The Algebraic Weak Factorisation System for Delta Lenses

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Delta lenses are functors equipped with a suitable choice of lifts, and are used to model bidirectional transformations between systems. In this paper, we construct an algebraic weak factorisation system whose R-algebras are delta lenses. Our approach extends a semi-monad for delta lenses previously introduced by Johnson and Rosebrugh, and generalises to any suitable category equipped with an orthogonal factorisation system and an idempotent comonad. We demonstrate how the framework of an algebraic weak factorisation system provides a natural setting for understanding the lifting operation of a delta lens, and also present an explicit description of the free delta lens on a functor.

1 Introduction

Delta lenses were first introduced by Diskin, Xiong, and Czarnecki [17] as an algebraic framework for *bidirectional transformations* [1, 13] between systems, particularly in the context of *model-driven engineering* [16, 28]. The original motivation behind delta lenses came from adapting the classical notion of a lens [19] from a "state-based" setting to a "delta-based" setting. Instead of treating a system as a mere *set* of states, it should be regarded as a *category*, whose objects are the states of the system and whose morphisms are the updates (or deltas) between them. The purpose of delta lenses is to model the notion of *synchronisation* between systems through specifying how certain updates between states are propagated.

A delta lens is a functor $f\colon A \to B$ equipped with a *lifting operation*, see (1), that satisfies certain axioms. The lifting operation specifies, for each object a in A and for each morphism $u\colon fa \to b$ in B, a morphism $\varphi(a,u)\colon a \to a'$, often called the *chosen lift*, such that $f\varphi(a,u) = u$. The axioms placed on the lifting operation ensure that it respects identities and composition. Thus a delta lens is a functor equipped with additional *algebraic structure*, and it is natural to wonder if delta lenses arise as algebras for a monad. In this paper, we provide an answer in the affirmative.

$$\begin{array}{ccc} \{0\} & \xrightarrow{\quad a \quad} & A \\ \downarrow & \overset{\varphi(a,u)}{\dashrightarrow} & \downarrow f \\ \{0 \to 1\} & \xrightarrow{\quad u \quad} & B \end{array} \tag{1}$$

The question of asking whether certain kinds of lenses are algebras for a monad is not new. Classical state-based lenses [19] were characterised by Johnson, Rosebrugh, and Wood [26] as algebras for a monad on the slice category $\mathcal{S}\mathrm{et}/B$. The same authors later introduced the notion of a *c-lens* [25], better known as a *split opfibration*, and characterised them as algebras for a monad, first introduced by Street [29], on the slice category $\mathcal{C}\mathrm{at}/B$. Delta lenses generalise state-based lenses and split opfibrations [24], however they were only shown by Johnson and Rosebrugh [23] to be certain algebras for a *semi-monad* (a monad without a unit) on $\mathcal{C}\mathrm{at}/B$. One of the contributions of the current paper is resolve this gap in the literature.

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Although it is generally useful to know when a mathematical structure arises as an algebra for a monad, in isolation this result provides limited benefit towards a deeper understanding of lenses. One reason is that we wish to study lenses as *morphisms* of a category, rather than *objects* in a category of algebras. The knowledge that lenses are morphisms with algebraic structure does not provide any information of how to sequentially *compose* them, nor justification for why this algebraic structure encodes a notion of *lifting*.

Cofunctors[1] are a natural kind of morphism between categories [2, 22] which fundamentally involve a lifting operation and admit a straightforward sequential composition. The characterisation of delta lenses as a compatible functor and cofunctor [3, 7], together with related characterisations of state-based lenses and split opfibrations [8], provides a clear understanding of their composition and lifting, and has led to several fruitful developments in the study of lenses in applied category theory [6, 10, 14]. However the question remains: why do lenses frequently arise as algebras for a monad?

An *algebraic weak factorisation system* [4], also known as a *natural weak factorisation system* [21], generalises the notion of an orthogonal factorisation system (OFS) on a category. An algebraic weak factorisation system (AWFS) on a category $\mathcal{C}$ consists of a comonad (L, ε, Δ) and a monad (R, η, μ) on $\mathcal{C}^{\mathbf{2}}$ that are suitably compatible. The categories of *L-coalgebras* and *R-algebras* of an AWFS (L,R) replace the usual *left* and *right* classes of morphisms of an OFS. In particular, every morphism factors into a cofree L-coalgebra followed by free R-algebra, and every lifting problem (2), where (f, p) is a L-coalgebra and (g,q) is a R-algebra, admits a chosen lift $\varphi_{f,g}\langle h,k\rangle$ making the diagram commute. Crucially, these chosen lifts also induce a canonical composition of R-algebras [4, Section 2.8]. Both classical state-based lenses and split opfibrations arise as R-algebras for an AWFS on $\mathcal{S}\mathrm{et}$ and $\mathcal{C}\mathrm{at}$, respectively.

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & C \\ {\scriptstyle (f,p)}\downarrow & \overset{\varphi_{f,g}\langle h,k\rangle}{\nearrow} & \downarrow{\scriptstyle (g,q)} \\ B & \xrightarrow[\;k\;]{} & D \end{array} \tag{2}$$

The main contribution of this paper is to construct an algebraic weak factorisation system (L,R) on $\mathcal{C}\mathrm{at}$ whose R-algebras are precisely delta lenses. The principal benefit is a new framework for understanding lenses as algebras for a monad that naturally incorporates the fundamental aspects of composition and lifting. In addition, we are able to generalise the notion of delta lens to any suitable category equipped with an orthogonal factorisation system and idempotent comonad, as well as present an explicit description of the free delta lens on the functor. This approach to lenses as algebras for a monad also highlights an interesting duality with their recent characterisation as coalgebras for a comonad [9].

Overview of the paper. In Section 2 we review the necessary background material on delta lenses and factorisation systems. In particular, we recall two important structures on $\mathcal{C}\mathrm{at}$, the *comprehensive factorisation system* (Example 6) and the *discrete category comonad* (Example 12), which are generalised in our main constructions to an orthogonal factorisation system and an idempotent comonad, respectively. In Section 3 we utilise these structures on a category $\mathcal{C}$ to build a semi-monad on $\mathcal{C}^{\mathbf{2}}$ (Proposition 13), and show that when $\mathcal{C} = \mathcal{C}\mathrm{at}$ (Example 15), we recover delta lenses as certain algebras for this semi-monad (Theorem 17 and Appendix A). In Section 4 we enhance this construction to a monad (Theorem 19) using pushouts in $\mathcal{C}$, and prove that when $\mathcal{C} = \mathcal{C}\mathrm{at}$, the algebras for this monad are delta lenses (Theorem 23). We also describe the free delta lens on a functor (Example 27). Section 5 completes the construction of an algebraic weak factorisation system on $\mathcal{C}$ (Theorem 29) and shows how delta lenses lift against the L-coalgebras when $\mathcal{C} = \mathcal{C}\mathrm{at}$. Section 6 presents some concluding remarks and avenues for future work.

[1] The term *retrofunctor* proposed by Di Meglio [14] is preferred, but not yet in widespread use.

2 Background

2.1 Delta lenses

We introduce the category $\mathcal{L}\mathrm{ens}$ whose objects are delta lenses, which we will later show is the category of algebras for a monad on $\mathcal{C}\mathrm{at}^{\mathbf{2}}$. For further details and examples, we refer the reader to [11, Chapter 2].

