\Gamma_2;Y;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;B;\Gamma_2;\mathsf{F}Y;\Gamma_3 \vdash_{\mathcal{L}} A}\,\textsc{Fl}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Delta;\Gamma_2;Y;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;\Delta;\Gamma_2;Y;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\Delta;\Gamma_2;\mathsf{F}Y;\Gamma_3 \vdash_{\mathcal{L}} A}\,\textsc{Fl}$$

- Case 3:

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;Y;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{F}Y;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A}\,\textsc{Fl}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;Y;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;Y;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{F}Y;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A}\,\textsc{Fl}$$

- Case 4:

$$\begin{array}{c}\Pi_1\\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Gamma_1;Y;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{F}Y;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\,\textsc{Fl}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;Y;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma_1;Y;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{F}Y;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\,\textsc{Fl}$$

A.6.17 Right introduction of the functor G (with low priority)

$$\begin{array}{c}\Pi_1\\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi\\ \Psi_1;X;\Psi_2 \vdash_{\mathcal{L}} A\end{array}}{\Psi_1,X,\Psi_2 \vdash_{\mathcal{C}} \mathsf{G}A}\,\textsc{Gr}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi'\\ \Psi_1;\Phi;\Psi_2 \vdash_{\mathcal{L}} A\end{array}}{\Psi_1,\Phi,\Psi_2 \vdash_{\mathcal{C}} \mathsf{G}A}\,\textsc{Gr}$$

A.6.18 Left introduction of the functor G (with low priority)

- Case 1:

$$\begin{array}{c}\Pi_1 \\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;X;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;X;\Gamma_2;\mathsf{G}B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Phi;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;\Phi;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\Phi;\Gamma_2;\mathsf{G}B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

- Case 2:

$$\begin{array}{c}\Pi_1 \\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;B;\Gamma_2;C;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;B;\Gamma_2;\mathsf{G}C;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;\Delta;\Gamma_2;C;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;\Delta;\Gamma_2;C;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\Delta;\Gamma_2;\mathsf{G}C;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

- Case 3:

$$\begin{array}{c}\Pi_1 \\ \Phi \vdash_{\mathcal{C}} X\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;B;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{G}B;\Gamma_2;X;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |X|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;B;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |X|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |X|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;B;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{G}B;\Gamma_2;\Phi;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

- Case 4:

$$\begin{array}{c}\Pi_1 \\ \Delta \vdash_{\mathcal{L}} B\end{array} \qquad \Pi_2: \dfrac{\begin{array}{c}\pi \\ \Gamma_1;C;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{G}C;\Gamma_2;B;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

By assumption, $c(\Pi_1), c(\Pi_2) \leq |B|$. By induction on Π_1 and π, there is a proof Π' for sequent $\Gamma_1;C;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A$ s.t. $c(\Pi') \leq |B|$. Therefore, the proof Π can be constructed as follows with $c(\Pi) = c(\Pi') \leq |B|$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Gamma_1;C;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A\end{array}}{\Gamma_1;\mathsf{G}C;\Gamma_2;\Delta;\Gamma_3 \vdash_{\mathcal{L}} A}\ \text{GL}$$

