Effective dimension in some general metric spaces

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We introduce the concept of effective dimension for a general metric space. Effective dimension was defined by Lutz in (Lutz 2003) for Cantor space and has also been extended to Euclidean space. Our extension to other metric spaces is based on a supergale characterization of Hausdorff dimension. We present here the concept of constructive dimension and its characterization in terms of Kolmogorov complexity. Further research directions are indicated.

1 Introduction

Effective dimension in Cantor space was defined by Lutz in [8, 9] in order to quantitatively study complexity classes [7]. The connections of effective dimension with Information Theory [11], in particular with Kolmogorov complexity and compression algorithms, some of them suspected even before the definition of effective dimension itself ([12, 13, 15, 16, 1] and more recently for other spaces [14]), have lead to very fruitful areas of research including those within Algorithmic Information theory [3].

In this paper we shall explore the definition of effective dimension for more general metric spaces. The long term purpose of this line of research is to find more and easier dimension bound proofs in those spaces, while the connections with Information Theory already suggest further developments.

The original definition of effective dimension was done in Cantor space which is the set of infinite binary sequences with the usual longest-common-prefix-based metric. The spaces of infinite sequences over other finite alphabets have been also explored, for instance the case of Finite-State effectivity is particularly interesting with this variation [2]. Finally, the Euclidean space $\mathbb{R}^n$ has been explored by several papers that go back to fractal geometry, starting in [10].

Gales and supergales, introduced by Lutz in [8], are intuitively betting strategies in a guessing game on the elements of Cantor space. They allow the interpretation of Hausdorff dimension in terms of prediction and provide natural effectivizations of dimension by restricting the computability and resource-bounds used in the computation of these betting strategies.

We introduce here the concept of nice cover of a metric space. A nice cover can simulate very closely any of the covers required in the definition of Hausdorff dimension, while it allows simple representations of the points in the space and the use of gales as betting games on those representations.

We then characterize Hausdorff dimension using supergales for any metric space with a nice cover. Spaces with nice covers can be fairly general (they are not even required to be locally separable). This characterization allows the definition of effective dimension by restricting the family of supergales that can be used.

In this paper we present an initial step in this direction by introducing the definition of constructive dimension on a metric space. We then characterize constructive dimension in terms of Kolmogorov

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complexity and sketch further properties such as absolute stability (that is, the fact that constructive dimension can be pointwise defined) and a correspondence principle (that is, the fact that constructive dimension coincides with Hausdorff dimension for an interesting family of sets). We finish with a list of topics for further development.

2   Preliminaries

Let (X, ρ) be a metric space. (From now on we shall omit ρ when referring to space (X, ρ)).

Definition. The *diameter* of a set $A \subseteq X$ is

$$\mathrm{diam}(A) = \sup \left\{ \rho(x, y) \mid x, y \in A \right\}.$$

Notice that the diameter of a set can be infinite.

Definition. Let $A \subseteq X$. A *cover of A* is $\mathscr{C} \subseteq \mathscr{P}(X)$ such that $A \subseteq \cup_{U \in \mathscr{C}} U$.

Definition. Let $A \subseteq X$. A is *separable* if there exists a countable set $S \subseteq A$ that is dense in A, that is, for every $x \in A, \delta > 0$ there is an $s \in S$ such that $\rho(x, s) < \delta$.

Definition. The *ball* of radius $r > 0$ about $x \in X$ is the set $B(x, r) = \{ y \in X \mid \rho(y, x) < r \}$.

Definition. An *isolated point in X* is $x \in X$ such that there is a $\delta > 0$ with $B(x, \delta) \cap X = \{x\}$.

We shall be interested in metric spaces that have no isolated points. Notice that metric spaces consisting only of isolated points have little interest for Hausdorff dimension (cf. definition below), while Hausdorff dimension in general spaces can be analyzed by restricting to non isolated points in the space.

