Semantic Structures for Spatially-Distributed Multi-Agent Systems

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Spatial constraint systems (scs) are semantic structures for reasoning about spatial and epistemic information in concurrent systems. They have been used to reason about beliefs, lies, and group epistemic behaviour inspired by social networks. They have also been used for proving new results about modal logics and giving semantics to process calculi. In this paper we will discuss the theory and main results about scs.

1 Introduction

Epistemic, mobile and spatial behavior are common place in today's distributed systems. The intrinsic *epistemic* nature of these systems arises from social behavior. Most people are familiar with digital systems where *agents* (users) share their *beliefs, opinions* and even intentional *lies* (hoaxes). Also, systems modeling decision behavior must account for those decisions' dependance on the results of interactions with others within some social context. The courses of action stemming from some agent decision result not only from the rational analysis of a particular situation but also from the agent beliefs or information that sprang from the interactions with other participants involved in that situation. Appropriate performance within these social contexts requires the agent to form beliefs about the beliefs of others. Spatial and mobile behavior is exhibited by apps and data moving across (possibly nested) spaces defined by, for example, friend circles and shared folders. We therefore believe that a solid understanding of the notion of *space* and *spatial mobility* as well as the flow of epistemic information is relevant in any model of today's distributed systems.

The notion of group is also fundamental in distributed systems. Since the early days of multi-user operating systems, information was categorized into that available to one user, some group of users, or everyone. Information was thus separated into "spaces" with boundaries defined by accessibility. In these systems we could say that, from the restrictive point of view of information "permissions", the notion of group was *reified* as another agent of the system.

In current distributed systems such as social networks, actors behave more as members of a certain *group* than as isolated individuals. Information, opinions, and beliefs of a particular actor are frequently the result of an evolving process of interchanges with other actors in a group. This suggests a reified notion of group as a single actor operating within the context of the collective information of its members. It also conveys two notions of information, one spatial and the other epistemic. In the former, information is localized in compartments associated with a user or group. In the latter, it refers to something known or believed by a single agent or collectively by a group.

Furthermore, in many real life multi-agent systems, the agents are unknown in advance. New agents can subscribe to the system in unpredictable ways. Thus, there is usually no a-priori bound on the number

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of agents in the system. It is then often convenient to model the group of agents as an infinite set. In fact, in models from economics and epistemic logic [17, 16], groups of agents have been represented as infinite, even uncountable, sets. This raises interesting issues about the distributed information of such groups. In particular, that of *group compactness*: information that when obtained by an infinite group can also be obtained by one of its finite subgroups.

Spatial constraint systems (scs)¹ are semantic structures for the epistemic behaviour of multi-agent systems. These structures single out the notions we previously discussed: Namely, space, beliefs, and distributed information of potentially infinite groups. In this paper we will describe the theory of scs and highlight its main results from [8, 10, 11, 12, 9].

2 Overview

In this section we will give a brief description and motivate scs in the context of space, extrusion, and distributed information.

Declarative formalisms of concurrency theory such as process calculi for *concurrent constraint programming* (ccp) [24] were designed to give explicit access to the concept of partial information and, as such, have close ties with logic. This makes them ideal for the incorporation of epistemic and spatial concepts by expanding the logical connections to include *multi-agent modal logic* [19]. In fact, the sccp calculus [18] extends ccp with the ability to define local computational spaces where agents can store epistemic information and run processes.

Constraint systems (cs) are algebraic structures for the semantics of ccp [24, 2, 18, 5, 22, 20]. They specify the domain and elementary operations and partial information upon which programs (processes) of these calculi may act.

A cs can be formalized as a complete lattice (Con, \sqsubseteq) . The elements of *Con* represent partial information and we shall think of them as being *assertions*. They are traditionally referred to as *constraints* since they naturally express partial information (e.g., x > 42). The order \sqsubseteq corresponds to entailment between constraints, $c \sqsubseteq d$, often written $d \sqsupseteq c$, means c can be derived from d, or that d represents as much information as c. The join \sqcup , the bottom *true*, and the top *false* of the lattice correspond to conjunction, the empty information, and the join of all (possibly inconsistent) information, respectively.

Constraint systems provide the domains and operations upon which the semantic foundations of ccp calculi are built. As such, ccp operations and their logical counterparts typically have a corresponding elementary construct or operation on the elements of the constraint system. In particular, parallel composition and conjunction correspond to the *join* operation, and existential quantification and local variables correspond to a cylindrification operation on the set of constraints [24].

Space. Similarly, the notion of computational space and the epistemic notion of belief in sccp [18] correspond to a family of join-preserving maps $\mathfrak{s}_i : Con \to Con$ called *space functions*. A cs equipped with space functions is called a *spatial constraint system* (scs). From a *computational point of view* $\mathfrak{s}_i(c)$ can be interpreted as an assertion specifying that *c* resides within the space of agent *i*. From an *epistemic point of view*, $\mathfrak{s}_i(c)$ specifies that *i* considers *c* to be true. An alternative epistemic view is that *i* interprets *c* as $\mathfrak{s}_i(c)$. All these interpretations convey the idea of *c* being local or subjective to agent *i*.

In the spatial ccp process calculus *sccp* [18], scs are used to specify the spatial distribution of information in configurations $\langle P, c \rangle$ where *P* is a process and *c* is a constraint, called *the store*, representing the current partial information. E.g., a reduction $\langle P, \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b) \rangle \longrightarrow \langle Q, \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \sqcup c) \rangle$ means

¹For simplicity we use *scs* for both *spatial constraint system* and its plural form.

that P, with a in the space of agent 1 and b in the space of agent 2, can evolve to Q while adding c to the space of agent 2.

Extrusion. An extrusion function for the space \mathfrak{s}_i is a map $\mathfrak{e}_i : Con \to Con$ that satisfies $\mathfrak{s}_i(\mathfrak{e}_i(c)) = c$. This means that we think of extrusion as the *right inverse* of space. Intuitively, within a space context $\mathfrak{s}_i(\cdot)$, the assertion $\mathfrak{e}_i(c)$ specifies that *c* must be posted outside of agent *i*'s space. The computational interpretation of \mathfrak{e}_i is that of a process being able to extrude any *c* from the space \mathfrak{s}_i . The extruded information *c* may not necessarily be part of the information residing in the space of agent *i*. For example, using properties of space and extrusion functions we shall see that $\mathfrak{s}_i(d \sqcup \mathfrak{e}_i(c)) = \mathfrak{s}_i(d) \sqcup c$ specifying that *c* is extruded (while *d* is still in the space of *i*). The extruded *c* could be inconsistent with *d* (i.e., $c \sqcup d = false$), it could be related to *d* (e.g., $c \sqsubseteq d$), or simply unrelated to *d*. From an epistemic perspective, we can use \mathfrak{e}_i to express *utterances* by agent *i* and such utterances could be intentional lies (i.e., inconsistent with their beliefs), informed opinions (i.e., derived from the beliefs), or simply arbitrary statements (i.e., unrelated to their beliefs).

