Deductive Systems and Coherence for Skew Prounital Closed Categories

Tarmo Uustalu
Reykjavik University, Reykjavik, Iceland
Tallinn University of Technology, Tallinn, Estonia
tarmo@ru.is

Niccolò Veltri
Tallinn University of Technology, Tallinn, Estonia
niccolo@cs.ioc.ee

Noam Zeilberger
École Polytechnique, Palaiseau, France
noam.zeilberger@lix.polytechnique.fr

In this paper we develop the proof theory of skew prounital closed categories. These are variants of the skew closed categories of Street where the unit is not represented. Skew closed categories in turn are a weakening of the closed categories of Eilenberg and Kelly where no structural law is required to be invertible. The presence of a monoidal structure in these categories is not required. We construct several equivalent presentations of the free skew prounital closed category on a given set of generating objects: a categorical calculus (Hilbert-style system), a cut-free sequent calculus and a natural deduction system corresponding to a variant of planar (= non-commutative linear) typed lambda-calculus. We solve the coherence problem for skew prounital closed categories by showing that the sequent calculus admits focusing and presenting two reduction-free normalization procedures for the natural deduction calculus: normalization by evaluation and hereditary substitutions. Normal natural deduction derivations ($\beta\eta$-long forms) are in one-to-one correspondence with derivations in the focused sequent calculus. Unexpectedly, the free skew prounital closed category on a set satisfies a left-normality condition which makes it lose its skew aspect. This pitfall can be avoided by considering the free skew prounital closed category on a skew multicategory instead. The latter has a presentation as a cut-free sequent calculus for which it is easy to see that the left-normality condition generally fails.

The whole development has been fully formalized in the dependently typed programming language Agda.

1 Introduction

Proof theory and category theory have gone hand in hand since the pioneering works of Lambek [17, 18, 19] and a number of researchers that followed immediately, like Lawvere [20], Szabo [31, 32], Mann [24], Mints [26]. Category theory helps the proof theorist with mathematical models for logical proof systems, which should help tackling problems like analysis of the connections between different types of proof systems, e.g., sequent calculi and natural deduction [41]. On the other hand, proof theory provides the category theorist with a toolbox for identifying the internal language of categories and for solving problems of combinatorial nature such as Mac Lane's coherence problem [22, 14].

Given a certain notion of category with structure, it is natural to ask whether there exists deductive systems (with good proof-theoretic properties) presenting the "canonical" category with that structure. For Cartesian closed categories, such systems are given by, e.g., the sequent calculus of intuitionistic logic and its natural deduction system, which we also know as typed lambda-calculus. For symmetric monoidal closed categories, some such systems are the sequent calculus of intuitionistic linear logic (with

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$\mathsf{I}, \otimes, \multimap$) and the linear variant of typed lambda-calculus [7, 34]. Formally, the "canonical" category with structure arises from a *free* construction. E.g., simply-typed lambda-calculus (with $1, \times, \Rightarrow$) and with atomic types taken from a set At, is a presentation of the free Cartesian monoidal closed category on At.

In recent work, we have investigated the deductive systems associated to *skew monoidal categories* [35, 37]. These are a weakening, first studied by Szlachányi [33], of *monoidal categories* [6, 22] in which the unitors and associator are not required to be invertible, they are merely natural transformations in a particular direction. These categories are not uncommon, e.g., they appear in the study of relative monads [3] and quantum categories [16]. The free skew monoidal category on a set At can be constructed as a sequent calculus with sequents of the form $S \mid \Gamma \longrightarrow C$, where the antecedent is split into an optional formula S, called the *stoup*, and a list of formulae Γ, the *context*. This sequent calculus has some peculiarities: left rules apply only to the formula in the special stoup position, while the tensor right rule forces the formula in the stoup of the conclusion to be the formula in the stoup of the first premise. This sequent calculus enjoys cut elimination and a focused subsystem, defining a root-first proof search strategy attempting to build a derivation of a sequent. The focused calculus finds exactly one representative of each equivalence class of derivations and is thus a concrete presentation of the free skew monoidal category, as such solving the coherence problem for skew monoidal categories.

In this paper, we perform a similar proof-theoretic analysis of *skew prounital closed categories* [36]. These are the skew variant of *prounital closed category* of Shulman [28, Rev. 49], which in turn is a relaxation of the notion of *closed category* by Eilenberg and Kelly [11, 21]. Intuitively, a prounital closed category is a category with an internal hom object $A \multimap B$ for any two objects A and B. There is no requirement for a unit object I, nor for a tensor product $\otimes$. But the unit is implicitly present to a degree thanks to the presence of a functor $J : \mathbb{C} \to \mathbf{Set}$, with JA playing the role of the set of maps from the non-represented unit to the object A. In other words, for a closed category $\mathbb{C}$, we have $JA = \mathbb{C}(\mathsf{I}, A)$. Related categories where the unit is "half-there" appear in the study of categorical models of classical linear logic [12]. Yet weaker are Lambek's *residuated categories* [17] (with one implication) where the unit is completely absent. *Skew closed categories*, a variant of closed categories where no structural law is required to be invertible, were first considered by Street [30].

We present several deductive systems giving different but equivalent presentations of the free skew prounital closed category on a set At: a categorical calculus (Hilbert-style system), a cut-free sequent calculus and a natural deduction calculus. Similarly to the skew monoidal case [35], sequents in sequent calculus have the form $S \mid \Gamma \longrightarrow C$ and the left implication rule only applies to the formula in the stoup position. The natural deduction calculus is, under Curry-Howard correspondence, a variant of planar typed lambda-calculus [1, 39]. Lambda-terms in this calculus have all free and bound variables used exactly once and in the order of their declaration.

We give two equivalent calculi of normal forms: a focused sequent calculus and normal natural deduction derivations, corresponding to canonical representatives of $\beta\eta$-equality. We show three reduction-free (in the sense that we do not use techniques from rewriting theory) normalization procedures: focusing [5], sending a sequent calculus derivation to a focused derivation; normalization by hereditary substitutions [38, 13], sending a natural deduction derivation to a focused derivation; normalization by evaluation [8, 4], sending a natural deduction derivation to a normal natural deduction derivation.

Using our sequent calculus, it is possible to show that the free skew prounital closed category on a set satisfies a *left-normality* condition. The structural law $\widehat{j}$, that we are not asking to be invertible, turns out to be invertible anyway. This degeneracy implies that our sequent calculus and natural deduction calculus admit a stoup-free presentation. In particular, the natural deduction calculus is equivalent, under Curry-Howard correspondence, to (non-skew) planar typed lambda-calculus. From a category-theoretic point of view, there is no reason to construct the free skew prounital closed category on a *set* instead of a more interesting category. We conclude this paper by discussing two equivalent presentations of the free skew prounital closed category on a *skew multicategory* [10] instead of a set: a Hilbert-style calculus and a cut-free sequent calculus. These constructions generalize the free construction on a set and do not generally entail the left-normality condition.

As explained by Shulman [28, Rev. 49], prounital closed categories are the natural notion of category with internal hom as the only required connective (when we do not have/do not want a monoidal structure $\mathsf{I}, \otimes$ in the category), since they form an essential, in the sense of minimal, class of models for planar typed lambda-calculus. This is the case because, formally, they are equivalent to *closed multicategories* [23]. Multicategories are models of deductive systems with only identity and composition as basic operations, while closed multicategories are also able to model implication. Standard approaches to denotational semantics of typed lambda-calculus, adapted to the planar case, would exclude prounital closed categories as valid models. This is because these approaches usually require the presence of a Cartesian monoidal (just monoidal in the planar case) structure complementing the closed structure. We believe there is no good reason to discard models not interpreting the non-existing connectives $\mathsf{I}, \otimes$ and prounital closed categories are the right notion of categorical model for planar typed lambda calculus. Analogously, skew prounital closed categories are the correct notion of category with skew internal hom as the only required connective, since they are equivalent to *closed skew multicategories* [10], a skew variant of closed multicategories.

It is worth mentioning that there is another way of weakening closed categories by simply dropping all references to the unit altogether, that is, by only asking for internal hom objects equipped with the extranatural transformation L and pentagon equation (c5) described below. These may be called *nonunital closed categories,* or *semi-closed categories* after Bourke [9], and are of some interest in providing interpretations for planar lambda terms with no closed subterms (a condition analogous to that of *bridgelessness* in graph theory [39]). We do not treat nonunital closed categories explicitly here, although we expect that our results may be adapted from the prounital to the nonunital case in a straightforward way.

We fully formalized the results presented in the paper in the dependently typed programming language Agda. The formalization uses Agda version 2.6.0. and it is available at `https://github.com/niccoloveltri/skew-prounital-closed-cats`.

2 Skew Prounital Closed Categories

A *skew prounital closed category* [36] is a category $\mathbb{C}$ equipped with functors $J : \mathbb{C} \to \mathbf{Set}$ (the *element set* functor) and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ (the *internal hom* functor) and (extra)natural transformations j, i, L typed

$$j_A \in J(A \multimap A) \qquad i_{A,B} : JA \to \mathbb{C}(A \multimap B, B) \qquad L_{A,B,C} \in \mathbb{C}(B \multimap C, (A \multimap B) \multimap (A \multimap C))$$

satisfying the following equations where we write $\circ_0 : \mathbb{C}(A,B) \times JA \to JB$ for J as a left action, i.e., $f \circ_0 e = J\, f\, e$:

(c1) $e = i_{A,A}\, e \circ_0 j_A \in JA$ for $e \in JA$;

(c2) $i_{A \multimap A, A \multimap C}\, j_A \circ L_{A,A,C} = \mathsf{id}_{A \multimap C} \in \mathbb{C}(A \multimap C, A \multimap C)$;

(c3) $L_{A,B,B} \circ_0 j_B = j_{A \multimap B} \in J((A \multimap B) \multimap (A \multimap B))$;

(c4) $i_{A,B}\, e \multimap C = ((A \multimap B) \multimap i_{A,C}\, e) \circ L_{A,B,C} \in \mathbb{C}(B \multimap C, (A \multimap B) \multimap C)$ for $e \in JA$;

(c5) $(B \multimap C) \multimap L_{A,B,D} \circ L_{B,C,D} = L_{A,B,C} \multimap ((A \multimap B) \multimap (A \multimap D)) \circ L_{A \multimap B, A \multimap C, A \multimap D} \circ L_{A,C,D}$ $\in \mathbb{C}(C \multimap D, (B \multimap C) \multimap ((A \multimap B) \multimap (A \multimap D)))$.

