

# A standardisation proof for algebraic pattern calculi

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This work gives some insights and results on standardisation for call-by-name pattern calculi. More precisely, we define standard reductions for a pattern calculus with constructor-based data terms and patterns. This notion is based on reduction steps that are needed to match an argument with respect to a given pattern. We prove the Standardisation Theorem by using the technique developed by Takahashi [14] and Crary [2] for  $\lambda$ -calculus. The proof is based on the fact that any development can be specified as a sequence of head steps followed by internal reductions, i.e. reductions in which no head steps are involved.

## 1 Introduction

**Pattern Calculi:** Several calculi, called *pattern calculi*, have been proposed in order to give a formal description of pattern matching; i.e. the ability to analyse the different possible forms of the argument of a function in order to decide among different alternative definition clauses.

The *pattern matching* operation is the kernel of the evaluation mechanism of all these formalisms, basically because reduction can only be fired when the argument passed to a given function matches its pattern specification. An analysis of various pattern calculi based on different notions of pattern matching operations and different sets of allowed patterns can be found in [8].

**Standardisation:** A fundamental result in the  $\lambda$ -calculus is the *Standardisation Theorem*, which states that if a term  $M$   $\beta$ -reduces to a term  $N$ , then there is a *standard*  $\beta$ -reduction sequence from  $M$  to  $N$  which can be seen as a canonical way to reduce terms. This result has several applications, e.g. it is used to prove the non-existence of reduction between given terms. One of its main corollaries is the quasi-leftmost-reduction theorem, which in turn is used to prove the non-existence of a normal form for a given term.

A first study on standardisation for call-by-name  $\lambda$ -calculus appears in [3]. Subsequently, several standardisation methods have been devised, for example [1] Section 11.4, [14], [9] and [13].

While leftmost-outermost reduction gives a standard strategy for call-by-name  $\lambda$ -calculus, more refined notions of reductions are necessary to define standard strategies for call-by-value  $\lambda$ -calculus [13], first-order term rewriting systems [6, 15], Proof-Nets [4], etc.

All standard reduction strategies require the definition of some *selected* redex by means of a partial function from terms to redexes; they all give priority to the selected step, if possible. This selected redex is sometimes called *external* [11], but we will refer here to it as the *head redex* of a term.

It is also worth mentioning a generic standardisation proof [12] that can uniformly treat call-by-name and call-by-value  $\lambda$ -calculus. It is parameterized over the set of values that allow to fire the beta-reduction rule. However, the set of values are defined there in a global sense, while in pattern calculi being a value strongly depends on the form of the given pattern.

**Standardisation in Pattern Calculi:** For call-by-name  $\lambda$ -calculus, any term of the form  $(\lambda x.M)N$  is a redex, and the head redex for such a term is the whole term. In pattern calculi any term of the form

$(\lambda p.M)N$  is a redex candidate, but not necessarily a redex. The parameter  $p$  in such terms can be more complex than a single variable, and the whole term is not a redex if the argument  $N$  does not match  $p$ , i.e., if  $N$  does not verify the structural conditions imposed by  $p$ . In this case we will choose as head a reduction step lying inside  $N$  (or even inside  $p$ ) which makes  $p$  and  $N$  be closer to a possible match. While this situation bears some resemblance with *call-by-value*  $\lambda$ -calculus [13], there is an important difference: both the fact of  $(\lambda p.M)N$  being a redex, and whether a redex inside  $N$  could be useful to get  $p$  and  $N$  closer to a possible match, depend on *both*  $N$  and  $p$ .

The aim of this contribution is to analyse the existence of a standardisation procedure for pattern calculi in a direct way, i.e. without using any complicated encoding of such calculi into some general computational framework [10]. This direct approach aims to put to evidence the fine interaction between reduction and pattern matching, and gives a standardisation *algorithm* which is specified in terms of the combination of computations of independent terms with partial computations of terms depending on some pattern. We hope to be able to extend this algorithmic approach to more sophisticated pattern calculi handling open and dynamic patterns [7].

The paper is organized as follows. Section 2 introduces the calculus, Section 3 gives the main concepts needed for the standardisation proof and the main results, Section 4 presents some lemmas used in the main proofs, Sections 5 and 6 show the main results used in the Standardisation Theorem proof and then the theorem itself; finally, Section 7 concludes and gives future research directions.

## 2 The calculus

We will study a very simple form of pattern calculus, consisting of the extension of standard  $\lambda$ -calculus with a set of constructors and allowing constructed patterns. This calculus appears for example in Section 4.1 in [8].

**Definition 2.1 (Syntax)** *The calculus is built upon two different enumerable sets of symbols, the variables  $x, y, z, w$  and the constants  $c, a, b$ ; its syntactical categories are:*

$$\begin{array}{ll} \text{Terms} & M, N, Q, R ::= x \mid c \mid \lambda p.M \mid MM \\ \text{Patterns} & p, q ::= x \mid d \end{array} \quad \begin{array}{ll} \text{DataTerms} & D ::= c \mid DM \\ \text{DataPatterns} & d ::= c \mid dp \end{array}$$

Free and bound variables of terms are defined as expected as well as  $\alpha$ -conversion.

**Definition 2.2 (Substitution)** *A substitution  $\theta$  is a function from variables to terms with finite domain, where  $\text{dom}(\theta) = \{x : \theta(x) \neq x\}$ . The extension of  $\theta$  to terms is defined as expected. We denote  $\theta ::= \{x_1/M_1, \dots, x_n/M_n\}$  whenever  $\text{dom}(\theta) \subseteq \{x_1, \dots, x_n\}$ . Moreover, for  $\theta, \nu$  substitutions,  $X$  a set of variables, we define*

$$\begin{aligned} \text{var}(\theta) & ::= \text{dom}(\theta) \cup \left( \bigcup_{x \in \text{dom}(\theta)} \text{fv}(\theta x) \right) \\ \nu \theta & ::= \left( \bigcup_{x \in \text{dom}(\theta)} \{x/\nu(\theta x)\} \right) \cup \left( \bigcup_{x \in (\text{dom}(\nu) - \text{dom}(\theta))} \{x/\nu x\} \right) \\ \theta \upharpoonright_X & ::= \bigcup_{x \in X \cap \text{dom}(\theta)} \{x/\theta x\} \end{aligned}$$

**Definition 2.3 (Matching)** *Let  $p$  be a pattern and  $M$  a term which do not share common variables. Matching on  $p$  and  $M$  is a partial function yielding a substitution and defined by the following rules ( $\uplus$  on substitutions denotes disjoint union with respect to their domains, being undefined if the domains have a non-empty intersection):*

$$\frac{}{x \ll^{\{x/N\}} N} \quad \frac{}{c \ll^{\emptyset} c} \quad \frac{d \ll^{\theta_1} D \quad p \ll^{\theta_2} N \quad \theta_1 \uplus \theta_2 \text{ defined}}{dp \ll^{\theta_1 \uplus \theta_2} DN}$$

We write  $p \ll M$  iff  $\exists \theta p \ll^{\theta} M$ . Remark that  $p \ll M$  implies that  $p$  is linear.

