

# On Logics of Perfect Parafinite Algebras

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The present study shows how to enrich De Morgan algebras with a *perfection operator* that allows one to express the Boolean properties of negation-consistency and negation-determinedness. The variety of *perfect parafinite algebras* thus obtained (*PP-algebras*) is shown to be term-equivalent to the variety of involutive Stone algebras, introduced by R. Cignoli and M. Sagastume, and more recently studied from a logical perspective by M. Figallo-L. Cantú and by S. Marcelino-U. Rivieccio. This equivalence plays an important role in the investigation of the 1-assertional logic and of the order-preserving logic associated to PP-algebras. The latter logic (here called  $\mathcal{PP}_{\leq}$ ) is characterized by a single 6-valued matrix and is shown to be a Logic of Formal Inconsistency and Formal Undeterminedness. We axiomatize  $\mathcal{PP}_{\leq}$  by means of an analytic finite Hilbert-style calculus, and we present an axiomatization procedure that covers the logics corresponding to other classes of De Morgan algebras enriched by a perfection operator.

## 1 Introduction

The variety of De Morgan algebras consists of all bounded distributive lattices equipped with a primitive involutive negation operation  $\sim$  satisfying the well-known De Morgan laws. Such negation need not be Boolean, that is, it may fail to satisfy the equations  $x \vee \sim x \approx \top$  and  $x \wedge \sim x \approx \perp$ , respectively expressing the classical ‘negation-determinedness’ and the ‘negation-consistency’ assumptions. Involutive Stone algebras (henceforth referred to as IS-algebras) are obtained by enriching De Morgan algebras with a further unary operation  $\nabla$  that allows for the definition of a pseudo-complement operator  $\neg$  satisfying the Stone equation  $\neg x \vee \neg\neg x \approx \top$  (as well as its dual,  $\neg x \wedge \neg\neg x \approx \perp$ ).

While the order-preserving logic canonically induced by De Morgan algebras, namely Dunn-Belnap’s 4-valued logic [5], has been extensively studied over the last four decades, the logic similarly induced by IS-algebras (which we call  $IS_{\leq}$ ) has only recently attracted due attention [7, 8, 20]. The latter studies make (but do not pursue to any significant length) an observation that we shall take as the starting point of our present work, namely that, by replacing  $\nabla$  with a unary ‘consistency operator’ (here denoted by  $\circ$ ), it is possible to view  $IS_{\leq}$  as a Logic of Formal Inconsistency ([10]).

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From the point of view of non-classical logics, some of the most prominent features of  $IS_{\leq}$  are the facts that it is parafinite [2] (it is, indeed, at once  $\sim$ -paraconsistent and  $\sim$ -paracomplete, both properties being inherited from the Dunn-Belnap logic), and yet, with the help of the single connective  $\circ$ , it may be seen to be expressive enough so as to fully recover the ‘lost perfection’ of classical negation, by being at once  $\sim$ -gently explosive and  $\sim$ -gently implosive [22]; in other words,  $IS_{\leq}$  may be seen as a Logic of Formal Inconsistency (**LFI**) and a Logic of Formal Undeterminedness (**LFU**), in the sense of [23]. These features, however, are somehow concealed by the usual presentation of IS-algebras in terms of  $\nabla$ , an algebraic operator whose significance and philosophical motivations are unclear.<sup>1</sup>

Logics that allow for the internalization of the very notions of negation-consistency and negation-determinedness at the object-language level have been extensively studied in the last two decades (cf. [4], for example, for the so-called ‘classicality’, ‘restoration’, ‘recapture’, or ‘recovery’ operators). In order to establish a fruitful dialogue with the logical study of negation, we thus propose an alternative rendition of IS-algebras in terms of structures that we christen ‘perfect parafinite algebras’ (or more briefly PP-algebras), obtained by replacing  $\nabla$  with a primitive perfection operation  $\circ$ . The significance of such an approach, we believe, is not only nor primarily technical; instead, it lies mainly in a clarification of the intuitive meanings associated to the propositional connectives employed to present  $IS_{\leq}$ . As explained below, another important consequence of our work will be the possibility of singling out new meaningful logical axioms (expressed in the alternative language we propose for  $IS_{\leq}$ ) that present more general logics than (i.e. weakenings of)  $IS_{\leq}$ .

The equational characterization we present for PP-algebras will not only guarantee that the corresponding variety is term-equivalent to the variety of IS-algebras but also highlight the expressive parafinite character of the order-preserving logic thereby induced ( $PP_{\leq}$ ). The latter logic will be shown, more specifically, to constitute a fully self-extensional and non-protoalgebraic member of the families of logics known as **C**-systems and **D**-systems (detailed explanations and discussions about the latter classes may be found in [22]). A procedure for constructing a PP-algebra using a De Morgan algebra as material is introduced and the logic  $PP_{\leq}$  is shown to be characterizable, like  $IS_{\leq}$ , by a single six-element logical matrix. Lastly, we also provide a well-behaved symmetrical Hilbert-style calculus for the SET-SET logics determined by logical matrices based on De Morgan algebras enriched with  $\circ$ , as well as conventional Hilbert-style calculi for the SET-FMLA logics determined by logical matrices with prime filters based on De Morgan algebras enriched with  $\circ$  (and, in particular, an analytical proof system for the logic  $PP_{\leq}$  itself).

In distinction to what has been done in the study of logics associated to IS-algebras [7, 8, 20], in the present work we take the more general path of first obtaining results on the SET-SET order-preserving logic (denoted by  $PP_{\leq}^{\triangleright}$ ) and on the 1-assertional logic (denoted by  $PP_{\top}^{\triangleright}$ ) associated to PP-algebras, and then specializing them to the corresponding SET-FMLA logics. From a proof-theoretical viewpoint, the present paper may thus also be viewed as providing another illustration (additional to the one in [20]) of the wide range of applicability of the machinery of SET-SET Hilbert-style calculi, which has recently been further developed in [6, 21]. Indeed, having established that  $PP_{\leq}^{\triangleright}$  is characterizable by a matrix that is finite and sufficiently expressive, the problem of obtaining a finite and analytical SET-SET calculus for it can be solved by an application of the algorithm of [21], which we used via the implementation of [16]. In order to make the present paper self-contained, we have, though, also included here the proofs of completeness and analyticity of the SET-SET calculus thus obtained. The conventional SET-FMLA

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<sup>1</sup>For the 3-valued case, such a ‘possibility’ operator is known at least since [19], where J. Łukasiewicz notes it has been first defined during one of his 1921 seminars by a student called Tarski. The lack of a robust *modal* reading for such an operator, however, has caused it to have largely fallen by the wayside over the following decades.

(non-analytic) axiomatization of  $\mathcal{PP}_{\leq}$  is then obtained from the SET-SET axiomatization following the general procedure laid out in [29, Th.5.37].

The remainder of the paper is organized as follows. Section 2 provides the basic notions on algebras and logics, the latter considered from the semantical perspective of logical matrices and from the proof-theoretical viewpoint through Hilbert-style deductive systems. Section 3 introduces the variety of perfect parafinite algebras and discusses the semantical aspects of the associated logics. More specifically, we prove that these algebras are term-equivalent to the involutive Stone algebras (Theorem 3.6), also enjoying, thus, the property of being generated by a single, six-element algebra (Proposition 3.8). We then show that the order-preserving and 1-assertional logics associated to this variety are respectively generated by a 6-valued and a 3-valued logical matrix (Theorem 3.11 and Proposition 3.14). Subsequently, we provide a recipe to endow a De Morgan algebra with a perfection operator (Definition 3.15) and study the lattice of extensions of the corresponding SET-FMLA order-preserving logic, showing that it has at least the cardinality of the continuum (Corollary 3.24). We close the section by studying parafinite extensions of Belnap-Dunn's 4-valued logic, showing how to recover classical reasoning via assumptions expressed with the help of the perfection operator (Theorem 3.28). Section 4 provides analytic SET-SET Hilbert-style deductive systems for logical matrices based on De Morgan algebras endowed with a consistency operator (Theorem 4.1), as well as SET-FMLA Hilbert-style systems for logics determined by matrices based on De Morgan algebras and prime filters (Theorem 4.6). Finally, Section 5 contains some concluding remarks and outlines future directions of research.

## 2 Algebraic and logical preliminaries

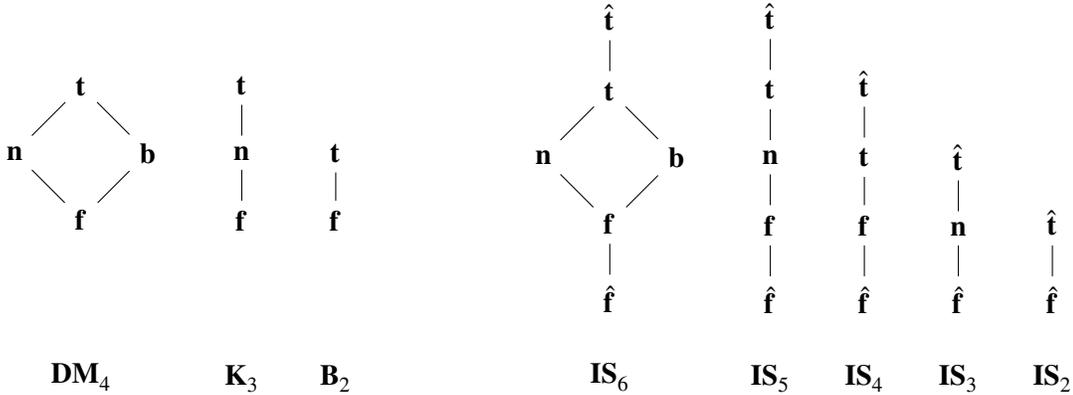
A *propositional signature* is a family  $\Sigma := \{\Sigma_k\}_{k \in \omega}$ , where each  $\Sigma_k$  is a collection of  $k$ -ary *connectives*. A  $\Sigma$ -*algebra* is a structure  $\mathbf{A} := \langle A, \cdot^{\mathbf{A}} \rangle$ , where  $A$  is a non-empty set called the *carrier* of  $\mathbf{A}$  and, for each  $\odot \in \Sigma_k$ ,  $\odot^{\mathbf{A}} : A^k \rightarrow A$  is the *interpretation* of  $\odot$  in  $\mathbf{A}$ . Given a denumerable set  $P \supseteq \{p, q, r, x, y\}$ , the absolutely free algebra over  $\Sigma$  freely generated by  $P$ , or simply the *language* over  $\Sigma$  (generated by  $P$ ), is denoted by  $\mathbf{L}_{\Sigma}(P)$ , and its members are called  $\Sigma$ -*formulas*. The collection of all propositional variables occurring in a formula  $\varphi \in L_{\Sigma}(P)$  is denoted by  $\text{props}(\varphi)$ , and we let  $\text{props}(\Phi) := \bigcup_{\varphi \in \Phi} \text{props}(\varphi)$ , for all  $\Phi \subseteq L_{\Sigma}(P)$ . Given  $\Sigma' \subseteq \Sigma$  (that is,  $\Sigma'_k \subseteq \Sigma_k$  for all  $k \in \omega$ ), the  $\Sigma'$ -*reduct* of a  $\Sigma$ -algebra  $\mathbf{A}$  is the  $\Sigma'$ -algebra over the same carrier of  $\mathbf{A}$  that agrees with  $\mathbf{A}$  on the interpretation of the connectives in  $\Sigma'$ . The collection of homomorphisms between two  $\Sigma$ -algebras  $\mathbf{A}$  and  $\mathbf{B}$  is denoted by  $\text{Hom}(\mathbf{A}, \mathbf{B})$ , and the collection of mappings that are structure-preserving over  $\Sigma' \subseteq \Sigma$  is denoted by  $\text{Hom}_{\Sigma'}(\mathbf{A}, \mathbf{B})$ . Furthermore, the set of endomorphisms on  $\mathbf{A}$  is denoted by  $\text{End}(\mathbf{A})$  and each one of the members  $\sigma \in \text{End}(\mathbf{L}_{\Sigma}(P))$  is called a *substitution*. The elements of  $\text{Hom}(\mathbf{L}_{\Sigma}(P), \mathbf{A})$  will sometimes be referred to as *valuations on  $\mathbf{A}$* . Given  $h, h' \in \text{Hom}(\mathbf{L}_{\Sigma}(P), \mathbf{A})$ , we shall say that  $h'$  *agrees with  $h$  on  $\Phi \subseteq L_{\Sigma}(P)$*  provided that  $h'(\varphi) = h(\varphi)$  for all  $\varphi \in \Phi$ . In case  $p_1, \dots, p_n$  are the only propositional variables occurring in  $\varphi \in L_{\Sigma}(P)$ , we say that  $\varphi$  is  $n$ -ary and denote by  $\varphi^{\mathbf{A}}$  the  $n$ -ary operation on  $A$  such that, for all  $a_1, \dots, a_n \in A$ ,  $\varphi^{\mathbf{A}}(a_1, \dots, a_n) = h(\varphi)$ , for an  $h \in \text{Hom}(\mathbf{L}_{\Sigma}(P), \mathbf{A})$  with  $h(p_i) = a_i$  for each  $1 \leq i \leq n$ . Also, if  $\psi_1, \dots, \psi_n \in L_{\Sigma}(P)$ , we let  $\varphi(\psi_1, \dots, \psi_n)$  denote the formula  $\varphi^{\mathbf{L}_{\Sigma}(P)}(\psi_1, \dots, \psi_n)$ . A  $\Sigma$ -*equation* is a pair  $(\varphi, \psi)$  of  $\Sigma$ -formulas that we will denote by  $\varphi \approx \psi$ , and a  $\Sigma$ -algebra  $\mathbf{A}$  is said to *satisfy*  $\varphi \approx \psi$  if  $h(\varphi) = h(\psi)$  for every  $h \in \text{Hom}(\mathbf{L}_{\Sigma}(P), \mathbf{A})$ . We call  $\Sigma$ -*variety* the class of all  $\Sigma$ -algebras that satisfy the same given collection of  $\Sigma$ -equations; an equation is said to be *valid* in a given variety if it is satisfied by each algebra in this variety. The variety generated by a class  $\mathbf{K}$  of  $\Sigma$ -algebras, denoted by  $\mathbb{V}(\mathbf{K})$ , is the closure of  $\mathbf{K}$  under homomorphic images, subalgebras and direct products. We write  $\text{Cng } \mathbf{A}$  to refer to the collection of all congruence relations on  $\mathbf{A}$ , which is known to form a complete lattice under inclusion.

In what follows, we assume the reader is familiar with basic notations and terminology of lattice theory [12]. We denote by  $\Sigma^{\text{bL}}$  the signature containing but two binary connectives,  $\wedge$  and  $\vee$ , and two nullary connectives  $\top$  and  $\perp$ , and by  $\Sigma^{\text{DM}}$  the extension of the latter signature by the addition of a unary connective  $\sim$ . Moreover, we let  $\Sigma^{\text{IS}}$  and  $\Sigma^{\text{PP}}$  be the signatures obtained from  $\Sigma^{\text{DM}}$  by adding unary connectives  $\nabla$  and  $\circ$ , respectively. We provide below the definitions and some examples of De Morgan algebras and of involutive Stone algebras.

**Definition 2.1.** Given a  $\Sigma^{\text{DM}}$ -algebra whose  $\Sigma^{\text{bL}}$ -reduct is a bounded distributive lattice, we say that it constitutes a De Morgan algebra if it satisfies the equations:

$$\text{(DM1)} \quad \sim\sim x \approx x \qquad \text{(DM2)} \quad \sim(x \wedge y) \approx \sim x \vee \sim y$$

**Example 2.2.** Let  $\mathcal{V}_4 := \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$  and let  $\mathbf{DM}_4 := \langle \mathcal{V}_4, \cdot^{\mathbf{DM}_4} \rangle$  be the  $\Sigma^{\text{DM}}$ -algebra known as the Dunn-Belnap lattice, whose interpretations for the lattice connectives are those induced by the Hasse diagram in Figure 1a, and the interpretation for  $\sim$  is such that  $\sim^{\mathbf{DM}_4} \mathbf{f} := \mathbf{t}$ ,  $\sim^{\mathbf{DM}_4} \mathbf{t} := \mathbf{f}$  and  $\sim^{\mathbf{DM}_4} a := a$ , for  $a \in \{\mathbf{n}, \mathbf{b}\}$ ; as expected, for the nullary connectives, we have  $\top^{\mathbf{DM}_4} := \mathbf{t}$  and  $\perp^{\mathbf{DM}_4} := \mathbf{f}$ . In Figure 1a, besides depicting the lattice structure of  $\mathbf{DM}_4$ , we also show its subalgebras  $\mathbf{K}_3$  and  $\mathbf{B}_2$ , which coincide with the three-element Kleene algebra and the two-element Boolean algebra. These three algebras are the only subdirectly irreducible De Morgan algebras [3].