Definition 1. A *delta lens* $(f, \varphi)\colon A \to B$ consists of a functor $f\colon A \to B$ together with a *lifting operation*

$$(a \in A, u\colon fa \to b \in B) \qquad \longmapsto \qquad \varphi(a,u)\colon a \to p(a,u),$$

where $p(a,u) = \mathrm{cod}(\varphi(a,u))$, that satisfies the following three axioms:

(L1) $f\varphi(a,u) = u$

(L2) $\varphi(a, 1_{fa}) = 1_a$

(L3) $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$

Example 2. A *discrete opfibration* is a functor $f\colon A \to B$ such that for each pair $(a \in A, u\colon fa \to b \in B)$ there is a unique morphism $\bar{f}(a,u)\colon a \to a'$ in A for which $f\bar{f}(a,u) = u$. Thus each discrete opfibration f admits a unique lifting operation $\bar{f}$ such that the pair $(f, \bar{f})$ is a delta lens. Conversely, the underlying functor f of a delta lens (f, φ) is a discrete opfibration if $\varphi(a, fw) = w$ for all morphisms $w\colon a \to a'$ in A.

Definition 3. Let $\mathcal{L}\mathrm{ens}$ denote the category whose objects are delta lenses and whose morphisms $\langle h, k\rangle$ from (f, φ) to (g, ψ) consist of a pair of functors h and k such that $k \circ f = g \circ h$ and $h\varphi(a,u) = \psi(ha, ku)$.

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle (f,\varphi)}\downarrow & & \downarrow{\scriptstyle (g,\psi)} \\ B & \xrightarrow[k]{} & D \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} \{0\} & \xrightarrow{a} & A & \xrightarrow{h} & C \\ \downarrow & {\scriptstyle \varphi(a,u)} \nearrow & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \{0 \to 1\} & \xrightarrow{u} & B & \xrightarrow{k} & D \end{array} \quad = \quad \begin{array}{ccc} \{0\} & \xrightarrow{ha} & C \\ \downarrow & {\scriptstyle \psi(ha,ku)} \nearrow & \downarrow{\scriptstyle g} \\ \{0 \to 1\} & \xrightarrow{ku} & D \end{array}$$

Let $U\colon \mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{at}^{\mathbf{2}}$ denote the canonical forgetful functor that sends (f, φ) to f.

2.2 Factorisation systems

We recall two related notions of factorisation system on a category: *orthogonal factorisations systems* [20] and *algebraic weak factorisation systems* [4, 21]. For a full account, we refer the reader to [4].

Definition 4. An *orthogonal factorisation system* $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$ consists of two classes of morphisms $\mathcal{E}$ and $\mathcal{M}$, both containing the isomorphisms and closed under composition, such that:

(i) **Factorisation**: Every morphism f of $\mathcal{C}$ admits a factorisation $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$;

(ii) **Orthogonality**: For each solid commutative square in $\mathcal{C}$ below such that $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique morphism h such that $f = h \circ e$ and $g = m \circ h$.

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle e}\downarrow & \exists! \ \nearrow {\scriptstyle h} & \downarrow{\scriptstyle m} \\ B & \xrightarrow[g]{} & D \end{array}$$

Notation 5. As an aid when diagram-chasing, the morphisms in the left class $\mathcal{E}$ and the right class $\mathcal{M}$ of an orthogonal factorisation system on $\mathcal{C}$ will be decorated in the remainder of the paper as follows.

$$\bullet \xrightarrow{e \in \mathcal{E}} \!\!\!\!\!\rightarrow \bullet \qquad\qquad \bullet \rightarrowtail^{m \in \mathcal{M}} \bullet$$

Example 6. A functor $f\colon A \to B$ is called *initial* if, for each object $b \in B$, the comma category f/b is connected. The *comprehensive factorisation system* [31] is an orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}\mathrm{at}$ in which $\mathcal{E}$ is the class of initial functors and $\mathcal{M}$ is the class of discrete opfibrations.

Lemma 7. *If $(\mathcal{E}, \mathcal{M})$ be an orthogonal factorisation system on $\mathcal{C}$, then the following properties hold:*

(1) The class $\mathcal{E}$ is stable under pushouts in $\mathcal{C}$.

(2) If $g \circ f$ and f are in $\mathcal{E}$, then g is in $\mathcal{E}$. Dually, if $g \circ f$ and g are in $\mathcal{M}$, then f is in $\mathcal{M}$.

Definition 8. A *functorial factorisation* (L, E, R) on a category $\mathcal{C}$ is a section $(L, E, R)\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ to the composition functor $\mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$. The factorisation of a morphism in $\mathcal{C}^{\mathbf{2}}$ is denoted as follows.

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & D
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle Lg} \\
Ef & \xrightarrow{E\langle h,k\rangle} & Eg \\
{\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle Rg} \\
B & \xrightarrow[k]{} & D
\end{array}
\quad (f = Rf \circ Lf,\ g = Rg \circ Lg)
$$

Remark 9. Each functorial factorisation (L, E, R) on $\mathcal{C}$ induces a copointed endofunctor (L, ε) and a pointed endofunctor (R, η) on $\mathcal{C}^{\mathbf{2}}$, where the components of $\varepsilon\colon L \Rightarrow 1$ and $\eta\colon 1 \Rightarrow R$ at f are given below.

$$
\begin{array}{ccc}
A & = & A \\
{\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle f} \\
Ef & \xrightarrow[Rf]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{Lf} & Ef \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\
B & = & B
\end{array}
\tag{3}
$$

Definition 10. [4, Section 2.2] An *algebraic weak factorisation system* (L, R) on a category $\mathcal{C}$ consists of:

(i) A functorial factorisation (L, E, R) on $\mathcal{C}$;

(ii) An extension of (L, ε) to a comonad (L, ε, Δ) on $\mathcal{C}^{\mathbf{2}}$;

(iii) An extension of (R, η) to a monad (R, η, μ) on $\mathcal{C}^{\mathbf{2}}$;

(iv) A distributive law $\lambda\colon LR \Rightarrow RL$ of the comonad L over the monad R with $\lambda_f = \langle \Delta_f, \mu_f \rangle$.

2.3 Idempotent comonads and weak equivalences

Given an idempotent comonad (M, ι) on a category $\mathcal{C}$, let $\mathcal{W} = \{f \in \mathcal{C} \mid Mf \text{ is invertible}\}$ denote the class of morphisms in $\mathcal{C}$ whose members are called *weak equivalences*. This class satisfies the *2-out-of-3 property*, and contains the isomorphisms, thus making $\mathcal{C}$ a *category with weak equivalences* [18]. Since the comonad M is idempotent, each counit component ι_A is inverted by M and therefore a morphism of $\mathcal{W}$. If M preserves pushouts, the morphisms in $\mathcal{W}$ are stable under pushout along morphisms in $\mathcal{C}$.