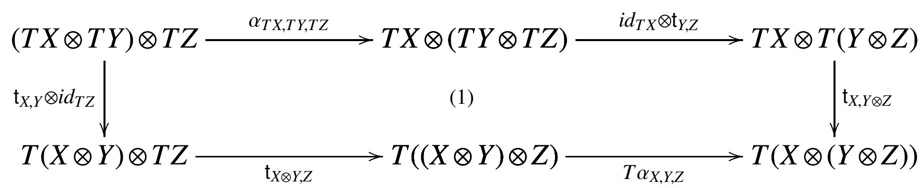
B Proof For Lemma 9

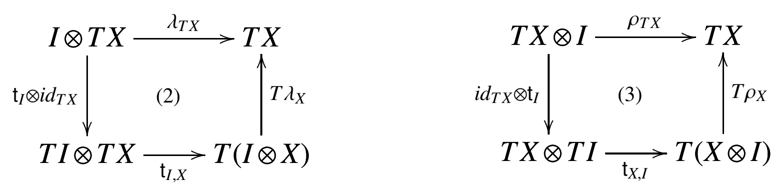
Let $(C,\mathcal{L},F,G,\eta,\varepsilon)$ be a LAM. We define the monad $(T,\eta : id_C \to T,\mu : T^2 \to T)$ on C as $T = GF$, $\eta_X : X \to GFX$, and $\mu_X = G\varepsilon_{FX} : GFGFX \to GFX$. Since (F,m) and (G,n) are monoidal functors, we have

$$\mathsf{t}_{X,Y} = G\mathsf{m}_{X,Y} \circ \mathsf{n}_{FX,FY} : TX \otimes TY \to T(X \otimes Y) \qquad \text{and} \qquad \mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI.$$

The monad T being monoidal means:

1. T is a monoidal functor, i.e. the following diagrams commute:





We write GF instead of T in the proof for clarity.
By replacing $\mathsf{t}_{X,Y}$ with its definition, diagram (1) above commutes by the following commutative diagram, in which the two hexagons commute because G and F are monoidal functors, and the two quadrilaterals commute by the naturality of n.

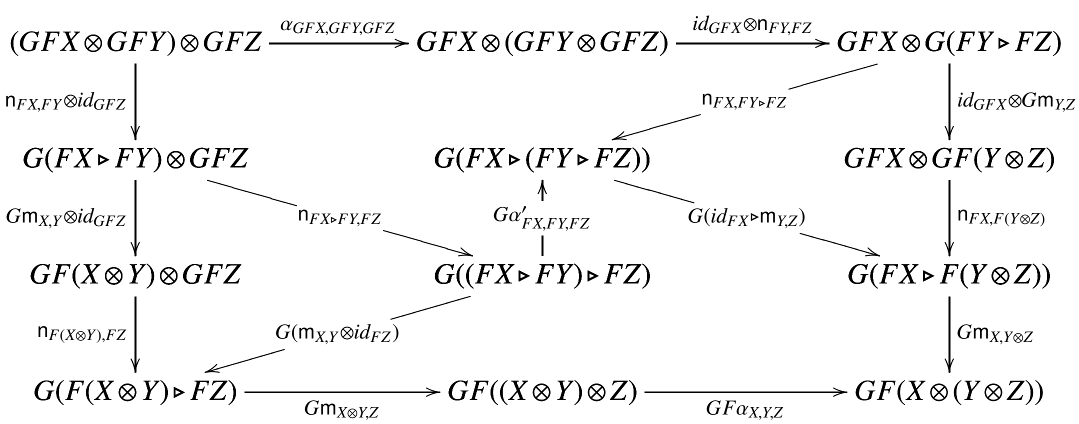
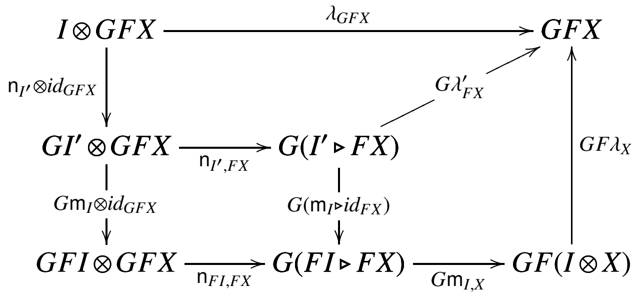
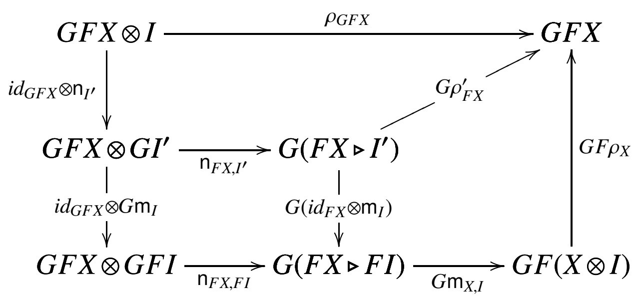


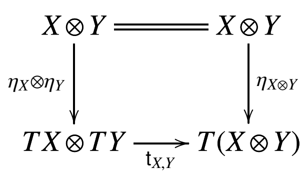
Diagram (2) commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.

