

Termination Analysis of Programs with Multiphase Control-Flow*

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Programs with multiphase control-flow are programs where the execution passes through several (possibly implicit) phases. Proving termination of such programs (or inferring corresponding runtime bounds) is often challenging since it requires reasoning on these phases separately. In this paper we discuss techniques for proving termination of such programs, in particular: (1) using multiphase ranking functions, where we will discuss theoretical aspects of such ranking functions for several kinds of program representations; and (2) using control-flow refinement, in particular partial evaluation of Constrained Horn Clauses, to simplify the control-flow allowing, among other things, to prove termination with simpler ranking functions.

1 Introduction

Proving that a program will eventually terminate, i.e., that it does not go into an infinite loop, is one of the most fundamental tasks of program verification, and has been the subject of voluminous research. Perhaps the best known, and often used, technique for proving termination is that of *ranking functions*, which has already been used by Alan Turing in his early work on program verification [33]. This consists of finding a function ρ that maps program states into the elements of a well-founded ordered set, such that $\rho(s) \succ \rho(s')$ holds for any consecutive states s and s' . This implies termination since infinite descent is impossible in a well-founded order. Besides proving termination, Turing [33] mentions that ranking functions can be used to bound the length computations as well. This is useful in applications such as cost analysis and loop optimisation [16, 3, 1, 11].

Unlike termination of programs in general, which is undecidable, the algorithmic problems of detection (deciding the existence) or generation (synthesis) of a ranking function can well be solvable, given certain choices of the program representation, and the class of ranking functions. There is a considerable amount of research in this direction, in which different kinds of ranking functions for different kinds of program representations were considered. In some cases, the algorithmic problems have been completely settled, and efficient algorithms provided, while other cases remain open.

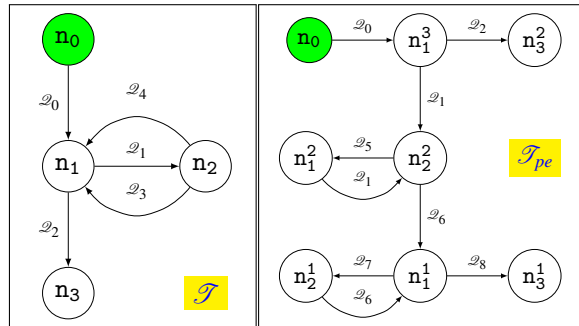
A common program representation in this context is *Single-path Linear-Constraint* (SLC) loops, where a state is described by the values of numerical variables, and the effect of a transition (one iteration) is described by a conjunction of *linear constraints*. Here is an example of this loop representation; primed