Notation 11. As a visual aid when diagram-chasing, the morphisms in the class $\mathcal{W}$ of weak equivalences of a category $\mathcal{C}$ will be decorated in the remainder of the paper as follows.

$$\bullet \xrightarrow[\sim]{w \in \mathcal{W}} \bullet$$

Example 12. Let $(-)_0\colon \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ denote the idempotent comonad that assigns a category A to its corresponding *discrete category* A_0 with counit component $\iota_A\colon A_0 \to A$. The endofunctor $(-)_0$ has a right adjoint (the *codiscrete category* monad) and therefore preserves all colimits. A functor $f\colon A \to B$ is called *bijective-on-objects* if f_0 is invertible; these are the weak equivalences with respect to $(-)_0$.

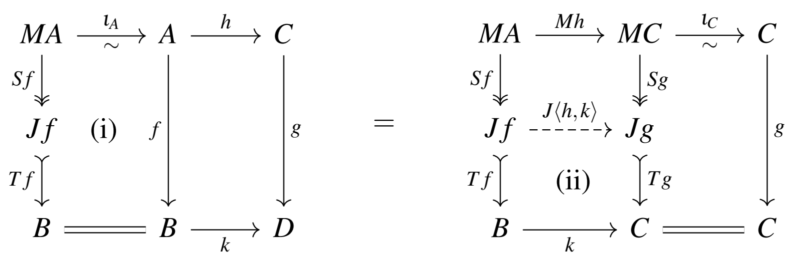
3 Delta lenses as certain algebras for a semi-monad

Throughout this section, let $(\mathcal{E},\mathcal{M})$ be an orthogonal factorisation system on a category $\mathcal{C}$, and let (M,ι) be an idempotent comonad on $\mathcal{C}$ with corresponding class $\mathcal{W}$ of weak equivalences.

3.1 Constructing a semi-monad for delta lenses

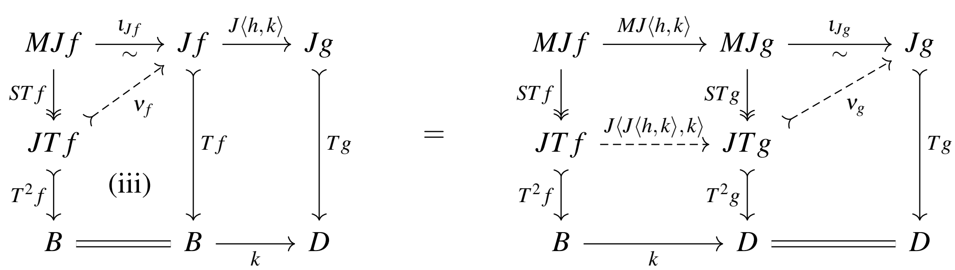
We now construct a semi-monad (T,ν) on the category $\mathcal{C}^{\mathbf{2}}$, for a category $\mathcal{C}$ equipped with an idempotent comonad (M,ι) and an orthogonal factorisation system $(\mathcal{E},\mathcal{M})$. We show that when $\mathcal{C} = \mathcal{C}\mathrm{at}$ equipped with the discrete category comonad and the comprehensive factorisation system, this specialises to the semi-monad defined on $\mathcal{C}\mathrm{at}^{\mathbf{2}}$ by Johnson and Rosebrugh [23, Section 6].

We begin by constructing an endofunctor $T\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$. Given a morphism $f\colon A \to B$ in $\mathcal{C}$, we first pre-compose with the counit component $\iota_A\colon MA \to A$ and then choose an $(\mathcal{E},\mathcal{M})$-factorisation of the resulting morphism as depicted in commutative square (i) below; this defines the action of T on objects in $\mathcal{C}^{\mathbf{2}}$. Given a morphism $\langle h,k\rangle\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$, there exists a unique morphism $J\langle h,k\rangle\colon Jf \to Jg$ in $\mathcal{C}$ by applying the orthogonality property; the action of T on the morphism $\langle h,k\rangle$ is given by the commutative square (ii) depicted below. Note that the equation (4) holds by naturality of $\iota\colon M \Rightarrow 1$ at the morphism h.



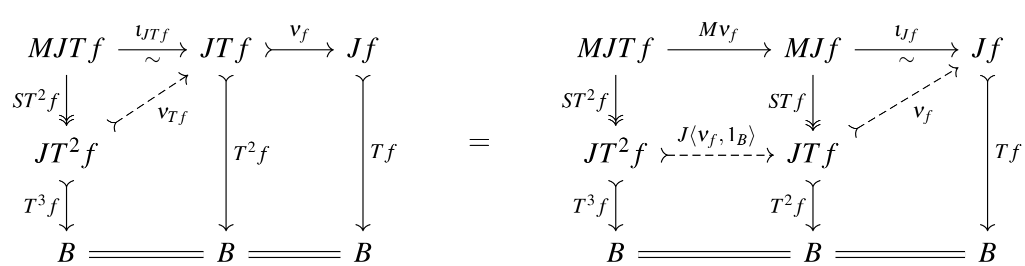
(4)

Applying the functor T to the morphism $Tf\colon Jf \to B$ and using the orthogonality property, we obtain the component ν_f of the multiplication $\nu\colon T^2 \Rightarrow T$ at f as depicted in the commutative square (iii) below. Naturality of ν at follows from noticing in (5) that $J\langle h,k\rangle \circ \nu_f = \nu_g \circ J\langle J\langle h,k\rangle,k\rangle$ by orthogonality.



(5)

The associative law for ν follows from observing in (6) that $\nu_f \circ \nu_{Tf} = \nu_f \circ J\langle \nu_f, 1_B\rangle$ by orthogonality.



(6)

We have thus constructed an endofunctor $T\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ with an associative multiplication $\nu\colon T^2 \Rightarrow T$.

Proposition 13. *The pair (T,ν) is a semi-monad on $\mathcal{C}^2$.*
Corollary 14. *The semi-monad (T,ν) on $\mathcal{C}^{\mathbf{2}}$ restricts to a semi-monad in the 2-category $\mathcal{C}\mathrm{AT}/\mathcal{C}$ on the codomain functor* $\mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$. *In particular, (T,ν) induces a semi-monad on each slice category $\mathcal{C}/B$.*
Example 15. Consider the category $\mathcal{C}\mathrm{at}$ equipped with the comprehensive factorisation system and the discrete category comonad. Given a functor $f\colon A \to B$, the category Jf defined in (4) is given by the coproduct $\sum_{a\in A_0} fa/B$ of the coslice categories indexed by the discrete category A_0. The objects in Jf are pairs $(a \in A, u\colon fa \to b \in B)$, while morphisms $\langle 1_a, v\rangle\colon (a,u_1) \to (a,u_2)$ are given by morphisms $v \in B$ such that $u_2 = v \circ u_1$. The functor $Sf\colon A_0 \twoheadrightarrow J_f$ has an assignment on objects $a \mapsto (a, 1_{fa})$, and is an *initial functor* since each slice category $Sf/(a,u)$ is isomorphic to the terminal category and hence connected. The functor $Tf\colon Jf \rightarrowtail B$ is given by the codomain projection with assignment on objects $(a,u) \mapsto \mathrm{cod}(u)$, and is a *discrete opfibration*. In this setting, restricting the semi-monad (T,ν) to the slice categories $\mathcal{C}\mathrm{at}/B$ coincides with semi-monad for delta lenses defined by Johnson and Rosebrugh [23].

