Kleene Algebras, Regular Languages and Substructural Logics

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We introduce the two substructural propositional logics KL, KL^+, which use disjunction, fusion and a unary, (quasi-)exponential connective. For both we prove strong completeness with respect to the interpretation in Kleene algebras and a variant thereof. We also prove strong completeness for language models, where each logic comes with a different interpretation. We show that for both logics the cut rule is admissible and both have a decidable consequence relation.

1 Introduction and Motivation

We introduce two substructural logics, the logic KL and the logic KL^+, by providing a Gentzen-style sequent calculus. KL and KL^+ have the same syntax: they are propositional, and thus consist of a countable set of propositional variables together with propositional connectives. They have the binary connectives $\vee, \bullet$, where $\vee$ is the classical *or*, and $\bullet$ is the *fusion* operator well-known from Lambek calculus, which is non-commutative and non-monotonic in both directions. Moreover, they have the unary connective ?, which is similar to the right exponential from linear logic, in the sense that it allows weakening and contraction on the right hand side of $\vdash$ and can be introduced only under strict conditions on the left hand side.[1]

We show that KL can be interpreted in Kleene algebras (we use this term in the sense of the axiomatization of [9]) in a natural way, and that this results in a strongly complete semantics We can thus interpret formulas as languages with the standard interpretation of regular expressions, consequence as set-theoretic inclusion, and still have a strongly complete semantics. A similar approach has be taken by [11], but our calculus differs substantially: in particular, it has a pure Gentzen-style presentation, no structural rules, and apart from the (admissible) cut rule, its syntactic decidability can be neatly read off. KL^+ differs minimally, but its semantics is much less obvious: KL^+ is a fragment of KL, that is, every valid sequent of KL^+ is valid in KL, but not vice versa. We show that we can interpret it in a closely related variant of Kleene algebras, and what is more interesting: we can interpret expressions of these algebras (and thus KL^+ formulas) as languages in the usual sense, with the only difference that ? is interpreted as Kleene plus instead of star, and get a strongly complete language-theoretic semantics. From this we can easily conclude that both logics are decidable (because the inclusion problem for regular languages is decidable), and that the cut rule is admissible for both logics: for every provable sequent, there is a cut-free proof.

There are two main motivations for KL: as it is sound and complete for Kleene algebras, we hardly need to explain its many possible interpretations. What is firstly interesting about KL is that it connects logical questions with language-theoretic questions and regular expressions (as has been done, in a rather different setting, by [1]). We thus can use language-theoretic techniques to check theoremhood and consequence; and conversely, we can use our cut-free sequent calculus to check whether the language

[1] See [5] and [14] for treatment of the full Lambek calculus; [7] for linear logic.

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denoted by one regular expression is a subset of a language denoted by another one (which is still a "hot topic" in computer science, see e.g. [8]). The main motivation for KL^+ is somewhat philosophical, but also has a computational content: if we want to interpret KL as a logic of events/processes and ? as a *progressive aspect* of a process (saying it is going on in some interval rather than being completed in this interval), some of the KL rules seem to be too strong, whereas those of KL^+ seem to be reasonable. It is interesting and instructive that many nice results from KL – cut-free sequent calculus, decidable and complete algebraic and language-theoretic semantics – can be transferred to KL^+; in particular, the language-theoretic semantics of the latter is far from obvious.

2 Syntax and Sequent Calculus of KL

We now present the syntax of both KL and KL^+. The set of formulas is defined as follows: let *Var* be a countably infinite set of variables. Then we define:

1. If $\alpha \in Var$, then $\alpha \in Form(Var)$.
2. If $\alpha \in Form(Var)$, then $\alpha? \in Form(Var)$.
3. If $\alpha, \beta \in Form(Var)$, then $\alpha \bullet \beta \in Form(Var)$.
4. If $\alpha, \beta \in Form(Var)$, then $\alpha \vee \beta \in Form(Var)$.

This defines the set of formulas. I first say a word on the intuitive meaning of formulas. Atomic propositions might be best thought of as events or actions. The $\vee$ should be clear, representing a classical "or" in a non-classical context. The $\bullet$ can be read as "and then", meaning temporal sequence of events. The intuitive meaning of ? can be thought of: "is happening (or taking place) some arbitrary number (including 0) of times". This is very close to the intuitive interpretation of operations in Kleene algebras; for more explicit considerations consider [13].

We let lowercase Greek letters range over formulas, uppercase Greek letters over finite, possibly empty sequences of formulas, which we write in the usual fashion just separated by ",". $\mathsf{KL}, \mathsf{KL}^+$ are substructural logics, that is, the usual structural rules of weakening, contraction, monotonicity are not legitimate. If we present a sequence of formulas separated by ",", this means that the sequence is ordered: we cannot exchange neither order nor cardinality of its elements, nor can we add or take away anything without an additional rule. The "," is however associative, that is, for a sequence α, β, γ there is no precedence. Now we define the derivability relation $\vdash$ of the logic KL between sequences of formulas and formulas. A substructural consequence relation as the one we present here also gives rise to a structural consequence relation (see [5]), but we will not be concerned with this here.

$$\text{(ax)}\ \frac{}{\alpha \vdash \alpha} \qquad \text{(cut)}\ \frac{\Gamma, \alpha, \Delta \vdash \beta \quad \Theta \vdash \alpha}{\Gamma, \Theta, \Delta \vdash \beta}$$

$$(\vee\text{I})\ \frac{\Gamma, \alpha, \Delta \vdash \gamma \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \alpha \vee \beta, \Delta \vdash \gamma} \qquad (\text{I}\vee 1)\ \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta} \qquad (\text{I}\vee 2)\ \frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta}$$

$$(\text{I}\bullet)\ \frac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \bullet \beta} \qquad (\bullet\text{I})\ \frac{\Gamma, \alpha, \beta, \Delta \vdash \gamma}{\Gamma, \alpha \bullet \beta, \Delta \vdash \gamma}$$

These rules are the usual axioms of sequent calculi, and usual rules for $\vee, \bullet$ which should be familiar to anyone familiar with some substructural logic. In addition, we have to make sure that our logic satisfies the distributive law for $\vee$ and $\bullet$: (DIS1) $(\alpha \bullet \beta) \vee (\alpha \bullet \gamma) \vdash \alpha \bullet (\beta \vee \gamma)$ and (DIS2) $\alpha, \beta \vee \gamma \vdash (\alpha \bullet \beta) \vee (\alpha \bullet \gamma)$. (DIS1) is derivable from the above rules, but (DIS2) is not (see [14] for background), so we have to add a rule to make sure (DIS2) holds:

$$\text{(D)}\ \frac{\Gamma \vdash \alpha \bullet (\beta \vee \gamma)}{\Gamma \vdash (\alpha \bullet \beta) \vee (\alpha \bullet \gamma)}$$

