

# Collages of String Diagrams

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We introduce collages of string diagrams as a diagrammatic syntax for gluing multiple monoidal categories. Collages of string diagrams are interpreted as pointed bimodular profunctors. As the main examples of this technique, we introduce string diagrams for bimodular categories, string diagrams for functor boxes, and string diagrams for internal diagrams.

## 1 Introduction

String diagrams are a convenient and intuitive, sound and complete syntax for monoidal categories [26]. Monoidal categories are algebras of processes composing in parallel and sequentially [30]; string diagrams formalize the process diagrams of engineering [5, 7]. Formalization is not only of conceptual interest: it means we can sharpen our reasoning, scale our diagrams, or explain them to a computer [37].

However, the formal syntax of monoidal categories is not enough for all applications and, sometimes, we need to extend it. Functor boxes allow us to reason about translations between theories of processes [13, 32], ownership [34], higher-order processes [1], or programming effects [38]. Quantum combs not only model some classes of supermaps [10, 14, 21], but they coincide with the monoidal lenses of functional programming [4, 11, 45] and compositional game theory [20, 6]. Premonoidal categories, which appear in Moggi’s semantics of programming effects [33, 27, 46], are now within the realm of string diagrammatic reasoning [42]. Internal diagrams extend the syntax of monoidal categories allowing us to draw diagrams inside tubular cobordisms and reason about topological quantum field theories [3], but also coends [41] and traces [24].

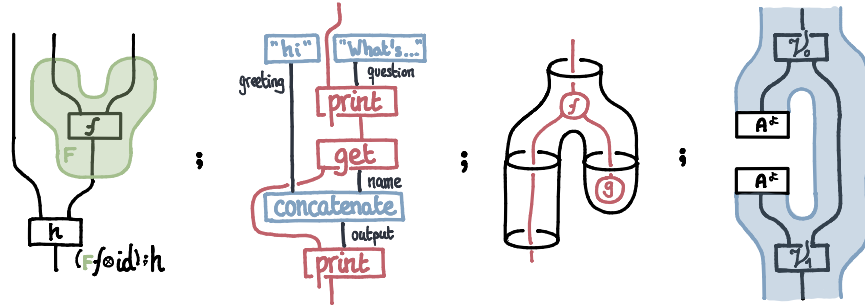


Figure 1: Examples from the literature. From left to right: functor boxes [32], premonoidal categories [42], internal diagrams [3], and combs or optics [10, 11, 21].

The extensions showcase the expressive power of string diagrams on surprisingly diverse application domains. At the same time, these different ideas could be regarded as separate ad-hoc extensions: they belong to different fields; they use different categorical formalisms. The overhead of learning and combining each one of them prevents the exchange of ideas between the different domains of application: e.g. an idea about topological quantum field diagrams does not transfer to premonoidal diagrams.

**Collages.** This manuscript claims that this division is only apparent and that all these extensions are particular instances of the same encompassing idea: that of glueing multiple string diagrams into what we call a *collage of string diagrams*. We introduce a formal notion of collage (Section 4.4) and employ string diagrammatic syntaxes for them, based on the calculus of bicategories (Sections 2.1, 3.1 and 5).

Even though collages of string diagrams are our novel contribution, collages are not yet another new concept to category theory. “Collage” was Bob Walters’ term for a lax colimit in a module-like category [47]. This can be considered as a glueing of objects together along the action of a scalar. For example, given two sets  $A$  and  $B$ , with an action of a monoid  $M$ , we can construct their tensor product  $A \otimes_M B$ , where  $(a \cdot m) \otimes b = a \otimes (m \cdot b)$  for any scalar  $m \in M$ . Categorifying this idea in a possible direction we obtain monoidal categories acting on *bimodular categories*. The following is the takeaway of this work.

Collages of string diagrams consist of multiple string diagrams of different monoidal categories glued together. Collages can be interpreted as *pointed bimodular profunctors* between *bimodular categories*.

A *bimodular category*, sometimes referred to as a *biactegory* [9], is to a bimodule what a monoidal category is to a monoid. This is, a plain category  $\mathbb{A}$  endowed with a left action of a monoidal category  $(\triangleright): \mathbb{M} \times \mathbb{A} \rightarrow \mathbb{A}$  and a right action of another, possibly different, monoidal category  $(\triangleleft): \mathbb{A} \times \mathbb{N} \rightarrow \mathbb{A}$ . We can *collage* two bimodular categories along a common monoidal category that acts on both. Later on the paper, exploiting a second axis of categorification, we pass from bimodular categories to *bimodular profunctors*, which are a kind of 2-dimensional bimodule, and we define their collage. This structure facilitates glueing categories together in 2-dimensions: we can represent complexes of morphisms from different categories and glue them together. Collages of string diagrams are the syntactic representations of this glueing, in the same sense that ordinary string diagrams represent tensors in monoidal categories.

We observe that collages of bimodular categories embed into a tricategory of pointed bimodules. This provides a versatile setting where we can interpret many syntaxes already present in the literature.

**Contributions.** We introduce string diagrams of *bimodular categories* and we prove they construct the free bimodular category on a signature (Theorem 2.6). We introduce novel string diagrammatic syntax for *functor boxes* and we prove it constructs the free lax monoidal functor on a suitable signature (Theorem 3.4). We describe the tricategory of pointed bimodular profunctors (Definition 4.6) and, in terms of it, we explain the semantics of functor boxes (Proposition 4.9) and internal diagrams (Theorem 5.3), for which we also provide a novel explicit formal syntax (Definition 5.2).

## 2 String Diagrams of Bimodular Categories

We introduce string diagrams for *bimodular categories* in terms of the better-known string diagrams of bicategories. In algebra, a *bimodule* is a structure with a compatible left and right action. *Bimodular categories are to bimodules what monoidal categories are to monoids*. Explicitly this means a category,  $\mathbb{C}$ , acted on by two monoidal categories,  $\mathbb{M}$  and  $\mathbb{N}$  [48]. Bimodular categories have also been known as “biactegories” [9, 31], while the name “bimodule category” typically refers to actions of certain vector enriched categories [17]. For our purposes, we consider a bimodular category  $\mathbb{C}$ , as glueing together the two acting categories,  $\mathbb{M}$  and  $\mathbb{N}$ .