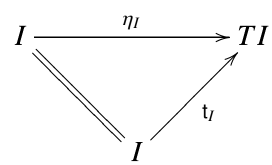


Similarly, diagram (3) commutes as follows:

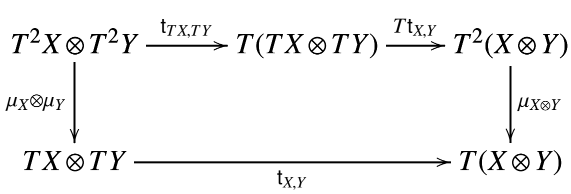


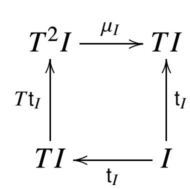
2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal by definition and thus the following two diagrams commute.





3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ. Thus the following diagrams commute.

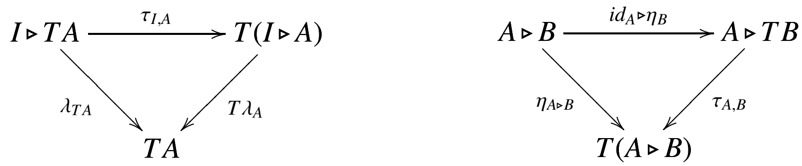


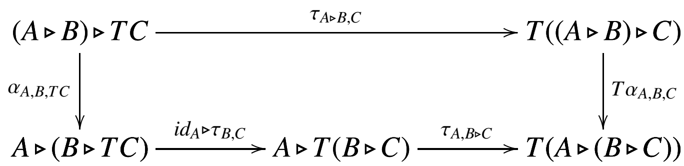


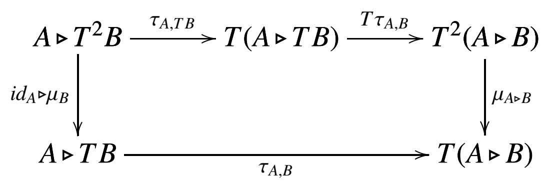
C Proof For Lemma 10

Definition 21. *Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on $\mathcal{M}$. T is a* ***strong monad*** *if there is natural transformation τ, called the* ***tensorial strength****, with components $\tau_{A,B} : A \triangleright TB \to$*

$T(A \triangleright B)$ such that the following diagrams commute:



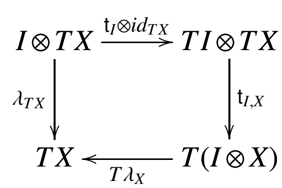




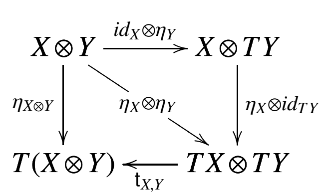
The proof for Lemma 10 goes as follows. Let $(\mathcal{C},\mathcal{L},F,G,\eta,\varepsilon)$ be a LAM, where $(\mathcal{C},\otimes,I,\alpha,\lambda,\rho)$ is symmetric monoidal closed, and
$(\mathcal{L},\triangleright,I',\alpha',\lambda',\rho')$ is Lambek. In Lemma 9, we have proved that the monad $(T = GF,\eta,\mu)$ is monoidal with the natural transformation $\mathsf{t}_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ and the morphism $\mathsf{t}_I : I \to TI$. We define the tensorial strength $\tau_{X,Y} : X \otimes TY \to T(X \otimes Y)$ as

$$\tau_{X,Y} = \mathsf{t}_{X,Y} \circ (\eta_X \otimes id_{TY}).$$

Since η is a monoidal natural transformation, we have $\eta_I = G\mathsf{m}_I \circ \mathsf{n}_{I'}$, and thus $\eta_I = \mathsf{t}_I$. The following diagram commutes because T is monoidal, where the composition $\mathsf{t}_{I,X} \circ (\mathsf{t}_I \otimes id_{TX})$ is the definition of $\tau_{I,X}$. So the first triangle in Definition 21 commutes.