**Definition 2.4 (Reduction step)** We consider the following reduction steps modulo  $\alpha$ -conversion:

$$\frac{M \rightarrow M'}{MN \rightarrow M'N} \text{SAppL} \quad \frac{N \rightarrow N'}{MN \rightarrow MN'} \text{SAppR} \quad \frac{p \ll^{\theta} N}{(\lambda p.M)N \rightarrow \theta M} \text{SBeta} \quad \frac{M \rightarrow M'}{\lambda p.M \rightarrow \lambda p.M'} \text{SAbs}$$

By working modulo  $\alpha$ -conversion we can always assume in rule (SBeta) that  $p$  and  $N$  do not share common variables in order to compute matching.

**Lemma 2.5 (Basic facts about the calculus)**

- (data pattern/term structure) Let  $d \in \mathbf{DataPatterns}$  (resp.  $D \in \mathbf{DataTerms}$ ), then  $d = cp_1 \dots p_n$  (resp.  $D = cM_1 \dots M_n$ ) for some  $n \geq 0$ .
- (data patterns only match data terms) Let  $d \in \mathbf{DataPatterns}$ ,  $M$  a term, such that  $d \ll M$ . Then  $M \in \mathbf{DataTerms}$ .
- (minimal matches) If  $p \ll^{\theta} M$  then  $\text{dom}(\theta) = \text{fv}(p)$ .
- (uniqueness of match) If  $p \ll^{\theta_1} M$  and  $p \ll^{\theta_2} M$ , then  $\theta_1 = \theta_2$ .

Crucial to the standardisation proof is the concept of development, we formalize it through the relation  $\triangleright$ , meaning  $M \triangleright N$  iff there is a development (not necessarily complete) with source  $M$  and target  $N$ .

**Definition 2.6 (Term and substitution development)** We define the relation  $\triangleright$  on terms and a corresponding relation  $\blacktriangleright$  on substitutions. The relation  $\triangleright$  is defined by the following rules:

$$\frac{}{M \triangleright M} \text{DRefl} \quad \frac{M \triangleright M'}{\lambda p.M \triangleright \lambda p.M'} \text{DAbs}$$

$$\frac{M \triangleright M' \quad N \triangleright N'}{MN \triangleright M'N'} \text{DApp} \quad \frac{M \triangleright M' \quad \theta \blacktriangleright \theta' \quad p \ll^{\theta} N}{(\lambda p.M)N \triangleright \theta'M'} \text{DBeta}$$

and  $\blacktriangleright$  is defined as follows:  $\theta \blacktriangleright \theta'$  iff  $\text{dom}(\theta) = \text{dom}(\theta')$  and  $\forall x \in \text{dom}(\theta) . \theta x \triangleright \theta' x$

## 2.1 Head step

The definition of head step will take into account the terms  $(\lambda p.M)N$  even if  $p \not\ll N$ . In such cases, the head redex will be inside  $N$  as the patterns in this calculus are always normal forms (this will not be the case for more complex pattern calculi).

The selection of the head redex inside  $N$  depends on both  $N$  and  $p$ . This differs from standard call-by-value  $\lambda$ -calculus, where the selection depends only on  $N$ .

We show this phenomenon with a simple example. Let  $a, b, c$  be constants and  $N = (aR_1)R_2$ , where  $R_1$  and  $R_2$  are redexes. The redexes in  $N$  needed to achieve a match with a certain pattern  $p$ , and thus the selection of the head redex, depend on the pattern  $p$ .

Take for example different patterns  $p_1 = (ax)(by)$ ,  $p_2 = (abx)y$ ,  $p_3 = (abx)(cy)$ ,  $p_4 = (ax)y$ , and consider the term  $Q = (\lambda p.M)N$ . If  $p = p_1$ , then it is not necessary to reduce  $R_1$  (because it already matches

$x$ ) but it is necessary to reduce  $R_2$ , because no redex can match the pattern  $by$ ; hence  $R_2$  will be the head redex in this case. Analogously, for  $p_2$  it is necessary to reduce  $R_1$  but not  $R_2$ , for  $p_3$  both are needed (in this case we will choose the leftmost one) and  $p_4$  does match  $N$ , hence the whole  $Q$  is the head redex. This observation motivates the following definition.

**Definition 2.7 (Head step)** *The relations  $\rightarrow_h$  (head step) and  $\rightsquigarrow_p$  (preferred needed step to match pattern  $p$ ) are defined as follows:*

$$\begin{array}{ccc} \frac{M \rightarrow_h M'}{MN \rightarrow_h M'N} \text{HApp1} & \frac{p \ll^\theta N}{(\lambda p.M)N \rightarrow_h \theta M} \text{HBeta} & \frac{N \rightsquigarrow_p N'}{(\lambda p.M)N \rightarrow_h (\lambda p.M)N'} \text{HPat} \\ \\ \frac{M \rightarrow_h M'}{M \rightsquigarrow_d M'} \text{PatHead} & \frac{D \rightsquigarrow_d D'}{DM \rightsquigarrow_{dp} D'M} \text{Pat1} & \frac{M \rightsquigarrow_p M' \quad d \ll D}{DM \rightsquigarrow_{dp} DM'} \text{Pat2} \end{array}$$

The rule PatHead is intended for data patterns only, not being valid for variable patterns; we point this by writing a  $d$  (data pattern) instead of a  $p$  (any pattern) in the arrow subscript inside the conclusion.

We observe that the rule analogous to HPat in the presentation of standard reduction sequences for call-by-value  $\lambda$ -calculus in both [13] and [2] reads

$$\frac{N \rightarrow_h N'}{(\lambda p.M)N \rightarrow_h (\lambda p.M)N'}$$

reflecting the  $N$ -only-dependency feature aforementioned.

We see also that a head step in a term like  $(\lambda p.M)N$  determined by rule HPat will lie inside  $N$ , but the same step will not necessarily be considered head if we analyse  $N$  alone.

It is easy to check that if  $M \rightsquigarrow_p M'$  then  $p \not\ll M$ , avoiding any overlap between HBeta and HPat and also between Pat1 and Pat2. This in turn implies that all terms have at most one head redex. We remark also that the head step depends not only on the pattern structure but also on the match or lack of match between pattern and argument.

**Lemma 2.8 (Basic facts about head steps)**

- a. (head reduction only if abstraction in head) Let  $M$  be a term such that  $M \rightarrow_h M'$  for some  $M'$ . Then  $M = (\lambda p.M_{01})M_1 \dots M_n$  with  $n \geq 1$ .
- b. (head reduction only if no match) Let  $M$  be a term such that  $M \rightarrow_h M'$  for some  $M'$ ,  $d \in \mathbf{DataPatterns}$ . Then  $d \not\ll M$ .
- c. ( $\rightsquigarrow_p$  only if  $\rightarrow_h$  or data term) Let  $p$  be a pattern and let  $M$  be a term such that  $M \rightsquigarrow_p M'$  for some  $M'$ . Then either  $M \in \mathbf{DataTerms}$  or  $M \rightarrow_h M'$ .

