

Analyzing Innermost Runtime Complexity Through Tuple Interpretations

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Time complexity in rewriting is naturally understood as the number of steps needed to reduce terms to normal forms. Establishing complexity bounds to this measure is a well-known problem in the rewriting community. A vast majority of techniques to find such bounds consist of modifying termination proofs in order to recover complexity information. This has been done for instance with semantic interpretations, recursive path orders, and dependency pairs. In this paper, we follow the same program by tailoring tuple interpretations to deal with innermost complexity analysis. A tuple interpretation interprets terms as tuples holding upper bounds to the cost of reduction and size of normal forms. In contrast with the full rewriting setting, the strongly monotonic requirement for cost components is dropped when reductions are innermost. This weakened requirement on cost tuples allows us to prove the innermost version of the compatibility result: if all rules in a term rewriting system can be strictly oriented, then the innermost rewrite relation is well-founded. We establish the necessary conditions for which tuple interpretations guarantee polynomial bounds to the runtime of compatible systems and describe a search procedure for such interpretations.

1 Introduction

In the step-by-step computational model induced by rewriting, time complexity is naturally understood as the number of rewriting steps needed to reach normal forms. Usually, the cost of firing a redex (i.e., performing a computational step) is assumed constant. So the intricacies of a low-level rewriting realization (e.g., a concrete rewriting engine implementation) are ignored. This assumption does not pose a problem as long as the low-level time complexity needed to apply a rule is kept low. Additionally, this abstract approach has the advantage of being independent of the specific hardware platform evaluating the rewriting system at hand.

In this rewriting setting, a complexity function bounds the length of rewrite sequences and is parametrized by the size of the starting term of the derivation. Two distinct complexity notions are commonly considered in the literature: derivational and runtime complexity, and they differ by the restrictions imposed on the initial term of derivations. On the one hand, derivational complexity imposes no restriction on the set of initial terms. Intuitively, it captures the worst-case behavior of reducing a term to normal form. On the other hand, runtime complexity requires basic initial terms which, conceptually, are terms where a single function call is performed on data (e.g., integers, lists, and trees) as arguments.

If programs are expressed by rewriting, their execution time is closely related to the runtime complexity of the associated rewrite system. Similarly related are programs using call-by-value evaluation strategy and innermost rewrite systems. Therefore, by combining these two concepts, we obtain a connection

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between the cost analysis of call-by-value programs and the runtime complexity analysis of innermost term rewriting. More importantly, due to the abstract nature of rewriting, it is feasible to forgo any specific programming language detail and still derive useful term rewriting results that may carry over to programs. For an overview of the applicability of rewriting to program complexity the reader is referred to [1, 19].

Therefore, a rewriting approach to program complexity allows us to fully concentrate on finding techniques to establish bounds to the derivational or runtime complexity functions. A natural way to determine these bounds is adapting the proof techniques used to show termination to deduce the complexity naturally induced by the method. There is a myriad of works following this program. To mention a few, see [2, 4, 6, 13, 14, 20] for interpretation methods, [5, 12, 24] for lexicographic and path orders, and [11, 21] for dependency pairs. In this paper, we follow the same idea and concentrate on investigating the existence of upper bounds to the innermost runtime complexity for applicative systems. The termination method on which we base our complexity analysis framework upon is tuple interpretations [16].

Tuple interpretations are an instance of the interpretation method. Thus, we seek to interpret terms in such a way that the rewrite relation can be embedded in a well-founded ordering. More precisely, we choose an interpretation domain A which is a set together with a well-founded order $>$ over A and interpret terms as elements of A compositionally. This interpretation of terms is such that whenever a rewriting is fired, i.e., $s \rightarrow t$, the interpretations $\llbracket s \rrbracket$ and $\llbracket t \rrbracket$ of s and t satisfy $\llbracket s \rrbracket > \llbracket t \rrbracket$. Hence, a rewriting step on terms implies a strict decrease on A . The well-foundedness of such domains together with this compatibility requirement on reduction guarantee that no infinite reduction sequence on terms exists.

The defining characteristic of tuple interpretations is to allow for a split of the complexity measure into abstract notions of cost and size. When distilled into its essence, the ingredient we need to express the concepts of cost and size is a product $\mathcal{C} \times \mathcal{S}$ of a well-founded set \mathcal{C} — the cost set — and a quasi-ordered set \mathcal{S} — the size set. Intuitively, the cost tuples in \mathcal{C} bound the number of rewriting steps needed to reach normal forms, which is in line with the aforementioned rewriting cost model. Meanwhile, the size tuples in \mathcal{S} are more general. We can use integers, reals, and terms themselves as size. Following the treatment in [16], the construction of cost–size products is done inductively on the structure of types. So we map each type σ to a cost–size product $\mathcal{C}_\sigma \times \mathcal{S}_\sigma$. Hence, in this paper, our first-order term formalism follows a type discipline.

In order to extend the usability of our techniques, we would like to not only exhibit bounds to the runtime complexity function but also determine sufficient conditions for its feasibility, that is, the existence of polynomial upper bounds. In the eighties Huet and Oppen [15] conjectured that polynomial interpretations are sufficient to evince feasibility, which was disproved by Lautemann [17] in the same decade. Indeed, polynomial interpretations induce a double exponential upper bound on the derivation length, as shown by the seminal work of Hofbauer and Lautemann [14]. Feasibility can be recovered by imposing additional conditions on interpretations. To the best of our knowledge, Cichon and Lescanne [6] were the first to propose such conditions even though their setting is restricted to number theoretic functions only. Similar results are proved in [4], where the authors provide rewriting characterizations of complexity classes using bounds for the interpretation of data constructors. These same conditions appear in the higher-order setting, see [2, 16]. In the present paper, we follow a similar approach to that in [4] and show that we can recover those classical results by bounding size tuples in interpretations.

Tuple interpretations do not provide a complete termination proof method: there are terminating systems for which interpretations cannot be found. Consequently, it does not induce a complete complexity analysis framework either. Notwithstanding, it has the potential to be very powerful if we choose the cost–size sets wisely. A second limitation is that the search for interpretations is undecidable in general, which is expected already in the polynomial case [18]. Undecidability never hindered computer scientists' efforts on mechanizing difficult problems, however. Indeed, several proof search methods have been

developed over the years to find interpretations automatically [3, 7, 8, 13, 25].

Contribution. We provide a formal definition of cost–size products (Definition 1) and use it to interpret types in Definition 3. Cost–size products provide an interpretation domain for cost–size tuple algebras, Definition 6. In Lemmas 2 and 4 we show the soundness of this approach. In Definition 5 we introduce a type-safe application operator on cost–size products and prove its strong monotonicity, an important ingredient to show the Compatibility Theorem 1. We establish the termination of Toyama’s system in Example 3, showing that Theorem 1 correctly captures innermost termination in our setting. We provide sufficient conditions so that feasible bounds on innermost runtime complexity can be achieved in Lemmas 7 and 8.

Outline. In Section 2, we fix notation and recall basic notions of rewriting syntax, basic terminology on the complexity of rewriting, and review our notation for sets, orders, and functions. In Section 3, we tailor tuple interpretations to the innermost setting and prove the innermost version of the compatibility theorem. We proceed to establish complexity bounds to the innermost runtime complexity in Section 4. In Section 5, we present preliminary work on automation techniques to find cost–size tuple interpretations. We conclude the paper in Section 6.

2 Preliminaries

TRSs and Innermost Rewriting. We consider simply typed first-order term rewriting systems in curried notation. Fix a set \mathcal{B} , whose elements are called *sorts*. The set $\mathcal{T}_{\mathcal{B}}$ of *types* is generated by the grammar $\mathcal{T}_{\mathcal{B}} ::= \mathcal{B} \mid \mathcal{B} \Rightarrow \mathcal{T}_{\mathcal{B}}$. Each type is written as $\iota_1 \Rightarrow \dots \Rightarrow \iota_m \Rightarrow \kappa$ where all ι_i and κ are sorts. A *signature* is a set \mathcal{F} of symbols together with an arity function ar which associates to each $f \in \mathcal{F}$ a type $\sigma \in \mathcal{T}_{\mathcal{B}}$. We call the triple $(\mathcal{B}, \mathcal{F}, \text{ar})$ a *syntax signature*. For each sort ι , we postulate a set \mathcal{X}_{ι} of countably many variables and assume that $\mathcal{X}_{\iota} \cap \mathcal{X}_{\iota'} = \emptyset$ if $\iota \neq \iota'$. Let \mathcal{X} denote $\bigcup_{\iota} \mathcal{X}_{\iota}$ and assume that $\mathcal{F} \cap \mathcal{X} = \emptyset$.