We include the basic definitions of Hausdorff dimension. We refer the reader to [4] for a complete introduction and motivation.

For each $A \subseteq X$ and $\delta > 0$, we define the set of countable δ-covers of A

$$\mathscr{H}_\delta(A) = \left\{ \mathscr{U} \mid \mathscr{U} \text{ is a countable cover of } A \text{ and } \mathrm{diam}(U) < \delta \text{ for every } U \in \mathscr{U} \right\}.$$

We can now define $H_\delta^s(A)$ and $H^s(X)$

$$
\begin{aligned}
H_\delta^s(A) &= \inf_{\mathscr{U} \in \mathscr{H}_\delta(A)} \sum_{U \in \mathscr{U}} \mathrm{diam}(U)^s. \\
H^s(A) &= \lim_{\delta \to 0} H_\delta^s(A).
\end{aligned}
$$

Notice that $H_\delta^s(X)$ is monotone as $\delta \to 0$ so $H^s(X)$ is well defined. It is routine to verify that H^s is an outer measure [4], H^s is called the *s-Hausdorff measure*.

Definition. (Hausdorff [5]). The *Hausdorff dimension* of $A \subseteq X$ is

$$\dim_{\mathrm{H}}(A) = \inf \left\{ s \in [0, \infty) \mid H^s(A) = 0 \right\}.$$

Let Σ be a finite set. We denote as Σ^* the set of finite strings over Σ.

3   A supergale characterization of dimension in some metric spaces

3.1   Nice covers

We introduce the concept of a nice cover for a metric space. A nice cover allows well behaving representations of all points in the space, and it will be the key to the gale characterization of Hausdorff

dimension in the next subsection. Intuitively, a nice cover of A is a sequence of covers of A that can closely simulate any Hausdorff cover of A.

Let X be a metric space without isolated points.

Definition. Let $c \in \mathbb{N}$. A *c-nice cover of X* is a sequence $(\mathcal{B}_n)_{n \in \mathbb{N}}$ with $\mathcal{B}_n \subseteq \mathcal{P}(X)$ for every n and such that the following hold

1. (Decreasing monotonicity) For every $n \in \mathbb{N}$, for every $U \in \mathcal{B}_n$, $|\{V \in \mathcal{B}_{n+1}, V \subseteq U\}| < \infty$.

2. (Increasing monotonicity) For every $n \in \mathbb{N}$, $U \in \mathcal{B}_n$, $m < n$, there is a unique $V \in \mathcal{B}_m$ such that $U \subseteq V$.

3. (c-cover) For every $r \in \mathbb{N}$ there is an $\varepsilon > 0$ such that for every $A \subseteq X$ with $0 < \operatorname{diam}(A) < \varepsilon$ there exists $\{U_1, \ldots, U_c\} \subseteq \cup_{n>r}\mathcal{B}_n$ a cover of A, with $\operatorname{diam}(U_i) < c \cdot \operatorname{diam}(A)$ for every i.

Definition. A *nice cover of X* is a c-nice cover for some $c \in \mathbb{N}$.

Notice that the above definition does not require the elements of each cover $\mathcal{B}_n$ to be open or disjoint.

Theorem 3.1 *If X has a countable nice cover then X is separable.*

Notice that both examples mentioned in the introduction (Cantor space and Euclidean space) correspond to metric spaces with countable and very simple nice covers.

3.2 Supergale characterization of Hausdorff dimension

In this subsection we prove a supergale characterization of Hausdorff dimension for X with a nice cover. Notice that each nice cover gives an equivalent characterization of dimension.

The concept of gale we introduce here is the natural extension of the gales introduced in [8] to spaces with nice covers, while the flexibility on the metric spaces makes the proof of this characterization quite more involved than the case of Cantor spaces proven in [8]. For instance we cannot assume anything about the diameters of the covers used.