We typically write $\mathbb{C}(-,B)$ for JB. Let S be an *optional object*, i.e., S is either nothing (denoted $S = -$) or it is an object of $\mathbb{C}$. We define $\mathbb{C}(S,B)$ as JB if $S = -$ and as $\mathbb{C}(A,B)$ if $S = A$. The use of this "enhanced" notion of homset with an optional object as domain allows the unification of the two notions of composition $\circ$ and $\circ_0$. We overload the composition symbol $\circ$: given $f \in \mathbb{C}(S,B)$ and $g \in \mathbb{C}(B,C)$, we write $g \circ f \in \mathbb{C}(S,C)$, which is equal to $g \circ_0 f$ when $S = -$ and it is equal to the usual composition of maps $g \circ f$ when $S = A$. With the new notation, the types of structural laws j and i become

$$j_A \in \mathbb{C}(-,A \multimap A) \qquad i_{A,B} : \mathbb{C}(-,A) \to \mathbb{C}(A \multimap B,B)$$

We note that it is not strictly necessary to require that $\multimap$ is a functor $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$. It suffices to require that $A \multimap\ : \mathbb{C} \to \mathbb{C}$ is a functor for every A, since the functorial action $\multimap B$ can for any B be defined from the rest of the structure as $f \multimap B = i_{A \multimap A', A \multimap B}\,((A \multimap f) \circ j_A) \circ L_{A,A',B}$ for $f : A \to A'$ and proved to preserve identity and composition and be natural in B from the equations governing it. Under such an alternative definition of skew prounital closed category, the equations (c4)–(c5) above have to be suitably adjusted and an equation for bifunctoriality of $\multimap$ added.

Skew prounital closed categories differ from Shulman's prounital closed categories in that the derivable map

$$\begin{array}{l} \widehat{j}_{A,B} : \mathbb{C}(A,B) \to \mathbb{C}(-,A \multimap B) \\ \widehat{j}_{A,B}\, f = (A \multimap f) \circ j_A \end{array} \tag{1}$$

is not required to be a natural isomorphism. A skew prounital closed category in which $\widehat{j}$ is invertible is called *left-normal*.

We should note that Shulman's prounital closed categories, although more normal than skew prounital closed categories, are nonetheless partially skew. One could also require *right-normality* and *associative-normality*, corresponding to invertibility of certain derivable maps $\widehat{\imath}$ and $\widehat{L}$ [36]. Eilenberg and Kelly's closed categories are partially skew in that they are not associative-normal.

Example. A simple example of a prounital closed category (adapted from de Schipper [29]) is obtained by taking a suitable full subcategory of the skeletal version $\mathbb{F}$ of the Cartesian monoidal closed category of finite sets and functions (i.e., exactly one set of each finite cardinality). Namely, we keep only cardinalities from $M \subseteq \mathbb{N}$ given inductively by: $3 \in M$ and $n^m \in M$ for all $m,n \in M$. Now $1 \notin M$ and $3 \times 3 = 9 \notin M$, so we have lost the original unit $\mathsf{I} = 1$ and the tensor $p \otimes m = p \times m$, but we still have the internal hom given by $m \multimap n = n^m$. No other candidate unit or tensor can work since we need to have $m \cong \mathsf{I} \multimap m$ and $\mathbb{C}(p \otimes m, n) \cong \mathbb{C}(p, m \multimap n)$. Nevertheless, this category is prounital with the J functor given by the composition of inclusions $M \hookrightarrow \mathbb{F} \hookrightarrow \mathbf{Set}$, and j_m and $i_{m,n}$ defined in the evident way. This category is left-normal, but neither right-normal nor associative-normal.

To skew a closed category, one can use any left-strong monad on it [36]; the same construction works for a prounital closed category (the concept of left-strength of a monad has to be adjusted to this setting). We consider the reader monad given by $Tm = m^k$ for some fixed $k \in M$. The Kleisli category is skew prounital closed with the internal hom defined by $m \multimap^T n = m \multimap Tn = n^{k \times m}$. This category is neither left-normal nor right-normal or associative-normal.

Alternatively, we can begin with $\mathbb{F}$ and take the full subcategory corresponding to $M = \mathbb{N} \setminus \{0,1\}$. This time we get a prounital monoidal closed category. We can skew it as before and we still get a skew prounital nonmonoidal closed category since the Kleisli construction destroys the tensor.

A *strict prounital closed functor* between skew prounital closed categories $\mathbb{C}$ and $\mathbb{D}$ consists of a functor $F : \mathbb{C} \to \mathbb{D}$ such that $F\,(A \multimap B) = F\,A \multimap F\,B$, $F\,(f \multimap g) = F\,f \multimap F\,g$, and the structural laws j, i and L are preserved on the nose. In particular, the functor F is asked to map $\mathbb{C}(S,B)$ to $\mathbb{D}(F\,S, F\,B)$, with

$F\,S = -$ if $S = -$ and $F\,S = F\,A$ if $S = A$, and preserve the enhanced notion of composition $\circ : \mathbb{C}(B,C) \times \mathbb{C}(S,B) \to \mathbb{C}(S,C)$. Skew prounital closed categories and strict prounital closed functors between them form a category.

3 The Free Skew Prounital Closed Category on a Set

We now look at different presentations of the free skew prounital closed category on a set of generating objects. Let us first make explicit the definition of this free construction.

The *free* skew prounital closed category on a set At is a skew prounital closed category $\mathbf{FSkPCl}(\mathsf{At})$ equipped with an inclusion $\iota : \mathsf{At} \to \mathbf{FSkPCl}(\mathsf{At})$, interpreting elements of At as objects of $\mathbf{FSkPCl}(\mathsf{At})$. For any other skew prounital closed category $\mathbb{C}$ with a function $G : \mathsf{At} \to \mathbb{C}$, there must exist a unique strict prounital closed functor $\bar{G} : \mathbf{FSkPCl}(\mathsf{At}) \to \mathbb{C}$ compatible with ι.

The existence of the free skew prounital closed category $\mathbf{FSkPCl}(\mathsf{At})$ entails the existence of a left adjoint to the forgetful functor between the category of skew prounital closed categories and strict prounital closed functors and the category of sets and functions.

3.1 Categorical Calculus

The first presentation consists of a deductive system that we call the *categorical calculus* since it is directly derived from the definition of skew prounital closed category. (We could also think of it as a Hilbert-style calculus of sorts, or under the Curry-Howard correspondence, a combinatory logic.) Objects are *formulae* inductively generated as follows: a formula is either an element X of At (an *atomic* formula) or of the form $A \multimap B$, where A,B are formulae. We write Fma for the set of formulae.

Maps between an optional formula S and a formula C are *derivations* of the sequent $S \Longrightarrow C$, inductively generated by the following inference rules:

$$\frac{}{A \Longrightarrow A}\ \mathsf{id} \qquad \frac{S \Longrightarrow B \quad B \Longrightarrow C}{S \Longrightarrow C}\ \mathsf{comp} \qquad \frac{C \Longrightarrow A \quad B \Longrightarrow D}{A \multimap B \Longrightarrow C \multimap D}\ \multimap$$
$$\frac{}{\Longrightarrow A \multimap A}\ j \qquad \frac{\Longrightarrow A}{A \multimap B \Longrightarrow B}\ i \qquad \frac{}{B \multimap C \Longrightarrow (A \multimap B) \multimap (A \multimap C)}\ L \tag{2}$$

Derivations are *identified* up to a congruence relation $\doteq$ that is inductively generated by the following pairs of derivations:

$$\begin{array}{ll}
\text{(category laws)} & \mathsf{id} \circ f \doteq f \qquad f \doteq f \circ \mathsf{id} \qquad (f \circ g) \circ h \doteq f \circ (g \circ h) \\
(\multimap\ \text{functorial}) & \mathsf{id} \multimap \mathsf{id} \doteq \mathsf{id} \qquad (f \circ h) \multimap (k \circ g) \doteq (h \multimap k) \circ (f \multimap g) \\
 & f \multimap \mathsf{id} \circ j \doteq \mathsf{id} \multimap f \circ j \\
(j,i,L\ \text{(extra)nat. trans.}) & g \circ i(e) \circ h \multimap \mathsf{id} \doteq i(h \circ e) \circ \mathsf{id} \multimap g \\
 & (f \multimap g) \multimap (\mathsf{id} \multimap h) \circ L \doteq \mathsf{id} \multimap (f \multimap \mathsf{id}) \circ L \circ g \multimap h \\
 & i(e) \circ j \doteq e \qquad i(j) \circ L \doteq \mathsf{id} \\
\text{(c1-c5)} & L \circ j \doteq j \qquad \mathsf{id} \multimap i(e) \circ L \doteq i(e) \multimap \mathsf{id} \\
 & \mathsf{id} \multimap L \circ L \doteq L \multimap \mathsf{id} \circ L \circ L
\end{array} \tag{3}$$

In the term notation for derivations, we write $g \circ f$ for $\mathsf{comp}\, f\, g$ to agree with the standard categorical notation.

The categorical calculus defines the free skew prounital closed category on At in a straightforward way. Given another skew prounital closed category $\mathbb{C}$ with function $G : \mathsf{At} \to \mathbb{C}$, we can easily define mappings $\bar{G}_0 : \mathsf{Fma} \to \mathbb{C}_0$ and $\bar{G}_1 : S \Longrightarrow C \to \mathbb{C}(\bar{G}_0(S), \bar{G}_0(C))$ by induction. These specify a strict prounital closed functor, in fact the only existing one satisfying $\bar{G}_0(X) = G(X)$.

3.2 Cut-Free Sequent Calculus

The second presentation of **FSkPCl**(At) is a sequent calculus. Sequents are triples of the form $S \mid \Gamma \longrightarrow C$. The succedent C is a formula in Fma. The antecedent is split in two parts: the *stoup* S is an optional formula, i.e. it is either empty or it is a single formula; the *context* Γ is a list of formulae.