The set \mathbb{T} of *pre-terms* is generated by the grammar $\mathbb{T} ::= \mathcal{F} \mid \mathcal{X} \mid (\mathbb{T} \mathbb{T})$. The set $T(\mathcal{F}, \mathcal{X})$ of *terms* consists of pre-terms which can be typed as follows: (i) $f : \sigma$ if $\text{ar}(f) = \sigma$, (ii) $x : \iota$ if $x \in \mathcal{X}_{\iota}$, and (iii) $(s t) : \tau$ if $s : \iota \Rightarrow \tau$ and $t : \iota$. Application of terms is left-associative, so we write $s t u$ for $((s t) u)$. Let $\text{vars}(s)$ be the set of variables occurring in s . A *ground term* is a term s such that $\text{vars}(s) = \emptyset$. A symbol $f \in \mathcal{F}$ is called the *head symbol* of s if $s = f s_1 \dots s_k$. A *subterm* of s is a term t (we write $s \supseteq t$) such that (i) $s = t$, or (ii) t is a subterm of s' or s'' when $s = s' s''$. A *proper subterm* of s is a subterm of s which is not equal to s . A *substitution* γ is a type-preserving map from variables to terms such that the set $\text{dom}(\gamma) = \{x \in \mathcal{X} \mid \gamma(x) \neq x\}$ is finite. Every substitution γ extends to a type-preserving map from terms to terms, whose image on s is written as $s\gamma$, as follows: (i) $f\gamma = f$, (ii) $x\gamma = \gamma(x)$, and (iii) $(s t)\gamma = (s\gamma) (t\gamma)$.

A relation \rightarrow on terms is *monotonic* if $s \rightarrow s'$ implies $t s \rightarrow t s'$ and $s u \rightarrow s' u$ for all terms t and u of appropriate types. A *rewrite rule* $\ell \rightarrow r$ is a pair of terms of the same type such that $\ell = f \ell_1 \dots \ell_k$ and $\text{vars}(\ell) \supseteq \text{vars}(r)$. A *term rewriting system* (TRS) \mathcal{R} is a set of rewrite rules. The *rewrite relation* $\rightarrow_{\mathcal{R}}$ induced by \mathcal{R} is the smallest monotonic relation on terms such that $\ell\gamma \rightarrow_{\mathcal{R}} r\gamma$ for all rules $\ell \rightarrow r \in \mathcal{R}$ and substitutions γ . A *reducible expression* (redex) is a term of form $\ell\gamma$ for some rule $\ell \rightarrow r$ and substitution γ . A term is in *normal form* if none of its subterms is a redex. A TRS \mathcal{R} is *terminating* if no infinite rewrite sequence $s \rightarrow_{\mathcal{R}} s' \rightarrow_{\mathcal{R}} s'' \rightarrow_{\mathcal{R}} \dots$ exists.

Every rewrite rule $\ell \rightarrow r$ *defines* a symbol f , namely, the head symbol of ℓ . For each $f \in \mathcal{F}$, let \mathcal{R}_f denote the set of rewrite rules that define f in \mathcal{R} . A symbol $f \in \mathcal{F}$ is a *defined symbol* if $\mathcal{R}_f \neq \emptyset$; otherwise,

f is called a *constructor*. Let \mathcal{D} be the set of defined symbols and \mathcal{C} the set of constructors. So $\mathcal{F} = \mathcal{D} \cup \mathcal{C}$. A *data term* is a term of the form $c d_1 \dots d_k$ where c is a constructor and each d_i is a data term. A *basic term* is a term of type ι and of form $f d_1 \dots d_m$ where ι is a sort, f is a defined symbol and all d_1, \dots, d_m are data terms. We let $T_b(\mathcal{F})$ denote the set of all basic terms.

Example 1 We fix `nat` and `list` for the sorts of natural numbers and lists of natural numbers, respectively. In the below TRS, `0` : `nat`, `s` : `nat` \Rightarrow `nat`, `nil` : `list` and `cons` : `nat` \Rightarrow `list` \Rightarrow `list` are constructors while `add`, `minus`, `quot` : `nat` \Rightarrow `nat` \Rightarrow `nat`, `append` : `list` \Rightarrow `list` \Rightarrow `list`, `sum` : `list` \Rightarrow `nat` and `rev` : `list` \Rightarrow `list` are defined symbols.

$$\begin{array}{ll}
\text{add } x \ 0 \rightarrow x & \text{sum nil} \rightarrow 0 \\
\text{add } x \ (s \ y) \rightarrow s \ (\text{add } x \ y) & \text{sum } (\text{cons } x \ q) \rightarrow \text{add } (\text{sum } q) \ x \\
\text{append nil } l \rightarrow l & \text{rev nil} \rightarrow \text{nil} \\
\text{append } (\text{cons } x \ q) \ l \rightarrow \text{cons } x \ (\text{append } q \ l) & \text{rev } (\text{cons } x \ q) \rightarrow \text{append } (\text{rev } q) \ (\text{cons } x \ \text{nil}) \\
\text{minus } x \ 0 \rightarrow x & \text{quot } 0 \ (s \ y) \rightarrow 0 \\
\text{minus } 0 \ y \rightarrow 0 & \text{quot } (s \ x) \ (s \ y) \rightarrow s \ (\text{quot } (\text{minus } x \ y) \ (s \ y)) \\
\text{minus } (s \ x) \ (s \ y) \rightarrow \text{minus } x \ y &
\end{array}$$

We restrict our attention to innermost rewriting: only redexes with no reducible proper subterms might be reduced. More precisely, the *innermost rewrite relation* $\rightarrow_{\mathcal{R}}^i$ induced by \mathcal{R} is defined as follows:

- (i) $\ell\gamma \rightarrow_{\mathcal{R}}^i r\gamma$ if $\ell \rightarrow r \in \mathcal{R}$ and all proper subterms of $\ell\gamma$ are in normal form,
- (ii) $s \ t \rightarrow_{\mathcal{R}}^i s' \ t'$ if $s \rightarrow_{\mathcal{R}}^i s'$, and
- (iii) $s \ t \rightarrow_{\mathcal{R}}^i s \ t'$ if $t \rightarrow_{\mathcal{R}}^i t'$.

In this paper we only analyze innermost rewriting. So we write \rightarrow for $\rightarrow_{\mathcal{R}}^i$ whenever no ambiguity arises.

Derivation Height and Complexity. Given a well-founded and finitely branching relation \rightarrow on terms, we write $s \xrightarrow{n} t$ if there is a sequence $s = s_0 \rightarrow \dots \rightarrow s_n = t$ of length n . The *derivation height* $\text{dh}(s, \rightarrow)$ of a term s with respect to \rightarrow is the length of the longest \rightarrow -sequence of starting with s , i.e., $\text{dh}(s, \rightarrow) = \max\{n \mid \exists t \in T(\mathcal{F}, \mathcal{X}) : s \xrightarrow{n} t\}$. The *absolute size* of a term s , denoted by $|s|$, is 1 if s is a symbol in \mathcal{F} or a variable, and $|s_1| + |s_2|$ if $s = s_1 s_2$. In order to express various complexity notions in the rewriting setting, we define the *complexity function* as follows: $\text{comp}(n, \rightarrow, \mathcal{T}) = \max\{\text{dh}(s, \rightarrow) \mid s \in \mathcal{T} \text{ and } |s| \leq n\}$. Intuitively, $\text{comp}(n, \rightarrow, \mathcal{T})$ is the length of the longest \rightarrow -sequence starting with a term whose absolute size is at most n from \mathcal{T} . We summarize four particular instances in the following table:

| | derivational | runtime |
|-----------|--|---|
| full | $\text{dc}_{\mathcal{R}}(n) = \text{comp}(n, \rightarrow_{\mathcal{R}}, T(\mathcal{F}, \mathcal{X}))$ | $\text{rc}_{\mathcal{R}}(n) = \text{comp}(n, \rightarrow_{\mathcal{R}}, T_b(\mathcal{F}))$ |
| innermost | $\text{idc}_{\mathcal{R}}(n) = \text{comp}(n, \rightarrow_{\mathcal{R}}^i, T(\mathcal{F}, \mathcal{X}))$ | $\text{irc}_{\mathcal{R}}(n) = \text{comp}(n, \rightarrow_{\mathcal{R}}^i, T_b(\mathcal{F}))$ |

Ordered Sets and Monotonic Functions. A *quasi-ordered set* (A, \sqsupseteq) consists of a nonempty set A and a quasi-order (reflexive and transitive) \sqsupseteq on A . An *extended well-founded set* $(A, >, \geq)$ is a nonempty set A together with a well-founded order $>$ and a quasi-order \geq on A such that \geq is compatible with $>$, i.e., $x > y$ implies $x \geq y$ and $x > y \geq z$ implies $x > z$. Below we refer to an extended well-founded set simply as a *well-founded set*.

