Improving Angular Speed Uniformity by Piecewise Radical Reparameterization

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For a rational parameterization of a curve, it is desirable that its angular speed is as uniform as possible. Hence, given a rational parameterization, one wants to find *re*-parameterization with better uniformity. One natural way is to use *piecewise* rational reparameterization. However, it turns out that the piecewise rational reparameterization does *not* help when the angular speed of the given rational parameterization is zero at some points on the curve. In this paper, we show how to overcome the challenge by using piecewise *radical* reparameterization.

1 Introduction

Parametric curves and surfaces are fundamental objects that are most frequently used in computer aided geometric design. A given curve or surface may have many different parameterizations, of which some may possess better properties and thus are more suitable for certain applications than the others. Thus, one often needs to convert one parameterization into another, i.e., to *re*-parameterize the given parameterization (see, e.g., [1, 2, 3, 5, 6, 7, 8, 9, 10]). In this paper, we focus our investigation on an important class of parameterizations, called uniform (angular-speed) parameterizations, where the distribution of points are determined by the local curvature and show how to construct such reparameterizations for a specific class of curves.

Uniform parameterization has been studied in a series of papers (see [4, 6, 8, 11, 14, 13, 12] and references therein). The authors have defined a function of angular speed uniformity to measure the quality of any given parameterization of a plane curve and proposed a method to compute its uniform reparameterization. However, the computed reparameterization is irrational in most cases (with straight lines as exceptions). For the sake of efficiency, a framework has been proposed for the computation of rational approximations of uniform parameterizations [4]. Four different methods of reparameterization (i.e., optimal reparameterization with fixed degree, C^0 and C^1 optimal piecewise reparameterization, and nearly optimal C^1 piecewise reparameterization) have been integrated into this framework. They have also been generalized to compute uniform quasi-speed reparameterizations of parametric curves in n-dimensional space.

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However, there is still a major challenge: all the above-mentioned methods do *not* work well when the angular speed of the given rational parameterization is zero at some points on the curve. This is due to an intrinsic property of the angular speed function [13]:

Let ω_p be an angular speed function of a curve p and r be a proper transformation. Then

$$\omega_{p\circ r} = (\omega_p \circ r)\cdot r'. \tag{1}$$

Uniformizing the angular speed can be seen as modifying the angular speed value at each point under the constraint (1) iteratively until all the values are equal to the average. However, the constraint indicates that $\omega_{p\circ r}$ will never reach the average value for any rational r' when $\omega_p(t) = 0$ for some t.

In this paper, we propose to overcome the challenge by using *radical* transformations instead of rational ones. We show that radical transformations allow one to increase the angular speed toward the average value at the points where the angular speed is zero. Then we adapt the idea of piecewise Möbius transformation from [11] and the strategies in [12] to optimally improve the uniformity of angular speed.

Experiments show that the proposed approach can improve the angular speed uniformity significantly when the angular speed of the given parameterization vanishes at some point on the curve.

The rest of the paper is structured as follows. In Section 2, we formulate the problem precisely. For this, we also introduce all the needed notations and notions. In Section 3, we develop mathematical theory to tackle the problem. In particular, we show how to use piecewise radical transformation to transform an angular speed function with zeros into one without zero in such a way that the parameters involved are also optimized. In Section 4, we summarize the theoretical results into an algorithm and illustrate its performance on an example. In Section 5, we briefly discuss implementational issues/suggestions when floating point arithmetic is used.

2 Problem

Consider a regular parametric curve

$$p = (x_1(t), \ldots, x_n(t)) : \mathbb{R} \mapsto \mathbb{R}^n.$$

Its *angular speed* ω_p is given by the following expression (see [4]):[1]

$$\omega_p = \frac{\sqrt{\sum_{1\le i<j\le n} \begin{vmatrix} x_i'' & x_j'' \\ x_i' & x_j' \end{vmatrix}^2}}{\sum_{i=1}^{n} x_i'^2}. \tag{2}$$

Recall that the mean μ_p and the variation σ_p^2 of ω_p are given by

$$\mu_p = \int_0^1 \omega_p(t)\,dt, \quad \text{and} \quad \sigma_p^2 = \int_0^1 (\omega_p(t) - \mu_p)^2\,dt.$$

Definition 1 *The* angular speed uniformity u_p *of a parameterization* p *is defined as*

$$u_p = \begin{cases} \dfrac{1}{1+\sigma_p^2/\mu_p^2} & \text{if } \mu_p \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

[1] The concept of angular speed is defined in the same manner as the one in physics but for $p'(t)$. The reason is to make the angular speed independent of the origin.

Example 2 (Running) *Consider the parametric curve $p = (t,t^3)$. Then*

$$\omega_p = \frac{6t}{9t^4+1},$$

$\mu_p \doteq 1.249$ *and* $u_p \doteq 0.846$*. The goal is to find a proper parameter transformation r over* $[0,1]$ *in order to increase the uniformity.*

Recall the following results from [4]. For any proper parameter transformation r over $[0,1]$, we have

$$\omega_{p\circ r}(s) = (\omega_p \circ r)(s) \cdot r'(s) \tag{4}$$

and

$$u_{p\circ r} = \frac{\mu_p^2}{\eta_{p,r}}, \quad \text{where} \quad \eta_{p,r} = \int_0^1 \frac{\omega_p^2}{(r^{-1})'}(t)\,dt. \tag{5}$$

By [13, Theorem 2], one can construct a uniform reparameterization from p, but such a reparameterization is irrational in most cases. Therefore, we proposed several methods in [4] to improve the angular speed uniformity by computing piecewise rational reparameterizations. However, those methods are not applicable to curves whose angular speed may vanish over $[0,1]$. Intuitively speaking, uniformizing the angular speed over $[0,1]$ can be viewed as getting all the values of $\omega_p(t)$ (for all $t \in [0,1]$) as close to μ_p as possible.

