Unavoidable Sets of Partial Words of Uniform Length

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A set X of partial words over a finite alphabet A is called unavoidable if every two-sided infinite word over A has a factor compatible with an element of X. Unlike the case of a set of words without holes, the problem of deciding whether or not a given finite set of n partial words over a k-letter alphabet is avoidable is NP-hard, even when we restrict to a set of partial words of uniform length. So classifying such sets, with parameters k and n, as avoidable or unavoidable becomes an interesting problem. In this paper, we work towards this classification problem by investigating the maximum number of holes we can fill in unavoidable sets of partial words of uniform length over an alphabet of any fixed size, while maintaining the unavoidability property.

1 Introduction

The study of combinatorics on partial words has been developing in recent years (see, e.g., [3]). A partial word is a finite sequence over a finite alphabet A, a sequence that may have some undefined positions, called *holes* and denoted by $\diamond$'s, where the $\diamond$ symbol is compatible with every letter of A. For example, $a\diamond\diamond c\diamond\diamond\diamond\diamond b$ is a partial word with six holes over the alphabet $\{a,b,c\}$. Now let w be a two-sided infinite word and u be a partial word. Then, *w meets u* if w has a factor compatible with u; otherwise, *w avoids u*. A set X of partial words over A is *unavoidable* if every two-sided infinite word over A meets an element of X; otherwise, it is *avoidable*. It is important to note that if X is unavoidable, then every infinite unary word has a factor compatible with a member of X. Unavoidable sets of partial words were introduced in [5]. In the context of *total* words, those without holes, this concept of unavoidable sets has been extensively studied (see, e.g., [1, 9, 10, 12, 13, 14, 15, 17, 18, 19]).

There are two major problems that have been identified in the context of unavoidable sets of partial words. The first one is the problem of deciding whether a given finite set of partial words over a k-letter alphabet is avoidable, where $k \geq 2$. Unlike for total words, this problem is NP-hard [8] (see [11, 16] for an algorithm that efficiently decides the avoidability of sets of total words). While several variations of this problem are NP-hard, others are efficiently decidable [2, 7]. The second problem is the one of characterizing the unavoidable sets of n partial words over an alphabet of size k. As shown in [5], it is enough to consider the case where $k \leq n$ and when $k \geq 3$, the case where $k < n$. The $n = 1$ and $k = 1$ cases being trivial, the $n = 2, k = 2$ case was completely characterized by coloring Cayley graphs [4]. So the next step is to study the $n = 3, k = 2$ case.

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A problem, related to the characterization problem, we are concerned with is "What is the minimum number of holes in an m-uniform unavoidable set of partial words (summed over all partial words in the set)?" By m-uniform here, we mean each element in the set has constant length m. In [6], it was proved that for $m \geq 4$, the minimum number of holes in an m-uniform unavoidable set of size three over a binary alphabet is $2m-5$ if m is even, and $2m-6$ if m is odd. An easier way to think of it is the following.

Theorem 1. *[6] Let $m \geq 4$ and let $X = \{a\diamond^{m-2}a, b\diamond^{m-2}b, a\diamond^{m-2}b\}$ be an unavoidable set over $\{a,b\}$. Then the maximum number of holes we can fill in X, while maintaining the unavoidability property, is $m-1$ if m is even, and m if m is odd.*

In this paper, given a k-letter alphabet $A_k = \{a_1,\ldots,a_k\}$, we consider subsets of $X_0 = \{a_i\diamond^{m-2}a_j \mid i \leq j\}$. We denote by $H^k_{m,n}$ the minimum number of holes in any unavoidable m-uniform set (summed over all partial words in the set) of size n over A_k. Thus Theorem 1 states that for $m \geq 4$, $H^2_{m,3} = 2m-5$ if m is even, and $H^2_{m,3} = 2m-6$ if m is odd. Without loss of generality, we require that $0, m-1$ are defined positions, i.e., $0, m-1$ are not holes, in each partial word in any unavoidable m-uniform set.

The contents of our paper are as follows. In Section 2, we review some background material on unavoidable sets of partial words. We also give the $k+\binom{k}{2}$ lower bound on the size of an m-uniform unavoidable set over A_k. In Section 3, we give results on m-uniform unavoidable sets over A_3 which are useful to show our main result. In Section 4, we calculate the minimum number of holes in an m-uniform unavoidable set X over A_k, where X has size exactly $k+\binom{k}{2}$. In Section 5, we conclude with some remarks.

2 Preliminaries on unavoidable sets

An *alphabet* A is a non-empty finite set of *letters*. A *finite word* over A is a finite sequence of elements from A; in other words, it is a function $w : \{0,\ldots,|w|-1\} \to A$, where $|w|$ denotes the length of w. We write $w(i)$ for the letter at position i of w (positions are indexed starting at 0).

A *two-sided infinite word* over A is a function $w : \mathbb{Z} \to A$. It is called p-periodic, or has period p, if p is a positive integer such that $w(i) = w(i+p)$ for all $i \in \mathbb{Z}$. For a non-empty finite word v, we write $v^{\mathbb{Z}}$ for the unique two-sided infinite $|v|$-periodic word w such that $w(0)\cdots w(|v|-1) = v$, and we write $v^{\mathbb{N}}$ for the unique one-sided infinite $|v|$-periodic word w such that $w(0)\cdots w(|v|-1) = v$. A finite word u is a *factor* of a two-sided infinite word w if $w(i)\cdots w(i+|u|-1) = u$ for some $i \in \mathbb{Z}$.

A *(finite) partial word* over A is a function $u : \{0,\ldots,|u|-1\} \to A_\diamond$, where $A_\diamond = A \cup \{\diamond\}$ with $\diamond \notin A$. For $0 \leq i < |u|$, if $u(i) \in A$ then $i \in D(u)$ or i is defined in u; otherwise, i is a hole in u. We write $h(u)$ for the number of holes in u. We say u is a *total* word when $h(u) = 0$. Letting u and v be two partial words of equal length, u is *compatible* with v, denoted $u \uparrow v$, if $u(i) = v(i)$ whenever $i \in D(u) \cap D(v)$.

To *strengthen* a partial word is to replace a $\diamond$ with a letter in A, while to *weaken* a partial word is to set $u(i) = \diamond$ for some $i \in D(u)$. For example, $aa\diamond cb$ is a strengthening of $aa\diamond\diamond b$ and $a\diamond\diamond\diamond b$ is a weakening of $aa\diamond\diamond b$. We say that we have "filled a hole" or "inserted a letter" in a partial word u to mean that we have strengthened u. We also say that the partial word v is a *strengthening* of the partial word u, denoted $v \succ u$, if v has a factor strengthening u. We similarly define *weakening*.

We extend these notions to sets X, Y of partial words as follows. The set X is a strengthening of Y, denoted $X \succ Y$, if for every $x \in X$ there exists $y \in Y$ such that $x \succ y$. Similarly for X is a weakening of Y. It is important to note that if an infinite word w meets a set X, then it also meets every weakening of X, while if w avoids X then it avoids any strengthening of X. This means that if X is unavoidable, so are all weakenings of X, while if X is avoidable, so are all strengthenings of X.

If X is a set of partial words and Y is the set resulting from performing operations on X called factoring (if there exist partial words $x, y \in X$ such that y is a weakening of a factor of x, then $Y = X \setminus \{x\}$), prefix-suffix (if there exists a partial word $x = ya \in X$ with $a \in A$ such that for every $b \in A$ there exists a suffix z of y and a partial word $v \in X$ with v a weakening of zb, then $Y = (X \setminus \{x\}) \cup \{y\}$), hole truncation (if $x\diamond^n \in X$ for some positive integer n, then $Y = (X \setminus \{x\diamond^n\}) \cup \{x\}$), and expansion ($Y = (X \setminus \{x\}) \cup \{x_1, x_2, \dots, x_n\}$, where $\{x_1, x_2, \dots, x_n\}$ is a partial expansion on $x \in X$), then X is avoidable if and only if Y is avoidable [5]. If $u = u_1\diamond u_2\diamond \dots u_{n-1}\diamond u_n$, then $\{u_1a_1u_2a_2\dots u_{n-1}a_{n-1}u_n \mid a_1, a_2, \dots, a_{n-1} \in A\}$ is called a *partial expansion* on u (note that $u_1, u_2, \dots, u_n$ are partial words that may contain holes, and also note that u is a weakening of v for every member v of a partial expansion on u).

Letting p be a prime and $q, m \in \mathbb{N}$, we write $p^q \| m$ if p^q *maximally divides* m, i.e., p^q divides m, but p^{q+1} does not divide m.

We end this section by establishing a lower bound on the size of an m-uniform unavoidable set over a k-ary alphabet.