Given quasi-ordered sets (A, \sqsubseteq) and (B, \sqsubseteq) , a function $f : A \rightarrow B$ is said to be *weakly monotonic* if $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$. Let $A \Longrightarrow B$ denote the set of weakly monotonic functions from A to B . The comparison operator \sqsubseteq on B induces pointwise comparison on $A \Longrightarrow B$ as follows: $f \sqsubseteq g$ if $f(x) \sqsubseteq g(x)$ for all $x \in A$. This way $(A \Longrightarrow B, \sqsubseteq)$ is also a quasi-ordered set. Given well-founded sets $(A, >, \geq)$ and $(B, >, \geq)$, a function $f : A \rightarrow B$ is said to be *strongly monotonic* if $x > y$ implies $f(x) > f(y)$ and $x \geq y$ implies $f(x) \geq f(y)$.

3 Tuple Interpretations

In this section, we introduce the notion of tuple algebras in the context of innermost rewriting. We start by interpreting types as cost–size products, give interpretation of terms as cost–size tuples, and finally, prove the innermost version of the compatibility theorem.

3.1 Types as Cost–Size Products

We start by constructing a cost–size denotational semantics to types in \mathcal{T}_B . The goal is to define a function $\llbracket \cdot \rrbracket$ that maps each type $\sigma \in \mathcal{T}_B$ to a well-founded set $\llbracket \sigma \rrbracket$, the cost–size interpretation of σ .

Definition 1 (Cost–Size Products) Given a well-founded set $(C, >, \geq)$, called the *cost set*, and a quasi-ordered set (S, \sqsubseteq) , called the *size set*, we call $C \times S$ the *cost–size product* of $(C, >, \geq)$ and (S, \sqsubseteq) , and its elements *cost–size tuples*.

Given a cost–size product $C \times S$, the well-foundedness of C and quasi-ordering on S naturally induce an ordering structure on the cartesian product $C \times S$ as follows.

Definition 2 (Product Order) Let $(C, >, \geq) \times (S, \sqsubseteq)$ be a cost–size product. Then we define the relations \succ, \succcurlyeq over $C \times S$ as follows: for all $\langle x, y \rangle$ and $\langle x', y' \rangle$ in $C \times S$,

- (i) $\langle x, y \rangle \succ \langle x', y' \rangle$ if $x > x'$ and $y \sqsubseteq y'$, and
- (ii) $\langle x, y \rangle \succcurlyeq \langle x', y' \rangle$ if $x \geq x'$ and $y \sqsubseteq y'$.

Next, we show that cost–size products ordered as above form a well-founded set.

Lemma 1 *The triple $(C \times S, \succ, \succcurlyeq)$ is a well-founded set.*

PROOF It follows immediately from Definition 1 that \succ, \succcurlyeq are transitive and \succcurlyeq is reflexive. To prove that \succ is well-founded, note that the existence of $\langle x_1, y_1 \rangle \succ \langle x_2, y_2 \rangle \succ \dots$ would imply $x_1 > x_2 > \dots$ which cannot be the case since $>$ is well-founded.

We still need to check that \succcurlyeq is compatible with \succ .

- Suppose $\langle x, y \rangle \succ \langle x', y' \rangle$. Since $x > x'$ implies $x \geq x'$, we have $\langle x, y \rangle \succcurlyeq \langle x', y' \rangle$.
- Suppose $\langle x, y \rangle \succ \langle x', y' \rangle \succcurlyeq \langle x'', y'' \rangle$. Since $x > x' \geq x''$ implies $x > x''$ and \sqsubseteq is transitive, we have $\langle x, y \rangle \succ \langle x'', y'' \rangle$. ■

Now we interpret types as a particular kind of cost–size products.

Definition 3 (Interpretation of Types) Let \mathcal{B} denote the set of sorts. An *interpretation key* \mathcal{J}_B for \mathcal{B} maps each sort ι to a quasi-ordered set $(\mathcal{J}_B(\iota), \sqsubseteq)$ with a minimum. For each type $\sigma \in \mathcal{T}_B$, we define the cost–size interpretation of σ as the product $\llbracket \sigma \rrbracket = \mathcal{C}_\sigma \times \mathcal{S}_\sigma$ with

$$\begin{aligned} \mathcal{C}_\sigma &= \mathbb{N} \times \mathcal{F}_\sigma^c \\ \mathcal{F}_\iota^c &= \text{unit} & \mathcal{S}_\iota &= \mathcal{J}_B(\iota) \\ \mathcal{F}_{\iota \Rightarrow \tau}^c &= \mathcal{S}_\iota \Longrightarrow \mathcal{C}_\tau & \mathcal{S}_{\iota \Rightarrow \tau} &= \mathcal{S}_\iota \Longrightarrow \mathcal{S}_\tau \end{aligned}$$

where $\text{unit} = \{\mathcal{U}\}$ is quasi-ordered by \geq with $\mathcal{U} \geq \mathcal{U}$. All $\mathcal{F}_{\iota \Rightarrow \tau}^c$ and $\mathcal{S}_{\iota \Rightarrow \tau}$ are ordered by pointwise comparison. The set \mathcal{C}_σ is ordered as follows: $(n, f) > (m, g)$ if $n > m$ and $f \geq g$, and $(n, f) \geq (m, g)$ if $n \geq m$ and $f \geq g$. This definition requires that all $(\mathcal{C}_\sigma, \geq)$ and $(\mathcal{S}_\sigma, \sqsupseteq)$ are quasi-ordered sets, which is guaranteed by the following lemma.

Lemma 2 *For any type σ , $(\mathcal{C}_\sigma, >, \geq)$ is a well-founded set and $(\mathcal{S}_\sigma, \sqsupseteq)$ is a quasi-ordered set with a minimum. Therefore, (σ) is a cost–size product.*

PROOF When σ is a sort, $\mathcal{C}_\sigma = \mathbb{N} \times \text{unit} \cong \mathbb{N}$ and $\mathcal{S}_\sigma = \mathcal{J}_B(\sigma)$, so the statement is trivially true. When $\sigma = \iota \Rightarrow \tau$, we have $\mathcal{C}_\sigma = \mathbb{N} \times \mathcal{F}_{\iota \Rightarrow \tau}^c$, $\mathcal{F}_{\iota \Rightarrow \tau}^c = \mathcal{J}_B(\iota) \implies \mathcal{C}_\tau$ and $\mathcal{S}_\sigma = \mathcal{J}_B(\iota) \implies \mathcal{S}_\tau$. By induction, (\mathcal{C}_τ, \geq) and $(\mathcal{S}_\tau, \sqsupseteq)$ are quasi-ordered sets. So are $(\mathcal{F}_{\iota \Rightarrow \tau}^c, \geq)$ and $(\mathcal{S}_\sigma, \sqsupseteq)$, which are ordered by pointwise comparison. By Lemma 1, $(\mathcal{C}_\sigma, >, \geq)$ is a well-founded set. One minimum of $(\mathcal{S}_\sigma, \sqsupseteq)$ is the constant function $\lambda x. \perp$ where \perp is a minimum of $(\mathcal{S}_\tau, \sqsupseteq)$. ■

The cost component \mathcal{C}_σ of (σ) holds information about the cost of reducing a term of type σ to its normal form. It has two parts: one is numeric; the other is functional. The functional part \mathcal{F}_σ^c degenerates to unit when σ is just a sort and is indeed a functional space when $\sigma = \iota \Rightarrow \tau$ is a function type. In the latter case, $\mathcal{F}_\sigma^c = \mathcal{S}_\iota \implies \mathcal{C}_\tau$ consists of weakly monotonic functions with domain \mathcal{S}_ι , the size component of (ι) . This is very much in line with the standard complexity notion based on Turing Machines in which time complexity is parametrized by the input’s size.