Distributed Information. Let us consider again the sccp reduction $\langle P, \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b) \rangle \longrightarrow \langle Q, \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \sqcup c) \rangle$. Assume that *d* is some piece of information resulting from the combination (join) of the three constraints above, i.e., $d = a \sqcup b \sqcup c$, but strictly above the join of any two of them. We are then in the situation where neither agent has *d* in their spaces, but as a group they could potentially have *d* by combining their information. Intuitively, *d* is distributed in the spaces of the group $I = \{1, 2\}$. Being able to predict the information that agents 1 and 2 may derive as group is a relevant issue in multi-agent concurrent systems, particularly if *d* represents unwanted or conflicting information (e.g., d = false).

In [9] we introduced the theory of group space functions $\Delta_I : Con \to Con$ to reason about information distributed among the members of a potentially infinite group *I*. We refer to Δ_I as the *distributed space* of group *I*. In our theory $c \supseteq \Delta_I(e)$ holds exactly when we can derive from *c* that *e* is distributed among the agents in *I*. E.g., for *d* above, we should have $\mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \sqcup c) \supseteq \Delta_{\{1,2\}}(d)$ meaning that from the information $\mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \sqcup c)$ we can derive that *d* is distributed among the group $I = \{1,2\}$. Furthermore, $\Delta_I(e) \supseteq \Delta_J(e)$ holds whenever $I \subseteq J$ since if *e* is distributed among a group *I*, it should also be distributed in a group that includes the agents of *I*.

Distributed information of infinite groups can be used to reason about multi-agent computations with unboundedly many agents. For example, a *computation* in sccp is a possibly infinite reduction sequence γ of the form $\langle P_0, c_0 \rangle \longrightarrow \langle P_1, c_1 \rangle \longrightarrow \cdots$ with $c_0 \sqsubseteq c_1 \sqsubseteq \cdots$. The *result* of γ is $\bigsqcup_{n \ge 0} c_n$, the join of all the stores in the computation. In sccp all fair computations from a configuration have the same result [18]. Thus, the *observable behaviour* of *P* with initial store *c*, written $\mathcal{O}(P,c)$, is defined as the result of any fair computation starting from $\langle P, c \rangle$. Now consider a setting where in addition to their sccp capabilities in [18], processes can also create new agents. Hence, unboundedly many agents, say agents 1,2,..., may be created during an infinite computation. In this case, $\mathcal{O}(P,c) \sqsupseteq \Delta_{\mathbb{N}}(false)$, where \mathbb{N} is the set of natural numbers, would imply that some (finite or infinite) set of agents in any fair computation from $\langle P, c \rangle$ may reach contradictory local information among them. Notice that from the above-mentioned properties of distributed spaces, the existence of a finite set of agents $H \subseteq \mathbb{N}$ such that $\mathcal{O}(P,c) \sqsupseteq \Delta_H(false)$ implies $\mathcal{O}(P,c) \sqsupseteq \Delta_{\mathbb{N}}(false)$. The converse of this implication will be called *group compactness* and we will discuss meaningful sufficient conditions for it to hold.

In the next sections we will describe the above spatial and epistemic notions in more detail.

3 Background

We presuppose basic knowledge of domain and order theory [3, 1, 7] and use the following notions. Let **C** be a poset (Con, \sqsubseteq) , and let $S \subseteq Con$. We use $\bigsqcup S$ to denote the least upper bound (or *supremum* or *join*) of the elements in *S*, and $\bigsqcup S$ is the greatest lower bound (glb) (*infimum* or *meet*) of the elements in *S*. An element $e \in S$ is the greatest element of *S* iff for every element $e' \in S$, $e' \sqsubseteq e$. If such *e* exists, we denote it by *max S*. As usual, if $S = \{c,d\}$, $c \sqcup d$ and $c \sqcap d$ represent $\bigsqcup S$ and $\bigsqcup S$, respectively. If $S = \emptyset$, we denote $\bigsqcup S = true$ and $\bigsqcup S = false$. We say that **C** is a *complete lattice* iff each subset of *Con* has a supremum in *Con*. The poset **C** is *distributive* iff for every $a, b, c \in Con$, $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$. A non-empty set $S \subseteq Con$ is *directed* iff for every pair of elements $x, y \in S$, there exists $z \in S$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, or iff every *finite* subset of *S* has an upper bound in *S*. Also $c \in Con$ is *a* function *f* from *Con* to *Con*. Let (Con, \sqsubseteq) be a complete lattice. The self-map on *Con* is a function *f* a set $S \subseteq Con$ iff $f(\bigsqcup S) = \bigsqcup \{f(c) \mid c \in S\}$. A self-map that preserves the join of finite sets is called *join-homomorphism*. A self-map *f* on *Con* is *monotonic* if $a \sqsubseteq b$ implies $f(a) \sqsubseteq f(b)$. We say that *f distributes* over joins (or that *f preserves* joins) iff it preserves the join of arbitrary sets. A self-map *f* on *Con* is *continuous* iff it preserves the join of any directed set.

Constraint systems [24] are semantic structures to specify partial information. They can be formalized as complete lattices [2].

Definition 3.1 (Constraint Systems [2]). A constraint system (*cs*) \mathbb{C} *is a complete lattice* (*Con*, \sqsubseteq). *The elements of Con are called* constraints. *The symbols* \sqcup , *true and false will be used to denote the least upper bound (lub) operation, the bottom, and the top element of* \mathbb{C} .