Derivations in the sequent calculus are inductively generated by the following inference rules:

$$\frac{A \mid \Gamma \longrightarrow C}{- \mid A,\Gamma \longrightarrow C}\ \mathsf{pass} \qquad \frac{S \mid \Gamma, A \longrightarrow B}{S \mid \Gamma \longrightarrow A \multimap B}\ {\multimap}\mathsf{R}$$
$$\frac{}{A \mid\ \longrightarrow A}\ \mathsf{ax} \qquad \frac{- \mid \Gamma \longrightarrow A \quad B \mid \Delta \longrightarrow C}{A \multimap B \mid \Gamma, \Delta \longrightarrow C}\ {\multimap}\mathsf{L} \tag{4}$$

(pass for 'passivate', L, R for introduction on the left (in the stoup) resp. right) and identified up to the congruence $\circeq$ induced by the equations:

$$(\eta\text{-conversion}) \quad \mathsf{ax}_{A \multimap B} \circeq {\multimap}\mathsf{R}\ ({\multimap}\mathsf{L}\ (\mathsf{pass}\ \mathsf{ax}_A, \mathsf{ax}_B))$$

$$\begin{array}{ll} \text{(commutative conversions)} & \\ \mathsf{pass}\ ({\multimap}\mathsf{R}\ f) \circeq {\multimap}\mathsf{R}\ (\mathsf{pass}\ f) & (\text{for } f : A' \mid \Gamma, A \longrightarrow B) \\ {\multimap}\mathsf{L}\ (f, {\multimap}\mathsf{R}\ g) \circeq {\multimap}\mathsf{R}\ ({\multimap}\mathsf{L}\ (f, g)) & (\text{for } f : - \mid \Gamma \longrightarrow A',\ g : B' \mid \Delta, A \longrightarrow B) \end{array} \tag{5}$$

There are no primitive cut rules in this sequent calculus, but two forms of cut are admissible:

$$\frac{S \mid \Gamma \longrightarrow A \quad A \mid \Delta \longrightarrow C}{S \mid \Gamma, \Delta \longrightarrow C}\ \mathsf{scut} \qquad \frac{- \mid \Gamma \longrightarrow A \quad S \mid \Delta_0, A, \Delta_1 \longrightarrow C}{S \mid \Delta_0, \Gamma, \Delta_1 \longrightarrow C}\ \mathsf{ccut} \tag{6}$$

Notice that the left rule ${\multimap}$L acts only on the implication $A \multimap B$ in the stoup. Another left rule ${\multimap}$C acting on implication in the passive context is derivable from cut:

$$\frac{- \mid \Gamma \xrightarrow{f} A \quad S \mid \Delta_0, B, \Delta_1 \xrightarrow{g} C}{S \mid \Delta_0, A \multimap B, \Gamma, \Delta_1 \longrightarrow C}\ {\multimap}\mathsf{C} \quad = \quad \frac{\dfrac{\dfrac{- \mid \Gamma \xrightarrow{f} A \quad \dfrac{}{B \mid\ \longrightarrow B}\ \mathsf{ax}}{A \multimap B \mid \Gamma \longrightarrow B}\ {\multimap}\mathsf{L}}{- \mid A \multimap B, \Gamma \longrightarrow B}\ \mathsf{pass} \quad S \mid \Delta_0, B, \Delta_1 \xrightarrow{g} C}{S \mid \Delta_0, A \multimap B, \Gamma, \Delta_1 \longrightarrow C}\ \mathsf{ccut} \tag{7}$$

Soundness. Sequent calculus derivations can be turned into categorical calculus derivations using a function $\mathsf{sound} : (S \mid \Gamma \longrightarrow C) \to (S \Longrightarrow [\![\Gamma|C]\!])$, where the formula $[\![\Gamma|C]\!]$ is inductively defined as

$$[\![\ |C]\!] = C \qquad\qquad [\![A,\Gamma|C]\!] = A \multimap [\![\Gamma|C]\!]$$

Given $f : S \mid \Gamma, A \longrightarrow B$, define $\mathsf{sound}({\multimap}\mathsf{R}\ f)$ simply as $\mathsf{sound}(f)$. Given $f : A \mid \Gamma \longrightarrow C$, define $\mathsf{sound}(\mathsf{pass}\ f)$ as

$$\frac{\dfrac{\dfrac{}{\Longrightarrow A \multimap A}\ j \quad \dfrac{\dfrac{}{A \Longrightarrow A}\ \mathsf{id} \quad A \overset{\mathsf{sound}(f)}{\Longrightarrow} [\![\Gamma|C]\!]}{A \multimap A \Longrightarrow A \multimap [\![\Gamma|C]\!]}\ \multimap}{\Longrightarrow A \multimap [\![\Gamma|C]\!]}\ \mathsf{comp}}{\Longrightarrow [\![A,\Gamma|C]\!]}$$

The double-line rule corresponds to the application of the equality $[\![A,\Delta|C]\!] = A \multimap [\![\Delta|C]\!]$. Given $f : - \mid \Gamma \longrightarrow A$ and $g : B \mid \Delta \longrightarrow C$, define $\mathsf{sound}({\multimap}\mathsf{L}(f,g))$ as

$$\frac{\dfrac{\dfrac{}{A \Longrightarrow A}\ \mathsf{id} \quad B \overset{\mathsf{sound}(g)}{\Longrightarrow} [\![\Delta|C]\!]}{A \multimap B \Longrightarrow A \multimap [\![\Delta|C]\!]}\ \multimap \quad \dfrac{\dfrac{}{A \multimap [\![\Delta|C]\!] \Longrightarrow [\![\Gamma|A]\!] \multimap [\![\Gamma,\Delta|C]\!]}\ L^\star \quad \dfrac{\overset{\mathsf{sound}(f)}{\Longrightarrow} [\![\Gamma|A]\!]}{[\![\Gamma|A]\!] \multimap [\![\Gamma,\Delta|C]\!] \Longrightarrow [\![\Gamma,\Delta|C]\!]}\ i}{A \multimap [\![\Delta|C]\!] \Longrightarrow [\![\Gamma,\Delta|C]\!]}\ \mathsf{comp}}{A \multimap B \Longrightarrow [\![\Gamma,\Delta|C]\!]}\ \mathsf{comp}$$

where the operation $L^\star$, defined by induction on Γ, performs iterated applications of the structural law L. The function sound is well-defined, in the sense that it sends $\circeq$-equivalent derivations to $\doteq$-related derivations.

Completeness. Derivations in the categorical calculus can be turned into sequent calculus derivations via a function $\mathsf{cmplt} : (S \Longrightarrow [\![\Gamma|C]\!]) \to (S \mid \Gamma \longrightarrow C)$. The ax rule models the identity map, while sequential composition is interpreted using scut. Functoriality of $\multimap$ is modelled using $\multimap\mathsf{R}$ and $\multimap\mathsf{L}$. The function cmplt sends the structural laws j, i, L of skew prounital closed categories to the following derivations in the sequent calculus:

$$(j) \qquad \dfrac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\,\mathsf{ax}}{- \mid A \longrightarrow A}\,\mathsf{pass}}{- \mid \ \longrightarrow A \multimap A}\,{\multimap}\mathsf{R}$$

$$(i) \qquad \dfrac{- \mid \ \longrightarrow A \quad \dfrac{}{B \mid \ \longrightarrow B}\,\mathsf{ax}}{A \multimap B \mid \ \longrightarrow B}\,{\multimap}\mathsf{L}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\,\mathsf{ax}}{- \mid A \longrightarrow A}\,\mathsf{pass} \quad \dfrac{}{B \mid \ \longrightarrow B}\,\mathsf{ax}}{A \multimap B \mid A \longrightarrow B}\,{\multimap}\mathsf{L}}{- \mid A \multimap B, A \longrightarrow B}\,\mathsf{pass} \quad \dfrac{}{C \mid \ \longrightarrow C}\,\mathsf{ax}}{B \multimap C \mid A \multimap B, A \longrightarrow C}\,{\multimap}\mathsf{L}}{B \multimap C \mid A \multimap B \longrightarrow A \multimap C}\,{\multimap}\mathsf{R}}{B \multimap C \mid \ \longrightarrow (A \multimap B) \multimap (A \multimap C)}\,{\multimap}\mathsf{R} \qquad (L)$$

The function cmplt is well-defined, in the sense that it sends $\doteq$-equivalent derivations to $\circeq$-related derivations. Moreover it is possible to prove that cmplt is the inverse of sound (up to the equivalence relations $\doteq$ and $\circeq$). This shows that the sequent calculus is a presentation of the free skew prounital closed category $\mathbf{FSkPCl}(\mathsf{At})$.