Proposition 1. *There is no non-trivial unavoidable m-uniform set of size less than $k + \binom{k}{2}$ over A_k (we call* trivial *any set of partial words containing the empty word or $\diamond^n$ for some positive integer n).*

Proof. Let X be an m-uniform unavoidable set over A_k. None of the two-sided infinite words $w_i = a_i^{\mathbb{Z}}$ and $w_{i,j} = (a_i^{m-1}a_j^{m-1})^{\mathbb{Z}}$, $i < j$, can avoid X. Therefore, X must contain an element compatible with a length m factor of w_i (by our convention, that element starts and ends with a_i), for each i, and an element compatible with a length m factor of $w_{i,j}$ (by our convention, that element starts with a_i and ends with a_j or vice versa), for each $i < j$. Since these elements are distinct, we deduce that $|X| \geq k + \binom{k}{2}$. □

3 Uniform unavoidable sets over the ternary alphabet

In examining the minimum number of holes in m-uniform unavoidable sets over $\{a, b, c\}$, we must consider sets of size at least $3 + \binom{3}{2} = 6$. As mentioned earlier, we restrict our attention to sets of size exactly six. By the proof of Proposition 1, $a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c$ must be in the set, as well as one of $a\diamond^{m-2}b$ or $b\diamond^{m-2}a$, one of $a\diamond^{m-2}c$ or $c\diamond^{m-2}a$, and one of $b\diamond^{m-2}c$ or $c\diamond^{m-2}b$. There result eight possible sets:

$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, a\diamond^{m-2}b, a\diamond^{m-2}c, b\diamond^{m-2}c\},$$
$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, a\diamond^{m-2}b, a\diamond^{m-2}c, c\diamond^{m-2}b\},$$
$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, a\diamond^{m-2}b, c\diamond^{m-2}a, c\diamond^{m-2}b\},$$
$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, b\diamond^{m-2}a, a\diamond^{m-2}c, b\diamond^{m-2}c\},$$
$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, b\diamond^{m-2}a, c\diamond^{m-2}a, b\diamond^{m-2}c\},$$
$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, b\diamond^{m-2}a, c\diamond^{m-2}a, c\diamond^{m-2}b\},$$
$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, a\diamond^{m-2}b, c\diamond^{m-2}a, b\diamond^{m-2}c\},$$
$$\{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c, b\diamond^{m-2}a, a\diamond^{m-2}c, c\diamond^{m-2}b\}.$$

The last two are avoidable by $(a^{m-1}c^{m-1}b^{m-1})^{\mathbb{Z}}$ and $(b^{m-1}c^{m-1}a^{m-1})^{\mathbb{Z}}$ respectively, while the six others are equivalent up to renamings of letters (in fact, there is an unavoidable m-uniform set of minimal size for any total order on the alphabet). So we define the basic m-uniform unavoidable set of minimal size over $\{a, b, c\}$ as $X_0 = T_0 \cup T_0'$, where

$$T_0 = \{a\diamond^{m-2}a, b\diamond^{m-2}b, c\diamond^{m-2}c\} \text{ and } T_0' = \{a\diamond^{m-2}b, a\diamond^{m-2}c, b\diamond^{m-2}c\}.$$

The set T_0 contains only the words whose endpoints are the same, while T_0' contains only those whose endpoints are different. We begin by filling in the holes in X_0 one at a time to classify which strengthenings preserve unavoidability. In the rest of the paper, the notation X_i refers to a set created by filling in i holes in X_0.

3.1 Filling in holes in T_0

When we attempt to strengthen one of the T_0 words, we need to consider the following two cases: we can either insert a into $a\diamond^{m-2}a$, say, to obtain $a\diamond^{x_1}a\diamond^{x_2}a$, or we can insert b, say, into $a\diamond^{m-2}a$ to obtain $a\diamond^{x_1}b\diamond^{x_2}a$. However, filling in one hole of $a\diamond^{m-2}a$ with an a is equivalent to filling in any number of holes in $a\diamond^{m-2}a$ with a if that is the only word we strengthen.

Proposition 2. *1. For all $q \in \mathbb{N}$, the m-uniform set $X_q = (X_0 \setminus \{a\diamond^{m-2}a\}) \cup \{a\diamond^{x_1}a\diamond^{x_2}a\cdots a\diamond^{x_q}a\}$ is unavoidable.*

2. The m-uniform set $X_1 = (X_0 \setminus \{a\diamond^{m-2}a\}) \cup \{a\diamond^{x_1}b\diamond^{x_2}a\}$ is avoidable.

Now, let us fill in two holes in two words of T_0. Recall that strengthenings of avoidable sets are avoidable. We just noticed that if we want to preserve unavoidability, we cannot fill any of the holes in $a\diamond^{m-2}a$ with b. Furthermore, filling in any number of the holes in $a\diamond^{m-2}a$ with a's preserves unavoidability. The only remaining way to fill in two of the holes in T_0 is when one hole from $a\diamond^{m-2}a$ is filled with an a and one hole from $b\diamond^{m-2}b$ is filled with a b.

Proposition 3. *Let $X_2 = (X_0 \setminus \{a\diamond^{m-2}a, b\diamond^{m-2}b\}) \cup \{a\diamond^{x_1}a\diamond^{x_2}a, b\diamond^{y_1}b\diamond^{y_2}b\}$ be an m-uniform set. Furthermore, let $2^r \| x_1+1$, $2^s \| y_1+1$, and $2^t \| m-1$. Then, X_2 is avoidable if and only if m is odd and $r = s < t$.*

Next, we address what happens when we fill in one hole in each of the three words in T_0. From [5, Lemma 3], there exists a two-sided infinite word w_i over $\{a,b,c\}$ with period $m-1$ that avoids $\{a\diamond^i a, b\diamond^i b, c\diamond^i c\}$ for every $i \le \lfloor \frac{m-3}{2} \rfloor$. However, since w_i has period $m-1$, w_i avoids the set

$$Z = T_0' \cup \{a\diamond^i a\diamond^{m-i-3}a, b\diamond^i b\diamond^{m-i-3}b, c\diamond^i c\diamond^{m-i-3}c\}.$$

Proposition 4. *Any m-uniform set of the form $X_3 = T_3 \cup T_0'$, where $T_3 = \{a\diamond^{x_1}a\diamond^{x_2}a, b\diamond^{y_1}b\diamond^{y_2}b, c\diamond^{z_1}c\diamond^{z_2}c\}$, is avoidable.*

3.2 Filling in holes in T_0'

First, there are two cases to consider when filling a hole in $a\diamond^{m-2}b$: the added letter is distinct from both a and b, or it is one of a or b. For the former case, the m-uniform set $(X_0 \setminus \{a\diamond^{m-2}b\}) \cup \{a\diamond^{x_1}c\diamond^{x_2}b\}$ is avoidable by the infinite word $(a^{m-1}b^{m-1})^{\mathbb{Z}}$. For the latter case, the following proposition holds.

Proposition 5. *1. If $x_2+1 \not\equiv 0 \pmod{x_1+1}$, the m-uniform set $X_1 = (X_0 \setminus \{a\diamond^{m-2}c\}) \cup \{a\diamond^{x_1}c\diamond^{x_2}c\}$ is avoidable. Otherwise, $x_2 \ge x_1$ and $w = (a^{x_1+1}b^{x_2+1}c^{x_1+1}a^{x_2+1}b^{x_1+1}c^{x_2+1})^{\mathbb{Z}}$ avoids X_1.*

2. If $x_1+1 \not\equiv 0 \pmod{x_2+1}$, then the m-uniform set $X_1 = (X_0 \setminus \{a\diamond^{m-2}c\}) \cup \{a\diamond^{x_1}a\diamond^{x_2}c\}$ is avoidable. Otherwise, $x_1 \ge x_2$ and $w = (a^{x_1+1}b^{x_2+1}c^{x_1+1})^{\mathbb{Z}}$ avoids X_1.

Consequently, if $(X_0 \setminus \{a\diamond^{m-2}c\}) \cup \{x\}$, where $x \uparrow a\diamond^{m-2}c$, is unavoidable, then x has no interior defined positions.

Filling holes in the words $a\diamond^{m-2}b$ and $b\diamond^{m-2}c$ is not as simple as filling holes in $a\diamond^{m-2}c$ while maintaining unavoidability. We know that inserting a letter different from the endpoints of the word into which it was inserted makes the resulting set avoidable. Thus we only consider the case when we insert a letter that is the same as one of the endpoints. If we insert one letter into $a\diamond^{m-2}b$, then no c can appear in an avoiding word w. This is because w must avoid $a\diamond^{m-2}c$, $b\diamond^{m-2}c$, and $c\diamond^{m-2}c$. Thus if w were to contain a c, there would be no possible letter for the position $m-1$ spaces before the c. Likewise, if we insert one letter into $b\diamond^{m-2}c$, any word w which contains an a must meet one of $a\diamond^{m-2}a$, $a\diamond^{m-2}b$, or $a\diamond^{m-2}c$. So in both of these cases, we are reduced to the use of a binary alphabet.