3.2 Delta lenses as certain semi-monad algebras

An *algebra* (f,p) for the semi-monad (T,ν) on the codomain functor $\mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ (or, equivalently, on the slice category $\mathcal{C}/B$) consists of a pair of morphisms $f\colon A \to B$ and $p\colon Jf \to A$ such that the following diagrams commute.

$$\begin{array}{ccc} Jf & \xrightarrow{\;p\;} & A \\ {\scriptstyle Tf}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & = & B \end{array} \qquad\qquad \begin{array}{ccc} JTf & \xrightarrow{\;J\langle p, 1_B\rangle\;} & Jf \\ {\scriptstyle \nu_f}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Jf & \xrightarrow[\;p\;]{} & A \end{array} \tag{7}$$

Johnson and Rosebrugh (JR) introduced an additional condition on the algebras for the semi-monad (T,ν) on $\mathcal{C}\mathrm{at}/B$ which we now adapt to our more general setting under the name *JR-algebra*. The intuition is that this additional condition replaces the missing "unit law" that an algebra for a monad would satisfy.
Definition 16. A *JR-algebra* is an algebra (f,p) for the semi-monad (T,ν) on the codomain functor $\mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ such that the following diagram commutes.

$$\begin{array}{ccc} MA & \xrightarrow[\sim]{\;\iota_A\;} & A \\ {\scriptstyle Sf}\big\downarrow & \nearrow_{p} & \\ Jf & & \end{array} \tag{8}$$

A *morphism* $\langle h,k\rangle\colon (f,p) \to (g,q)$ of algebras for the semi-monad (T,ν) consists of a pair of morphisms h and k such that the following equation in $\mathcal{C}^{\mathbf{2}}$ holds.

$$\begin{array}{ccccc} Jf & \xrightarrow{\;p\;} & A & \xrightarrow{\;h\;} & C \\ {\scriptstyle Tf}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ B & = & B & \xrightarrow[\;k\;]{} & D \end{array} \quad = \quad \begin{array}{ccccc} Jf & \xrightarrow{\;J\langle h,k\rangle\;} & Jg & \xrightarrow{\;q\;} & C \\ {\scriptstyle Tf}\big\downarrow & & \big\downarrow{\scriptstyle Tg} & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[\;k\;]{} & D & = & D \end{array} \tag{9}$$

Let $\mathcal{A}\mathrm{lg}(T,\nu)$ denote the category of algebras for the semi-monad (T,ν) on the codomain functor $\mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$, and let $\mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$ denote the full subcategory of JR-algebras.
Theorem 17. *If $\mathcal{C} = \mathcal{C}\mathrm{at}$ equipped with the discrete category comonad and the comprehensive factorisation system, then there is an isomorphism of categories $\mathcal{L}\mathrm{ens} \cong \mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$.*

Proof. This result is due to Johnson and Rosebrugh [23]. See Appendix A for a proof in our notation. □

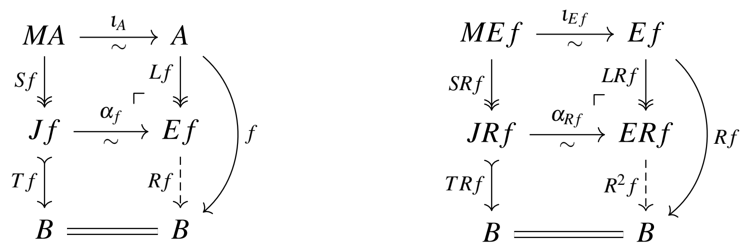
4 Delta lenses as algebras for a monad

Throughout this section, let $(\mathcal{E},\mathcal{M})$ be an orthogonal factorisation system on a category $\mathcal{C}$ with (chosen) pushouts, and let (M,ι) be an idempotent comonad on $\mathcal{C}$ such that $M\colon \mathcal{C}\to\mathcal{C}$ preserves pushouts.

4.1 Constructing a monad for delta lenses

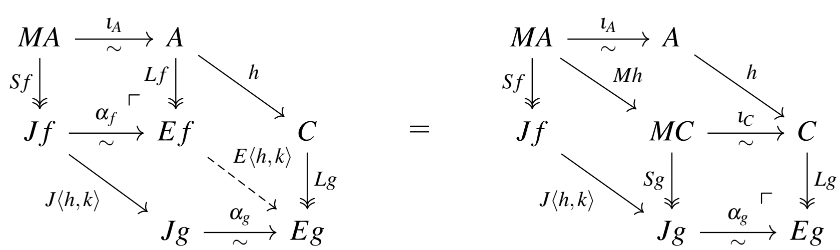
We now extend the semi-monad (T,ν) to a monad (R,η,μ) on $\mathcal{C}^{\mathbf{2}}$, for a category $\mathcal{C}$ as described above. Our approach is to utilise the universal properties of pushouts and orthogonal factorisation systems, as well as properties of the class of weak equivalences for the idempotent comonad, to construct the necessary data for the monad from that of the semi-monad (T,ν).

We begin by constructing an endofunctor $R\colon \mathcal{C}^{\mathbf{2}}\to\mathcal{C}^{\mathbf{2}}$. Given a morphism $f\colon A\to B$ in $\mathcal{C}$, first construct the pushout of ι_A along Sf from (4), and then use the universal property of the pushout to define $Rf\colon Ef\to B$ as depicted on the left below; this defines the action of R on objects in $\mathcal{C}^{\mathbf{2}}$.



(10)

Given a morphism $\langle h,k\rangle\colon f\to g$ in $\mathcal{C}^{\mathbf{2}}$, there exists a unique morphism $E\langle h,k\rangle\colon Jf\to Jg$ in $\mathcal{C}$, as depicted below, by the universal property of the pushout, where $J\langle h,k\rangle$ is defined in (4). It is not difficult to show through diagram-chasing that $Rg\circ E\langle h,k\rangle = k\circ Rf$, thus defining the action of R on morphisms of $\mathcal{C}^{\mathbf{2}}$.

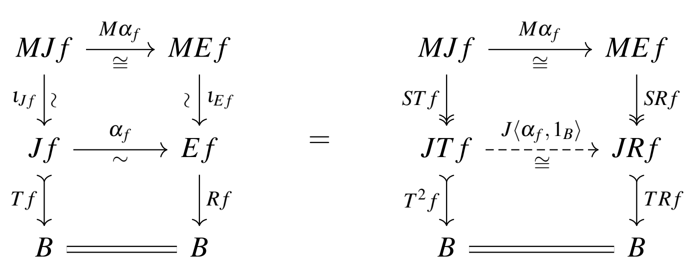


(11)

Lemma 18. *The triple* (L,E,R) *constructed in* (10) *and* (11) *is functorial factorisation on* $\mathcal{C}$.