3.3 Natural Deduction Calculus

The third presentation of $\mathbf{FSkPCl}(\mathsf{At})$ is a natural deduction calculus. Sequents are triples $S \mid \Gamma \longrightarrow_{\mathsf{nd}} C$ as in the sequent calculus of Section 3.2, but the left rule $\multimap\mathsf{L}$ for $\multimap$ is replaced by the elimination rule $\multimap\mathsf{e}$.

$$\dfrac{A \mid \Gamma \longrightarrow_{\mathsf{nd}} C}{- \mid A, \Gamma \longrightarrow_{\mathsf{nd}} C}\,\mathsf{pass} \qquad \dfrac{S \mid \Gamma, A \longrightarrow_{\mathsf{nd}} B}{S \mid \Gamma \longrightarrow_{\mathsf{nd}} A \multimap B}\,{\multimap}\mathsf{i}$$

$$\dfrac{}{A \mid \ \longrightarrow_{\mathsf{nd}} A}\,\mathsf{ax} \qquad \dfrac{S \mid \Gamma \longrightarrow_{\mathsf{nd}} A \multimap B \quad - \mid \Delta \longrightarrow_{\mathsf{nd}} A}{S \mid \Gamma, \Delta \longrightarrow_{\mathsf{nd}} B}\,{\multimap}\mathsf{e}$$

The two cut rules in (6) are admissible also in the natural deduction calculus. Derivations are identified by the congruence relation $\circeq_{\mathsf{nd}}$, a skew ordered variant of the usual $\beta\eta$-equivalence of simply typed lambda-calculus, induced by the equations

$$\begin{array}{lll}
(\beta\text{-conversion}) & {\multimap}\mathsf{e}\ ({\multimap}\mathsf{i}\ f, g) \circeq_{\mathsf{nd}} \mathsf{ccut}\ (g, f) & (\text{for } f : S \mid \Gamma, A \longrightarrow_{\mathsf{nd}} B,\ g : - \mid \Delta \longrightarrow_{\mathsf{nd}} A) \\
(\eta\text{-conversion}) & f \circeq_{\mathsf{nd}} {\multimap}\mathsf{i}\ ({\multimap}\mathsf{e}\ (f, \mathsf{pass}\ \mathsf{ax})) & (\text{for } f : S \mid \Gamma \longrightarrow_{\mathsf{nd}} A \multimap B) \\
(\text{commutative conversions}) & & \\
 & \mathsf{pass}\ ({\multimap}\mathsf{i}\ f) \circeq_{\mathsf{nd}} {\multimap}\mathsf{i}\ (\mathsf{pass}\ f) & (\text{for } f : A' \mid \Gamma, A \longrightarrow_{\mathsf{nd}} B) \\
 & \mathsf{pass}\ ({\multimap}\mathsf{e}\ (f, g)) \circeq_{\mathsf{nd}} {\multimap}\mathsf{e}\ (\mathsf{pass}\ f, g) & (\text{for } f : A' \mid \Gamma \longrightarrow_{\mathsf{nd}} A \multimap B,\ g : - \mid \Delta \longrightarrow_{\mathsf{nd}} A)
\end{array} \qquad (8)$$

This natural deduction calculus corresponds to a variant of the planar fragment of linear typed lambda-calculus [1, 39]. The formulae correspond to types. The derivations correspond to lambda terms in which all free and bound variables are used exactly once and in the order of their declaration. The equations axiomatize the appropriate variant of $\beta\eta$-equivalence.

It is possible to prove that the natural deduction calculus is equivalent to the sequent calculus (up to the equivalences of derivations $\circeq$ and $\circeq_{\mathsf{nd}}$), and it is therefore a presentation of **FSkPCl**(At). We do not follow this strategy here. Instead we construct reduction-free normalization procedures for the sequent calculus and the natural deduction calculus. The procedures target two calculi of normal forms: a focused subsystem of the sequent calculus and a calculus of $\beta\eta$-long normal forms wrt. $\circeq_{\mathsf{nd}}$. By showing that the calculi of normal forms are equivalent, we conclude that the sequent calculus and the natural deduction calculus are also equivalent up to the equivalences of derivations $\circeq$ and $\circeq_{\mathsf{nd}}$.

3.4 Focused Sequent Calculus Derivations

The congruence relation $\circeq$ on sequent calculus derivations can be considered as a term rewrite system, by directing every equation in (5) from left to right. The resulting rewrite system is weakly confluent and strongly normalizing, hence confluent with unique normal forms.

Derivations in normal form wrt. $\circeq$ can be described by a suitable *focused* subcalculus of the full sequent calculus, following the paradigm introduced by Andreoli [5]. Derivations in this subcalculus are inductively generated by the following inference rules:

$$
\begin{array}{c|c|c}
\dfrac{S \mid \Gamma, A \longrightarrow_{\mathsf{I}} B}{S \mid \Gamma \longrightarrow_{\mathsf{I}} A \multimap B}\ {\multimap}\mathsf{R}
&
\dfrac{A \mid \Gamma \longrightarrow_{\mathsf{P}} C}{- \mid A, \Gamma \longrightarrow_{\mathsf{P}} C}\ \mathsf{pass}
&
\dfrac{}{A \mid \ \longrightarrow_{\mathsf{F}} A}\ \mathsf{ax}
\\[2ex]
\dfrac{S \mid \Gamma \longrightarrow_{\mathsf{P}} X}{S \mid \Gamma \longrightarrow_{\mathsf{I}} X}\ \mathsf{P2I}
&
\dfrac{A \mid \Gamma \longrightarrow_{\mathsf{F}} C}{A \mid \Gamma \longrightarrow_{\mathsf{P}} C}\ \mathsf{F2P}
&
\dfrac{- \mid \Gamma \longrightarrow_{\mathsf{I}} A \quad B \mid \Delta \longrightarrow_{\mathsf{F}} C}{A \multimap B \mid \Gamma, \Delta \longrightarrow_{\mathsf{F}} C}\ {\multimap}\mathsf{L}
\end{array}
\tag{9}
$$

This is a sequent calculus with an additional phase annotation on sequents, for controlling root-first proof search. In phase I (for *inversion*), sequents have the form $S \mid \Gamma \longrightarrow_{\mathsf{I}} C$, where S is a general stoup and C is a general formula. During this phase, we eagerly apply the invertible rule ${\multimap}\mathsf{R}$ until the succedent is reduced to an atomic formula. In phase P (for *passivation*), we have the opportunity of applying the pass rule and can only go to the last phase F (for *focusing*) when the stoup has become a formula. During this phase we can finish the derivation using ax, which is now restricted to atomic formulae, or apply the ${\multimap}\mathsf{L}$ rule. If we apply the ${\multimap}\mathsf{L}$ rule, we are thrown back to the I phase in the first premise. One can observe that, in any I-derivation, the succedent of any P- or F-sequent must actually be an atom, but the generality of allowing any formula in the succedent in these phases (which we can have in derivations of P- or F-sequents) will be useful for us shortly in the discussion of hereditary substitutions below and also in Section 3.5 where we will relate focused sequent calculus derivations to normal natural deduction calculus derivations.

Focusing. The focused rules define a sound and complete root-first proof search strategy for the cut-free sequent calculus of Section 3.2. Soundness of the focused calculus is evident: focused derivations can be embedded into sequent calculus derivations via functions $\mathsf{emb}_k : (S \mid \Gamma \longrightarrow_k C) \to (S \mid \Gamma \longrightarrow C)$ for all phases $k \in \{\mathsf{I}, \mathsf{P}, \mathsf{F}\}$ that just erase all phase annotations and uses of the rules $\mathsf{P2I}$ and $\mathsf{F2P}$.

By the normalization property of the rewrite system associated to $\circeq$, we know that the focused calculus is also complete. This can also be established by constructing a reduction-free normalization function $\mathsf{focus} : (S \mid \Gamma \longrightarrow C) \to (S \mid \Gamma \longrightarrow_{\mathsf{I}} C)$ sending each derivation in the sequent calculus to a canonical representative of its $\circeq$-equivalence class in the focused calculus. This means in particular that focus maps $\circeq$-related derivations to equal focused derivations. For the definition of focus, we show that general pass, $\multimap$L and ax rules are admissible in phase I:

$$\frac{A \mid \Gamma \longrightarrow_{\mathsf{I}} C}{- \mid A,\Gamma \longrightarrow_{\mathsf{I}} C}\ \mathsf{pass}^{\mathsf{I}} \qquad \frac{- \mid \Gamma \longrightarrow_{\mathsf{I}} A \quad B \mid \Delta \longrightarrow_{\mathsf{I}} C}{A \multimap B \mid \Gamma,\Delta \longrightarrow_{\mathsf{I}} C}\ {\multimap}\mathsf{L}^{\mathsf{I}} \qquad \frac{}{A \mid \quad \longrightarrow_{\mathsf{I}} A}\ \mathsf{ax}^{\mathsf{I}}$$

This makes each sequent calculus inference rule matched by a focused calculus rule. Then the normalization procedure focus can be easily defined by induction on the input derivation. The function focus is the inverse of $\mathsf{emb}_{\mathsf{I}}$ up to $\circeq$: given a sequent calculus derivation $f : S \mid \Gamma \longrightarrow C$, we have $\mathsf{emb}_{\mathsf{I}}\ (\mathsf{focus}\ f) \circeq f$; given a focused derivation $f : S \mid \Gamma \longrightarrow_{\mathsf{I}} C$, we have $\mathsf{focus}\ (\mathsf{emb}_{\mathsf{I}}\ f) = f$.

The focused calculus solves the *coherence problem* for skew prounital closed categories, in the sense of giving an explicit characterization of the homsets of **FSkPCl**(At). It also solves two related algorithmic problems effectively:

- Duplicate-free enumeration of all maps $S \Longrightarrow C$ in the form of representatives of $\doteq$-equivalence classes of categorical calculus derivations: For this, find all focused derivations of $S \mid \quad \longrightarrow_{\mathsf{I}} C$, which is solvable by exhaustive proof search, which terminates, and translate them to the categorical calculus derivations.
- Finding whether two given maps of type $S \Longrightarrow C$, presented as categorical calculus derivations, are equal, i.e., $\doteq$-related as derivations: For this, translate them to focused derivations of $S \mid \quad \longrightarrow_{\mathsf{I}} C$ and check whether they are equal, which is decidable.

Monoidal and nonmonoidal closed categories, and prounital closed categories likewise, admit no simple coherence theorem like Mac Lane's for monoidal categories [22] (depending on an easy condition on the domain and codomain, no maps or just one in a homset). Enumeration of (presentations of) maps and equality checking are nontrivial [15, 21, 26, 32]. In our focused calculus, the sequent $(X \multimap Y) \multimap (X \multimap Z) \mid X \multimap Y, X \multimap X, X \longrightarrow_{\mathsf{I}} Z$ has two distinct focused derivations (we learned this example from Anupam Das).

Hereditary Substitutions. Focused sequent calculus derivations can also be used as normal forms for the natural deduction calculus of Section 3.3. We show this by describing a reduction-free normalization procedure that is typically called *normalization by hereditary substitution* [38, 13]. Normal forms for this procedure (at least in the case of simply-typed lambda-calculus) are typically defined in terms of a suitable spine calculus, but we have defined our focused sequent calculus liberally enough to serve this purpose.

The focused calculus defines a sound and complete root-first proof search strategy for the natural deduction calculus of Section 3.3. Focused derivations can easily be embedded into natural deduction derivations: there are functions $\mathsf{emb}^{\mathsf{nd}}_k : (S \mid \Gamma \longrightarrow_k C) \to (S \mid \Gamma \longrightarrow_{\mathsf{nd}} C)$ for all $k \in \{\mathsf{I}, \mathsf{P}, \mathsf{F}\}$.