This way, we make sure the distributive law holds for our logic. Now comes the most interesting group of axioms and rules, namely the ones for ?.

$$\text{(ax?)}\ \frac{}{\vdash \alpha?} \qquad \text{(I?1)}\ \frac{\Delta \vdash \alpha \quad \Gamma \vdash \alpha?}{\Delta, \Gamma \vdash \alpha?} \qquad \text{(I?2)}\ \frac{\Delta \vdash \alpha \quad \Gamma \vdash \alpha?}{\Gamma, \Delta \vdash \alpha?}$$

(I?1),(I?2) form a symmetric group; we use it in order to introduce formulas to the left of the turnstile, if on the right there is a formula α?. By (ax?), the problem is not to get a formula α? on the right, but rather introduce material on its left. The next group is responsible for introducing the ? on the left side of the turnstile.

$$\text{(?I1)}\ \frac{\alpha, \beta \vdash \beta \quad \Gamma \vdash \beta}{\alpha?, \Gamma \vdash \beta} \qquad \text{(?I2)}\ \frac{\beta, \alpha \vdash \beta \quad \Gamma \vdash \beta}{\Gamma, \alpha? \vdash \beta}$$

This is sufficient for ? We now introduce the constants $1, 0$ as follows; note that we can also do without, and the extension we thereby introduce is conservative.

$$\text{(1I)}\ \frac{\Gamma, \Delta \vdash \alpha}{\Gamma, 1, \Delta \vdash \alpha} \qquad \text{(I1)}\ \frac{}{\vdash 1} \qquad \text{(0I)}\ \frac{}{\Gamma, 0, \Delta \vdash \alpha}$$

Note that our 1 is weak, in the sense that it is not necessarily derivable from any sequence; our 0 on the other side is strong: everything is derivable from any sequence containing it. So the two are not duals, and we do not have a strong 1 (usually written $\top$) or a weak 0 (the strong 0 is usually denoted by $\perp$). From the axioms it follows that 1 is the "neutral element" (we already speak in algebraic terms) of $\bullet$, and 0 the "neutral element" of $\vee$. Moreover, 0? is logically equivalent to 1, and 1? equivalent to 1. The cut rule goes without comment, but we already presage that we will later show that it is admissible, that is, anything derivable with cut is also derivable without it. We define a KL-**proof tree** as usual in logic: it is a (binary) labelled tree where each leaf is labelled by an axiom, and each elementary subtree (node with its two daughters) is labelled according to one of our inference rules. We say a sequent $\Gamma \vdash \alpha$ is derivable (in KL) by our proof calculus, if there is a KL-proof tree such that its root is labelled by $\Gamma \vdash \alpha$. If such a sequent is derivable, we write $\Vdash_{\mathsf{KL}} \Gamma \vdash \alpha$.

3 Kleene Algebras, Semantics of KL and its Completeness

For the semantics of KL we use a well-known class of structures (see [9], and [4] as classical reference). A **Kleene algebra** is an algebra $(K, +, \cdot, *, 0, 1)$, where $+, \cdot$ are binary operators, $*$ is unary, $0, 1$ are constants in K and K is a set closed under the former operations, and satisfying the following equations:

(K1) $(a+b)+c = a+(b+c)$

(K2) $a+b=b+a$
(K3) $a+a=a$
(K4) $a+0=a$
(K5) $a\cdot(b\cdot c)=(a\cdot b)\cdot c$
(K6) $a\cdot 1=1\cdot a=a$
(K7) $0\cdot a=a\cdot 0=0$
(K8) $a\cdot(b+c)=a\cdot b+a\cdot c$
(K9) $(b+c)\cdot a=b\cdot a+c\cdot a$

This means: $(K,+,\cdot,0,1)$ is an idempotent semiring. We write ab for $a\cdot b$; as usual, $\cdot$ has precedence over $+$. We also define a partial order on K as usual: $a\leq b$ if and only if $a+b=b$. To get a Kleene algebra, we still need two inequations and two quasi-equations for $*$:

(K10) $1+aa^*\leq a^*$
(K11) $1+a^*a\leq a^*$
(K12) If $b+ac\leq c$, then $a^*b\leq c$
(K13) If $b+ca\leq c$, then $ba^*\leq c$.

These axioms say that a^*b is the unique smallest solution for x in the inequation $b+ax\leq x$. As usual, $*$ has precedence over both $+,\cdot$. By $\mathscr{K}$ we denote the class of all Kleene algebras, that is, all algebras satisfying (K1)–(K13). Let $\mathbf{K}$ be a Kleene algebra. If an equation (or inequation) $a=b$ ($a\leq b$) holds in $\mathbf{K}$, we sometimes write $a=_{\mathbf{K}}b$ (or $a\leq_{\mathbf{K}}b$) for reasons of clarity. If a certain (in)equation is valid in all Kleene algebras, we write $a=_{\mathscr{K}}b$ ($a\leq_{\mathscr{K}}b$) (equivalently, if it is valid in the free Kleene algebra, see [2] for algebraic background). Two properties of Kleene algebras are extremely important for us: 1. $\leq_{\mathbf{K}}$ is transitive for any Kleene algebra $\mathbf{K}$ (and so is $\leq_{\mathscr{K}}$), and secondly, $\leq_{\mathscr{K}}$ respects concatenation: if $a\leq b, c\leq d$, then $ac\leq bd$. This follows from distributivity: $bd=(a+b)(c+d)=a(c+d)+b(c+d)=ac+ad+bc+bd=ac+ac+ad+bc+bd$. This is equivalent to saying that from $a\leq b$ it follows that $cad\leq cbd$, a property to which we refer as **strong transitivity** (this is also known as monotonicity of $\cdot$, but we prefer to use monotonicity in connection with weakening).

An **interpretation** $\overline{\sigma}:Form(Var)\to\mathbf{K}$ is defined as follows: let $\sigma:Var\to K$ be an arbitrary map from variables to K; we obtain $\overline{\sigma}$ inductively as follows:

1. $\overline{\sigma}(\alpha)=\sigma(\alpha)$, if $\alpha\in Var$.
2. $\overline{\sigma}(\alpha\vee\beta)=\overline{\sigma}(\alpha)+\overline{\sigma}(\beta)$
3. $\overline{\sigma}(\alpha\bullet\beta)=\overline{\sigma}(\alpha)\cdot\overline{\sigma}(\beta)$
4. $\overline{\sigma}(\alpha?)=(\overline{\sigma}(\alpha))^*$
5. $\overline{\sigma}(1)=1$
6. $\overline{\sigma}(0)=0$

We extend $\overline{\sigma}$ also to *sequences* of formulas: for $\Gamma=\gamma_1,...,\gamma_i$, put $\overline{\sigma}(\Gamma):=\overline{\sigma}(\gamma_1\bullet...\bullet\gamma_i)$; that is, the "," of sequences is interpreted in the same manner as "$\bullet$". Having defined $\overline{\sigma}$, we define a **model** as a tuple $(\mathbf{K},\sigma)$. We say a sequent $\Gamma\vdash\alpha$ of KL is **true** in $(\mathbf{K},\sigma)$, in symbols $(\mathbf{K},\sigma)\models\Gamma\vdash\alpha$, if $\overline{\sigma}(\Gamma)\leq\overline{\sigma}(\alpha)$ holds in $\mathbf{K}$ (we also write: $\overline{\sigma}(\Gamma)\leq_{\mathbf{K}}\overline{\sigma}(\alpha)$). In case we have a sequent of the form $\vdash\alpha$, where Γ is the empty sequence, we write $(\mathbf{K},\sigma)\models\ \vdash\alpha$ if $1\leq_{\mathbf{K}}\overline{\sigma}(\alpha)$.