The elements of the lattice, the *constraints*, represent (partial) information. A constraint c can be viewed as an *assertion*. The lattice order \sqsubseteq is meant to capture entailment of information: $c \sqsubseteq d$, alternatively written $d \sqsupseteq c$, means that the assertion d represents at least as much information as c. We think of $d \sqsupseteq c$ as saying that d entails c or that c can be derived from d. The operator \sqcup represents join of information; $c \sqcup d$ can be seen as an assertion stating that both c and d hold. We can think of \sqcup as representing conjunction of assertions. The top element represents the join of all, possibly inconsistent, information, hence it is referred to as *false*. The bottom element *true* represents *empty information*. We say that c is *consistent* if $c \neq false$, otherwise we say that c is *inconsistent*. Similarly, we say that c is consistent with d if $c \sqcup d$ is consistent/inconsistent.

Constraint Frames. One can define a general form of implication by adapting the corresponding notion from Heyting Algebras to cs. A *Heyting implication* $c \rightarrow d$ in our setting corresponds to the *weakest constraint* one needs to join c with to derive d.

Definition 3.2 (Constraint Frames [8]). A constraint system (Con, \sqsubseteq) is said to be a constraint frame *iff* its joins distribute over arbitrary meets. More precisely, $c \sqcup \Box S = \Box \{c \sqcup e \mid e \in S\}$ for every $c \in Con$ and $S \subseteq Con$. Define $c \to d$ as $\Box \{e \in Con \mid c \sqcup e \sqsupseteq d\}$.

The following properties of Heyting implication correspond to standard logical properties (with \rightarrow , \sqcup , and \supseteq interpreted as implication, conjunction, and entailment).

Proposition 3.3 ([8]). Let (Con, \sqsubseteq) be a constraint frame. For every $c, d, e \in Con$ the following holds: (1) $c \sqcup (c \to d) = c \sqcup d$, (2) $(c \to d) \sqsubseteq d$, (3) $c \to d = true$ iff $c \sqsupseteq d$.

4 Space and Beliefs

The authors of [18] extended the notion of cs to account for distributed and multi-agent scenarios with a finite number of agents, each having their own space for local information and their computations. The extended structures are called spatial cs (scs). Here we adapt scs to reason about possibly infinite groups of agents.

A group G is a set of agents. Each $i \in G$ has a space function $\mathfrak{s}_i : Con \to Con$ satisfying some structural conditions. Recall that constraints can be viewed as assertions. Thus given $c \in Con$, we can then think of the constraint $\mathfrak{s}_i(c)$ as an assertion stating that c is a piece of information residing within a space of agent i. Some alternative epistemic interpretations of $\mathfrak{s}_i(c)$ is that it is an assertion stating that agent i believes c, that c holds within the space of agent i, or that agent i interprets c as $\mathfrak{s}_i(c)$. All these interpretations convey the idea that c is local or subjective to agent i.

In [18] scs are used to specify the spatial distribution of information in configurations $\langle P, c \rangle$ where *P* is a process and *c* is a constraint. E.g., a reduction $\langle P, \mathfrak{s}_i(c) \sqcup \mathfrak{s}_j(d) \rangle \longrightarrow \langle Q, \mathfrak{s}_i(c) \sqcup \mathfrak{s}_j(d \sqcup e) \rangle$ means that *P* with *c* in the space of agent *i* and *d* in the space of agent *j* can evolve to *Q* while adding *e* to the space of agent *j*.

We now introduce the notion of space function.

Definition 4.1 (Space Functions). A space function over $a \ cs \ (Con, \sqsubseteq)$ is a continuous self-map $f : Con \rightarrow Con \ s.t.$ for every $c, d \in Con \ (S.1) \ f(true) = true, \ (S.2) \ f(c \sqcup d) = f(c) \sqcup f(d)$. We shall use $\mathscr{S}(\mathbf{C})$ to denote the set of all space functions over $\mathbf{C} = (Con, \sqsubseteq)$.

The assertion f(c) can be viewed as saying that c is in the space represented by f. Property S.1 states that having an empty local space amounts to nothing. Property S.2 allows us to join and distribute the information in the space represented by f.

In [18] space functions were not required to be continuous. Nevertheless, continuity comes naturally in the intended phenomena we wish to capture: modelling information of possibly *infinite* groups. In fact, in [18] scs could only have finitely many agents.

In [9] we extended scs to allow arbitrary, possibly infinite, sets of agents. A *spatial cs* is a cs with a possibly infinite group of agents each having a space function.

Definition 4.2 (Spatial Constraint Systems). A spatial cs (scs) *is a cs* $\mathbf{C} = (Con, \sqsubseteq)$ *equipped with a possibly infinite tuple* $\mathfrak{s} = (\mathfrak{s}_i)_{i \in G}$ *of space functions from* $\mathscr{S}(\mathbf{C})$.

We shall use $(Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ to denote an scs with a tuple $(\mathfrak{s}_i)_{i \in G}$. We refer to G and \mathfrak{s} as the group of agents and space tuple of \mathbb{C} and to each \mathfrak{s}_i as the space function in \mathbb{C} of agent i. Subsets of G are also referred to as groups of agents (or sub-groups of G).

Let us illustrate a simple scs that will be used throughout the paper.

Example 4.3. The scs $(Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in \{1,2\}})$ in Fig.1 is given by the complete lattice \mathbf{M}_2 and two agents. We have $Con = \{p \lor \neg p, p, \neg p, p \land \neg p\}$ and $c \sqsubseteq d$ iff *c* is a logical consequence of *d*. The top element *false* is $p \land \neg p$, the bottom element *true* is $p \lor \neg p$, and the constraints *p* and $\neg p$ are incomparable with each other. The set of agents is $\{1,2\}$ with space functions \mathfrak{s}_1 and \mathfrak{s}_2 : For agent 1, $\mathfrak{s}_1(p) = \neg p$, $\mathfrak{s}_1(\neg p) = p, \mathfrak{s}_1(false) = false, \mathfrak{s}_1(true) = true$, and for agent 2, $\mathfrak{s}_2(p) = false = \mathfrak{s}_2(false), \mathfrak{s}_2(\neg p) = \neg p$, $\mathfrak{s}_2(true) = true$. The intuition is that the agent 2 sees no difference between *p* and *false* while agent 1 interprets $\neg p$ as *p* and vice versa.

More involved examples of scs include meaningful families of structures from logic and economics such as Kripke structures and Aumann structures (see [18]). We illustrate scs with infinite groups in the next section.

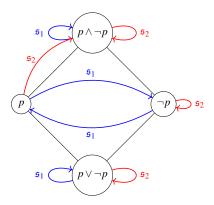


Figure 1: Cs given by lattice M_2 ordered by implication and space functions \mathfrak{s}_1 and \mathfrak{s}_2 .