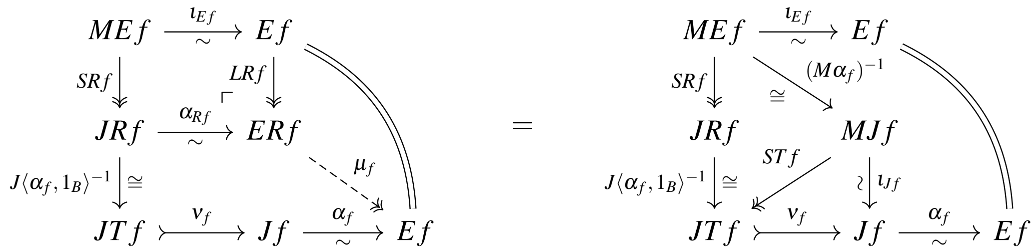
By Remark 9, this functorial factorisation induces a pointed endofunctor (R,η) on $\mathcal{C}^{\mathbf{2}}$ where the component of η at f is given by the morphism $Lf\colon A\to Ef$ as depicted in (3). To extend this pointed endofunctor to a monad, all that remains is to define a suitable multiplication $\mu\colon R^2\Rightarrow R$.

To construct this multiplication, we first observe that the morphism $\alpha_f\colon Jf\to Ef$ constructed in (10) is a weak equivalence, and therefore the morphism $M\alpha_f\colon MJf\to MEf$ is invertible. It follows from the orthogonality property that the morphism $J\langle\alpha_f,1_B\rangle\colon JTf\to JRf$ is invertible as depicted below.



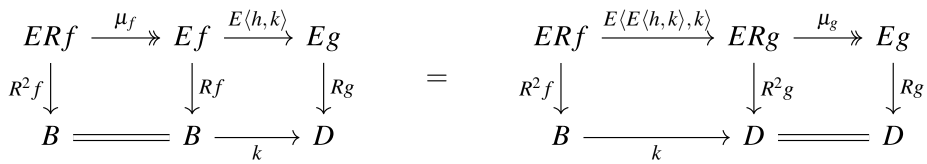
(12)

Using the universal property of the pushout, the morphism ν_f defined in (5), and the morphism $J\langle \alpha_f, 1_B\rangle^{-1}$ defined in (12), we obtain the component μ_f of the multiplication $\mu\colon R^2 \Rightarrow R$ at f as depicted below.

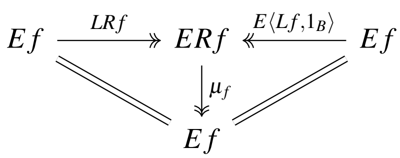


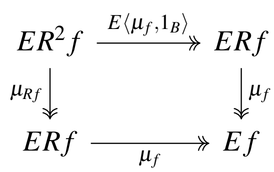
(13)

A tedious, yet routine, exercise in diagram-chasing using the morphisms defined in (11) and (13), and applying the universal property of the pushout shows that $Rf \circ \mu_f = R^2 f$ and that μ is natural as depicted below.



Showing that the diagrams below commute, and thus establishing that the multiplication μ is unital and associative, is also a straightforward application of definitions and the universal property of the pushout.





Theorem 19. *The triple* (R, η, μ) *is a monad on* $\mathcal{C}^{\mathbf{2}}$.

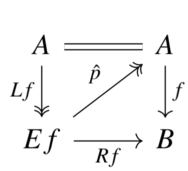
Corollary 20. *The monad* (R, η, μ) *on* $\mathcal{C}^{\mathbf{2}}$ *restricts to a monad in the* 2*-category* $\mathcal{C}\mathrm{AT}/\mathcal{C}$ *on the codomain functor* $\mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$. *In particular,* (R, η, μ) *induces a monad on each slice category* $\mathcal{C}/B$.

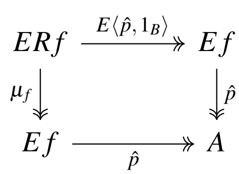
Remark 21. The morphisms α_f defined as pushout injections in (10) assemble into a natural transformation $\alpha\colon T \Rightarrow R$ which underlies a morphism of semi-monads $(T, \nu) \to (R, \mu)$. We conjecture that (R, η, μ) is actually the *free monad* on the semi-monad (T, ν), in a suitable sense, however leave this for future work.

4.2 Delta lenses as monad algebras

We now construct the algebras for the monad (R, η, μ) on $\mathcal{C}^{\mathbf{2}}$ and show they are the same as JR-algebras for the semi-monad (T, ν). When $\mathcal{C} = \mathcal{C}\mathrm{at}$ equipped with the comprehensive factorisation system and the discrete category comonad, this result establishes that delta lenses are algebras for the monad (R, η, μ).

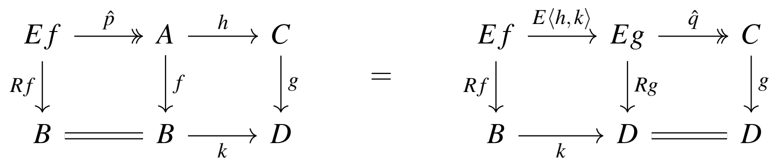
An *algebra* $(f, \hat{p})$ for the monad (R, η, μ) on $\mathcal{C}^{\mathbf{2}}$ consists of a pair of morphisms $f\colon A \to B$ and $\hat{p}\colon Ef \to A$ such that the following diagrams commute:





(14)

A *morphism* $\langle h,k\rangle : (f,\hat{p}) \to (g,\hat{q})$ of algebras for the monad (R,η,μ) consists of a pair of morphisms h and k such that the following equation in $\mathcal{C}^{\mathbf{2}}$ holds.

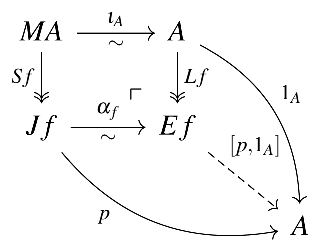


(15)

Let $\mathcal{A}\mathrm{lg}(R,\eta,\mu)$ denote the category of algebras for the monad (R,η,μ).

Proposition 22. *There is an isomorphism of categories* $\mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu) \cong \mathcal{A}\mathrm{lg}(R,\eta,\mu)$.

Proof. Let $(f : A \to B, p : Jf \to A)$ be a JR-algebra for the semi-monad (T,ν). Using the diagram (8) and the universal property of the pushout, we obtain a morphism $[p,1_A] : Ef \to A$ as depicted below.



Using the universal property of the pushout and the axioms for the JR-algebra (f,p), it is straightforward to prove that the pair $(f,[p,1_A])$ is an algebra for the monad (R,η,μ).

Now consider an algebra $(f : A \to B, \hat{p} : Ef \to A)$ for the monad (R,η,μ). Pre-composing the structure map of the algebra with α_f we obtain a morphism $\hat{p}\circ\alpha_f : Jf \to B$. Using the axioms for the algebra $(f,\hat{p})$ and appropriate pasting of commutative diagrams, one may easily show that the pair $(f,\hat{p}\circ\alpha_f)$ is a JR-algebra for the semi-monad (T,ν).

The JR-algebras for (T,ν) and the algebra for (R,η,μ) are in bijective correspondence with each other, since $[\hat{p}\circ\alpha_f, 1_A] = \hat{p}$ by the universal property of the pushout, and $[p,1_A]\circ\alpha_f = p$ by construction. One may extend this correspondence to the morphisms (9) and (15) of the respective categories and show it is functorial, thus establishing the stated isomorphism of categories. □

The following theorem establishes a key result of the paper: delta lenses are algebras for a monad.

Theorem 23. *There is an isomorphism of categories* $\mathcal{L}\mathrm{ens} \cong \mathcal{A}\mathrm{lg}(R,\eta,\mu)$.