Proposition 6. *If any of the following conditions 1–4 hold, then the m-uniform set X_1 is unavoidable if and only if $r \leq s$:*

1. $X_1 = (X_0 \setminus \{a\diamond^{m-2}b\}) \cup \{a\diamond^{x_1}b\diamond^{x_2}b\}$, *and* $2^s \| m-1$ *and* $2^r \| x_1+1$.
2. $X_1 = (X_0 \setminus \{a\diamond^{m-2}b\}) \cup \{a\diamond^{x_1}a\diamond^{x_2}b\}$, *and* $2^s \| m-1$ *and* $2^r \| x_2+1$.
3. $X_1 = (X_0 \setminus \{b\diamond^{m-2}c\}) \cup \{b\diamond^{x_1}b\diamond^{x_2}c\}$, *and* $2^s \| m-1$ *and* $2^r \| x_2+1$.
4. $X_1 = (X_0 \setminus \{b\diamond^{m-2}c\}) \cup \{b\diamond^{x_1}c\diamond^{x_2}c\}$, *and* $2^s \| m-1$ *and* $2^r \| x_1+1$.

Second, let us fill in two holes in T_0'. We have seen that inserting any letter into $a\diamond^{m-2}c$ causes X_0 to become avoidable, but that inserting a letter into only one of $a\diamond^{m-2}b$ or $b\diamond^{m-2}c$ only sometimes causes X_0 to become avoidable. Inserting a letter in $a\diamond^{m-2}b$ or $b\diamond^{m-2}c$ that is different from both endpoints makes X_0 avoidable. Thus, in examining what happens when we fill in two holes in T_0' we have two cases to consider. The first is when we fill two of the holes in either $a\diamond^{m-2}b$ or $b\diamond^{m-2}c$ with letters that match the endpoints. The second case to consider is we fill one hole from $a\diamond^{m-2}b$ and one hole from $b\diamond^{m-2}c$ with letters matching one of the endpoints of their respective partial words. We consider the first case first, for which results from [6] prove useful.

Let X_2 be the set created from X_0 by filling in two of the holes in the same word in T_0'. If we have filled in two holes in $a\diamond^{m-2}b$, then, as before, any word avoiding X_2 must be a word over $\{a,b\}$. Similarly, if we have filled in two holes in $b\diamond^{m-2}c$, any word avoiding X_2 must be a word over $\{b,c\}$. Let Y be the set created by removing all of the elements of X_2 that contain the letter that cannot be contained in X_2's avoiding word. In either case, X_2 has the same avoidability as Y, since any word avoiding Y automatically avoids X_2 and vice versa. The avoidability of Y is completely characterized in [6].

Theorem 2. *[6] Let $Y = \{a\diamond^{m-2}a, b\diamond^{m-2}b, a\diamond^{x_1}b\diamond^{x_2}b\diamond^{x_3}b\}$ be an m-uniform set over $\{a,b\}$. Let $2^s \| m-1$, $2^t \| x_1+1$, $2^r \| x_1+x_2+2$. Then Y is unavoidable if and only if $s \geq t, r$ holds in addition to one of (i) $x_1 = x_2$, (ii) $x_1 = x_3$, or (iii) $m = 7(x_1+1)+1$ and $x_2+1 \in \{2(x_1+1), 4(x_1+1)\}$.*

Theorem 3. *[6] Let $i_1 < \cdots < i_s < j_1 < \cdots < j_r$ be elements of the set $\{1,\ldots,m-2\}$. Let x be defined as follows: $x(i) = a$ if $i \in \{0, i_1, \ldots, i_s\}$, $x(i) = b$ if $i \in \{j_1, \ldots, j_r, m-1\}$, and $x(i) = \diamond$ otherwise. Then $Y = \{a\diamond^{m-2}a, b\diamond^{m-2}b, x\}$ has the same avoidability as some set $Z = \{a\diamond^{m-2}a, b\diamond^{m-2}b, z\}$, where z is created by filling in $r+s$ of the holes in $a\diamond^{m-2}b$ with b's.*

We now focus on the set created by filling in one hole in $a\diamond^{m-2}b$ and one hole in $b\diamond^{m-2}c$. We define such a set, with $x_1+x_2 = y_1+y_2 = m-3$, as

$$X_2 = T_0 \cup \{a\diamond^{x_1}b\diamond^{x_2}b, b\diamond^{y_1}b\diamond^{y_2}c, a\diamond^{m-2}c\}. \tag{1}$$

Such set has the same avoidability as

$$Y_2 = T_0 \cup \{a\diamond^{y_2}b\diamond^{y_1}b, b\diamond^{x_2}b\diamond^{x_1}c, a\diamond^{m-2}c\}.$$

Proposition 7. *The m-uniform sets*

$$Y_2 = (X_0 \setminus \{a\diamond^{m-2}b, b\diamond^{m-2}c\}) \cup \{a\diamond^{y_2}b\diamond^{y_1}b, b\diamond^{x_2}b\diamond^{x_1}c\}$$

and X_2, defined by Eq. (1), have the same avoidability.

From Proposition 7, when considering X_2, defined by Eq. (1), we can assume without loss of generality that $x_1 \leq y_2$. If $y_2 < x_1$, X_2 has the same avoidability as the set Y_2 obtained by switching x_1 and y_2.

Proposition 8. *If $x_1 \leq x_2$, there exist integers $p,q > 0$, with $p+q = m-1$, such that the infinite word $w = (a^p c^q b^p)^{\mathbb{Z}}$ avoids X_2, defined by Eq. (1).*

Proof. Set $v = a^p c^q b^p$. If $w(i) = a$, we know $w(i+m-1) = b$. Thus in order for w to avoid X_2, we need $p \leq x_2+1$, to ensure that w avoids $a\diamond^{x_1}b\diamond^{x_2}b$. Since $x_1+x_2+2 = p+q = m-1$, $p \leq x_2+1$ implies $q \geq x_1+1$. Additionally, if $w(i) = c$, we need $q \leq p$ in order to ensure that $w(i+m-1)$ is an a and not a c. Finally, if $w(i) = b$, then $w(i+m-1) \in \{a,c\}$. In fact, $m-1$ spaces after the first $p-q$ b's in v is an a and $m-1$ spaces after the last q b's in v is a c. Thus to ensure that w avoids $b\diamond^{y_1}b\diamond^{y_2}c$, we need $q \leq y_2+1$. Consequently, w avoids X_2 if $x_1 \leq y_2$, which we have already assumed, and if we can find p,q such that $x_1+1 \leq q \leq p \leq x_2+1$. This occurs when $x_1 \leq x_2$. □

Thus the set X_2, defined by Eq. (1), is always avoidable except possibly when $y_1 \leq x_2 \leq x_1 \leq y_2$. Extensive computations yield the following conjecture.

Conjecture 1. *Set X_2, defined by Eq. (1), is avoidable when $y_1 \leq x_2 \leq x_1 \leq y_2$.*

We now discuss some results towards a proof of this conjecture. Table 1 gives specific examples of words that avoid sets defined by Eq. (1) under conditions on m, x_1, and y_1. We prove only the third item in Table 1, i.e, Proposition 9, as the proofs of the other items are analogous.

Proposition 9. *The infinite word $w = ((ab)^p a(bc)^q)^{\mathbb{Z}}$, where $p \geq 0, q > 0$, avoids X_2, defined by Eq. (1), if and only if the following conditions hold:*

1. $m \equiv 2 \pmod{2(p+q)+1}$;
2. $x_1 \equiv 2j-1 \pmod{2(p+q)+1}$ *for some* $j \in [0..q]$;
3. $y_1 \equiv 2k-1 \pmod{2(p+q)+1}$ *for some* $k \in [q..p+q+1]$.

Proof. For the remainder of the proof, assume all congruences are modulo $2(p+q)+1$. Suppose X_2 satisfies the above conditions. Since $m \equiv 2$ by Condition 1 and, thus, $m-1 \equiv 1$, $w(i) = a$ implies $w(i+m-1) = b$ since any letter after an a is a b. Similarly, $w(i) = b$ implies $w(i+m-1) \in \{a,c\}$, and $w(i) = c$ implies $w(i+m-1) \in \{a,b\}$. Thus w avoids $a\diamond^{m-2}a$, $b\diamond^{m-2}b$, $c\diamond^{m-2}c$, and $a\diamond^{m-2}c$.