(a) The subdirectly irreducible De Morgan algebras.

(b) The subdirectly irreducible IS-algebras.

Figure 1

**Definition 2.3.** Given a  $\Sigma^{\text{IS}}$ -algebra whose  $\Sigma^{\text{DM}}$ -reduct is a De Morgan algebra, we say that it constitutes an involutive Stone algebra (IS-algebra) if it satisfies the equations:

$$\text{(IS1)} \quad \nabla \perp \approx \perp \qquad \text{(IS2)} \quad x \wedge \nabla x \approx x \qquad \text{(IS3)} \quad \nabla(x \wedge y) \approx \nabla x \wedge \nabla y \qquad \text{(IS4)} \quad \sim \nabla x \wedge \nabla x \approx \perp$$

**Example 2.4.** Let  $\mathcal{V}_6 := \mathcal{V}_4 \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  and let  $\mathbf{IS}_6 := \langle \mathcal{V}_6, \cdot^{\mathbf{IS}_6} \rangle$  be the  $\Sigma^{\text{IS}}$ -algebra whose lattice structure is depicted in Figure 1b and interprets  $\sim$  and  $\nabla$  as per the following:

$$\sim^{\mathbf{IS}_6} a := \begin{cases} \sim^{\mathbf{DM}_4} a & a \in \mathcal{V}_4 \\ \hat{\mathbf{f}} & a = \hat{\mathbf{t}} \\ \hat{\mathbf{t}} & a = \hat{\mathbf{f}} \end{cases} \qquad \nabla^{\mathbf{IS}_6} a := \begin{cases} \hat{\mathbf{t}} & a \in \mathcal{V}_6 \setminus \{\hat{\mathbf{f}}\} \\ \hat{\mathbf{f}} & a = \hat{\mathbf{f}} \end{cases}$$

The subalgebras of  $\mathbf{IS}_6$  exhibited in Figure 1b constitute the only subdirectly irreducible IS-algebras [11].

We denote by  $\mathbb{IS}$  the variety of IS-algebras. The following result lists some equations satisfied by IS-algebras, which will be useful for proving the results in the next section.

**Lemma 2.5.** *The following equations are satisfied by IS-algebras:*

1.  $x \vee \nabla \sim x \approx \top$
2.  $x \wedge \sim \nabla x \approx \perp$
3.  $\sim \nabla (x \wedge \sim x) \wedge \sim x \approx \sim \nabla x$
4.  $\nabla \nabla x \approx \nabla x$
5.  $\nabla \sim \nabla x \approx \sim \nabla x$
6.  $\sim \nabla \sim (x \wedge y) \approx \sim \nabla \sim x \wedge \sim \nabla \sim y$

*Proof.* Equation 3 may be proved by using the usual De Morgan algebra equations together with  $\nabla x \vee x \approx \nabla x$ , an equation that is easily derivable from **(IS2)**. All other equations follow from Lemma 3.2 in [8].  $\square$

Here, a SET-FMLA logic (over  $\Sigma$ ) is a consequence relation  $\vdash$  on  $L_\Sigma(P)$  and a SET-SET logic (over  $\Sigma$ ) is a generalized consequence relation  $\triangleright$  on  $L_\Sigma(P)$  [17]. The SET-FMLA companion of a SET-SET logic  $\triangleright$  is the SET-FMLA logic  $\vdash_\triangleright$  such that  $\Phi \vdash_\triangleright \psi$  if, and only if,  $\Phi \triangleright \{\psi\}$ . We will write  $\Phi \triangleleft \triangleright \Psi$  when  $\Phi \triangleright \Psi$  and  $\Psi \triangleright \Phi$ . The complement of a given SET-SET logic  $\triangleright$  will be denoted by  $\blacktriangleright$ . We say that  $\vdash'$  extends  $\vdash$  when  $\vdash' \supseteq \vdash$ . It is worth recalling that the collection of all extensions of a given logic forms a complete lattice under inclusion. Given  $\Sigma \subseteq \Sigma'$ , a logic  $\vdash'$  over  $\Sigma'$  is a *conservative extension* of a logic  $\vdash$  over  $\Sigma$  when  $\vdash'$  extends  $\vdash$  and, for all  $\Phi \cup \{\psi\} \subseteq L_\Sigma(P)$ , we have  $\Phi \vdash' \psi$  iff  $\Phi \vdash \psi$ . These concepts may be extended to the SET-SET framework in the obvious way. We say, in addition, that a SET-FMLA logic  $\vdash$  over  $\Sigma$  has a *disjunction* provided that  $\Phi, \varphi \vee \psi \vdash \phi$  iff  $\Phi, \varphi \vdash \phi$  and  $\Phi, \psi \vdash \phi$  (for  $\vee$  a binary connective in  $\Sigma$ ).

A (logical)  $\Sigma$ -matrix  $\mathfrak{M}$  is a structure  $\langle \mathbf{A}, D \rangle$  where  $\mathbf{A}$  is a  $\Sigma$ -algebra and the members of  $D \subseteq A$  are called *designated values*. We will write  $\overline{D}$  to refer to  $A \setminus D$ . In case  $D = A$ , we say that  $\mathfrak{M}$  is *trivial*. Provided that  $\mathbf{A}$  has a lattice structure with underlying order  $\leq$ , we will often employ the notation  $\uparrow a := \{b \in A \mid a \leq b\}$  when specifying sets of designated values. For instance, over  $\mathbf{IS}_6$  we may consider the set of designated values  $\uparrow \mathbf{b} = \{\mathbf{b}, \mathbf{t}, \hat{\mathbf{t}}\}$  (see Figure 1b). The mappings in  $\text{Hom}(L_\Sigma(P), \mathbf{A})$  are called  $\mathfrak{M}$ -valuations. Every  $\Sigma$ -matrix determines a SET-SET logic  $\triangleright_{\mathfrak{M}}$  such that  $\Phi \triangleright_{\mathfrak{M}} \Psi$  iff  $h(\Phi) \cap \overline{D} \neq \emptyset$  or  $h(\Psi) \cap D \neq \emptyset$  as well as a SET-FMLA logic  $\vdash_{\mathfrak{M}}$  with  $\Phi \vdash_{\mathfrak{M}} \psi$  iff  $\Phi \triangleright_{\mathfrak{M}} \{\psi\}$  (notice that  $\vdash_{\mathfrak{M}}$  is the SET-FMLA companion of  $\triangleright_{\mathfrak{M}}$ ). Given a SET-SET logic  $\triangleright$  (resp. a SET-FMLA logic  $\vdash$ ), if  $\triangleright \subseteq \triangleright_{\mathfrak{M}}$  (resp.  $\vdash \subseteq \vdash_{\mathfrak{M}}$ ), we shall say that  $\mathfrak{M}$  is a *model* of  $\triangleright$  (resp.  $\vdash$ ), and if the converse also holds we shall say that  $\mathfrak{M}$  *characterises*  $\triangleright$  (resp.  $\vdash$ ). The SET-SET (resp. SET-FMLA) logic determined by a class  $\mathcal{M}$  of  $\Sigma$ -matrices is given by  $\bigcap \{\triangleright_{\mathfrak{M}} \mid \mathfrak{M} \in \mathcal{M}\}$  (resp.  $\bigcap \{\vdash_{\mathfrak{M}} \mid \mathfrak{M} \in \mathcal{M}\}$ ).

**Example 2.6.** *The  $\Sigma^{\text{DM}}$ -matrix  $\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle$  determines the logic known as the 4-valued Dunn-Belnap logic, or First-Degree Entailment (FDE) [5], which we hereby denote by  $\mathcal{B}$ . Extensions of  $\mathcal{B}$  are known as super-Belnap logics [27].*

**Example 2.7.** *Classical Logic, hereby denoted by  $\mathcal{CL}$ , is determined by the  $\Sigma^{\text{DM}}$ -matrix  $\langle \mathbf{B}_2, \{\mathbf{t}\} \rangle$ .*

Every  $\Sigma$ -variety  $\mathbf{K}$  such that each  $\mathbf{A} \in \mathbf{K}$  has a bounded lattice reduct with greatest element  $\top^{\mathbf{A}}$  and least element  $\perp^{\mathbf{A}}$  induces a finitary SET-SET order-preserving logic  $\triangleright_{\mathbf{K}}^{\leq}$  according to which  $\Psi$  follows from  $\Phi$  iff there exist finite  $\Phi' \subseteq \Phi$  and  $\Psi' \subseteq \Psi$  such that the equation  $\bigwedge \Phi' \approx \bigwedge \Phi' \wedge \bigvee \Psi'$  is valid in  $\mathbf{K}$  (as usual, we assume  $\bigwedge \emptyset = \top^{\mathbf{A}}$  and  $\bigvee \emptyset = \perp^{\mathbf{A}}$ ). The SET-FMLA companion of  $\triangleright_{\mathbf{K}}^{\leq}$  is usually referred to as the SET-FMLA order-preserving logic induced by  $\mathbf{K}$ , which we denote by  $\vdash_{\mathbf{K}}^{\leq}$ . Notice that, according to this logic,  $\Phi \vdash_{\mathbf{K}}^{\leq} \psi$  if, and only if, (i)  $\Phi = \emptyset$  and  $\psi \approx \top$  is valid in  $\mathbf{K}$  or (ii) there are  $\varphi_1, \dots, \varphi_n \subseteq \Phi$  ( $n \geq 1$ ) such that the equation  $\bigwedge_i \varphi_i \approx \bigwedge_i \varphi_i \wedge \psi$  is valid in  $\mathbf{K}$ . Furthermore, we associate to  $\mathbf{K}$  the 1-assertional logics  $\triangleright_{\mathbf{K}}^{\top}$  and  $\vdash_{\mathbf{K}}^{\top}$  corresponding respectively to the SET-SET and SET-FMLA logics determined by the class of  $\Sigma$ -matrices  $\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\} \rangle \mid \mathbf{A} \in \mathbf{K}\}$  (notice that  $\vdash_{\mathbf{K}}^{\top}$  is the SET-FMLA companion of  $\triangleright_{\mathbf{K}}^{\top}$ ).

A *lattice filter* of a  $\wedge$ -semilattice  $\mathbf{A}$  with a top element  $\top$  is a subset  $D \subseteq A$  with  $\top \in D$  and closed under  $\wedge^A$ ; moreover,  $D$  is a *proper* lattice filter of  $\mathbf{A}$  when  $D \neq A$ . If  $\mathbf{A}$  is a  $\vee$ -semilattice, a *prime filter* of  $\mathbf{A}$  is a proper lattice filter  $D$  of  $\mathbf{A}$  such that  $a \vee b \in D$  iff  $a \in D$  or  $b \in D$ , for all  $a, b \in A$ . In case every  $\mathbf{A} \in \mathbf{K}$  has a bounded distributive lattice reduct, as it happens with all varieties treated in the present work, the order-preserving logic induced by  $\mathbf{K}$  coincides with the logic determined by the class of matrices  $\{\langle \mathbf{A}, D \rangle \mid \mathbf{A} \in \mathbf{K}, D \subseteq A \text{ is a non-empty lattice filter of } \mathbf{A}\}$ .

Based on [29, 6], we define a *symmetrical (Hilbert-style) calculus*  $\mathbf{R}$  (or *SET-SET calculus*, for short) as a collection of pairs  $(\Phi, \Psi) \in \wp L_\Sigma(P) \times \wp L_\Sigma(P)$ , denoted by  $\frac{\Phi}{\Psi}$  and called (*symmetrical*) *inference rules*, where  $\Phi$  is the *antecedent* and  $\Psi$  is the *succedent* of the said rule. We will adopt the convention of omitting curly braces when writing sets of formulas and leaving a blank space instead of writing  $\emptyset$  when presenting inference rules and statements involving (generalized) consequence relations. We proceed to define what constitutes a proof in such calculi.

A *bounded rooted tree*  $t$  is a poset  $\langle \text{nds}(t), \leq^t \rangle$  with a single minimal element  $\text{rt}(t)$ , the *root* of  $t$ , such that, for each *node*  $n \in \text{nds}(t)$ , the set  $\{n' \in \text{nds}(t) \mid n' \leq^t n\}$  of *ancestors* of  $n$  (or the *branch* up to  $n$ ) is well-ordered under  $\leq^t$ , and every branch of  $t$  has a maximal element (a *leaf* of  $t$ ). We may assign a label  $l^t(n) \in \wp L_\Sigma(P) \cup \{*\}$  to each node  $n$  of  $t$ , in which case  $t$  is said to be *labelled*. Given  $\Psi \subseteq L_\Sigma(P)$ , a leaf  $n$  is  $\Psi$ -closed in  $t$  when  $l^t(n) = *$  or  $l^t(n) \cap \Psi \neq \emptyset$ . The tree  $t$  itself is  $\Psi$ -closed when all of its leaves are  $\Psi$ -closed. The immediate successors of a node  $n$  with respect to  $\leq^t$  are called the *children* of  $n$  in  $t$ .

Let  $\mathbf{R}$  be a symmetrical calculus. An  $\mathbf{R}$ -*derivation* is a labelled bounded rooted tree such that for every non-leaf node  $n$  of  $t$  there exists a rule of inference  $r = \frac{\Pi}{\Theta} \in \mathbf{R}$  and a substitution  $\sigma$  such that  $\sigma(\Pi) \subseteq l^t(n)$ , and the set of the children of  $n$  is either (i)  $\{n^\varphi \mid \varphi \in \sigma(\Theta)\}$ , in case  $\Theta \neq \emptyset$ , where  $n^\varphi$  is a node labelled with  $l^t(n) \cup \{\varphi\}$ , or (ii) a singleton  $\{n^*\}$  with  $l^t(n) = *$ , in case  $\Theta = \emptyset$ . We say that  $\Phi \triangleright_{\mathbf{R}} \Psi$  whenever there is a  $\Psi$ -closed derivation  $t$  such that  $\Phi \supseteq \text{rt}(t)$ ; such a tree consists in a *proof* that  $\Psi$  follows from  $\Phi$  in  $\mathbf{R}$ . As a matter of simplification when drawing such trees, we usually avoid copying the formulas inherited from the parent nodes (see Example 2.8 below). The relation  $\triangleright_{\mathbf{R}}$  so defined is a SET-SET logic and, when  $\triangleright_{\mathbf{R}} = \triangleright_{\mathfrak{M}}$ , we say that  $\mathbf{R}$  *axiomatizes*  $\mathfrak{M}$ . A rule  $\frac{\Phi}{\Psi}$  is *sound* with respect to  $\mathfrak{M}$  when  $\Phi \triangleright_{\mathfrak{M}} \Psi$ . It should be pointed out that such deductive formalism generalises the conventional (SET-FMLA) Hilbert-style calculi: the latter corresponds to symmetrical calculi whose rules have, each, a finite antecedent and a singleton as succedent. Given  $\Lambda \subseteq L_\Sigma(P)$ , we write  $\Phi \triangleright_{\mathbf{R}}^\Lambda \Psi$  whenever there is a proof of  $\Psi$  from  $\Phi$  using only formulas in  $\Lambda$ . We say that  $\mathbf{R}$  is  $\Xi$ -*analytic* when, for all  $\Phi, \Psi \subseteq L_\Sigma(P)$ , whenever  $\Phi \triangleright_{\mathbf{R}} \Psi$ , we have  $\Phi \triangleright_{\mathbf{R}}^{Y^\Xi} \Psi$ , with  $Y^\Xi := \text{sub}(\Phi \cup \Psi) \cup \{\sigma(\varphi) \mid \varphi \in \Xi \text{ and } \sigma : P \rightarrow \text{sub}(\Phi \cup \Psi)\}$ , which we shall dub the *generalized subformulas* of  $(\Phi, \Psi)$ . Intuitively, it means that a proof in  $\mathbf{R}$  that  $\Psi$  follows from  $\Phi$  may only use subformulas of  $\Phi \cup \Psi$  or substitution instances of the formulas in  $\Xi$  built with those same subformulas.