Similar to focusing completeness, normalization by hereditary substitution is also specified in two steps. First we need to show that ax and $\multimap$e rules are admissible in phase I:

$$\frac{}{A \mid \quad \longrightarrow_{\mathsf{I}} A}\ \mathsf{ax}^{\mathsf{I}} \qquad \frac{S \mid \Gamma \longrightarrow_{\mathsf{I}} A \multimap B \quad - \mid \Delta \longrightarrow_{\mathsf{I}} A}{S \mid \Gamma,\Delta \longrightarrow_{\mathsf{I}} B}\ {\multimap}\mathsf{e}^{\mathsf{I}}$$

(We already know that a general pass rule is admissible in phase I[1].) Focused derivations in phase I should correspond to normal forms, i.e., canonical representatives of $\circeq_{\mathsf{nd}}$-equivalence classes. In particular,

[1] We also already know from focusing that ax^{I} is admissible, but it is defined in terms of ${\multimap}\mathsf{L}^{\mathsf{I}}$. For normalization by hereditary substitutions, we define ax^{I} differently, avoiding the use of ${\multimap}\mathsf{L}^{\mathsf{I}}$ since $\multimap$L is not a primitive rule of the natural deduction calculus.

they should not contain any redex. This forces the rule $\multimap\mathsf{e}^{\mathsf{l}}$ to be simultaneously defined with 3 pairs of substitution rules (i.e., cut rules) in the focused calculus, one for each phase $k \in \{\mathsf{I},\mathsf{P},\mathsf{F}\}$:

$$\frac{S \mid \Gamma \longrightarrow_k A \quad A \mid \Delta \longrightarrow_k C}{S \mid \Gamma,\Delta \longrightarrow_k C}\ \mathsf{scut}^k \qquad \frac{- \mid \Gamma \longrightarrow_k A \quad S \mid \Delta_0,A,\Delta_1 \longrightarrow_k C}{S \mid \Delta_0,\Gamma,\Delta_1 \longrightarrow_k C}\ \mathsf{ccut}^k$$

The rule $\multimap\mathsf{e}^{\mathsf{l}}$ can then be defined as dictated by the β-conversion equation in the definition of $\stackrel{\circ}{=}_{\mathsf{nd}}$ as

$$\frac{\dfrac{S \mid \Gamma,A \xrightarrow{f}_{\mathsf{I}} B}{S \mid \Gamma \longrightarrow_{\mathsf{I}} A \multimap B}\ \multimap\mathsf{i} \quad - \mid \Delta \xrightarrow{g}_{\mathsf{I}} A}{S \mid \Gamma,\Delta \longrightarrow_{\mathsf{I}} B}\ \multimap\mathsf{e}^{\mathsf{l}} \quad = \quad \frac{- \mid \Delta \xrightarrow{g}_{\mathsf{I}} A \quad S \mid \Gamma,A \xrightarrow{f}_{\mathsf{I}} B}{S \mid \Gamma,\Delta \longrightarrow_{\mathsf{I}} B}\ \mathsf{ccut}^{\mathsf{l}}$$

Notice that the first premise is forced to be of the form $\multimap\mathsf{i}\ f$. This simultaneous substitution in canonical forms and reduction of redexes that appear from substitution is the main idea behind hereditary substitutions. We can then construct a normalization function $\mathsf{hered} : (S \mid \Gamma \longrightarrow_{\mathsf{nd}} C) \to (S \mid \Gamma \longrightarrow_{\mathsf{I}} C)$ sending each primitive rule of the natural deduction calculus to its admissible counterpart in the focused calculus. In particular, the function hered maps each natural deduction derivation to its normal form as rendered in the focused calculus (which is our spine calculus).

The function hered is the inverse of $\mathsf{emb}^{\mathsf{nd}}_{\mathsf{I}}$ up to $\stackrel{\circ}{=}_{\mathsf{nd}}$: given a natural deduction derivation $f : S \mid \Gamma \longrightarrow_{\mathsf{nd}} C$, we have $\mathsf{emb}^{\mathsf{nd}}_{\mathsf{I}}\ (\mathsf{hered}\ f) \stackrel{\circ}{=}_{\mathsf{nd}} f$; given a focused derivation $f : S \mid \Gamma \longrightarrow_{\mathsf{I}} C$, we get $\mathsf{hered}\ (\mathsf{emb}^{\mathsf{nd}}_{\mathsf{I}}\ f) = f$.

3.5 Normal Natural Deduction Derivations

The congruence relation $\stackrel{\circ}{=}_{\mathsf{nd}}$ on natural deduction calculus derivations also has normal forms, which correspond precisely to $\beta\eta$-long normal forms in the familiar terminology of lambda-calculus.

Derivations in normal form wrt. $\stackrel{\circ}{=}_{\mathsf{nd}}$ can be described by a suitable subcalculus of the full natural deduction calculus. Derivations in this subcalculus are inductively generated by the following inference rules:

$$\frac{S \mid \Gamma,A \longrightarrow_{\mathsf{nf}} B}{S \mid \Gamma \longrightarrow_{\mathsf{nf}} A \multimap B}\ \multimap\mathsf{i} \qquad \frac{A \mid \Gamma \longrightarrow_{\mathsf{p}} C}{- \mid A,\Gamma \longrightarrow_{\mathsf{p}} C}\ \mathsf{pass} \qquad \frac{}{A \mid \ \longrightarrow_{\mathsf{ne}} A}\ \mathsf{ax}$$

$$\frac{S \mid \Gamma \longrightarrow_{\mathsf{p}} X}{S \mid \Gamma \longrightarrow_{\mathsf{nf}} X}\ \mathsf{p2nf} \qquad \frac{A \mid \Gamma \longrightarrow_{\mathsf{ne}} C}{A \mid \Gamma \longrightarrow_{\mathsf{p}} C}\ \mathsf{ne2p} \qquad \frac{A' \mid \Gamma \longrightarrow_{\mathsf{ne}} A \multimap B \quad - \mid \Delta \longrightarrow_{\mathsf{nf}} A}{A' \mid \Gamma,\Delta \longrightarrow_{\mathsf{ne}} B}\ \multimap\mathsf{e}$$

Derivations are organized in an introduction phase and in an elimination phase [27]. In lambda-calculus jargon, we refer to derivations in these phases as (pure) *normal forms* and *neutrals*. Normal forms are derivations of sequents of the general form $S \mid \Gamma \longrightarrow_{\mathsf{nf}} C$. Similarly to the case of simply-typed lambda-calculus, a normal form is an iteration of λ-abstraction on a neutral term of an atomic type. Neutrals are derivations of sequents of the form $A \mid \Gamma \longrightarrow_{\mathsf{ne}} C$ where the stoup is required to be a formula. Intuitively, they correspond to terms which are stuck for $\stackrel{\circ}{=}_{\mathsf{nd}}$-conversion. A neutral is either a variable (declared in the stoup, in our case) or a function application that cannot compute due to the presence of another neutral in the function position. Due to the skew aspect of our natural deduction calculus, we also add an intermediate third phase p, with sequents of the form $A \mid \Gamma \longrightarrow_{\mathsf{p}} C$, in which we have the choice of applying the structural rule pass. This is analogous to the passivation phase of the focused sequent calculus of Section 3.4.

The normal natural deduction calculus defines a root-first proof search strategy for the natural deduction calculus. This procedure is sound. Normal forms can easily be embedded into natural deduction derivations: there are functions $\mathsf{emb}^{\mathsf{nd}}_k : (S \mid \Gamma \longrightarrow_k C) \to (S \mid \Gamma \longrightarrow_{\mathsf{nd}} C)$ for all $k \in \{\mathsf{nf},\mathsf{p},\mathsf{ne}\}$.

Normalization by Evaluation. The completeness of the normal natural deduction calculus, implying that normal natural deduction derivations are indeed $\beta\eta$-long normal forms, is proved via *normalization by evaluation* [8, 4]. This is a reduction-free procedure in which terms are first evaluated into a certain semantic domain, and their evaluations are then reified back into normal forms.

We begin by constructing two discrete categories: **Cxt** and **SCxt**. The category **Cxt** has lists of formulae as objects. It has a strict monoidal structure with the empty list as unit and concatenation of lists as tensor. The category **SCxt** has objects of the form $S \mid \Gamma$, with S an optional formula and Γ a list of formulae. It has a unit object $- \mid$ (in which both components are empty) and there exists an action of the monoidal category **Cxt** on **SCxt**: $(S \mid \Gamma) \cdot \Gamma' = S \mid \Gamma, \Gamma'$. There is a functor $E : \mathbf{Cxt} \to \mathbf{SCxt}$, sending each list Γ to the pair $- \mid \Gamma$.

We consider the two presheaf categories $\mathbf{Set}^{\mathbf{Cxt}}$ and $\mathbf{Set}^{\mathbf{SCxt}}$. The monoidal structure on **Cxt** lifts to the *Day convolution* monoidal closed structure on $\mathbf{Set}^{\mathbf{Cxt}}$:

$$\mathsf{I}_{\mathsf{cxt}}\,\Gamma = (\Gamma = ()) \qquad (P \otimes_{\mathsf{cxt}} Q)\,\Gamma = \textstyle\sum_{\Gamma_0,\Gamma_1} (\Gamma = \Gamma_0, \Gamma_1) \times P\,\Gamma_0 \times Q\,\Gamma_1$$
$$(Q \multimap_{\mathsf{cxt}} P)\,\Gamma = \textstyle\prod_{\Delta}\, Q\,\Delta \to P\,(\Gamma, \Delta)$$

(Here and below $()$ denotes the empty list.)