Suppose $w(i) = a$. Consider $w(i+x_1+1)$. By Condition 2, $x_1 \equiv 2j-1$ for some $j \in [0..q]$, which implies $w(i+x_1+1) = w(i+2j)$. Since any letter an even distance at most $2q$ spaces ahead of an a is in $\{a,c\}$, $w(i+x_1+1) \neq b$. Therefore, w avoids $a\diamond^{x_1}b$, which implies that w avoids $a\diamond^{x_1}b\diamond^{x_2}b$. Next, suppose $w(i) = b$. By Condition 3, $y_1 \equiv 2k-1$ for some $k \in [q..p+q+1]$. Equivalently, $2k-1+y_2+3 \equiv y_1+y_2+3 = m \equiv 2$. Thus, $y_2+1 \equiv 2r$ for some $r \in [0..p+1]$. Since any letter an even distance at most $2p+2$ spaces ahead of a b is in $\{a,b\}$, $w(i+y_2+1) = w(i+2r) \neq c$. Therefore, w avoids $b\diamond^{y_2}c$ and, thus, $b\diamond^{y_1}b\diamond^{y_2}c$. Therefore, if Conditions 1–3 are satisfied, then w avoids X_2.

Now, suppose w avoids X_2. We show that X_2 satisfies Conditions 1–3. Suppose for a contradiction that Condition 1 does not hold. Then either $m \equiv 2r+1$ for some $r \in [0..p+q]$ or $m \equiv 2r$ for some $r \in [2..p+q]$. Suppose $m \equiv 2r+1$ for some $r \in [0..p+q]$. Without loss of generality, suppose $w(i) = a$ begins a period of w and, thus, $w(i-2) = b$. Consider $w(i-2+m-1) = w(i+2r-2)$. Since any letter an even distance after the first letter in the period is in $\{a,c\}$, w meets either $a\diamond^{m-2}a$ or $a\diamond^{m-2}c$, a contradiction. Similarly, suppose $m \equiv 2r$ for some $r \in [2..p+q]$ and $w(i) = a$ once again begins a period of w. Consider $w(i+m-1) = w(i+2r)$. Since any letter an even distance after the first letter in the period is in $\{a,c\}$, w meets either $a\diamond^{m-2}a$ or $a\diamond^{m-2}c$, again a contradiction. Thus, Condition 1 holds.

Next, suppose for a contradiction that Condition 2 does not hold. There are two cases to consider. The first is that $x_1 \equiv 2r$ for some $r \in [0..p+q]$. Suppose $w(i) = a$ begins a period of w. Since any letter an odd distance after the first letter in the period is a b, $w(i+x_1+1) = w(i+2r+1) = b$. Furthermore, since w avoids X_2, if $w(i)=a$, then $w(i+m-1)=b$. Thus, w meets $a\diamond^{x_1}b\diamond^{x_2}b$, which is a contradiction. The second case is that $x_1+1\equiv 2r$ for some $r\in(q..p+q)$. Once again, let $w(i)=a$ begin a period of w, so $w(i-2q-1)=a$. Then $w(i-2q-1+x_1+1)=w(i+2r-2q-1)$. Since any letter an odd number of spaces after the first a in the period is a b, this means $w(i-2q-1+x_1+1)=b$. Since w avoids X_2, $w(i-2q-1+m-1)=b$. Thus w meets $a\diamond^{x_1}b\diamond^{x_2}b$, which is a contradiction. Thus, Condition 2 holds as well.

Finally, suppose for a contradiction that Condition 3 does not hold. This means $y_1\not\equiv 2k-1$ for any $k\in[q..p+q+1]$. By Condition 1, $m\equiv 2$ and since $y_1+y_2+1=m-2\equiv 0$, Condition 3 not holding is equivalent to $y_2+1\equiv 2r+1$ for some $r\in[0..p+q)$ or $y_2+1\equiv 2r$ for some $r\in[p+2..p+q]$. Suppose $y_2+1\equiv 2r+1$ for some $r\in[0..p+q)$. Now, let $w(i)=c$ be the last letter in a period of w. Thus $w(i-(y_2+1))=b$ and $w(i-(m-1))=b$, since any letter an odd number of spaces before the last c in the period is a b and since w avoids $a\diamond^{m-2}c$ and $c\diamond^{m-2}c$. This contradicts the assumption that w avoids $b\diamond^{y_1}b\diamond^{y_2}c$. Similarly, suppose $y_2+1\equiv 2r$ for some $r\in[p+2..p+q]$. Let $w(i)=a$ begin a period of w and $w(i-2)=b$. Consider $w(i-2+y_2+1)=w(i+2r-2)$. Since any letter an odd distance at least $2p+2$ spaces after the first letter in the period is a c, $w(i-2+y_2+1)=c$. Since w avoids $a\diamond^{m-2}c$ and $c\diamond^{m-2}c$, we have that $w(i-2+y_2+1-(m-1))=b$. This means w meets $b\diamond^{y_1}b\diamond^{y_2}c$, which is a contradiction. Thus, Condition 3 holds. □

The following proposition also provides conditions for X_2 to be avoidable.

Proposition 10. *Let X_2 be as defined by Eq. (1). Then X_2 is avoided by an infinite word of period at most m if one of the following conditions hold:*

1. *x_1,y_1 are even and $y_1\le x_2\le x_1$;*
2. *$y_1=0$ and $x_2\le x_1$.*

Tables 2, 3, and 4 summarize some sufficient conditions for patterns to avoid X_2, defined by Eq. (1), with respect to residues modulo 2, 3, and 4.

If X_2, defined by Eq. (1), is avoidable, then all sets that contain strengthenings of two of the T_0' words are avoidable.

Proposition 11. *Let X_2 be defined by Eq. (1) and let $X_2'=(X_2\setminus\{a\diamond^{x_1}b\diamond^{x_2}b\})\cup\{a\diamond^{x_2}a\diamond^{x_1}b\}$. Also let $Y_2'=(X_2'\setminus\{a\diamond^{x_2}a\diamond^{x_1}b,b\diamond^{y_1}b\diamond^{y_2}c\})\cup\{a\diamond^{y_2}b\diamond^{y_1}b,b\diamond^{x_1}c\diamond^{x_2}c\}$ and $Y_2=(Y_2'\setminus\{a\diamond^{y_2}a\diamond^{y_1}b\})\cup\{a\diamond^{y_1}b\diamond^{y_2}b\}$ be m-uniform sets.*

1. *If X_2 is avoidable, then X_2' is avoidable.*
2. *The sets X_2' and Y_2' have the same avoidability.*
3. *If Y_2' is avoidable, then Y_2 is avoidable.*

Third, Theorems 2 and 3 state that filling in two holes in the same word in T_0' only sometimes makes X_0 avoidable. We now prove that once we have filled in three holes in the same word in T_0', X_0 becomes avoidable.

Proposition 12. *If the m-uniform set $X_3=(X_0\setminus\{a\diamond^{m-2}b\})\cup\{x\}$ is unavoidable, where $x\uparrow a\diamond^{m-2}b$, then x has at most two interior defined positions.*

Proof. If more than two positions in x have been filled, we know that they have to be filled with a's or b's otherwise X_3 would be avoidable. However from [6, Corollary 4], $\{a\diamond^{m-2}a,\ b\diamond^{m-2}b,x\}$ can be avoided by an infinite word w over $\{a,b\}$. This means that w avoids $c\diamond^{m-2}c$, $a\diamond^{m-2}c$, and $b\diamond^{m-2}c$ as well. Thus w avoids all of X_3 and thus X_3 is avoidable. □

Table 1: Necessary and sufficient conditions for w to avoid sets defined by Eq. (1) when $y_1 \leq x_2 \leq x_1 \leq y_2$