A general method is introduced in [6, 21] for obtaining analytic calculi (in the sense of analyticity introduced in the above paragraph) for logics given by a  $\Sigma$ -matrix  $\langle \mathbf{A}, D \rangle$  whenever a certain expressiveness requirement (called ‘monadicity’ in [29]) is met: for every  $a, b \in A$ , there is a single-variable formula  $S$  (a so-called *separator*) such that  $S^A(a) \in D$  and  $S^A(b) \notin D$  or vice-versa. The following example illustrates a symmetrical calculus for  $\mathcal{B}$  generated by this method, as well as some proofs in this calculus.

**Example 2.8.** *The matrix  $\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle$  fulfills the above expressiveness requirement, with the following set of separators:  $S := \{p, \sim p\}$ . We may therefore apply the method introduced in [21] to obtain for  $\mathcal{B}$  the following  $S$ -analytic axiomatization we call  $\mathbf{R}_{\mathcal{B}}$ :*

$$\frac{}{\top} r_1 \quad \frac{\sim \top}{\sim} r_2 \quad \frac{}{\sim \perp} r_3 \quad \frac{\perp}{\perp} r_4 \quad \frac{p}{\sim \sim p} r_5 \quad \frac{\sim \sim p}{p} r_6$$

$$\begin{array}{cccccc}
\frac{p \wedge q}{p} r_7 & \frac{p \wedge q}{q} r_8 & \frac{p, q}{p \wedge q} r_9 & \frac{\sim p}{\sim(p \wedge q)} r_{10} & \frac{\sim q}{\sim(p \wedge q)} r_{11} & \frac{\sim(p \wedge q)}{\sim p, \sim q} r_{12} \\
\frac{p}{p \vee q} r_{13} & \frac{q}{p \vee q} r_{14} & \frac{p \vee q}{p, q} r_{15} & \frac{\sim p, \sim q}{\sim(p \vee q)} r_{16} & \frac{\sim(p \vee q)}{\sim p} r_{17} & \frac{\sim(p \vee q)}{\sim q} r_{18}
\end{array}$$

Figure 2 illustrates some derivations in  $\mathbf{R}_B$ .

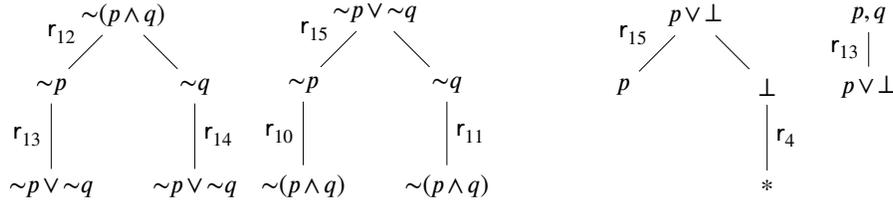


Figure 2: Proofs in  $\mathbf{R}_B$  witnessing that  $\sim(p \wedge q) \triangleleft_B \sim p \vee \sim q$  and  $p \vee \perp \triangleleft_B p, q$ .

Let  $\Sigma$  be any signature containing a unary connective  $\sim$ . A SET-SET logic  $\triangleright$  over  $\Sigma$  is said to be  $\sim$ -paraconsistent when we have  $p, \sim p \blacktriangleright q$ , and  $\sim$ -paracomplete when we have  $q \blacktriangleright p, \sim p$ , with  $p, q \in P$ . Moreover,  $\triangleright$  is  $\sim$ -gently explosive in case there is a collection  $\bigcirc(p) \subseteq L_\Sigma(P)$  of formulas on a single variable such that, for some  $\varphi \in L_\Sigma(P)$ , we have  $\bigcirc(\varphi), \varphi \blacktriangleright \emptyset$  and  $\bigcirc(\varphi), \sim\varphi \blacktriangleright \emptyset$ , and, for all  $\psi \in L_\Sigma(P)$ , we have  $\bigcirc(\psi), \psi, \sim\psi \blacktriangleright \emptyset$ . Dually,  $\triangleright$  is  $\sim$ -gently implosive in case there is a collection of formulas  $\star(p) \subseteq L_\Sigma(P)$  on a single variable such that, for some  $\varphi \in L_\Sigma(P)$ , we have  $\emptyset \blacktriangleright \varphi, \star(\varphi)$  and  $\emptyset \blacktriangleright \sim\varphi, \star(\varphi)$ , and, for all  $\psi \in L_\Sigma(P)$ , we have  $\emptyset \blacktriangleright \sim\psi, \psi, \star(\psi)$ . A SET-SET logic is  $\sim$ -paradefinite when it is both  $\sim$ -paraconsistent and  $\sim$ -paracomplete; is a *logic of formal inconsistency (LFI)* when it is  $\sim$ -paraconsistent yet  $\sim$ -gently explosive; and is a *logic of formal undeterminedness (LFU)* when it is  $\sim$ -paracomplete yet  $\sim$ -gently implosive. Furthermore, if  $\triangleright_1$  and  $\triangleright_2$  are logics over  $\Sigma_1 \supseteq \Sigma$  and  $\Sigma_2 \supseteq \Sigma$  respectively, we say that  $\triangleright_1$  is a *C-system based on  $\triangleright_2$*  with respect to  $\sim$  (or simply a *C-system*) when it is an LFI that agrees with  $\triangleright_2$  on statements involving formulas without  $\sim$  (that is,  $\Phi \triangleright_1 \Psi$  iff  $\Phi \triangleright_2 \Psi$  for all sets  $\Phi, \Psi$  of formulas without  $\sim$ ), and  $\bigcirc(p) = \{\circ p\}$ , for  $\circ$  a composite *consistency connective* in the language of  $\triangleright_1$ . We may dually define the notions of *D-system* and of *determinedness connective* [23]. It is worth pointing out that in the present paper we will have  $\star(p) = \{\sim\circ p\}$ .

**Example 2.9.** By exploiting the fact that  $\mathbf{n}, \mathbf{b} \in \mathcal{V}_4$  are fixpoints of  $\sim^{\mathbf{DM}_4}$ , one may easily notice that  $B$  is  $\sim$ -paraconsistent and  $\sim$ -paracomplete (thus  $\sim$ -paradefinite).

### 3 Perfect paradefinite algebras and their logics

#### 3.1 Involutive Stone and PP-algebras

We propose in this section to enrich De Morgan algebras by the addition of a perfection operator  $\circ$ , which will allow us to recover the classical properties of  $\sim$ -consistency and  $\sim$ -determinedness. In the sequel, we will prove that the variety of the algebras thus obtained is term-equivalent to the variety of IS-algebras.

**Definition 3.1.** Given a  $\Sigma^{\text{PP}}$ -algebra whose  $\Sigma^{\text{DM}}$ -reduct is a De Morgan algebra, we say that it constitutes a perfect paradefinite algebra (PP-algebra) if it satisfies the equations:

$$\begin{array}{llll}
(\text{PP1}) \circ\circ x \approx \top & (\text{PP2}) \circ x \approx \circ\sim x & (\text{PP3}) \circ\top \approx \top & (\text{PP4}) x \wedge \sim x \wedge \circ x \approx \perp \\
(\text{PP5}) \circ(x \wedge y) \approx (\circ x \vee \circ y) \wedge (\circ x \vee \sim y) \wedge (\circ y \vee \sim x)
\end{array}$$

**Example 3.2.** An example of PP-algebra is  $\mathbf{PP}_6 := \langle \mathcal{V}_6, \cdot^{\mathbf{PP}_6} \rangle$ , the  $\Sigma^{\mathbf{PP}}$ -algebra defined as  $\mathbf{IS}_6$  in Example 2.4, differing only in that, instead of containing an interpretation for  $\nabla$ , it interprets  $\circ$  as follows:

$$\circ^{\mathbf{PP}_6} a := \begin{cases} \hat{\mathbf{f}} & a \in \mathcal{V}_6 \setminus \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\} \\ \hat{\mathbf{t}} & a \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\} \end{cases}$$

Other examples are the algebras  $\mathbf{PP}_i$ , for  $2 \leq i \leq 5$ , the subalgebras of  $\mathbf{PP}_6$  having, respectively, the same lattice structures of the algebras  $\mathbf{IS}_i$  exhibited in Figure 1b.

As it occurs with IS-algebras, in the language of PP-algebras we may define, by setting  $\neg x := \circ x \wedge \sim x$ , a pseudo-complement satisfying the Stone equation. We denote by  $\mathbb{PP}$  the variety of PP-algebras. The following result illustrates some useful equations satisfied by the members of  $\mathbb{PP}$ .

**Lemma 3.3.** Every PP-algebra satisfies:

1.  $\sim \circ x \vee (x \vee \sim x) \approx \top$
2.  $\circ x \wedge \sim \circ x \approx \perp$
3.  $\circ x \approx \circ x \wedge (x \vee \sim x)$

*Proof.* Notice that 1 is a straightforward consequence of **(PP4)**, and 2 is a consequence of **(PP4)** using  $\circ x$  in place of  $x$  and invoking **(PP1)**. Finally, 3 may be easily proved using 1 and 2.  $\square$

Given  $\varphi \in L_{\Sigma^{\mathbf{IS}}}(P)$  (resp.  $\varphi \in L_{\Sigma^{\mathbf{PP}}}(P)$ ), let  $\varphi^\circ \in L_{\Sigma^{\mathbf{PP}}}(P)$  (resp.  $\varphi^\nabla \in L_{\Sigma^{\mathbf{IS}}}(P)$ ) be the result of applying the definition of  $\circ$  (resp. of  $\nabla$ ) given below, in Theorem 3.4 (resp. Theorem 3.5), over  $\varphi$ . Extend this notion to sets of formulas in the usual way. The subsequent results establish the term-equivalence between the varieties of involutive Stone algebras and of perfect parafinite algebras. We first provide ways of constructing PP-algebras from IS-algebras, and vice-versa.

**Theorem 3.4.** Let  $\mathbf{A} \in \mathbb{IS}$ . Then the  $\Sigma^{\mathbf{PP}}$ -algebra  $\mathbf{A}^\circ$  having the same  $\Sigma^{\mathbf{DM}}$ -reduct of  $\mathbf{A}$  and with  $\circ^{\mathbf{A}^\circ}$  being the operation induced by  $\sim \nabla (x \wedge \sim x)$  on  $\mathbf{A}$  is a PP-algebra.

*Proof.* We must check that  $\mathbf{A}^\circ$  satisfies each of the characteristic equations of PP-algebras:

$$\begin{aligned} \text{(PP1)} \quad \circ \circ x &\approx_{\text{def}} \sim \nabla ((\sim \nabla (x \wedge \sim x)) \wedge \sim (\sim \nabla (x \wedge \sim x))) \approx_{\text{(IS3)}} \sim \nabla \sim \nabla (x \wedge \sim x) \vee \sim \nabla \sim \nabla (x \wedge \sim x) \approx_{2.5.5} \\ &\sim \nabla (x \wedge \sim x) \vee \sim \nabla \sim \nabla (x \wedge \sim x) \approx_{\text{(DM1)}} \nabla (x \wedge \sim x) \vee \sim \nabla \nabla (x \wedge \sim x) \approx_{2.5.4} \nabla (x \wedge \sim x) \vee \sim \nabla (x \wedge \\ &\sim x) \approx_{\text{(IS4)}} \top. \end{aligned}$$

$$\text{(PP2)} \quad \circ x \approx_{\text{def}} \sim \nabla (x \wedge \sim x) \approx_{\text{(DM1)}} \sim \nabla (\sim \sim x \wedge \sim x) \approx_{\text{def}} \circ \sim x.$$

$$\text{(PP3)} \quad \circ \top \approx_{\text{def}} \sim \nabla (\top \wedge \sim \top) \approx \sim \nabla (\top \wedge \perp) \approx \sim \nabla \perp \approx_{\text{(IS1)}} \sim \perp \approx \top.$$

$$\text{(PP4)} \quad \circ x \wedge (\sim x \wedge x) \approx_{\text{def}} \sim \nabla (x \wedge \sim x) \wedge (\sim x \wedge x) \approx (\sim \nabla (x \wedge \sim x) \wedge \sim x) \wedge x \approx_{2.5.3} \sim \nabla x \wedge x \approx_{2.5.2} \perp.$$

$$\begin{aligned} \text{(PP5)} \quad \circ (x \wedge y) &\approx_{\text{def}} \sim \nabla ((x \wedge y) \wedge \sim (x \wedge y)) \approx_{\text{(IS3)}} \sim \nabla (x \wedge y) \vee \sim \nabla \sim (x \wedge y) \approx_{\text{(IS3)}} (\sim \nabla x \vee \sim \nabla y) \vee \sim \nabla \sim (x \wedge \\ &y) \approx_{2.5.6} (\sim \nabla x \vee \sim \nabla y) \vee (\sim \nabla \sim x \wedge \sim \nabla \sim y) \approx (\sim \nabla x \vee \sim \nabla y \vee \sim \nabla \sim x) \wedge (\sim \nabla x \vee \sim \nabla y \vee \sim \nabla \sim y) \approx_{2.5.3} \\ &(\sim \nabla x \vee (\sim \nabla (y \wedge \sim y) \wedge \sim y) \vee \sim \nabla \sim x) \wedge (\sim \nabla y \vee (\sim \nabla (x \wedge \sim x) \wedge \sim x) \vee \sim \nabla \sim y) \approx_{\text{(IS3)}} (\sim \nabla (x \wedge \sim x) \vee \\ &(\sim \nabla (y \wedge \sim y) \wedge \sim y)) \wedge (\sim \nabla (y \wedge \sim y) \vee (\sim \nabla (x \wedge \sim x) \wedge \sim x)) \approx_{\text{def}} (\circ x \vee (\circ y \wedge \neg y)) \wedge (\circ y \vee (\circ x \wedge \neg x)) \approx \\ &(\circ x \vee \circ y) \wedge (\circ x \vee \sim y) \wedge (\circ y \vee \sim x). \end{aligned} \quad \square$$

**Theorem 3.5.** Let  $\mathbf{A} \in \mathbb{PP}$ . Then the  $\Sigma^{\mathbf{IS}}$ -algebra  $\mathbf{A}^\nabla$  having the same  $\Sigma^{\mathbf{DM}}$ -reduct of  $\mathbf{A}$  and with  $\nabla^{\mathbf{A}^\nabla}$  being the operation induced by  $\sim \circ x \vee x$  on  $\mathbf{A}$  is an IS-algebra.