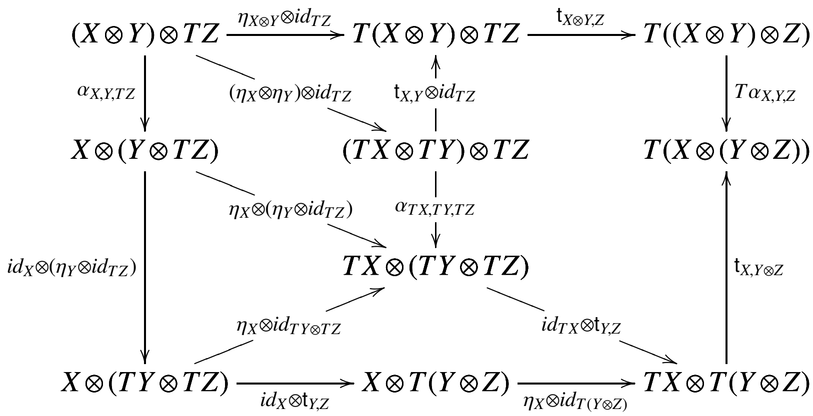


Similarly, by using the definition of τ, the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

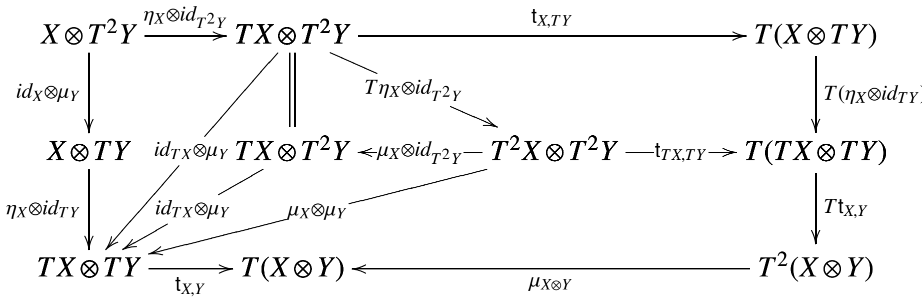


The first pentagon in the definition commutes by the following commutative diagrams, because η and α

are natural transformations and T is monoidal:



The last diagram in the definition commutes by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:



D Equivalence of Sequent Calculus and Natural Deduction Formalizations

We prove the equivalence of the sequence calculus formalization and the natural deduction formalization given in the paper by defining two mappings, one from the rules in natural deduction to proofs the sequent calculus, and the other is from the rules in sequent calculus to proofs in natural deduction.

D.1 Mapping from Natural Deduction to Sequent Calculus

Function $S : ND \to SC$ maps a rule in the natural deduction formalization to a proof of the same sequent in the sequent calculus. The function is defined as follows:

- The axioms map to axioms.
- Introduction rules map to right rules.
- Elimination rules map to combinations of left rules with cuts:
  - C-Unit_E:

$$\frac{\Phi \vdash_C t_1 : \mathsf{Unit} \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C \mathsf{let}\, t_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t_2 : Y} C\text{-}\mathsf{Unit}_E$$

  maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \qquad \dfrac{\Psi \vdash_{\mathcal{C}} t_2 : Y}{x : \mathsf{Unit}, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t_2 : Y}\,\mathcal{C}\text{-Unit}_L}{\Phi, \Psi \vdash_{\mathcal{C}} [t_1/x](\mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t_2) : Y}\,\mathcal{C}\text{-Cut}$$