**Proof** Item (a) is trivial. Item (b) uses Item (a) and L. 2.5:(b). Item (c) is trivial by definition of  $\rightsquigarrow_p$ .  $\square$

### 3 Main concepts and ideas needed for the standardisation proof

In order to build a standardisation proof for constructor based pattern calculi we chose to adapt the one in [14] for the call-by-name  $\lambda$ -calculus, later adapted to call-by-value  $\lambda$ -calculus in [2], over the classical presentation of [13].

The proof method relies on a **h-development** property stating that any development can be split into a leading sequence of head steps followed by a development in which no head steps are performed; this is our Corollary 5.4 which corresponds to the so-called “main lemma” in the presentations by Takahashi and Cray.

Even for a simple form of pattern calculus such as the one presented in this contribution, both the definitions (as we already mentioned when defining head steps) and the proofs are non-trivial extensions of the corresponding ones for standard  $\lambda$ -calculus, even in the framework of call-by-value. As mentioned before, the reason is the need to take into account, for terms involving the application of a function to an argument, the pattern of the function parameter when deciding whether a redex inside the argument should be considered as a head redex.

In order to formalize the notion of “development without occurrences of head steps”, an *internal development* relation will be defined. The dependency on both  $N$  and  $p$  when analysing the reduction steps from a term like  $(\lambda p.M)N$  is shown in the rule IApp2.

**Definition 3.1 (Internal development)** *The relations  $\triangleright^{int}$  (internal development) and  $\triangleright_p^{int}$  (internal development with respect to the pattern  $p$ ) are defined as follows:*

$$\begin{array}{c}
\frac{}{M \triangleright^{int} M} \text{IRefI} \quad \frac{M \triangleright M'}{\lambda p.M \triangleright \lambda p.M'} \text{IAbs} \quad \frac{M \neq \lambda p.M_1 \quad M \triangleright^{int} M' \quad N \triangleright N'}{MN \triangleright^{int} M'N'} \text{IApp1} \\
\\
\frac{M \triangleright M' \quad N \triangleright_p^{int} N'}{(\lambda p.M)N \triangleright^{int} (\lambda p.M')N'} \text{IApp2} \quad \frac{N \triangleright N' \quad p \ll N}{N \triangleright_p^{int} N'} \text{PMatch} \quad \frac{N \triangleright^{int} N'}{N \triangleright_c^{int} N'} \text{PConst} \\
\\
\frac{N \notin \mathbf{DataTerms} \quad N \triangleright^{int} N'}{N \triangleright_{dp}^{int} N'} \text{PNoCData} \quad \frac{D \triangleright_d^{int} D' \quad M \triangleright M' \quad d \ll D}{DM \triangleright_{dp}^{int} D'M'} \text{PCDataNo1} \\
\\
\frac{D \triangleright D' \quad M \triangleright_p^{int} M' \quad d \ll D \quad p \ll M}{DM \triangleright_{dp}^{int} D'M'} \text{PCDataNo2} \\
\\
\frac{D \triangleright D' \quad M \triangleright M' \quad d \ll D \quad p \ll M \quad dp \ll DM}{DM \triangleright_{dp}^{int} D'M'} \text{PCDataNo3}
\end{array}$$

Remark that rule PCDataNo3 is useful to deal with non-linear patterns.

Thus for example,  $ab((\lambda y.y)c) \triangleright_{axx}^{int} abc$  since  $ab \triangleright ab$ ,  $(\lambda y.y)c \triangleright c$ ,  $ax \ll ab$ ,  $x \ll (\lambda y.y)c$  but  $axx \not\ll ab((\lambda y.y)c)$ .

We observe also that if  $N \triangleright^{int} N'$  or  $N \triangleright_p^{int} N'$  then  $N \triangleright N'$ .

The following lemma analyses data / non-data preservation

**Lemma 3.2 (Development and data)**

- a. (internal development cannot create data terms) Let  $M \notin \mathbf{DataTerms}$ ,  $N$  such that  $M \triangleright^{\text{int}} N$ . Then  $N \notin \mathbf{DataTerms}$
- b. (development from data produces always data) Let  $M \in \mathbf{DataTerms}$ ,  $N$  such that  $M \triangleright N$ . Then  $N \in \mathbf{DataTerms}$

The formal description of the h-development condition takes a form of an additional binary relation. This relation corresponds to the one called *strong parallel reduction* in [2].

**Definition 3.3 (H-development)** We define the relations  $\triangleright_h$  and  $\blacktriangleright_h$ . Let  $M, N$  be terms;  $\nu, \theta$  substitutions.

- a.  $M \triangleright_h N$  iff (i)  $M \triangleright N$ , (ii)  $\exists Q$  s.t.  $M \xrightarrow[h]{*} Q \triangleright^{\text{int}} N$ , (iii)  $\forall p . \exists Q_p$  s.t.  $M \xrightarrow[p]{*} Q_p \triangleright_p^{\text{int}} N$ .
- b.  $\nu \blacktriangleright_h \theta$  iff (i)  $\text{Dom}(\nu) = \text{Dom}(\theta)$ , (ii)  $\forall x \in \text{Dom}(\nu) . \nu x \triangleright_h \theta x$ .

The clause (iii) in the definition of  $\triangleright_h$  shows the dependency on the patterns that was already noted in the definitions of head step and internal development.

This clause is needed when proving that all developments are h-developments; let's grasp the reason through a brief argument. Suppose we want to prove that a development inside  $N$  in a term like  $(\lambda p.M)N$  is an h-development. The rules to be used in this case are HPat (Def. 2.7) and IApp2 (Def. 3.1). Therefore we need to perform an analysis *relative to the pattern p*; and this is exactly expressed by clause (iii). Consequently the proof of clause (ii) for a term needs to consider clause (iii) (instantiated to a certain pattern) for a subterm; this is achieved by including clause (iii) in the definition and by performing an inductive reasoning on terms.

## 4 Auxiliary results

We collect in this section some results needed to complete the main proofs in this article.

**Lemma 4.1 (pattern-head reduction only if there is no match)**

Let  $M, N$  be terms,  $p$  a pattern, such that  $M \rightsquigarrow_p N$ . Then  $p \not\ll M$ .

**Proof** Using L. 2.8:(b). □

**Lemma 4.2 (development cannot lose matches)**

Let  $M, N$  be terms,  $p$  a pattern, such that  $M \triangleright N$  and  $p \ll^{\nu} M$ . Then  $p \ll^{\theta} N$  for some  $\theta$  such that  $\nu \blacktriangleright \theta$ .