*Proof.* We must check that  $\mathbf{A}^\nabla$  satisfies each of the characteristic equations of IS-algebras:

$$\text{(IS1)} \quad \nabla \perp \approx_{\text{def}} \sim \circ \perp \vee \perp \approx \sim \circ \perp \approx \sim \circ \sim \top \approx_{\text{(PP2)}} \sim \circ \top \approx_{\text{(PP3)}} \sim \top \approx \perp.$$

$$\text{(IS2)} \quad \text{By absorption and commutativity of } \vee, \text{ we have } x \wedge \nabla x \approx_{\text{def}} x \wedge (\sim \circ x \vee x) \approx x.$$

$$\begin{aligned}
\text{(IS3)} \quad & \nabla(x \wedge y) \approx_{def} \sim \circ(x \wedge y) \vee (x \wedge y) \approx_{\text{(PP5)}} (\sim \circ x \wedge \sim \circ y) \vee (\sim \circ x \wedge y) \vee (\sim \circ y \wedge x) \vee (x \wedge y) \approx (\sim \circ x \vee x) \wedge \\
& (\sim \circ y \vee y) \approx_{def} \nabla x \wedge \nabla y. \\
\text{(IS4)} \quad & \sim \nabla x \wedge \nabla x \approx_{def} \sim(\sim \circ x \vee x) \wedge (\sim \circ x \vee x) \approx_{\text{(DM2)}} (\circ x \wedge \sim x) \wedge (\sim \circ x \vee x) \approx (\circ x \wedge \sim x \wedge \sim \circ x) \vee (\circ x \wedge \\
& \sim x \wedge x) \approx_{\text{(PP4)}} (\circ x \wedge \sim x \wedge \sim \circ x) \vee \perp \approx \circ x \wedge \sim x \wedge \sim \circ x \approx \circ x \wedge \sim x \wedge \sim \circ x \wedge \top \approx_{\text{(PP1)}} \circ x \wedge \sim x \wedge \sim \circ x \wedge \\
& \circ \circ x \approx_{\text{(PP4)}} \perp \wedge \sim x \approx \perp. \quad \square
\end{aligned}$$

Then, for the announced term-equivalence, we just need to check that:

**Theorem 3.6.** *Given  $\mathbf{A} \in \mathbb{IS}$  and  $\mathbf{B} \in \mathbb{PP}$ , we have  $(\mathbf{A}^\circ)^\nabla = \mathbf{A}$  and  $(\mathbf{B}^\nabla)^\circ = \mathbf{B}$ .*

*Proof.* In order to prove that  $(\mathbf{A}^\circ)^\nabla = \mathbf{A}$ , it is enough to show that  $\sim(\sim \nabla(x \wedge \sim x)) \vee x \approx \nabla x$  holds in  $\mathbf{A}$ , that is, the operation induced by the term  $((\nabla x)^\circ)^\nabla$  coincides with the interpretation of  $\nabla$ . By the fact that  $\nabla x \vee x \approx \nabla x$ , we have  $\sim(\sim \nabla(x \wedge \sim x)) \vee x \approx_{\text{(DM1)}} \nabla(x \wedge \sim x) \vee x \approx_{\text{(IS3)}} (\nabla x \wedge \nabla \sim x) \vee x \approx (\nabla x \vee x) \wedge (\nabla \sim x \vee x) \approx_{\text{Lemma 2.5.1}} (\nabla x \vee x) \wedge \top \approx \nabla x \vee x \approx \nabla x$ . Similarly, for proving  $(\mathbf{B}^\nabla)^\circ = \mathbf{B}$ , it is enough to show that  $((\circ x)^\nabla)^\circ$  induces an operation that coincides with the interpretation of  $\circ$ , which amounts to proving that  $\sim(\sim \circ(x \wedge \sim x) \vee (x \wedge \sim x)) \approx \circ x$  holds in  $\mathbf{B}$ . Then, we have  $\sim(\sim \circ(x \wedge \sim x) \vee (x \wedge \sim x)) \approx_{\text{(DM2)}} \circ(x \wedge \sim x) \wedge (\sim x \vee x) \approx_{\text{(PP5)}} (\circ x \vee \circ \sim x) \wedge (\circ x \vee x) \wedge (\circ \sim x \vee \sim x) \wedge (\sim x \vee x) \approx_{\text{(PP2)}} (\circ x \vee \circ x) \wedge (\circ x \vee x) \wedge (\circ x \vee \sim x) \wedge (\sim x \vee x) \approx \circ x \wedge (\sim x \vee x) \approx_{\text{Lemma 3.3.3}} \circ x$ .  $\square$

By inspecting the interpretation induced by the definition of  $\circ$  in terms of  $\nabla$  given in Theorem 3.4, one may easily check the following result.

**Proposition 3.7.**  $\text{PP}_i = \text{IS}_i^\circ$ , for all  $2 \leq i \leq 6$ .

From the equivalence just presented and a similar result for IS-algebras [20], we may now conclude that the variety of PP-algebras is generated by  $\text{PP}_6$ :

**Proposition 3.8.**  $\mathbb{PP} = \mathbb{V}(\{\text{PP}_6\})$ .

### 3.2 Logics associated to PP-algebras

Recall that we denote by  $\mathcal{PP}_{\leq}^\triangleright$  and  $\mathcal{PP}_{\leq}$ , respectively, the SET-SET and SET-FMLA order-preserving logics induced by  $\mathbb{PP}$ . Also, we denote by  $\mathcal{PP}_{\top}^\triangleright$  and  $\mathcal{PP}_{\top}$ , respectively, the SET-SET and SET-FMLA 1-assertional logics induced by  $\mathbb{PP}$ . We will use the following auxiliary results together with analogous results for  $\text{IS}_{\leq}$  [20] (which smoothly generalizes to  $\text{IS}_{\leq}^\triangleright$ , the SET-SET order-preserving logic associated to  $\mathbb{IS}$ ) to prove some characterizations of the logics associated to  $\mathbb{PP}$  in terms of single finite logical matrices.

**Lemma 3.9.** *Given  $\mathbf{A} \in \mathbb{IS}$  and  $\mathbf{B} \in \mathbb{PP}$ ,*

1. *if  $h$  is a valuation on  $\mathbf{A}$ , then  $h((\varphi^\circ)^\nabla) = h(\varphi)$  for all  $\varphi \in L_{\Sigma^{\text{IS}}}(P)$ ;*
2. *if  $h$  is a valuation on  $\mathbf{B}$ , then  $h((\varphi^\nabla)^\circ) = h(\varphi)$  for all  $\varphi \in L_{\Sigma^{\text{PP}}}(P)$ ;*
3. *if  $h$  is a valuation on  $\mathbf{A}$ , then the mapping  $h^\circ \in \text{Hom}(L_{\Sigma^{\text{PP}}}(P), \mathbf{A}^\circ)$  such that  $h^\circ(p) = h(p)$  for all  $p \in P$  satisfies  $h^\circ(\varphi^\circ) = h(\varphi)$  for all  $\varphi \in L_{\Sigma^{\text{IS}}}(P)$ ;*
4. *if  $h$  is a valuation on  $\mathbf{B}$ , then the mapping  $h^\nabla \in \text{Hom}(L_{\Sigma^{\text{IS}}}(P), \mathbf{B}^\nabla)$  such that  $h^\nabla(p) = h(p)$  for all  $p \in P$  satisfies  $h^\nabla(\varphi^\nabla) = h(\varphi)$  for all  $\varphi \in L_{\Sigma^{\text{PP}}}(P)$ .*

*Proof.* We will first discuss the proofs of items 1 and 3, which may then be easily adapted, respectively, for proving items 2 and 4. Both proofs are by structural induction on the set of formulas. Starting with 1, when  $\varphi \in P$ , the result trivially holds, as propositional variables are not affected by translations. In case  $\varphi = \nabla \psi$ , if  $h((\psi^\circ)^\nabla) = h(\psi)$ , we will have  $h((\varphi^\circ)^\nabla) = h(((\nabla \psi)^\circ)^\nabla)$ . From the argument in the proof

of Theorem 3.6, we know that  $((\nabla\psi)^\circ)^\nabla$  and  $\nabla\psi$  induce the same operation on  $\mathbf{A}$ , thus  $h(((\nabla\psi)^\circ)^\nabla) = \nabla(h(\psi)) = h(\varphi)$ . The proof is analogous for the cases of  $\wedge, \vee, \sim, \top$  and  $\perp$ . Now, for item 3, the base case is again obvious, and, in case  $\varphi = \nabla\psi$ , we have  $h^\circ((\nabla\psi)^\circ) = h^\circ(\sim \circ \psi^\circ \vee \psi^\circ) = \sim^{\mathbf{A}^\circ} \circ \mathbf{A}^\circ h^\circ(\psi^\circ) \vee^{\mathbf{A}^\circ} h^\circ(\psi^\circ)$ , and, by the induction hypothesis, the latter is equal to  $\sim^{\mathbf{A}^\circ} \circ \mathbf{A}^\circ h(\psi) \vee^{\mathbf{A}^\circ} h(\psi)$ ; this is the same as  $h(\nabla\psi)$  in  $(\mathbf{A}^\circ)^\nabla$ , which coincides with  $\mathbf{A}$  by Theorem 3.6. The proof is again analogous for  $\wedge, \vee, \sim, \top$  and  $\perp$ .  $\square$

The former result allows us to prove the following auxiliary facts:

**Proposition 3.10.**

1.  $\Phi \triangleright_{\langle \mathbf{A}, D \rangle} \Psi$  iff  $\Phi^\nabla \triangleright_{\langle \mathbf{A}^\nabla, D \rangle} \Psi^\nabla$ , where  $\mathbf{A} \in \mathbb{P}\mathbb{P}$  and  $\langle \mathbf{A}, D \rangle$  is a  $\Sigma^{\mathbb{P}\mathbb{P}}$ -matrix
2.  $\Phi \triangleright_{\mathcal{M}} \Psi$  iff  $\Phi^\nabla \triangleright_{\mathcal{M}^\nabla} \Psi^\nabla$ , for  $\mathcal{M} = \{ \langle \mathbf{A}, D \rangle \mid \mathbf{A} \in \mathbb{P}\mathbb{P} \}$
3.  $\Phi \triangleright_{\mathcal{P}\mathcal{P}_\leq} \Psi$  iff  $\Phi^\nabla \triangleright_{\mathcal{I}\mathcal{S}_\leq} \Psi^\nabla$
4.  $\Phi \triangleright_{\mathbb{P}\mathbb{P}}^\top \Psi$  iff  $\Phi^\nabla \triangleright_{\mathbb{I}\mathbb{S}}^\top \Psi^\nabla$

*Proof.* We start by proving item 1. From the left to the right, suppose that there is a valuation  $h \in \text{Hom}(\mathbf{L}_{\Sigma^{\mathbb{P}\mathbb{P}}}(P), \mathbf{A}^\nabla)$  such that  $h(\Phi^\nabla) \subseteq D$  while  $h(\Psi^\nabla) \subseteq \overline{D}$ . By items 2 and 3 of Lemma 3.9, there is a valuation  $h^\circ \in \text{Hom}(\mathbf{L}_{\Sigma^{\mathbb{P}\mathbb{P}}}(P), (\mathbf{A}^\nabla)^\circ) = \text{Hom}(\mathbf{L}_{\Sigma^{\mathbb{P}\mathbb{P}}}(P), \mathbf{A})$  such that  $h^\circ((\Phi^\nabla)^\circ) = h^\circ(\Phi)$  and  $h^\circ((\Psi^\nabla)^\circ) = h(\Psi^\nabla)$ , thus  $h^\circ(\Phi) = h(\Phi^\nabla) \subseteq D$ . Similarly, we may conclude that  $h^\circ(\Psi) \subseteq \overline{D}$ , and we are done. The other direction is similar, but using item 4 of Lemma 3.9. Item 2, above, is a clear consequence of item 1, and items 3 and 4 follow directly from items 1 and 2, respectively.  $\square$

From this fact, we obtain that the order-preserving logics  $\mathcal{P}\mathcal{P}_\leq^\triangleright$  and  $\mathcal{P}\mathcal{P}_\leq$  are determined by a single 6-valued logical matrix:

**Theorem 3.11.**  $\mathcal{P}\mathcal{P}_\leq^\triangleright = \triangleright_{\langle \mathbb{P}\mathbb{P}_6, \uparrow \mathbf{b} \rangle}$ , and thus  $\mathcal{P}\mathcal{P}_\leq = \vdash_{\langle \mathbb{P}\mathbb{P}_6, \uparrow \mathbf{b} \rangle}$ .

*Proof.* By Proposition 3.10 and the fact that  $\triangleright_{\mathcal{I}\mathcal{S}_\leq}$  is characterized by the matrix  $\langle \mathbb{I}\mathbb{S}_6, \uparrow \mathbf{b} \rangle$  [20], we have  $\Phi \triangleright_{\langle \mathbb{P}\mathbb{P}_6, \uparrow \mathbf{b} \rangle} \Psi$  iff  $\Phi^\nabla \triangleright_{\langle \mathbb{I}\mathbb{S}_6, \uparrow \mathbf{b} \rangle} \Psi^\nabla$  iff  $\Phi^\nabla \triangleright_{\mathcal{I}\mathcal{S}_\leq} \Psi^\nabla$  iff  $\Phi \triangleright_{\mathcal{P}\mathcal{P}_\leq} \Psi$ .  $\square$

Furthermore, we have that the 1-assertional logics  $\mathcal{P}\mathcal{P}_\top^\triangleright$  and  $\mathcal{P}\mathcal{P}_\top$  are determined by a single 3-valued matrix:

**Proposition 3.12.**  $\mathcal{P}\mathcal{P}_\top^\triangleright = \triangleright_{\mathbb{V}(\mathbb{P}\mathbb{P}_3)}^\top = \triangleright_{\langle \mathbb{P}\mathbb{P}_3, \{\hat{\mathbf{t}} \rangle \rangle}$ , and thus  $\mathcal{P}\mathcal{P}_\top = \vdash_{\mathbb{V}(\mathbb{P}\mathbb{P}_3)}^\top = \vdash_{\langle \mathbb{P}\mathbb{P}_3, \{\hat{\mathbf{t}} \rangle \rangle}$ .

*Proof.* It is clear that  $\triangleright_{\mathbb{P}\mathbb{P}}^\top \subseteq \triangleright_{\mathbb{V}(\mathbb{P}\mathbb{P}_3)}^\top \subseteq \triangleright_{\langle \mathbb{P}\mathbb{P}_3, \{\hat{\mathbf{t}} \rangle \rangle}$ . The result then follows because  $\triangleright_{\mathbb{P}\mathbb{P}}^\top = \triangleright_{\langle \mathbb{P}\mathbb{P}_3, \{\hat{\mathbf{t}} \rangle \rangle}$ , as  $\Phi \triangleright_{\langle \mathbb{P}\mathbb{P}_3, \{\hat{\mathbf{t}} \rangle \rangle} \Psi$  iff  $\Phi^\nabla \triangleright_{\langle \mathbb{P}\mathbb{P}_3^\nabla, \{\hat{\mathbf{t}} \rangle \rangle} \Psi^\nabla$  (by Lemma 3.9) iff  $\Phi^\nabla \triangleright_{\langle \mathbb{I}\mathbb{S}_3, \{\hat{\mathbf{t}} \rangle \rangle} \Psi^\nabla$  (because  $\mathbb{I}\mathbb{S}_3 = \mathbb{P}\mathbb{P}_3^\nabla$ ) iff  $\Phi^\nabla \triangleright_{\mathbb{I}\mathbb{S}}^\top \Psi^\nabla$  (by [20, Prop. 4.5]) iff  $\Phi \triangleright_{\mathbb{P}\mathbb{P}}^\top \Psi$  (by Lemma 3.9).  $\square$

As the last item in this series of characterizations, we have, as it should be expected, that the 1-assertional logic associated to  $\{\mathbb{P}\mathbb{P}_2\}$  coincides with Classical Logic:

**Proposition 3.13.** For all  $\Phi, \Psi \subseteq L_{\Sigma^{\text{DM}}}(P)$ ,  $\Phi \triangleright_{\text{CL}} \Psi$  iff  $\Phi \triangleright_{\mathbb{P}\mathbb{P}_2}^\top \Psi$ .

*Proof.* The result follows from the clear isomorphism between  $\mathbb{P}\mathbb{P}_2$  and  $\mathbf{B}_2$ .  $\square$

Finally, we may explore the term-equivalence presented in the previous subsection (Theorem 3.6) to prove another important fact about  $\mathcal{P}\mathcal{P}_\leq$ . For the definitions of full self-extensionality, protoalgebraizability and algebraizability that appear in the following result, we refer the reader to [15, Definitions 5.25, 6.1 and 3.11, resp.].

**Proposition 3.14.**  $\mathcal{P}\mathcal{P}_\leq$  is fully self-extensional and non-protoalgebraic (hence non-algebraizable).