We need a concrete interpretation key in order to use Definition 3 to interpret types. In our examples, a particular kind of interpretation key maps each sort ι to size sets of the form $(\mathbb{N}^{K[\iota]}, \sqsupseteq)$, with $K[\iota] \geq 1$, and are ordered as follows: $\langle x_1, \dots, x_{K[\iota]} \rangle \sqsupseteq \langle y_1, \dots, y_{K[\iota]} \rangle$ if $x_i \geq y_i$ for all i . This class of interpretation key is used unless stated otherwise. We take a semantic approach (cf. [16]) to determine the number $K[\iota]$ for each sort ι . For instance nat is the sort of natural numbers in unary format, so a number $n \in \mathbb{N}$ is represented as the data term $s(\dots(s\ 0))$, that is, n successive applications of s to 0. With that in mind the number of occurrences of s in such terms is a reasonable measure for their size, so we let $K[\text{nat}] = 1$. A second example is that of list . To characterize the size of a list we may need information about the individual elements in addition to the length of the list. So we keep track of the length as well as the maximum size of their elements. This way $K[\text{list}] = 2$. In Example 2 we interpret nat , list constructors following this intuition.

Definition 4 Cost–size tuples in (σ) are written as $\langle (n, f^c), f^s \rangle$ where $n \in \mathbb{N}$, $f^c \in \mathcal{F}_\sigma^c$, and $f^s \in \mathcal{S}_\sigma$. When σ is a function type, we refer to f^c as the *cost function* and f^s as the *size function*.

In order to define the interpretation of terms (Definition 7), we need a notion of application for cost–size tuples. More precisely, given $\mathbf{f} \in (\iota \Rightarrow \tau)$ and $\mathbf{x} \in (\iota)$, our goal is to define $\mathbf{f} \cdot \mathbf{x} \in (\tau)$, the application of \mathbf{f} to \mathbf{x} . Let us illustrate how such an application should work with a concrete example. Consider the function $\text{append} : \text{list} \Rightarrow \text{list} \Rightarrow \text{list}$ from Example 1. It takes two lists q and l as input. The intended cost–size denotational semantics for append is a tuple $\mathbf{f} = \langle (n, f^c), f^s \rangle \in (\text{list} \Rightarrow \text{list} \Rightarrow \text{list})$, where

$$\begin{aligned} n &\in \mathbb{N}, \\ f^c &\in \overbrace{\mathcal{S}_{\text{list}}}^{\text{size of } q} \implies (\mathbb{N} \times (\overbrace{\mathcal{S}_{\text{list}}}^{\text{size of } l} \implies (\mathbb{N} \times \text{unit}))), \text{ and} \\ f^s &\in \overbrace{\mathcal{S}_{\text{list}}}^{\text{size of } q} \implies (\overbrace{\mathcal{S}_{\text{list}}}^{\text{size of } l} \implies \mathcal{S}_{\text{list}}). \end{aligned}$$

For the first list q , take a cost–size tuple $\mathbf{x} = \langle (m, \mathfrak{U}), x^s \rangle$ from $\langle \text{list} \rangle$. We apply f^c and f^s to x^s , and get $f^c(x^s) = (k, h) \in \mathbb{N} \times (\mathcal{S}_{\text{list}} \implies (\mathbb{N} \times \text{unit}))$ and $f^s(x^s) \in \mathcal{S}_{\text{list}} \implies \mathcal{S}_{\text{list}}$, respectively. Then we sum the numeric parts and collect all the data in the new cost–size tuple $\langle (n + m + k, h), f^s(x^s) \rangle$. This process is summarized in the following definition.

Definition 5 (Semantic Application) Given $\mathbf{f} = \langle (n, f^c), f^s \rangle \in \langle \iota \Rightarrow \tau \rangle$ and $\mathbf{x} = \langle (m, \mathfrak{U}), x^s \rangle \in \langle \iota \rangle$, the *semantic application* of \mathbf{f} to \mathbf{x} , denoted by $\mathbf{f} \cdot \mathbf{x}$, is $\langle (n + m + k, h), f^s(x^s) \rangle$ where $f^c(x^s) = (k, h)$.

Semantic application is left-associative, so $\mathbf{f} \cdot \mathbf{g} \cdot \mathbf{h}$ stands for $(\mathbf{f} \cdot \mathbf{g}) \cdot \mathbf{h}$. This definition conforms to the types, which is stated in the following lemma.

Lemma 3 *If $\mathbf{f} \in \langle \iota \Rightarrow \tau \rangle$ and $\mathbf{x} \in \langle \iota \rangle$, then $\mathbf{f} \cdot \mathbf{x} \in \langle \tau \rangle$.*

Remark 1 Because $\mathbb{N} \times \text{unit}$ is order-isomorphic to \mathbb{N} , we identify $\mathbb{N} \times \text{unit}$ with \mathbb{N} and (m, \mathfrak{U}) with m unless otherwise stated. So we write $\langle m, x^s \rangle$ for cost–size tuples in $\langle \iota \rangle$ where ι is a sort.

3.2 Cost–Size Tuple Algebras

An interpretation of a syntax signature $(\mathcal{B}, \mathcal{F}, \text{ar})$ interprets the types in $\mathcal{T}_{\mathcal{B}}$ and each $f : \sigma \in \mathcal{F}$ to an element of $\langle \sigma \rangle$. This is formally stated in the definition below.

Definition 6 A *cost–size tuple algebra* $(\langle \cdot \rangle, \mathcal{J})$ over a syntax signature $(\mathcal{B}, \mathcal{F}, \text{ar})$ consists of:

- (i) a family of cost–size products $\{\langle \sigma \rangle\}_{\sigma \in \mathcal{T}_{\mathcal{B}}}$, and
- (ii) an interpretation function $\mathcal{J} : \mathcal{F} \longrightarrow \biguplus_{\sigma} \langle \sigma \rangle$ that associates to each $f : \sigma$ an element $\mathcal{J}_f \in \langle \sigma \rangle$.

We extend the notion of interpretation to terms, where we use a valuation to map variables of type ι to elements of $\langle \iota \rangle$. With innermost rewriting we assume that variables have no cost.

Definition 7 Fix a cost–size tuple algebra $(\langle \cdot \rangle, \mathcal{J})$. A *valuation* $\alpha : \mathcal{X} \longrightarrow \biguplus_{\sigma} \langle \sigma \rangle$ is a function which maps each variable $x : \iota$ to a zero-cost tuple $\langle 0, x^s \rangle \in \langle \iota \rangle$. The interpretation of a term s under the valuation α , denoted by $\llbracket s \rrbracket_{\alpha}^{\mathcal{J}}$, is defined as follows:

$$\llbracket f \rrbracket_{\alpha}^{\mathcal{J}} = \mathcal{J}_f \qquad \llbracket x \rrbracket_{\alpha}^{\mathcal{J}} = \alpha(x) \qquad \llbracket s t \rrbracket_{\alpha}^{\mathcal{J}} = \llbracket s \rrbracket_{\alpha}^{\mathcal{J}} \cdot \llbracket t \rrbracket_{\alpha}^{\mathcal{J}}$$

We write $\llbracket s \rrbracket$ instead of $\llbracket s \rrbracket_{\alpha}^{\mathcal{J}}$ whenever α and \mathcal{J} are universally quantified or clear from the context. In both cases we may write $\llbracket x \rrbracket = x$ instead of $\llbracket x \rrbracket_{\alpha}^{\mathcal{J}} = \alpha(x)$. As a corollary of Lemma 3, the interpretation of terms conforms with types.