We write $\mathbf{K} \models \Gamma \vdash \alpha$, if for all maps $\sigma : Var \to K$, we have $(\mathbf{K}, \sigma) \models \Gamma \vdash \alpha$; and we write $\models_{\mathscr{K}} \Gamma \vdash \alpha$, if for all Kleene algebras $\mathbf{K}$ and valuations $\sigma : Var \to K$, we have $(\mathbf{K}, \sigma) \models \Gamma \vdash \alpha$. In that case, we also say that $\Gamma \vdash \alpha$ is **valid**. Our first main theorem is the following:

Theorem 1 $\Vdash_{\mathsf{KL}} \Gamma \vdash \alpha$ *if and only if* $\models_{\mathscr{K}} \Gamma \vdash \alpha$.

So we have soundness and strong completeness. We start with the only if direction, which is the usual soundness.

Proof. ***Soundness***: We perform the usual proof by induction; we only prove those cases which are not standard and straightforward. (ax), ($\vee$I), (I$\vee$1),(I$\vee$2), (I$\bullet$),($\bullet$I) are clear. (D) follows from the distributive law in $\mathscr{K}$.

(ax?) should be clear: it follows immediately from (K10) that $1 \leq_{\mathscr{K}} a^*$ holds.

(I?1) Assume $\overline{\sigma}(\Delta) \leq_{\mathscr{K}} \overline{\sigma}(\alpha)$, and $\overline{\sigma}(\Gamma) \leq_{\mathscr{K}} (\overline{\sigma}(\alpha))^*$. We have $aa^* \leq_{\mathscr{K}} a^*$ (by (K13)), and $\leq_{\mathscr{K}}$ is transitive and respects concatenation. Consequently, $\overline{\sigma}(\Delta)\overline{\sigma}(\Gamma) \leq_{\mathscr{K}} \overline{\sigma}(\alpha)(\overline{\sigma}(\alpha))^* \leq_{\mathscr{K}} (\overline{\sigma}(\alpha))^*$.

(I?2) is similar.

(?I1) Assume $\overline{\sigma}(\alpha)\overline{\sigma}(\beta) \leq_{\mathscr{K}} \overline{\sigma}(\beta)$, and $\overline{\sigma}(\Gamma) \leq_{\mathscr{K}} \overline{\sigma}(\beta)$. From the premises and the fact that from $a \leq_{\mathscr{K}} c, b \leq_{\mathscr{K}} c$ it follows that $a + b \leq_{\mathscr{K}} c$;[2] we thus have $\overline{\sigma}(\Gamma) + \overline{\sigma}(\alpha)\overline{\sigma}(\beta) \leq_{\mathscr{K}} \overline{\sigma}(\beta)$, and by (K12), $\overline{\sigma}(\alpha)^*\overline{\sigma}(\Gamma) \leq_{\mathscr{K}} \overline{\sigma}(\beta)$, and thus $\models_{\mathscr{K}} \alpha?, \Gamma \vdash \beta$.

(?I2) is similar.

(1I) clear for (K6) and the fact that "," matches $\bullet$.
(I1) is also obviously correct, as the empty antecedent is interpreted as 1.
(0I) follows from (K7) and (K4).
(cut) This corresponds to strong transitivity, which we have established for $\mathscr{K}$.

This completes the soundness direction.

We now prove ***completeness***. We proceed in the usual fashion: we construct the algebra $\mathbf{T}$ of KL-terms modulo KL-equivalence, where $\leq_T$ is $\vdash_{\mathsf{KL}}$ modulo logical equivalence. We prove that $\mathbf{T}$ is a Kleene algebra. From this we can conclude: if $\nVdash_{\mathsf{KL}} \Gamma \vdash \alpha$, then there is a Kleene algebra $\mathbf{T}$, assignment σ, such that $(\mathbf{T}, \sigma) \not\models \Gamma \vdash \alpha$, and consequently, $\not\models_{\mathscr{K}} \Gamma \vdash \alpha$. By contraposition, it follows that if $\models_{\mathscr{K}} \Gamma \vdash \alpha$, then $\Vdash_{\mathsf{KL}} \Gamma \vdash \sigma$ (alternatively, this goes by a direct argument: if something holds in all Kleene algebras, it holds in $\mathbf{T}$, therefore it is provable).

Definition of T, term algebra or Lindenbaum-Tarski construction.

Define $\sim_{\mathsf{KL}} \subseteq Form(Var) \times Form(Var)$ as follows: $\alpha \sim_{\mathsf{KL}} \beta$ if $\Vdash_{\mathsf{KL}} \alpha \vdash \beta$ and $\Vdash_{\mathsf{KL}} \beta \vdash \alpha$. This is obviously an equivalence relation, because $\vdash$ is reflexive and transitive. We now define $T := Form(Var)/\sim_{\mathsf{KL}}$, the set of equivalence classes of $Form(Var)$ under $\sim_{\mathsf{KL}}$, which is the generator set of $\mathbf{T}$. The operators and constants of our algebra are the constructors and constants of $Form(Var)$, so we get the algebra $\mathbf{T} = (Form(Var)/\sim_{\mathsf{KL}}, \vee, \bullet, ?, 0, 1)$, and for $a, b \in Form(Var)/\sim_{\mathsf{KL}}$, we define $a \leq_{\mathbf{T}} b$ iff there are $\alpha \in a, \beta \in b$, and $\Vdash_{\mathsf{KL}} \alpha \vdash \beta$; we define the equality $\mathbf{T}$ as usual: $a =_{\mathbf{T}} b$, if $a \leq_{\mathbf{T}} b$ and $b \leq_{\mathbf{T}} a$.

[2] Because then $a + c =_{\mathscr{K}} c =_{\mathscr{K}} b + c =_{\mathscr{K}} a + b + c$

As usually, we have to show that $\sim_{\mathsf{KL}}$ is a congruence over the constructors, in order to show that $\sim_{\mathsf{KL}}$ is a congruence and thus the operations on equivalence classes are independent of representatives. We indicate the usual inductive procedure:

1. If $a \sim_{\mathsf{KL}} b$, $c \sim_{\mathsf{KL}} d$, then $a \bullet c \sim_{\mathsf{KL}} b \bullet d$. This is fairly straightforward:

$$\dfrac{\dfrac{a \vdash b \quad c \vdash d}{a, c \vdash b \bullet d}}{a \bullet c \vdash b \bullet d}$$
, and the other way round.