The unit of **SCxt** lifts to a unit in $\mathbf{Set}^{\mathbf{SCxt}}$ given by $\mathsf{I}_{\mathsf{scxt}}\,(S \mid \Gamma) = (\Gamma = ()) \times (S = -)$. The action of **Cxt** on **SCxt** lifts to an action of $\mathbf{Set}^{\mathbf{Cxt}}$ on $\mathbf{Set}^{\mathbf{SCxt}}$:

$$(P \otimes_{\mathsf{scxt}} Q)\,(S \mid \Gamma) = \textstyle\sum_{\Gamma_0,\Gamma_1} (\Gamma = \Gamma_0, \Gamma_1) \times P\,(S \mid \Gamma_0) \times Q\,\Gamma_1$$

The functor $\otimes_{\mathsf{scxt}} Q$ has a right adjoint $Q \multimap_{\mathsf{scxt}}$ given by:

$$(Q \multimap_{\mathsf{scxt}} P)\,(S \mid \Gamma) = \textstyle\prod_{\Delta}\, Q\,\Delta \to P\,(S \mid \Gamma, \Delta)$$

The first step of normalization by evaluation is the interpretation of syntactic constructs, i.e. formulae and natural deduction derivations, as semantic entities in $\mathbf{Set}^{\mathbf{SCxt}}$. Formulae are modelled as presheaves over **SCxt**. Implication is modelled via the functor $\multimap_{\mathsf{scxt}}$. Notice the composition with the functor E, which is needed for the interpretation to be well-defined. The interpretation of an atomic formula X on an object $S \mid \Gamma$ is the set of normal forms of type $S \mid \Gamma \longrightarrow_{\mathsf{nf}} X$.

$$\{\!\{X\}\!\}\,(S \mid \Gamma) = S \mid \Gamma \longrightarrow_{\mathsf{nf}} X \qquad \{\!\{A \multimap B\}\!\}\,(S \mid \Gamma) = ((\{\!\{A\}\!\} \circ E) \multimap_{\mathsf{scxt}} \{\!\{B\}\!\})\,(S \mid \Gamma)$$

Lists of formulae can be interpreted as presheaves over **Cxt**.

$$\{\!\{\ \}\!\}\,\Delta = \mathsf{I}_{\mathsf{cxt}}\,\Delta \qquad \{\!\{A, \Gamma\}\!\}\,\Delta = ((\{\!\{A\}\!\} \circ E) \otimes_{\mathsf{cxt}} \{\!\{\Gamma\}\!\})\,\Delta$$

Finally, antecedents $S \mid \Gamma$ can be interpreted as presheaves over **SCxt**:

$$\{\!\{- \mid \Gamma\}\!\}\,(S \mid \Delta) = (\mathsf{I}_{\mathsf{scxt}} \otimes_{\mathsf{scxt}} \{\!\{\Gamma\}\!\})\,(S \mid \Delta) = (S = -) \times \{\!\{\Gamma\}\!\}\,\Delta$$
$$\{\!\{A \mid \Gamma\}\!\}\,(S \mid \Delta) = (\{\!\{A\}\!\} \otimes_{\mathsf{scxt}} \{\!\{\Gamma\}\!\})\,(S \mid \Delta)$$

The next step of normalization by evaluation is the interpretation of a derivation $f : S \mid \Gamma \longrightarrow_{\mathsf{nd}} C$ in the natural deduction calculus as a natural transformation between presheaves $\{\!\{S \mid \Gamma\}\!\}$ and $\{\!\{C\}\!\}$. Formally, we define an evaluation function by induction on the input derivation:

$$\mathsf{eval} : (S \mid \Gamma \longrightarrow_{\mathsf{nd}} C) \to \{\!\{S \mid \Gamma\}\!\}\,(S' \mid \Delta) \to \{\!\{C\}\!\}\,(S' \mid \Delta)$$

Subsequently, we extract a normal form from the evaluated term. The reification procedure sends a semantic element in $\{\!\{A\}\!\}\ (S \mid \Gamma)$ to a normal form in $S \mid \Gamma \longrightarrow_{\mathsf{nf}} A$. The latter is defined by mutual induction with a function reflecting neutrals in $A \mid \Gamma \longrightarrow_{\mathsf{ne}} C$ to semantic elements in $\{\!\{C\}\!\}\ (A \mid \Gamma)$.

$$\mathsf{reflect} : (A \mid \Gamma \longrightarrow_{\mathsf{ne}} C) \to \{\!\{C\}\!\}\ (A \mid \Gamma) \qquad \mathsf{reify} : \{\!\{A\}\!\}\ (S \mid \Gamma) \to (S \mid \Gamma \longrightarrow_{\mathsf{nf}} A)$$

Finally, a normalization procedure $\mathsf{nbe} : (S \mid \Gamma \longrightarrow_{\mathsf{nd}} C) \to (S \mid \Gamma \longrightarrow_{\mathsf{nf}} C)$ is defined as follows. Apply eval to a given derivation $f : S \mid \Gamma \longrightarrow_{\mathsf{nd}} C$ in the natural deduction calculus, obtaining a natural transformation $\mathsf{eval}\ f$ between presheaves $\{\!\{S \mid \Gamma\}\!\}$ and $\{\!\{C\}\!\}$. Take the component of $\mathsf{eval}\ f$ at $S \mid \Gamma$, which is a function of type $\{\!\{S \mid \Gamma\}\!\}\ (S \mid \Gamma) \to \{\!\{C\}\!\}\ (S \mid \Gamma)$. By induction on S and Γ, it is possible to define a canonical element $\gamma : \{\!\{S \mid \Gamma\}\!\}(S \mid \Gamma)$. This allows to obtain an element $\mathsf{eval}\ f\ \gamma : \{\!\{C\}\!\}\ (S \mid \Gamma)$, which can finally be reified into a normal form:

$$\mathsf{nbe}\ f = \mathsf{reify}\ (\mathsf{eval}\ f\ \gamma)$$

We formally verified that the function nbe is well-defined, i.e. it sends $\doteq_{\mathsf{nd}}$-related derivations to the same normal form. Moreover, nbe is the inverse up to $\doteq_{\mathsf{nd}}$ of the embedding $\mathsf{emb}^{\mathsf{nd}}_{\mathsf{nf}} : (S \mid \Gamma \longrightarrow_{\mathsf{nf}} C) \to (S \mid \Gamma \longrightarrow_{\mathsf{nd}} C)$ of normal forms into natural deduction derivations: given a natural deduction derivation $f : S \mid \Gamma \longrightarrow_{\mathsf{nd}} C$, we have $\mathsf{emb}^{\mathsf{nd}}_{\mathsf{nf}}\ (\mathsf{nbe}\ f) \doteq_{\mathsf{nd}} f$; given a normal form $f : S \mid \Gamma \longrightarrow_{\mathsf{nf}} C$, we have $\mathsf{nbe}\ (\mathsf{emb}^{\mathsf{nd}}_{\mathsf{nf}}\ f) = f$.

Comparing Normal Forms. We conclude this section by showing that focused sequent calculus derivations and normal natural deduction derivations are in one-to-one correspondence. Notice that we have already established a one-to-one correspondence indirectly: the correctness of normalization by hereditary substitution implies that the set of focused calculus derivations $S \mid \Gamma \longrightarrow_{\mathsf{I}} C$ is isomorphic to the set of natural deduction derivations $S \mid \Gamma \longrightarrow_{\mathsf{nd}} C$ quotiented by the equivalence relation $\doteq_{\mathsf{nd}}$, which is further isomorphic to the set of normal natural deduction derivations $S \mid \Gamma \longrightarrow_{\mathsf{nf}} C$ thanks to the correctness of normalization by evaluation. The goal of this section is to provide a simple direct comparison of the two classes of normal forms.

The crucial step of this comparison is the relation between neutrals and derivations in phase F. This is because normal forms in phase nf have the same primitive inference rules derivations in phase I, and similarly for derivations of the passivation phases of the two calculi. We simultaneously define six translations back and forth between the three pairs of corresponding phases of the two calculi. We only show the constructions of the translations $\mathsf{ne2F}$ and $\mathsf{F2ne}$ between neutrals and derivations in phase F. The functions $\mathsf{nf2I}$ and $\mathsf{I2nf}$ for translating between normal forms and derivations in phase I are trivially defined, similarly for the functions translating between the p and P phases. The definitions of translations $\mathsf{ne2F}$ and $\mathsf{F2ne}$ rely on two auxiliary functions $\mathsf{ne2F}'$ and $\mathsf{F2ne}'$.

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathsf{ne2F}' : (A \mid \Gamma \longrightarrow_{\mathsf{ne}} B) \to (B \mid \Delta \longrightarrow_{\mathsf{F}} C) \to (A \mid \Gamma, \Delta \longrightarrow_{\mathsf{F}} C)} \\
\mathsf{ne2F}'\ \mathsf{ax} & g = g \\
\mathsf{ne2F}'\ ({\multimap}\mathsf{e}\ (f, a)) & g = \mathsf{ne2F}'\ f\ ({\multimap}\mathsf{L}\ (\mathsf{nf2I}\ a, g))
\end{array}$$

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathsf{F2ne}' : (A \mid \Gamma \longrightarrow_{\mathsf{ne}} B) \to (B \mid \Delta \longrightarrow_{\mathsf{F}} C) \to (A \mid \Gamma, \Delta \longrightarrow_{\mathsf{ne}} C)} \\
\mathsf{F2ne}'\ f\ \mathsf{ax} & = f \\
\mathsf{F2ne}'\ f\ ({\multimap}\mathsf{L}\ (a, g)) & = \mathsf{F2ne}'\ ({\multimap}\mathsf{e}\ (f, \mathsf{I2nf}\ a))\ g
\end{array}$$

Remember that neutrals are lambda-terms of the form $x\ a_1\ \ldots\ a_n$ with x being a variable (the only one) declared in the stoup. The accumulator g in the definition of $\mathsf{ne2F}'$ is intended to collect the arguments $a_i, \ldots, a_n$ that have already been seen. So when a new argument a appears, which is a normal form, this is immediately translated to an l-phase derivation via $\mathsf{nf2l}$ and then pushed on top of the accumulator using the left rule $\multimap\mathsf{L}$. The accumulator f in the definition of $\mathsf{F2ne}'$ serves a similar purpose. The translations $\mathsf{ne2F}$ and $\mathsf{F2ne}$ are then easily definable:

$$\begin{array}{ll} \mathsf{ne2F} : (A \mid \Gamma \longrightarrow_{\mathsf{ne}} C) \to (A \mid \Gamma \longrightarrow_{\mathsf{F}} C) & \mathsf{F2ne} : (A \mid \Gamma \longrightarrow_{\mathsf{F}} C) \to (A \mid \Gamma \longrightarrow_{\mathsf{ne}} C) \\ \mathsf{ne2F}\ f = \mathsf{ne2F}'\ f\ \mathsf{ax} & \mathsf{F2ne}\ f = \mathsf{F2ne}'\ \mathsf{ax}\ f \end{array}$$

These translations form an isomorphism. The crucial lemma for proving this is: given $f : A \mid \Gamma \longrightarrow_{\mathsf{ne}} B$ and $g : B \mid \Delta \longrightarrow_{\mathsf{F}} C$, we have $\mathsf{ne2F}\ (\mathsf{F2ne}'\ f\ g) = \mathsf{ne2F}'\ f\ g$ and $\mathsf{F2ne}\ (\mathsf{ne2F}'\ f\ g) = \mathsf{F2ne}'\ f\ g$.