If r is a continuous rational function over $[0,1]$, then r' is bounded. Suppose that $\omega_p(t_0) = 0$ for some $t_0 \in [0,1]$ and $\mu_p \neq 0$. Then by (4), there must exist some $s_0 \in [0,1]$ such that $\omega_{p\circ r}(s_0) = 0$, which is not close to μ_p at all. This makes rational proper parameter transformations invalid. In what follows, we resort to radical transformations and develop a new approach to uniformize the angular speed of parametric curves which has zeros over $[0,1]$.

Let p be a parametric curve. Without loss of generality, we assume that

$$p'(t) = (x_1'(t), \ldots, x_n'(t)) = \Big(\frac{X_1(t)}{W(t)}, \cdots, \frac{X_n(t)}{W(t)}\Big),$$

where $X_i(t), W(t) \in \mathbb{R}[t]$ and $\gcd(X_1(t), \ldots, X_n(t), W(t)) = 1$. One can verify that

$$\omega_p = \frac{\sqrt{F}}{\sum_i X_i^2}, \quad \text{where} \quad F = \sum_{i\neq j} \begin{vmatrix} X_i' & X_j' \\ X_i & X_j \end{vmatrix}^2.$$

Let F be written as $F = \left(\prod_{i=0}^k (t-\tilde{t}_i)^{2\mu_i}\right)\zeta(t)$ for positive k. Note that $\{\tilde{t}_i : \tilde{t}_i < \tilde{t}_{i+1} \text{ for } 0 \le i < k\}$ contains all the zeros of F over $[0,1]$. It is allowed that some t_i's are not the roots of F. The positive integer $\mu_i \in \mathbb{N}$ is called the *multiplicity* of $\tilde{t}_i$ in ω_p and denoted by $\mathtt{mult}(\omega_p, \tilde{t}_i)$. If $\omega_p(\tilde{t}_i) \neq 0$, then $\mathtt{mult}(\omega_p, \tilde{t}_i) = 0$.

Let

$$T = (t_0, \ldots, t_N), \quad S = (s_0, \ldots, s_N), \quad Z = (z_0, \ldots, z_N), \quad \alpha = (\alpha_0, \ldots, \alpha_{N-1})$$

be sequences such that

- $0 = t_0 < \cdots < t_N = 1$, $0 = z_0 < \cdots < z_N = 1$, $0 = s_0 < \cdots < s_N = 1$, $0 < \alpha_i < 1$;
- at most one of $\omega_p(t_i) = 0$ and $\omega_p(t_{i+1}) = 0$ holds for $0 \le i < N$, that is, the successive appearance of two zeros of ω_p are not allowed;
- the multiplicity of t_i in ω_p is μ_i;

- $\omega_p(t) \neq 0$ for all $t \in (t_i, t_{i+1})$.

Definition 3 (Elementary Piecewise Radical Transformation) *Let p be a parametric curve with T, S defined above. Then φ is called an* elementary piecewise radical transformation *associated to p if φ has the following form:*

$$\varphi(s) = \begin{cases} \vdots \\ \varphi_i(s) & \text{if} \quad s \in [s_i, s_{i+1}], \\ \vdots \end{cases}$$

where

$$\varphi_i(s) = \begin{cases} t_i + \Delta t_i \sqrt[\mu_i+1]{\tilde{s}} & \text{if } \omega_p(t_i) = 0; \\ t_i + \Delta t_i (1 - \sqrt[\mu_{i+1}+1]{1-\tilde{s}}) & \text{if } \omega_p(t_{i+1}) = 0; \\ t_i + \Delta t_i \cdot \tilde{s} & \text{otherwise,} \end{cases} \tag{6}$$

and $\Delta t_i = t_{i+1} - t_i$, $\Delta s_i = s_{i+1} - s_i$, $\tilde{s} = (s - s_i)/\Delta s_i$.

Remark 4

1. *It can be verified that $\varphi(s_i) = t_i$ and $\varphi(s_{i+1}) = t_{i+1}$, which implies that φ is with C^0 continuity.*
2. *It is allowed that more than one intermediate point lie between two zeros of ω_p because it can reduce the number of radical pieces and thus enhance the efficiency of generating points with the new parameterization.*

It can be shown that $\omega_{p\circ\varphi}(s) \neq 0$ (see Theorem 7). Next let $q = p \circ \varphi$ and thus q has no inflation point. We adapt the reparameterization methods from [12] to increase the uniformity of ω_q to any value close to 1. For this purpose, we recall the following piecewise Möbius transformation.

Definition 5 (Piecewise Möbius Transformation) *Let p be a parametric curve with S, Z, α defined above. Then m is called a* piecewise Möbius transformation *associated to p if m has the following form:*

$$m(z) = \begin{cases} \vdots \\ m_i(z) & \text{if} \quad z \in [z_i, z_{i+1}], \\ \vdots \end{cases}$$

where

$$m_i(z) = s_i + \Delta s_i \cdot \frac{(1-\alpha_i)\tilde{z}}{(1-\alpha_i)\tilde{z} + \alpha_i(1-\tilde{z})} \tag{7}$$

and $\Delta z_i = z_{i+1} - z_i$, $\Delta s_i = s_{i+1} - s_i$, $\tilde{z} = (z - z_i)/\Delta z_i$.