5 Extrusion and Utterances

We can also equip each agent *i* with an *extrusion* function $\mathfrak{e}_i : Con \to Con$. Intuitively, within a space context $\mathfrak{s}_i(\cdot)$, the assertion $\mathfrak{e}_i(c)$ specifies that *c* must be posted outside of agent *i*'s space. This is captured by requiring the *extrusion* axiom (E.1) $\mathfrak{s}_i(\mathfrak{e}_i(c)) = c$. In other words, we view *extrusion/utterance* as the right inverse of *space/belief* (and thus space/belief as the left inverse of extrusion/utterance).

Definition 5.1 (Extrusion). *Given an scs* $(Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$, we say that \mathfrak{e}_i is an extrusion function for the space \mathfrak{s}_i *iff* \mathfrak{e}_i *is a right inverse of* \mathfrak{s}_i *, i.e., iff* $\mathfrak{s}_i(\mathfrak{e}_i(c)) = c$.

From the above definitions it follows that $\mathfrak{s}_i(c \sqcup \mathfrak{e}_i(d)) = \mathfrak{s}_i(c) \sqcup d$. From a spatial point of view, agent *i extrudes d* from its local space. From an epistemic view this can be seen as an agent *i* that believes *c* and *utters d* to the outside world. If *d* is inconsistent with *c*, i.e., $c \sqcup d = false$, we can see the utterance as an intentional *lie* by agent *i*: The agent *i* utters an assertion inconsistent with their own beliefs.

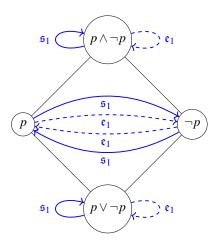
Example 5.2. Let $e = \mathfrak{s}_i(c \sqcup \mathfrak{e}_i(\mathfrak{s}_j(a))) \sqcup \mathfrak{s}_j(d)$. The constraint *e* specifies that agent *i* has *c* and wishes to transmit, via extrusion, *a* addressed to agent *j*. Agent *j* has *d* in their space. Indeed, with the help of E.1 and S.2, we can derive $e \sqsupseteq \mathfrak{s}_j(d \sqcup a)$ thus stating that *e* entails that *a* will be in the space of *j*.

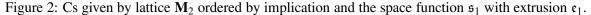
The Extrusion Problem. A legitimate question is: Given space \mathfrak{s}_i can we derive an extrusion function \mathfrak{e}_i for it ? From set theory we know that there is an extrusion function (i.e., a right inverse) \mathfrak{e}_i for \mathfrak{s}_i iff \mathfrak{s}_i is *surjective*. Recall that the *pre-image* of $y \in Y$ under $f : X \to Y$ is the set $f^{-1}(y) = \{x \in X \mid y = f(x)\}$. Thus the extrusion \mathfrak{e}_i can be defined as a function, called *choice* function, that maps each element *c* to some element from the pre-image of *c* under \mathfrak{s}_i .

The existence of the above-mentioned choice function assumes the *Axiom of Choice*. The next proposition from [8] gives some constructive extrusion functions. It also identifies a distinctive property of space functions for which a right inverse exists.

Proposition 5.3 ([8]). *Let* f *be a space function over* (Con, \sqsubseteq) *. Then*

- 1. If $f(false) \neq false$ then f does not have any right inverse.
- 2. If f is surjective then $g: c \mapsto \bigsqcup f(c)^{-1}$ is a right inverse of f that preserves arbitrary infima.
- 3. If f is surjective and preserves arbitrary infima then $h : c \mapsto \prod f(c)^{-1}$ is a right inverse of f that preserves arbitrary suprema.





The following example illustrates an application of Prop.5.3 to obtain an extrusion function for the space function \mathfrak{s}_1 from Ex.4.3. Notice that the space function \mathfrak{s}_2 from Ex.4.3 is not surjective thus it does not have an extrusion function.

Example 5.4. Fig.2 shows an extrusion function for the space function \mathfrak{s}_1 in Ex.4.3. This extrusion function can be obtained by applying Prop.5.3.2.

6 Groups and Distributed Knowledge

In [9] we introduced the notion of collective information of a group of agents. Roughly speaking, the *distributed (or collective) information* of a group *I* is the join of each piece of information that resides in the space of *some* $i \in I$. The distributed information of *I* w.r.t. *c* is the distributive information of *I* that can be derived from *c*. We wish to formalize whether a given *e* can be derived from the collective information of the group *I* w.r.t. *c*.

The following examples, which we will use throughout this section, illustrate the above intuition.

Example 6.1. Consider an scs $(Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ where $G = \mathbb{N}$ and (Con, \sqsubseteq) is a constraint frame. Let $c \stackrel{\text{def}}{=} \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(a \to b) \sqcup \mathfrak{s}_3(b \to e)$. The constraint c specifies the situation where $a, a \to b$ and $b \to e$ are in the spaces of agent 1, 2 and 3, respectively. Neither agent necessarily holds e in their space in c. Nevertheless, the information e can be derived from the collective information of the three agents w.r.t. c, since from Prop.3.3 we have $a \sqcup (a \to b) \sqcup (b \to e) \sqsupseteq e$. Let us now consider an example with infinitely many agents. Let $c' \stackrel{\text{def}}{=} \bigsqcup_{i \in \mathbb{N}} \mathfrak{s}_i(a_i)$ for some increasing chain $a_0 \sqsubseteq a_1 \sqsubseteq \ldots$. Take e' s.t. $e' \sqsubseteq \bigsqcup_{i \in \mathbb{N}} a_i$. Notice that unless e' is compact (see Section 3), it may be the case that no agent $i \in \mathbb{N}$ holds e' in their space; e.g., if $e' \sqsupset a_i$ for any $i \in \mathbb{N}$. Yet, from our assumption, e' can be derived from the collective information w.r.t. c' of all the agents in \mathbb{N} , i.e., $\bigsqcup_{i \in \mathbb{N}} a_i$.