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```

1 void phases1(int x, int y, int z) {
2   while( x >= 1 ){
3     if ( y <= z - 1 ) {
4       y = y + 1;
5     } else {
6       x = x - 1;
7     }
8   }
9 }

```



$$\begin{array}{ll}
\mathcal{Q}_0 \equiv \{ & x' = x, \quad y' = y, \quad z' = z \} \\
\mathcal{Q}_1 \equiv \{x \geq 1, & x' = x, \quad y' = y, \quad z' = z \} \\
\mathcal{Q}_2 \equiv \{x \leq 0, & x' = x, \quad y' = y, \quad z' = z \} \\
\mathcal{Q}_3 \equiv \{y \leq z - 1, & x' = x, \quad y' = y + 1, \quad z' = z \} \\
\mathcal{Q}_4 \equiv \{y \geq z, & x' = x - 1, \quad y' = y, \quad z' = z \} \\
\mathcal{Q}_5 \equiv \{x \geq 1, \quad y \leq z - 1, & x' = x, \quad y' = y + 1, \quad z' = z \} \\
\mathcal{Q}_6 \equiv \{x \geq 1, \quad y \geq z, & x' = x - 1, \quad y' = y, \quad z' = z \} \\
\mathcal{Q}_7 \equiv \{x \geq 1, \quad y \geq z, & x' = x, \quad y' = y, \quad z' = z \} \\
\mathcal{Q}_8 \equiv \{x \leq 0, \quad y \geq z, & x' = x, \quad y' = y, \quad z' = z \}
\end{array}$$

Figure 1: A loop with 2 phases, its corresponding TS \mathcal{T} , and the TS \mathcal{T}_{pe} after applying CFR.

variables x' and y' refer to the state following the transition:

$$\text{while } (x \leq y) \text{ do } x' = x + 1, y' \leq y \quad (1)$$

Note that $x' = x + 1$ is an equation, not an assignment. The description of a loop may involve linear inequalities rather than equations, such as $y' \leq y$ above, and consequently, be non-deterministic. Note that for a SLC loop with n variables, a transition can be seen as a point $(\begin{smallmatrix} x \\ x' \end{smallmatrix}) \in \mathbb{Q}^{2n}$, where its first n components correspond to \mathbf{x} and its last n components to \mathbf{x}' . We denote the set of all transitions by \mathcal{Q} , which is a polyhedron.

A more general program representation is *Transition Systems* (TSs), which are defined by *Control-Flow Graphs* (CFGs) with numerical variables, consisting of nodes representing program locations and edges annotated with linear constraints (polyhedra) describing how values of variables change when moving from one location to another. Figure 1 includes a program and its corresponding TS \mathcal{T} to its right. Primed variables in the linear constraints refer to the state following the transition, exactly as in the case of SLC loops.

In both program representations mentioned above, the domain of variables is also important as it typically affects the complexity of the underlying decision and synthesis problems. Although these program representations allow only numerical variables and linear constraints, data structures can be handled using *size abstractions*, e.g., length of lists, depth of trees, etc. [26, 23, 12, 32, 27, 19]. In such case, variables represent sizes of corresponding data structures.

Due to practical considerations, termination analysis tools typically focus on classes of ranking functions that can be synthesised efficiently. This, however, does not mean that theoretical aspects of such classes are put aside, as understanding theoretical limits and properties of the underlying problems is necessary for developing practical algorithms. The most popular class of ranking functions in

this context is probably that of *Linear Ranking Functions* (LRFs). A LRF is a function $\rho(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n + a_0$ such that any transition from \mathbf{x} to \mathbf{x}' satisfies (i) $\rho(\mathbf{x}) \geq 0$; and (ii) $\rho(\mathbf{x}) - \rho(\mathbf{x}') \geq 1$. For example, $\rho(x, y) = y - x$ is a LRF for Loop (1). Several polynomial-time algorithms to find a LRF using linear programming exist [3, 13, 16, 28, 29, 31]. These algorithms are complete¹ for TSs with rational-valued variables, but not with integer-valued variables. Ben-Amram and Genaim [5] showed how completeness for the integer case can be achieved, and also classified the corresponding decision problem as co-NP complete.

Despite their popularity, LRFs do not suffice for all programs, and a natural question is what to do when a LRF does not exist; and a natural answer is to try a richer class of ranking functions. Of particular importance is the class of *Lexicographic-Linear Ranking Functions* (LLRFs). These are tuples of linear functions that decrease lexicographically over a corresponding well-founded ordered set. For example, the program depicted in Figure 1 does not have a LRF, but it has the LLRF $\langle z - y, x \rangle$ since: for the *then* branch $z - y$ decreases and for the *else* branch x decreases while $z - y$ does not change. LLRFs might be necessary even for SLC loops. For example, the following SLC loop

$$\text{while } (x \geq 0) \text{ do } x' = x + y, y' = y + z, z' = z - 1 \quad (2)$$