2. If $a \sim_{\mathsf{KL}} b$, $c \sim_{\mathsf{KL}} d$, then $a \vee c \sim_{\mathsf{KL}} b \vee d$. Straightforward:

$$\dfrac{\dfrac{a \vdash b}{a \vdash b \vee d} \quad \dfrac{a \vdash d}{c \vdash b \vee d}}{a \vee c \vdash b \vee d}$$
, and the other way round.

3. If $a \sim_{\mathsf{KL}} b$, then $a? \sim_K Lb?$.

Take the following derivation:

$$\dfrac{\dfrac{a \vdash b \quad b? \vdash b?}{a, b? \vdash b?} \quad \vdash b?}{a? \vdash b?}$$

So we have shown that $\sim_{\mathsf{KL}}$ is a congruence over constructors, and thus the quotient algebra is well-formed and independent of representatives. Therefore, we will proceed in the sequel as if congruence classes of formulas were formulas, and not distinguish the two notationally. Now comes the crucial step, namely to show that **T** actually *is* a Kleene algebra. We prove that by going through the equations (K1)–(K13) one by one. Recall that for **T**, the equations consist of two inequations; while in principle, we have to prove both, we usually only prove one direction, as the other one is similar. Also, we omit some proofs, which are well-known from existing logics.

(K1) $(a+b)+c =_{\mathbf{T}} a+(b+c)$

Obvious, canonical proof.

(K2) $a+b =_{\mathbf{T}} b+a$

Obvious, canonical proof.

(K3) $a+a =_{\mathbf{T}} a$

Obvious.

(K4) $a+0 =_{\mathbf{T}} a$

$a \leq_{\mathbf{T}} a \vee 0$ follows from (I$\vee$1), $a \vee_{\mathbf{T}} 0 \leq_{\mathbf{T}} a$ holds because $a \leq_{\mathbf{T}} a, 0 \leq_{\mathbf{T}} a$.

(K5) $a \cdot (b \cdot c) =_{\mathbf{T}} (a \cdot b) \cdot c$

Standard, as sequences separated by “,” are associative.

(K6) $a \cdot 1 =_{\mathbf{T}} 1 \cdot a =_{\mathbf{T}} a$

Straightforward from (1I),(I1) and the the $\bullet$-rules.

(K7) $0 \cdot a =_{\mathbf{T}} a \cdot 0 = 0$

Clear from (0I).

(K8) $a \cdot (b+c) =_{\mathbf{T}} a \cdot b + a \cdot c$

$a(b \vee c) \leq_{\mathbf{T}} (ab) \vee (ac)$ follows from (D); $(ab) \vee (ac) \leq_{\mathbf{T}} a(b \vee c)$ follows from $ac \leq_{\mathbf{T}} a(b \vee c)$, $ab \leq_{\mathbf{T}} a(b \vee c)$ and ($\vee$I).

(K9) $(b+c) \cdot a =_{\mathbf{T}} b \cdot a + c \cdot a$

similar.

(K10) $1+aa^* \leq_{\mathbf{T}} a^*$
We have to show that $1 \vee (a \bullet a?) \leq_{\mathbf{T}} a?$. Here the derivation:

$$\dfrac{\dfrac{\overline{\vdash a?}}{1 \vdash a?} \qquad \dfrac{\dfrac{a \vdash a \quad a? \vdash a?}{a, a? \vdash a?}}{a \bullet a? \vdash a?}}{1 \vee a \bullet a? \vdash a?}$$

(K11) $1+a^*a \leq_{\mathbf{T}} a^*$
similar.

(K12) If $b+ac \leq_{\mathbf{T}} c$, then $a^*b \leq_{\mathbf{T}} c$

Assume we have $\Vdash_{\mathsf{KL}} b \vee (a \bullet c) \vdash c$. We first show that in this case, we must have $b \vdash c$, $a \bullet c \vdash c$. If the last rule of the derivation tree concerned the left-hand side, there is no choice: the rule was ($\vee$I), and the claim follows, because otherwise application was not legitimate.

So assume the last rule concerned the right-hand side; in that case, there are several possibilities. (I$\bullet$) can be excluded for the structure of the antecedent (it does not contain "$,$"); the ?-rules change the right hand side only in case of empty antecedent, so they can be excluded; so the only candidates are (D) and (I$\vee$1),(I$\vee$2). All of these rules have a very nice property: they are (1) unary, and (2) applicable regardless of the properties of left hand side of the antecedent. So we can safely assume that $b \vee (a \bullet c) \vdash c$ has been derived from $b \vee (a \bullet c) \vdash c'$, such that the last rule applied to derive $b \vee (a \bullet c) \vdash c'$ was ($\vee$I), and $b \vee (a \bullet c) \vdash c$ has been subsequently derived by applications of (D), (I$\vee$1) and (I$\vee$2) only. Consequently, we must have two valid derivations of $b \vdash c'$, and $a \bullet c \vdash c'$. Now, as the rules (D), (I$\vee$1), (I$\vee$2) do not care for the left hand side, we can thus also derive $b \vdash c$, and $a \bullet c \vdash c$.

We can apply exactly the same argument to show that if we can derive $a \bullet c \vdash c$, we can also derive $a, c \vdash c$. Thus we know that if $\Vdash_{\mathsf{KL}} b \vee (a \bullet c) \vdash c$ holds, then we also have $\Vdash_{\mathsf{KL}} a, c \vdash c$ and $\Vdash b \vdash c$. We can thus apply (?I1) to derive

$$\dfrac{a, c \vdash c \quad b \vdash c}{a?b \vdash c}$$

(K13) If $b+ca \leq_{\mathbf{T}} c$, then $ba^* \leq_{\mathbf{T}} c$.
similar.

This shows that **T** is a Kleene algebra and completes the proof of completeness. □

Note that for the completeness part of the proof, the cut-rule is not needed, in fact it is not even mentioned! There are a number of important consequences which follow from this result. But before we come to these, we first introduce the logic KL^+ and prove a similar result.