The problem addressed in this paper may be formulated as follows.

Problem 6 *Given a parametric curve p with $\omega_p(t) = 0$ for some $t \in [0,1]$, find a radical piecewise transformation φ and an optimal piecewise Möbius transformation m over $[0,1]$ such that*

- $u_{p\circ\varphi\circ m} \doteq 1$;
- $\forall s \in [0,1]$, $\omega_{p\circ\varphi\circ m}(s) \neq 0$.

3 Theory

3.1 Property of φ

Theorem 7 *For any $s \in [0,1]$, $\omega_{p\circ\varphi}(s) \neq 0$.*

Proof: Taking derivative of φ_i, we have

$$\varphi_i'(s) = \begin{cases} \dfrac{\Delta t_i}{\mu_i+1} \cdot \dfrac{1}{\sqrt[\mu_i+1]{\tilde{s}^{\mu_i}}} \cdot \dfrac{1}{\Delta s_i} & \text{if } \omega_p(t_i) = 0; \\ \dfrac{\Delta t_i}{\mu_{i+1}+1} \cdot \dfrac{1}{\sqrt[\mu_{i+1}+1]{(1-\tilde{s})^{\mu_{i+1}}}} \cdot \dfrac{1}{\Delta s_i} & \text{if } \omega_p(t_{i+1}) = 0; \\ \dfrac{\Delta t_i}{\Delta s_i} & \text{otherwise.} \end{cases}$$

Next we show that in the above three cases, $\omega_{p\circ\varphi}(s) \neq 0$.

Case 1: $\omega_p(t_i) = 0$.

Assume that $\mu_i = \texttt{mult}(\omega_p, t_i)$. Then ω_p can be written as

$$\omega_p = |t - t_i|^{\mu_i} \cdot \tilde{\zeta}(t),$$

where $\tilde{\zeta}(t) > 0$ for $t \in [t_i, t_{i+1}]$. Therefore,

$$\begin{aligned} \omega_{p\circ\varphi}(s) =\ & |\varphi_i(s) - t_i|^{\mu_i} \cdot (\tilde{\zeta} \circ \varphi_i)(s) \cdot \varphi_i'(s) \\ &= (\Delta t_i \tilde{s}^{\frac{1}{\mu_i+1}})^{\mu_i} \cdot (\tilde{\zeta} \circ \varphi_i)(s) \cdot \left[\frac{\Delta t_i}{\mu_i+1} \cdot \frac{1}{\sqrt[\mu_i+1]{\tilde{s}^{\mu_i}}} \cdot \frac{1}{\Delta s_i}\right] \\ &= \frac{\Delta t_i^{\mu_i+1}}{\mu_i+1} \cdot (\tilde{\zeta} \circ \varphi_i)(s) \cdot \frac{1}{\Delta s_i} \\ &= \frac{\Delta t_i^{\mu_i+1}}{\mu_i+1} \cdot \frac{1}{\Delta s_i} \cdot \tilde{\zeta}(t) \neq 0 \end{aligned}$$

for $s \in [s_i, s_{i+1}]$.

Case 2: $\omega_p(t_{i+1}) = 0$.

Assume that $\mu_{i+1} = \texttt{mult}(\omega_p, t_{i+1})$. Then ω_p can be written as

$$\omega_p = |t_{i+1} - t|^{\mu_{i+1}} \cdot \tilde{\zeta}(t),$$

where $\tilde{\zeta}(t) > 0$ for $t \in [t_i, t_{i+1}]$. Therefore,

$$\begin{aligned} \omega_{p\circ\varphi}(s) =\ & |t_{i+1} - \varphi_i(s)|^{\mu_{i+1}} \cdot (\tilde{\zeta} \circ \varphi_i)(s) \cdot \varphi_i'(s) \\ &= \frac{\Delta t_i}{\mu_{i+1}+1} \cdot \frac{1}{\sqrt[\mu_{i+1}+1]{(1-\tilde{s})^{\mu_{i+1}}}} \cdot \frac{1}{\Delta s_i} \\ &= [\Delta t_i (1-\tilde{s})^{\frac{1}{\mu_{i+1}+1}}]^{\mu_{i+1}} \\ &\quad \cdot (\tilde{\zeta} \circ \varphi_i)(s) \cdot \frac{\Delta t_i}{\mu_{i+1}+1} \cdot \frac{1}{\sqrt[\mu_{i+1}+1]{(1-\tilde{s})^{\mu_{i+1}}}} \cdot \frac{1}{\Delta s_i} \\ &= \frac{\Delta t_i^{\mu_{i+1}+1}}{\mu_{i+1}+1} \cdot (\tilde{\zeta} \circ \varphi_i)(s) \cdot \frac{1}{\Delta s_i} \\ &= \frac{\Delta t_i^{\mu_{i+1}+1}}{\mu_{i+1}+1} \cdot \tilde{\zeta}(t) \cdot \frac{1}{\Delta s_i} \neq 0. \end{aligned}$$

Case 3: $\omega_p(t_i)\omega(t_{i+1}) \neq 0$.