*Proof.* The result follows from [20, Prop. 4.2], [15, Theorem 7.18, item 4], and the term-equivalence of  $\mathbb{I}\mathbb{S}$  with  $\mathbb{P}\mathbb{P}$  given by Theorem 3.6.  $\square$

### 3.3 De Morgan algebras with a perfection operator

We now present a recipe for constructing a perfect parafinite algebra by endowing a De Morgan algebra with a perfection operator. This should be of particular interest, as we shall see in subsection 3.5, for an investigation on **LFIs** and **LFUs** when the De Morgan algebra at hand happens not to be Boolean. We will see in the next section how to axiomatize logics induced by PP-algebras produced through this recipe, starting from a calculus for the logic induced by a De Morgan algebra given as input.

**Definition 3.15.** Let  $\mathbf{A}$  be a  $\Sigma^{\text{DM}}$ -algebra. Given  $\hat{\mathbf{f}}, \hat{\mathbf{t}} \notin A$ , we define the  $\Sigma^{\text{PP}}$ -algebra  $\mathbf{A}^\circ := \langle A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}, \cdot^{\mathbf{A}^\circ} \rangle$  by letting:

$$\begin{aligned}
 a \wedge^{\mathbf{A}^\circ} b &:= \begin{cases} a \wedge^{\mathbf{A}} b & \text{if } a, b \in A \\ \hat{\mathbf{t}} & \text{if } a = b = \hat{\mathbf{t}} \\ \hat{\mathbf{f}} & \text{if } a = \hat{\mathbf{f}} \text{ or } b = \hat{\mathbf{f}} \\ c & \text{if } \{a, b\} = \{\hat{\mathbf{t}}, c\} \text{ with } c \in A \end{cases} & a \vee^{\mathbf{A}^\circ} b &:= \begin{cases} a \vee^{\mathbf{A}} b & \text{if } a, b \in A \\ \hat{\mathbf{f}} & \text{if } a = b = \hat{\mathbf{f}} \\ \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{t}} \text{ or } b = \hat{\mathbf{t}} \\ c & \text{if } \{a, b\} = \{\hat{\mathbf{f}}, c\} \text{ with } c \in A \end{cases} \\
 \sim^{\mathbf{A}^\circ} a &:= \begin{cases} \sim^{\mathbf{A}} a & \text{if } a \in A \\ \hat{\mathbf{f}} & \text{if } a = \hat{\mathbf{t}} \\ \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{f}} \end{cases} & \circ^{\mathbf{A}^\circ} a &:= \begin{cases} \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{f}} \text{ or } a = \hat{\mathbf{t}} \\ \hat{\mathbf{f}} & \text{otherwise} \end{cases} \\
 \perp^{\mathbf{A}^\circ} &:= \hat{\mathbf{f}} & \top^{\mathbf{A}^\circ} &:= \hat{\mathbf{t}}
 \end{aligned}$$

In addition, we define the  $\Sigma^{\text{IS}}$ -algebra  $\mathbf{A}^\nabla := \langle A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}, \cdot^{\mathbf{A}^\nabla} \rangle$  interpreting the connectives in  $\Sigma^{\text{DM}}$  as above, while letting  $\nabla^{\mathbf{A}^\nabla} a := \hat{\mathbf{f}}$  if  $a = \hat{\mathbf{f}}$  and  $\nabla^{\mathbf{A}^\nabla} a := \hat{\mathbf{t}}$  otherwise (cf. [20]).

**Proposition 3.16.** If  $\mathbf{A}$  is a De Morgan algebra, then  $\mathbf{A}^\circ$  is a PP-algebra.

*Proof.* When  $\mathbf{A}$  is a De Morgan algebra, it is clear that the  $\Sigma^{\text{DM}}$ -reduct of  $\mathbf{A}^\circ$  is also a De Morgan algebra. Moreover, the operation  $\circ^{\mathbf{A}^\circ}$  defined above satisfies all equations presented in Definition 3.1, as we confirm below:

- (PP1) By the definition of  $\circ^{\mathbf{A}^\circ}$ , we have either (1)  $\circ^{\mathbf{A}^\circ} a = \hat{\mathbf{t}}$  or (2)  $\circ^{\mathbf{A}^\circ} a = \hat{\mathbf{f}}$ . In both cases we have  $\circ^{\mathbf{A}^\circ} \circ^{\mathbf{A}^\circ} a = \hat{\mathbf{t}} = \top^{\mathbf{A}^\circ}$ .
- (PP2) By the definition of  $\sim^{\mathbf{A}^\circ}$ , we have that  $a \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  iff  $\sim^{\mathbf{A}^\circ} a \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ . Also, we have either (1)  $a \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  or (2)  $a \notin \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ . If (1) is the case, then  $\circ^{\mathbf{A}^\circ} a = \circ^{\mathbf{A}^\circ} \sim^{\mathbf{A}^\circ} a = \hat{\mathbf{t}}$ ; alternatively, if (2) is the case, then  $\circ^{\mathbf{A}^\circ} a = \circ^{\mathbf{A}^\circ} \sim^{\mathbf{A}^\circ} a = \hat{\mathbf{f}}$ .
- (PP3) By the definition of  $\circ^{\mathbf{A}^\circ}$  and  $\top^{\mathbf{A}^\circ}$ , we have that  $\circ^{\mathbf{A}^\circ} \top^{\mathbf{A}^\circ} = \circ^{\mathbf{A}^\circ} \hat{\mathbf{t}} = \hat{\mathbf{t}} = \top^{\mathbf{A}^\circ}$ .
- (PP4) By the definition of  $\circ^{\mathbf{A}^\circ}$ , we have either (1)  $\circ^{\mathbf{A}^\circ} a = \hat{\mathbf{t}}$  or (2)  $\circ^{\mathbf{A}^\circ} a = \hat{\mathbf{f}}$ . If (1) is the case, we have either (1.1)  $a = \hat{\mathbf{f}}$  or (1.2)  $a = \hat{\mathbf{t}}$ , then: If (1.2) is the case, then, by the definition of  $\sim^{\mathbf{A}^\circ}$ ,  $\sim^{\mathbf{A}^\circ} a = \hat{\mathbf{f}}$ . In all cases we have that at least one among  $a, \sim^{\mathbf{A}^\circ} a$  and  $\circ^{\mathbf{A}^\circ} a$  is  $\hat{\mathbf{f}}$ . Then, by the definition of  $\wedge^{\mathbf{A}^\circ}$ , we have  $a \wedge \sim^{\mathbf{A}^\circ} a \wedge^{\mathbf{A}^\circ} \circ^{\mathbf{A}^\circ} a = \hat{\mathbf{f}} = \perp^{\mathbf{A}^\circ}$ .
- (PP5) By the definition of  $\circ^{\mathbf{A}^\circ}$ , we have that either (1)  $\circ^{\mathbf{A}^\circ} (a \wedge^{\mathbf{A}^\circ} b) = \hat{\mathbf{t}}$  or (2)  $\circ^{\mathbf{A}^\circ} (a \wedge^{\mathbf{A}^\circ} b) = \hat{\mathbf{f}}$ . If (1) is the case, we have either (1.1)  $a \wedge^{\mathbf{A}^\circ} b = \hat{\mathbf{f}}$  or (1.2)  $a \wedge^{\mathbf{A}^\circ} b = \hat{\mathbf{t}}$ . If (1.1) is the case, then, by the definition of  $\wedge^{\mathbf{A}^\circ}$ , we have either (1.1.1)  $a = \hat{\mathbf{f}}$  or (1.1.2)  $b = \hat{\mathbf{f}}$ . If (1.1.1) is the case, then, by the definition of  $\circ^{\mathbf{A}^\circ}$  and  $\sim^{\mathbf{A}^\circ}$ , we have both  $\circ^{\mathbf{A}^\circ} a = \hat{\mathbf{t}}$  and  $\sim^{\mathbf{A}^\circ} a = \hat{\mathbf{t}}$ . The case (1.1.2) is similar to (1.1.1). If (1.2) is the case, then, by the definition of  $\wedge^{\mathbf{A}^\circ}$ , we have  $a = b = \hat{\mathbf{t}}$ . By the definition of  $\circ^{\mathbf{A}^\circ}$ , we have  $\circ^{\mathbf{A}^\circ} a = \circ^{\mathbf{A}^\circ} b = \hat{\mathbf{t}}$ . Hence, in all subcases of (1) we have, by the definition of  $\vee^{\mathbf{A}^\circ}$ :

$\circ^{\mathbf{A}^\circ} a \vee^{\mathbf{A}^\circ} \circ^{\mathbf{A}^\circ} b = \circ^{\mathbf{A}^\circ} a \vee^{\mathbf{A}^\circ} \sim^{\mathbf{A}^\circ} b = \circ^{\mathbf{A}^\circ} b \vee \sim^{\mathbf{A}^\circ} a = \hat{\mathbf{t}}$ . If (2) is the case, we have,  $a \wedge^{\mathbf{A}^\circ} b \notin \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ . Then, by the definition of  $\wedge^{\mathbf{A}^\circ}$ , we have either (2.1)  $a, b \notin \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  or (2.2) both  $a = \hat{\mathbf{t}}$  and  $b \notin \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  or (2.3) both  $b = \hat{\mathbf{t}}$  and  $a \notin \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ . If (2.1) is the case, then, by the definition of  $\circ^{\mathbf{A}^\circ}$ , we have  $\circ^{\mathbf{A}^\circ} a = \circ^{\mathbf{A}^\circ} b = \hat{\mathbf{f}}$ . If (2.2) is the case, then, by the definitions of  $\circ^{\mathbf{A}^\circ}$  and  $\sim^{\mathbf{A}^\circ}$ , we have  $\circ^{\mathbf{A}^\circ} b = \hat{\mathbf{f}}$  and  $\sim^{\mathbf{A}^\circ} a = \hat{\mathbf{f}}$ . If (2.3) is the case, then, by the definitions of  $\circ^{\mathbf{A}^\circ}$  and  $\sim^{\mathbf{A}^\circ}$ , we have  $\circ^{\mathbf{A}^\circ} a = \hat{\mathbf{f}}$  and  $\sim^{\mathbf{A}^\circ} b = \hat{\mathbf{f}}$ . Hence, in all subcases of (2) we have, by the definition of  $\vee^{\mathbf{A}^\circ}$ , that at least one among  $\circ^{\mathbf{A}^\circ} a \vee^{\mathbf{A}^\circ} \circ^{\mathbf{A}^\circ} b$ ,  $\circ^{\mathbf{A}^\circ} a \vee^{\mathbf{A}^\circ} \sim^{\mathbf{A}^\circ} b$  or  $\circ^{\mathbf{A}^\circ} b \vee \sim^{\mathbf{A}^\circ} a$  is  $\hat{\mathbf{f}}$ . In all cases we have that:  $\circ^{\mathbf{A}^\circ} (a \wedge^{\mathbf{A}^\circ} b) = (\circ^{\mathbf{A}^\circ} a \vee^{\mathbf{A}^\circ} \circ^{\mathbf{A}^\circ} b) \wedge (\circ^{\mathbf{A}^\circ} a \vee^{\mathbf{A}^\circ} \sim^{\mathbf{A}^\circ} b) \wedge (\circ^{\mathbf{A}^\circ} b \vee \sim^{\mathbf{A}^\circ} a)$ .  $\square$

**Example 3.17.** Comparing Figure 1a with Figure 1b, we see that  $\mathbf{IS}_6$ ,  $\mathbf{IS}_5$  and  $\mathbf{IS}_4$  coincide, respectively, with  $\mathbf{DM}_4^\circ$ ,  $\mathbf{K}_3^\circ$  and  $\mathbf{B}_2^\circ$ .

### 3.4 The lattice of extensions of $\mathcal{PP}_\leq$

Given a  $\Sigma^{\mathbf{DM}}$ -matrix  $\mathfrak{M} := \langle \mathbf{A}, D \rangle$ , let  $\mathfrak{M}^\circ := \langle \mathbf{A}^\circ, D \cup \{\hat{\mathbf{t}}\} \rangle$  be the  $\Sigma^{\mathbf{PP}}$ -matrix with the underlying (by Proposition 3.16, perfect parafinite) algebra  $\mathbf{A}^\circ$  given by Definition 3.15. We denote by  $\widehat{\mathfrak{M}^\circ}$  the  $\Sigma^{\mathbf{DM}}$ -reduct of  $\mathfrak{M}^\circ$ . Furthermore, given a class of  $\Sigma^{\mathbf{DM}}$ -matrices  $\mathcal{M}$ , we let  $\mathcal{M}^\circ := \{\mathfrak{M}^\circ \mid \mathfrak{M} \in \mathcal{M}\}$  and  $\widehat{\mathcal{M}^\circ} := \{\widehat{\mathfrak{M}^\circ} \mid \mathfrak{M} \in \mathcal{M}\}$ . Whenever  $\vdash$  is a super-Belnap logic, denote by  $\vdash^\circ$  the logic determined by the family of matrices  $\{\mathfrak{M}^\circ \mid \mathfrak{M} \text{ is a non-trivial model of } \vdash\}$ . The series of results presented in this section shows that the mapping  $(\cdot)^\circ$  just defined constitutes a lattice embedding from the lattice of super-Belnap logics into the lattice of extensions of the logic  $\mathcal{PP}_\leq$ . This allows us, in particular, to lift the result on the lower bound of the cardinality of extensions of  $\mathcal{B}$  to a corresponding one on the lower bound of the cardinality of extensions of  $\mathcal{PP}_\leq$ .

Before introducing the results, we recall some helpful definitions from abstract algebraic logic [15]. Given a  $\Sigma$ -matrix  $\mathfrak{M} = \langle \mathbf{A}, D \rangle$ , a congruence  $\theta \in \text{Cng } \mathbf{A}$  is said to be *compatible with  $\mathfrak{M}$*  when  $b \in D$  whenever both  $a \in D$  and  $a\theta b$ , for all  $a, b \in A$ . We denote by  $\Omega^\mathfrak{M}$  the *Leibniz congruence* associated to  $\mathfrak{M}$ , namely the greatest congruence of  $\mathbf{A}$  compatible with  $\mathfrak{M}$ . The matrix  $\mathfrak{M}^* = \langle \mathbf{A}/\Omega^\mathfrak{M}, D/\Omega^\mathfrak{M} \rangle$  is the *reduced version* of  $\mathfrak{M}$ . We say that  $\mathfrak{M}$  is *reduced* when it coincides with its own reduced version (or, equivalently, when its Leibniz congruence is the identity relation on  $A$ ). It is well known that  $\triangleright_\mathfrak{M} = \triangleright_{\mathfrak{M}^*}$  (and thus  $\vdash_\mathfrak{M} = \vdash_{\mathfrak{M}^*}$ ) and, since every logic is determined by a class of matrix models, we have that every logic coincides with the logic determined by its reduced matrix models. The class of all reduced matrix models for a logic  $\vdash$  is denoted by  $\text{Mat}^*(\vdash)$ .

**Lemma 3.18.** Let  $\mathfrak{M}$  be a non-trivial model of  $\mathcal{B}$ . Then  $\mathfrak{M}^* \cong (\widehat{\mathfrak{M}^\circ})^*$ .

*Proof.* We know from [20, Lemma 4.6] that  $\mathfrak{M}^* \cong (\widehat{\mathfrak{M}^\circ})^*$ . Clearly,  $\widehat{\mathfrak{M}^\circ}$  and  $\widehat{\mathfrak{M}^\circ}$  are isomorphic matrices under the identity mapping on  $A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ , and so are their reductions.  $\square$

**Corollary 3.19.** Where  $\mathfrak{M}$  is a non-trivial model of  $\mathcal{B}$ , we have  $\triangleright_\mathfrak{M} = \triangleright_{\widehat{\mathfrak{M}^\circ}}$  and  $\vdash_\mathfrak{M} = \vdash_{\widehat{\mathfrak{M}^\circ}}$ .

**Corollary 3.20.** Where  $\mathfrak{M}$  is a non-trivial model of  $\mathcal{B}$ , we have that  $\triangleright_{\mathfrak{M}^\circ}$  is a conservative extension of  $\triangleright_\mathfrak{M}$  and  $\vdash_{\mathfrak{M}^\circ}$  is a conservative extension of  $\vdash_\mathfrak{M}$ .

**Corollary 3.21.** Let  $\vdash$  be a super-Belnap logic determined by a class of non-trivial models of  $\mathcal{B}$ . Then  $\vdash^\circ$  is a conservative extension of  $\vdash$ . In particular,  $\mathcal{PP}_\leq$  is a conservative extension of  $\mathcal{B}$ .