The above example may suggest that the distributed information can be obtained by joining individual local information derived from c. Individual information of an agent i can be characterized as the *i*-projection of c defined thus:

Definition 6.2 (Agent and Join Projections [9]). Let $\mathbf{C} = (Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ be an scs. Given $i \in G$, the *i*-agent projection of $c \in Con$ is defined as $\pi_i(c) \stackrel{\text{def}}{=} \bigsqcup \{e \mid c \sqsupseteq \mathfrak{s}_i(e)\}$. We say that e is *i*-agent derivable

from c iff $\pi_i(c) \supseteq e$. Given $I \subseteq G$ the *I*-join projection of a group *I* of *c* is defined as $\pi_I(c) \stackrel{\text{def}}{=} \bigsqcup \{\pi_i(c) \mid i \in I\}$. We say that *e* is *I*-join derivable from *c* iff $\pi_I(c) \supseteq e$.

The *i*-projection of an agent *i* of *c* naturally represents the join of all the information of agent *i* in *c*. It turns out that projections are extrusion functions: If \mathfrak{s}_i admits extrusion then π_i is an extrusion function for the space \mathfrak{s}_i (see Def.5.1). More precisely,

Proposition 6.3 (Projection as extrusion). If \mathfrak{s}_i is surjective then $\mathfrak{s}_i(\pi_i(c)) = c$ for every $c \in Con$.

The *I*-join projection of group *I* joins individual *i*-projections of *c* for $i \in I$. This projection can be used as a sound mechanism for reasoning about distributed-information: If *e* is *I*-join derivable from *c* then it follows from the distributed-information of *I* w.r.t. *c*.

Example 6.4. Let *c* be as in Ex.6.1. We have $\pi_1(c) \supseteq a$, $\pi_2(c) \supseteq (a \to b)$, $\pi_3(c) \supseteq (b \to e)$. Indeed *e* is *I*-join derivable from *c* since $\pi_{\{1,2,3\}}(c) = \pi_1(c) \sqcup \pi_2(c) \sqcup \pi_3(c) \supseteq e$. Similarly we conclude that *e'* is *I*-join derivable from *c'* in Ex.6.1 since $\pi_{\mathbb{N}}(c') = \bigsqcup_{i \in \mathbb{N}} \pi_i(c) \supseteq \bigsqcup_{i \in \mathbb{N}} a_i \supseteq e'$.

Nevertheless, *I*-join projections do not provide a complete mechanism for reasoning about distributed information as illustrated below.

Example 6.5. Let $d \stackrel{\text{def}}{=} \mathfrak{s}_1(b) \sqcap \mathfrak{s}_2(b)$. Recall that we think of \sqcup and \sqcap as conjunction and disjunction of assertions: *d* specifies that *b* is present in the space of agent 1 or in the space of agent 2 though not exactly in which one. Thus from *d* we should be able to conclude that *b* belongs to the space of *some* agent in $\{1,2\}$. Nevertheless, in general *b* is not *I*-join derivable from *d* since from $\pi_{\{1,2\}}(d) = \pi_1(d) \sqcup \pi_2(d)$ we cannot, in general, derive *b*. To see this consider the scs in Fig.3a and take $b = \neg p$. We have $\pi_{\{1,2\}}(d) = \pi_1(d) \sqcup \pi_2(d) = true \sqcup true = true \not\supseteq b$. One can generalize the example to infinitely many agents: Consider the scs in Ex.6.1. Let $d' \stackrel{\text{def}}{=} \prod_{i \in \mathbb{N}} \mathfrak{s}_i(b')$. We should be able to conclude from *d'* that *b'* is in the space of *some* agent in \mathbb{N} but, in general, *b'* is not \mathbb{N} -join derivable from *d'*.

6.1 Distributed Spaces

In the previous section we illustrated that the *I*-join projection of c, $\pi_I(c)$, the join of individual projections, may not project all distributed information of a group *I*. To solve this problem we shall develop the notion of *I*-group projection of *c*, written as $\Pi_I(c)$. To do this we shall first define a space function Δ_I called the distributed space of group *I*. The function Δ_I can be thought of as a virtual space including all the information that can be in the space of a member of *I*. We shall then define an *I*-projection Π_I in terms of Δ_I much like π_i is defined in terms of \mathfrak{s}_i .

Recall that $\mathscr{S}(\mathbf{C})$ denotes the set of all space functions over a cs **C**. For notational convenience, we shall use $(f_I)_{I\subseteq G}$ to denote the tuple $(f_I)_{I\in\mathscr{P}(G)}$ of elements of $\mathscr{S}(\mathbf{C})$.

Set of Space Functions. We begin by introducing a new partial order induced by C. The set of space functions ordered point-wise.

Definition 6.6 (Space Functions Order). Let $\mathbf{C} = (Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ be an scs. Given $f, g \in \mathscr{S}(\mathbf{C})$, define $f \sqsubseteq_{\mathfrak{s}} g$ iff $f(c) \sqsubseteq g(c)$ for every $c \in Con$. We shall use $\mathbf{C}_{\mathfrak{s}}$ to denote the partial order $(\mathscr{S}(\mathbf{C}), \sqsubseteq_{\mathfrak{s}})$; the set of all space functions ordered by $\sqsubseteq_{\mathfrak{s}}$.

A very important fact for the design of our structure is that the set of space functions $\mathscr{S}(\mathbf{C})$ can be made into a complete lattice.

Lemma 6.7 ([9]). Let $\mathbf{C} = (Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ be an scs. Then $\mathbf{C}_{\mathcal{S}}$ is a complete lattice.

6.2 Distributed Spaces as Maximum Spaces

Let us consider the lattice of space functions $C_s = (\mathscr{S}(C), \sqsubseteq_s)$. Suppose that f and g are space functions in C_s with $f \sqsubseteq_s g$. Intuitively, every piece of information c in the space represented by g is also in the space represented by f since $f(c) \sqsubseteq g(c)$ for every $c \in Con$. This can be interpreted as saying that the space represented by g is included in the space represented by f; in other words the bigger the space, the smaller the function that represents it in the lattice C_s .

Following the above intuition, the order relation \sqsubseteq_s of C_s represents (reverse) space inclusion and the join and meet operations in C_s represent intersection and union of spaces. The biggest and the smallest spaces are represented by the bottom and the top elements of the lattice C_s , here called λ_{\perp} and λ_{\top} and defined as follows.

Definition 6.8 (Top and Bottom Spaces). For every $c \in Con$, define $\lambda_{\perp}(c) \stackrel{\text{def}}{=} true$, $\lambda_{\top}(c) \stackrel{\text{def}}{=} true$ if c = true and $\lambda_{\top}(c) \stackrel{\text{def}}{=} false$ if $c \neq true$.