– $\mathcal{C}$-$\otimes_E$:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \otimes Y \qquad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2 : Z}\,\mathcal{C}\text{-}\otimes_E$$

maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \otimes Y \qquad \dfrac{\Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, z : X \otimes Y, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2 : Z}\,\mathcal{C}\text{-}\otimes_L}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} [t_1/z](\mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2) : Z}\,\mathcal{C}\text{-Cut}$$

– $\mathcal{C}$-$\multimap_E$:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \multimap Y \qquad \Psi \vdash_{\mathcal{C}} t_2 : X}{\Phi, \Psi \vdash_{\mathcal{C}} t_1 t_2 : Y}\,\mathcal{C}\text{-}\multimap_E$$

maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \multimap Y \qquad \dfrac{\Psi \vdash_{\mathcal{C}} t_2 : X \qquad \dfrac{}{x : Y \vdash_{\mathcal{C}} x : Y}\,\mathcal{C}\text{-AX}}{y : X \multimap Y, \Psi \vdash_{\mathcal{C}} [y t_2/x]x : Y}\,\mathcal{C}\text{-}\multimap_L}{\Phi, \Psi \vdash_{\mathcal{C}} [t_1/y][y t_2/x]x : Y}\,\mathcal{C}\text{-Cut}$$

– $\mathcal{LC}$-Unit$_E$:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{Unit} \qquad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi; \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\,\mathcal{LC}\text{-Unit}_E$$

maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{Unit} \qquad \dfrac{\Gamma \vdash_{\mathcal{L}} s : A}{x : \mathsf{Unit}; \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\,\mathcal{LC}\text{-Unit}_L}{\Phi; \Psi \vdash_{\mathcal{L}} [t/x](\mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s) : A}\,\mathcal{LC}\text{-Cut}$$

– $\mathcal{L}$-Unit$_E$:

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2 : A}\,\mathcal{L}\text{-Unit}_E$$

maps to

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A}{x : \mathsf{Unit}; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2 : A}\,\mathcal{L}\text{-Unit}_L}{\Gamma; \Delta \vdash_{\mathcal{L}} [s_1/x](\mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2) : A}\,\mathcal{L}\text{-Cut}$$

– $\mathcal{LC}$-$\otimes_E$:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : X \otimes Y \qquad \Gamma_1; x : X; y : Y; \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1; \Phi; \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, t : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\,\mathcal{LC}\text{-}\otimes_E$$

maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : X \otimes Y \qquad \dfrac{\Gamma_1; x : X; y : Y; \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1; z : X \otimes Y; \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\,\mathcal{LC}\text{-}\otimes_L}{\Gamma_1; \Phi; \Gamma_2 \vdash_{\mathcal{L}} [t/z](\mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s) : A}\,\mathcal{LC}\text{-Cut}$$

– $\mathcal{L}$-$\otimes_E$:

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \Delta_1; x : A; y : B; \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1; \Gamma; \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\,\mathcal{L}\text{-}\otimes_E$$

maps to

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \dfrac{\Delta_1; x : A; y : B; \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1; z : A \triangleright B; \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\,\mathcal{L}\text{-}\otimes_L}{\Delta_1; \Gamma; \Delta_2 \vdash_{\mathcal{L}} [s_1/z](\mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2) : C}\,\mathcal{L}\text{-Cut}$$

– $\mathcal{L}$-$\rightharpoonup_E$:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma;\Delta \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B}\,\mathcal{L}\text{-}\rightharpoonup_E$$

maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A \qquad \dfrac{}{x : B \vdash_{\mathcal{L}} x : B}\,\text{AX}}{y : A \rightharpoonup B;\Delta \vdash_{\mathcal{L}} [\mathsf{app}_r\, y\, s_2/x]x : B}\,\mathcal{L}\text{-}\rightharpoonup_L}{\Gamma;\Delta \vdash_{\mathcal{L}} [s_1/y][\mathsf{app}_r\, y\, s_2/x]x : B}\,\mathcal{L}\text{-Cut}$$