Let X be a metric space with a nice cover, fix a nice cover $(\mathcal{B}_n)_{n \in \mathbb{N}}$. Let $\mathcal{B} = \cup_n \mathcal{B}_n$. For $n \in \mathbb{N}$, let $\mathcal{B}_{\geq n} = \cup_{m \geq n}\mathcal{B}_m$.

Definition. Given $x \in X$, a *$\mathcal{B}$-representation of x* is a sequence $(w_n)_{n \in \mathbb{N}}$ such that $w_n \in \mathcal{B}_n$ and $x \in \cap_n w_n$.

We denote with $\mathcal{R}(x)$ the set of $\mathcal{B}$-representations of $x \in X$.

A supergale is intuitively a strategy in a betting game on a representation $(w_n)_{n \in \mathbb{N}}$ of an unknown $x \in X$.

Definition. Let $s \in [0, \infty)$. An *s-supergale d* is a function $d : \mathcal{B} \to [0, \infty)$ such that the following hold

- $\sum_{U \in \mathcal{B}_0} d(U)\operatorname{diam}(U)^s < \infty$,

- for every $n \in \mathbb{N}$, for every $U \in \mathcal{B}_n$ the following inequality holds

$$d(U)\operatorname{diam}(U)^s \geq \sum_{V \in \mathcal{B}_{n+1}, V \subseteq U} d(V)\operatorname{diam}(V)^s. \tag{1}$$

An *s-gale* is an s-supergale for which equation (1) holds with equality.

Definition. An *s-supergale d succeeds on $x \in X$* if there is a $(w_n)_{n \in \mathbb{N}} \in \mathcal{R}(x)$, such that

$$\limsup_n d(w_n) = \infty.$$

Definition. Let d be an s-supergale. The success set of d is

$$S^\infty[d] = \{x \in X \mid d \text{ suceeds on } x\}.$$

Definition. $\hat{\mathcal{G}}(A) = \{s \mid \text{there is an } s\text{-supergale } d \text{ with } A \subseteq S^\infty[d]\}.$

Theorem 3.2 *(Supergale characterization) Let X be a metric space that has a nice cover, let $A \subseteq X$. Then*

$$\dim_{\mathrm{H}}(A) = \inf \hat{\mathcal{G}}(A).$$

Proof. Let $s > \dim_{\mathrm{H}}(A)$. Then for any $k \in \mathbb{N}$ there is a countable cover of A, $\mathscr{C}_k$, such that $\sum_{U \in \mathscr{C}_k} \mathrm{diam}(U)^s < 2^{-k}$ and $\mathrm{diam}(U) > 0$ for each $U \in \mathscr{C}_k$. (If necessary substitute each $U_n \in \mathscr{C}_k$ with $\mathrm{diam}(U_n) = 0$ by a ball of radius $2^{-k/s - n/s - 1}$).

Let $r \in \mathbb{N}$ and fix ε as in property (3) of nice covers, let $k = k_r > k_{r-1}$ be such that $2^{-k} < \varepsilon$. Using property (3) of nice covers we can get a cover $\mathscr{E}_k \subseteq \mathscr{B}_{\geq r}$ of A such that

$$\sum_{W \in \mathscr{E}_k} \mathrm{diam}(W)^s < c^{1+s} \cdot 2^{-k}.$$

Let $\mathscr{D}_k = \{U \mid U \in \mathscr{E}_k \text{ and no proper superset of } U \text{ is in } \mathscr{E}_k\}$. Then $\mathscr{D}_k$ is a cover of A and

$$\sum_{W \in \mathscr{D}_k} \mathrm{diam}(W)^s < c^{1+s} \cdot 2^{-k}.$$

Define $d_k : \mathscr{B} \to [0, \infty)$ as follows,
For $U \in \mathscr{B}$, if $\mathrm{diam}(U) = 0$ then $d(U) = 1$.
If $\mathrm{diam}(U) > 0$, $U \in \mathscr{B}_n$ for $n > 0$, and there is $V \in \mathscr{B}_{n-1} - \mathscr{B}_n$ and $W \in \mathscr{D}_k$ with $U \subseteq V \subseteq W$ then

$$d_k(U) = \frac{d_k(V) \, \mathrm{diam}(V)^s}{\sum_{U' \subseteq V, U' \in \mathscr{B}_n} \mathrm{diam}(U')^s}.$$