Lemma 4 *If $s : \sigma$ then $\llbracket s \rrbracket \in \langle \sigma \rangle$.*

Let σ be $\iota_1 \Rightarrow \dots \Rightarrow \iota_m \Rightarrow \kappa$ where all ι_i and κ are sorts. Elements of \mathcal{C}_{σ} can be written as

$$\begin{aligned} & (e_0, \lambda x_1. \\ & \quad (e_1, \lambda x_2. \\ & \quad \quad \dots \\ & \quad \quad (e_{m-1}, \lambda x_m. \\ & \quad \quad \quad (e_m, \mathfrak{U}) \dots)). \end{aligned} \tag{1}$$

When $e_0 = e_1 = \dots = e_{m-1} = 0$, we write $(\lambda x_1 \dots x_m. e_m)$ as a shorthand.

Example 2 Let \mathcal{S}_{nat} and $\mathcal{S}_{\text{list}}$ be \mathbb{N} and $\mathbb{N} \times \mathbb{N}$, respectively. Recall that the size of a natural number is the number of occurrences of s , and the size of a list is a pair $q = (q_1, q_m)$ where q_1 is the length and q_m is the maximum size of the elements. We interpret the constructors as follows:

$$\begin{aligned} \mathcal{J}_0 &= \langle 0, 0 \rangle & \mathcal{J}_s &= \langle (\lambda x.0), \lambda x.x + 1 \rangle \\ \mathcal{J}_{\text{nil}} &= \langle 0, (0, 0) \rangle & \mathcal{J}_{\text{cons}} &= \langle (\lambda xq.0), \lambda xq.(q_1 + 1, \max(x, q_m)) \rangle \end{aligned}$$

Both 0 and nil have no cost because they are constructors without a function type. With innermost rewriting, constructors with a function type, such as s and cons , have $e_0 = \dots = e_m = 0$ for cost, of form (1).

Remark 2 In Definition 7 we require that valuations interpret variables as zero-cost tuples. This is an important but subtle requirement that only works when reductions are innermost. Indeed, if reduction is unrestricted we can instantiate variables on the left-hand side of rules to terms containing redexes for which the cost should be accounted. Hence, not accounting for the cost of variables in full rewriting would lead to unsound analysis. Additionally, zero-cost tuples allow us to prove the innermost termination of the TRS \mathcal{R} in Example 3, which is non-terminating in full rewriting.

3.3 Compatibility Theorem

Roughly, the compatibility theorem (Theorem 1) states that if \mathcal{R} is compatible with a tuple algebra \mathcal{A} , then the innermost rewrite relation $\rightarrow_{\mathcal{R}}^i$ is embedded in the well-founded order on cost-size products. The next two lemmas are technical results needed in order to prove it. Lemma 5 states that interpretations are closed under substitution and Lemma 6 provides strong monotonicity to semantic application.

Definition 8 Fix a cost-size tuple algebra $(\langle \cdot \rangle, \mathcal{J})$. A substitution γ is *zero-cost* under valuation α if $\llbracket \gamma(x) \rrbracket_{\alpha}^{\mathcal{J}}$ is a zero-cost tuple for each variable x . Given a valuation α and a zero-cost substitution γ , the function $\alpha^{\gamma} = \llbracket \cdot \rrbracket_{\alpha}^{\mathcal{J}} \circ \gamma = \llbracket \gamma(\cdot) \rrbracket_{\alpha}^{\mathcal{J}}$ is thus a valuation.

Lemma 5 (Substitution) *If γ is a zero-cost substitution under valuation α , $\llbracket s\gamma \rrbracket_{\alpha}^{\mathcal{J}} = \llbracket s \rrbracket_{\alpha^{\gamma}}^{\mathcal{J}}$ for any term s .*

Lemma 6 *The application functional $\text{App}(\mathbf{f}, \mathbf{x}) = \mathbf{f} \cdot \mathbf{x}$ is strongly monotonic on both arguments.*

PROOF We need to prove (i) if $\mathbf{f} \succ \mathbf{g}$ and $\mathbf{x} \succcurlyeq \mathbf{y}$, then $\text{App}(\mathbf{f}, \mathbf{x}) \succ \text{App}(\mathbf{g}, \mathbf{y})$; (ii) if $\mathbf{f} \succcurlyeq \mathbf{g}$ and $\mathbf{x} \succ \mathbf{y}$, then $\text{App}(\mathbf{f}, \mathbf{x}) \succ \text{App}(\mathbf{g}, \mathbf{y})$; (iii) if $\mathbf{f} \succcurlyeq \mathbf{g}$ and $\mathbf{x} \succcurlyeq \mathbf{y}$, then $\text{App}(\mathbf{f}, \mathbf{x}) \succcurlyeq \text{App}(\mathbf{g}, \mathbf{y})$. Consider cost-size tuples $\mathbf{f}, \mathbf{g} \in \langle t \Rightarrow \tau \rangle$ and $\mathbf{x}, \mathbf{y} \in \langle t \rangle$. Let $\mathbf{f} = \langle \langle n, f^c \rangle, f^s \rangle$, $\mathbf{g} = \langle \langle m, g^c \rangle, g^s \rangle$, $\mathbf{x} = \langle x^c, x^s \rangle$, and $\mathbf{y} = \langle y^c, y^s \rangle$. We proceed to show (i) and observe that (ii) and (iii) follow similar reasoning. Indeed, if $\mathbf{f} \succ \mathbf{g}$ and $\mathbf{x} \succcurlyeq \mathbf{y}$ we have that $n > m$, $f^c \geq g^c$, $f^s \sqsupseteq g^s$, $x^c \geq y^c$, and $x^s \sqsupseteq y^s$. Hence, by letting $f^c(x^s) = (k, h)$ and $g^c(y^s) = (k', h')$, we get:

$$\text{App}(\mathbf{f}, \mathbf{x}) = \langle \langle n, f^c \rangle, f^s \rangle \cdot \langle x^c, x^s \rangle = \langle \langle n + x^c + k, h \rangle, f^s(x^s) \rangle > \langle \langle m + y^c + k', h' \rangle, g^s(y^s) \rangle = \text{App}(\mathbf{g}, \mathbf{y})$$

■

Definition 9 A TRS \mathcal{R} is said to be *compatible* with a cost-size tuple algebra $(\langle \cdot \rangle, \mathcal{J})$ if $\llbracket \ell \rrbracket_{\alpha}^{\mathcal{J}} \succ \llbracket r \rrbracket_{\alpha}^{\mathcal{J}}$ for all rules $\ell \rightarrow r \in \mathcal{R}$ and valuations α .

Theorem 1 (Compatibility) *Let \mathcal{R} be a TRS compatible with a cost-size tuple algebra $(\langle \cdot \rangle, \mathcal{J})$. Then, for any pair of terms s and t , whenever $s \rightarrow_{\mathcal{R}}^i t$ we have $\llbracket s \rrbracket_{\alpha}^{\mathcal{J}} \succ \llbracket t \rrbracket_{\alpha}^{\mathcal{J}}$.*

PROOF We proceed by induction on $\rightarrow_{\mathcal{R}}^i$. For the base case, $s \rightarrow_{\mathcal{R}}^i t$ by $\ell\gamma \rightarrow r\gamma$ and all subterms of $\ell\gamma$ are in $\rightarrow_{\mathcal{R}}$ normal form. Therefore, since $\llbracket \ell \rrbracket_{\alpha}^{\mathcal{J}} \succ \llbracket r \rrbracket_{\alpha}^{\mathcal{J}}$ by hypothesis, Lemma 5 gives us that $\llbracket \ell\gamma \rrbracket_{\alpha}^{\mathcal{J}} \succ \llbracket r\gamma \rrbracket_{\alpha}^{\mathcal{J}}$.