4 KL^+

We now present a new logic KL^+, which is a fragment of KL. Our motivation is the following: we would like to intuitively interpret $\alpha?$ as a sort of progressive aspect of the event α, that is: if α means "(in some interval) α is completed", then $\alpha?$ should mean: "(in some interval) α is going on". Obviously, then (ax?) is way too strong: we cannot assert that just anything is going on. What we rather can assert is the weaker implication: if something happens (possibly several times) in an interval, then it is happening in this interval: if in some interval I ate a pizza, I was eating a pizza in this interval, though not the converse, for my starting and finishing the pizza might be laying outside the interval.[3] In terms of logical consequence, the progressive $\alpha?$ of an atomic event α can be characterized as follows: it follows from any (non-zero) number of iterations (in terms of $\cdot$) of α and $\alpha?$, and from nothing else, except for transitivity and logical laws which govern the other connectives.[4] We devise KL^+ in order to agree with our intuition on the progressive aspect of events; one of our main results will be that whereas the ? of KL can be interpreted as Kleene star, the ? of KL^+ can be interpreted as Kleene plus.

As we said, the syntax of KL^+ is identical to KL. Regarding its consequence relation and sequent calculus, we can re-use also most of the rules and axioms of KL. So we just say which rules are discarded, and which ones are new. To obtain the rules for KL^+, we take away the axiom (ax?) and substitute it with the weaker axiom (+?) It is clear that (+?) is derivable in KL. Note that (+?) *cannot* derive a sequent of the form $\vdash \alpha?$ or $\alpha \vdash \alpha\alpha?$. This is our intention; but as a consequence we also need to reconsider the rules (?I1) and (?I2), which we replace by (?+1),(?+2):

$$(+?)\ \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha?} \qquad (?{+}1)\ \frac{\alpha,\beta \vdash \beta \quad \alpha,\Gamma \vdash \beta}{\alpha?,\Gamma \vdash \beta} \qquad (?{+}2)\ \frac{\beta,\alpha \vdash \beta \quad \Gamma,\alpha \vdash \beta}{\Gamma,\alpha? \vdash \beta}$$

Regarding these rules, we have to say the following: given the cut rule, (?I1) and (?I2) are derivable from (?+1) and (?+2), respectively (just assume you have the premises of (?I1), $\alpha,\beta \vdash \beta$, $\Gamma \vdash \beta$; by cut, you get $\alpha,\Gamma \vdash \beta$). The converse is not true: we cannot derive (?+1) and (?+2) without (ax?). In particular, it is easy to show that without these rules, we cannot derive the sequent $\alpha?,\alpha \vdash \alpha \bullet (\alpha?)$: just ask what was the last rule applied to this sequent: the (I$\bullet$) rule was not applicable to derive this sequent, for $\alpha? \not\vdash \alpha$, and any other rule is out of the question for the syntactic form of the sequent. As long as we have the cut rule, we can moreover derive (I?1),(I?2). What if the cut rule is lacking? I do not see how to derive the two, but from cut admissibility, which we prove later on for $\mathsf{KL},\mathsf{KL}^+$, it follows that the two do not allow us to derive anything we could not derive without them in KL^+. On a related note, note that (?+1),(?+2) are also admissible in KL, that is, they would not add anything new to the calculus, though I do not see how they can be derived. As we have the cut rule, we could thus substitute (I?1),(I?2) by (?+1),(?+2) in KL, thereby giving a more uniform treatment of KL and KL^+. The reason we have not chosen this presentation is the following: for the alternative presentation, (as far as I can see) we need the cut-rule to prove that its term-algebra is a Kleene-algebra, so our simple semantic proof of cut admissibility (see section 9) for "standard" KL would no longer work.

It is clear that KL^+ is a fragment of KL, as its axioms and inference rules are derivable in KL. The question is: what exactly is the expressive power of KL^+? We will show the following: we can give it a strongly complete semantics in terms of (slightly modified) Kleene algebras and in terms of language models, by only a minor change in interpretation: we interpret the connector ? as Kleene *plus* instead of star. So what changes with the new axiom is essentially the meaning of ?, and nothing else. The proof of this is however slightly more complicated, as we have to make several steps consisting in algebraic embeddings. It is well known that we can define a^+ as aa^* (or a^*a). But nonetheless we cannot work directly with Kleene algebras, putting $\overline{\sigma}(\alpha?) = \overline{\sigma}(\alpha)^*\overline{\sigma}(\alpha)$, because we have $a^*a =_{\mathscr{K}} aa^*$, but none of the corresponding sequents is provable, and thus completeness fails.

As we do not work directly with Kleene algebras, we have to work with a variant and two embeddings. We interpret KL^+ in a class of algebras which we call $\mathscr{K}^{\#}$. We show strong completeness for this

[3]This is known as the imperfective paradox, see [12]; if I crossed the street, I was crossing it, but not vice versa.

[4]Of course, this is a gross simplification; linguistically speaking there is much more to it. See for example [6].

semantics, where we first go the usual way: we show that the term algebra $\mathbf{T}^+$ of KL^+ is a $\mathscr{K}^\#$ algebra, such that if $\not\Vdash_{\mathsf{KL}^+} \Gamma \vdash \alpha$, then $\not\models_{\mathscr{K}^\#} \Gamma \vdash \alpha$.

We then have to show that strong completeness also holds for the language-theoretic semantics. To this aim, we devise two maps i, j, which map $\mathscr{K}$ terms onto $\mathscr{K}^\#$ terms and vice versa. We show some validity-preserving properties of these maps, which allow us to extend language-completeness results from $\mathscr{K}$ to $\mathscr{K}^\#$, without having to perform a complicated proof from scratch as in [10].

5 Algebraic Semantics: $\mathscr{K}^\#$-algebras

We now define a variant of Kleene algebras, namely $\mathscr{K}^\#$ or Kleene plus algebras. We have the connectives $+, \cdot, \#$, where # is unary, and constants 0,1. We have (K1)–(K9) as in $\mathscr{K}$; then things change. We list the new axioms (K14+)–(K17+), together with the more expectable, but "wrong" axioms (K10+)–(K13+), just to show how the latter can be derived from the former, but not vice versa:

(K14+) $a + aa^\# \leq a^\#$.
(K15+) $a + a^\# a \leq a^\#$.
(K16+) If $ab + ac \leq c$, then $a^\# b \leq c$.
(K17+) If $ba + ca \leq c$, then $ba^\# \leq c$.