To simplify the presentation, we limit ourselves to considering only strict structure, but we expect that all of the results considered hold analogously in the weaker setting. In the following, we assume all monoidal, bimodular, and 2-categories to be strict, along with the associated functors between them.

**Definition 2.1.** A *bimodular category*  $(\mathbb{C}, \mathbb{M}, \mathbb{N})$  is a category  $\mathbb{C}$  endowed with a left monoidal action  $(\triangleright): \mathbb{M} \times \mathbb{C} \rightarrow \mathbb{C}$ , and a right monoidal action  $(\triangleleft): \mathbb{C} \times \mathbb{N} \rightarrow \mathbb{C}$ , which are compatible in that  $M \triangleright (X \triangleleft N) = (M \triangleright X) \triangleleft N$ .

Bimodular categories over arbitrary monoidal categories form a category, **Bimod**. The morphisms  $(F, H, K): (\mathbb{C}, \mathbb{M}, \mathbb{N}) \rightarrow (\mathbb{D}, \mathbb{P}, \mathbb{Q})$  consist of two monoidal functors  $H: \mathbb{M} \rightarrow \mathbb{P}$  and  $K: \mathbb{N} \rightarrow \mathbb{Q}$  and a functor  $F: \mathbb{C} \rightarrow \mathbb{D}$  that strictly preserves monoidal actions according to  $H$  and  $K$ .

Every monoidal category  $(\mathbb{C}, \otimes, I)$  is a  $(\mathbb{C}, \mathbb{C})$ -bimodular category with its own tensor product defining the two actions.

## 2.1 Signature of a Bimodular Category

The next sections exhibit a sound and complete string diagram syntax for bimodular categories. Bimodular string diagrams consist of two monoidal regions glued by a bimodular wire. We begin by defining a notion of bimodular signature and then construct an adjunction (Theorem 2.7) using the notion of *collages*.

**Definition 2.2.** A *bimodular graph*  $(\mathcal{A}, \mathcal{M}, \mathcal{N})$  (the bimodular analogue of a multigraph [43]) is given by three sets of objects  $(\mathcal{A}_{obj}, \mathcal{M}_{obj}, \mathcal{N}_{obj})$  and three different types of edges:

- the left-acting edges, a set  $\mathcal{M}(\vec{M}; \vec{P})$  for each pair of lists of objects  $\vec{M}, \vec{P} \in \mathcal{M}_{obj}^*$ ,
- the right-acting edges, a set  $\mathcal{N}(\vec{N}; \vec{Q})$  for  $\vec{N}, \vec{Q} \in \mathcal{N}_{obj}^*$ ;
- the *central edges*, a set of edges  $\mathcal{A}(\vec{M}, A, \vec{N}; \vec{P}, B, \vec{Q})$ , for each  $\vec{M}, \vec{P} \in \mathcal{M}_{obj}^*$ ; each  $\vec{N}, \vec{Q} \in \mathcal{N}_{obj}^*$  and each  $A, B \in \mathcal{A}_{obj}$ .

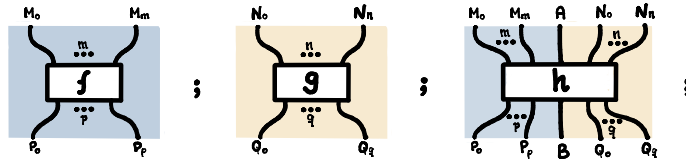


Figure 2: Left, right, and central edges of a bimodular graph.

**Proposition 2.3.** *Bimodular graphs form a category **bmGraph**. We define a morphism of bimodular graphs  $(l, f, g): (\mathcal{A}, \mathcal{M}, \mathcal{N}) \rightarrow (\mathcal{A}', \mathcal{M}', \mathcal{N}')$  to be a triple of functions on objects,  $(l_{obj}, f_{obj}, g_{obj})$ , that extend to the morphism sets. There exists a forgetful functor  $U: \mathbf{Bimod} \rightarrow \mathbf{bmGraph}$ .*

This provides a syntactic presentation of bimodular categories. We would like to additionally, construct a free model from a syntactic presentation. We make use of the well-known similar result for 2-categories, exhibiting bimodular categories as certain bicategories: explicitly, those which are the *collage* of a bimodular category.

## 2.2 The Collage of a Bimodular Category

Each profunctor induces a *collage category*; analogously a bimodular category induces a *collage 2-category*. This section proves that constructing the collage of a bimodular category is left adjoint to considering the bimodular hom-category between any two cells of a 2-category.

**Definition 2.4.** The *collage* of an  $(\mathbb{M}, \mathbb{N})$ -bimodular category  $\mathbb{C}$  is a 2-category,  $\text{Coll}_{\mathbb{C}}$ . This 2-category has two 0-cells,  $M$  and  $N$ . The hom-categories are given by  $\text{Coll}_{\mathbb{C}}(M, M) = \mathbb{M}$ ,  $\text{Coll}_{\mathbb{C}}(N, N) = \mathbb{N}$ , and  $\text{Coll}_{\mathbb{C}}(M, N) = \mathbb{C}$ ; and finally  $\text{Coll}_{\mathbb{C}}(N, M)$  is the empty category. The composition of 1-cells is given by the monoidal products and actions.

**Definition 2.5.** The category of bipointed 2-categories,  $\mathbf{2Cat}_2$ , has as objects  $(\mathbb{A}, M, N)$ , 2-categories  $\mathbb{A}$  with two chosen 0-cells on it,  $M \in \mathbb{A}$  and  $N \in \mathbb{A}$ . A morphism of  $\mathbf{2Cat}_2$  is a 2-functor preserving the chosen 0-cells.