Proof. Follows directly from Theorem 17 and interpreting Proposition 22 in the setting of $\mathcal{C} = \mathcal{C}\mathrm{at}$ equipped with the discrete category comonad and the comprehensive factorisation system. □

Corollary 24. *The forgetful functor* $U : \mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{at}^{\mathbf{2}}$ *is strictly monadic.*

4.3 The free delta lens on a functor

We now construct a left adjoint to the functor $U : \mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{at}^{\mathbf{2}}$ which defines the *free delta lens* on a functor $f : A \to B$. This amounts to providing an explicit description of the category Ef together with a lifting operation on the functor $Rf : Ef \to B$. First we recall [7, Corollary 20] the following result which represents an delta lens as a certain commutative diagram (see [11, Section 2.4] for a detailed proof).

Proposition 25. *Each delta lens $(f,\varphi)\colon A \to B$ determines a commutative diagram in $\mathcal{C}\mathrm{at}$, as depicted on the left below, such that φ is bijective-on-objects and $f\varphi$ is a discrete opfibration.*

$$
\begin{array}{ccc}
 & \Lambda(f,\varphi) & \\
 {}^{\varphi}\swarrow\!\sim & & \searrow^{f\varphi} \\
A & \xrightarrow[f]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & X & \\
 {}^{g}\swarrow\!\sim & & \searrow^{fg} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

Conversely, each commutative diagram on the right above, where g is bijective-on-objects and fg is a discrete opfibration, uniquely determines a delta lens structure on f.

Remark 26. The above result may be understood as a consequence of an equivalence of double categories [11, Section 3.4], however the details are outside the scope of this paper.

Using Proposition 25, the *free delta lens* on a functor $f\colon A \to B$ corresponds to the following commutative diagram in $\mathcal{C}\mathrm{at}$ constructed in (10). An immediate benefit of this presentation of the free delta lens is that it condenses the three commutative diagrams (14) for the (free) R-algebra to a single diagram.

$$
\begin{array}{ccc}
 & Jf & \\
 {}^{\alpha_f}\swarrow\!\sim & & \searrow^{Tf} \\
Ef & \xrightarrow[Rf]{} & B
\end{array}
$$

In Example 15, we unpacked the definition of the category Jf and the discrete opfibration Tf. We now provide an explicit characterisation of the category Ef and the delta lens structure on $Rf\colon Ef \to B$.

Example 27. The objects of Ef are pairs $(a \in A, u\colon fa \to b \in B)$. The morphisms are generated by pairs $\langle w, fw\rangle\colon (a, 1_{fa}) \to (a', 1_{fa'})$ and $\langle 1_a, v\rangle\colon (a,u) \to (a, v\circ u)$ for $w \in A$ and $v \in B$, respectively, as depicted below. The identity morphisms are well-defined since $f(1_a) = 1_{fa}$. As Jf has the same objects as Ef and consists of morphisms of the form $\langle 1_a, v\rangle$, the functor $\alpha_f\colon Jf \to Ef$ is identity-on-objects and faithful.

$$
\begin{array}{ccc}
a & \xrightarrow{w} & a' \\
 & & \\
fa & \xrightarrow{f(w)} & fa' \\
{\scriptstyle 1_{fa}}\downarrow & & \downarrow{\scriptstyle 1_{fa'}} \\
fa & \xrightarrow{fw} & fa'
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xrightarrow{1_a} & a \\
 & & \\
fa & \xrightarrow{f(1_a)} & fa \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v\circ u} \\
b & \xrightarrow{v} & b'
\end{array}
\tag{16}
$$

The functor $Rf\colon Ef \to B$ is projection in the second component; on the generators this is given by $Rf\langle w, fw\rangle = fw$ and $Rf\langle 1_a, v\rangle = v$. The lifting operation on Rf takes an object (a,u) in Ef and a morphism $v\colon \mathrm{cod}(u) \to b$ in B to the chosen lift $\langle 1_a, v\rangle\colon (a,u) \to (a, v\circ u)$ in Ef.

Although, in principle, the morphisms in Ef are finite sequences of the generators (16), one may show that each morphism $(a_1,u_1) \to (a_2,u_2)$ is actually just one of the following two kinds depicted below: either a retraction v of u_1 followed by morphism $w\colon a_1 \to a_2$, or a morphism $v\colon \mathrm{cod}(u_1) \to \mathrm{cod}(u_2)$ such that $v\circ u_1 = u_2$. The functor Rf sends these morphisms to $u_2 \circ fw \circ v$ and v, respectively.

$$
\begin{array}{ccccccc}
a_1 & = & a_1 & \xrightarrow{w} & a_2 & = & a_2 \\
 & & & & & & \\
fa_1 & = & fa_1 & \xrightarrow{f(w)} & fa_2 & = & fa_2 \\
{\scriptstyle u_1}\downarrow & \circlearrowright & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u_2} \\
b_1 & \xrightarrow{v} & fa_1 & \xrightarrow{fw} & fa_2 & \xrightarrow{u_2} & b_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
a_1 & = & a_2 \\
 & & \\
fa_1 & = & fa_2 \\
{\scriptstyle u_1}\downarrow & \circlearrowright & \downarrow{\scriptstyle u_2} \\
b_1 & \xrightarrow{v} & b_2
\end{array}
$$

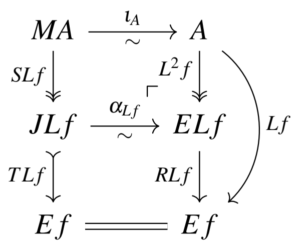
5 Delta lenses as the R-algebras of an algebraic weak factorisation system

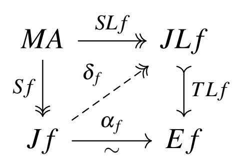
In this section, let $(\mathcal{E},\mathcal{M})$ be an orthogonal factorisation system on a category $\mathcal{C}$ with (chosen) pushouts, and let (M,ι) be an idempotent comonad on $\mathcal{C}$ such that $M\colon \mathcal{C}\to\mathcal{C}$ preserves pushouts.

5.1 Constructing the AWFS for delta lenses

Thus far we have constructed a functorial factorisation (L,E,R) on $\mathcal{C}$ (Lemma 18), and extended the pointed endofunctor (R,η) to a monad (R,η,μ) on $\mathcal{C}^{\mathbf{2}}$ (Theorem 19). We now show that the copointed endofunctor (L,ε) extends to a comonad (L,ε,Δ), therefore completing the data required to describe an algebraic weak factorisation system on $\mathcal{C}$. For $\mathcal{C}=\mathcal{C}at$ equipped with the comprehensive factorisation system and the discrete category monad, this yields an AWFS whose R-algebras are precisely delta lenses.