Avoiding word w	Necessary and sufficient conditions
$(a^pb^p)^{\mathbb{Z}}$	$m \equiv p+1 \pmod{2p}$ $x_1 \equiv -1 \pmod{2p}$
$(b^pc^p)^{\mathbb{Z}}$	$m \equiv p+1 \pmod{2p}$ $y_1 \equiv p-1 \pmod{2p}$
$((ab)^pa(bc)^q)^{\mathbb{Z}}$ $p \geq 0, q > 0$	$m \equiv 2 \pmod{2(p+q)+1}$ $x_1 \equiv 2j-1 \pmod{2(p+q)+1}$, $j \in [0..q]$ $y_1 \equiv 2k-1 \pmod{2(p+q)+1}$, $k \in [q..q+p+1]$
$(ab((ab)^pa(bc)^q)^r)^{\mathbb{Z}}$ $p \geq 0, q > 0$	$m \equiv 2 \pmod{r(2p+2q+1)+2}$ $x_1 \equiv (2p+2q+1)j+2k-1 \pmod{r(2p+2q+1)+2}$ $j \in [0..r], k \in [1..r]$ $y_1 \equiv (2q+2r+1)s+2t+1 \pmod{r(2p+2q+1)+2}$ $s \in [0..r), t \in [q..p+q) \cup \{0\}$
$(((ab)^pa(cb)^q)^r(ab)^pa(cb)^{q-1})^{\mathbb{Z}}$ $p \geq 0, q > 0$ $r \geq 0$	$m \equiv 0 \pmod{(r+1)(2p+2q+1)-2}$ $x_1 \equiv (2p+2q+1)j+2k \pmod{(r+1)(2p+2q+1)-2}$ $j \in [0..r], k \in [p..p+q)$ $y_1 \equiv (2p+2q+1)s+2t \pmod{(r+1)(2p+2q+1)-2}$ $s \in [0..r], t \in [-1..p]$
$(a^pb^qc^r)^{\mathbb{Z}}$ $1 \leq p \leq q$, $1 \leq r \leq q$, $q \leq p+r$	$m \equiv q+1 \pmod{p+q+r}$ $x_1 \equiv \{p+q-1, \ldots, p+q+r-1\} \pmod{p+q+r}$ $y_1 \equiv \{q-1, \ldots, p+q-1\} \pmod{p+q+r}$
$(a^pc^rb^q)^{\mathbb{Z}}$ $1 \leq p \leq q$, $1 \leq r \leq q$, $q \leq p+r$	$m \equiv p+r+1 \pmod{p+q+r}$ $x_1 \equiv \{-1, \ldots, r-1\} \pmod{p+q+r}$ $y_1 \equiv \{r-1, \ldots, p+r-1\} \pmod{p+q+r}$
$(a^{p+1}b^{q-1}c^ra^pb^qc^{r-1})^{\mathbb{Z}}$ $0 \leq p < q$, $1 \leq r \leq q$, $q \leq p+r$	$m \equiv p+r+2q \pmod{2p+2q+2r-1}$ $x_1 \equiv \{p+q-1, \ldots, p+q+r-2, 2p+2q+r-1,$ $\ldots, 2p+2q+2r-2\} \pmod{2p+2q+2r-1}$ $y_1 \equiv \{q-1, \ldots, p+q-1, p+2q+r-2,$ $\ldots, 2p+2q+r-2 \pmod{2p+2q+2r-1}$
$(a^{p-1}c^{r+1}b^{q-1}a^pc^rb^q)^{\mathbb{Z}}$ $1 \leq p \leq q$, $0 \leq r < q$, $q \leq p+r$	$m \equiv p+r+1 \pmod{2p+2q+2r-1}$ $x_1 \equiv \{1, \ldots, r-1, p+r+q-1, \ldots, p+2r+q-1\}$ $\pmod{2p+2q+2r-1}$ $y_1 \equiv \{r-1, \ldots, p+r-1, 2p+r, \ldots, 2p+2r+q-2\}$ $\pmod{2p+2q+2r-1}$
$(a^r(b^qc^q)^p)^{\mathbb{Z}}$ $1 \leq r \leq q$, $p > 0$	$m \equiv q+1 \pmod{2pq+r}$ $x_1 \equiv \{-1, 2qj+k\} \pmod{2pq+r}$, $j \in [0..p), k \in [q+r-1..2q)$ $y_1 \equiv q-1 \pmod{2pq+r}$
$(a^r(c^qb^q)^p)^{\mathbb{Z}}$ $1 \leq r \leq q$, $p > 0$	$m \equiv -q+1 \pmod{2pq+r}$ $x_1 \equiv \{-1, 2qj+k\} \pmod{2pq+r}$, $j \in [0..p), k \in [r-1..r+q-2]$ $y_1 \equiv \{-q-1, q-1\} \pmod{2pq+r}$
$(a^qb^qc^q(c^qb^{2q}c^q)^p)^{\mathbb{Z}}$ $p \geq 0, q > 0$	$m \equiv -2q+1 \pmod{(4p+3)q}$ $x_1 \equiv \{-1, 4qj+k-1\} \pmod{(4p+3)q}$, $j \in [0..p], k \in [2q..3q]$ $y_1 \equiv \{-2q-1, 2q-1\} \pmod{(4p+3)q}$
$((a^pc^rb^q)^tb)^{\mathbb{Z}}$ $1 \leq p \leq q$, $p+r = q+1$	$m \equiv q+2 \pmod{t(p+q+r)+1}$ $x_1 \equiv (2q+1)j+k-1 \pmod{t(p+q+r)+1}$ $j \in [0..t], k \in [1..r]$ $y_1 \equiv (2q+1)h+i \pmod{t(p+q+r)+1}$ $h \in [0..t], i \in [r..q]$

Table 2: Sufficient conditions on residues modulo 2 for w to avoid sets defined by Eq. (1)

Avoiding word w	m	x_1	y_1
$(ab)^{\mathbb{Z}}$	0	1	0,1
$(bc)^{\mathbb{Z}}$	0	0,1	0
$((ab)^p a(cb)^q)^{\mathbb{Z}}$	1	0	0

Table 3: Sufficient conditions on residues modulo 3 for w to avoid sets defined by Eq. (1)

Avoiding word w	m	x_1	y_1
$(abc)^{\mathbb{Z}}$	2	1,2	0,1
$(acb)^{\mathbb{Z}}$	0	0,2	0,1
$(ab(abc)^p)^{\mathbb{Z}}$	1	1	$0,=1$
$((acb)^p b)^{\mathbb{Z}}$	1	0	$=0,1$

3.3 Filling in holes in T_0 and T_0'

Since filling in any of the holes in $a\diamond^{m-2}c$ results in an avoidable set, the only strengthenings of T_0' we need to consider are strengthenings of $a\diamond^{m-2}b$ and $b\diamond^{m-2}c$. Furthermore, in order to preserve unavoidability, we must fill in a word in T_0 with the same letter as its two endpoints. Thus when filling in one hole in T_0 and one hole in T_0', there are two possible cases to consider: the endpoints of the T_0 word are the same as one of the endpoints of the T_0' word or the endpoints of the T_0 word are different from the two endpoints of the T_0' word. We now focus on the m-uniform set

$$X_2 = (X_0 \setminus \{a\diamond^{m-2}a, b\diamond^{m-2}c\}) \cup \{a\diamond^{x_1}a\diamond^{x_2}a, b\diamond^{y_1}c\diamond^{y_2}c\}. \quad (2)$$

When considering it, we can assume without loss of generality that $x_1 \leq x_2$. Indeed, it is easy to show that the m-uniform set X_2, defined by Eq. (2), is avoidable if and only if the m-uniform set $X_2' = (X_0 \setminus \{a\diamond^{m-2}a, b\diamond^{m-2}c\}) \cup \{a\diamond^{x_2}a\diamond^{x_1}a, b\diamond^{y_1}c\diamond^{y_2}c\}$ is avoidable. It is also easy to show that if the m-uniform set X_2, defined by Eq. (2), is unavoidable, then so is $Y_2 = (X_0 \setminus \{a\diamond^{m-2}a, b\diamond^{m-2}c\}) \cup \{a\diamond^{x_1}a\diamond^{x_2}a, b\diamond^{y_2}b\diamond^{y_1}c\}$.

Table 5 gives some of the recurring patterns of words avoiding sets defined by Eq. (2). For instance, the last item in Table 5 translates as Proposition 13.

Proposition 13. *Let $u = (b^{y_2+1}c^{y_2+1})^{\mathbb{N}}$. If $y_2 \leq x_1 \leq x_2 \leq y_1$, there exist integers $p,q > 0$, $p+q = m-1$ such that the infinite word $w = v^{\mathbb{Z}}$ avoids X_2, defined by Eq. (2), where $v = a^p u_q a^p \overline{u_q}$ (here, u_q denotes the q-length prefix of u and $\overline{u_q}$ denotes the complement of u_q, where $\overline{b} = c$ and $\overline{c} = b$).*

Proposition 14. *Let X_2, defined by Eq. (2), and*

$$Y_2' = (X_0 \setminus \{a\diamond^{m-2}a, b\diamond^{m-2}c\}) \cup \{a\diamond^{x_1}a\diamond^{x_2}a, b\diamond^{y_2}b\diamond^{y_1}c\}$$

Table 4: Sufficient conditions on residues modulo 4 for w to avoid sets defined by Eq. (1)

Avoiding word w	m	x_1	y_1
$(a^2b^2)^{\mathbb{Z}}$	3	3	0,1,2,3
$(b^2c^2)^{\mathbb{Z}}$	3	0,1,2,3	1

Table 5: Conditions for w to avoid sets defined by Eq. (2); here, u_q denotes the q-length prefix of u, $\overline{u_q}$ denotes the complement of u_q where $\overline{b} = c$ and $\overline{c} = b$, and $p,q > 0$ are integers such that $p+q = m-1$

Conditions	Avoiding word w
m even, y_1 even	$(a(bc)^{\frac{m-2}{2}}a(cb)^{\frac{m-2}{2}})^{\mathbb{Z}}$
m odd, x_1 even, y_1 even	$((ab)^{\frac{m-1}{2}}(ac)^{\frac{m-1}{2}})^{\mathbb{Z}}$
$y_1 \leq x_1 \leq x_2 \leq y_2$	$(a^p b^q a^p c^q)^{\mathbb{Z}}$
$y_2 \leq x_1 \leq x_2 \leq y_1$	$(a^p u_q a^p \overline{u_q})^{\mathbb{Z}}$, where $u = (b^{y_2+1}c^{y_2+1})^{\mathbb{N}}$

be m-uniform sets. Furthermore, let $2^s \| x_1+1$ *and* $2^t \| y_1+1$. *If* $y_1 = y_2$ *and* $s \neq t$, *then* X_2 *and* Y_2' *are unavoidable.*

Proof. Suppose $y_1 = y_2$ and $s \neq t$. From now on, we refer to $y_1 = y_2$ just as y. By performing the operations of factoring, prefix-suffix, hole truncation, and expansion on X_2 from [5], we obtain the set

$$Y = \{a\diamond^{x_1}a, b\diamond^{y}b, b\diamond^{y}c, c\diamond^{y}b, c\diamond^{y}c, a\diamond^{y}a\diamond^{y}b, a\diamond^{y}a\diamond^{y}c, b\diamond^{y}a\diamond^{y}b, c\diamond^{y}a\diamond^{y}c\},$$

which has the same avoidability as X_2.