does not have a LRF, but can be proved terminating using the LLRF $\langle z, y, x \rangle$.

There are several definitions for LLRFs in the literature [3, 6, 10, 22] and they have different power, i.e., some can prove termination of a program while others fail. They have corresponding polynomial-time synthesis algorithms for the case of rational variables, and the underlying decision problems are co-NP complete for the case of integer variables [6, 7]. The definition of Larraz et al. [22] is the most general in this spectrum of LLRFs, its complexity classification is not known yet and corresponding complete synthesis algorithms do not exist.

The LLRF $\langle z, y, x \rangle$ used above for Loop (2) belongs to a class of LLRFs that is known as *Multiphase-Linear Ranking Functions* (M Φ RFs). Ranking functions in this class are characterised by the following behaviour: the first component *always* decreases, and when it becomes negative the second component starts to decrease (and will keep decreasing during the rest of the execution), and when it becomes negative the third component starts to decrease, and so on. This behaviour defines phases through which executions pass. This is different from LLRFs in general since once a component becomes negative it cannot be used anymore. Also the LLRF $\langle z - y, x \rangle$ for the program of Figure 1 induces a similar multiphase behaviour, since once $z - y$ becomes negative it cannot be used anymore. However, it is not a M Φ RF since $z - y$ stops decreasing once we move to the second phase.

In the rest of this paper we overview works on termination analysis of multiphase programs, in particular: in Section 2 we discuss techniques for proving termination using M Φ RFs; and in Section 3 we discuss techniques for proving termination using *Control-Flow Refinement* (CFR). Section 4 ends the paper with some concluding remarks and discusses further research directions.

2 Termination analysis using multiphase ranking functions

A M Φ RF is a tuple $\langle \rho_1, \dots, \rho_d \rangle$ of linear functions that define phases of the program that are linearly ranked, where d is the *depth* of the M Φ RF, intuitively the number of phases. The decision problem *Existence of a M Φ RF* asks to determine whether a program has a M Φ RF. The *bounded* decision problem restricts the search to M Φ RFs of depth d , where d is part of the input.

¹Complete means that if there is a LRF, they will find one.

For the case of SLC loops, the complexity and algorithmic aspects of the bounded version of the $M\Phi RF$ problem were settled by Ben-Amram and Genaim [8]. The decision problem is PTIME for SLC loops with rational-valued variables, and coNP-complete for SLC loops with integer-valued variables; synthesising $M\Phi RF$ s, when they exist, can be performed in polynomial and exponential time, respectively. We note that the proof of the rational case is done by showing that $M\Phi RF$ s and *nested* ranking functions [24] (a strict subclass of $M\Phi RF$ s for which a polynomial-time algorithm exists) have the same power for SLC loops. Besides, they show that for SLC loops $M\Phi RF$ s have the same power as general lexicographic-linear ranking functions, and that $M\Phi RF$ s induce linear iteration bounds. The problem of deciding if a given SLC loop admits a $M\Phi RF$, without a given bound on the depth, is still open.

In practice, termination analysis tools search for $M\Phi RF$ s incrementally, starting by depth 1 and increase the depth until they find one, or reach a predefined limit, after which the returned answer is *don't know*. Finding a theoretical upper-bound on the depth of a $M\Phi RF$, given the loop, would also settle this problem, however, as shown by Ben-Amram and Genaim [8] such bound must depend not only on the number of constraints or variables, as for other classes of LLRFs [3, 6, 10], but also on the coefficients used in the corresponding constraints. Yuan et al. [34] proposed an incomplete method to bound the depth of $M\Phi RF$ s for SLC loops.

Ben-Amram et al. [4] have done a significant progress towards solving the problem of *existence of a $M\Phi RF$* for SLC loop, i.e., seeking a $M\Phi RF$ without a given bound on the depth. In particular, they present an algorithm for seeking $M\Phi RF$ s that reveals novel insights on the structure of these ranking functions. In a nutshell, the algorithm starts from the set of transitions of the given SLC loop, which is a polyhedron, and iteratively removes transitions $(\begin{smallmatrix} \mathbf{x} \\ \mathbf{x}' \end{smallmatrix})$ such that $\rho(\mathbf{x}) - \rho(\mathbf{x}') > 0$ for some function $\rho(\mathbf{x}) = \vec{a} \cdot \mathbf{x} + b$ that is *non-negative on all enabled states*. The process continues