*Proof.* It follows from Corollary 3.20 and the fact that  $\mathcal{PP}_\leq$  is characterized by the matrix  $\langle \mathbf{PP}_6, \uparrow \mathbf{b} \rangle$ , which is obtained from the matrix  $\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle$  by the construction introduced in Definition 3.15.  $\square$

**Corollary 3.22.** *Let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be non-trivial models of  $\mathcal{B}$ . If  $\triangleright_{\mathfrak{M}_1^\circ} = \triangleright_{\mathfrak{M}_2^\circ}$ , then  $\triangleright_{\mathfrak{M}_1} = \triangleright_{\mathfrak{M}_2}$ ; analogously, if  $\vdash_{\mathfrak{M}_1^\circ} = \vdash_{\mathfrak{M}_2^\circ}$ , then  $\vdash_{\mathfrak{M}_1} = \vdash_{\mathfrak{M}_2}$ .*

**Corollary 3.23.** *Let  $\vdash_1$  and  $\vdash_2$  be super-Belnap logics. Then  $\vdash_1 \subseteq \vdash_2$  iff  $\vdash_1^\circ \subseteq \vdash_2^\circ$ .*

*Proof.* From the left to the right, assuming  $\vdash_1 \subseteq \vdash_2$  gives that  $\text{Mat}^*(\vdash_2) \subseteq \text{Mat}^*(\vdash_1)$ , so  $(\text{Mat}^*(\vdash_2))^\circ \subseteq (\text{Mat}^*(\vdash_1))^\circ$ , which clearly entails that  $\vdash_1^\circ \subseteq \vdash_2^\circ$ . Conversely, suppose that  $\vdash_1^\circ \subseteq \vdash_2^\circ$  and that  $\Phi \vdash_1 \psi$ . Hence  $\Phi \vdash_1^\circ \psi$ , and then  $\Phi \vdash_2^\circ \psi$ , which gives  $\Phi \vdash_2 \psi$  by Corollary 3.21.  $\square$

**Corollary 3.24.** *The map given by  $\vdash \mapsto \vdash^\circ$  is an embedding (that is, an injective homomorphism) of the lattice of super-Belnap logics into the lattice of extensions of  $\mathcal{PP}_\leq$ . This, the latter lattice has (at least) the cardinality of the continuum.*

*Proof.* By Corollary 3.23 and [27, Theorem 4.13].  $\square$

### 3.5 On the recovery of classical reasoning

The following result shows that parafinite extensions of  $\mathcal{B}$ , when extended with  $\circ$  in the way we propose, result in logics which are at once **C**-systems and **D**-systems. This result applies, in particular, to the logic  $\mathcal{PP}_\leq$ .

**Proposition 3.25.** *Let  $\mathcal{M}$  be a class of non-trivial models of  $\mathcal{B}$  that determines a parafinite logic. Then the SET-SET logic determined by  $\mathcal{M}^\circ$  is a **C**-system and a **D**-system.*

*Proof.* That parafiniteness is preserved when passing from  $\mathcal{M}$  to  $\mathcal{M}^\circ$  follows by Corollary 3.20. As it is well-known that the negation-free fragments of  $\mathcal{CL}$  and  $\mathcal{B}$  coincide, by taking  $\circ$  as the consistency connective and  $\sim\circ$  as the determinedness connective, we may straightforwardly use the values  $\hat{\mathbf{f}}$  and  $\hat{\mathbf{t}}$  to build suitable valuations for showing that the logic determined by  $\mathcal{M}^\circ$  is at once a **C**-system and a **D**-system.  $\square$

**Corollary 3.26.**  *$\mathcal{PP}_\leq$  is a **C**-system and a **D**-system.*

A unary connective  $\odot$  is said to constitute a *classical negation* in a SET-FMLA logic  $\vdash$  based on  $\Sigma$  when, for all  $\varphi, \psi \in L_\Sigma(P)$ , we have that (i):  $\Phi, \varphi \vdash \psi$  and  $\Phi, \odot(\varphi) \vdash \psi$  imply  $\Phi \vdash \psi$ , and (ii):  $\varphi, \odot(\varphi) \vdash \psi$ . In case  $\vdash$  has a disjunction, we may equivalently replace (i) by (iii):  $\emptyset \vdash \varphi \vee \odot(\varphi)$  in this characterization. We prove in what follows that in  $\mathcal{PP}_\leq$  no composite unary connective may be defined that simultaneously satisfies both (i) and (iii). Since  $\mathcal{PP}_\leq$  has a disjunction, this entails that a classical negation is not definable in this logic.

**Proposition 3.27.** *There is no unary formula  $\varphi \in L_{\Sigma^{\mathcal{PP}}}(P)$  such that  $p, \varphi(p) \vdash_{\mathcal{PP}_\leq} q$  and  $\emptyset \vdash_{\mathcal{PP}_\leq} p \vee \varphi(p)$ .*

*Proof.* Let  $\varphi \in L_{\Sigma^{\mathcal{PP}}}(P)$  be a unary formula and suppose that  $p, \varphi(p) \vdash_{\mathcal{PP}_\leq} q$  and  $\emptyset \vdash_{\mathcal{PP}_\leq} p \vee \varphi(p)$ . Then, since  $\mathcal{PP}_\leq$  is an order-preserving logic, we have, for all  $h \in \text{Hom}(L_{\Sigma^{\mathcal{PP}}}(P), \mathbf{PP}_6)$ , that  $h(p) \vee^{\mathbf{PP}_6} \varphi^{\mathbf{PP}_6}(h(p)) = \hat{\mathbf{t}}$  (the greatest element of  $\mathbf{PP}_6$ ) and  $h(p) \wedge^{\mathbf{PP}_6} \varphi^{\mathbf{PP}_6}(h(p)) = \hat{\mathbf{f}}$  (the least element of  $\mathbf{PP}_6$ ), which is to say that  $\varphi^{\mathbf{PP}_6}(a)$  is a Boolean complement of  $a$ , for every element  $a$  of  $\mathbf{PP}_6$ . This is absurd, since, by the definition of  $\wedge^{\mathbf{PP}_6}$  and  $\vee^{\mathbf{PP}_6}$ , only  $\hat{\mathbf{t}}$  and  $\hat{\mathbf{f}}$  have Boolean complements in  $\mathbf{PP}_6$ .  $\square$

On what concerns the previous result, it is worth observing that a similar phenomenon, concerning the undefinability of a classical negation, is observed concerning several **LFIs** and **LFUs** with a modal character ([18, Theorem 6.1.2]) built on top of complete distributive lattices.

As argued in [23], the ability to recover negation-consistent (resp. negation-determined) reasoning is the most fundamental feature of **LFIs** (resp. **LFUs**). This feature may be expressed in terms of a convenient *Derivability Adjustment Theorem* (DAT) with respect to Classical Logic, which states, in the present case, that classical reasoning may be fully recovered as long as premises restoring the lost ‘perfection’ and establishing the ‘classicality’ of a certain set of formulas are available. The result presented below is a DAT that applies to any super-Belnap logic determined by a class of non-trivial models of  $\mathcal{B}$  extended with the perfection operator  $\circ$  considered in this paper. As a corollary, we will, in particular, have a DAT for the logics  $\mathcal{PP}_{\leq}^{\triangleright}$  and  $\mathcal{PP}_{\leq}$ .

**Theorem 3.28.** *Let  $\mathcal{M}$  be a class of non-trivial models of  $\mathcal{B}$ . Then, for all  $\Phi, \Psi \subseteq L_{\Sigma^{\text{DM}}}(P)$ , we have*

$$\Phi \triangleright_{\mathcal{CL}} \Psi \text{ iff } \Phi, \circ p_1, \dots, \circ p_n \triangleright_{\mathcal{M}^\circ} \Psi,$$

with  $\{p_1, \dots, p_n\} = \text{props}(\Phi \cup \Psi)$ .

*Proof.* Let  $\mathcal{M}$  be a class of non-trivial models of  $\mathcal{B}$ . Notice that  $\langle \mathbf{PP}_2, \{\hat{\mathbf{t}}\} \rangle$  is a submatrix of  $\mathfrak{M}^\circ$  for all  $\mathfrak{M}^\circ \in \mathcal{M}^\circ$ .

From the left to the right, contrapositively, suppose that  $\Phi, \circ p_1, \dots, \circ p_n \not\triangleright_{\mathcal{M}^\circ} \Psi$ . Then, there are  $\mathfrak{M}^\circ = \langle \mathbf{A}^\circ, D \cup \{\hat{\mathbf{t}}\} \rangle \in \mathcal{M}^\circ$  and  $h \in \text{Hom}(\mathbf{L}_{\Sigma^{\text{PP}}}(P), \mathbf{A}^\circ)$  such that (a)  $h(\Phi \cup \{\circ p_1, \dots, \circ p_n\}) \subseteq D \cup \{\hat{\mathbf{t}}\}$  and (b)  $h(\Psi) \subseteq \overline{D} \cup \{\hat{\mathbf{f}}\}$ . The interpretation of  $\circ$  given in Definition 3.15 and (a) entail that  $h(p_i) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  for all  $1 \leq i \leq n$ . As  $\mathbf{PP}_2$  is a subalgebra of  $\mathbf{A}^\circ$ , we may define an  $h' : \{p_1, \dots, p_n\} \rightarrow \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  by setting  $h'(p_i) := h(p_i)$ ; this extends to the full language and, in view of Definition 3.15, agrees with  $h$  on the set  $\Phi \cup \Psi$ . Thus, by (a),  $h'(\Phi) \subseteq \{\hat{\mathbf{t}}\}$  (as  $\hat{\mathbf{f}} \notin D$ ), while  $h'(\Psi) \subseteq \{\hat{\mathbf{f}}\}$  by (b), meaning that  $\Phi \not\triangleright_{\mathbf{PP}_2}^{\top} \Psi$ . Hence, by Proposition 3.13, we have  $\Phi \not\triangleright_{\mathcal{CL}} \Psi$ .

From the right to the left, again contrapositively, assume that  $\Phi \not\triangleright_{\mathcal{CL}} \Psi$ . Thus, by Proposition 3.13, we have  $\Phi \not\triangleright_{\mathbf{PP}_2}^{\top} \Psi$ . Then there is  $h \in \text{Hom}(\mathbf{L}_{\Sigma^{\text{PP}}}(P), \mathbf{PP}_2)$  such that  $h(\Phi) \subseteq \{\hat{\mathbf{t}}\}$  and  $h(\Psi) \subseteq \{\hat{\mathbf{f}}\}$ . Notice that, if  $\mathfrak{M}^\circ = \langle \mathbf{A}^\circ, D \cup \{\hat{\mathbf{t}}\} \rangle \in \mathcal{M}^\circ$ , then we may define  $h' : P \rightarrow A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$  with  $h'(p) = h(p)$ , for all  $p \in P$ . As  $\mathbf{PP}_2$  is a subalgebra of  $\mathbf{A}^\circ$ ,  $h'$  extends to the full language and agrees with  $h$  on it. Moreover, as  $h'(p_i) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ , we have, by Definition 3.15,  $h'(\circ p_i) = \hat{\mathbf{t}}$ , for all  $1 \leq i \leq n$ . Hence  $h'(\Phi \cup \{\circ p_1, \dots, \circ p_n\}) \subseteq \{\hat{\mathbf{t}}\}$ , while  $h'(\Psi) \subseteq \{\hat{\mathbf{f}}\}$ . Therefore,  $\Phi, \circ p_1, \dots, \circ p_n \triangleright_{\mathfrak{M}^\circ} \Psi$  for each  $\mathfrak{M}^\circ \in \mathcal{M}^\circ$ , and, in particular, we obtain  $\Phi, \circ p_1, \dots, \circ p_n \triangleright_{\mathcal{M}^\circ} \Psi$ .  $\square$

**Corollary 3.29.** *For all  $\Phi, \Psi \subseteq L_{\Sigma^{\text{DM}}}(P)$ , we have*

$$\Phi \triangleright_{\mathcal{CL}} \Psi \text{ iff } \Phi, \circ p_1, \dots, \circ p_n \triangleright_{\mathbf{PP}}^{\leq} \Psi,$$

with  $\{p_1, \dots, p_n\} = \text{props}(\Phi \cup \Psi)$ .

*Proof.* Follows by Theorem 3.28, together with the facts that  $\mathbf{DM}_4$  is non-trivial and that  $\mathcal{PP}_{\leq}^{\triangleright}$  is determined by the single matrix  $\mathbf{PP}_6$ , which coincides with  $\mathbf{DM}_4^\circ$ .  $\square$

## 4 Axiomatizing Logics of De Morgan Algebras Enriched with Perfection

In the first part of this section, we provide a general recipe for producing a symmetrical Hilbert-style calculus for the SET-SET logic determined by any class  $\mathcal{M}$  of  $\Sigma^{\text{DM}}$ -matrices expanded with the perfection

operator  $\circ$  according to the mechanism set up in the previous section. Our approach is based on adding some rules governing  $\circ$  to a given axiomatization of  $\mathcal{M}$ , resulting in what we call a *relative axiomatization* of  $\mathcal{M}^\circ$  by the added rules with respect to the SET-SET logic determined by  $\mathcal{M}$ . In the sequel, we will show, for a particular class of matrices, how to turn the given SET-SET relative axiomatizations into SET-FMLA axiomatizations, using the fact proved in [29, Theorem 5.37] that a symmetrical calculus  $R$  can be transformed into a SET-FMLA calculus provided that  $\vdash_R$  has a disjunction. If  $R$  axiomatizes a class  $\mathcal{M}$  of  $\Sigma^{\text{DM}}$ -matrices, a sufficient condition for the latter property to hold is that all members of  $\mathcal{M}$  have prime filters as sets of designated values. For this reason, the provided SET-FMLA Hilbert-style calculi will consist in axiomatizations for logics determined by classes of  $\Sigma^{\text{DM}}$ -matrices whose designated values form prime filters.

#### 4.1 Analyticity-preserving symmetrical calculi

In what follows, if  $\triangleright_1$  and  $\triangleright_2$  are SET-SET logics over  $\Sigma$ , we set  $\triangleright_1 \simeq \triangleright_2$  iff  $\triangleright_1 \cup \{(L_\Sigma(P), \emptyset)\} = \triangleright_2 \cup \{(L_\Sigma(P), \emptyset)\}$ . It is clear that two SET-SET logics satisfying this condition induce the same SET-FMLA logic.<sup>2</sup> We will employ this weaker relation instead of the equality relation to make the results in this section more general and simpler to prove. The first result below provides a generic recipe for axiomatizing the SET-SET logic determined by the class  $\mathcal{M}^\circ$ , assuming we have a calculus  $R$  that axiomatizes the SET-SET logic determined by  $\widehat{\mathcal{M}}^\circ$  (namely, the family of the  $\Sigma^{\text{DM}}$ -reducts of the matrices in  $\mathcal{M}^\circ$ ). The rules listed in this result are among those obtained by running the axiomatization algorithm described in [21] on the matrix  $\langle \mathbf{PP}_6, \uparrow \mathbf{b} \rangle$ , using  $\{p, \circ p, \sim p\}$  as set of separators, and then streamlining the resulting calculus. What we will see now is that  $R$  together with these very rules axiomatizes the SET-SET logic determined by  $\mathcal{M}^\circ$ .