**Proof** Induction on  $p \ll^{\nu} M$ . The axioms can be checked trivially. For the rule, let  $M = M_1 M_2$ ,  $N = N_1 N_2$ ,  $p = p_1 p_2$  and  $\nu = \nu_1 \uplus \nu_2$ ;  $p$  is linear since it matches a term. The only rules applicable for  $M \triangleright N$  are DRefl or DApp; DBeta is not applicable because  $M_1 \in \mathbf{DataTerms}$ . If DRefl was used, the lemma holds trivially taking  $\theta = \nu$ . If DApp was used, we apply the IH on both hypotheses obtaining  $p_i \ll^{\theta_i} N_i$  with  $\nu_i \blacktriangleright \theta_i$ ; by L. 2.5:(c) and the linearity of  $p$  we know  $\theta = \theta_1 \uplus \theta_2$  is well-defined; it is easy to check that  $\theta$  satisfies the lemma conditions. □

**Lemma 4.3 ( $\triangleright_p^{\text{int}}$  cannot create match)**

Let  $M, N$  be terms,  $p$  a pattern, such that  $M \triangleright_p^{\text{int}} N$ . Then  $p \not\ll M$  implies  $p \not\ll N$ .

**Proof** Induction on  $M \triangleright_p^{int} N$  by rule analysis

PMatch not applicable as  $p \not\ll M$ .

PConst in this case the condition  $p \not\ll M$  implies  $p \not\ll N$  equates to  $M \neq p$  implies  $N \neq p$ , as  $p$  is a constant.

The rule premise reads  $M \triangleright^{int} N$ : if rule lRefl was used then  $N \neq p$  by hypothesis, else the  $\triangleright^{int}$  rule conclusions exclude the possibility of  $N$  being a constant.

PNoCData  $M \notin \mathbf{DataTerms}$  and  $M \triangleright^{int} N$  by rule hyp., then  $N \notin \mathbf{DataTerms}$  by L. 3.2:(a), finally  $p \not\ll N$  by L. 2.5:(b).

PCDataNo1 By the IH, as rule hyp. includes both  $D \triangleright_d^{int} D'$  and  $d \not\ll D$  being  $M = DT$  and  $p = dp'$ .

PCDataNo2 Similar to the former considering  $p = dp'$  and using  $T \triangleright_{p'}^{int} T'$  and  $p' \not\ll T$ .

PCDataNo3 In this case  $M = DM'$ ,  $p = dp'$ ,  $d \ll^\theta D$ ,  $p' \ll^{\theta'} M'$  and  $dp' \not\ll DM'$ . We necessarily have that  $\theta \uplus \theta'$  is not defined hence  $p$  is not linear so that  $p \not\ll N$  also holds.

□

#### Lemma 4.4 (left-pattern-head implies whole-pattern-head)

Let  $p_1, p_2$  be patterns and  $M_1, N_1, M_2$  be terms such that  $M_1 \rightsquigarrow_{p_1} N_1$ . Then  $M_1 M_2 \rightsquigarrow_{p_1 p_2} N_1 M_2$ .

**Proof** It is clear that  $p_1 \notin \mathit{Var}$ , because there is no  $N_1$  such that  $M_1 \rightsquigarrow_x N_1$  if  $x \in \mathit{Var}$ .

If PatHead applied in  $M_1 \rightsquigarrow_{p_1} N_1$ , then  $M_1 \xrightarrow{h} N_1$ , by HApp1  $M_1 M_2 \xrightarrow{h} N_1 M_2$ , and finally by PatHead  $M_1 M_2 \rightsquigarrow_{p_1 p_2} N_1 M_2$ .

If either Pat1 or Pat2 applied in  $M_1 \rightsquigarrow_{p_1} N_1$ , then  $M_1$  is clearly a data term, Then  $M_1 M_2 \rightsquigarrow_{p_1 p_2} N_1 M_2$  by Pat1. □

#### Lemma 4.5 (matching is compatible with substitution)

Let  $M$  be a term,  $p$  a pattern and  $\theta$  a substitution such that  $p \ll^\theta M$ . Then for any substitution  $\nu$ , the following holds:  $p \ll^\gamma \nu M$  where  $\gamma = \nu \theta \upharpoonright_{\mathit{fv}(p)}$ .

**Proof** By induction on the match. The axioms can be checked trivially given L. 2.5:(c).

We analyze the rule applied in this context

$$\frac{d \ll^{\theta_1} M_1 \quad p' \ll^{\theta_2} M_2}{dp' = p \ll^{\theta = \theta_1 \uplus \theta_2} M = M_1 M_2}$$

Applying the IH on both hypotheses and then using the rule gives  $dp' \ll^{(\nu \theta_1) \upharpoonright_{\mathit{fv}(d)} \uplus (\nu \theta_2) \upharpoonright_{\mathit{fv}(p)}} M_1 M_2$ ; an easy check of  $(\nu \theta_1) \upharpoonright_{\mathit{fv}(d)} \uplus (\nu \theta_2) \upharpoonright_{\mathit{fv}(p)} = (\nu(\theta_1 \uplus \theta_2)) \upharpoonright_{\mathit{fv}(dp')}$  concludes the proof. □

#### Lemma 4.6 (development is compatible with substitution)

Let  $M, N$  be terms and  $\nu, \theta$  substitutions, such that  $M \triangleright N$  and  $\nu \blacktriangleright \theta$ . Then  $\nu M \triangleright \theta N$

**Proof** By induction on  $M \triangleright N$  by rule analysis.

For DRefl the thesis amounts to  $\nu M \triangleright \theta M$ , which can be checked by a simple induction on  $M$ . DAbs and DApp can be simply verified by the IH.

For DBeta first we mention a technical result which will be used. Let  $\theta, \tau$  be substitutions such that  $\text{dom}(\tau) \cap \text{var}(\theta) = \emptyset$ , then

$$((\theta\tau) \upharpoonright_{\text{dom}(\tau)}) \theta = \theta\tau \quad (1)$$

this can be easily checked comparing the effect of applying both substitutions to an arbitrary variable.

Let's analyze the rule premises and conclusion applied in this context

$$\frac{M_1 \triangleright M'_1 \quad \tau \blacktriangleright \tau' \quad p \ll^{\tau} M_2}{M = (\lambda p.M_1)M_2 \triangleright \tau'M'_1 = N}$$

As we can freely choose the variables appearing in  $p$ , we assume  $\text{fv}(p) \cap (\text{var}(v) \cup \text{var}(\theta)) = \emptyset$ . By L. 2.5:(c) we know  $\text{dom}(\tau) = \text{dom}(\tau') = \text{fv}(p)$ .

We apply the IH on  $M_1 \triangleright M'_1$  and also on  $\tau x \triangleright \tau'x$  for each  $x \in \text{dom}(\tau)$  to conclude  $vM_1 \triangleright \theta M'_1$  and  $(v\tau) \upharpoonright_{\text{dom}(\tau)} \blacktriangleright (\theta\tau') \upharpoonright_{\text{dom}(\tau)}$  respectively. Furthermore, from  $p \ll^{\tau} M_2$  and L. 4.5 we conclude  $p \ll^{(v\tau \upharpoonright_{\text{dom}(\tau)})} vM_2$ .