Otherwise, if $U \in \mathscr{B}_n - \mathscr{B}_{n-1}$ for $n > 0$ or $U \in \mathscr{B}_n$ for $n = 0$,

$$d_k(U) = \sum_{W \in \mathscr{D}_k \cap \mathscr{B}_{\geq n}, W \subseteq U} \frac{\mathrm{diam}(W)^s}{\mathrm{diam}(U)^s}.$$

Claim 3.3 d_k *is an s-supergale.*

Proof of Claim 3.3. Let $V \in \mathscr{B}_{n-1} - \mathscr{B}_n$ with $\mathrm{diam}(V) > 0$ and $\sum_{U' \subseteq V, U' \in \mathscr{B}_n} \mathrm{diam}(U') > 0$.
If there is $W \in \mathscr{D}_k$ such that $V \subseteq W$ then

$$\sum_{U \subseteq V, U \in \mathscr{B}_n} d_k(U) \, \mathrm{diam}(U)^s \; = \; \sum_{U \subseteq V, U \in \mathscr{B}_n} \frac{d_k(V) \, \mathrm{diam}(V)^s}{\sum_{U' \subseteq V, U' \in \mathscr{B}_n} \mathrm{diam}(U')^s} \mathrm{diam}(U)^s$$

$$= \; d_k(V) \, \mathrm{diam}(V)^s.$$

If for any $W \in \mathscr{D}_k$, $V \nsubseteq W$ then

$$d_k(V) = \sum_{W \in \mathscr{D}_k \cap \mathscr{B}_{\geq n-1}, W \subseteq V} \frac{\mathrm{diam}(W)^s}{\mathrm{diam}(V)^s}.$$

Therefore,

$$\sum_{U \subseteq V, U \in \mathscr{B}_n} d_k(U) \operatorname{diam}(U)^s = \sum_{U \subseteq V, U \in \mathscr{B}_n} \sum_{W \in \mathscr{D}_k \cap \mathscr{B}_{\geq n}, W \subseteq U} \frac{\operatorname{diam}(W)^s}{\operatorname{diam}(U)^s} \operatorname{diam}(U)^s$$

$$= \sum_{U \subseteq V, U \in \mathscr{B}_n} \sum_{W \in \mathscr{D}_k \cap \mathscr{B}_{\geq n}, W \subseteq U} \operatorname{diam}(W)^s$$

$$\leq \sum_{W \in \mathscr{D}_k \cap \mathscr{B}_{\geq n-1}, W \subseteq V} \operatorname{diam}(W)^s = d_k(V) \operatorname{diam}(V)^s,$$

where the last inequality follows from property (2) of nice covers.

For every $U \in \mathscr{B}_0$, we use the second part in the definition of d_k (since there is no $n > 0$ with $U \in \mathscr{B}_n$ and $V \in \mathscr{B}_{n-1} - \mathscr{B}_n$ with $U \subseteq V$). Therefore, using property (2) of nice covers,

$$\sum_{U \in \mathscr{B}_0} d_k(U) \operatorname{diam}(U)^s \leq \sum_{W \in \mathscr{D}_k} \operatorname{diam}(W)^s < c^{1+s} \cdot 2^{-k} < \infty.$$

$\square$

Claim 3.4 *If $W \in \mathscr{D}_k$, $d_k(W) = 1$.*

Proof of Claim 3.4. If $\operatorname{diam}(W) > 0$ and $W \in \mathscr{B}_n$, since all sets in $\mathscr{D}_k$ are incomparable, we use the second part in the definition of d_k and

$$d_k(W) = \sum_{W' \in \mathscr{D}_k \cap \mathscr{B}_{\geq n}, W' \subseteq W} \frac{\operatorname{diam}(W')^s}{\operatorname{diam}(W)^s} = 1.$$