In the inductive step we use Lemma 6 combined with the (IH) as follows. Suppose $s \rightarrow_{\mathcal{R}}^i t$ by $s = s'u \rightarrow_{\mathcal{R}}^i s''u$ with $s' \rightarrow_{\mathcal{R}}^i s''$. Hence, $\llbracket s'u \rrbracket_{\alpha}^{\mathcal{J}} = \llbracket s' \rrbracket_{\alpha}^{\mathcal{J}} \cdot \llbracket u \rrbracket_{\alpha}^{\mathcal{J}} = \text{App}(\llbracket s' \rrbracket_{\alpha}^{\mathcal{J}}, \llbracket u \rrbracket_{\alpha}^{\mathcal{J}})$, henceforth the induction hypothesis gives $\llbracket s' \rrbracket_{\alpha}^{\mathcal{J}} \succ \llbracket s'' \rrbracket_{\alpha}^{\mathcal{J}}$, which combined with Lemma 6 implies $\llbracket s \rrbracket_{\alpha}^{\mathcal{J}} = \text{App}(\llbracket s' \rrbracket_{\alpha}^{\mathcal{J}}, \llbracket u \rrbracket_{\alpha}^{\mathcal{J}}) \succ \text{App}(\llbracket s'' \rrbracket_{\alpha}^{\mathcal{J}}, \llbracket u \rrbracket_{\alpha}^{\mathcal{J}}) = \llbracket t \rrbracket_{\alpha}^{\mathcal{J}}$. When $s \rightarrow_{\mathcal{R}}^i t$ with $s = s'u \rightarrow_{\mathcal{R}}^i s'u'$ the proof is analogous. ■

Example 3 Let $0, 1: \iota, g: \iota \Rightarrow \iota \Rightarrow \iota$, and $f: \iota \Rightarrow \iota \Rightarrow \iota \Rightarrow \iota$. The rewrite system introduced by Toyama [23] and defined by $\mathcal{R} = \{gxy \rightarrow x, gxy \rightarrow y, f01z \rightarrow fzzz\}$ was given to show that termination is not modular for disjoint unions of TRSs. Indeed, it admits the infinite rewriting sequence $f01(g01) \rightarrow_{\mathcal{R}} f(g01)(g01)(g01) \rightarrow_{\mathcal{R}}^+ f01(g01)$. However, the innermost relation $\rightarrow_{\mathcal{R}}^i$ is terminating. In order to prove it, we introduce a non-numeric notion of size. Let $\mathcal{J}_{\mathcal{B}}(\iota) = \mathcal{P}(T(\mathcal{F}, \mathcal{X}))$, i.e., the set of all subsets of $T(\mathcal{F}, \mathcal{X})$. This set is partially ordered by set inclusion, so $x \sqsupseteq y$ iff $x \supseteq y$, which is a quasi-order. Consider the following interpretation:

$$\mathcal{J}_0 = \langle 0, \{0\} \rangle \quad \mathcal{J}_1 = \langle 0, \{1\} \rangle \quad \mathcal{J}_g = \langle (\lambda xy.1), \lambda xy.x \cup y \rangle \quad \mathcal{J}_f = \langle (\lambda xyz.H(x,y)), \lambda xyz.\emptyset \rangle,$$

where H is a helper function defined by $H(x,y) = \text{if } x \sqsupseteq \{0\} \wedge y \sqsupseteq \{1\} \text{ then } 1 \text{ else } 0$. Notice that H is weakly monotonic and all terms in normal form are interpreted as sets of size ≤ 1 . Checking compatibility is straightforward: $\llbracket gxy \rrbracket = \langle 1, x \cup y \rangle \succ \langle 0, x \rangle = \llbracket x \rrbracket$ and $\llbracket gxy \rrbracket = \langle 1, x \cup y \rangle \succ \langle 0, y \rangle = \llbracket y \rrbracket$; and $\llbracket f01z \rrbracket = \langle 1, \emptyset \rangle \succ \langle 0, \emptyset \rangle = \llbracket fzzz \rrbracket$, because any instantiation of z is necessarily in normal form, so it cannot include both 0 and 1.

This example, albeit artificial, is interesting from a termination point of view. It shows that tuple interpretations can be used to deal with rewrite systems that only terminate via the innermost strategy.

4 Polynomial Bounds for Innermost Runtime Complexity

In this section, we study the applications of tuple interpretations to complexity analysis of compatible TRSs, i.e., rewriting systems that admit an interpretation in a tuple algebra $(\langle \cdot \rangle, \mathcal{J})$. Even though cost and size are split in our setting, they are intertwined concepts (in a sense we make precise in this section) that constitute what we intuitively call “complexity” of a TRS.

4.1 Additive Tuple Interpretations

In order to establish upper bounds to $\text{irc}_{\mathcal{R}}(n)$, it suffices to give upper bounds to the cost component $\llbracket s \rrbracket^c$ of all terms s where $|s| \leq n$. Furthermore, since basic terms are of the form $f d_1 \dots d_m$, the size of data terms plays an important role in our analysis. In what follows, we use the default choice for interpretation key when interpreting types; that is, $\mathcal{J}_{\mathcal{B}}(\iota) = \mathbb{N}^{K[\iota]}$, with $K[\iota] \geq 1$ for each $\iota \in \mathcal{B}$.

Given $\sigma = \iota_1 \Rightarrow \dots \Rightarrow \iota_m \Rightarrow \kappa$, the size component of (σ) is $\mathcal{S}_{\sigma} = \mathbb{N}^{K[\iota_1]} \Longrightarrow \dots \Longrightarrow \mathbb{N}^{K[\iota_m]} \Longrightarrow \mathbb{N}^{K[\kappa]}$. Size functions $f^s \in \mathcal{S}_{\sigma}$ when fully applied can be written in terms of functional components. Hence, $f^s(x_1, \dots, x_m) = \langle f_1^s(x_1, \dots, x_m), \dots, f_{K[\kappa]}^s(x_1, \dots, x_m) \rangle$.

Definition 10 Let σ be a type and $f^s \in \mathcal{S}_{\sigma}$. The size function f^s is *linearly bounded* if each one of its component functions $f_1^s, \dots, f_{K[\kappa]}^s$ is upper-bounded by a positive linear polynomial, i.e., there is a positive constant $a \in \mathbb{N}$ such that for all $1 \leq l \leq m$, $f_l^s(x_1, \dots, x_m) \leq a(1 + \sum_{i=1}^m \sum_{j=1}^{K[\iota_i]} x_{ij})$. Analogously, we say f^s is *additive* if there is a constant $a \in \mathbb{N}$ such that $\sum_{l=1}^{K[\kappa]} f_l^s(x_1, \dots, x_m) \leq a + \sum_{i=1}^m \sum_{j=1}^{K[\iota_i]} x_{ij}$.

Notice that by this definition linearly bounded (or additive) size functions are not required to be linear (or additive) but to be upper-bounded by a linear (additive) function. So this permits us to use for instance $\min(x, 2y)$, whereas xy cannot be used. Size interpretations do not necessarily bound the absolute size of data terms. For instance, we may interpret a data constructor $c: \iota \Rightarrow \kappa$ with $\mathcal{J}_c^s = \lambda x. \lfloor x/2 \rfloor$ which would give us $|d| \geq \llbracket d \rrbracket^s$. This is especially useful when dealing with sublinear interpretations.

The next lemma ensures that by interpreting constructors additively, the size interpretation of data terms is proportional to their absolute size:

Lemma 7 *Let \mathcal{R} be a TRS compatible with a cost–size tuple algebra $(\langle \cdot \rangle, \mathcal{J})$.*

- (i) *Assume \mathcal{J}_c^s is additive for all data constructors c , then for all data terms d : if $|d| \leq n$, then there exists a constant $b > 0$ such that $\llbracket d \rrbracket_l^s \leq bn$, for each size-component $\llbracket d \rrbracket_l^s$ of $\llbracket d \rrbracket$.*
- (ii) *Assume \mathcal{J}_c^s is linearly bounded for all data constructors c , then for all data terms d : if $|d| \leq n$, then there exists a constant $b > 0$ such that $\llbracket d \rrbracket_l^s \leq 2^{bn}$, for each size-component $\llbracket d \rrbracket_l^s$ of $\llbracket d \rrbracket$.*

The bound in (ii) is sharp. Indeed, define (when interpreting \mathcal{R}_{add}): $\mathcal{J}_0 = \langle 0, 1 \rangle$, $\mathcal{J}_s = \langle \lambda x. 0, \lambda x. 2x + 1 \rangle$, and $\mathcal{J}_{\text{add}} = \langle \langle \lambda xy. y + 1, \lambda xy. x + y \rangle \rangle$. In this case, for a data term $n = s^n(0)$ its size interpretation is exactly $\llbracket n \rrbracket^s = 2^n + n \leq 2^{|n|}$. However, whereas this choice is compatible with \mathcal{R}_{add} , and hence proving its termination, it induces an exponential overhead on $\text{irc}_{\mathcal{R}_{\text{add}}}$, which is linearly bounded (see Example 4). Such a huge overestimation is not desirable in a complexity analysis setting. This behavior sets a strict upper-bound to the interpretation of data constructors; namely, we seek to bound the size interpretations of constructors additively. It is easy to show that size components for `nat` and `list` in Example 2 are additive.