We use DBeta from the three conclusions above to obtain

$$vM = (\lambda p.vM_1)(vM_2) \triangleright ((\theta\tau') \upharpoonright_{\text{dom}(\tau)})(\theta M'_1)$$

To check  $\theta N = \theta(\tau'M'_1) = ((\theta\tau') \upharpoonright_{\text{dom}(\tau)})(\theta M'_1)$  it is enough to verify  $\theta\tau' = ((\theta\tau') \upharpoonright_{\text{dom}(\tau)})\theta$ , the latter can be easily checked by (1). □

#### Lemma 4.7 (head reduction is compatible with substitution)

(i) Let  $M, N$  be terms and  $v$  a substitution such that  $M \xrightarrow[h]{\rightarrow} N$ . Then  $vM \xrightarrow[h]{\rightarrow} vN$ .

(ii) Let  $M, N$  be terms,  $p$  a pattern and  $v$  a substitution such that  $M \xrightarrow[p]{\rightsquigarrow} N$ . Then  $vM \xrightarrow[p]{\rightsquigarrow} vN$ .

**Proof** (sketch)

Both items are proved by simultaneous induction on  $M \xrightarrow[h]{\rightarrow} N$  and  $M \xrightarrow[p]{\rightsquigarrow} N$ .

We use L. 4.5 for case HBeta, the IH and L. 4.5 for case Pat2, and just the IH for the remaining cases. □

## 5 H-developments

The aim of this section is to prove that all developments are h-developments.

We found easier to prove separately that the h-development condition is compatible with the language constructs, diverging from the structure of the proofs in [2].

#### Lemma 5.1 ( $\triangleright$ is compatible with abstraction)

Let  $M, N$  be terms such that  $M \triangleright_h N$ . Then  $\lambda q.M \triangleright_h \lambda q.N$  for any pattern  $q$ .

**Proof** Part (i) trivially holds by hyp. (i) and DAbs.

Part (ii): by hyp. (i) and lAbs we get  $\lambda q.M \overset{int}{\triangleright} \lambda q.N$ . Then  $Q = \lambda q.M$ .

Part (iii): if  $p \in \text{Var}$  then PMatch applies, if  $p$  is a constant or a compound data pattern then PConst or PNoCData apply respectively as  $(\lambda q.M) \overset{int}{\triangleright} (\lambda q.N)$ . In all cases we obtain  $(\lambda q.M) \overset{int}{\triangleright}_p (\lambda q.N)$ . Then  $Q = \lambda q.M$ . □



**Lemma 5.2** ( $\triangleright$  is compatible with application)

Let  $M_1, M_2, N_1, N_2$  be terms such that  $M_1 \triangleright_h N_1$  and  $M_2 \triangleright_h N_2$ . Then  $M_1 M_2 \triangleright_h N_1 N_2$ .

**Proof** Part (i) is immediate by the hypotheses (i) and DApp.

Let's prove part (ii).

We first use hypothesis (ii) on  $M_1 \triangleright_h N_1$  to obtain  $M_1 \xrightarrow{h}^* Q_1 \triangleright^{int} N_1$  and subsequently apply HApp1 to  $M_1 \xrightarrow{h}^* Q_1$  to get

$$M_1 M_2 \xrightarrow{h}^* Q_1 M_2 \quad (2)$$

Either  $Q_1$  is an abstraction or not.

Assume  $Q_1$  is not an abstraction. Since  $Q_1 \triangleright^{int} N_1$  and  $M_2 \triangleright_h N_2$ , we apply lApp1 so that  $Q_1 M_2 \triangleright^{int} N_1 N_2$ ; this together with (2) gives the desired result.

Now assume  $Q_1 = \lambda p. Q_{12}$ . We use the hyp. (iii) on  $M_2 \triangleright_h N_2$ , obtaining  $M_2 \xrightarrow{p}^* Q_2 \triangleright_p^{int} N_2$  and then we apply HPat to get

$$Q_1 M_2 \xrightarrow{h}^* Q_1 Q_2 \quad (3)$$

Moreover, as  $Q_1 = \lambda p. Q_{12} \triangleright^{int} N_1$ , the only applicable rules are lRef1 or lAbs, and in both cases  $N_1 = \lambda p. N_{12}$  and  $Q_{12} \triangleright N_{12}$ .

We now use lApp2 with premises  $Q_{12} \triangleright N_{12}$  and  $Q_2 \triangleright_p^{int} N_2$  to get

$$Q_1 Q_2 = (\lambda p. Q_{12}) Q_2 \triangleright^{int} (\lambda p. N_{12}) N_2 = N_1 N_2 \quad (4)$$

The desired result is obtained by (2), (3) and (4).

Let's prove part (iii).

If  $p \in Var$  we are done by (i) and PMatch; we thus get  $M_1 M_2 \triangleright_p^{int} N_1 N_2$  so that  $Q = M_1 M_2$ .

If  $p = c$  then using (ii) we obtain  $M_1 M_2 \xrightarrow{h}^* Q \triangleright^{int} N_1 N_2$  for some  $Q$ ; we apply PatHead and PConst to get  $M_1 M_2 \xrightarrow{c}^* Q$  and  $Q \triangleright_c^{int} N_1 N_2$  respectively, concluding the proof for this case.

Consider  $p = p_1 p_2$  with  $p_1$  a data pattern and  $p_2$  a pattern.

We use the hyp. (iii) on  $M_1 \triangleright_h N_1$ , getting  $M_1 \xrightarrow{p_1}^* Q_1 \triangleright_{p_1}^{int} N_1$ . Let us define  $R_1$  as follows: if there is a data term in the sequence  $M_1 \xrightarrow{p_1}^* Q_1$  then  $R_1$  is the first of such terms; otherwise  $R_1$  is  $Q_1$ . In both cases  $M_1 \xrightarrow{p_1}^* R_1 \xrightarrow{p_1}^* Q_1$ . We necessarily have  $M_1 \xrightarrow{h}^* R_1$  by PatHead, then  $M_1 M_2 \xrightarrow{h}^* R_1 M_2$  by HApp1 and subsequently  $M_1 M_2 \xrightarrow{p}^* R_1 M_2$  by PatHead.

We conclude  $M_1 M_2 \xrightarrow{p}^* Q_1 M_2$ , trivially if  $Q_1 = R_1$ , and applying Pat1 to  $R_1 \xrightarrow{p_1}^* Q_1$  to obtain  $R_1 M_2 \xrightarrow{p}^* Q_1 M_2$  otherwise.

If  $Q_1 = (\lambda q.Q'_1)$  then we use the hyp. (iii) on  $M_2 \triangleright N_2$  getting  $M_2 \overset{*}{\rightsquigarrow}_q Q_2 \overset{int}{\triangleright}_q N_2$ .

We apply HPat to  $M_2 \overset{*}{\rightsquigarrow}_q Q_2$  getting  $Q_1 M_2 \xrightarrow{h}^* Q_1 Q_2$ ; therefore we obtain  $Q_1 M_2 \overset{*}{\rightsquigarrow}_p Q_1 Q_2$  by PatHead.

In the other side  $Q_1 = (\lambda q.Q'_1) \triangleright N_1$ , therefore  $N_1 = (\lambda q.N'_1)$  and  $Q'_1 \triangleright N'_1$ .