$\square$

Claim 3.5 *For every $k \in \mathbb{N}$, $U \in \mathscr{B}$, with $\operatorname{diam}(U) > 0$, $d_k(U) \leq \frac{c^{1+s} \cdot 2^{-k}}{\operatorname{diam}(U)^s}$.*

Proof of Claim 3.5.

We prove by induction on $n - m$ that for every $n, m \in \mathbb{N}$ with $m < n$, $U \subseteq V$ with $\operatorname{diam}(U) > 0$, $U \in \mathscr{B}_n$ and $V \in \mathscr{B}_m$,

$$d_k(U) \leq \frac{d_k(V) \operatorname{diam}(V)^s}{\operatorname{diam}(U)^s}.$$

By the definition of supergale, if $U \in \mathscr{B}_n$, $d_k(U) \leq \frac{d_k(U') \operatorname{diam}(U')^s}{\operatorname{diam}(U)^s}$ for $U' \in \mathscr{B}_{n-1}$ with $U \subseteq U'$. By induction $d_k(U') \leq \frac{d_k(V) \operatorname{diam}(V)^s}{\operatorname{diam}(U')^s}$ and therefore $d_k(U) \leq \frac{d_k(V) \operatorname{diam}(V)^s}{\operatorname{diam}(U)^s}$.

For every $W \in \mathscr{B}_0$ with $\operatorname{diam}(W) > 0$, we use the second part in the definition of d_k and so $d_k(W) \leq \frac{c^{1+s} \cdot 2^{-k}}{\operatorname{diam}(W)^s}$.

Since for every $U \in \mathscr{B}$ there is a $W \in \mathscr{B}_0$ with $U \subseteq W$ we have that

$$d_k(U) \leq \frac{d_k(W) \operatorname{diam}(W)^s}{\operatorname{diam}(U)^s} \leq \frac{c^{1+s} \cdot 2^{-k}}{\operatorname{diam}(U)^s}.$$

$\square$

We define next an s-supergale $d(U) = \sum_r 2^{k_r} d_{2k_r}(U)$. By Claim 3.5 d is well-defined. By Claim 3.4, if $W \in \mathscr{D}_k$, $d(W) \geq 2^k$. Since for every r, $\mathscr{D}_{k_r} \subseteq \mathscr{B}_{\geq r}$ is a cover of A, we have that $A \subseteq S^{\infty}[d]$ and $s \in \mathscr{G}(A)$.

For the other direction, let $s \in \hat{\mathscr{G}}(A)$. Then there exists an s-supergale d such that $A \subseteq S^{\infty}[d]$.

Claim 3.6 *The set $\mathscr{R} = \{U \,|\, d(U)\mathrm{diam}(U) > 0\}$ is countable.*

Claim 3.7 *Let d be an s-supergale. Then for every $\mathscr{E} \subseteq \mathscr{B} \cap \mathscr{R}$ such that all sets in $\mathscr{E}$ are incomparable we have that*

$$\sum_{W \in \mathscr{B}_0} d(W)\,\mathrm{diam}(W)^s \geq \sum_{V \in \mathscr{E}} d(V)\,\mathrm{diam}(V)^s.$$

For each $k \in \mathbb{N}$ let

$$\mathscr{C}_k = \left\{ U \,\middle|\, \mathrm{diam}(U) > 0, d(U) > 2^k \cdot \sum_{W \in \mathscr{B}_0} d(W)\,\mathrm{diam}(W)^s \right\},$$

let $\mathscr{D}_k = \{U \,|\, U \in \mathscr{C}_k$ and no proper superset of U is in $\mathscr{C}_k\}$. Then, using Claim 3.7, $\sum_{U \in \mathscr{D}_k} \mathrm{diam}(U)^s \leq 2^{-k}$.