4 Losing Skewness and How to Restore It

The free skew prounital closed category $\mathbf{FSkPCl}(\mathsf{At})$ on a set of atoms At is left normal, which means that its skew aspect is superfluous. In other words, $\mathbf{FSkPCl}(\mathsf{At})$ is also the free (non-skew) prounital closed category on At. An advantage of our proof theoretic analysis is that left-normality can be proved in any one of the equivalent calculi of Section 3. Left-normality is a simple observation in the sequent calculus, while it is not clear how to derive it directly in the categorical calculus of Section 3.1.

First we notice that left-normality, defined as the invertibility of the derivable map $\widehat{j}$ of (1), is translated to the invertibility of the passivation rule pass in the sequent calculus of Section 3.2. In other words, $\widehat{j}$ is invertible up to $\doteq$ in the categorical calculus if and only if pass is invertible up to $\stackrel{\circ}{=}$ in the sequent calculus. Then we show that pass has as inverse the admissible rule act:

$$\frac{- \mid A, \Gamma \longrightarrow C}{A \mid \Gamma \longrightarrow C}\ \mathsf{act} \tag{10}$$

This is defined by induction on the given derivation $f : - \mid A, \Gamma \longrightarrow C$. There are only two possible cases: if $f = \mathsf{pass}\ f'$, define $\mathsf{act}\ f = f'$; if $f = \multimap\mathsf{R}\ f'$, define $\mathsf{act}\ f = \multimap\mathsf{R}\ (\mathsf{act}\ f')$.

An important consequence of left-normality is that all calculi described in Sections 3.2–3.5 admit a presentation without the stoup and the pass rule. In particular, the natural deduction calculus of Section 3.3 is equivalent to (non-skew) planar simply-typed lambda-calculus [1, 39]. The categorical calculus of 3.1 also admits a stoup-free version where sequents take the form $\Longrightarrow A$ where A is a formula. The inference rules are

$$\frac{\Longrightarrow B \quad \Longrightarrow B \multimap C}{\Longrightarrow C}\ \mathsf{comp}'$$

$$\frac{}{\Longrightarrow A \multimap A}\ j \qquad \frac{\Longrightarrow A}{\Longrightarrow (A \multimap B) \multimap B}\ i' \qquad \frac{}{\Longrightarrow (B \multimap C) \multimap ((A \multimap B) \multimap (A \multimap C))}\ L' \tag{11}$$

Under the Curry-Howard correspondence, this is the combinatory logic capturing planar lambda-calculus: comp' is application, j is the I-combinator, and L' is the B-combinator, while the operation i' replaces the C-combinator of BCI combinatory logic [25] and is needed in the absence of symmetry.

A natural question arises: why did we bother including the stoup in our calculi in the first place? There are two reasons behind our choice to include the stoup.

First, in the future we plan to extend the skew calculi described in this paper with other connectives, such as unit and tensor. We already know from previous work on the proof theory of skew monoidal categories [35] that the extended calculi will not be left-normal, so we will not be able to discard the stoup. We believe that a thorough investigation of the normalization procedures of Sections 3.4 and 3.5, which work in the presence of the stoup, is a stepping stone towards the development of normalization functions for more involved calculi with additional connectives.

Second, the left-normality of **FSkPCl**(At) arises from the fact that we are considering the free skew prounital closed category on a *set*. In other words, it corresponds to a left adjoint to the forgetful functor between the category of skew prounital closed categories and strict prounital closed functors and the category of sets and functions. From a categorical point of view, there is no good reason to privilege the category of sets and functions in this picture. The next subsection is devoted to the study of the free skew prounital closed category **FSkPCl**$(\mathbb{M})$ on a skew multicategory $\mathbb{M}$. The category **FSkPCl**(At) arises as a particular instance of the latter more general construction. Crucially, **FSkPCl**$(\mathbb{M})$ is generally not left-normal.

4.1 The Free Skew Prounital Closed Category on a Skew Multicategory

We start by recollecting Bourke and Lack's notion of skew multicategory [10]. We slightly reformulate Bourke and Lack's definition to make its relationship to the sequent calculus of Section 3.2 more direct. Skew multicategories are similar to the multicategories of Lambek [18] (also known as colored (non-symmetric) operads), but instead use an optional object paired with a list of objects as the domain of a multimap, rather than just a list of objects.

A *skew multicategory* $\mathbb{M}$ consists of a set $\mathbb{M}_0$ of objects and, for any optional object S, list of objects Γ and object C in $\mathbb{M}_0$, a set $\mathbb{M}(S|\Gamma;C)$ of multimaps whereby a multimap is called *loose* if S is empty and *tight* if S is an object. For any object A, there is an identity multimap $\mathsf{id} \in \mathbb{M}(A|\ ;A)$. There are two composition operations $\mathsf{s}\circ : \mathbb{M}(A|\Delta;C) \times \mathbb{M}(S|\Gamma;A) \to \mathbb{M}(S|\Gamma,\Delta;C)$ and $\mathsf{c}\circ : \mathbb{M}(S|\Delta_0,A,\Delta_1;C) \times \mathbb{M}(-|\Gamma;A) \to \mathbb{M}(S|\Delta_0,\Gamma,\Delta_1;C)$ and a loosening operation $\mathsf{loosen} : \mathbb{M}(A|\Gamma;C) \to \mathbb{M}(-|A,\Gamma;C)$ satisfying a large number of equations, expressing unitality of identity wrt. composition, associativity of composition, commutativity of parallel cuts and commutativity of composition and loosening. See the whole list of equations in our previous work [35].

A *skew multifunctor* G between skew multicategories $\mathbb{M}$ and $\mathbb{M}'$ consists of a function G_0 sending objects of $\mathbb{M}$ to objects of $\mathbb{M}'$ and a function $G_1 : \mathbb{M}(S|\Gamma;C) \to \mathbb{M}'(G_0 S|G_0\Gamma;G_0 C)$ preserving identity, composition and loosening. Here G_0 is extended to optional objects and lists of objects by $G_0\,S = -$ if $S = -$ and $G_0\,S = G_0\,A$ if $S = A$. Similarly, $G_0(A_1,\dots,A_n) = G_0 A_1,\dots,G_0 A_n$. Skew multicategories and skew multifunctors form a category. There exists a forgetful functor U between the category of skew prounital closed categories and the latter category. Given a skew prounital closed category $\mathbb{C}$, we define $U\,\mathbb{C}$ as the skew multicategory with the same objects as $\mathbb{C}$ and with the multihomset $(U\,\mathbb{C})(S|\Gamma;C)$ given by $\mathbb{C}(S,[\![\Gamma|C]\!])$. From the structure of $\mathbb{C}$, using properties of the interpretation $[\![\Gamma|C]\!]$, one defines the identity, composition and loosening of $U\,\mathbb{C}$.

The *free* skew prounital closed category on a skew multicategory $\mathbb{M}$ is then a skew prounital closed category **FSkPCl**$(\mathbb{M})$ equipped with a skew multifunctor $\iota : \mathbb{M} \to U(\mathbf{FSkPCl}(\mathbb{M}))$. For any other skew prounital closed category $\mathbb{C}$ with a skew multifunctor $G : \mathbb{M} \to U\,\mathbb{C}$, there must exist a unique strict prounital closed functor $\bar{G} : \mathbf{FSkPCl}(\mathbb{M}) \to \mathbb{C}$ compatible with ι.

Let $\mathbb{M}$ be a skew multicategory. We construct a categorical calculus presenting the free skew prounital closed category on $\mathbb{M}$. We then proceed to describe an equivalent cut-free sequent calculus.

Categorical Calculus The formulae are given by objects $X \in \mathbb{M}_0$ (atomic formulae) and $A \multimap B$ for any formulae A, B. The inference rules are the same as in (2), supplemented with an additional inference rule

$$\frac{\mathbb{M}(T|\Phi;Z)}{T \Longrightarrow [\![\Phi|Z]\!]}\ \iota$$

where T is an optional atom, Φ is a list of atoms and Z is an atom. The equational theory $\doteq$ from (3) is extended with new generating equations expressing the fact that ι is a skew multifunctor between $\mathbb{M}$ and $U(\mathbf{FSkPCl}(\mathbb{M}))$.