**Theorem 4.1.** *Let  $\mathcal{M}$  be a class of  $\Sigma^{\text{DM}}$ -matrices. If  $\triangleright_{\widehat{\mathcal{M}}^\circ} \simeq \triangleright_R$ , then  $\triangleright_{\mathcal{M}^\circ} = \triangleright_{R_{\mathcal{R}_\circ}}$ , where  $R_\circ$  consists of the following inference rules:*

$$\begin{array}{cccccc}
\frac{}{\circ \perp} r_1 & \frac{}{\circ \top} r_2 & \frac{}{\circ \circ p} r_3 & \frac{\circ p}{\circ \sim p} r_4 & \frac{\circ \sim p}{\circ p} r_5 & \frac{\circ p}{p, \sim p} r_6 & \frac{\circ p, p, \sim p}{\circ p} r_7 \\
\frac{\circ p}{\circ(p \wedge q), p} r_8 & \frac{\circ q}{\circ(p \wedge q), q} r_9 & \frac{\circ(p \wedge q), q}{\circ p} r_{10} & \frac{\circ(p \wedge q), p}{\circ q} r_{11} & \frac{\circ p, \circ q}{\circ(p \wedge q)} r_{12} & \frac{\circ(p \wedge q)}{\circ p, \circ q} r_{13} \\
\frac{\circ p, \circ q}{\circ(p \vee q)} r_{14} & \frac{\circ(p \vee q)}{\circ p, \circ q} r_{15} & \frac{\circ p, p}{\circ(p \vee q)} r_{16} & \frac{\circ q, q}{\circ(p \vee q)} r_{17} & \frac{\circ(p \vee q)}{\circ p, q} r_{18} & \frac{\circ(p \vee q)}{\circ q, p} r_{19}
\end{array}$$

*Proof.* Checking the soundness of those rules is routine; we provide only a couple of examples. Let  $v$  be an  $\mathfrak{M}^\circ$ -valuation. The rule  $r_3$  is sound in  $\mathfrak{M}^\circ$ , given that  $v(\circ\varphi) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ , so we have that  $v(\circ\circ\varphi) = \hat{\mathbf{t}}$ . On what concerns rule  $r_8$ , we have that, if  $v(\circ\varphi) = \hat{\mathbf{t}}$ , then either (i)  $v(\varphi) = \hat{\mathbf{f}}$  or (ii)  $v(\varphi) = \hat{\mathbf{t}}$ . Soundness is obvious in case (ii). In case (i),  $v(\varphi \wedge \psi) = \hat{\mathbf{f}}$ , so  $v(\circ(\varphi \wedge \psi)) = \hat{\mathbf{t}}$ .

For completeness, assume  $\Phi \blacktriangleright_{R_\circ} \Psi$ . Then, by cut for sets, there is a partition  $\langle T, F \rangle$  of  $L_{\Sigma^{\text{PP}}}(P)$  such that  $\Phi \subseteq T$  and  $\Psi \subseteq F$  and  $T \blacktriangleright_{R_\circ} F$ . Note that (by  $r_3, r_6$  and  $r_7$ ) for each  $\varphi$ , we have either  $\circ\varphi \in T$  or  $\sim\circ\varphi \in T$ , but never both. In particular,  $F$  is never empty. Also, by  $r_6$  and  $r_7$ , if we have  $\circ\varphi \in T$ , we have either  $\varphi \in T$  or  $\sim\varphi \in T$ , but never both. Hence, each  $\varphi$  must belong to exactly one of three cases: (a)  $\sim\circ\varphi \in T$ , (b)  $\circ\varphi, \varphi \in T$  or (c)  $\circ\varphi, \sim\varphi \in T$ .

<sup>2</sup>This has been observed by R. Carnap, already in the 1940s [9]. It might seem that extending a logic this way would imply that a semantics characterising the extended logic would have to provide ‘models for contradictory formulas’. However, such a model, in this case, would be trivial, for it would make all formulas equally true. As argued in [24], this is not the kind of models that a paraconsistent logician is interested upon. This explains, by the way, why our definition of paraconsistency, presented towards the end of Section 2, has been formulated in terms of  $p, \sim p \blacktriangleright q$  rather than  $p, \sim p \blacktriangleright \emptyset$ .

Since  $R \subseteq RUR_o$ , we also have  $T \triangleright_R F$ . From the fact that  $\triangleright_R \simeq \triangleright_{\widehat{\mathfrak{M}}}$  and  $F \neq \emptyset$  we know that  $T \triangleright_{\widehat{\mathfrak{M}}} F$ . We may therefore pick some  $\widehat{\mathfrak{M}}$ -valuation  $v$ , for some  $\mathfrak{M} \in \mathcal{M}$ , such that  $v(T) \subseteq D$  and  $v(F) \subseteq \overline{D}$ . Consider now the mapping  $v' : L_{\Sigma^{PP}}(P) \rightarrow \mathfrak{M}^\circ$  defined by:

$$v'(\varphi) := \begin{cases} v(\varphi_i) & \text{if } \varphi = \varphi_1 \wedge \varphi_2 \text{ and } \sim \circ \varphi, \circ \varphi_{3-i} \in T \text{ for } i \in \{1, 2\} \\ v(\varphi_i) & \text{if } \varphi = \varphi_1 \vee \varphi_2 \text{ and } \sim \circ \varphi, \circ \varphi_i \in T \text{ for } i \in \{1, 2\} \\ v(\varphi) & \text{if } \sim \circ \varphi \in T \\ \hat{\mathbf{t}} & \text{if } \circ \varphi, \varphi \in T \\ \hat{\mathbf{f}} & \text{if } \circ \varphi, \sim \varphi \in T \end{cases}$$

We will check that  $v'$  is an  $\mathfrak{M}^\circ$ -valuation:

1.  $v'(\circ\varphi) = \circ v'(\varphi)$ : If (i)  $\sim \circ \varphi \in T$  then, by  $r_3$ ,  $\circ \circ \varphi \in T$  (so  $v'(\circ\varphi) = \hat{\mathbf{f}}$ ). Thus  $v'(\circ\varphi) = \hat{\mathbf{f}} = \circ(v'(\varphi))$ . If (ii)  $\circ \varphi, \varphi \in T$ , then, by  $r_3$ ,  $\circ \circ \varphi \in T$  (so  $v'(\circ\varphi) = \hat{\mathbf{t}}$ ). So  $v'(\circ\varphi) = \hat{\mathbf{t}} = \circ(v'(\varphi))$ . Case (iii) is analogous to (ii).
2.  $v'(\sim\varphi) = \sim v'(\varphi)$ : If (i)  $\sim \circ \sim \varphi \in T$ , then, by  $r_3$  and  $r_7$ ,  $\circ \sim \varphi \notin T$ . Then, by  $r_4$ ,  $\circ \varphi \notin T$ . Thus, by  $r_3$  and  $r_6$ ,  $\sim \circ \varphi \in T$  (so  $v'(\varphi) = v(\varphi)$ ). So  $v'(\sim\varphi) = v(\sim\varphi) = \sim v(\varphi) = \sim(v'(\varphi))$ . If (ii)  $\circ \sim \varphi, \sim \varphi \in T$ , by  $r_5$ ,  $\circ \varphi \in T$  (so  $v'(\varphi) = \hat{\mathbf{f}}$ ). Then  $v'(\sim\varphi) = \hat{\mathbf{t}} = \sim(v'(\varphi))$ . Case (iii) is analogous to (ii).
3.  $v'(\varphi \wedge \psi) = v'(\varphi) \wedge v'(\psi)$ : If (i)  $\sim \circ(\varphi \wedge \psi) \in T$ , then, by  $r_3$  and  $r_7$ , we have that  $\circ(\varphi \wedge \psi) \notin T$ . By  $r_{12}$ , we have that (a)  $\circ \varphi, \circ \psi \notin T$ , (b)  $\circ \varphi \in T$  and  $\circ \psi \notin T$  or (c)  $\circ \varphi \notin T$  and  $\circ \psi \in T$ . So:
  - (a) By  $r_3$  and  $r_6$ ,  $\sim \circ \varphi, \sim \circ \psi \in T$  (so  $v'(\varphi) = v(\varphi)$  and  $v'(\psi) = v(\psi)$ ). So  $v'(\varphi \wedge \psi) = v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi) = v'(\varphi) \wedge v'(\psi)$ .
  - (b) By  $r_3$  and  $r_6$ ,  $\sim \circ \psi \in T$  (so  $v'(\psi) = v(\psi)$ ). By  $r_8$ ,  $\varphi \in T$  (so  $v'(\varphi) = \hat{\mathbf{t}}$ ). Therefore  $v'(\varphi \wedge \psi) = v(\psi) = v'(\psi) = \hat{\mathbf{t}} \wedge v'(\psi) = v'(\varphi) \wedge v'(\psi)$ .
  - (c) This case is analogous to the previous one, but now using  $r_9$ .
 If (ii)  $\circ(\varphi \wedge \psi), \varphi \wedge \psi \in T$ , then  $\varphi, \psi \in T$ . By  $r_{10}$  and  $r_{11}$ ,  $\circ \varphi, \circ \psi \in T$ . (so  $v'(\varphi) = v'(\psi) = \hat{\mathbf{t}}$ ) hence  $v'(\varphi \wedge \psi) = \hat{\mathbf{t}} = v'(\varphi) \wedge v'(\psi)$ . If (iii)  $\circ(\varphi \wedge \psi), \sim(\varphi \wedge \psi) \in T$ , then either  $\sim \varphi \in T$  or  $\sim \psi \in T$ . By  $r_{13}$ , we have that (a)  $\circ \varphi, \circ \psi \in T$ , (b)  $\circ \varphi \in T$  and  $\circ \psi \notin T$  or (c)  $\circ \varphi \notin T$  and  $\circ \psi \in T$ . So:
  - (a) Here, we have that  $v'(\varphi) = \hat{\mathbf{f}}$  or  $v'(\psi) = \hat{\mathbf{f}}$ . So  $v'(\varphi \wedge \psi) = \hat{\mathbf{f}} = v'(\varphi) \wedge v'(\psi)$ .
  - (b) By  $r_{11}$  and  $r_6$   $\sim \varphi \in T$  (so  $v'(\varphi) = \hat{\mathbf{f}}$ ). So  $v'(\varphi \wedge \psi) = \hat{\mathbf{f}} = v'(\varphi) \wedge v'(\psi)$ .
  - (c) This case is analogous to the previous one, using  $r_{10}$ .
4.  $v'(\varphi \vee \psi) = v'(\varphi) \vee v'(\psi)$ : analogous to the case of  $\wedge$ .
5.  $v'(\perp) = \hat{\mathbf{f}}$  and  $v'(\top) = \hat{\mathbf{t}}$ : directly from rules  $r_1$  and  $r_2$ . □

Given  $\Xi \subseteq L_{\Sigma^{PP}}(P)$ , let  $\Xi^\circ := \Xi \cup \{\sim p, \circ p, \sim \circ p\}$ . The theorem below shows that the recipe presented above preserves analyticity.

**Theorem 4.2.** *Let  $\mathcal{M}$  be a class of  $\Sigma^{\text{DM}}$ -matrices. If  $R$  is a  $\Xi$ -analytic axiomatization of  $\triangleright_{\widehat{\mathcal{M}}}$ , then  $RUR_o$  is a  $\Xi^\circ$ -analytic axiomatization of  $\triangleright_{\mathcal{M}^\circ}$ .*

*Proof.* Let  $Y := \text{sub}(\Phi \cup \Psi)$  and  $\Lambda := Y_{\Xi^\circ}$ . Assume that  $\Phi \triangleright_{RUR_o}^\Lambda \Psi$ . Then, by cut for sets, there is a partition  $\langle T, F \rangle$  of  $\Lambda$  such that  $\Phi \subseteq T$  and  $\Psi \subseteq F$  and  $T \triangleright_{RUR_o}^\Lambda F$ . Since  $R \subseteq RUR_o$ , we also have  $T \triangleright_R^\Lambda F$ . From the fact that  $R$  axiomatizes  $\triangleright_{\widehat{\mathcal{M}}}$  and  $F \neq \emptyset$  we know that  $T \triangleright_{\widehat{\mathcal{M}}} F$ . We may therefore

pick  $v \in \text{Hom}_{\Sigma^{\text{DM}}}(\mathbf{L}_{\Sigma^{\text{PP}}}(P), \widehat{\mathfrak{M}}^\circ)$ , for some  $\mathfrak{M} \in \mathcal{M}$ , such that  $v(T) \subseteq D$  and  $v(F) \subseteq \overline{D}$ . Since, for each  $\varphi \in \Upsilon$ , we have  $\sim\varphi, \circ\varphi, \sim\circ\varphi \in \Lambda$ , we may use the same construction given in Theorem 4.1 to define a certain mapping  $v' : \Upsilon \rightarrow \mathfrak{M}^\circ$ . That  $v'$  respects all the connectives follows from the fact that in the proof of Theorem 4.1 we only used instances of the rules employing formulas present in  $\Lambda$ . This, together with the fact that  $\Upsilon$  is closed under subformulas, implies that  $v'$  is a partial  $\mathfrak{M}^\circ$ -valuation. Hence,  $v'$  may be extended to a total  $\mathfrak{M}^\circ$ -valuation, witnessing the fact that  $\Phi \blacktriangleright_{\mathcal{M}^\circ} \Psi$ , thus concluding the proof.  $\square$

From Theorem 3.15 and Theorem 4.2, it follows that:

**Corollary 4.3.** *Let  $S := \{p, \sim p\}$ . The calculus presented in Example 2.8 together with the rules of  $\mathcal{R}_\circ$  is an  $S^\circ$ -analytic axiomatization of  $\mathcal{PP}_\leq$ .*

As explained in [21], analytic calculi as those we have been discussing are associated to a proof-search algorithm and a countermodel-search algorithm, and consequently to a decision procedure for the corresponding SET-SET logics. Briefly put, if we want to know whether  $\Phi \triangleright_{\mathcal{R}} \Psi$ , where  $\mathcal{R}$  is a  $\Xi$ -analytic symmetrical calculus, obtaining a proof when the answer is positive and a countermodel otherwise, we may attempt to build a derivation in the following way: start from a single node labelled with  $\Phi$  and search for a rule instance of  $\mathcal{R}$  not used in the same branch with formulas in the set  $\Upsilon^\Xi$  (namely, the set of generalized subformulas of  $(\Phi, \Psi)$ , as defined in Section 2) whose premises are in  $\Phi$ . If there is one, expand that node by creating a child node labelled with  $\Phi \cup \{\varphi\}$  for each formula  $\varphi$  in the succedent of the chosen rule instance and repeat this step for each new node. In case it fails in finding a rule instance for applying to some node, we may conclude that no proof exists, and from each non- $\Psi$ -closed branch we may extract a countermodel. In case every branch eventually gets  $\Psi$ -closed, the resulting tree is a proof of the desired statement. The following example illustrates how this works.

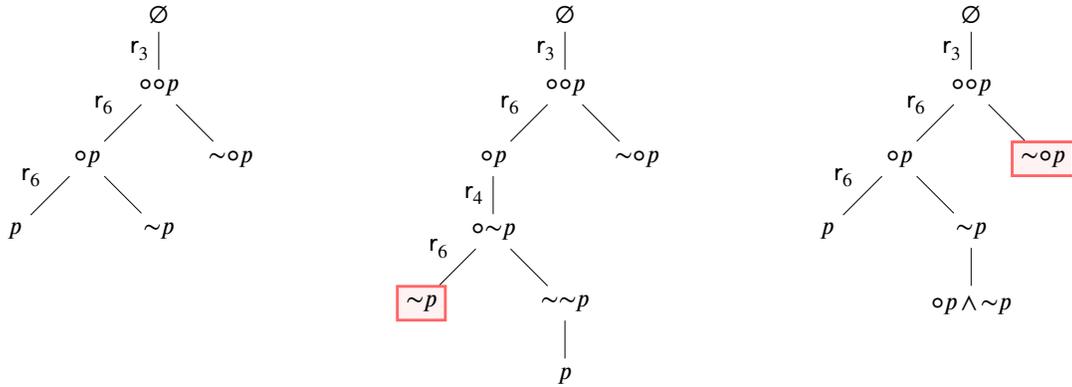


Figure 3:

Outputs of the proof-search and of the countermodel-search algorithm induced by our analytic symmetrical calculus, witnessing that  $\emptyset \triangleright_{\mathcal{R} \cup \mathcal{R}^\circ} p, \sim p, \sim \circ p$ ; that  $\emptyset \blacktriangleright_{\mathcal{R}_B \cup \mathcal{R}_B^\circ} p, \sim \circ p$  and that  $\emptyset \blacktriangleright_{\mathcal{R}_B \cup \mathcal{R}_B^\circ} p, \circ p \wedge \sim p$ .