**Theorem 2.6.** *There exists an adjunction  $\text{Coll}_{\mathbb{C}} : \mathbf{Bimod} \rightleftarrows \mathbf{2Cat}_2 : \text{Chosen}$  given by the collage, and by picking the hom-category between the chosen 0-cells. Moreover, the unit of this adjunction is a natural isomorphism.*

### 2.3 String Diagrams of Bimodular Categories, via Collages

We have the two ingredients for bimodular string diagrams: sound complete string diagrams for 2-categories, and an embedding of bimodular categories into 2-categories by taking the collage. We combine results to provide an adjunction from bimodular graphs to bimodular categories.

$$\begin{array}{ccc}
 \mathbf{2Graph} & & \mathbf{2Graph}_2 \xrightleftharpoons[\text{Coll}]{\text{Chosen}} \mathbf{Bimod} \\
 \text{Str} \left( \downarrow \dashv \uparrow \right) U & & \text{Str}_2 \left( \downarrow \dashv \uparrow \right) U_2 \xrightleftharpoons[\text{Coll}]{\text{Chosen}} \mathbf{Bimod} \\
 \mathbf{2Cat} & & \mathbf{2Cat}_2 \xrightleftharpoons[\text{Coll}]{\text{Chosen}} \mathbf{Bimod}
 \end{array}$$

Figure 3: Summary of adjunctions for the string diagrams of bimodular categories.

**Theorem 2.7.** *There exists an adjunction between bimodular graphs and bimodular categories. The left adjoint finds the bimodular category whose collage is the free 2-category on the bimodular graph,  $\text{bmStr} : \mathbf{bmGraph} \rightarrow \mathbf{Bimod}$ . The right adjoint is the forgetful functor  $U : \mathbf{Bimod} \rightarrow \mathbf{bmGraph}$ .*

This result provides a basis for a graphical syntax for bimodular categories. We now sketch an example of how these string diagrams can be of interest, but a larger class of examples come from premonoidal and effectful categories [42].

### 2.4 Example: Shared State

In the same way that premonoidal categories are particularly well-suited to describe stateful computations, bimodular categories are particularly well-suited to describe shared state between two processes. These processes can be different and even live in different categories. As an example, consider the generators in Figure 4. They represent two different process theories that access a common state with get and put operations.



Figure 4: Signature generators for the bimodular theory of shared state.

In the same way that monoidal categories are a good setting for defining monoids and comonoids, **bimodular categories** are a good setting for defining bimodules. To capture interacting shared state, the generators of Figure 4 are quotiented by the equations of a pair of semifrobenius modules with compatible comonoid actions and semimonoid actions.

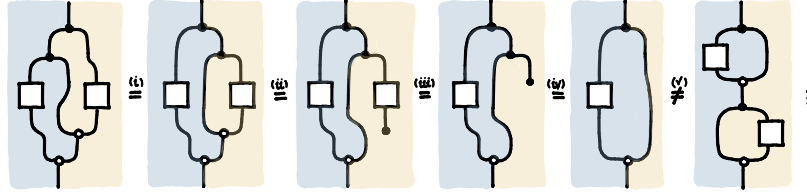


Figure 5: Race condition in bimodular string diagrams.

This setup is enough to exhibit one of the most salient features of shared state: *race conditions*. Race conditions were first studied by Huffman in 1954, who used diagrams to show how the behavior of a shared state is dependent on the relative timing of the actions of the parties [25]. We employ string diagrams of **bimodular categories** to show how two different timings of the actions – the leftmost and rightmost sides of the equation in Figure 5 – result in two different executions: even when the two get statements are compatible (i), the two put statements interact causing the earlier of the two to be discarded (ii, iii, iv); this causes the discrepancy with the intended protocol (v).

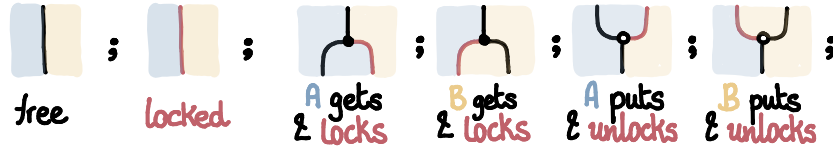


Figure 6: Binary semaphore in bimodular string diagrams.

Race conditions have a commonly accepted workaround: the *binary semaphore* [44]. Dijkstra described general semaphores with the aid of flow diagrams [16]; we instead use bimodular categories to model a binary semaphore (Figure 6). We consider a signature with two object generators, free and locked, for our bimodular category. Each operation must suitably lock or unlock the semaphore, rendering race conditions ill-typed, and leaving most of the interaction equations of the theory of shared state unnecessary.

String diagrams of bimodular categories model a pair of interacting monoidal categories. We can also model an arbitrary number of interacting monoidal categories via a general collage construction. These collages can be subsumed into a ‘universe of collages’ that we describe in Section 4: the tricategory of pointed bimodular profunctors. To motivate this, we study a second example: the syntax of functor boxes.

### 3 String Diagrams of Functor Boxes

Functor boxes are an extension of the string diagrammatic notation that represents plain functors, lax, oplax and strong monoidal functors. Functor boxes were introduced by Cockett and Seely [13] and later studied by Melliès [32]. We introduce here a syntactic presentation of (op)lax functor boxes that has the

advantage of treating each piece of the box as a separate entity in a 2-category and applying the string diagrammatic calculus of 2-categories.

### 3.1 Functor box signatures

**Definition 3.1.** A *functor box signature*  $\mathcal{F} = (\mathcal{A}, \mathcal{X}, \mathcal{F}_\bullet, \mathcal{F}^\bullet)$  consists of a pair of sets,  $\mathcal{A}_{obj}$  and  $\mathcal{X}_{obj}$ , and four different types of edges:

- the plain edges,  $\mathcal{A}(A_0, \dots, A_n; B_0, \dots, B_m)$  for any objects  $A_0, \dots, A_n, B_0, \dots, B_m \in \mathcal{A}_{obj}$ ;
- the functor box edges,  $\mathcal{X}(X_0, \dots, X_n; Y_0, \dots, Y_m)$  for any objects  $X_0, \dots, X_n, Y_0, \dots, Y_m \in \mathcal{X}_{obj}$ ;
- the in-box edges,  $\mathcal{F}_\bullet(A_0, \dots, A_n; Y_0, \dots, Y_m)$  for any  $A_0, \dots, A_n \in \mathcal{A}_{obj}$  and  $Y_0, \dots, Y_m \in \mathcal{X}_{obj}$ ;
- the out-box edges,  $\mathcal{F}^\bullet(X_0, \dots, X_n; B_0, \dots, B_m)$  for any  $B_0, \dots, B_m \in \mathcal{A}_{obj}$  and  $X_0, \dots, X_n \in \mathcal{X}_{obj}$ .