Definition 11 We say an interpretation \mathcal{J} is additive if for each $c \in \mathcal{C}$, \mathcal{J}_c^s is additive.

4.2 Cost-Bounded Tuple Interpretations

In what follows, we consider rewriting systems with additive interpretations.

Definition 12 Let σ be a type and $f^c \in \mathcal{C}_\sigma$. We say f^c , written as in form (1), is linearly (additively) bounded whenever each e_i , $0 \leq i \leq m$, is linearly (additively) bounded. Additionally, \mathcal{J}_f is bounded by a functional f if both \mathcal{J}_f^c and \mathcal{J}_f^s are bounded by f .

In the next lemma, we collect the appropriate induced upper-bounds on innermost runtime complexity given that we can provide bounds to the cost–size components of interpretations.

Lemma 8 *Suppose \mathcal{R} is a TRS compatible with a tuple algebra $(\langle \cdot \rangle, \mathcal{J})$, then:*

- (i) *if, for all $f \in \mathcal{F}$, \mathcal{J}_f^s is logarithmically and \mathcal{J}_f^c is additively bounded, then $\text{irc}_{\mathcal{R}}(n) \in \mathcal{O}(\log n)$;*
- (ii) *if, for all $f \in \mathcal{F}$, \mathcal{J}_f is additively bounded, then $\text{irc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$; and*
- (iii) *if, for all defined symbols f and constructors c , \mathcal{J}_c is additively and \mathcal{J}_f is polynomially bounded, then $\text{irc}_{\mathcal{R}}(n) \in \mathcal{O}(n^k)$, for some $k \in \mathbb{N}$.*

Example 4 Let us illustrate this behavior by interpreting functions from Example 1. Interpretation for constructors was given in Example 2.

$$\begin{aligned}
 \mathcal{J}_{\text{add}} &= \langle \langle \lambda xy. y + 1, \lambda xy. x + y \rangle \rangle & \mathcal{J}_{\text{sum}} &= \langle \langle \lambda q. 2q_l + q_l q_m, \lambda q. q_l q_m \rangle \rangle \\
 \mathcal{J}_{\text{minus}} &= \langle \langle \lambda xy. y + 1, \lambda xy. x \rangle \rangle & \mathcal{J}_{\text{rev}} &= \langle \langle \lambda q. q_l + \frac{q_l(q_l+1)}{2} + 1, \lambda q. q \rangle \rangle \\
 \mathcal{J}_{\text{quot}} &= \langle \langle \lambda xy. x + xy + 1, \lambda xy. x \rangle \rangle \\
 \mathcal{J}_{\text{append}} &= \langle \langle \lambda ql. q_l + 1, \lambda ql. \langle q_l + l_l, \max(q_m, q_m) \rangle \rangle \rangle
 \end{aligned}$$

Checking the compatibility of this interpretation is straightforward. Notice that in each set of rules defining a function f in Example 1 size components are additively and cost components are polynomially bounded. By case (b) of Lemma 8, we have that $\text{irc}_{\mathcal{R}_{\text{add}}}$, $\text{irc}_{\mathcal{R}_{\text{append}}}$, and $\text{irc}_{\mathcal{R}_{\text{minus}}}$ are linear. Quadratic bounds can be derived to $\text{irc}_{\mathcal{R}_{\text{quot}}}$, $\text{irc}_{\mathcal{R}_{\text{sum}}}$, and $\text{irc}_{\mathcal{R}_{\text{rev}}}$.

Recall the semantic meaning given to size components, see Example 2, one can observe that the cost component of interpretations do not only bound the innermost runtime complexity of \mathcal{R}_f but also provide additional information on the role each size component plays in the rewriting cost. For instance: the cost of adding two numbers depends solely on the size of add 's second argument; the cost of summing every element of a list has a linear dependency on its length and non-linear dependency on its length and maximum element. This is particularly useful in program analysis since one can detect a possible costly operation by analyzing the shape of interpretations themselves.

5 Automation

In this section, we limn a procedure for finding cost–size tuple interpretations. Our goal is to find interpretations that guarantee polynomial bounds to the runtime complexity of the rewriting system at hand. Hence, we have the following conditions: (i) the interpretation key chosen is over \mathbb{N} , (ii) the size interpretation of constructors is additively bounded, and (iii) the interpretation of function symbols is polynomially bounded.

Parametric Interpretations. Recall that previously in the paper we assigned an intuitive meaning for size components. In a fully automated setting, where no human guidance is allowed, all sorts ι start with $K[\iota] = 1$ and go up to a predefined bound K . This maximum bound K is needed to limit the search space and guarantee that the procedure terminates.

Roughly, the procedure works as follows. The interpretation of data constructors is set to be additive. So if $c : \iota_1 \Rightarrow \dots \Rightarrow \iota_m \Rightarrow \kappa$ is a data constructor, its size interpretation is $\lambda x_1 \dots x_m. a + \sum_{i=1}^m \sum_{j=1}^{K[\iota_i]} x_{ij}$, where a is a parameter to be determined by the search procedure. We say such an *interpretation shape* is parametrized by the coefficient a . The next step is to choose (parametric) interpretations for defined symbols $f \in \mathcal{F}$. In contrast with constructors where the cost components are zero-valued functions and size components are additive, we can choose any function that is polynomially bounded for cost and size components of a defined symbol $f \in \mathcal{F}$.

However, the class of functions from which we can choose interpretations of defined symbols is too big. So we restrict our search space to a limited class of polynomially bounded functions: max-polynomials, i.e., functions that combine polynomial terms and the max function. For instance, the interpretations of cons in Example 2 and append in Example 4 are max-polynomials. We then choose generic max-polynomials for the cost and size components which are parametrized by their coefficients. Recall that we wish for finding interpretations that satisfy the compatibility condition, i.e., $\llbracket \ell \rrbracket_\alpha \succ \llbracket r \rrbracket_\alpha$, for any α . Therefore, if we pick max-polynomials parametrized by their coefficients, those give rise to a set of constraints that must be solved in order to determine valid interpretations.

Example 5 Let us illustrate the ideas above with a simple system defining the function dbl over natural numbers. So we consider the system with rules $\text{dbl } 0 \rightarrow 0$ and $\text{dbl } (s\ x) \rightarrow s\ (s\ (\text{dbl } x))$. Let us choose the following parametric interpretation

$$\mathcal{J}_0 = \langle 0, a_0 \rangle \quad \mathcal{J}_s = \langle (\lambda x.0), \lambda x.x + b_0 \rangle \quad \mathcal{J}_{\text{dbl}} = \langle (\lambda x.c_1x + c_0), \lambda x.d_1x + d_0 \rangle,$$

which satisfy conditions (i)-(iii) above. The interpretation above is *parametric* in the sense that the coefficients $a_0, b_0, c_0, c_1, d_0, d_1$ are to be determined. The compatibility condition for the first rule gives:

$$\llbracket \text{dbl } 0 \rrbracket \succ \llbracket 0 \rrbracket \implies \langle (c_1 a_0 + c_0), d_1 a_0 + d_0 \rangle \succ \langle 0, a_0 \rangle,$$

which in consequence requires the validity of $C_0 = (c_1 a_0 + c_0 > 0) \wedge (d_1 a_0 + d_0 \geq a_0)$. The compatibility condition for the second rule, on the other hand, gives us the following:

$$\llbracket \text{dbl } (s \ x) \rrbracket \succ \llbracket s \ (s \ (\text{dbl } x)) \rrbracket \implies \langle (c_1 x + c_1 b_0 + c_0), d_1 x + d_1 b_0 + d_0 \rangle \succ \langle (c_1 x + c_0), d_1 x + d_0 + 2b_0 \rangle,$$

which in consequence requires the validity of the formula

$$C_1 = (c_1 x + c_1 b_0 + c_0 > c_1 x + c_0) \wedge (d_1 x + d_1 b_0 + d_0 \geq d_1 x + d_0 + 2b_0).$$

Hence, we seek to find witnesses for the constraints C_0, C_1 over \mathbb{N} . For which we can use an SMT solver.