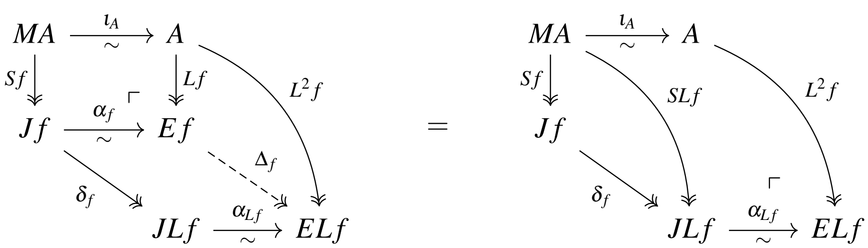
First we construct the morphism $L^2f\colon A\to ELf$ as on the left below. Using this diagram and (10), it follows that $TLf\circ SLf = Lf\circ\iota_A = \alpha_f\circ Sf$ and there is solid commutative diagram as on the right below. By the orthogonality property, there exists a unique morphism $\delta_f\colon Jf\to JLf$ as shown.



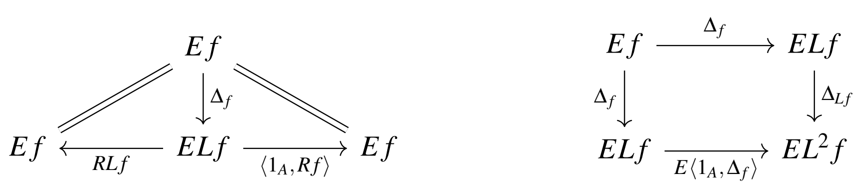


(17)

Using the diagrams (17) and the universal property of the pushout, we obtain the component Δ_f of the comultiplication $\Delta\colon L\Rightarrow L^2$ at f as depicted below. For each morphism $\langle h,k\rangle\colon f\to g$ in $\mathcal{C}^{\mathbf{2}}$, we may show that $\Delta_g\circ E\langle h,k\rangle = E\langle h,E\langle h,k\rangle\rangle\circ\Delta_f$, providing us with a well-defined transformation $\Delta\colon L\Rightarrow L^2$.



Showing that the diagrams below commute, and thus establishing that the comultiplication Δ is counital and coassociative, is a straightforward application of definitions and the universal property of a pushout.



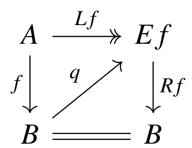
Proposition 28. *The triple (L,ε,Δ) is a comonad on $\mathcal{C}^{\mathbf{2}}$.*

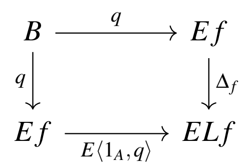
Theorem 29. *The pair (L,R) is an algebraic weak factorisation system on $\mathcal{C}$.*

Proof. The data of the algebraic weak factorisation system follows from Lemma 18, Theorem 19, and Proposition 28. Checking that there is a distributive law $\lambda\colon LR\Rightarrow RL$ of the comonad L over the monad R with components $\lambda_f=\langle\Delta_f,\mu_f\rangle$ involves routine diagram-chasing and applying universal properties. □

5.2 Coalgebras and lifting

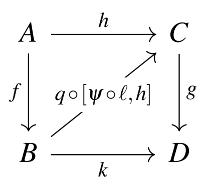
A *coalgebra* (f,q) for the comonad (L,ε,Δ) consists of a pair of morphisms $f\colon A\to B$ and $q\colon B\to Ef$ such that the following diagrams commute:

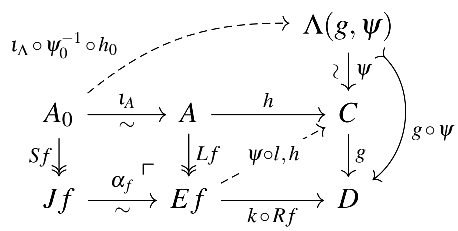




Remark 30. In contrast to the algebras for the monad (R,η,μ), the coalgebras above cannot be easily simplified since q is a morphism *into* a pushout. For $\mathcal{C}=\mathcal{C}\mathrm{at}$, one may show that for a functor f to admit a coalgebra structure, it must be a left-adjoint-right-inverse (LARI) and is therefore also injective-on-objects and fully faithful. A complete characterisation of the L-coalgebras is left for future work.

We now provide a simple diagrammatic proof that delta lenses, in the form of Proposition 25 rather than as R-algebras, lift against L-coalgebras. Consider a morphism $\langle h,k\rangle\colon f\to g$ such that (f,q) is an L-coalgebra and (g,ψ) is a delta lens. Since ψ is bijective-on-objects, ψ_0 is invertible, and there is a morphism $\iota_\Lambda\circ\psi_0^{-1}\circ h_0\colon A_0\to\Lambda(g,\psi)$ making the diagram, depicted below, commute. Then by the orthogonality property, there exists a unique morphism $\ell\colon Jf\to\Lambda(g,\psi)$ such that $\ell\circ Sf=\iota_\Lambda\circ\psi_0^{-1}\circ h_0$ and $g\circ\psi\circ\ell=k\circ Rf\circ\alpha_f$. Finally, by the universal property of the pushout, there exists a unique morphism $[\psi\circ\ell,h]\colon Ef\to C$. Thus, there is a specified morphism $q\circ[\psi\circ\ell,h]\colon B\to C$ as on the left below.





Therefore we have shown that delta lenses lift against functors with the structure of a L-coalgebra, which is stronger than one would expect from their simple axiomatic definition. It also demonstrates how the notion of lifting is intrinsic to delta lenses as the R-algebras of an AWFS. The sequential composition of delta lenses as R-algebras may also be defined from this notion of lifting against L-coalgebras, providing further clarification of this essential operation.

6 Concluding remarks and future work

In this paper, we have shown that delta lenses are algebras for a monad (R,η,μ), and that this monad arises from an algebraic weak factorisation system on $\mathcal{C}\mathrm{at}$. Moreover, we have shown that this AWFS exists on any suitable category equipped with an orthogonal factorisation system and an idempotent comonad which preserves pushouts. These results generalise immediately to *internal lenses* [7, 8] using the internal comprehensive factorisation system [30], however an analogous result for *enriched lenses* [12] or *weighted lenses* [27] is unknown. There are many avenues for future work. One example is the relationship between the *proxy pullbacks* [14] of delta lenses and the canonical pullback of R-algebras [4]. Another is the connection between spans of delta lenses [9] and the categories of weak maps for an

AWFS [5]. The *double category of delta lenses* [11], which is naturally induced by the AWFS, provides a rich setting studying the properties of delta lenses previously considered in a 1-categorical setting [6, 15].

Acknowledgements. This research first appeared in Chapter 6 of my PhD thesis [11], and I would like to thank my supervisor, Michael Johnson, for his helpful feedback during my PhD studies. I am also very grateful to Richard Garner who first sketched the construction of the free delta lens, and also suggested the approach using AWFS, after I presented this work at the Australian Category Seminar. I would also like to thank John Bourke, Matthew Di Meglio, Tim Hosgood, and Noam Zeilberger for sharing useful insights which contributed to the development of this research.

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A Appendix

In this section, we provide a proof of Theorem 17. The correspondence between JR-algebras and delta lenses was first shown by Johnson and Rosebrugh [23, Proposition 3]; we reprove this correspondence in our notation, and extend it to an isomorphism of categories. We refer the reader to Example 15 for an explicit description of the category Jf and the functor $Tf\colon Jf \to B$.