A *functor box signature morphism*  $(h, k, l): (\mathcal{A}, \mathcal{X}, \mathcal{F}) \rightarrow (\mathcal{B}, \mathcal{Y}, \mathcal{G})$  is a pair of functions between the object sets,  $h_{obj}: \mathcal{A}_{obj} \rightarrow \mathcal{B}_{obj}$  and  $k_{obj}: \mathcal{X}_{obj} \rightarrow \mathcal{Y}_{obj}$ , that extend to a function between the edge sets;

- $h: \mathcal{A}(A_0, \dots, A_n; B_0, \dots, B_m) \rightarrow \mathcal{B}(h(A_0), \dots, h(A_n); h(B_0), \dots, h(B_m))$ ;
- $k: \mathcal{X}(X_0, \dots, X_n; Y_0, \dots, Y_m) \rightarrow \mathcal{Y}(k(X_0), \dots, k(X_n); k(Y_0), \dots, k(Y_m))$ ;
- $l_\bullet: \mathcal{F}_\bullet(A_0, \dots, A_n; Y_0, \dots, Y_m) \rightarrow \mathcal{G}_\bullet(h(A_0), \dots, h(A_n); k(Y_0), \dots, k(Y_m))$ ;
- $l^\bullet: \mathcal{F}^\bullet(X_0, \dots, X_n; B_0, \dots, B_m) \rightarrow \mathcal{G}^\bullet(k(X_0), \dots, k(X_n); h(B_0), \dots, h(B_m))$ .

Functor box signatures and homomorphisms form a category, **Fbox**.

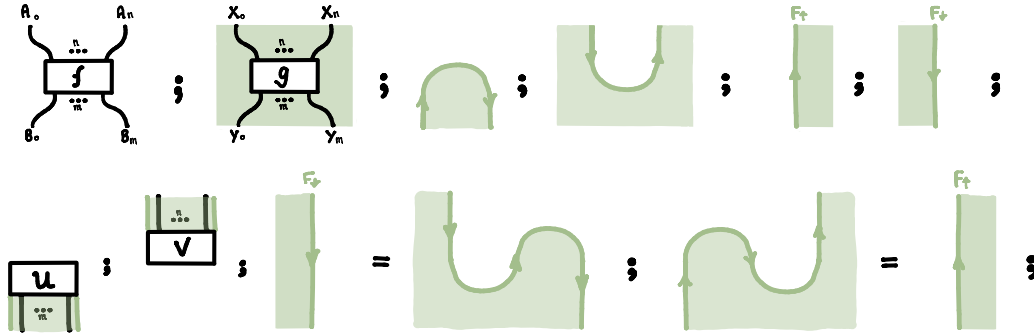


Figure 7: Syntactic 2-category of a lax monoidal functor box signature.

**Definition 3.2.** The syntactic 2-category of a functor box signature  $\mathcal{F} = (\mathcal{A}, \mathcal{X}, \mathcal{F}_\bullet, \mathcal{F}^\bullet)$  is the 2-category freely presented by Figure 7, which we call  $\mathbb{S}_{\mathcal{A}, \mathcal{X}, \mathcal{F}}$ .

In other words, the 2-category  $\mathbb{S}_{\mathcal{A}, \mathcal{X}, \mathcal{F}}$  contains exactly two 0-cells, labelled  $\mathcal{A}$  and  $\mathcal{X}$ ; it contains a 1-cell  $A: \mathcal{A} \rightarrow \mathcal{A}$  for each  $A \in \mathcal{A}_{obj}$ , a 1-cell  $X: \mathcal{X} \rightarrow \mathcal{X}$  for each  $X \in \mathcal{X}_{obj}$  and, moreover, a pair of adjoint 1-cells  $F^\uparrow: \mathcal{A} \rightarrow \mathcal{X}$  and  $F^\downarrow: \mathcal{X} \rightarrow \mathcal{A}$ . Finally, it contains a pair of 2-cells witnessing the adjunction  $F^\uparrow \dashv F^\downarrow$ , given by  $n: \text{id} \rightarrow F^\uparrow \circ F^\downarrow$  and  $e: F^\downarrow \circ F^\uparrow \rightarrow \text{id}$  which additionally satisfy the snake equations; and it also contains

- a 2-cell,  $f \in \mathbb{S}(\mathcal{A}, \mathcal{A})(A_0 \circ \dots \circ A_n; B_0 \circ \dots \circ B_m)$ , for each *plain edge*;
- a 2-cell,  $g \in \mathbb{S}(\mathcal{X}, \mathcal{X})(X_0 \circ \dots \circ X_n; Y_0 \circ \dots \circ Y_m)$ , for each *functor box edge*;
- a 2-cell,  $u \in \mathbb{S}(\mathcal{A}, \mathcal{A})(A_0 \circ \dots \circ A_n; F^\uparrow \circ Y_0 \circ \dots \circ Y_m \circ F^\downarrow)$  for each *in-box edge*; and
- a 2-cell,  $v \in \mathbb{S}(\mathcal{A}, \mathcal{A})(F^\uparrow \circ X_0 \circ \dots \circ X_n \circ F^\downarrow; B_0 \circ \dots \circ B_m)$  for each *out-box edge*.