We apply IApp2 to  $Q'_1 \triangleright N'_1$  and  $Q_2 \overset{int}{\triangleright}_q N_2$  to obtain  $Q_1 Q_2 \overset{int}{\triangleright} N_1 N_2$ , therefore  $Q_1 Q_2 \overset{int}{\triangleright}_p N_1 N_2$  by PNoCData. We thus get the desired result taking  $Q_p = Q_1 Q_2$ .

If  $Q_1$  is not an abstraction and  $Q_1 \notin \mathbf{DataTerms}$ , then only PConst or PNoCData can justify  $Q_1 \overset{int}{\triangleright}_{p_1} N_1$ , thus implying  $Q_1 \overset{int}{\triangleright} N_1$ ; this together with the hypothesis (i)  $M_2 \triangleright N_2$  gives  $Q_1 M_2 \overset{int}{\triangleright} N_1 N_2$  by IApp1, hence  $Q_1 M_2 \overset{int}{\triangleright}_p N_1 N_2$  by PNoCData. We get the desired result by taking  $Q_p = Q_1 M_2$ .

If  $Q_1 \in \mathbf{DataTerms}$  we analyse the different alternatives for the matching between  $p_1 p_2$  and  $Q_1 M_2$ .

Assume  $p_1 \not\ll Q_1$ . In this case we apply PDataNo1 to  $Q_1 \overset{int}{\triangleright}_{p_1} N_1$  and  $M_2 \triangleright N_2$  to obtain  $Q_1 M_2 \overset{int}{\triangleright}_p N_1 N_2$  and thus the desired result holds by taking  $Q_p = Q_1 M_2$ .

Assume  $p_1 \ll Q_1$  and  $p_2 \not\ll M_2$ . In this case we use the hyp. (iii) on  $M_2 \triangleright N_2$  to get  $M_2 \overset{*}{\rightsquigarrow}_{p_2} Q_2 \overset{int}{\triangleright}_{p_2} N_2$ , then apply Pat2 to get  $Q_1 M_2 \overset{*}{\rightsquigarrow}_p Q_1 Q_2$ . Finally from  $Q_1 \overset{int}{\triangleright}_{p_1} N_1$  and  $Q_2 \overset{int}{\triangleright}_{p_2} N_2$  we obtain  $Q_1 Q_2 \overset{int}{\triangleright}_p N_1 N_2$  by either PDataNo2, PDataNo3 or PMatch. We get the desired result by taking  $Q_p = Q_1 Q_2$ .

Finally assume  $p_1 \ll Q_1$  and  $p_2 \ll Q_2$ . In this case the hypotheses imply in particular  $Q_1 \triangleright N_1$  and  $M_2 \triangleright N_2$ . We thus conclude  $Q_1 M_2 \overset{int}{\triangleright}_p N_1 N_2$  using either PMatch or PDataNo3 (depending on whether  $p \ll Q_1 M_2$  or not), getting the desired result by taking  $Q_p = Q_1 M_2$ . □

Now we proceed with the proof of the h-development property. The generalization of the statement involving  $\blacktriangleright_h$  is needed to conclude the proof<sup>1</sup>, as can be seen in the DBeta case below.

### Lemma 5.3 (Generalized h-developments property)

Let  $M, N$  be terms and  $\nu, \theta$  substitutions, such that  $M \triangleright N$  and  $\nu \blacktriangleright_h \theta$ .

Then  $\nu M \overset{int}{\triangleright}_h \theta N$

**Proof** By induction on  $M \triangleright N$  analyzing the rule used in the last step of the derivation.

**DRefl** in this case  $N = M$ , we proceed by induction on  $M$

- $M = x \in \text{Dom}(\nu)$ , in this case  $\nu M = \nu x \overset{int}{\triangleright}_h \theta x = \theta N$  by hypothesis.
- $M = x \notin \text{Dom}(\nu)$ , in this case  $\nu M = x \overset{int}{\triangleright}_h x = \theta N$ .
- $M = M_1 M_2$ , in this case  $\nu M_1 \overset{int}{\triangleright}_h \theta M_1$  and  $\nu M_2 \overset{int}{\triangleright}_h \theta M_2$  hold by the IH. The desired result is obtained by L. 5.2.
- $M = \lambda p.M_1$ , in this case  $\nu M_1 \overset{int}{\triangleright}_h \theta M_1$  holds by the IH. The desired result is obtained by L. 5.1.

<sup>1</sup>In [2] the compatibility of h-development with substitutions is stated as a separate lemma; for pattern calculi we could not find a proof of compatibility with substitution independent of the main h-development result.

**DAbs** in this case  $M = \lambda p.M_1, N = \lambda p.N_1, M_1 \triangleright N_1$ .

Using the IH on  $M_1 \triangleright N_1$  we obtain  $vM_1 \triangleright_h \theta N_1$ , the desired result is obtained by L. 5.1.

**DApp** in this case  $M = M_1 M_2, N = N_1 N_2, M_i \triangleright N_i$ .

Using the IH on both rule premises we obtain  $vM_i \triangleright_h \theta N_i$ , the desired result is obtained by L. 5.2.

**DBeta** Let's write down the rule instantiation

$$\frac{M_{12} \triangleright N_{12} \quad \tau \blacktriangleright \tau' \quad q \ll^{\tau} M_2}{M = (\lambda q.M_{12})M_2 \triangleright \tau' N_{12} = N}$$

(i) can be obtained by hypotheses  $M \triangleright N$  and  $v \blacktriangleright_h \theta$ , and then L. 4.6.

For [ (iii) if  $p \in Var$  ] we are done by (i) and PMatch.

For [ (iii) if  $p = d$  ] and also for (ii) : we know both  $M \xrightarrow{h} \tau M_{12}$  and  $M \rightsquigarrow_p \tau M_{12}$ , then by L. 4.7

$$vM \xrightarrow{h} v(\tau M_{12}) \quad \text{and} \quad vM \rightsquigarrow_p v(\tau M_{12}) \quad (5)$$

We apply the IH on each  $\tau x \triangleright \tau' x$ , obtaining  $(v\tau)x = v(\tau x) \triangleright_h \theta(\tau' x) = (\theta\tau')x$  for all  $x \in Dom(\tau)$ .  
Moreover, if  $x \in Dom(v) - Dom(\tau)$  then  $(v\tau)x = vx \triangleright_h \theta x = (\theta\tau')x$  by hypothesis.

Consequently,  $v\tau \blacktriangleright_h \theta\tau'$ . Now we use the IH on  $M_{12} \triangleright N_{12}$  taking  $v\tau \blacktriangleright_h \theta\tau'$  as second hypothesis to obtain

$$v(\tau M_{12}) = (v\tau)M_{12} \triangleright_h (\theta\tau')N_{12} = \theta(\tau' N_{12}) = \theta N$$

This result along with (5) concludes the proof for both parts. □

#### Corollary 5.4 (H-development property)

Let  $M, N$  be terms such that  $M \triangleright N$ . Then  $M \triangleright_h N$ .

## 6 Standardisation

The part of the standardisation proof following the proof of the h-development property coincides in structure with the proof given in [2].