Assume for contradiction that Y is avoidable. This implies there exists an infinite word w that avoids Y. It is clear that w cannot contain only a's since w must avoid $a\diamond^{x_1}a$. Thus, w must contain a b or a c. Without loss of generality let us assume that w contains a b since the argument if w contains a c is identical. Now without loss of generality, assume $w(y+1) = b$. Since w avoids $b\diamond^{y}b$ and $b\diamond^{y}c$, this means $w(2(y+1)) = a$. Since w avoids $b\diamond^{y}a\diamond^{y}b$, $w(3(y+1)) \neq b$. If $w(3(y+1)) = a$, $w(4(y+1)) = a$ since w avoids $a\diamond^{y}a\diamond^{y}b$ and $a\diamond^{y}a\diamond^{y}c$. But this means that $w(5(y+1)) = a$ and so on. Thus inductively, if $w(3(y+1)) = a$, then $w(p(y+1)) = a$ for all $p \geq 2$. If $w(3(y+1)) = c$, then $w(4(y+1)) = a$ because w must avoid $c\diamond^{y}b$ and $c\diamond^{y}c$. Since $w(4(y+1)) = a$ and w avoids $c\diamond^{y}a\diamond^{y}c$, then either w can degenerate into a repeating string of a's as before, or $w(5(y+1)) = b$ and the sequence repeats. Thus it is easy to see that w must be made up of two possible strings of letters:

$$a\underbrace{}_{y}a\underbrace{}_{y}a\underbrace{}_{y}a\underbrace{}_{y}a\underbrace{}_{y}a,$$

$$a\underbrace{}_{y}b\underbrace{}_{y}a\underbrace{}_{y}c\underbrace{}_{y}a\underbrace{}_{y}b.$$

Thus w must be $4(y+1)$-periodic. Since the period of w must avoid $a\diamond^{x_1}a$, the period of w cannot contain all a's. Thus the second string must occur in the period of w.

Without loss of generality assume $w(0) = a$, $w(y+1) = b$, $w(2(y+1)) = a$, and $w(3(y+1)) = c$. This implies for $k \geq 1$ that $w(k(y+1) = a$ if k is even and $w(k(y+1)) \in \{b,c\}$ if k is odd.

Since $w(0) = a$ and w avoids $a\diamond^{x_1}a$, $w(x_1+1) \in \{b,c\}$. This means that $w(x_1+y+1) = a$. Now assume $w(n(x_1+1)+n(y+1)) = a$ and consider $w((n+1)(x_1+1)+(n+1)(y+1))$. Since $w(n(x_1+1)+n(y+1)) = a$, $w((n+1)(x_1+1)+n(y+1)) \in \{b,c\}$ because w avoids $a\diamond^{x_1}a$. This means that $w((n+1)(x_1+1)+(n+1)(y+1)) = a$. So by induction, $w(n(x_1+1)+n(y+1)) = a$ for all $n \in \mathbb{N}$.

Now consider $w(p(x_1+1)+q(y+1))$ for $p,q \in \mathbb{N}$ with one of p,q even and the other odd. We know $p \pm r = q$ for some odd $r \in \mathbb{N}$. Thus, $w(p(x_1+1)+q(y+1)) = w(p(x_1+1)+p(y+1) \pm r(y+1)) \in \{b,c\}$.

Similarly, if we consider $w(p(x_1+1)+q(y+1))$ for $p,q \in \mathbb{N}$ with both of p,q even or both of p,q odd, $p \pm r = q$ for some even $r \in \mathbb{N}$. Thus, $w(p(x_1+1)+q(y+1)) = w(p(x_1+1)+p(y+1)\pm r(y+1)) = a$.

Now, let l be the least common multiple of x_1+1 and $y+1$. Since $s \neq t$ the power of two that maximally divides l is the same as the power of two that maximally divides one of x_1+1 and $y+1$ and is greater than the power of two that maximally divides the other. Thus l is even and $l = \alpha(x_1+1)$ and $l = \beta(y+1)$ where one of α,β is odd and the other is even. This implies $w(\alpha(x_1+1)+\beta(y+1)) \in \{b,c\}$. However, $w(\alpha(x_1+1)+\beta(y+1)) = w(2l) = w(2\beta(y+1)) = a$, which is a contradiction.

Thus, Y is unavoidable and so is X_2. The set Y_2' is then unavoidable. To see this, assume for contradiction that there exists an infinite word w that avoids Y_2'. Since X_2 is unavoidable, w must meet an element of X_2. This means w meets $b\diamond^{y_1}c\diamond^{y_2}c$. Suppose $w(i)=b, w(i+y_1+1)=c$, and $w(i+y_1+1+y_2+1)=c$. Since w avoids $a\diamond^{m-2}c$ and $c\diamond^{m-2}c$, this means $w(i+y_1+1-(m-1)) = w(i-(y_2+1)) = b$. Thus, $w(i-(y_2+1)) = b$, $w(i)=b$, and $w(i+y_1+1)=c$. This contradicts the fact that w avoids $b\diamond^{y_2}b\diamond^{y_1}c$. □

4 Minimum number of holes in uniform unavoidable sets

We now consider the minimum number of holes in an m-uniform unavoidable set of size $k+\binom{k}{2}$ over A_k. To do this, our results from Section 3 prove useful. As discussed in Section 3, there is an unavoidable m-uniform set of minimal size for any total order on the alphabet and these sets are equivalent up to renamings of letters. So we define the basic m-uniform unavoidable set of minimal size over A_k as $X_0 = T_0 \cup T_0'$, where $T_0 = \{a_i\diamond^{m-2}a_i \mid 1 \leq i \leq k\}$ and $T_0' = \{a_i\diamond^{m-2}a_j \mid 1 \leq i < j \leq k\}$.

Proposition 15. *Let $X_2 = (X_0 \setminus \{a_{i_1}\diamond^{m-2}a_{i_2}, a_{i_3}\diamond^{m-2}a_{i_4}\}) \cup \{x,y\}$ where the integers i_1,i_2,i_3,i_4 are all distinct and where $x \uparrow a_{i_1}\diamond^{m-2}a_{i_2}$ and $y \uparrow a_{i_3}\diamond^{m-2}a_{i_4}$. If x and y both have at least one defined interior position, then X_2 is avoidable.*

Proof. If we fill in $a_{i_1}\diamond^{m-2}a_{i_2}$ or $a_{i_3}\diamond^{m-2}a_{i_4}$ with letters different from their endpoints, we know that X_2 is avoidable by an infinite word over a ternary alphabet. Thus, we must fill in $a_{i_1}\diamond^{m-2}a_{i_2}$ and $a_{i_3}\diamond^{m-2}a_{i_4}$ with letters that are the same as their respective endpoints. For ease of notation, we let $a_{i_1}=a, a_{i_2}=b, a_{i_3}=c, a_{i_4}=d$. Without loss of generality, assume $x = a\diamond^{x_1}b\diamond^{x_2}b$ and $y = c\diamond^{y_1}d\diamond^{y_2}d$. Filling in more holes in x and y is just a strengthening of X_2. Furthermore, filling in $a\diamond^{m-2}b$ with an a instead of a b or $c\diamond^{m-2}d$ with a c instead of a d yield an equivalent proof. We thus have eight cases:

$$x_1 \leq y_1 \leq y_2 \leq x_2; \tag{3}$$
$$x_1 \leq y_2 \leq y_1 \leq x_2; \tag{4}$$
$$y_1 \leq x_1 \leq x_2 \leq y_2; \tag{5}$$
$$y_2 \leq x_1 \leq x_2 \leq y_1; \tag{6}$$
$$x_2 \leq y_1 \leq y_2 \leq x_1; \tag{7}$$
$$x_2 \leq y_2 \leq y_1 \leq x_1; \tag{8}$$
$$y_1 \leq x_2 \leq x_1 \leq y_2; \tag{9}$$
$$y_2 \leq x_2 \leq x_1 \leq y_1. \tag{10}$$