Combining $\omega_p(t) \neq 0$ for $t \in [t_i, t_{i+1}]$, $\Delta t_i > 0$ and $\Delta s_i > 0$, we have

$$\omega_{p\circ\varphi}(s) = (\omega_p \circ \varphi)(s) \cdot \varphi'(s) = \omega_p(t) \cdot \frac{\Delta t_i}{\Delta s_i} \neq 0.$$

To sum up, we have $\omega_{p\circ\varphi}(s) \neq 0$ when $s \in [s_i, s_{i+1}]$. □

Example 8 (Continued from Example 2) *For the cubic curve $p = (t, t^3)$ whose angular speed is $\omega_p = \frac{6t}{9t^4+1}$, it is easy to see that $t = 0$ is a zero of ω_p with multiplicity* 1. *Let $T = (0,1)$ and $S = (0,1)$. Then the constructed φ is $\varphi(s) = \sqrt{s}$. It follows that*

$$\omega_{p\circ\varphi}(s) = (\omega_p \circ \varphi)(s) \cdot \varphi'(s) = \frac{6\sqrt{s}}{9s^2+1} \cdot \frac{1}{2\sqrt{s}} = \frac{3}{9s^2+1}$$

which is nonzero over $[0,1]$.

Remark 9 *It may be further deduced that $\omega_{p\circ\varphi}(s)$ is discontinuous at $s = s_i$.*

3.2 Choice of T

By Definition 3, T should contain all the zeros of ω_p over $[0,1]$ and some intermediate points in the subintervals separated by the zeros of ω_p. One question is how to choose intermediate points to make the uniformity improvement as significant as possible. In this subsection, we present a strategy similar to the one introduced in [12] for determining such points.

Recall [13, Theorem 2] which states that the uniformizing parameter transformation r_p of p satisfies

$$(r_p)^{-1} = \int_0^t \omega_p(\gamma) d\gamma / \mu_p.$$

Let φ be a piecewise radical transformation associated to p. If $(r_p)^{-1}$ and φ^{-1} share some common properties, we say informally that r_p and φ are similar to each other.

First of all, the following can be derived:

$$\varphi^{-1}(t) = \begin{cases} s_i + \Delta s_i \cdot \tilde{t}^{\mu_i+1} & \text{if } \omega_p(t_i) = 0; \\ s_i + \Delta s_i \cdot [1 - (1-\tilde{t})^{\mu_{i+1}+1}] & \text{if } \omega_p(t_{i+1}) = 0; \\ s_i + \Delta s_i \cdot \tilde{t} & \text{otherwise,} \end{cases} \tag{8}$$

where $\tilde{t} = (t - t_i)/\Delta t_i$. Furthermore,

$$\left[\varphi^{-1}\right]' = \begin{cases} \frac{\Delta s_i}{\Delta t_i} \cdot (\mu_i + 1) \cdot \tilde{t}^{\mu_i} & \text{if } \omega_p(t_i) = 0; \\ \frac{\Delta s_i}{\Delta t_i} \cdot (\mu_{i+1} + 1) \cdot (1-\tilde{t})^{\mu_{i+1}} & \text{if } \omega_p(t_{i+1}) = 0; \\ \frac{\Delta s_i}{\Delta t_i} & \text{otherwise;} \end{cases}$$

$$[\varphi^{-1}]'' = \begin{cases} \dfrac{\Delta s_i}{\Delta t_i^2} \cdot (\mu_i + 1)\mu_i \cdot \tilde{t}^{\mu_i - 1} & \text{if } \omega_p(t_i) = 0; \\ -\dfrac{\Delta s_i}{\Delta t_i^2} \cdot (\mu_{i+1} + 1)\mu_{i+1} \cdot (1 - \tilde{t})^{\mu_{i+1} - 1} & \text{if } \omega_p(t_{i+1}) = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Note that φ has the properties listed below.

- $\varphi^{-1}(0) = 0$, $\varphi^{-1}(1) = 1$.
- φ_i^{-1} is monotonic over (t_i, t_{i+1}) because $(\varphi_i^{-1})'(t) \geq 0$ for all $t \in (t_i, t_{i+1})$; since φ^{-1} is continuous over $[0,1]$, φ^{-1} is monotonic over $[0,1]$.
- $[\varphi_i^{-1}]'$ is monotonic over (t_i, t_{i+1}) because $[\varphi_i^{-1}]''$ has a constant sign over (t_i, t_{i+1}).

The above properties indicate that φ is composed of some monotonically increasing convex or concave pieces. Moreover, it can be verified that

- $r_p^{-1}(0) = \int_0^0 \omega_p(\gamma)\,d\gamma/\mu_p = 0$, $r_p^{-1}(1) = \int_0^1 \omega_p(\gamma)\,d\gamma/\mu_p = 1$;
- r_p^{-1} is monotonic over $[0,1]$ because $(r_p^{-1})'(t) = \omega_p(t)/\mu_p \geq 0$.

One may observe that φ shares the first two properties with r_p. If r_p possesses the third property of φ, then r_p and φ are expected to be similar. This inspires us to divide $[0,1]$ into some monotonic intervals of $(r_p^{-1})'(t)$ (i.e., ω_p). Thus we may try to choose the intermediate t_i in T by solving

$$\omega_p(t_i)\omega_p'(t_i) = 0.$$

Note that $\omega_p(t)$ is nonnegative. Thus 0 is the local minimum value of ω_p. In this sense, T consists of all the local extreme points of ω_p and the two boundary points of the unit interval.

With the above operation, r_p is divided into some monotonically increasing/decreasing convex or concave pieces with each piece having a corresponding one in φ. Therefore, T can be obtained by collecting and inserting the zeros of ω_p and ω_p' into $[0,1]$ in order.