First we will prove that we can get, for any reduction involving head steps that follows an internal development, another reduction in which the head steps are at the beginning. The name given to the Lemma 6.1 was taken from [2].

This proof needs again to consider explicitly the relations relative to patterns, for similar reasons to those described when introducing h-development in section 3.

#### Lemma 6.1 (Postponement)

(i) if  $M \overset{int}{\triangleright} N \xrightarrow{h} R$  then there exists some term  $N'$  such that  $M \xrightarrow{h} N' \triangleright R$

(ii) for any pattern  $p$ , if  $M \overset{int}{\triangleright}_p N \rightsquigarrow R$  then there exists some term  $N'_p$  such that  $M \rightsquigarrow_p N'_p \triangleright R$

**Proof** For (i), if the rule used in  $M \triangleright^{int} N$  is lRefl, then the result is immediate taking  $N' = R$ . Therefore, in the following we will ignore this case.

We prove (i) and (ii) by simultaneous induction on  $M$  taking into account the previous observation.

**variable** in this case it must be  $N = M$  for both (i) and (ii) and neither  $M \xrightarrow[h]{\triangleright} R$  nor  $M \xrightarrow[p]{\rightsquigarrow} R$  for any  $p, R$ .

**abstraction** in this case  $N$  must also be an abstraction for both (i) and (ii) and neither  $N \xrightarrow[h]{\triangleright} R$  nor  $N \xrightarrow[p]{\rightsquigarrow} R$  for any  $p, R$ .

**application** in this case  $M = M_1M_2$

We prove (i) first, analysing the possible forms of  $M_1$

- Assume  $M_1$  is not an abstraction

In this case lApp1 applies, so we know  $N = N_1N_2$ ,  $M_1 \triangleright^{int} N_1$ , and  $M_2 \triangleright N_2$ .

Since  $M_1 \triangleright^{int} N_1$ ,  $N_1$  is not an abstraction, then the only applicable rule for  $N \xrightarrow[h]{\triangleright} R$  is HApp1, hence  $R = R_1N_2$  and  $N_1 \xrightarrow[h]{\triangleright} R_1$ .

Now we use the IH on  $M_1 \triangleright^{int} N_1 \xrightarrow[h]{\triangleright} R_1$  to get  $M_1 \xrightarrow[h]{\triangleright} N'_1 \triangleright R_1$ , then we obtain  $M = M_1M_2 \xrightarrow[h]{\triangleright} N'_1M_2$  by HApp1.

Finally we apply DApp to  $N'_1 \triangleright R_1$  and  $M_2 \triangleright N_2$  to get  $N'_1M_2 \triangleright R_1N_2 = R$ , which concludes the proof for this case.

- Now assume  $M_1 = \lambda p.M_{12}$  and  $p \not\ll M_2$

Since  $M = (\lambda p.M_{12})M_2 \triangleright^{int} N$ , the only rule that applies is lApp2, then  $N = (\lambda p.N_{12})N_2$ ,  $M_{12} \triangleright N_{12}$ , and  $M_2 \triangleright_p^{int} N_2$ . By L. 4.3 we obtain  $p \not\ll N_2$ , so the only applicable rule in  $N = (\lambda p.N_{12})N_2 \xrightarrow[h]{\triangleright} R$  is HPat, then  $R = (\lambda p.N_{12})R_2$  and  $N_2 \rightsquigarrow_p R_2$ .

Now we use the IH (ii) on  $M_2 \triangleright_p^{int} N_2 \rightsquigarrow_p R_2$ , to get  $M_2 \rightsquigarrow_p N'_2 \triangleright R_2$ .

We obtain  $M = (\lambda p.M_{12})M_2 \xrightarrow[h]{\triangleright} (\lambda p.M_{12})N'_2$  by HPat, then we get  $(\lambda p.M_{12}) \triangleright (\lambda p.N_{12})$  by DAbs on  $M_{12} \triangleright N_{12}$ , finally we apply DApp to the previous result and  $N'_2 \triangleright R_2$  to obtain  $(\lambda p.M_{12})N'_2 \triangleright (\lambda p.N_{12})R_2 = R$  which concludes the proof for this case.

- Finally, assume  $M_1 = \lambda p.M_{12}$  and  $p \ll^v M_2$

Again, the only rule that applies in  $M = (\lambda p.M_{12})M_2 \triangleright^{int} N$  is lApp2, then  $N = (\lambda p.N_{12})N_2$ ,  $M_{12} \triangleright N_{12}$ , and  $M_2 \triangleright_p^{int} N_2$ . Now, by L. 4.2 we obtain  $p \ll^\theta N_2$  for some substitution  $\theta$  such that  $v \triangleright \theta$ , then the applied rule in  $N \xrightarrow[h]{\triangleright} R$  is HBeta (the case HPat being excluded by L. 4.1), hence  $R = \theta N_{12}$

It is clear that  $M \xrightarrow[h]{\triangleright} vM_{12}$ . By L. 4.6 we obtain  $vM_{12} \triangleright \theta N_{12} = R$ , which concludes the proof for this case.

For (ii) we proceed by a case analysis of  $p$

If  $p \in \text{Var}$  then there is no  $R$  such that  $N \rightsquigarrow_p R$  for any term  $N$ .

If  $p \ll M$  then by L. 4.2  $p \ll N$ , and therefore by L. 4.1 there can be no  $R$  such that  $N \rightsquigarrow_p R$ .

If  $p = c$  then  $p \not\ll M$ , hence  $M \triangleright_p^{int} N \rightsquigarrow R$  implies  $M \triangleright_h^{int} N \rightarrow R$  as PConst and PatHead are the only possibilities for this case respectively. We use part (i) to obtain  $M \rightarrow_h N' \triangleright R$ , and  $M \rightsquigarrow_p N'$  by PatHead which concludes the proof for this case.

If  $p = d p_2$  and  $M \notin \mathbf{DataTerms}$ , then the only possibilities for  $M \triangleright_p^{int} N \rightsquigarrow R$  are PNoCData and PatHead respectively, then  $M \triangleright_h^{int} N \rightarrow R$ . We use part (i) to obtain  $M \rightarrow_h N' \triangleright R$ , and  $M \rightsquigarrow_p N'$  by PatHead which concludes the proof for this case.

Now assume  $p = d p_2$ ,  $M \in \mathbf{DataTerms}$ , and  $p \not\ll M$ . We must analyse three possibilities

- $d \not\ll M_1$ .

In this case only PCDataNo1 applies for  $M \triangleright_p^{int} N$ , therefore  $N = N_1 N_2$  with  $M_1 \triangleright_d^{int} N_1$  and  $M_2 \triangleright N_2$ . By L. 4.3 we know  $d \not\ll N_1$  and moreover  $N_1$  is a data term (as can be seen by L. 3.2) thus not having head redexes, so the only possible rule for  $N \rightsquigarrow_p R$  is Pat1, then  $R = R_1 N_2$  with  $N_1 \rightsquigarrow_d R_1$ .