Notice that for every k, $\mathscr{D}_k$ is a $2^{-k/s}$-cover of $S^\infty[d]$, so $\dim_\mathrm{H}(A) \leq s$. This completes our proof.

$\square$

4   Constructive dimension

In this section we take a first step in the effectivization of Hausdorff dimension by considering constructive dimension. We consider spaces that have computable nice covers (defined below). Computable nice covers have a flavor similar to computable metric spaces, although we conjecture they are incomparable to those.

Then we characterize constructive dimension in terms of Kolmogorov complexity using the concept of Kolmogorov complexity of $x \in X$ at precision $r \in \mathbb{N}$ inspired by [10]. This characterization, together with the absolute stability and correspondence principle sketched below allows a full Theory of Information view of Hausdorff dimension in some general metric spaces.

Definition.   Let X be a metric space with a nice cover $(\mathscr{B}_n)_{n \in \mathbb{N}}$. We say that *$X$ has a computable nice cover* if the following hold,

4. (Small size) There is an $0 < \zeta < 1$ such that for every $n \in \mathbb{N}$, for every $U \in \mathscr{B}_n$, $\mathrm{diam}(U) < \zeta^n$.

5. (Computable diameter) $\mathscr{B} = \cup_n \mathscr{B}_n$ is countable and there is a surjective $\delta : \Sigma^* \to \mathscr{B}$ for a finite Σ such that $\mathrm{diam} \circ \delta$ is computable.

6. (Computable cover) For each $n \in \mathbb{N}$, $U \in \mathbb{N}$, $P_U = \{V \,|\, V \in \mathscr{B}_{n+1}, V \subseteq U\}$ can be computed from $\delta^{-1}(U)$.

Fix a space X with a computable nice cover. Fix δ as in the definition above.

Definition.   Let d be a supergale. Then d is *contructive* if $d \circ \delta$ is lower semicomputable.

Definition.   Let $A \subseteq X$,

$$\hat{\mathscr{G}}_\mathrm{constr}(A) = \{s \,|\, \text{there is a constructive } s\text{-supergale } d \text{ with } A \subseteq S^\infty[d]\}.$$

Definition.   Let $A \subseteq X$. We define the constructive dimension of A as $\mathrm{cdim}(A) = \inf \hat{\mathscr{G}}_\mathrm{constr}(A)$.

Constructive dimension can be characterized in terms of Kolmogorov complexity as follows. Let $K(w)$ denote the usual self-delimiting Kolmogorov complexity of $w \in \Sigma^*$.

Definition.   Let $x \in X$, let $r \in \mathbb{N}$. The *Kolmogorov complexity of x at precision r* is

$$K_r(x) = \inf \left\{ K(w) \,\middle|\, x \in \delta(w), 2^{-r} < \mathrm{diam}(\delta(w)) \leq 2^{-r+1} \right\}.$$

Theorem 4.1 *Let X be a metric space with a computable nice cover. Let $x \in X$,*

$$\mathrm{cdim}(x) = \liminf_r \frac{\mathrm{K}_r(x)}{r}.$$

Proof. Let s, s' be rational such that $s > s' > s'' > \liminf_r \frac{\mathrm{K}_r(x)}{r}$. Let

$$A = \left\{ w \,\middle|\, \mathrm{K}(w) \leq s'(-\log(\mathrm{diam}(\delta(w)))) \right\}.$$

Then A is computably enumerable.