### 3.2 Lax Monoidal Functor Semantics

**Definition 3.3** (Lax functors category). An object of the *lax functors category*, **Lax**, is a pair of monoidal categories  $(\mathbb{A}, \mathbb{X})$  together with a lax monoidal functor between them,  $(F, \varepsilon, \mu)$ ; that is, a functor  $F: \mathbb{X} \rightarrow \mathbb{A}$  endowed with two natural transformations  $\varepsilon: I \rightarrow FI$ , and  $\mu: FX \otimes FY \rightarrow F(X \otimes Y)$ , satisfying associativity  $(\mu \otimes id) \circ \mu = (id \otimes \mu) \circ \mu$ , left unitality  $(\varepsilon \otimes id) \circ \mu = id$  and right unitality  $(id \otimes \varepsilon) \circ \mu = id$ .

A morphism of the *lax functors category*, from  $(\mathbb{A}, \mathbb{X}, F, \varepsilon_F, \mu_F)$  to  $(\mathbb{B}, \mathbb{Y}, G, \varepsilon_G, \mu_G)$  is a pair of monoidal functors  $H: \mathbb{X} \rightarrow \mathbb{A}$  and  $K: \mathbb{A} \rightarrow \mathbb{B}$  such that  $F \circ K = H \circ G$  and such that  $K(\varepsilon_F) = \varepsilon_G$  and  $K(\mu_F) = \mu_G$ .

**Theorem 3.4.** *There exists an adjunction between the category of functor box signatures, **Fbox**, and the category of pairs of monoidal categories with a lax monoidal functor between them, **Lax**. The free side of this adjunction is given by the syntax of Figure 7.*

Collages, by themselves, explained the 2-region diagrams of **bimodular categories**; collages will also explain the two-region diagrams of functor boxes in Section 4.5. However, as currently defined, collages are only sufficient to encode the vertical boundaries. To additionally represent boundaries along the horizontal axis we can make use of profunctors between bimodular categories and extend our notion of collage to these structures. Following this thread we find that collages embed into a tricategory of pointed bimodular profunctors, described in the next section, which we consider a universe of interpretation for all of the graphical theories described.

## 4 Bimodular Profunctors

Where can we interpret all these string diagrams and provide compositional semantics for them? In this section, we introduce a single structure where all the previous calculi take semantics.

We will need two different ingredients: *coends* and *bimodularity*. Coends and profunctors [29, 30], far from being obscure concepts from category theory, can be seen as the right tool to glue together morphisms from different categories [15, 41]; we follow an explicitly *pointed* version of coend calculus, which keeps track of the transformation between profunctors we are constructing (Section 4.3). In a similar sense, **bimodular categories** tensor together objects from different monoidal categories. Both ideas combine into the calculus of pointed bimodular profunctors.

### 4.1 Bimodular Profunctors

Consider  $\mathbb{C}$  and  $\mathbb{D}$ , both  $(\mathbb{M}, \mathbb{N})$ -bimodular categories. A natural notion of morphism between them is a functor  $\mathbb{C} \rightarrow \mathbb{D}$  which preserves both actions. However, there is another notion of morphism between them, which is a generalization of a profunctor between categories to this bimodular setting. **Bimodular profunctors** are a generalized reformulation of the Tambara modules of Pastro and Street [36].

**Definition 4.1.** Let  $\mathbb{M}$  and  $\mathbb{N}$  be two monoidal categories and let  $\mathbb{C}$  and  $\mathbb{D}$  be two  $(\mathbb{M}, \mathbb{N})$ -bimodular categories. A *bimodular profunctor* from  $\mathbb{C}$  to  $\mathbb{D}$  is a profunctor  $T: \mathbb{C}^{op} \times \mathbb{D} \rightarrow \mathbf{Set}$  with a natural family of strengths,

$$t_M: T(X, Y) \rightarrow T(M \triangleright X, M \triangleright Y), \quad \text{and} \quad t^N: T(X, Y) \rightarrow T(X \triangleleft N, Y \triangleleft N),$$

such that the actions are associative,  $t_M \circ t_{M'} = t_{M \otimes M'}$  and  $t_N \circ t_{N'} = t_{N \otimes N'}$ , unital  $t_I = id$  and  $t^I = id$ , and compatible,  $t_M \circ t^N = t^N \circ t_M$ , up to the coherence isomorphisms of the monoidal category.

**Proposition 4.2.** *For any pair of monoidal categories,  $\mathbb{M}$  and  $\mathbb{N}$ , there is a 2-category  ${}_{\mathbb{M}}\mathbf{Mod}_{\mathbb{N}}$  of  $(\mathbb{M}, \mathbb{N})$ -bimodular categories, bimodular profunctors, and natural transformations between them.*

These will form the hom-bicategories of the tricategory we later define. The other significant piece of data we require is a family of tensors  $\otimes : {}_{\mathbb{M}}\mathbf{Mod}_{\mathbb{N}} \times {}_{\mathbb{N}}\mathbf{Mod}_{\mathbb{O}} \rightarrow {}_{\mathbb{M}}\mathbf{Mod}_{\mathbb{O}}$ , which we now study.

## 4.2 Tensor of Bimodular Profunctors

The tensor of bimodular categories is similar to the tensor of modules over a monoid in classical algebra: we consider pairs of elements and we quotient out the action of a common scalar [35]. In this case, the quotienting is substituted by an appropriate structural isomorphism: the *equilibrator*.

**Definition 4.3** (Tensor of bimodular categories, [35]). Let  $\mathbb{C}$  be a  $(\mathbb{M}, \mathbb{N})$ -bimodular category and let  $\mathbb{D}$  be a  $(\mathbb{N}, \mathbb{O})$ -bimodular category. Their tensor product,  $\mathbb{C} \otimes_{\mathbb{N}} \mathbb{D}$ , is a category with the same objects as  $\mathbb{C} \times \mathbb{D}$ : we write them as  $X \otimes_{\mathbb{N}} Y$ . The category is presented by the morphisms of  $\mathbb{C} \times \mathbb{D}$  and a free family of natural isomorphisms, called the *equilibrators*,

$$\tau_{X,N,Y} : (X \triangleleft N) \otimes_N Y \rightarrow X \otimes_N (N \triangleright Y), \text{ for each } N \in \mathbb{N}, X \in \mathbb{C}, Y \in \mathbb{D},$$

which are additionally quotiented by the following equations up to the structure isomorphisms of the monoidal actions,  $\tau_{X,M \otimes N,Y} = \tau_{X \triangleleft M,N,Y} \circ \tau_{X,M,N \triangleright Y}$ , and  $\tau_{X,I,Y} = \text{id}$ .