In any infinite word w that avoids X_2, if $w(i)=a$, $w(i+m-1)=b$ and $w(i+2(m-1))=a$ and similarly if $w(i)=c$, $w(i+m-1)=d$ and $w(i+2(m-1))=c$. So let $\overline{a}=b$, $\overline{b}=a$, $\overline{c}=d$, and $\overline{d}=c$. Furthermore, given a one-sided infinite word v, let v_i denote the prefix of v of length i. Now, let

$v=(a^{x_2+1}b^{x_2+1})^{\mathbb{N}}$ and $u=(c^{y_2+1}d^{y_2+1})^{\mathbb{N}}$. Define the infinite word $w=(v_pu_q\overline{v_pu_q})^{\mathbb{Z}}$ where $p,q>0$ and $p+q=m-1$. The word w avoids X_2 as long as $q>x_2$ and $p>y_2$. This is because in w, $m-1$ spaces after every a is a b and $m-1$ spaces after every b is an a and similarly for c and d. Furthermore, as long as $q>x_2$ and $p>y_2$, if $w(i)=b$, then $w(i-(m-1))=a$ and if $w(i)=d$, then $w(i-(m-1))=c$. Since $p+q=x_1+x_2+2=y_1+y_2+2=m-1$, w avoids X_2 in Cases (6), (7), (8), and (10).

Let us now define the infinite word $w'=(a^pc^qb^pd^q)^{\mathbb{Z}}$ for some $p,q>0$ such that $p+q=m-1$. We claim that w' avoids X_2 as long as $p\leq x_2+1$ and $q\leq y_2+1$. If $w'(i)=a$, then $w'(i+m-1)=b$, if $w'(i)=b$, then $w'(i+m-1)=a$, and similarly for c and d. Furthermore, as long as $p\leq x_2+1$, w' avoids $a\diamond^{x_1}b\diamond^{x_2}b$ and as long as $q\leq y_2+1$, w' avoids $c\diamond^{y_1}d\diamond^{y_2}d$. Since $p+q=x_1+x_2+2=y_1+y_2+2=m-1$, w' avoids X_2 in Cases (3), (4), (5), and (9).

We have thus found infinite words that avoid X_2 for all eight cases and so X_2 is avoidable. □

Proposition 16. *Let $X_1=(X_0\setminus\{a_i\diamond^{m-2}a_{i+p}\})\cup\{x\}$ where $k\geq i+p\geq i+2$, $x\uparrow a_i\diamond^{m-2}a_{i+p}$, and x has at least one defined interior position. Then X_1 is avoidable.*

Proof. Suppose an infinite word w avoids X_1 and contains only the letters a_i, a_{i+1}, and a_{i+p}. If $w(j)=a_i$, then $w(j+m-1)=a_{i+p}$ since w must avoid $a_i\diamond^{m-2}a_i$ and $a_i\diamond^{m-2}a_{i+1}$. If $w(j)=a_{i+1}$, then $w(j+m-1)=a_i$ since w must avoid $a_{i+1}\diamond^{m-2}a_{i+1}$ and $a_{i+1}\diamond^{m-2}a_{i+p}$. Finally, if $w(j)=a_{i+p}$, then $w(j+m-1)=a_i$ or $w(j+m-1)=a_{i+1}$ since w must avoid $a_{i+p}\diamond^{m-2}a_{i+p}$. Therefore, the conditions on a_i,a_{i+1}, and a_{i+p} are identical to the conditions on the letters a,b, and c when we considered the avoidability over $\{a,b,c\}$ of $\{a\diamond^{m-2}a,b\diamond^{m-2}b,c\diamond^{m-2}c,a\diamond^{m-2}b,b\diamond^{m-2}c,x\}$, where $x\uparrow a\diamond^{m-2}c$ and x contains only a's and c's. Thus, the proof that we can generate such an avoiding word is identical to the proof of Proposition 5. □

To prove our main result, we show that X_0 becomes avoidable once we fill in more than $m-1$ holes if m is even and m holes if m is odd.

Theorem 4. *For $m\geq 4$, if Conjecture 1 is true, then the maximum number of holes we can fill into an m-uniform unavoidable set of size $k+\binom{k}{2}$ over A_k is $m-1$ if m is even and m if m is odd. In other words, $H^k_{m,k+\binom{k}{2}}=(k+\binom{k}{2})(m-2)-(m-1)$ if m is even, and $H^k_{m,k+\binom{k}{2}}=(k+\binom{k}{2})(m-2)-m$ if m is odd.*

Proof. When we fill in holes in T_0, say we fill in a hole in $a_i\diamond^{m-2}a_i$, the letter we fill in must be a_i or else the infinite word $a_i^{\mathbb{Z}}$ avoids X_0 (see Proposition 2). Additionally, filling in holes in more than two words in T_0 makes X_0 avoidable. This is because by Proposition 4, if we fill in holes in three words in T_0, there exists an infinite word w that avoids X_0 and that contains three distinct letters. Since w does not contain any of the letters that make up the other elements of X_0, w avoids all of the elements of X_0 and thus X_0 is avoidable. Thus we can fill holes into at most two of the words in T_0.

Using Proposition 11 we prove that if Conjecture 1 is true, then filling in holes in two T_0' words that have an endpoint in common makes X_0 avoidable. To prove this, it is enough to consider the 3-letter alphabet $\{a,b,c\}$. Let $Z_2=(X_0\setminus\{a\diamond^{m-2}b,b\diamond^{m-2}c\})\cup\{x,y\}$ where $x\uparrow a\diamond^{m-2}b$, $y\uparrow b\diamond^{m-2}c$, and x and y each have at least one defined interior position. We show that if Conjecture 1 is true, then Z_2 is avoidable. Indeed, we know that if the defined interior letter in either x or y is different from the endpoints of its respective word, then Z_2 is avoidable. Thus,

$$\begin{array}{rcccl} X_2 &=& (X_0\setminus\{a\diamond^{m-2}b,b\diamond^{m-2}c\}) &\cup& \{a\diamond^{x_1}b\diamond^{x_2}b,b\diamond^{y_1}b\diamond^{y_2}c\},\\ X_2' &=& (X_0\setminus\{a\diamond^{m-2}b,b\diamond^{m-2}c\}) &\cup& \{a\diamond^{x_2}a\diamond^{x_1}b,b\diamond^{y_1}b\diamond^{y_2}c\},\\ Y_2 &=& (X_0\setminus\{a\diamond^{m-2}b,b\diamond^{m-2}c\}) &\cup& \{a\diamond^{y_1}b\diamond^{y_2}b,b\diamond^{x_1}c\diamond^{x_2}c\},\\ Y_2' &=& (X_0\setminus\{a\diamond^{m-2}b,b\diamond^{m-2}c\}) &\cup& \{a\diamond^{y_2}a\diamond^{y_1}b,b\diamond^{x_1}c\diamond^{x_2}c\} \end{array}$$

represent the only remaining cases to consider. If Conjecture 1 is true, then X_2 is avoidable for all $x_1, x_2, y_1, y_2 > 0$. However if X_2 is avoidable for all $x_1, x_2, y_1, y_2 > 0$, this implies X_2' is avoidable for all $x_1, x_2, y_1, y_2 > 0$, which then implies Y_2' is avoidable for all $x_1, x_2, y_1, y_2 > 0$, which implies Y_2 is avoidable for all $x_1, x_2, y_1, y_2 > 0$.

Using Proposition 15, filling in holes in two T_0' words whose endpoints are all distinct also makes X_0 avoidable. Thus we can fill in holes in at most one word in T_0'. Furthermore, we know that the letter we fill in must be the same as one of the endpoints. From Proposition 16, the word we fill in must be of the form $a_i\diamond^{m-2}a_{i+1}$ and from Proposition 12, we cannot fill in more than two holes in any word in T_0'.