Example 10 (Continued from Example 8) *One may compute that*

$$\omega_p'(t) = -\frac{6\,(27t^4 - 1)}{(9t^4 + 1)^2}.$$

Then the solution of $\omega_p(t)\omega_p'(t) = 0$ *over* $[0,1]$ *gives us a partition of* $[0,1]$, *i.e.,*

$$T \doteq (0, 0.439, 1).$$

Furthermore, one may check that the multiplicities of t_0, t_1, t_2 *as roots of* ω_p *are* $1, 0$ *and* 0*, respectively.*

3.3 Determination of S

Once a partition T of $[0,1]$ is obtained, one can compute the sequence S in various ways. In this subsection, we present an optimization strategy for the computation of S.

When T is fixed, $u_{p\circ\varphi}$ becomes a function of s_i $(i = 1, \ldots, N-1)$. The following theorem provides a formula for computing the optimal values for s_i's.

Theorem 11 *The uniformity $u_{p\circ\varphi}$ reaches the maximum when*

$$s_i = s_i^* = \frac{\sum_{k=0}^{i-1}\sqrt{L_k}}{\sum_{k=0}^{N-1}\sqrt{L_k}}, \tag{9}$$

where

$$L_k = \begin{cases} \Delta t_k \int_{t_k}^{t_{k+1}} \dfrac{\omega_p^2(t)}{(\mu_k+1)\tilde{t}^{\mu_k}}\,dt & \text{if} \quad \omega_p(t_k)=0; \\ \Delta t_k \int_{t_k}^{t_{k+1}} \dfrac{\omega_p^2(t)}{(\mu_{k+1}+1)(1-\tilde{t})^{\mu_{k+1}}}\,dt & \text{if} \quad \omega_p(t_{k+1})=0; \\ \Delta t_k \int_{t_k}^{t_{k+1}} \omega_p^2(t)\,dt & \text{otherwise.} \end{cases} \tag{10}$$

The maximum value of $u_{p\circ\varphi}$ is

$$u_{p\circ\varphi}^* = \mu_p^2/\eta_{p,\varphi}^*, \quad \text{where} \quad \eta_{p,\varphi}^* = \left(\sum_{i=0}^{N-1}\sqrt{L_i}\right)^2.$$

Proof: Recall (5). Since μ_p is a constant for any given p, the problem of maximizing $u_{p\circ\varphi}$ can be reduced to that of minimizing

$$\eta_{p,\varphi} = \int_0^1 \frac{\omega_p^2}{(\varphi^{-1})'}(t)\,dt = \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \frac{\omega_p^2}{(\varphi_i^{-1})'}(t)\,dt.$$

We first simplify each component in the above equation. Denote $\int_{t_i}^{t_{i+1}} \dfrac{\omega_p^2}{(\varphi_i^{-1})'}(t)\,dt$ by I_i. Note that

$$\frac{1}{(\varphi_i^{-1})'(t)} = \begin{cases} \dfrac{\Delta t_i}{\Delta s_i}\cdot\dfrac{1}{\mu_i+1}\cdot\dfrac{1}{\tilde{t}^{\mu_i}} & \text{if} \quad \omega_p(t_i)=0; \\ \dfrac{\Delta t_i}{\Delta s_i}\cdot\dfrac{1}{\mu_{i+1}+1}\cdot\dfrac{1}{(1-\tilde{t})^{\mu_{i+1}}} & \text{if} \quad \omega_p(t_{i+1})=0; \\ \dfrac{\Delta t_i}{\Delta s_i} & \text{otherwise.} \end{cases}$$

When $\omega_p(t_i)=0$,

$$I_i = \frac{\Delta t_i}{\Delta s_i}\cdot\frac{1}{\mu_i+1}\int_{t_i}^{t_{i+1}} \frac{\omega_p^2}{\tilde{t}^{\mu_i}}\,dt = L_i/\Delta s_i.$$

Similarly, when $\omega_p(t_{i+1})=0$,

$$I_i = \frac{\Delta t_i}{\Delta s_i}\cdot\frac{1}{\mu_{i+1}+1}\int_{t_i}^{t_{i+1}} \frac{\omega_p^2}{(1-\tilde{t})^{\mu_{i+1}}}\,dt = L_i/\Delta s_i.$$

When $\omega_p(t_i)\cdot\omega_p(t_{i+1})\neq 0$,

$$I_i = \frac{\Delta t_i}{\Delta s_i}\cdot\int_i^{t_{i+1}} \omega_p^2\,dt = L_i/\Delta s_i.$$

It is obvious that $\eta_{p,\varphi} = \sum_{i=0}^{N-1} I_i > 0$; it increases to $+\infty$ when s_i approaches the boundary of the feasible set of parameters. Now we compute the extrema of $\eta_{p,\varphi}$. Let

$$\frac{\partial \eta_{p,\varphi}}{\partial s_i} = 0,$$

i.e.,

$$\frac{L_i}{\Delta s_i^2} - \frac{L_{i-1}}{\Delta s_{i-1}^2} = 0,$$

where L_i is as in (10). Solving the above equation, we obtain

$$\Delta s_i = \Delta s_i^* = \Delta s_0^* \sqrt{L_i/L_0}.$$

Note that $\sum_{i=0}^{N-1} \Delta s_i^* = 1$. Thus

$$\Delta s_0^* = \left(\sum_{k=0}^{N-1} \sqrt{L_k/L_0} \right)^{-1}, \quad s_i^* = \sum_{k=0}^{i-1} \Delta s_k^* = \frac{\sum_{k=0}^{i-1} \sqrt{L_k/L_0}}{\sum_{k=0}^{N-1} \sqrt{L_k/L_0}} = \frac{\sum_{k=0}^{i-1} \sqrt{L_k}}{\sum_{k=0}^{N-1} \sqrt{L_k}}.$$