**Definition 4.4.** Let  $\mathbb{C}$  and  $\mathbb{C}'$  be two  $(\mathbb{M}, \mathbb{N})$ -bimodular categories and let  $\mathbb{D}$  and  $\mathbb{D}'$  be a  $(\mathbb{N}, \mathbb{O})$ -bimodular categories. Given two bimodular profunctors,  $T : \mathbb{C} \rightarrow \mathbb{C}'$  and  $R : \mathbb{D} \rightarrow \mathbb{D}'$ , their tensor is a bimodular profunctor,  $T \otimes_{\mathbb{N}} R : \mathbb{C} \otimes_{\mathbb{N}} \mathbb{D} \rightarrow \mathbb{C}' \otimes_{\mathbb{N}} \mathbb{D}'$ , defined by

$$(T \otimes_{\mathbb{N}} R)(X \otimes_{\mathbb{N}} Y; X' \otimes_{\mathbb{N}} Y') = T(X; X') \times R(Y; Y') / (\sim),$$

where  $(\sim)$  is the equivalence relation generated by  $(t_N(x), y) \sim (x, t_N(y))$ .

## 4.3 Pointed Profunctors

Profunctors deal with families of morphisms, and their natural isomorphisms determine correspondences between these families. However, when we use profunctors for the semantics of string diagrams, we most often want to single out a particular morphism between a particular pair of objects. A simple technique to achieve this is to use *pointed profunctors* instead of simply profunctors: this technique was explicitly described by this second author [41] although it has implicit appearances in the literature [3, 24].

**Definition 4.5.** A pointed profunctor  $(P, p) : (\mathbb{A}, X) \rightarrow (\mathbb{B}, Y)$  between two pointed categories with a chosen object  $X \in \mathbb{A}_{obj}$  and  $Y \in \mathbb{B}_{obj}$  is a profunctor  $P : \mathbb{A} \rightarrow \mathbb{B}$  together with an element  $p \in P(A, B)$  of the profunctor evaluated on the chosen object of the categories.

From now on, we work using pointed profunctors instead of plain profunctors.

## 4.4 The Tricategory of Pointed Bimodular Profunctors

We call *collages of string diagrams* to the diagrams of the tricategory of pointed bimodular profunctors.

**Definition 4.6.** The tricategory of pointed bimodular profunctors,  $\mathbb{BmProf}_{pt}$ , has as 0-cells the monoidal categories,  $\mathbb{M}, \mathbb{N}, \mathbb{O}, \dots$ . The 1-cells between two monoidal categories  $\mathbb{M}$  and  $\mathbb{N}$  are *pointed bimodular categories*,  $(\mathbb{A}, \triangleright, \triangleleft, A)$ , consisting of a  $(\mathbb{M}, \mathbb{N})$ -bimodular category with two actions  $(\mathbb{A}, \triangleright, \triangleleft)$  and some object of that category,  $A \in \mathbb{A}$ . Pointed bimodular categories compose by the tensor of bimodular categories,

$$(\mathbb{A}, \triangleright, \triangleleft, A) \otimes_{\mathbb{N}} (\mathbb{B}, \triangleright, \triangleleft, B) = (\mathbb{A} \otimes_{\mathbb{N}} \mathbb{B}, \triangleright, \triangleleft, A \otimes_{\mathbb{N}} B).$$

The 2-cells between two pointed bimodular categories  $(\mathbb{A}, \triangleright, \triangleleft, A)$  and  $(\mathbb{B}, \triangleright, \triangleleft, B)$  are *pointed bimodular profunctors*  $(P, t, p)$ , consisting of a profunctor  $P: \mathbb{A} \rightarrow \mathbb{B}$  together with a point  $p \in P(A, B)$  that are moreover bimodular with compatible natural transformations  $t_M: P(A; B) \rightarrow P(M \triangleright A; M \triangleright B)$ , and  $t_N: P(A; B) \rightarrow P(A \triangleleft N; B \triangleleft N)$ . These 2-cells compose by profunctor composition and by the tensor of bimodular profunctors.

Finally, the 3-cells between two pointed bimodular profunctors  $(P, t, p)$  and  $(Q, r, q)$  are bimodular natural transformations that preserve the point, consisting of a natural transformation  $\alpha: P \rightarrow Q$  such that the  $\alpha(p) = q$  and, moreover,  $t_M \circ \alpha = \alpha \circ r_M$  and  $t_N \circ \alpha = \alpha \circ r_N$ .

*Remark 4.7.* At the moment of writing, it is unclear to the authors whether a string diagrammatic calculus for tricategories, described by transformations of the string diagrammatic calculus of bicategories, has been fully described and proved sound and complete. However, there seems to be a consensus that this would be the right language for tricategories: much literature assumes it. Let us close this section by tracking explicitly the assumptions we need to employ a diagrammatic syntax for bimodular profunctors.

**Conjecture 4.8.** *The previous data satisfies all coherence conditions of a tricategory. Moreover, we can reason with tricategories using the calculus of deformations of string diagrams, extending the string diagrams for quasistrict monoidal 2-categories of Bartlett [2].*

## 4.5 Functor Boxes via Collages of String Diagrams

The following Figure 8 details how to interpret functor boxes as collages of string diagrams. The colored region represents the domain of the lax monoidal functor; the white region represents the codomain. Morphisms of both categories are interpreted as elements of their respective hom-profunctors, and the laxators are used to merge colored regions. The only element that we will explicitly detail is the bimodular category that appears in the closing and opening wires of a functor box.