Thus if we want to preserve the unavoidability of X_0, we can fill in holes in at most two of the T_0 words and one of the T_0' words. Therefore, filling in holes in X_0 is equivalent to filling in holes in subsets of X_0 of size three, where each subset contains two words from T_0 and one word from T_0'. So given a word u in T_0', either none of the two T_0 words share endpoints with u, both of the two T_0 words share endpoints with u, or one of the two T_0 words shares an endpoint with u. Without loss of generality, these subsets are of three possible forms:

$$\begin{aligned} Q &= \{a_i\diamond^{m-2}a_i, a_j\diamond^{m-2}a_j, a_l\diamond^{m-2}a_{l+1}\}, \\ R &= \{a_i\diamond^{m-2}a_i, a_{i+1}\diamond^{m-2}a_{i+1}, a_i\diamond^{m-2}a_{i+1}\}, \\ S &= \{a_i\diamond^{m-2}a_i, a_j\diamond^{m-2}a_j, a_i\diamond^{m-2}a_{i+1}\}. \end{aligned}$$

We first consider Q. Let $Z = (X_0 \setminus Q) \cup \{y, z, a_l\diamond^{x_1}d\diamond^{x_2}a_{l+1}\}$ where $y \uparrow a_i\diamond^{m-2}a_i$, $z \uparrow a_j\diamond^{m-2}a_j$, and $d \in \{a_l, a_{l+1}\}$. We now show that if $h(y)+h(z) = m-2$ then Z is avoidable. So suppose $h(y)+h(z) = m-2$. If $h(y) = 0$, then $(a_j^{m-2}a_la_j^{m-2}a_{l+1})^{\mathbb{Z}}$ avoids Z, and similarly, if $h(z) = 0$, then $(a_i^{m-2}a_la_i^{m-2}a_{l+1})^{\mathbb{Z}}$ avoids Z. Thus, suppose $h(y), h(z) \geq 1$. Let $h(y) = n-2$ and $h(z) = m-n$. If the $m-n$ holes in z are not consecutive, then the $(m-1)$-periodic word $(a_j^{n-1}a_i^{m-n})^{\mathbb{Z}}$ avoids Z, while if the $m-n$ holes in z are consecutive, then the $(m-1)$-periodic word $(a_j^{n-2}a_ia_ja_i^{m-n-1})^{\mathbb{Z}}$ avoids Z. Filling in a second hole in $a_l\diamond^{m-2}a_{l+1}$ for a total of m holes filled is just a strengthening of Z and thus is also avoidable. Thus, filling in more than $m-1$ holes in Q makes X_0 avoidable.

We now consider S. Let $Y = (X_0 \setminus S) \cup \{x, y, z\}$ where $x \uparrow a_i\diamond^{m-2}a_i$, $y \uparrow a_j\diamond^{m-2}a_j$, and $z \uparrow a_i\diamond^{m-2}a_{i+1}$. We show that filling in more than $m-1$ holes in S makes Y avoidable (and thus X_0 avoidable). As discussed above, we can assume that x contains only the letter a_i and y contains only the letter a_j. If $h(y) = 0$, then filling in any of the holes in S is equivalent to filling in holes in R, which we do below. Thus, we assume $h(y) \geq 1$.

Let $Y' = \{x, y, a_i\diamond^{x_1}a_i\diamond^{x_2}a_{i+1}, a_{i+1}\diamond^{m-2}a_{i+1}, a_j\diamond^{m-2}a_i, a_j\diamond^{m-2}a_{i+1}\}$. We now show that if $h(x)+h(y) = m-2$, then Y' is avoidable. If $h(x) = 0$, then $({a_j}^{m-2}a_i{a_j}^{m-2}a_{i+1})^{\mathbb{Z}}$ avoids Y'. Thus, assume $h(x) \geq 1$. Let $h(x) = m-n$ and $h(y) = n-2$. First, suppose the $m-n$ holes in x do not appear in a contiguous block. Then the $(m-1)$-periodic word $w = ({a_i}^{n-1}{a_j}^{m-n})^{\mathbb{Z}}$ avoids Y'. Since w is $(m-1)$-periodic, it does not meet $a_j\diamond^{m-2}a_i$. Since w does not contain any a_{i+1}'s it avoids $a_{i+1}\diamond^{m-2}a_{i+1}$, $a_j\diamond^{m-2}a_{i+1}$, and $a_i\diamond^{x_1}a_i\diamond^{x_2}a_{i+1}$. Let u be an m-length factor of w such that $u(0) = a_j$. We know that u contains $n-1$ consecutive occurrences of a_i. Since $h(y) = n-2$, there is at least one instance where u has an a_i in a position where y has an a_j. Thus, w does not meet y. Similarly, let v be an m-length factor of w such that $v(0) = a_i$. This means v contains a contiguous block of $m-n$ a_j's. However, $v \not\uparrow x$ since the holes in x do not form a contiguous block. Now, suppose the $m-n$ holes in x appear in a contiguous block. Then the $(m-1)$-periodic word $w' = ({a_i}^{n-2}a_ja_i{a_j}^{m-n-1})^{\mathbb{Z}}$ avoids Y'. It avoids $a_j\diamond^{m-2}a_i$, $a_j\diamond^{m-2}a_{i+1}$, $a_i\diamond^{x_1}a_i\diamond^{x_2}a_{i+1}$, $a_{i+1}\diamond^{m-2}a_{i+1}$, and y for the same reasons that w does. However, since the $m-n$ holes in x appear in a contiguous block, and there are $m-n$ a_j's in w' that are not situated in a contiguous block, w' avoids x. Thus we have shown that filling in $m-2$ holes in T_0 and filling in a hole with a_i

in $a_i\diamond^{m-2}a_{i+1}$ makes Y' avoidable. If we fill in a second hole with a_i in $a_i\diamond^{m-2}a_{i+1}$, for a total of m holes filled, this is just a strengthening of the previous case and thus is also avoidable. Furthermore, by Theorem 3 substituting a_{i+1}'s for the a_i's would yield the same avoidability.

We finally consider R. Suppose an infinite word w avoids $X = (X_0 \setminus R) \cup \{x,y,z\}$ where $x \uparrow a_i\diamond^{m-2}a_i$, $y \uparrow a_{i+1}\diamond^{m-2}a_{i+1}$, and $z \uparrow a_i\diamond^{m-2}a_{i+1}$. Since we want to show that we can fill in $m-1$ holes, suppose at least two of x,y,z have some defined interior positions. We prove that w must be over the binary alphabet $\{a_i,a_{i+1}\}$ by considering two cases. First, suppose $a_i = a_1$ (the proof is similar if $a_{i+1} = a_k$). If $w(m-1) = a_k$, then no letter works for $w(0)$ since w must avoid $a_j\diamond^{m-2}a_k$ for all $j \in \{1,\ldots,k\}$; thus, w does not contain any a_k's. Similarly if $w(m-1) = a_{k-1}$, then no letter works for $w(0)$ since w must avoid $a_j\diamond^{m-2}a_{k-1}$, for all $j \in \{1,\ldots,k-1\}$, and w does not contain any a_k's; thus, w cannot contain any a_{k-1}'s. We can continue eliminating potential letters from w until we are left with only a_1 and a_2. If $w(m-1) = a_2$, then $w(0) \in \{a_1,a_2\}$ depending on which of x,y,z have defined interior positions. Similarly, if $w(m-1) = a_1$, then $w(0) \in \{a_1,a_2\}$. Thus, w is over $\{a_1,a_2\}$. Now, suppose $a_i \neq a_1$ and $a_{i+1} \neq a_k$. If $w(m-1) = a_k$, then as above we can show that w cannot contain any of $a_{i+2},\ldots,a_k$, and if $w(0) = a_1$, that w cannot contain any of $a_1,\ldots,a_{i-1}$. If $w(0) = a_i$, then $w(m-1) \in \{a_i,a_{i+1}\}$ depending on which of x,y,z have defined interior positions. Similarly, if $w(0) = a_{i+1}$, then $w(m-1) \in \{a_i,a_{i+1}\}$.

We have shown that any infinite word that avoids X must be over the alphabet $\{a_i,a_{i+1}\}$. Thus, by Theorem 1, the maximum number of holes we can fill in X while maintaining the unavoidability property is $m-1$ if m is even and m if m is odd. □

5 Conclusion

In this paper, we have considered m-uniform unavoidable sets of partial words over an arbitrary alphabet $A_k = \{a_1,\ldots,a_k\}$. We have formulated a conjecture, Conjecture 1, that states that the sets defined by Eq. (1) are avoidable when $y_1 \leq x_2 \leq x_1 \leq y_2$ and a,b,c are distinct letters. If Conjecture 1 is true, for $m \geq 4$, we have exhibited a formula that calculates the maximum number of holes we can fill in any m-uniform unavoidable set of partial words over A_k, while maintaining the unavoidability property.

We believe that Conjecture 1 is true and have tested it for all m-uniform sets defined by Eq. (1) up to $m = 100$ that satisfy $y_1 \leq x_2 \leq x_1 \leq y_2$. We have found that these sets are all avoidable. In fact, all of the sets we tested have an avoiding word with period less than $2m$. Of the $41,650$ such sets, only 4 were found to require avoiding words that did not match any of our patterns. Furthermore, only 77 of the roughly 42 million sets for $m \leq 1000$ are not covered by our patterns. However, we are doubtful that a small number of similar patterns could be shown to cover the remaining cases.

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