We define d as follows, let $U \in \mathscr{B}_n$ with $\mathrm{diam}(U) > 0$,

$$d(U) = \sum_{V \subseteq U, V \in \delta(A) \cap \mathscr{B}_{\geq n}} \frac{\mathrm{diam}(V)^{s'}}{\mathrm{diam}(U)^s}.$$

d is well defined since $\sum_{V \in \delta(A)} \mathrm{diam}(V)^{s'} \leq \sum_w 2^{-\mathrm{K}(w)} < \infty$.
d is an s-supergale since for $W \in \mathscr{B}_{n-1}$,

$$\sum_{U \subseteq W, U \in \mathscr{B}_n} d(U)\, \mathrm{diam}(U)^s = \sum_{U \subseteq W, U \in \mathscr{B}_n} \sum_{V \subseteq U, V \in \delta(A) \cap \mathscr{B}_{\geq n}} \mathrm{diam}(V)^{s'} \leq$$

$$\leq \sum_{V \subseteq W, V \in \delta(A) \cap \mathscr{B}_{\geq n-1}} \mathrm{diam}(V)^{s'} = d(W)\mathrm{diam}(W)^s.$$

If $U \in \delta(A)$ then $d(U) \geq \mathrm{diam}(U)^{s'-s}$. Since $\mathrm{K}_r(x) \leq rs''$ for infinitely many r, for those r there is a w with $\mathrm{K}(w) \leq rs''$, $x \in \delta(w)$ and $2^{-r} < \mathrm{diam}(\delta(w)) \leq 2^{-r+1}$. Therefore $w \in A$ and $d(w) \geq \mathrm{diam}(\delta(w))^{s'-s} \geq 2^{(r-1)(s-s')}$.

By condition (4) of computable nice covers, $\delta(w) \in \mathscr{B}_{>ar}$ (for $a = 1/(-\log(\zeta))$) and $x \in S^\infty[d]$.

For the other direction, let $s > \mathrm{cdim}(x)$. Let d be a constructive s-supergale such that $x \in S^\infty[d]$. For each $k \in \mathbb{N}$, let

$$A_k = \left\{ w \,\middle|\, d(\delta(w)) \geq 2^k \Big(\sum_{W \in \mathscr{B}_0} d(W)\, \mathrm{diam}(W)^s \Big) \right\}.$$

Then the number of $w \in A_k$ such that $2^{-r} < \mathrm{diam}(\delta(w)) \leq 2^{-r+1}$ is at most 2^{-k+rs}. Therefore for $w \in A_k$ with $2^{-r} < \mathrm{diam}(\delta(w)) \leq 2^{-r+1}$, $\mathrm{K}(w) \leq rs - k + O(\log k) + O(\log r)$, and

$$\liminf_r \frac{\mathrm{K}_r(x)}{r} \leq \frac{rs - k + O(\log k) + O(\log r)}{r} \leq s.$$

$\square$

We next state without proof the property of total stability of constructive dimension (cf. [9] for the corresponding version in Cantor space).

Theorem 4.2 *Let X be a metric space with a computable nice cover. Let $A \subseteq X$. Then*

$$\mathrm{cdim}(A) = \sup_{x \in A} \mathrm{cdim}(x).$$

Finally we remark (again without detailing a proof in this initial paper) that for arbitrary unions of Π_1^0 sets Hausdorff dimension and constructive dimension coincide. Cf. [6] for the Cantor space version.

Definition. For each $U \in \mathcal{B}$, let the U-cylinder be defined as $\mathbf{C}_U = \{x \in X \mid x \in U\}$. For $A \subseteq X$, $A \in \Sigma_1^0$ if there is a computable $h : \mathbb{N} \to \Sigma^*$ such that

$$A = \bigcup_{i \in \mathbb{N}} \mathbf{C}_{\delta(h(i))}.$$

For $A \subseteq X$, $A \in \Pi_1^0$ if $A^c \in \Sigma_1^0$. For $A \subseteq X$, $A \in \Sigma_2^0$ if there is a computable $h : \mathbb{N} \times \mathbb{N} \to \Sigma^*$ such that

$$A = \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} \mathbf{C}_{\delta(h(i,j))}^c.$$

Theorem 4.3 *Let X be a metric space with a computable nice cover. Let $A \subseteq X$ be a union of Π_1^0 sets. Then* $\mathrm{cdim}(A) = \dim_{\mathrm{H}}(A)$.