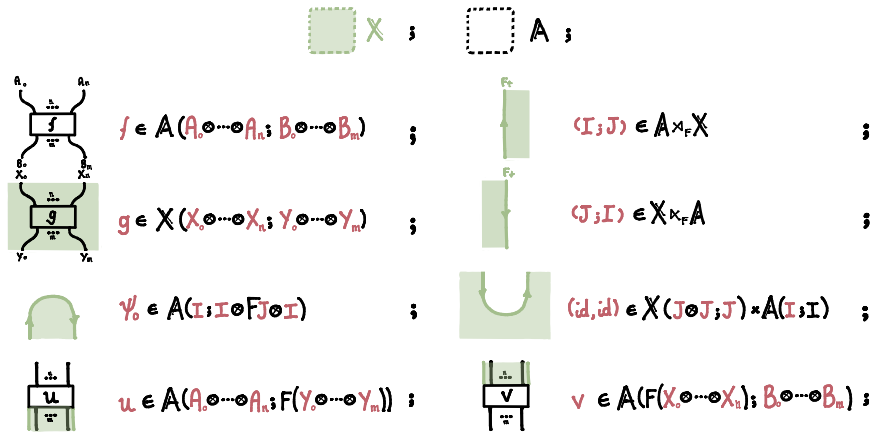


Figure 8: Semantics for functor boxes in terms of pointed bimodular profunctors.

**Proposition 4.9** (Bimodular categories of a lax monoidal functor). *Let  $\mathbb{X}$  and  $\mathbb{A}$  be two monoidal categories and let  $F: \mathbb{X} \rightarrow \mathbb{A}$  be a monoidal functor between them, endowed with natural transformations  $\psi_0: J \rightarrow FI$  and  $\psi_2: FX \otimes FY \rightarrow F(X \otimes Y)$ . The following profunctors,  $\mathbb{A} \rtimes_F \mathbb{X}: \mathbb{A} \times \mathbb{X} \rightarrow \mathbb{A} \times \mathbb{X}$  and*

$\mathbb{X} \times_F \mathbb{A}: \mathbb{X} \times \mathbb{A} \rightarrow \mathbb{X} \times \mathbb{A}$  determine two promonads, and therefore two Kleisli categories.

$$\begin{aligned}\mathbb{A} \rtimes_F \mathbb{X}(A, X; B, Y) &= \int^{M \in \mathbb{X}} \mathbb{A}(A; B \otimes FM) \times \mathbb{X}(M \otimes X; Y); \\ \mathbb{X} \times_F \mathbb{A}(X, A; Y, B) &= \int^{M \in \mathbb{X}} \mathbb{A}(A; FM \otimes B) \times \mathbb{X}(M \otimes A; B);\end{aligned}$$

These two Kleisli categories are  $(\mathbb{A}, \mathbb{X})$  and  $(\mathbb{X}, \mathbb{A})$ -bimodular, respectively.

## 5 String Diagrams of Internal Diagrams

The tubular 3-dimensional cobordisms of internal diagrams are first described as a Frobenius algebra by Bartlett, Douglas, Schommer-Pries and Vicary [3]. We are indebted to this first introduction, which made internal diagrams into a convenient graphical notation in topological quantum field theory [3]. Internal diagrams themselves were later given explicit semantics in a monoidal bicategory of pointed profunctors; this was the subject of this second author's contribution to *Applied Category Theory 2020* [40]. An important aspect of the syntax of internal diagrams is their 3-dimensional nature: the syntax not only contains string diagrams but also reductions between them.

We introduce here a novel syntactic presentation of *internal diagrams* that has the advantage of treating each piece of an internal diagram (including the closing and opening of tubes) as a separate entity in a tricategory. That is, the identity tube or the multiplication and comultiplication tubes are constructed out of smaller pieces in Figure 9. As a consequence, we are later able to introduce, for the first time, a more refined semantics in terms of a tricategory of *pointed bimodular profunctors*.

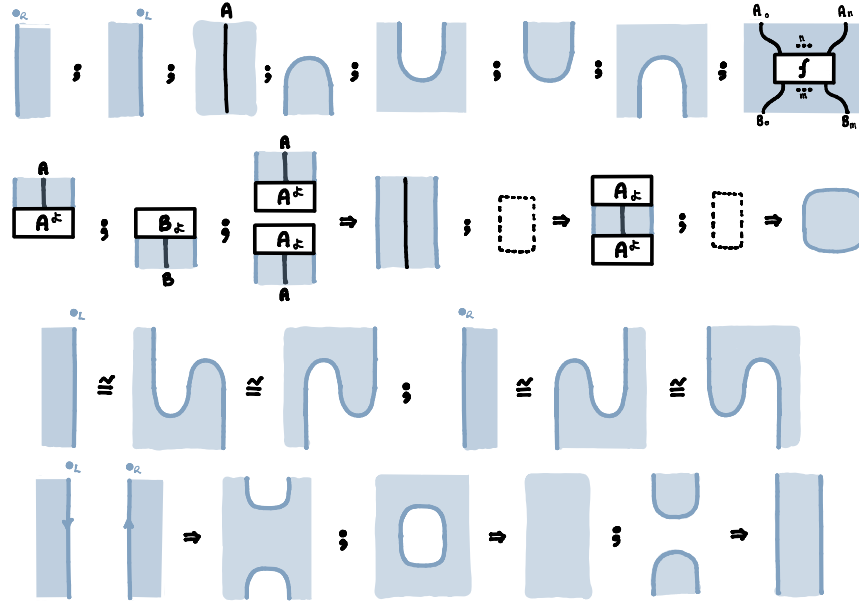


Figure 9: Syntax for open internal diagrams.

**Definition 5.1.** A *polygraph*,  $\mathcal{G}$ , is the signature for the string diagrams of a monoidal category. It consists of a set of objects,  $\mathcal{G}_{obj}$ , and a set of morphisms  $\mathcal{G}(A_0, \dots, A_n; B_0, \dots, B_m)$  between any two lists of objects,  $A_0, \dots, A_n, B_0, \dots, B_m \in \mathcal{G}_{obj}$ .