– $\mathcal{L}$-$\leftharpoonup_E$:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta;\Gamma \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B}\,\mathcal{L}\text{-}\leftharpoonup_E$$

maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \qquad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A \qquad \dfrac{}{x : B \vdash_{\mathcal{L}} x : B}\,\mathcal{L}\text{-AX}}{\Delta; y : B \leftharpoonup A \vdash_{\mathcal{L}} [\mathsf{app}_l\, y\, s_2/x]x : B}\,\mathcal{L}\text{-}\leftharpoonup_L}{\Delta;\Gamma \vdash_{\mathcal{L}} [s_1/y][\mathsf{app}_l\, y\, s_2/x]x : B}\,\mathcal{L}\text{-Cut}$$

– $\mathcal{L}$-F_E:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{F}X \quad \Delta_1; x : X; \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}\,x\, \mathsf{in}\, s_2 : A}\,\mathcal{L}\text{-}\mathsf{F}_E$$

maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} y : \mathsf{F}X \qquad \dfrac{\Delta_1; x : X; \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1; z : \mathsf{F}X; \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}\,x\, \mathsf{in}\, s : A}\,\mathcal{L}\text{-}\mathsf{F}_L}{\Delta_1;\Gamma;\Delta_2 \vdash_{\mathcal{L}} [y/z](\mathsf{let}\, y : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}\,x\, \mathsf{in}\, s) : A}\,\mathcal{L}\text{-Cut}$$

– $\mathcal{L}$-G_E:

$$\frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\, t : A}\,\mathcal{L}\text{-}\mathsf{G}_E$$

maps to

$$\frac{\dfrac{\dfrac{}{x : A \vdash_{\mathcal{L}} x : A}\,\mathcal{L}\text{-AX}}{y : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{G}A\, \mathsf{be}\, \mathsf{G}\,x\, \mathsf{in}\, x : A}\,\mathcal{L}\text{-}\mathsf{G}_L \qquad \Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} [t/y](\mathsf{let}\, y : \mathsf{G}A\, \mathsf{be}\, \mathsf{G}\,x\, \mathsf{in}\, x) : A}\,\mathcal{LC}\text{-Cut}$$

D.2 Mapping from Sequent Calculus to Natural Deduction

Function $N : SC \rightarrow ND$ maps a rule in the sequent calculus to a proof of the same sequent in the natural deduction. The function is defined as follows:

- Axioms map to axioms.
- Right rules map to introductions.
- Left rules map to eliminations modulo some structural fiddling.
  – $\mathcal{C}$-Unit_L:

$$\frac{\Phi,\Psi \vdash_{\mathcal{C}} t : X}{\Phi, x : \mathsf{Unit}, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t : X}\,\mathcal{C}\text{-}\mathsf{Unit}_L$$

maps to

$$\frac{\dfrac{}{x : \mathsf{Unit} \vdash_{\mathcal{C}} x : \mathsf{Unit}}\,\mathcal{C}\text{-AX} \qquad \Phi,\Psi \vdash_{\mathcal{C}} t : X}{\Phi,\Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t : X}\,\mathcal{C}\text{-Cut}$$

– $\mathcal{C}$-$\otimes_L$:

$$\dfrac{\Phi, x : X, y : Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t : Z}\ \mathcal{C}\text{-}\otimes_L$$

maps to

$$\dfrac{\dfrac{}{z : X \otimes Y \vdash_{\mathcal{C}} z : X \otimes Y}\ \mathcal{C}\text{-AX} \qquad \Phi, x : X, y : Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t : Z}\ \mathcal{C}\text{-}\otimes_E$$