((K10+) $1 + a(1 + a^\#) \leq 1 + a^\#$)
((K11+) $1 + (1 + a^\#)a \leq 1 + a^\#$)
((K12+) If $b + ac \leq c$, then $((1 + a^\#)b) = b + a^\# b \leq c$)
((K13+) If $b + ca \leq c$, then $(b(1 + a^\#)) = b + ba^\# \leq c$)

(K10+)–(K13+) are very "expectable": they just consist in (K10)–(K13), where each time, a^* is replaced by $1 + a^\#$. If we "read" $a^\#$ as aa^* (or a^*a), then they are valid in $\mathscr{K}$, because then we have $a^* =_{\mathscr{K}} 1 + a^\#$. But the reader has to keep in mind that we have a different algebra here, where $*$ does not exist as a connective. (K14+) and (K15+) seem to be redundant with (K10+),(K11+), and in fact they create redundancy: we can derive (K10+) from (K14+) and (K11+) from (K15+), but not vice versa (at least I do not see how): we can easily derive $1 + a + aa^\# \leq 1 + a^\#$ from (K10+); but we still have to get rid of the 1. Conversely, we can derive (K10+),(K11+) from (K14+),(K15+): if $aa^\# \leq a^\#$, then $aa^\# \leq 1 + a^\#$; this means: $aa^\# + 1 + a^\# = 1 + aa^\# + 1 + a^\# = 1 + a^\#$ iff $1 + aa^\# \leq 1 + a^\#$. Same holds for the pairs (K12+),(K13+) and (K16+),(K17+): we can derive (K12+) from (K16+), because if $b + ac \leq c$, then by strong transitivity $ab \leq c$, and thus $a^\# b \leq c$. The converse does not hold, and in particular, (K12+),(K13+) do not allow us to derive $a^\# \leq a + a^\# a$. With (K14+),(K16+) it is an easy exercise (put $b := 1, a := a, c := a + a^\# a$). So we axiomatize $\mathscr{K}^\#$ by (K1)–(K9),(K14+)–(K17+); we leave the other axioms for illustration and because they turn out to be useful for proving properties of our later embeddings.

Most basic properties of $\mathscr{K}$ transfer to $\mathscr{K}^\#$: $\leq_{\mathscr{K}^\#}$ is defined over $+$ in the usual fashion and is thus reflexive, transitive and antisymmetric. The same holds for the fact that $\leq_{\mathscr{K}^\#}$ respects concatenation, because of distributivity, and we thus have strong transitivity. We now devise an **interpretation** of KL^+ in $\mathscr{K}^\#$. We define σ as usual, and $\overline{\sigma}$ as before, the expectable exception that

$$\overline{\sigma}(\alpha?) := \overline{\sigma}(\alpha)^\#.$$

6 Completeness of the Algebraic Semantics

We take the usual definitions, and prove soundness and completeness of KL^+ for $\mathscr{K}^\#$:

Theorem 2 *We have* $\models_{\mathscr{K}^\#} \Gamma \vdash \alpha$ *if and only if* $\Vdash_{\mathsf{KL}^+} \Gamma \vdash \alpha$.

Proof. We start with the *if* direction (***soundness***). Most of the axioms can be skipped, as the old soundness arguments remain valid (as KL^+ is a fragment of KL, and most equations of $\mathscr{K}$ are valid in $\mathscr{K}^\#$). We only need to prove that the inference rules regarding ? are sound with respect to $\mathscr{K}^\#$.

(+?) We have to show that if $\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\alpha)$, then $\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\alpha)^\#$. That is straightforward with (K14+), according to which $a \leq_{\mathscr{K}^\#} a^\#$, and transitivity of $\leq_{\mathscr{K}^\#}$.

(?+1) We make the usual induction. Assume the premises are satisfied, that is, we have $\models_{\mathscr{K}^\#} \alpha,\Gamma \vdash \beta$ and $\models_{\mathscr{K}^\#} \alpha,\beta \vdash \beta$; consequently, $\overline{\sigma}(\alpha)\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\beta)$ and $\overline{\sigma}(\alpha)\overline{\sigma}(\beta) \leq_{\mathscr{K}^\#} \overline{\sigma}(\beta)$. Consequently, by the order definition, we have $\overline{\sigma}(\alpha)\overline{\sigma}(\Gamma)+\overline{\sigma}(\alpha)\overline{\sigma}(\beta) \leq_{\mathscr{K}^\#} \overline{\sigma}(\beta)$. By (K16+) it follows that $\overline{\sigma}(\alpha)^\#\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\beta)$, and thus $\models_{\mathscr{K}^\#} \alpha?,\Gamma \vdash \beta$.

(?+2) similar.

(I?1) Assume the premises hold. Then we have $\overline{\sigma}(\Delta) \leq_{\mathscr{K}^\#} \overline{\sigma}(\alpha)$ and $\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\alpha)^\#$. We have to show that $\overline{\sigma}(\Delta)\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\alpha)^\#$. Because $\leq_{\mathscr{K}^\#}$ respects concatenation, we know that $\overline{\sigma}(\Delta)\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\alpha)\overline{\sigma}(\alpha)^\#$. From $(K14+)$ and transitivity it follows that $\overline{\sigma}(\Delta)\overline{\sigma}(\Gamma) \leq_{\mathscr{K}^\#} \overline{\sigma}(\alpha)^\#$.

(I?2) similar.

This completes the soundness direction.

Completeness.

We do the same construction as before, now call $\mathbf{T}^+$ the algebra $(T,\vee,\bullet,?,0,1)$; we define T to be the set of $Form(Var)/{\sim_{\mathsf{KL}^+}}$, that is, the set of congruence classes of KL^+ formulas under the congruence $\sim_{\mathsf{KL}^+}$, which is logical equivalence in KL^+. We put $a \leq_{\mathbf{T}^+} b$ iff $\Vdash_{\mathsf{KL}^+} \alpha \vdash \beta$ for some $\alpha \in a, \beta \in b$, and put $a =_{\mathbf{T}^+} b$ iff $a \leq_{\mathbf{T}^+} b$ and $b \leq_{\mathbf{T}^+} a$. We skip the proof that $\sim_{\mathsf{KL}^+}$ is a congruence over connectives, which is straightforward, and write congruence classes as if they were formulas.

To show completeness, we have to show that $\mathbf{T}^+$ is a $\mathscr{K}^\#$-algebra. Again, we can skip most of the equations as the proof is identical to the one for KL; we check only those in which # (which is ? in $\mathbf{T}^+$) occurs, and which are not derivable.

(K14+) $a + aa^\# \leq_{\mathbf{T}^+} a^\#$.
It is obvious that $a \leq_{\mathbf{T}^+} a?$; we need to show that $aa? \leq_{\mathbf{T}^+} a?$. That is also easy:

$$\frac{a \vdash a \quad a? \vdash a?}{aa? \vdash a?};$$

and consequently $a \vee aa? \leq_{\mathbf{T}^+} a?$.

(K15+) similar.

(K16+) If $ab+ac \leq_{\mathbf{T}^+} c$, then $a^\# b \leq_{\mathbf{T}^+} c$.

Assume we have $(ab) \vee (ac) \leq_{\mathbf{T}^+} c$. Then by the same argument as for KL, we can conclude that we have both $\Vdash_{\mathsf{KL}^+} a, b \vdash c$ and $\Vdash_{\mathsf{KL}^+} a, c \vdash c$. Then by (?+1), we derive

$$\frac{ab \vdash c \quad ac \vdash c}{a?b \vdash c}$$

(K17+) is similar.

This completes the proof. □

Note again that the cut rule is not needed for the completeness proof. But this is not as satisfying as the other completeness theorem, because we do not know nearly as much about $\mathscr{K}^{\#}$ algebras as about $\mathscr{K}$ algebras. We now establish some results showing the algebraic correlation between $\mathscr{K}$ and $\mathscr{K}^{\#}$.