Theorem 31. *If $\mathcal{C} = \mathcal{C}\mathrm{at}$ equipped with the discrete category comonad and the comprehensive factorisation system, then there is an isomorphism of categories $\mathcal{L}\mathrm{ens} \cong \mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$.*

We prove this theorem in two parts: first defining the functor $\mathcal{L}\mathrm{ens} \to \mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$, then defining the functor $\mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu) \to \mathcal{L}\mathrm{ens}$ and showing that they are mutually inverse.

Proof. We begin by constructing a functor $\mathcal{L}\mathrm{ens} \to \mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$.

Given a delta lens $(f,\varphi)\colon A \to B$ as in Definition 1, we define a functor $p\colon Jf \to B$ whose assignment on morphisms $\langle 1_a, v\rangle\colon (a,u_1) \to (a,u_2)$ is given below, where we recall that $p(a,u) = \mathrm{cod}(\varphi(a,u))$.

$$\begin{array}{ccc} a & = & a \\ & & \\ fa & = & fa \\ {\scriptstyle u_1}\downarrow & \circlearrowright & \downarrow{\scriptstyle u_2} \\ b_1 & \xrightarrow{\;v\;} & b_2 \end{array} \qquad \longmapsto \qquad p(a,u_1) \xrightarrow{\varphi(p(a,u_1),v)} p(a,u_2) \tag{18}$$

This functor preserves identities and composition by the axioms (L2) and (L3) of a delta lens, respectively. Moreover, the equation $f \circ p = Tf$ from the left diagram of (7) is satisfied by axiom (L1). The equation $p \circ Sf = \iota_A$ from the diagram (8) also holds since $Sf(a) = (a, 1_{fa})$ and $p(a, 1_{fa}) = a$ by axiom (L2).

To verify the remaining condition for a JR-algebra given by the right diagram of (7), we first describe the category JTf and the functors $\nu_f, \langle p, 1_B\rangle\colon JTf \to Jf$.

The category JTf has objects given by triples $(a \in A, u\colon fa \to b, u'\colon b \to b')$ and morphisms given by triples $\langle 1_a, 1_b, v\rangle$ as depicted below. The functor ν_f has an assignment on objects $(a,u,u') \mapsto (a, u' \circ u)$ and an assignment on morphisms $\langle 1_a, 1_b, v\rangle \mapsto \langle 1_a, v\rangle$, while the functor $\langle p, 1_B\rangle$ has corresponding assignments on objects and morphisms given by $(a,u,u') \mapsto (p(a,u),u')$ and $\langle 1_a, 1_b, v\rangle \mapsto \langle 1_{p(a,u)}, v\rangle$ which are well-defined by (L1). The equation $p \circ \mu_f = p \circ \langle p, 1_B\rangle$ holds since $p(a, u' \circ u) = p(p(a,u),u')$ by axiom (L3). Therefore, we have a JR-algebra (f,p) and the functor $\mathcal{L}\mathrm{ens} \to \mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$ is well-defined on objects.

$$\begin{array}{ccc} a & = & a \\ & & \\ fa & = & fa \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ b & = & b \\ {\scriptstyle u_1'}\downarrow & \circlearrowright & \downarrow{\scriptstyle u_2'} \\ b_1' & \xrightarrow{\;v\;} & b_2' \end{array} \tag{19}$$

Consider a pair of delta lenses $(f,\varphi)\colon A \to B$ and $(g,\psi)\colon C \to D$ with corresponding JR-algebras (f,p) and (g,q), respectively. Given a morphism of delta lenses $\langle h,k\rangle\colon (f,\varphi) \to (g,\psi)$, we want to show that there is a morphism of JR-algebras $\langle h,k\rangle\colon (f,p) \to (g,q)$. First note that the functor

$J\langle h,k\rangle\colon Jf \to Jg$ has an assignment on objects $(a,u) \mapsto (ha,ku)$ and an assignment on morphisms $\langle 1_a, v\rangle \mapsto \langle 1_{ha}, kv\rangle$. As we have $h\varphi(a,u) = \psi(ha,ku)$ by the definition of a morphism of delta lenses, it follows that $hp(a,u) = \mathrm{cod}(h\varphi(a,u)) = \mathrm{cod}(\psi(ha,ku)) = q(ha,ku)$. A similar argument on morphisms of Ef establishes that $q \circ J\langle h,k\rangle = h \circ p$ and thus the equation (9) for a morphism of JR-algebras holds. □

Proof. We now construct a functor $\mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu) \to \mathcal{L}\mathrm{ens}$ and show that it is inverse to $\mathcal{L}\mathrm{ens} \to \mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$.

Given a JR-algebra determined by the pair of functors $f\colon A \to B$ and $p\colon Jf \to A$, we define a delta lens $(f,\varphi)\colon A \to B$ whose lifting operation φ is given below, where $p(a,1_{fa}) = a$ by (8).

$$\begin{array}{ccc} a & = & a \\ & & \\ fa & = & fa \\ {\scriptstyle 1_{fa}}\downarrow & & \downarrow{\scriptstyle u} \\ fa & \xrightarrow{u} & b \end{array} \qquad \longmapsto \qquad p(a,1_{fa}) = a \xrightarrow{p\langle 1_a,u\rangle} p(a,u) \tag{20}$$

By (8) on morphisms, it follows that axiom (L2) for a delta lens holds. By the left diagram of (7), it is also immediate that axiom (L1) holds. For axiom (L3) to hold, we need to show that

$$p\langle 1_a, v\circ u\rangle = p\langle 1_a, v\rangle \circ p\langle 1_a, u\rangle = p\langle 1_{p(a,u)}, v\rangle \circ p\langle 1_a, u\rangle.$$

This amounts to proving that the morphism $p\langle 1_a, v\rangle\colon p(a,u) \to p(a, v\circ u)$ is equal to the morphism $p\langle 1_{p(a,u)}, v\rangle\colon p(p(a,u),1_b) \to p(p(a,u),v)$, which follows directly from the right diagram in (7).

Given a morphism of JR-algebras $\langle h,k\rangle\colon (f,p) \to (g,q)$, we have $hp\langle 1_a,u\rangle = q\langle 1_{ha}, ku\rangle$ from (9). Therefore there is a well-defined morphism $\langle h,k\rangle$ between the corresponding delta lenses.

To show that the functors $\mathcal{L}\mathrm{ens} \to \mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu)$ and $\mathcal{A}\mathrm{lg}_{\mathsf{JR}}(T,\nu) \to \mathcal{L}\mathrm{ens}$ are inverse, it is enough to show it holds on the objects as the morphisms consist of the same data.

First consider a delta lens (f,φ) and define a functor $p\colon Jf \to B$ as in (18). Applying this functor at a morphism $\langle 1_a, u\rangle\colon (a,1_{fa}) \to (a,u)$ in Jf, we obtain $\varphi(p(a,1_{fa}),u) = \varphi(a,u)$ by (L2) as desired. Now consider a JR-algebra (f,p) and define a lifting operation φ for a delta lens as in (20). Defining a functor $\hat{p}\colon Jf \to A$ from this delta as in (18) and applying it to a morphism $\langle 1_a, v\rangle\colon (a,u_1) \to (a,u_2)$ we find that $\hat{p}\langle 1_a, v\rangle = p\langle 1_{p(a,u)}, v\rangle = p\langle 1_a, v\rangle$ by the right diagram in (7) as desired. This completes the proof. □