Now we use the IH on the derivation  $M_1 \triangleright_d^{int} N_1 \rightsquigarrow_d R_1$  to get  $M_1 \rightsquigarrow_d N'_1 \triangleright R_1$ , therefore  $M = M_1 M_2 \rightsquigarrow_p N'_1 M_2$  by Pat1.

Moreover as  $N'_1 \triangleright R_1$  and  $M_2 \triangleright N_2$  hence  $N'_1 M_2 \triangleright R_1 N_2 = R$ , which concludes the proof for this case.

- $d \ll M_1$  and  $p_2 \not\ll M_2$ .

In this case only PCDataNo2 applies for  $M \triangleright_p^{int} N$ , therefore  $N = N_1 N_2$  with  $M_1 \triangleright N_1$  and  $M_2 \triangleright_{p_2}^{int} N_2$ . By L. 4.2 and L. 4.3 respectively, we obtain both  $d \ll N_1$  and  $p_2 \not\ll N_2$ . Moreover  $N$  is a data term (as can be seen by L. 3.2) thus not having head redexes. Hence the only possibility for  $N \rightsquigarrow_p R$  is Pat2, then  $R = N_1 R_2$  with  $N_2 \rightsquigarrow_{p_2} R_2$

We now use the IH on  $M_2 \triangleright_{p_2}^{int} N_2 \rightsquigarrow_{p_2} R_2$  to get  $M_2 \rightsquigarrow_{p_2} N'_2 \triangleright R_2$ , and by Pat2  $M = M_1 M_2 \rightsquigarrow_p M_1 N'_2$

We also use DApp on  $M_1 \triangleright N_1$  and  $N'_2 \triangleright R_2$  to get  $M_1 N'_2 \triangleright N_1 R_2 = R$ , which concludes the proof for this case.

- $d \ll M_1$ ,  $p_2 \ll M_2$  and  $d p_2 \not\ll M_1 M_2$ .

$d \ll M_1$  implies (L 2.5:(b))  $M_1 \in \mathbf{DataTerms}$  so that from  $M = M_1 M_2 \triangleright_p^{int} N$  we can only have  $N = N_1 N_2$  with  $M_1 \triangleright N_1$  and  $M_2 \triangleright N_2$ . L. 4.2 gives  $d \ll N_1$  and  $p_2 \ll N_2$ . L. 3.2:(b) gives  $N \in \mathbf{DataTerms}$ . To show  $N \rightsquigarrow_p R$  we have three possibilities: PatHead is not possible since  $N \in \mathbf{DataTerms}$  (c.f. L 2.8:(a)), Pat1 is not possible since  $d \ll M_1$  (c.f. L 4.1), Pat2 is not possible since  $p_2 \ll N_2$  (c.f. L 4.1).

□

### Corollary 6.2

Let  $M, N, R$  be terms such that  $M \triangleright_h^{int} N \rightarrow R$ . Then  $\exists N'$  s.t.  $M \rightarrow_h^* N' \triangleright_h^{int} R$ .

**Proof** Immediate by L. 6.1 and Corollary 5.4. □

Now we generalize the h-development concept to a sequence of developments. The name given to Lemma 6.3 was taken from [2].

**Lemma 6.3 (Bifurcation)**

Let  $M, N$  be terms such that  $M \triangleright^* N$ . Then  $M \xrightarrow[h]{*} R \triangleright^* N$  for some term  $R$ .

**Proof** Induction on the length of  $M \triangleright^* N$ . If  $M = N$  the result holds trivially.

Assume  $M \triangleright Q \triangleright^* N$ . By C. 5.4 and IH respectively, we obtain  $M \xrightarrow[h]{*} S \triangleright^* Q$  and  $Q \xrightarrow[h]{*} T \triangleright^* N$  for some terms  $S$  and  $T$ . Now we use Corollary 6.2 (many times) on  $S \triangleright^* Q \xrightarrow[h]{*} T$  to get  $S \xrightarrow[h]{*} R \triangleright^* T$ .

Therefore  $M \xrightarrow[h]{*} S \xrightarrow[h]{*} R \triangleright^* T \triangleright^* N$  as we desired.  $\square$

Using the previous results, the standardisation theorem admits a very simple proof.

**Definition 6.4 (Standard reduction sequence)** *The standard reduction sequences are the sequences of terms  $M_1; \dots; M_n$  which can be generated using the following rules.*

$$\frac{M_2; \dots; M_k \quad M_1 \xrightarrow[h]{*} M_2}{M_1; \dots; M_k} \text{StdHead} \qquad \frac{M_1; \dots; M_k}{(\lambda p.M_1); \dots; (\lambda p.M_k)} \text{StdAbs}$$

$$\frac{M_1; \dots; M_j \quad N_1; \dots; N_k}{(M_1 N_1); \dots; (M_j N_1); (M_j N_2); \dots; (M_j N_k)} \text{StdApp} \qquad \frac{}{x} \text{StdVar}$$

**Theorem 6.5 (Standardisation)**

Let  $M, N$  be terms such that  $M \triangleright^* N$ . Then there exists a standard reduction sequence  $M; \dots; N$ .

**Proof** By L. 6.3 we have  $M \xrightarrow[h]{*} R \triangleright^* N$ ; we observe that it is enough to obtain a standard reduction sequence  $R; \dots; N$ , because we subsequently apply StdHead many times.

Now we proceed by induction on  $N$

- $N \in \text{Var}$ ; in this case  $R = N$  and we are done.
- $N = \lambda p.N_1$ ; in this case  $R = \lambda p.R_1$  and  $R_1 \triangleright^* N_1$ . By IH we obtain a standard reduction sequence  $R_1; \dots; N_1$ , then by StdAbs so is  $R = \lambda p.R_1; \dots; \lambda p.N_1 = N$ .
- $N = N_1 N_2$ , so  $R = R_1 R_2$  and  $N_i \triangleright^* R_i$ . We use the IH on both reductions to get two standard reduction sequences  $N_i; \dots; R_i$ , then we join them using StdApp.

$\square$

## 7 Conclusion and further work

We have presented an elegant proof of the Standardisation Theorem for constructor-based pattern calculi.

We aim to generalize both the concept of standard reduction and the structure of the Standardisation Theorem proof presented here to a large class of pattern calculi, including both open and closed variants as the Pure Pattern Calculus [7]. It would be interesting to have sufficient conditions for a pattern calculus

to enjoy the standardisation property. This will be close in spirit with [8] where an abstract confluence proof for pattern calculi is developed.

The kind of calculi we want to deal with imposes challenges that are currently not handled in the present contribution, such as open patterns, reducible (dynamic) patterns, and the possibility of having `fail` as a decided result of matching. Furthermore, the possibility of decided `fail` combined with compound patterns leads to the convenience of studying forms of *inherently parallel* standard reduction strategies.

The abstract axiomatic Standardisation Theorem developed in [5] could be useful for our purpose. However, while the axioms of the abstract formulation of standardisation are assumed to hold in the proof of the standardisation result, they need to be defined and verified for each language to be standardised. This could be nontrivial, as in the case of TRS [6, 15], where a meta-level matching operation is involved in the definition of the rewriting framework. We leave this topic as further work.

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