**Example 4.4.** *The first tree in Figure 3 proves that  $\emptyset \triangleright_{\mathcal{R} \cup \mathcal{R}^\circ} p, \sim p, \sim \circ p$  in any  $\Xi^\circ$ -analytic calculus  $\mathcal{R} \cup \mathcal{R}^\circ$  obtained from Theorem 4.1, and may be easily built by the algorithm described above. If we consider the calculus  $\mathcal{R}_B$  given in Example 2.8, the second tree in the same figure shows an output of the described algorithm when we search for a countermodel witnessing  $\emptyset \blacktriangleright_{\mathcal{R}_B \cup \mathcal{R}_B^\circ} p, \sim \circ p$ . In this tree, the leftmost branch is a non- $\Psi$ -closed branch for which no rule instance based only on subformulas of  $\Phi \cup \Psi$  and not used yet in the same branch is available. This implies that, for  $\Theta = \{\circ \circ p, \circ p, \circ \sim p, \sim p\}$ , which are the formulas in the leaf of this non- $\Psi$ -closed branch, we have  $\Theta \blacktriangleright_{\mathcal{R}_B \cup \mathcal{R}_B^\circ} \Upsilon^{\Xi^\circ} \setminus \Theta$ . As the semantical*

counterpart of this calculus is the matrix  $\langle \mathbf{PP}_6, \uparrow \mathbf{b} \rangle$ , a valuation  $v$  such that  $v(\Theta) \subseteq \uparrow \mathbf{b}$  necessarily sets  $v(p) = \hat{\mathbf{f}}$ , since  $\{\sim p, \circ p\} \subseteq \Theta$ . A similar situation occurs in the third tree, which constitutes evidence for  $\emptyset \triangleright_{\mathbf{R}_B \cup \mathbf{R}_B^\circ} p, \circ p \wedge \sim p$ , meaning that the pseudo-complement given by  $\neg x := \circ x \wedge \sim x$  is non-implosive and, thus, not a classical negation in  $\mathcal{PP}_\leq$ . (This is not surprising, in view of Proposition 3.27, but it is worth contrasting this with what happens in many other LFI's [22], in which the latter definition of  $\neg$  does correspond to a classical negation.)

## 4.2 SET-FMLA Hilbert-style calculi for logics of De Morgan algebras with prime filters

We may extend the recipe given in Theorem 4.1, which delivers a symmetrical Hilbert-style calculus, to provide a SET-FMLA Hilbert-style calculus for the class  $\mathcal{M}^\circ$  when  $\mathcal{M}$  itself is axiomatized by a SET-FMLA Hilbert-style calculus. Before showing how, we will define a collection of such conventional Hilbert-style inference rules associated to a given collection of symmetrical rules. In what follows, when  $\Phi = \{\varphi_1, \dots, \varphi_n\} \subseteq L_\Sigma(P)$  ( $n \geq 1$ ), let  $\bigvee \Phi := (\dots (\varphi_1 \vee \varphi_2) \vee \dots) \vee \varphi_n$ . Also, let  $\Phi \vee \psi := \{\varphi \vee \psi \mid \varphi \in \Phi\}$ .

**Definition 4.5.** Let  $\mathbf{R}$  be a symmetrical calculus. Define the set  $\mathbf{R}^\vee := \left\{ \frac{p}{p \vee q}, \frac{p \vee q}{q \vee p}, \frac{p \vee (q \vee r)}{(p \vee q) \vee r} \right\} \cup \{r^\vee \mid r \in \mathbf{R}\}$  where  $r^\vee$  is  $\frac{\emptyset}{\varphi}$  if  $r = \frac{\emptyset}{\varphi}$ ,  $\frac{\Phi \vee p_0}{(\bigvee \Psi) \vee p_0}$  if  $r = \frac{\Phi}{\Psi}$ , and  $\frac{\Phi \vee p_0}{p_0}$  if  $r = \frac{\Phi}{\emptyset}$ , where  $p_0$  is a propositional variable not occurring in the rules that belong to  $\mathbf{R}$ .

The following result states that, when  $\mathbf{R}$  is the calculus given by Theorem 4.1, the calculus  $\mathbf{R}^\vee$  is the SET-FMLA Hilbert-style calculus we are looking for.

**Theorem 4.6.** Let  $\mathcal{M}$  be a class of  $\Sigma^{\text{DM}}$ -matrices whose designated sets are prime filters, and let  $\mathbf{R}$  be a SET-FMLA Hilbert-style calculus. If  $\vdash_{\mathbf{R}} = \vdash_{\mathcal{M}} = \vdash_{\widehat{\mathcal{M}}^\circ}$ , then  $\vdash_{(\mathbf{R}_{\mathbf{R}_0})^\vee} = \vdash_{\mathcal{M}^\circ}$ .

*Proof.* If  $\vdash_{\mathbf{R}} = \vdash_{\widehat{\mathcal{M}}^\circ}$  then  $\triangleright_{\mathbf{R}} \simeq \triangleright_{\widehat{\mathcal{M}}^\circ}$ , so, by Theorem 4.1, we have that  $\triangleright_{\mathcal{M}^\circ} = \triangleright_{\mathbf{R}_{\mathbf{R}_0}}$ , and thus  $\vdash_{\mathcal{M}^\circ} = \vdash_{\mathbf{R}_{\mathbf{R}_0}}$ . Given that  $\mathcal{M}$  is a class of  $\Sigma^{\text{DM}}$ -matrices whose designated sets are prime filters and  $\vdash_{\mathbf{R}} = \vdash_{\mathcal{M}}$ , we have  $p \vdash_{\mathbf{R}} p \vee q$ ,  $q \vdash_{\mathbf{R}} p \vee q$  and  $p \vee q \vdash_{\mathbf{R}} p, q$ . Since  $\widehat{\mathcal{M}}^\circ$  preserves the latter inferences, then  $p \vdash_{(\mathbf{R}_{\mathbf{R}_0})} p \vee q$ , and  $q \vdash_{(\mathbf{R}_{\mathbf{R}_0})} p \vee q$ , and  $p \vee q \vdash_{(\mathbf{R}_{\mathbf{R}_0})} p, q$ . The latter statements guarantee that  $\vdash_{\mathbf{R}}$  has a disjunction, so by [29, Theorem 5.37] we have that  $\vdash_{\mathbf{R}_{\mathbf{R}_0}} = \vdash_{(\mathbf{R}_{\mathbf{R}_0})^\vee}$ . Therefore,  $\vdash_{\mathcal{M}^\circ} = \vdash_{(\mathbf{R}_{\mathbf{R}_0})^\vee}$ .  $\square$

**Example 4.7.** Consider a SET-FMLA Hilbert calculus that axiomatizes  $\vdash_{\mathcal{B}}$ . Since  $\mathcal{B} = \vdash_{\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle} = \vdash_{\langle \widehat{\mathbf{DM}_4}, \uparrow \mathbf{b} \rangle}^\circ$  (cf. [20]), we obtain a conventional Hilbert-style axiomatization for  $\mathcal{PP}_\leq = \vdash_{\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle}^\circ = \vdash_{\langle \mathbf{PP}_6, \uparrow \mathbf{b} \rangle}$  by adding to that calculus the  $\mathbf{R}_0^\vee$  rules. We illustrate, below, with some of the resulting rules:

$$\frac{\circ p \vee r}{(p \vee \sim p) \vee r} r_6^\vee \quad \frac{\circ p \vee r, p \vee r, \sim p \vee r}{r} r_7^\vee \quad \frac{\circ p \vee r}{(\circ(p \wedge q) \vee p) \vee r} r_8^\vee \quad \frac{\circ p \vee r, p \vee r}{\circ(p \vee q) \vee r} r_{16}^\vee$$

In what follows we consider a few extensions of  $\mathcal{PP}_\leq$ , illustrating how our methods may be used to axiomatize them. The following result, which is an immediate consequence of Theorem 4.6, shows that Example 4.7 smoothly generalises to all super-Belnap logics.

**Proposition 4.8.** Let  $\mathcal{M}$  be a class of models of  $\mathcal{B}$  whose designated sets are prime filters. If  $\vdash_{\mathcal{M}}$  is axiomatized relative to  $\mathcal{B}$  by a set  $\mathbf{R}$  of SET-FMLA rules, then  $\vdash_{\mathcal{M}^\circ}$  is also axiomatized by  $\mathbf{R}$  relative to  $\mathcal{B}^\circ$ .

Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be two classes of models of  $\mathcal{B}$  such that  $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$ . Then  $\vdash_{\mathcal{M}_1}$  and  $\vdash_{\mathcal{M}_2}$  are axiomatized by the same set  $\mathbf{R}$  of singleton-succedent rules. Hence,  $\vdash_{\mathcal{M}_1^\circ} = \vdash_{\mathcal{M}_2^\circ}$  is axiomatized by the set  $\mathbf{R}_0^\vee$  defined above. This entails, in particular, that, if a super-Belnap logic  $\vdash$  is finitary, then  $\vdash^\circ$  (described in Lemma 3.21) is also finitary. Since the lattice of super-Belnap logics contains continuum-many finitary logics [26, Corollary 8.17], we obtain the following sharpening of Corollary 3.24:

**Proposition 4.9.** *There are continuum-many finitary extensions of  $\mathcal{PP}_{\leq}$ .*

The super-Belnap logics (see [1] for further details) considered below for the sake of illustration are the Asenjo-Priest Logic of Paradox  $\mathcal{LP}$ , the two logics  $\mathcal{K}_{\leq}$  and  $\mathcal{K}_1$  named after S. C. Kleene, and Classical Logic  $\mathcal{CL}$ . In the remaining results of this section, we use the notation  $\mathcal{L} + (R)$  to refer to the SET-FMLA Hilbert-style calculus resulting from adding rule  $(R)$  to a Hilbert-style system for  $\mathcal{L}$ . In addition, we will write  $\text{Log } \mathcal{M}$  for  $\vdash_{\mathcal{M}}$ . The next result establishes that each of these logics can be axiomatized, relative to  $\mathcal{B}$ , by a combination of the rules given below. In the sequel, we show, in a similar way, how some logics characterized by matrices over PP-algebras can be axiomatized relatively to  $\mathcal{PP}_{\leq}$ .

$$\frac{p \wedge (\sim p \vee q)}{q} \text{ (DS)} \quad \frac{(p \wedge \sim p) \vee q}{q} \text{ (K}_1\text{)} \quad \frac{(p \wedge \sim p) \vee r}{q \vee \sim q \vee r} \text{ (K}_{\leq}\text{)} \quad \frac{}{p \vee \sim p} \text{ (EM)}$$

**Proposition 4.10.** ([1, Theorem 3.4])

- (i)  $\mathcal{LP} = \text{Log}\langle \mathbf{K}_3, \uparrow \mathbf{n} \rangle = \mathcal{B} + (EM)$
- (ii)  $\mathcal{K}_1 = \text{Log}\langle \mathbf{K}_3, \{\mathbf{t}\} \rangle = \mathcal{B} + (K_1)$
- (iii)  $\mathcal{K}_{\leq} = \text{Log}\{\langle \mathbf{K}_3, \uparrow \mathbf{n} \rangle, \langle \mathbf{K}_3, \{\mathbf{t}\} \rangle\} = \mathcal{B} + (K_{\leq})$
- (iv)  $\mathcal{CL} = \text{Log}\langle \mathbf{B}_2, \{\mathbf{t}\} \rangle = \mathcal{B} + (DS) + (EM)$

**Theorem 4.11.** *For logics above  $\mathcal{PP}_{\leq}$  we have the following relative axiomatizations:*

- (i)  $\text{Log}\langle \mathbf{PP}_5, \uparrow \mathbf{n} \rangle = \mathcal{LP}^{\circ} = \mathcal{PP}_{\leq} + (EM)$
- (ii)  $\text{Log}\langle \mathbf{PP}_5, \uparrow \mathbf{t} \rangle = \mathcal{K}_1^{\circ} = \mathcal{PP}_{\leq} + (K_1)$
- (iii)  $\text{Log}\{\langle \mathbf{PP}_5, \uparrow \mathbf{n} \rangle, \langle \mathbf{PP}_5, \uparrow \mathbf{t} \rangle\} = \mathcal{K}_{\leq}^{\circ} = \mathcal{PP}_{\leq} + (K_{\leq})$
- (iv)  $\text{Log}\langle \mathbf{PP}_4, \uparrow \mathbf{t} \rangle = \mathcal{CL}^{\circ} = \mathcal{PP}_{\leq} + (DS) + (EM)$

*Proof.* This follows directly from Proposition 4.8 and Proposition 4.10, taking into account that  $\langle \mathbf{B}_2, \{\mathbf{t}\} \rangle^{\circ} = \langle \mathbf{PP}_4, \uparrow \mathbf{t} \rangle$ , and for  $x \in \{\mathbf{t}, \mathbf{n}\}$ , we have  $\langle \mathbf{K}_3, \uparrow x \rangle^{\circ} = \langle \mathbf{PP}_5, \uparrow x \rangle$ .  $\square$

## 5 Final remarks

We have seen how to endow with a perfection connective logics characterized by matrices having a De Morgan algebraic reduct, offering two possible directions: either by appropriately expanding the corresponding matrices or by adding new rules of inference to an existing Hilbert-style axiomatization. In particular, by so enriching Dunn-Belnap's 4-valued logic we obtained the 6-valued order-preserving logic  $\mathcal{PP}_{\leq}$ , associated to the variety of expanded algebras, which we called 'perfect parafinite algebras' and proved to be term-equivalent with the variety of involutive Stone algebras. It is worth mentioning that the one-one correspondence between both varieties can be used to introduce back-and-forth functors that establish a categorical equivalence between the corresponding algebraic categories.

By providing a Derivability Adjustment Theorem for  $\mathcal{PP}_{\leq}$  and its extensions, we have also shown that Boolean reasoning is fully recovered using De Morgan negation and the perfection operator. Notice, indeed, that adding the equation  $\circ x \approx \top$  to a perfect parafinite algebra, intuitively stating that every element is Boolean, what results in an algebra that is (term-equivalent to) a Boolean algebra.

The equational basis of the variety of PP-algebras studied here was conceived having in mind the expected term-equivalence with the variety of IS-algebras. A natural path for future work is to drop this constraint and study De Morgan algebras enriched with perfection operators satisfying weaker equations, and the corresponding logics. Within such more general algebraic structures and the logics based thereon,

and taking also into account the proposed comparison between  $\mathcal{PP}_{\leq}$  and other **C**-systems and **D**-systems in the literature, one may for instance consider two distinct negations (not necessarily respecting all De Morgan laws) instead of a single one that is at once paraconsistent and paracomplete, and possibly also two separate ‘recovery connectives’, as, e.g., in [13, 18].

Yet another direction for future investigation would be to enrich  $\mathcal{PP}_{\leq}$  with an implication connective; similar paths have recently been explored in [14], but considering involutive distributive residuated lattices instead of De Morgan algebras. A promising starting point for this research could be provided by the following observation. The algebra  $\mathbf{PP}_6$ , as a finite distributive lattice, has an implicitly definable (and unique) intuitionistic implication given by the relative pseudo-complement. By adding this operation to the propositional language and considering the corresponding logical matrix, one thus obtains a conservative extension of  $\mathcal{PP}_{\leq}$  by the intuitionistic implication. We expect this logic to be algebraizable, but not necessarily self-extensional. On the other hand, a (weaker) self-extensional conservative extension of  $\mathcal{PP}_{\leq}$  may be obtained by considering the logic determined by the family of all matrices of type  $\langle \mathbf{PP}_6, D \rangle$  based on  $\mathbf{PP}_6$  (endowed with the relative pseudo-complement operation), with  $D$  a lattice filter. Logics obtained in this way will have as algebraic counterparts (subclasses of) algebras that carry both a De Morgan negation and an intuitionistic implication: these structures have been studied in the literature under the names of *symmetric Heyting algebras* (A. Monteiro [25]) and *De Morgan-Heyting algebras* (H.P. Sankappanavar [28]). From a technical point of view, an advantage of the proposed approach is thus one may hope to be able to import results from the well-developed theory of the above-mentioned classes of algebras.

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