7 Algebraic Relations between $\mathscr{K}$ and $\mathscr{K}^{\#}$

We now prove three lemmas, which together give us a very powerful result. Let $(K, +, \cdot, *, 0, 1)$ be a $\mathscr{K}$ algebra, $(K', +, \cdot, \#, 0, 1)$ be a $\mathscr{K}^{\#}$ algebra. We define the map i as follows:

1. for a atomic, $i(a) = a$;
2. $i(a+b) = i(a) + i(b)$;
3. $i(a \cdot b) = i(a) \cdot i(b)$;
4. $i(a^*) = 1 + (i(a))^{\#}$.

i is a map from $\mathscr{K}$-terms to $\mathscr{K}^{\#}$-terms. Its most important property is the following, which is intuitively clear, but still needs to be proved.

Lemma 3 *If $a \leq_{\mathscr{K}} b$, then $i(a) \leq_{\mathscr{K}^{\#}} i(b)$.*[5]

Proof. We have $a^* =_{\mathscr{K}} 1 + aa^* =_{\mathscr{K}} 1 + a^*a$. Consider a class of algebras $\mathscr{K}^{\#'}$, which consists of all algebras which are i-images of $\mathscr{K}$-algebras. Its axioms are exactly the i-images of axioms (K1+)–(K13+), but the terms have a different syntax: $(1 + a^{\#})$ is just a single constructor over a, to which the distributive, commutative, associative laws etc. do not apply. It is immediately clear (by the purely syntactic translation) that $a \leq_{\mathscr{K}} b$ if and only if $i(a) \leq_{\mathscr{K}^{\#'}} i(b)$. Now, as we have shown before, all axioms (K1+)–(K13+) are valid in $\mathscr{K}^{\#}$. Moreover, the syntactic difference of $\mathscr{K}^{\#}$ and $\mathscr{K}^{\#'}$ needs not bother us, as long as we are trying to prove that $\mathscr{K}^{\#'}$ inequations are valid in $\mathscr{K}^{\#}$: in the latter, nothing prevents us from treating $(1 + a^{\#})$ as a unit, that is, not applying any of distributive, commutative or associative laws to it. So the inequations valid in $\mathscr{K}^{\#'}$ are a (proper) subset of the inequations valid in $\mathscr{K}^{\#}$; and so, as $i(a) \leq_{\mathscr{K}^{\#'}} i(b)$, we have $i(a) \leq_{\mathscr{K}^{\#}} i(b)$. □

Now we define a sort of inverse of i; define the map j from $\mathscr{K}^{\#}$ to $\mathscr{K}$ terms as follows:

1. $j(a) = a$ for atomic a;
2. $j(a+b) = j(a) + j(b)$;
3. $j(a \cdot b) = j(a) \cdot j(b)$;
4. $j(a^{\#}) = j(a)(j(a))^*$.

[5]The inverse implication can also be proved in much the same way as lemma 4, just using lemma 5 instead of lemma 3. This result does however not play a role in the sequel, so we do not explicitly state it.

Lemma 4 *If $j(a) \leq_{\mathscr{K}} j(b)$, then $a \leq_{\mathscr{K}^\#} b$.*

Proof. Assume we have $j(a) \leq_{\mathscr{K}} j(b)$. As j is trivial for all connectives but #, we only treat terms of the form $a^\#$. It follows from the the map j that in $j(a), j(b)$ all occurrences a^* occur in subterms (aa^*). Now form the images $i(j(a)), i(j(b))$, which are $\mathscr{K}^\#$-terms. By lemma 3, we have $i(j(a)) \leq_{\mathscr{K}^\#} i(j(b))$. Here, all occurrences of # are in subterms $(a(1+a^\#))$. By the usual laws, we have $(a(1+a^\#)) =_{\mathscr{K}^\#} (a1+aa^\#) =_{\mathscr{K}^\#} (a+aa^\#)$. Moreover, we have $a^\# =_{\mathscr{K}^\#} a+aa^\#$; so we have $i(j(a)) =_{\mathscr{K}^\#} j^{-1}(j(a)) =_{\mathscr{K}^\#} a$; same for b; and thus $a \leq_{\mathscr{K}^\#} b$. □

Lemma 5 *If $a \leq_{\mathscr{K}^\#} b$, then $j(a) \leq_{\mathscr{K}} j(b)$.*

Proof. We show this in the same fashion as lemma 3: construct an intermediate class of algebras $\mathscr{K}'$, where aa^* is a single constructor over a, to which associative, distributive laws do not apply, and where all terms have the form $j(a)$ for some $\mathscr{K}^\#$-term a, and all $\mathscr{K}^\#$-axioms in their translation under j are valid. We show that all these j-images of $\mathscr{K}^\#$-axioms are also valid (under the different syntactic reading of $\mathscr{K}$!) within $\mathscr{K}$:

(j(K14+)) $a + aaa^* \leq aa^*$.

That is obviously derivable by $1 \leq_{\mathscr{K}} a^*$, $a \leq_{\mathscr{K}} a^*$, $a \leq_{\mathscr{K}} a$, and strong transitivity. (K15+) is similar.

(j(K16+)) If $ab+ac \leq c$, then $aa^*b \leq c$.

Just put $b := ab$ in the quasi-inequation "if $b+ac \leq_{\mathscr{K}} c$, then $a^*b \leq_{\mathscr{K}} c$", and we obtain $a^*ab \leq c$ from the premise. As we have $aa^* =_{\mathscr{K}} a^*a$, the consequence is valid in $\mathscr{K}$ if the premises are. (K17+) is similar.

Again, all axioms of $\mathscr{K}'$ are valid in $\mathscr{K}$; the possible manipulations of $\mathscr{K}'$ are also a subset thereof, so everything valid in $\mathscr{K}'$ is valid in $\mathscr{K}$; if $a \leq_{\mathscr{K}^\#} b$, then $j(a) \leq_{\mathscr{K}'} j(b)$, and thus $j(a) \leq_{\mathscr{K}} j(b)$. □

These results are of course of some value on their own, because they show us how the intuitive correlation of $\mathscr{K}$ and $\mathscr{K}^\#$ corresponds to formal notions. Their importance for KL^+ reveals itself in the next section, where we consider language models.