Cut-Free Sequent Calculus The inference rules are those given in (4) minus the rules ax and pass plus two new rules

$$\frac{\mathbb{M}(T|\Phi;Z)}{T \mid \Phi \longrightarrow Z}\ \iota \qquad\qquad \frac{- \mid \Gamma \longrightarrow A \quad S \mid \Delta_0, B, \Delta_1 \longrightarrow C}{S \mid \Delta_0, A \multimap B, \Gamma, \Delta_1 \longrightarrow C}\ \multimap\mathsf{C}$$

The rule $\multimap\mathsf{C}$ was derivable using cut in the sequent calculus of Section 3.2, as we showed in (7). Here it is needed as a primitive rule to achieve cut admissibility. From the presence of a map $f \in \mathbb{M}(X|Y;Z)$ in the base skew multicategory $\mathbb{M}$, we need to be able to derive, e.g., the sequent $X \mid A \multimap Y, A \longrightarrow Z$, which in the categorical calculus is derivable as follows:

$$\frac{\dfrac{\mathbb{M}(X|Y;Z)}{X \Longrightarrow Y \multimap Z}\ \iota \quad \dfrac{}{Y \multimap Z \Longrightarrow (A \multimap Y) \multimap (A \multimap Z)}\ L}{X \Longrightarrow (A \multimap Y) \multimap (A \multimap Z)}\ \mathsf{comp}$$

The equational theory on derivations is obtained from the congruence $\stackrel{\circ}{=}$ of (5) by adding the following generating equations:

(preservation of id and loosen by ι)

$\mathsf{ax}_X \stackrel{\circ}{=} \iota\ (\mathsf{id}_X)$

$\mathsf{pass}\ (\iota\ f) \stackrel{\circ}{=} \iota\ (\mathsf{loosen}\ f)$ (for $f \in \mathbb{M}(X|\Phi;Z)$)

(commutative conversions of $\multimap\mathsf{C}$)

$\multimap\mathsf{C}\ (f, \multimap\mathsf{R}\ g) \stackrel{\circ}{=} \multimap\mathsf{R}\ (\multimap\mathsf{C}\ (f,g))$ (for $f : - \mid \Gamma \longrightarrow A', g : S \mid \Delta_0, B', \Delta_1, A \longrightarrow B$)

$\mathsf{pass}\ (\multimap\mathsf{C}\ (f,g)) \stackrel{\circ}{=} \multimap\mathsf{C}\ (f, \mathsf{pass}\ g)$ (for $f : - \mid \Gamma \longrightarrow A, g : A' \mid \Delta_0, B, \Delta_1 \longrightarrow C$)

$\mathsf{pass}\ (\multimap\mathsf{L}\ (f,g)) \stackrel{\circ}{=} \multimap\mathsf{C}\ (f, \mathsf{pass}\ g)$ (for $f : - \mid \Gamma \longrightarrow A, g : B \mid \Delta \longrightarrow C$)

$\multimap\mathsf{C}\ (f, \multimap\mathsf{L}(g,h)) \stackrel{\circ}{=} \multimap\mathsf{L}\ (g, \multimap\mathsf{C}\ (f,h))$ (for $f : - \mid \Gamma \longrightarrow A, g : - \mid \Gamma' \longrightarrow A', h : B' \mid \Delta_0, B, \Delta_1 \longrightarrow C$)

$\multimap\mathsf{C}\ (f, \multimap\mathsf{L}(g,h)) \stackrel{\circ}{=} \multimap\mathsf{L}\ (\multimap\mathsf{C}\ (f,g), h)$ (for $f : - \mid \Gamma \longrightarrow A, g : - \mid \Delta_0, B, \Delta_1 \longrightarrow A', h : B' \mid \Delta \longrightarrow C$)

$\multimap\mathsf{C}\ (f, \multimap\mathsf{C}(g,h)) \stackrel{\circ}{=} \multimap\mathsf{C}\ (g, \multimap\mathsf{C}\ (f,h))$ (for $f : - \mid \Gamma \longrightarrow A, g : - \mid \Gamma' \longrightarrow A', h : S \mid \Delta_0, B, \Delta_1, B', \Delta_2 \longrightarrow C$)

$\multimap\mathsf{C}\ (f, \multimap\mathsf{C}(g,h)) \stackrel{\circ}{=} \multimap\mathsf{C}\ (\multimap\mathsf{C}\ (f,g), h)$ (for $f : - \mid \Gamma \longrightarrow A, g : - \mid \Delta_0, B, \Delta_1 \longrightarrow A', h : S \mid \Delta_2, B', \Delta_3 \longrightarrow C$)

Thanks to the presence of the primitive rule $\multimap\mathsf{C}$, the two cut rules in (6) are admissible in this sequent calculus. In this case, they need to be defined by mutual induction with another cut rule

$$\frac{A' \mid \Gamma \longrightarrow A \quad S \mid \Delta_0, A, \Delta_1 \longrightarrow C}{S \mid \Delta_0, A', \Gamma, \Delta_1 \longrightarrow C}\ \mathsf{ccut}_{\mathsf{Fma}}$$

In the sequent calculus of Section 3.2, the rule $\mathsf{ccut}_{\mathsf{Fma}}$ is definable by first applying pass to the first premise and then using ccut. In the new sequent calculus of the current section, we have to define it simultaneously with scut and ccut because of the added cases for the added primitive rules. It is possible to prove that the embedding ι is a skew multifunctor, in particular it preserves the cut operations.

Notice that the sequent calculus is generally not left-normal. In fact, an attempt to prove the admissibility of the rule act of (10) fails when the premise is of the form $\iota\ f$ for some $f \in \mathbb{M}(-|X,\Phi;Z)$. E.g., we may well have a map in $\mathbb{M}(-|X;Z)$ for some X and Z without there being any map in $\mathbb{M}(X|;Z)$. Therefore the stoup cannot be discarded.

The categorical calculus and the sequent calculus are equivalent. It is possible to construct functions sound and cmplt translating between the two calculi, and show that they form an isomorphism up to the extended equivalence relations $\doteq$ and $\stackrel{\circ}{=}$.

Focused derivations The focused subcalculus uses that the ax and pass rules of the sequent calculus are admissible from id and loosen (crucially because the presence of $\multimap$C makes it possible to commute pass and $\multimap$L). It has the inference rules from (9) minus the rules pass and ax plus two new rules

$$\frac{\mathbb{M}(T|\Phi;Z)}{T \mid \Phi \longrightarrow_{\mathsf{F}} Z}\ \iota \qquad \frac{- \mid \Gamma \longrightarrow_{\mathsf{I}} A \quad T \mid \Psi, B, \Delta \longrightarrow_{\mathsf{F}} C}{T \mid \Psi, A \multimap B, \Gamma, \Delta \longrightarrow_{\mathsf{F}} C}\ \multimap\mathsf{C}$$

Notice that, in the rule $\multimap$C, T is restricted to be an optional atom and Ψ a list of atoms. Notice also that the passivation phase is trivial because we have removed the rule pass.

4.2 Starting From a Skew Multigraph

The free skew prounital closed category **FSkPCl**($\mathbb{M}$) over a multicategory $\mathbb{M}$ is special in that we can have a cut-free sequent calculus where the use of the generating multimaps is confined to "direct import" by ι. In fact, no structural rules (neither any cut rules nor pass or ax) are needed beyond the degree that they are readily available to us in the form of composition, loosening and identity in the base skew multicategory $\mathbb{M}$ where they also satisfy the skew multicategory equations. This is possible because the cut rules happen to be admissible from the sound rule $\multimap$C that we may choose to take as primitive.

This approach is not robust for extensions with further connectives; we cannot have a similar cut-free sequent calculus for the free skew (unital) closed category **FskCl**($\mathbb{M}$): from $\mathbb{M}(-|\,;Y)$ and $\mathbb{M}(X|Y;Z)$, we must be able to derive $X \mid \mathsf{I} \longrightarrow Z$, but for this we need ccut as a primitive rule (together with pass) since, differently from $\multimap$, it is unsound to introduce I into the passive context. However, as soon as we introduce primitive cut rules (it suffices to take scut and ccut as primitive), we also need to introduce (i) equations stating that ι preserves compositions as cuts and (ii) also the skew multicategory equations for scut, ccut, ax and pass. The equations (i) can be dispensed with if we start with a *skew multigraph* $(\mathsf{At}, \mathsf{DC})$ (of *atoms* and *definite clauses*) rather than a skew multicategory $\mathbb{M}$, so that composition as well as the identities and loosening are only available in terms of scut, ccut, ax and pass. We conjecture that the equations (ii) can then also be avoided in a focused subcalculus with all the inference rules from (9) plus the rule

$$\frac{- \mid \Gamma_1 \longrightarrow_{\mathsf{I}} Y_1 \quad \dots \quad - \mid \Gamma_n \longrightarrow_{\mathsf{I}} Y_n \quad \mathsf{DC}(T|Y_1,\dots,Y_n;X) \quad X \mid \Delta \longrightarrow_{\mathsf{F}} C}{T \mid \Gamma_1,\dots,\Gamma_n,\Delta \longrightarrow_{\mathsf{F}} C}\ \iota'$$

which packages a particular combination of ι and scut and ccut inferences.

5 Conclusions and Future Work

We presented several equivalent presentations of the free skew prounital closed category on a set At. We showed that these correspond to a skew variant of the planar fragment of linear typed lambda-calculus. We constructed two calculi of normal forms: a focused sequent calculus and a normal natural deduction calculus. These solve the coherence problem for skew prounital closed categories by fully characterizing the homsets of **FSkPCl**(At). The latter category is left-normal, meaning that its skew aspect is redundant. We restored the skewness by studying deductive systems for the free skew prounital closed category on a skew multicategory and showing that the latter is generally not left-normal.

The development presented in the paper has been fully formalized in the dependently typed programming language Agda. Our Agda formalization also includes a similar proof theoretic analysis of the free *skew closed category* on a set, in which the element set functor J is replaced by a unit object I [30].

The primitive rules of the cut-free sequent calculus of skew closed categories also include left and right introduction rule for the unit I, where again the left rule acts only on the unit in the stoup. Similarly, the natural deduction calculus has introduction and elimination rules for I. Our reduction-free normalization procedures can be adapted to the skew closed case without much difficulty.

In the future, we plan to extend the work of this paper and our previous work on the sequent calculus of the Tamari order [40] and of skew monoidal categories [35, 37] to a proof theoretic investigation of skew monoidal closed categories, i.e. including unit I, tensor $\otimes$ and internal hom $\multimap$ related by an adjunction $- \otimes B \dashv B \multimap -$. We already know from our previous work that the corresponding sequent calculus would not be left-normal. We conjecture that the free skew monoidal closed category on At corresponds to a skew variant of the $(\mathsf{I}, \otimes, \multimap)$ fragment of noncommutative intuitionistic linear logic [2]. It is currently not clear how to extend the normalization procedures of this paper to the skew monoidal closed case, in particular normalization by evaluation, which has not been studied in the planar (or even linear) fragment of lambda-calculus. Inspiration could come from the normalization by hereditary substitution algorithm of Watkins et al. [38] for the propositional fragment of their concurrent logical framework.

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