Therefore,

$$\Delta s_i^* = \frac{\sqrt{L_i}}{\sum_{k=0}^{N-1} \sqrt{L_k}}.$$

Moreover, the optimal value of $\eta_{p,\varphi}$ is

$$\eta_{p,\varphi} = \eta_{p,\varphi}^* = \sum_{i=0}^{N-1} \frac{L_i}{\Delta s_i^*} = \sum_{k=0}^{N-1} \frac{L_i}{\dfrac{\sqrt{L_i}}{\sum_{i=0}^{N-1} \sqrt{L_k}}} = \left(\sum_{k=0}^{N-1} \sqrt{L_k} \right)^2,$$

from which it follows that the optimal value of $u_{p\circ\varphi}$ is

$$u_{p\circ\varphi} = u_{p\circ\varphi}^* = \frac{\mu_p^2}{\eta_{p,\varphi}^*} = \frac{\mu_p^2}{\left(\sum_{k=0}^{N-1} \sqrt{L_k}\right)^2}.$$

The proof is completed. □

Example 12 (Continued from Example 10) *By using* (10)*, we compute the values of L_0 and L_1 and obtain*

$$L_0 \doteq 0.439 \int_0^{0.439} \frac{\left(\dfrac{6t}{9t^4+1}\right)^2}{2 \cdot \dfrac{t-0}{0.439}} \, dt \doteq 0.276,$$
$$L_1 \doteq (1-0.439) \int_{0.439}^{1} \left(\frac{6t}{9t^4+1}\right)^2 dt \doteq 0.590.$$

By (9)*, we have $s_1 = 0.406$. Thus $S \doteq (0, 0.406, 1)$. Furthermore, one may calculate the optimal value of $u_{p\circ\varphi}$ and obtain $u_{p\circ\varphi}^* \doteq 0.932$.*

3.4 Determination of Z and α

Once a partition S of $[0,1]$ is obtained, one can compute the sequence Z. In this subsection, we give explicit formulae for the optimal values of Z and α which are directly computed from the sequence T. For this purpose, we first recall the the following result from [11].

Theorem 13 *Let q be a rational parameterization such that $\omega_q(s) \neq 0$ over $[0,1]$ and m be a piecewise Möbius transformation determined by S, Z and α. For a given sequence S, the uniformity $u_{q\circ m}$ reaches the maximum when*

$$\alpha_i = \alpha_i^* = \frac{1}{1+\sqrt{C_i/A_i}}, \quad z_i = z_i^* = \frac{\sum_{k=0}^{i-1}\sqrt{M_k}}{\sum_{k=0}^{N-1}\sqrt{M_k}}, \tag{11}$$

where

$$A_i = \int_{s_i}^{s_{i+1}} \omega_q^2 \cdot (1-\tilde{s})^2 ds, \qquad B_i = \int_{s_i}^{s_{i+1}} \omega_q^2 \cdot 2\tilde{s}(1-\tilde{s}) ds,$$
$$C_i = \int_{s_i}^{s_{i+1}} \omega_q^2 \cdot \tilde{s}^2 ds, \qquad M_k = \Delta s_k \left(2\sqrt{A_k C_k} + B_k\right).$$

Let m^* be the piecewise Möbius transformation determined by S, Z^* and α^*. Then the maximum value of $u_{q\circ m}$ is $u_{q\circ m^*} = \mu_q^2/\eta_{q,m^*}$ where

$$\eta_{q,m^*} = \left(\sum_{i=0}^{N-1}\sqrt{M_k}\right)^2.$$

Remark 14 *Let φ be an elementary radical transformation as in Definition 3 and $q = p\circ\varphi$. Note that*

$$\mu_q = \int_0^1 \omega_q ds = \int_0^1 \omega_{p\circ\varphi} ds = \int_0^1 (\omega_p\circ\varphi)(s)\cdot\varphi'(s) ds = \int_0^1 \omega_p dt = \mu_p.$$

Thus

$$u_{p\circ\varphi\circ m^*} = u_{q\circ m^*} = \mu_q^2/\eta_{q,m^*} = \mu_p^2 \Big/ \left(\sum_{i=0}^{N-1}\sqrt{M_k}\right)^2.$$

Let φ and q be defined as before. By Theorem 7, $\omega_q \neq 0$ over $[0,1]$. One may compute the optimal values of S, α and Z by Theorems 11 and 13. However, $p\circ\varphi$ is a composition of radical function and rational function and the composition will cause an increase of complexity because ω_q is radical. In what follows, we simplify the formulae for A_i, B_i and C_i with the goal of computing the values of A_i, B_i and C_i directly from p.