8 Completeness of Language-theoretic Semantics

As is well known, we can interpret $\mathscr{K}$-terms as regular expressions; atomic terms are interpreted as letters (or as languages), $+$ as union, $\cdot$ as concatenation, and $*$ as Kleene star, that is, union of all finite iterations. Let a be a $\mathscr{K}$ term; we denote the language it denotes under this interpretation by $\|a\|$. If a is a term over the set of (atomic) generators A, then $\|a\| \subseteq A^*$ (the set of all finite strings of A-symbols). The following fundamental theorem for Kleene algebras was proved by Kozen in [10]:

Theorem 6 *(Kozen) We have $a \leq_{\mathscr{K}} b$ if and only if $\|a\| \subseteq \|b\|$.*

From this theorem and the lemmas of the preceding section we can easily derive a similar result for $\mathscr{K}^\#$ algebras. Let a be a $\mathscr{K}^\#$ term. We can interpret $\mathscr{K}^\#$ terms as languages as follows:

1. $\|a\|^\# = \{a\}$, for atomic a;
2. $\|a+b\|^\# = \|a\|^\# \cup \|b\|^\#$;
3. $\|a \cdot b\|^\# = \|a\|^\# \cdot \|b\|^\#$;
4. $\|a^\#\|^\# = \|a\|^\# \bigcup_{n \in \mathbb{N}_0} (\|a\|^\#)^n$.

Thus the # is interpreted as Kleene plus. We now show the following:

Theorem 7 *We have $a \leq_{\mathscr{K}^\#} b$ if and only if $\|a\|^\# \subseteq \|b\|^\#$.*

Proof. *If*: Assume $\|a\|^{\#} \subseteq \|b\|^{\#}$. We can read a, b as regular expressions with Kleene plus. By definition of the Kleene plus in terms of the star, we have $\|a\|^{\#} = \|j(a)\|$, same for b. Consequently, $\|j(a)\| \subseteq \|j(b)\|$, and by theorem 6, $j(a) \leq_{\mathscr{K}} j(b)$. By lemma 4, we obtain $a \leq_{\mathscr{K}^{\#}} b$.

Only if: Assume $a \leq_{\mathscr{K}^{\#}} b$. By lemma 5, we then have $j(a) \leq_{\mathscr{K}} j(b)$, and thus $\|j(a)\| \subseteq \|j(b)\|$. So the claim follows the fact that $\|j(c)\| = \|c\|^{\#}$. □

From this follows that both KL and KL^+ have a complete language-theoretic semantics:

Theorem 8 *We have*

1. $\Vdash_{\mathsf{KL}} \Gamma \vdash \alpha$ *if and only if* $\|\overline{\sigma}(\Gamma)\| \subseteq \|\overline{\sigma}(\alpha)\|$*; and*
2. $\Vdash_{\mathsf{KL}^+} \Gamma \vdash \alpha$ *if and only if* $\|\overline{\sigma}(\Gamma)\|^{\#} \subseteq \|\overline{\sigma}(\alpha)\|^{\#}$.

Now, as under this interpretation, formulas of both KL and KL^+ denote regular languages, and the problem whether one regular language (represented, e.g., as a regular expression) is a subset of another one is decidable, we immediately get the following:

Theorem 9 $\mathsf{KL}, \mathsf{KL}^+$ *are decidable, that is, for any sequent* $\Gamma \vdash \alpha$*, we can effectively decide whether* $\Vdash_{\mathsf{KL}} \Gamma \vdash \alpha$, $\Vdash_{\mathsf{KL}+} \Gamma \vdash \alpha$ *hold.*

So in order to decide whether a sequent is valid, we can just go over the language-theoretic interpretation of formulas as regular expressions. A more direct way to establish the decidability of $\mathsf{KL}, \mathsf{KL}^+$ is by showing that for every derivable sequent, there is a proof which does not make use of the cut rule.

9 Cut Admissibility

We now show that in $\mathsf{KL}, \mathsf{KL}^+$ cut is admissible, that is, for every proof of a sequent in the two calculi there is a cut-free proof. [6] This is important for the following reason: the cut rule is the only rule in which there is material in the antecedents, which is not in the consequent; so when we want to check whether a sequent is derivable in our calculi, it is the only rule which makes the search space infinite; for all other rules, we know how the antecedents have to look like (in the sense of: there is a finite number of possible choices). So from cut admissibility follows that $\mathsf{KL}, \mathsf{KL}^+$ are decidable also "inside the calculus", without a detour over semantics.

How does this result follow? We established the soundness of the cut rule. But in order to show the completeness direction of the algebraic semantics, proving the term algebra of $\mathsf{KL}, \mathsf{KL}^+$ is a $\mathscr{K}, \mathscr{K}^{\#}$ algebra, respectively, we did not make any use of the cut rule. Now let $\mathsf{KL}_{cf}, \mathsf{KL}^+_{cf}$ be the logics which result from taking all axioms and inference rules of KL and KL^+, respectively, except for the cut rule. To prove their soundness and completeness for $\mathscr{K}, \mathscr{K}^{\#}$, we can take over the proofs for $\mathsf{KL}, \mathsf{KL}^+$ without any change, except that we can do away with soundness of cut. So we have $\Vdash_{\mathsf{KL}_{cf}} \Gamma \vdash \alpha$ iff $\models_{\mathscr{K}} \Gamma \vdash \alpha$ iff $\Vdash_{\mathsf{KL}} \Gamma \vdash \alpha$, same for KL^+ and KL^+_{cf}. The only problem with this reasoning is that the interpretation $\overline{\sigma}$ is a homomorphism rather than a bijection, as it maps both "," and "$\bullet$" to "$\cdot$". So in addition we need a proof of the fact that $\Vdash_{\mathsf{KL}} \gamma_1, ..., \gamma_i \vdash \alpha$ iff $\Vdash_{\mathsf{KL}} \gamma_1 \bullet ... \bullet \gamma_i \vdash \alpha$, and the same for KL_{cf}. This is however easy to show (cf. [5], Proposition 7.1). From this follows:

Theorem 10 *In both* $\mathsf{KL}, \mathsf{KL}^+$*, the cut rule is admissible.*

[6]Contrary to some other usage, we do not speak of cut elimination, because we only show the existence of a cut free proof, whereas cut elimination means that from a proof using cut we can effectively construct a cut free proof.

10 Conclusion

We have presented two propositional substructural logics, strongly inspired by Kleene algebras and some considerations on processes. We have proved strong completeness theorems for algebraic semantics as well as for language models. We have also proved cut admissibility, from which follows the decidability of the calculus by purely syntactic means (though the cut admissibility proof itself is semantic). By our strong completeness, proof search in the calculus of KL can be reduced to the validity of an inequation in all Kleene algebras (PSPACE-complete). There remains the question whether for some subclass of formulas, we can do substantially better (this is an approach commonly taken, see [8]): there might fall off a good algorithm for checking the inclusion relation of languages denoted by two regular expressions. Another question is the following: as can be seen from already existing work (see e.g. [3]), if we enrich a substructural logic having an exponential as ? with implication (or vice versa), undecidability strikes rather quickly. Still, the decidability results for $\mathsf{KL}, \mathsf{KL}^+$ seem robust; in particular, we conjecture that the *external* consequence relations (the smallest structural consequence relations generated by the two, see e.g. [5]) of the two logics are decidable. These seem to us the most natural and interesting questions to ask.

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