The example above is very simple in nature but uses the main ideas of our procedure. Essentially, we choose parametric interpretations for function symbols in \mathcal{F} and solve the constraints that arise from the compatibility condition. As we have seen in Example 4, cost–size interpretations may become complicated, so more interpretation shapes are needed in the search procedure. We describe such a procedure below. It is modular in the sense that it is parametrized by a *selector strategy* \mathcal{S} and constraint solver. A selector strategy is an algorithm to choose a parametric interpretation for each function symbol in \mathcal{F} . For instance, in the example above we have chosen *linear parametric shapes* for all function symbol.

Main Procedure

Parameter: A selector algorithm \mathcal{S} and a constraint solver over non-linear integer arithmetic.

Data Input: A TRS \mathcal{R} over a syntax signature $(\mathcal{B}, \mathcal{F}, \text{ar})$.

Output: YES, if a cost–size tuple interpretation satisfying compatibility can be found and MAYBE, if all steps below were executed and no interpretation could be found¹.

1. Split \mathcal{F} into two disjoint sets of constructors and defined symbols, i.e., $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$.
2. For each constructor $c: t_1 \Rightarrow \dots \Rightarrow t_m \Rightarrow \kappa$, choose its cost interpretation as the zero-valued cost function; size interpretations are additive.
3. Split \mathcal{D} into sets $\mathcal{D}_1, \dots, \mathcal{D}_n$ such that for each $f \in \mathcal{D}_i$, with $1 \leq i \leq n$, all function symbols occurring in the rules defining f are either constructors or in $\mathcal{D}_1 \cup \dots \cup \mathcal{D}_i$.
4. For each $1 \leq i \leq n$, choose an *interpretation shape* for the symbols in \mathcal{D}_i based on the selector strategy \mathcal{S} (to be defined below).
 - Mark the chosen interpretation shape on \mathcal{S} , so we don't choose the same again in case this step fails.
 - If no choice can be made by \mathcal{S} , stop and return MAYBE.
5. If $f \ell_1 \dots \ell_k \rightarrow r$ is a rule of type t with $f \in \mathcal{D}_1 \cup \dots \cup \mathcal{D}_i$. *Simplify* $\llbracket f \ell_1 \dots \ell_k \rrbracket \succ \llbracket r \rrbracket$ so that the result is a set of inequality constraints C that does not depend on any interpreted variable (we shall define this simplification step below).
 - If this simplification step fails, then we return to step 4 to choose another interpretation shape.

¹Notice that in our setting we cannot possibly return NO.

6. Check if C holds.

- If all constraints in C hold and $i < n$, it means that we could orient all rules headed by function symbols in \mathcal{D}_i , so we go to step 4 with $i := i + 1$.
- If all constraints in C hold and $i = n$, then we could orient all rules \mathcal{R} , stop return YES.
- Otherwise, increase $K[t]$ by one, update the additive size interpretation for the constructors, and return to step 4 choosing another interpretation shape.

Two key aspects of the procedure above remain to be defined. The strategy \mathcal{S} for selecting interpretation shapes and the constraint solver, Step 6.

Strategy-based Search for Tuple Interpretations. Intuitively, a selector strategy \mathcal{S} is an algorithm for choosing parametric interpretations for defined symbols in \mathcal{D}_i . For instance, we could randomly pick an interpretation shape from a list (the **blind** strategy); we could incrementally select interpretations from a list of possible attempts (the **progressive** strategy); or we could select interpretations based on their syntax patterns (the **pattern** strategy).

The definition below lists some interpretation shapes we consider. They are based on the classes studied in [9, 22] Parametric interpretations are built by considering the type of defined symbols.

Definition 13 (Interpretation Shapes) Let $\sigma = \iota_1 \Rightarrow \dots \Rightarrow \iota_m \Rightarrow \kappa$ and each f_{ij} appearing in the shapes below be an additively bounded weakly monotonic function over S_σ . We write $f(\vec{x})$ for the application of f to each argument x_1, \dots, x_m .

- The *additive class* contains additively bounded cost–size functionals of the following form:

$$\lambda x_1 \dots x_m. \sum_{i=1}^m \sum_{j=1}^{K[t_i]} x_{ij} + b_0 + f(\vec{x})$$

- The *linear class* contains cost–size functionals written as:

$$\lambda x_1 \dots x_m. \sum_{i=1}^m \sum_{j=1}^{K[t_i]} a_{ij} x_{ij} f_{ij}(\vec{x})$$

- The *simple class* contains cost–size functionals written as:

$$\lambda x_1 \dots x_m. \sum_{i=1}^m \sum_{j=1}^{K[t_i]} a_{ij} x_{ij}^{k_{ij}} f_{ij}(\vec{x}), \text{ such that each } k_{ij} \in \{0, 1\}$$

- Finally, the *quadratic class* contains cost–size functionals where we allow general products of variables with degree at maximum 2:

$$\lambda x_1 \dots x_m. \sum_{i=1}^m \sum_{j=1}^{K[t_i]} a_{ij} x_{ij}^{k_{ij}} f_{ij}(\vec{x}), \text{ such that each } k_{ij} \in \{0, 1, 2\}$$

- The *simple quadratic class* contains cost–size functionals built as a sum of a simple functional plus a quadratic component:

$$\lambda x_1 \dots x_m. \sum_{i=1}^m \sum_{j=1}^{K[t_i]} a_{ij} x_{ij}^{k_{ij}} f_{ij}(\vec{x}) + \sum_{i=1}^m \sum_{j=1}^{K[t_i]} a_{ij} x_{ij}^{l_{ij}} f_{ij}(\vec{x}),$$

with $k_{ij} \in \{0, 1\}$ and $l_{ij} \in \{0, 1, 2\}$.

Hence, the blind strategy randomly selects one of the shapes above. The incremental strategy chooses interpretations in order, from additive ones to quadratic ones. The pattern strategy is slightly more difficult to realize since we need heuristic analysis on the shape of rules. For instance, every rule of the form $f\ x_1 \dots x_m \rightarrow x_i$ have constant cost functions $(\lambda x_1 \dots x_m.1)$ and additive size components. Rules that duplicate variables, as in the pattern $C[x] \rightarrow D[x, x]$, induce at least quadratic bound on cost. Notice that this is the case for all quadratic complexities in this paper. The concrete implementation of a selector algorithm determines the efficiency of the main procedure for finding interpretations.

In order to simplify constraints $\llbracket \ell \rrbracket \succ \llbracket r \rrbracket$ we have to simplify inequalities between polynomials (max-polynomials). To simplify polynomial (max-polynomial) shapes, we need to compare polynomials $P_\ell^c > R_r^c$ and $P_{\ell_1}^s \sqsupseteq P_{r_1}^s \wedge \dots \wedge P_{\ell_{k[\tau]}}^s \sqsupseteq P_{r_{k[\tau]}}^s$. These conditions are then reduced to formulas in QF-NIA (Quantifier-Free Non-Linear Integer Arithmetic) and sent to an SMT solver, see [10]. Max-polynomials are simplified using the rules $\max(x, y) + z \rightsquigarrow \max(x + z, y + z)$ and $\max(x, y)z \rightsquigarrow \max(xz, yz)$. The result has the form $\max_i P_i$ where each P_i is a polynomial without max occurrences [7].

6 Conclusion

In this paper we showed that cost–size tuple pairs can be adapted to handle innermost rewriting. The type-aware algebraic interpretation style provided the machinery necessary to deal with innermost termination and a mechanism to establish upper bounds to the innermost runtime complexity of compatible TRSs. We presented sufficient conditions for feasible (polynomial) bounds on $\text{irc}_{\mathcal{R}}$ of compatible systems, which are in line with related works in the literature. This line of investigation is far from over. Since searching for interpretations can be cumbersome, our immediate future work is to develop new strategies and interpretation shapes. For instance, we seek to expand the class of interpretations beyond max-polynomials such as logarithmic functionals. This has the potential to drastically improve the efficiency of our tooling.

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