The formula of A_i $(0 \le i \le N-1)$ is derived via the following steps:

$$\begin{aligned}
A_i &= \int_{s_i}^{s_{i+1}} \omega_q^2 \cdot (1-\tilde{s})^2 ds \\
&= \int_{s_i}^{s_{i+1}} [(\omega_p\circ\varphi)(s)]^2 \cdot [\varphi'(s)]^2 \cdot (1-\tilde{s})^2 ds \\
&= \int_{s_i}^{s_{i+1}} [(\omega_p\circ\varphi)(s)]^2 \cdot [\varphi'(s)] \cdot (1-\tilde{s})^2 \left[\varphi'(s) ds\right] \\
&= \int_{t_i}^{t_{i+1}} \frac{\omega_p^2}{(\varphi^{-1})'}(t) \cdot \left(1 - \frac{\varphi^{-1}-s_i}{\Delta s_i}\right)^2 dt && \text{by (8)} \\
&= \begin{cases}
\int_{t_i}^{t_{i+1}} \frac{\omega_p^2}{(\varphi^{-1})'}(t) \cdot (1-\tilde{t}^{\mu_i+1})^2 dt & \text{if } \omega(t_i) = 0; \\
\int_{t_i}^{t_{i+1}} \frac{\omega_p^2}{(\varphi^{-1})'}(t) \cdot (1-\tilde{t})^{2(\mu_{i+1}+1)} dt & \text{if } \omega(t_{i+1}) = 0; \\
\int_{t_i}^{t_{i+1}} \frac{\omega_p^2}{(\varphi^{-1})'}(t) \cdot (1-\tilde{t})^2 dt & \text{otherwise;}
\end{cases}
\end{aligned}$$

$$= \begin{cases} \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_i+1)\tilde{t}^{\mu_i}} \cdot (1-\tilde{t}^{\mu_i+1})^2 dt & \text{if } \omega(t_i)=0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{\mu_{i+1}+1} \cdot (1-\tilde{t})^{\mu_{i+1}+2} dt & \text{if } \omega(t_{i+1})=0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot (1-\tilde{t})^2 dt & \text{otherwise.} \end{cases}$$

Similarly, we have

$$B_i = \begin{cases} \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{\mu_i+1} \cdot 2\tilde{t}(1-\tilde{t}^{\mu_i+1}) dt & \text{if } \omega(t_i)=0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{\mu_{i+1}+1} \cdot 2[1-(1-\tilde{t})^{\mu_{i+1}+1}](1-\tilde{t}) dt & \text{if } \omega(t_{i+1})=0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot (1-\tilde{t})^2 dt & \text{otherwise;} \end{cases}$$

$$C_i = \begin{cases} \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{\mu_i+1} \cdot \tilde{t}^{\mu_i+2} dt & \text{if } \omega(t_i)=0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_{i+1}+1)(1-\tilde{t})^{\mu_{i+1}}} \cdot [1-(1-\tilde{t})^{\mu_{i+1}+1}]^2 dt & \text{if } \omega(t_{i+1})=0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot (1-\tilde{t})^2 dt & \text{otherwise.} \end{cases}$$

Example 15 (Continued from Example 12) *With the above formulae and T, S as in Examples 10 and 12, one may obtain the following:*

$$\begin{array}{lll} A_0 \doteq 0.258, & B_0 \doteq 0.229, & C_0 \doteq 0.193, \\ A_1 \doteq 0.518, & B_1 \doteq 0.317, & C_1 \doteq 0.159. \end{array}$$

Thus

$$M_0 \doteq 0.406(2\sqrt{0.258 \cdot 0.193} + 0.229) \doteq 0.274,$$
$$M_1 \doteq (1-0.406)(2\sqrt{0.518 \cdot 0.159} + 0.317) \doteq 0.529.$$

By Theorem 13, we obtain

$$\alpha_0 \doteq 1/(1+\sqrt{0.193/0.258}) \doteq 0.536,$$
$$\alpha_1 \doteq 1/(1+\sqrt{0.159/0.518}) \doteq 0.643,$$

and $z_1^* \doteq 0.419$. Thus $\alpha \doteq (0.536, 0.643)$ and $Z \doteq (0, .419, 1)$. One may further calculate

$$u_{q \circ m^*} \doteq \frac{1.249^2}{(\sqrt{0.274} + \sqrt{0.529})^2} \doteq 0.997.$$

4 Algorithm

In this section, we summarize the above ideas and results as Algorithm 1 and illustrate how the algorithm works for the cubic curve in Example 2.

Algorithm 1: Optimal_Radical_Transformation

Input: p, a rational parameterization of a plane curve.

Output: r, the optimal piecewise radical transformation of p such that $u_{p\circ r} > u_p$.

1. Compute ω_p and μ_p using (2), u_p using (3) and ω_p'.
2. Solve $\omega_p \omega_p' = 0$ and get T.
3. Compute S, Z, α and u using (9) and (11).
4. Construct φ with T, S and m with S, Z, α using (6) and (7).
5. $r \leftarrow \varphi \circ m$.
6. Return r.

Example 16 (Continued from Example 15) *Given $p = (t, t^3)$, after the above calculation, one may obtain*

$$T \doteq (0, 0.439, 1), \quad S \doteq (0, 0.406, 1), \quad Z \doteq (0, .419, 1), \quad \alpha \doteq (0.536, 0.643).$$

Then one may construct φ with T and S, and m with S, Z and α, and obtain

$$\varphi \doteq \begin{cases} 0.688\sqrt{s} & \text{if} \quad 0.000 \le s \le 0.406; \\ 0.055 + 0.945s & \text{if} \quad 0.406 \le s \le 1.000; \end{cases}$$

$$m \doteq \begin{cases} \dfrac{-0.450z}{0.172z - 0.536} & \text{if} \quad 0.000 \le z \le 0.049; \\ -\dfrac{0.165z + 0.192z}{0.492z - 0.849} & \text{if} \quad 0.419 \le z \le 1.000. \end{cases}$$

Then the optimal transformation r is constructed below.