5 Further directions

This paper intended to give an initial view of effective dimension on arbitrary metric spaces. A number of issues have not been addressed here including the following.

- The definition of resource-bounded dimension for resource-bounds other than lower semicomputability.

- The role of different (computable) nice covers in effectivization and condition for their equivalence within it. For instance Finite-State dimension in Euclidean space depends heavily on the choice of nice cover [2].

- The exact relationship between computable nice covers and computable metric spaces.

- The effectivization of packing dimension, a dual of Hausdorff dimension for which a gale characterization exists (proof not included in this initial paper).

References

[1] J. Cai & J. Hartmanis (1994): *On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line*. Journal of Computer and Systems Sciences 49, pp. 605–619, doi:10.1016/S0022-0000(05)80073-X.

[2] J. J. Dai, J. I. Lathrop, J. H. Lutz & E. Mayordomo (2004): *Finite-State Dimension*. Theoretical Computer Science 310, pp. 1–33, doi:10.1016/S0304-3975(03)00244-5.

[3] R. Downey & D. Hirschfeldt (2010): *Algorithmic randomness and complexity*. Theory and Applications of Computability. Springer-Verlag, doi:10.1007/978-0-387-68441-3.

[4] K. Falconer (2003): *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, doi:10.1002/0470013850.

[5] F. Hausdorff (1919): *Dimension und äußeres Maß*. Mathematische Annalen 79, pp. 157–179, doi:10.1007/BF01457179.

[6] J. M. Hitchcock (2005): *Correspondence Principles for Effective Dimensions*. Theory of Computing Systems 38, pp. 559–571, doi:10.1007/s00224-004-1122-1.

[7] J. M. Hitchcock, J. H. Lutz & E. Mayordomo (2005): *The Fractal Geometry of Complexity Classes*. *SIGACT News Complexity Theory Column* 36, pp. 24–38, doi:10.1145/1086649.1086662.

[8] J. H. Lutz (2003): *Dimension in complexity classes*. *SIAM Journal on Computing* 32, pp. 1236–1259, doi:10.1137/S0097539701417723.

[9] J. H. Lutz (2003): *The dimensions of individual strings and sequences*. *Information and Computation* 187, pp. 49–79, doi:10.1016/S0890-5401(03)00187-1.

[10] J.H. Lutz & E. Mayordomo (2008): *Dimensions of Points in Self-Similar Fractals*. *SIAM Journal on Computing* 38, pp. 1080–1112, doi:10.1137/070684689.

[11] E. Mayordomo (2008): *Effective Fractal Dimension in Algorithmic Information Theory*. In: S. B. Cooper, B. Löwe, A. Sorbi (eds.), *New Computational Paradigms: Changing Conceptions of What is Computable*, Springer-Verlag, pp. 259–285, doi:10.1007/978-0-387-68546-5_12.

[12] B. Ya. Ryabko (1984): *Coding of combinatorial sources and Hausdorff dimension*. *Soviets Mathematics Doklady* 30, pp. 219–222.

[13] B. Ya. Ryabko (1986): *Noiseless coding of combinatorial sources*. *Problems of Information Transmission* 22, pp. 170–179.

[14] S.G. Simpson (2011): *Symbolic dynamics: entropy = dimension = complexity*. Submitted.

[15] L. Staiger (1993): *Kolmogorov Complexity and Hausdorff Dimension*. *Information and Computation* 103, pp. 159–94, doi:10.1006/inco.1993.1017.

[16] L. Staiger (1998): *A Tight Upper Bound on Kolmogorov Complexity and Uniformly Optimal Prediction*. *Theory of Computing Systems* 31, pp. 215–29, doi:10.1007/s002240000086.