The distributed space Δ_I of a group *I* can be viewed as the function that represents the smallest space that includes all the local information of the agents in *I*. From the above intuition, Δ_I should be the *greatest space function* below the space functions of the agents in *I*. The existence of such a function follows from completeness of $(\mathscr{S}(\mathbf{C}), \sqsubseteq_{\mathbf{S}})$ (Lemma 6.7).

Definition 6.9 (Distributed Spaces [9]). Let **C** be an scs $(Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. The distributed spaces of **C** is given by $\Delta = (\Delta_I)_{I \subseteq G}$ where $\Delta_I \stackrel{\text{def}}{=} max\{f \in \mathscr{S}(\mathbf{C}) \mid f \sqsubseteq_{\mathfrak{s}} \mathfrak{s}_i \text{ for every } i \in I\}$. We shall say that e is distributed among $I \subseteq G$ w.r.t. c iff $c \sqsupseteq \Delta_I(e)$. We shall refer to each Δ_I as the (distributed) space of the group I.

It follows from Lemma 6.7 that $\Delta_I = \prod \{ \mathfrak{s}_i \mid i \in I \}$ (where \prod is the meet in the complete lattice $(\mathscr{S}(\mathbb{C}), \sqsubseteq_{\mathfrak{s}})$). Fig.3b illustrates an scs and its distributed space $\Delta_{\{1,2\}}$.

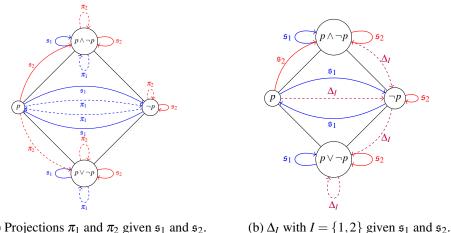
Compositionality. Distributed spaces have pleasant compositional properties. They capture the intuition that the *distributed information* of a group *I* can be obtained from the the distributive information of its subgroups.

Theorem 6.10 ([9]). Let $(\Delta_I)_{I\subseteq G}$ be the distributed spaces of an scs $(Con, \sqsubseteq, (\mathfrak{s}_i)_{i\in G})$. Suppose that $K, J \subseteq I \subseteq G$. (1) $\Delta_I = \lambda_{\top}$ if $I = \emptyset$, (2) $\Delta_I = \mathfrak{s}_i$ if $I = \{i\}$, (3) $\Delta_J(a) \sqcup \Delta_K(b) \sqsupseteq \Delta_I(a \sqcup b)$, and (4) $\Delta_J(a) \sqcup \Delta_K(a \to c) \sqsupseteq \Delta_I(c)$ if (Con, \sqsubseteq) is a constraint frame.

Recall that λ_{\top} corresponds to the empty space (see Def.6.8). The first property realizes the intuition that the empty subgroup \emptyset *does not* have any information whatsoever distributed w.r.t. a consistent *c*: for if $c \supseteq \Delta_{\emptyset}(e)$ and $c \neq false$ then e = true. Intuitively, the second property says that the function Δ_I for the group of one agent must be the agent's space function. The third property states that a group can join the information of its subgroups. The last property uses constraint implication, hence the constraint frame condition, to express that by joining the information *a* and $a \rightarrow c$ of their subgroups, the group *I* can obtain *c*.

Let us illustrate how to derive information of a group from smaller ones using Thm.6.10.

Example 6.11. Let $c = \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(a \to b) \sqcup \mathfrak{s}_3(b \to e)$ as in Ex.6.1. We want to prove that *e* is distributed among $I = \{1, 2, 3\}$ w.r.t. *c*, i.e., $c \sqsupseteq \Delta_{\{1,2,3\}}(e)$. Using Properties 2 and 4 in Thm.6.10 we obtain $c \sqsupseteq \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(a \to b) = \Delta_{\{1\}}(a) \sqcup \Delta_{\{2\}}(a \to b) \sqsupseteq \Delta_{\{1,2\}}(b)$, and then $c \sqsupseteq \Delta_{\{1,2\}}(b) \sqcup \mathfrak{s}_3(b \to e) = \Delta_{\{1,2\}}(b) \sqcup \Delta_{\{3\}}(b \to e) \sqsupseteq \Delta_{\{1,2,3\}}(e)$ as wanted.



(a) Projections π_1 and π_2 given \mathfrak{s}_1 and \mathfrak{s}_2 .

Figure 3: Projections (a) and Distributed Space function (b) over lattice M_2 .

Remark 6.1 (Continuity). The example with infinitely many agents in Ex.6.1 illustrates well why we require our spaces to be continuous in the presence of possibly infinite groups. Clearly $c' = \bigsqcup_{i \in \mathbb{N}} \mathfrak{s}_i(a_i) \supseteq$ $\bigsqcup_{i \in \mathbb{N}} \Delta_{\mathbb{N}}(a_i)$. By continuity, $\bigsqcup_{i \in \mathbb{N}} \Delta_{\mathbb{N}}(a_i) = \Delta_{\mathbb{N}}(\bigsqcup_{i \in \mathbb{N}} a_i)$ which indeed captures the idea that each a_i is in the distributed space $\Delta_{\mathbb{N}}$.

We conclude this subsection with an important family of scs from mathematical economics: Aumann structures. We illustrate that the notion of distributed knowledge in these structures is an instance of a distributed space.

Example 6.12. Aumann Constraint Systems. Aumann structures [16] are an event-based approach to modelling knowledge. An Aumann structure is a tuple $\mathscr{A} = (S, \mathscr{P}_1, \dots, \mathscr{P}_n)$ where S is a set of states and each \mathcal{P}_i is a partition on S for agent i. The partitions are called *information sets*. If two states t and u are in the same information set for agent i, it means that in state t agent i considers state u possible, and vice versa. An *event* in an Aumann structure is any subset of S. Event e holds at state t if $t \in e$. The set $\mathcal{P}_i(s)$ denotes the information set of \mathcal{P}_i containing s. The event of agent i knowing e is defined as $K_i(e) = \{s \in S \mid \mathscr{P}_i(s) \subseteq e\}$, and the distributed knowledge of an event e among the agents in a group I is defined as $D_I(e) = \{s \in S \mid \bigcap_{i \in I} \mathscr{P}_i(s) \subseteq e\}.$

An Aumann structure can be seen as a spatial constraint system $\mathbf{C}(\mathscr{A})$ with events as constraints, i.e., $Con = \{e \mid e \text{ is an event in } \mathscr{A}\}$, and for every $e_1, e_2 \in Con$, $e_1 \sqsubseteq e_2$ iff $e_2 \subseteq e_1$. The operators join (\sqcup) and meet (\sqcap) are intersection (\cap) and union (\cup) of events, respectively; *true* = S and *false* = \emptyset . The space functions are the knowledge operators, i.e., $\mathfrak{s}_i(c) = \mathsf{K}_i(c)$. From these definitions and since meets are unions one can easily verify that $\Delta_I(c) = D_I(c)$ which shows the correspondence between distributed information and distributed knowledge.