– $\mathcal{C}$-$\multimap_L$:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \qquad \Psi_1, x : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, y : X \multimap Y, \Phi, \Psi_2 \vdash_{\mathcal{C}} [y t_1 / x] t_2 : Z}\ \mathcal{C}\text{-}\multimap_L$$

maps to

$$\dfrac{\dfrac{\dfrac{}{y : X \multimap Y \vdash_{\mathcal{C}} y : X \multimap Y}\ \mathcal{C}\text{-AX} \qquad \Phi \vdash_{\mathcal{C}} t_1 : X}{y : X \multimap Y, \Phi \vdash_{\mathcal{C}} y t_1 : Y}\ \mathcal{C}\text{-}\multimap_E \qquad \Psi_1, x : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, y : X \multimap Y, \Phi, \Psi_2 \vdash_{\mathcal{C}} [y t_1 / x] t_2 : Z}\ \mathcal{C}\text{-Cut}$$

– $\mathcal{LC}$-Unit_L:

$$\dfrac{\Gamma; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; x : \mathsf{Unit}; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \mathcal{LC}\text{-}\mathsf{Unit}_L$$

maps to

$$\dfrac{\dfrac{}{x : \mathsf{Unit} \vdash_{\mathcal{C}} x : \mathsf{Unit}}\ \mathcal{C}\text{-AX} \qquad \Gamma; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; x : \mathsf{Unit}; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \mathcal{LC}\text{-}\mathsf{Unit}_E$$

– $\mathcal{L}$-Unit_L:

$$\dfrac{\Gamma; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; x : \mathsf{Unit}; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \mathcal{L}\text{-}\mathsf{Unit}_L$$

maps to

$$\dfrac{\dfrac{}{x : \mathsf{Unit} \vdash_{\mathcal{L}} x : \mathsf{Unit}}\ \mathcal{L}\text{-AX} \qquad \Gamma; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; x : \mathsf{Unit}; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \mathcal{L}\text{-}\mathsf{Unit}_E$$

– $\mathcal{LC}$-$\otimes_L$:

$$\dfrac{\Gamma; x : X; y : Y; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; z : X \otimes Y; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\ \mathcal{LC}\text{-}\otimes_L$$

maps to

$$\dfrac{\dfrac{}{z : X \otimes Y \vdash_{\mathcal{C}} z : X \otimes Y}\ \mathcal{C}\text{-AX} \qquad \Gamma; x : X; y : Y; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; z : X \otimes Y; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\ \mathcal{LC}\text{-}\otimes_E$$

– $\mathcal{L}$-$\otimes_L$:

$$\dfrac{\Gamma; x : A; y : B; \Delta \vdash_{\mathcal{L}} s : C}{\Gamma; z : A \triangleright B; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s : C}\ \mathcal{L}\text{-}\otimes_L$$

maps to

$$\dfrac{\dfrac{}{z : A \triangleright B \vdash_{\mathcal{L}} z : A \triangleright B}\ \mathcal{L}\text{-AX} \qquad \Gamma; x : A; y : B; \Delta \vdash_{\mathcal{L}} s : C}{\Gamma; z : A \triangleright B; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s : C}\ \mathcal{L}\text{-}\otimes_E$$

– $\mathcal{L}$-$\multimap_L$:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : X \qquad \Gamma; x : Y; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; y : X \multimap Y; \Phi; \Delta \vdash_{\mathcal{L}} [y t / x] s : A}\ \mathcal{L}\text{-}\multimap_L$$

maps to

$$\dfrac{\dfrac{\dfrac{}{y : X \multimap Y \vdash_{\mathcal{C}} y : X \multimap Y}\ \mathcal{L}\text{-AX} \qquad \Phi \vdash_{\mathcal{C}} t : X}{y : X \multimap Y, \Phi \vdash_{\mathcal{C}} yt : Y}\ \mathcal{C}\text{-}\multimap_E \qquad \Gamma; x : Y; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; y : X \multimap Y; \Phi; \Delta \vdash_{\mathcal{L}} [yt/x]s : A}\ \mathcal{LC}\text{-Cut}$$

– $\mathcal{L}\text{-}\rightharpoonup_L$:

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad \Delta_1; x : B; \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1; y : A \rightharpoonup B; \Gamma; \Delta_2 \vdash_{\mathcal{L}} [\mathsf{app}_r\, y\, s_1/x]s_2 : C}\ \mathcal{L}\text{-}\rightharpoonup_L$$

maps to

$$\dfrac{\dfrac{\dfrac{}{y : A \rightharpoonup B \vdash_{\mathcal{L}} y : A \rightharpoonup B}\ \mathcal{L}\text{-AX} \qquad \Gamma \vdash_{\mathcal{L}} s_1 : A}{y : A \rightharpoonup B; \Gamma \vdash_{\mathcal{L}} \mathsf{app}_r\, y\, s_1 : B}\ \mathcal{L}\text{-}\rightharpoonup_E \qquad \Delta_1; x : B; \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1; y : A \rightharpoonup B; \Gamma; \Delta_2 \vdash_{\mathcal{L}} [\mathsf{app}_r\, y\, s_1/x]s_2 : C}\ \mathcal{L}\text{-Cut}$$

– $\mathcal{L}\text{-}\leftharpoonup_L$:

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad \Delta_1; x : B; \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1; \Gamma; y : B \leftharpoonup A; \Delta_2 \vdash_{\mathcal{L}} [\mathsf{app}_l\, y\, s_1/x]s_2 : C}\ \mathcal{L}\text{-}\leftharpoonup_L$$

maps to

$$\dfrac{\dfrac{\dfrac{}{y : B \leftharpoonup A \vdash_{\mathcal{L}} y : B \leftharpoonup A}\ \mathcal{L}\text{-AX} \qquad \Gamma \vdash_{\mathcal{L}} s_1 : A}{\Gamma; y : B \leftharpoonup A \vdash_{\mathcal{L}} \mathsf{app}_l\, y\, s_1 : B}\ \mathcal{L}\text{-}\leftharpoonup_E \qquad \Delta_1; x : B; \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1; \Gamma; y : B \leftharpoonup A; \Delta_2 \vdash_{\mathcal{L}} [\mathsf{app}_l\, y\, s_1/x]s_2 : C}\ \mathcal{L}\text{-Cut}$$

– $\mathcal{L}\text{-}\mathsf{F}_L$:

$$\dfrac{\Gamma; x : X; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; y : \mathsf{F}X; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}x\, \mathsf{in}\, s : A}\ \mathcal{L}\text{-}\mathsf{F}_L$$

maps to

$$\dfrac{\dfrac{}{y : \mathsf{F}X \vdash_{\mathcal{L}} y : \mathsf{F}X}\ \mathcal{L}\text{-AX} \qquad \Gamma; x : X; \Delta \vdash_{\mathcal{L}} s : A}{\Gamma; y : \mathsf{F}X; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\, \mathsf{be}\, y\, \mathsf{in}\, s : A}\ \mathcal{L}\text{-}\mathsf{F}_E$$

– $\mathcal{L}\text{-}\mathsf{G}_L$:

$$\dfrac{\Gamma; x : A; \Delta \vdash_{\mathcal{L}} s : B}{\Gamma; y : \mathsf{G}A; \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{G}A\, \mathsf{be}\, \mathsf{G}x\, \mathsf{in}\, s : B}\ \mathcal{L}\text{-}\mathsf{G}_L$$

maps to

$$\dfrac{\dfrac{\dfrac{}{y : \mathsf{G}A \vdash_{\mathcal{C}} y : \mathsf{G}A}\ \mathcal{L}\text{-AX}}{y : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{derelict}\, y : A}\ \mathcal{L}\text{-}\mathsf{G}_E \qquad \Gamma; x : A; \Delta \vdash_{\mathcal{L}} s : B}{\Gamma; y : \mathsf{G}A; \Delta \vdash_{\mathcal{L}} [\mathsf{derelict}\, y/x]s : B}\ \mathcal{L}\text{-Cut}$$