iteratively, since after removing some transitions, more functions ρ may satisfy the non-negativity condition, and they may eliminate additional transitions in the next iteration. When all transitions are eliminated in a finite number of iterations, one can construct a $M\Phi RF$ using the ρ functions; and when reaching a situation in which no transition can be eliminated, the remaining set of transitions, which is a polyhedron, is actually a recurrent set that witnesses non-termination. The algorithm always finds a $M\Phi RF$ if one exists, and in many cases, it finds a recurrent set when the loop is non-terminating, however, it is not a decision procedure as it diverges in some cases. Nonetheless, the algorithm provides important insights into the structure of $M\Phi RF$ s. Apart from revealing a relation between seeking $M\Phi RF$ s and seeking recurrent sets, these insights are useful for finding classes of SLC loops for which, when terminating, there is always a $M\Phi RF$ and thus have linear run-time bound. This result was proven for two kinds of SLC loops, both considered in previous work, namely *octagonal relations* and *affine relations with the finite-monoid property* – for both classes, termination has been proven decidable [21].

Ben-Amram et al. [4] have also suggested a new representation for SLC loops, called the *displacement* representation, that provides new tools for studying termination of SLC loops in general, and the existence of a $M\Phi RF$ in particular. In this representation, a transition $(\begin{smallmatrix} \mathbf{x} \\ \mathbf{x}' \end{smallmatrix})$ is represented as $(\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix})$ where $\mathbf{y} = \mathbf{x}' - \mathbf{x}$. Using this representation the algorithm described above can be formalised in a simple way that avoids computing the ρ functions mentioned above, and reduces the existence of a $M\Phi RF$ of depth d to unsatisfiability of a certain linear constraint system. Moreover, any satisfying assignment for this linear constraint system is a witness that explains why the loop has no $M\Phi RF$ of depth d . As an evidence on the usefulness of this representation in general, they showed that some non-trivial observations on termination of bounded SLC loops (i.e., the set of transitions is a bounded polyhedron) are made straightforward in this representation, while they are not easy to see in the normal representation, in particular: a bounded SLC loop terminates if and only if it has a LRF, and it does not terminate if and only if it has a fixpoint transition $(\begin{smallmatrix} \mathbf{x} \\ \mathbf{x} \end{smallmatrix})$.

The works discussed above are limited to the case of SLC loops. The case of general TSs has been considered by Leike and Heizmann [24] and Li et al. [25], where both translate the existence of a $M\Phi$ RF of a given depth d to solving a corresponding non-linear constraint problem, which is complete for the case of rational variables. However, while complete, these approaches do not provide any insights on the complexity of the underlying decision problems. The technique of Borralleras et al. [9] can be used to infer, among other things, $M\Phi$ RFs for TSs without a given bound on the depth. It is based on solving corresponding safety problems using Max-SMT, and it is not complete.

3 Termination analysis using control-flow refinement

As we have seen in Section 1, there are TSs with multiphase behaviour that do not admit $M\Phi$ RFs, but rather a different notion of LLRF that does not require the components to keep decreasing after turning negative. To prove termination of such TSs one can use other classes of LLRFs [3, 6, 10, 22], however, this might not be enough due to complex control-flow where abstract properties of several implicit execution paths are merged. To overcome this imprecision, not only for termination but for program analysis in general, one approach is to simplify the control-flow in order to make implicit execution paths explicit, and thus makes it possible to infer the desired properties with a weaker version of the analysis. This is known as *Control-Flow Refinement* (CFR).

Let us see how CFR can simplify the kind of ranking functions needed to prove termination, and thus improve the precision of the underlying termination analyser. Consider the program depicted Figure 1 again, and recall that we failed to prove its termination when using only LRFs. Examining this program carefully, we can see that any execution passes in two phases: in the first one, y is incremented until it reaches the value of z , and in the second phase x is decremented until it reaches 0. Let us transform the program into a semantically equivalent one such that the two phases are separate and explicit:

```

1 while (x >= 0 and y <= z-1) y = y + 1;
2 while (x >= 0 and y >= z) x = x - 1;
```

Now we can prove termination of this program using LRFs only: for the first loop $z - y$ is a LRF, and for the second loop x is a LRF. Moreover, cost analysis tools that are based on bounding loop iterations using LRFs [1] would infer a linear bound for this program while they would fail on the original one.

Apart from simplifying the termination proof, there are also cases where it is not possible to prove termination without such transformations even when using LLRFs. In addition, CFR can help in inferring more precise invariants, without the need for expensive disjunctive abstract domains, which can benefit any analysis that uses such invariants, e.g., termination and cost analysis.

CFR has been considered by Gulwani et al. [20] and Flores-Montoya and Hähnle [17] to improve the precision of cost analysis, and by Sharma et al. [30] to improve the precision of invariants in order to prove program assertions. While all these techniques can automatically obtain the transformed program above, they are developed from scratch and tailored to some analysis of interest. Recently, CFR has also been considered by Albert et al. [2] to improve cost analysis as well, but from a different perspective that uses termination witnesses to guide CFR.

Since CFR is, in principle, a program transformation that specialises programs to distinguish different execution scenarios, Domenech et al. [15] explored the use of general-purpose specialisation techniques for CFR, in particular the techniques of Gallagher [18] for partial evaluation of Constrained Horn Clauses (CHCs). Basing CFR on partial evaluation has the clear advantage that soundness comes for free because partial evaluation guarantees semantic equivalence between the original program and its

transformed version. Moreover, this way we obtain a CFR procedure that is not tailored for a particular purpose, but rather can be tuned depending on the application domain.

Domenech et al. [15] developed such a CFR procedure for TSs, by transforming TSs into CHCs and using the partial evaluator of Gallagher [18], and integrated it in a termination analysis algorithm in a way that allows applying CFR at different levels of granularity, and thus controlling the trade-off between precision and performance. This is done by suggesting different schemes for applying CFR, not only as a preprocessing step but also on specific parts of the TS which we could not prove terminating. Moreover, they developed heuristics for automatically configuring partial evaluation (i.e., inferring properties to guides specialisation) in order to achieve the desired CFR. Experimental evaluation provides a clear evidence to that their CFR procedure significantly improves the precision of termination analysis, cost analysis, and invariants generation.

Let us demonstrate this CFR procedure on the TS \mathcal{T} that is depicted in Figure 1. In a first step, the TS \mathcal{T} is translated into the following (semantically) equivalent CHC program:

$$\begin{aligned} q_{n_0}(x, y, z) &\leftarrow q_{n_1}(x, y, z). \\ q_{n_1}(x, y, z) &\leftarrow \{x \geq 1\}, \quad q_{n_2}(x, y, z). \\ q_{n_1}(x, y, z) &\leftarrow \{x \leq 0\}, \quad q_{n_3}(x, y, z). \\ q_{n_2}(x, y, z) &\leftarrow \{y \leq z - 1, \quad y' = y + 1\}, \quad q_{n_1}(x, y', z). \\ q_{n_2}(x, y, z) &\leftarrow \{y \geq z, \quad x' = x - 1\}, \quad q_{n_1}(x', y, z). \end{aligned}$$

Then the partial evaluator of Gallagher [18] is applied, using the (automatically inferred) properties $\{x \geq 1, y \geq z\}$ for the loop head predicate $q_{n_1}(x, y, z)$, which results in the following CHC program:

$$\begin{aligned} q_{n_0}(x, y, z) &\leftarrow q_{n_1^3}(x, y, z). \\ q_{n_1^3}(x, y, z) &\leftarrow \{x \leq 0\}, \quad q_{n_2^3}(x, y, z). \\ q_{n_1^3}(x, y, z) &\leftarrow \{x \geq 1\}, \quad q_{n_2^2}(x, y, z). \\ q_{n_2^2}(x, y, z) &\leftarrow \{x \geq 1, \quad y \leq z - 1, \quad y' = y + 1\}, \quad q_{n_1^2}(x, y', z). \\ q_{n_2^2}(x, y, z) &\leftarrow \{x \geq 1, \quad y \geq z, \quad x' = x - 1\}, \quad q_{n_1^1}(x', y, z). \\ q_{n_1^2}(x, y, z) &\leftarrow \{x \geq 1\}, \quad q_{n_2^2}(x, y, z). \\ q_{n_1^1}(x, y, z) &\leftarrow \{x \leq 0, \quad y \geq z\}, \quad q_{n_3^1}(x, y, z). \\ q_{n_1^1}(x, y, z) &\leftarrow \{x \geq 1, \quad y \geq z\}, \quad q_{n_2^1}(x, y, z). \\ q_{n_1^1}(x, y, z) &\leftarrow \{x \geq 1, \quad y \geq z, \quad x' = x - 1\}, \quad q_{n_1^1}(x', y, z). \end{aligned}$$

Now translating this CHC program into a TS results in the TS \mathcal{T}_{pe} that is depicted in Figure 1. Note that the two phases are now separated into two different strongly connected components, which can be proven terminating using only LRFs.

4 Concluding Remarks

In this paper we have discussed techniques for proving termination of programs with multiphase control-flow, where the execution passes through several (possibly implicit) phases. In particular, we have discussed techniques that correspond to our recent work: (1) the use of multiphase ranking functions [8, 4]; and (2) the use of control-flow refinement [15]. As a byproduct of our research, we have developed an open-source termination analyser called `iRANKFINDER`² that implements all these techniques as well as other state-of-the-art techniques. Some of the components of `iRANKFINDER` can be used independently, in particular the CFR component that can be used to incorporate CFR in static analysers with little effort.

²<http://irankfinder.loopkiller.com>

Future Work. For $M\Phi RF$ s, an obvious future direction is to study the problem of deciding whether a TS has a $M\Phi RF$, both from algorithmic and theoretical complexity perspectives. In initial unpublished work, we have proven that the corresponding decision problem is NP-hard for the rational setting, but we could not obtain a further classification. Further exploration of the $M\Phi RF$ problem for SLC loops is also required since it is not solved for the general case yet. For CFR, we have developed some heuristics for the automatic inference of properties, which is crucial for obtaining the desired transformations, but further research in this direction is required. We concentrated on CFR of TSs, and in a future direction one could apply our CFR techniques for program representations that allow recursion as well. Technically, this would not require much work since the partial evaluation techniques of Gallagher [18] specialise CHCs that include recursion already. We also concentrated on numerical programs, and a possible future direction can concentrate on using CFR for program analysis tools where the data is not necessarily numerical. Here one should also adapt the partial evaluation techniques to support such specialisations, which seems doable for the partial evaluation techniques of Gallagher [18] since it is based on using abstract properties like those used in abstract domains of program analysis.

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