6.3 **Group Projections**

As promised in Section 6.1 we now give a definition of *Group Projection*. The function $\Pi_I(c)$ extracts exactly all information that the group I may have distributed w.r.t. c.

Definition 6.13 (Group Projection [9]). Let $(\Delta_I)_{I \subseteq G}$ be the distributed spaces of an scs $\mathbf{C} = (Con, \sqsubseteq$ $(\mathfrak{s}_i)_{i\in G}$. Given the set $I \subseteq G$, the *I*-group projection of $c \in Con$ is defined as $\Pi_I(c) \stackrel{\mathsf{def}}{=} \bigsqcup \{e \mid c \sqsupseteq \Delta_I(e)\}$. We say that e is I-group derivable from c iff $\Pi_I(c) \supseteq e$.

Much like space functions and agent projections, group projections and distributed spaces also form a pleasant correspondence: a Galois connection [3].

Proposition 6.14 ([9]). Let $(\Delta_I)_{I \subseteq G}$ be the distributed spaces of $\mathbf{C} = (Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. For every $c, e \in Con$, (1) $c \sqsupseteq \Delta_I(e)$ iff $\Pi_I(c) \sqsupseteq e$, (2) $\Pi_I(c) \sqsupseteq \Pi_J(c)$ if $J \subseteq I$, and (3) $\Pi_I(c) \sqsupseteq \pi_I(c)$.

The first property in Prop.6.14, a Galois connection, states that we can conclude from c that e is in the distributed space of I exactly when e is I-group derivable from c. The second says that the bigger the group, the bigger the projection. The last property says that whatever is I-join derivable is I-group derivable, although the opposite is not true as shown in Ex.6.5.

6.4 Group Compactness

Suppose that an *infinite* group of agents *I* can derive *e* from *c* (i.e., $c \supseteq \Delta_I(e)$). A legitimate question is whether there exists a *finite* sub-group *J* of agents from *I* that can also derive *e* from *c*. The following theorem provides a positive answer to this question provided that *e* is a compact element and *I*-join derivable from *c*.

Theorem 6.15 (Group Compactness [9]). Let $(\Delta_I)_{I \subseteq G}$ be the distributed spaces of an scs $\mathbb{C} = (Con, \sqsubseteq$, $(\mathfrak{s}_i)_{i \in G}$). Suppose that $c \sqsupseteq \Delta_I(e)$. If e is compact and I-join derivable from c then there exists a finite set $J \subseteq I$ such that $c \sqsupseteq \Delta_J(e)$.

We conclude this section with the following example of group compactness.

Example 6.16. Consider the example with infinitely many agents in Ex.6.1. We have $c' = \bigsqcup_{i \in \mathbb{N}} \mathfrak{s}_i(a_i)$ for some increasing chain $a_0 \sqsubseteq a_1 \sqsubseteq \ldots$ and e' s.t. $e' \sqsubseteq \bigsqcup_{i \in \mathbb{N}} a_i$. Notice that $c' \sqsupseteq \Delta_{\mathbb{N}}(e')$ and $\pi_{\mathbb{N}}(c') \sqsupseteq e'$. Hence e' is \mathbb{N} -join derivable from c'. If e' is compact, by Thm.6.15 there must be a finite subset $J \subseteq \mathbb{N}$ such that $c' \sqsupseteq \Delta_J(e')$.

7 Computing Distributed Information

Let us consider a *finite* scs $\mathbf{C} = (Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ with distributed spaces $(\Delta_I)_{I \subseteq G}$. By finite scs we mean that *Con* and *G* are finite sets. Let us consider the problem of computing Δ_I : Given a set $\{\mathfrak{s}_i\}_{i \in I}$ of space functions, we wish to find the greatest space function *f* such that $f \sqsubseteq \mathfrak{s}_i$ for all $i \in I$ (see Def.6.9).

Because of the finiteness assumption, the above problem can be rephrased in simpler terms: *Given* a finite lattice L and a finite set S of join-homomorphisms on L, find the greatest join-homomorphism below all the elements of S. Even in small lattices with four elements and two space functions, finding such greatest function may not be immediate, e.g., for $S = \{\mathfrak{s}_1, \mathfrak{s}_2\}$ and the lattice in Fig.1 the answer is given Fig.3b.

A brute force approach would be to compute $\Delta_I(c)$ by generating the set $\{f(c) \mid f \in \mathscr{S}(\mathbb{C}) \text{ and } f \sqsubseteq \mathfrak{s}_i$ for all $i \in I\}$ and taking its join. This approach works since $(\bigsqcup S)(c) = \bigsqcup \{f(c) \mid f \in S\}$. However, the number of such functions in $\mathscr{S}(\mathbb{C})$ can be at least factorial in the size of *Con*. For distributive lattices, the size of $\mathscr{S}(\mathbb{C})$ can be non-polynomial in the size of *Con*.

Proposition 7.1 ([9]). For every $n \ge 2$, there exists a lattice $\mathbf{C} = (Con, \sqsubseteq)$ such that $|\mathscr{S}(\mathbf{C})| \ge (n-2)!$ and n = |Con|. For every $n \ge 1$, there exists a distributed lattice $\mathbf{C} = (Con, \sqsubseteq)$ such that $|\mathscr{S}(\mathbf{C})| \ge n^{\log_2 n}$ and n = |Con|.

Nevertheless, we can exploit order theoretical results and compositional properties of distributive spaces to compute Δ_I in polynomial time in the size of *Con*.