$$r = \varphi \circ m \doteq \begin{cases} 0.462\sqrt{-\dfrac{z}{0.172z - 0.536}} & \text{if} \quad 0.000 \le z \le 0.419; \\ \dfrac{-0.129z - 0.228}{0.492z - 0.849} & \text{if} \quad 0.419 \le z \le 1.000. \end{cases}$$

With the optimal radical transformation r, one may construct $p \circ r$ and obtain

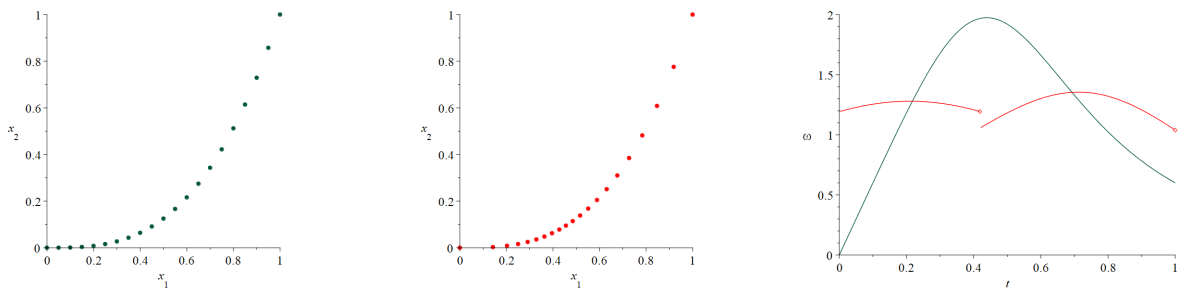
$$p \circ r \doteq \begin{cases} \left(\dfrac{-0.079\sqrt{z}(z - 3.116)}{(-0.172z + 0.5359)^{3/2}}, \dfrac{0.098z^{3/2}}{(-0.172z + 0.536)^{3/2}} \right) & \text{if} \quad 0.000 \le z \le 0.419; \\ \left(\dfrac{-0.129z - 0.228}{0.492z - 0.849}, -\dfrac{0.002\,(z + 1.771)^3}{(0.492z - 0.849)^3} \right) & \text{if} \quad 0.419 \le z \le 1.000. \end{cases}$$

The angular speed function of $p \circ r$ is

$$w_{p\circ r} \doteq \begin{cases} \dfrac{0.781}{z^2 - 0.420z + 0.655} & \text{if} \quad 0.000 \le z \le 0.4187; \\ \dfrac{-1.379\,(z + 1.771)(z - 1.725)}{(z^2 - 1.456z + 0.899)(z^2 - 4.876z + 9.888)} & \text{if} \quad 0.4187 \le z \le 1.000. \end{cases}$$

Furthermore, one may calculate its uniformity $u_{p\circ r} \doteq 0.997$.

The plots of p *and* $p \circ r$ *as well as the behavior of their angular speed functions are shown below:*



where

- *the left plot shows the equi-sampling of the original parameterization* p *(green);*
- *the middle plot shows the equi-sampling of the optimal piecewise radical reparameterization* $p \circ r$ *(red);*
- *the right plot shows the angular speed functions of* p *and* $p \circ r$.

It is seen that the angular speed uniformity is greatly improved by the piecewise radical reparameterization.

5 Implementational Issues/Suggestions

If one chooses to implement the proposed algorithm using floating-point arithmetic, then, as usual, one should be careful to avoid numerical instability.

For instance, if L_i is computed by (10) naively, then it leads to instability. For example, $t_i = 1/\sqrt{2}$ is a zero of ω_p with multiplicity 1, so

$$\omega_p(t) = |t - 1/\sqrt{2}| \cdot m(t),$$

where $m(1/\sqrt{2}) \neq 0$. The numeric solution over $[0,1]$ is $t_i = 0.707$. Thus

$$L_i = \Delta t_i \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_i+1)\tilde{t}^{\mu_i}}\, dt = \Delta t_i^2 \int_{t_i}^{t_{i+1}} \frac{(t-1/\sqrt{2})^2 \cdot m^2(t)}{2\,(t-0.707)}\, dt.$$

During integration, it is necessary to evaluate the integral at $t = 0.707$. When t approaches 0.707, the integral quickly increases to $+\infty$, causing numerical instability.

To avoid such cases, one could adopt a technique from symbolic computation to represent algebraic numbers. Suppose that $t = \gamma$ is a zero of $\omega_p(t)$ with multiplicity μ_i and t_i is its numerical approximation. By (2), $\omega_p^2(t)$ is a rational function. Let G and H be its numerator and denominator. Then $t = \gamma$ is a zero of G with multiplicity $2\,\mu_i$. Carrying out the Euclidean division with G as the dividend and $(t-\gamma)^{2\,\mu_i}$ as the divisor, we obtain

$$G(t) = (t-\gamma)^{2\,\mu_i} Q(t,\gamma) + R(t,\gamma).$$

Since $t = \gamma$ is a zero of G with multiplicity $2\,\mu_i$, it is also a zero of $R(t,\gamma)$ with multiplicity at least $2\,\mu_i$. Given that $\deg(R,t) < 2\,\mu_i$, $R(t,\gamma)$ must be zero, which leads to the following conclusion:

$$L_i = \Delta t_i \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_i+1)\,\tilde{t}^{\mu_i}}\, dt \doteq \frac{\Delta t_i^{\mu_i+1}}{\mu_i+1} \cdot \int_{t_i}^{t_{i+1}} \frac{Q(t,t_i)(t-t_i)^{\mu_i}}{H(t)}\, dt.$$

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