Theorem 7.2 ([9]). Suppose that (Con, \sqsubseteq) is a distributed lattice. Let J and K be two sets such that $I = J \cup K$. Then the following equalities hold:

1.
$$\Delta_I(c) = \prod \{ \Delta_J(a) \sqcup \Delta_K(b) \mid a, b \in Con \text{ and } a \sqcup b \sqsupseteq c \}.$$
 (1)

2.
$$\Delta_I(c) = \bigcap \{ \Delta_J(a) \sqcup \Delta_K(a \to c) \mid a \in Con \}.$$
 (2)

3.
$$\Delta_I(c) = \prod \{ \Delta_J(a) \sqcup \Delta_K(a \to c) \mid a \in Con \text{ and } a \sqsubseteq c \}.$$
 (3)

The above theorem characterizes the information of a group from that of its subgroups. It bears witness to the inherent compositional nature of our notion of distributed space, and realizes the intuition that by joining the information a and $a \rightarrow c$ of their subgroups, the group I can obtain c. This compositional nature is exploited by the algorithms below.

Given a finite scs $\mathbf{C} = (Con, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$, the recursive function DELTAPART3(I, c) in Algorithm 1 computes $\Delta_I(c)$ for any given *c* in *Con*. Its correctness, assuming that (Con, \sqsubseteq) is a distributed lattice, follows from Thm.7.2(3). Termination follows from the finiteness of \mathbf{C} and the fact the sets *J* and *K* in the recursive calls form a partition of *I*. Notice that we select a partition (in halves) rather than any two sets *K*, *J* satisfying the condition $I = J \cup K$ to avoid significant recalculation.

| Algorithm 1 Function DELTAPART3 (I, c) computes $\Delta_I(c)$ | | |
|---|--|--|
| 1: function DeltaPart3(I, c) | | \triangleright Computes $\Delta_I(c)$ |
| 2: | if $I = \{i\}$ then | |
| 3: | return $\mathfrak{s}_i(c)$ | |
| 4: | else | |
| 5: | $\{J, K\} \leftarrow \text{Partition}(I)$ | ▷ returns a partition $\{J, K\}$ of <i>I</i> s.t., $ J = \lfloor I /2 \rfloor$ |
| 6: | return \prod {DELTAPART3 (J, a) \sqcup DELTAPART3 ($K, a \rightarrow c$) $a \in Con$ and $a \sqsubseteq c$ }. | |

Algorithms. Notice DELTAPART3(I,c) computes $\Delta_I(c)$ using Thm.7.2(3). By modifying Line 6 with the corresponding meet operations, we obtain two variants of DELTAPART3 that use, instead of Thm.7.2(3), the Properties Thm.7.2(1) and Thm.7.2(2). We call them DELTAPART1 and DELTAPART2.

Worst-case time complexity. We assume that binary distributive lattice operations \sqcap , \sqcup , and \rightarrow are computed in O(1) time. We also assume a fixed group I of size m = |I| and express the time complexity for computing Δ_I in terms of n = |Con|, the size of the set of constraints. The above-mentioned algorithms compute the value $\Delta_I(c)$. The worst-case time complexity for computing the function Δ_I is in $O(mn^{1+2\log_2 m})$ using DELTAPART1, and $O(mn^{1+\log_2 m})$ using DELTAPART2 and DELTAPART3 [9].

8 Conclusions and Related Work

We have highlighted some results about scs as semantic structures for spatially-distributed systems exhibiting epistemic behaviour. Our work in scs have been inspired by the seminal work on epistemic logic for knowledge and group knowledge in [15, 6, 16]. Meaningful families of structures from logic and economics such as Kripke structures and Aumann structures have been shown to be instances of scs [18]. Furthermore scs have been used to give semantics to modal logics and process calculi [18, 8, 12].

In [18] we introduced a spatial and epistemic process calculus, called sccp, for reasoning about spatial information and knowledge distributed among the agents of a system. In this work scs were introduced as the domain-theoretical structures to represent spatial and epistemic information. These structures are

also used in the denotational and operational semantics of sccp processes. In [18] we also provided operational and denotational techniques for reasoning about the potentially infinite behaviour of spatial and epistemic processes.

In [8, 12] we developed the theory of spatial constraint systems (scs) with extrusion to specify information and processes moving from a space to another. In [11, 10] scs with extrusion are used to give a novel algebraic characterization of the notion of normality in modal logic and to derive right inverse/reverse operators for modal languages. These results were applied to derive new expressiveness results for bisimilarity and well-established modal languages such as Hennessy-Milner logic, and lineartime temporal logic.

In [8, 10, 11, 12] scs are used to reason about beliefs, lies and other epistemic utterances but also restricted to a finite number of agents and individual, rather than group, behaviour of agents.

In [9] we developed semantic foundations and provided algorithms for reasoning about the distributed information of possibly infinite groups in multi-agents systems. We plan to develop similar techniques for reasoning about other group phenomena in multi-agent systems from social sciences and computer music such as group polarization [4] and group improvisation [23].

We have recently learnt that the fundamental operations of dilation and erosion from digital images and mathematical morphology [25] are space and projection functions, respectively. Dilations are applied to figures. Intuitively, figures that are very lightly drawn get thick when dilated. We are currently studying potential applications of distributed spaces in mathematical morphology: E.g., for computing the greatest dilation under a given set of dilations. Similarly, we are also studying scs interpretations of other fundamental operations from mathematical morphology such as opening and closing.

We conclude with some applications of scs in the development of ccp tools and languages. In [14, 13] we described D-SPACES, an implementation of scs that provides property-checking methods as well as an implementation of a specific type of constraint systems (boolean algebras). In [21] we used rewriting logic for specifying and analyzing ccp processes combining spatial and real-time behavior. These processes can run processes in different computational spaces while subject to real-time requirements. The real-time rewriting logic semantics is fully executable in Maude with the help of rewriting modulo SMT: partial information (i.e., constraints) in the specification is represented by quantifier-free formulas on the shared variables of the system that are under the control of SMT decision procedures. The approach is used to symbolically analyze existential real-time reachability properties of process calculi in the presence of spatial hierarchies for sharing information and knowledge. We also developed dspacenet, a multi-agent spatial and reactive ccp language for programming academic forums². The fundamental structure of dspacenet is that of *space*: A space may contain spatial and reactive ccp programs or other spaces. The fundamental operation of dspacenet is that of *program posting*: In each time unit, agents (users) can post spatial reactive ccp programs in the spaces they are allowed to do so. Currently dspacenet is used at Univ. Javeriana Cali for teaching spatial reactive declarative programming.

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²You can try dspacenet at http://www.dspacenet.com.

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