**Definition 5.2.** The *syntactic 3-category of internal diagrams* over a polygraph  $\mathcal{G}$  is the 3-category  $\mathbf{G}$  presented by the cells in Figure 9. In other words, it contains two 0-cells,  $\mathcal{I}$  and  $\mathcal{G}$ , in white and blue in the figure, respectively. It contains a 1-cell  $A: \mathcal{G} \rightarrow \mathcal{G}$  for each object  $A \in \mathcal{G}_{obj}$  and two 1-cells,  $L_\bullet: \mathcal{I} \rightarrow \mathcal{G}$  and  $R_\bullet: \mathcal{G} \rightarrow \mathcal{I}$  forming two 2-adjunctions  $(L_\bullet) \dashv (R_\bullet)$  and  $(R_\bullet) \dashv (L_\bullet)$  up to a 3-cell. It contains the following 2-cells,

- two 2-cells  $n_1: \text{id} \rightarrow L_\bullet \circ R_\bullet$  and  $e_1: R_\bullet \circ L_\bullet \rightarrow \text{id}$  witnessing the 2-adjunction  $(L_\bullet) \dashv (R_\bullet)$  and two 2-cells  $n_2: 1 \rightarrow R_\bullet \circ L_\bullet$  and  $e_2: L_\bullet \circ R_\bullet \rightarrow \text{id}$  witnessing the 2-adjunction  $(R_\bullet) \dashv (L_\bullet)$  – see Vicary and Heunen [22] for a reference on 2-adjunctions and the swallowtail equations;
- two 2-cells,  $A^\vee: L_\bullet \circ A \circ R_\bullet \rightarrow \text{id}$  and  $A_\vee: \text{id} \rightarrow L_\bullet \circ A \circ R_\bullet$ , forming an adjunction  $A^\vee \dashv A_\vee$  for each object  $A \in \mathcal{G}_{obj}$ ; and a 2-cell,  $f: A_0 \circ \dots \circ A_n \rightarrow B_0 \circ \dots \circ B_m$ , for each edge  $f \in \mathcal{G}(A_0, \dots, A_n; B_0, \dots, B_m)$ .

Finally, it contains the following 3-cells,

- two invertible 3-cells,  $\alpha_1: (1 \otimes n_1) \circ (e_1 \otimes 1) \rightarrow 1$  and  $\beta_1: (n_1 \otimes 1) \circ (1 \otimes e_1) \rightarrow 1$ , witnessing the 2-adjunction  $(L_\bullet) \dashv (R_\bullet)$  and satisfying the swallowtail equations; and two invertible 3-cells,  $\alpha'_2: (1 \otimes n_2) \circ (e_2 \otimes 1) \rightarrow 1$  and  $\beta'_2: (n_2 \otimes 1) \circ (1 \otimes e_1) \rightarrow 1$ , witnessing the 2-adjunction  $(R_\bullet) \dashv (L_\bullet)$  and satisfying the swallowtail equations;
- two 3-cells,  $c: A^\vee \circ A_\vee \rightarrow 1$  and  $i: 1 \rightarrow A_\vee \circ A^\vee$ , witnessing the adjunction  $A^\vee \dashv A_\vee$  and satisfying the snake equations;
- two 3-cells,  $u_i: n_1 \circ e_2 \rightarrow 1$  and  $v_i: 1 \rightarrow e_2 \circ n_1$  witnessing an adjunction  $e_2 \dashv n_1$  and satisfying the snake equations; two 3-cells  $u_j: 1 \rightarrow n_2 \circ e_1$  and  $v_j: e_1 \circ n_2 \rightarrow 1$  witnessing an adjunction  $n_2 \dashv e_1$  and satisfying the snake equations.

**Theorem 5.3.** For any interpretation of a polygraph into a monoidal category, there exists a 3-functor from the syntactic tricategory of internal diagrams into pointed bimodular profunctors that preserves this interpretation.

*Remark 5.4.* This syntax can be exemplified by evaluating a quantum comb [10], or a monoidal lens [39] with a morphism, in terms of internal string diagrams [24], see Figure 10. It has been used more generally to reason about coends in monoidal categories [41] and topological quantum field theory [3].

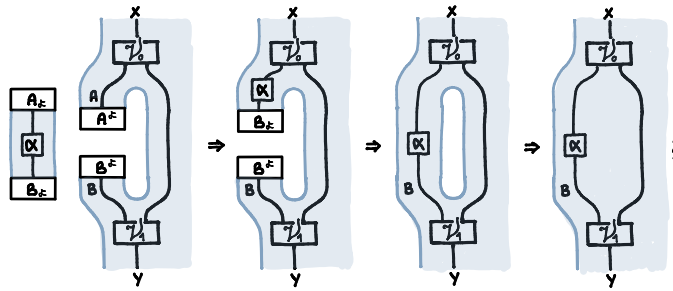


Figure 10: Evaluating a comb in terms of internal string diagrams.

## 6 Conclusions

Collages of string diagrams provide an abundant graphical calculus. Functor boxes, tensors of bimodular categories and internal diagrams all exist in the graphical calculus of collages. Their technical underpinning is complex: we characterized them as diagrams of pointed bimodular profunctors, but these arrange themselves into a tricategory, which may be difficult to reason about.

Apart from introducing the technique of collages and formalizing multiple extensions to string diagrams, we would like to call attention to the techniques we use: most of our results on soundness and completeness of diagrams are arranged into adjunctions, which allows us to prove them by reusing the better-known results on soundness and completeness for monoidal categories and bicategories.

**Related work.** An important line of research revolves around *module categories* and *fusion categories*, some specific enriched categories with actions with applications in topological quantum field theories [17, 18, 35]. Especially relevant and recent is Hoek’s work, which constructs diagrams for a bimodule category [23, Theorem 3.5.2]. We follow the more elementary notion of *bimodular category*, called “*biactegory*” in the taxonomy of Capucci and Gavranović [9]. Cockett and Pastro [12] have used instead *linear actions* for concurrency, and even when we take inspiration from their work, their approach is more sophisticated and expressive than our toy example demonstrating bimodular categories (Figure 5).

Most work has been presented for some particular cases of collages: functor boxes have been extensively employed, but never reduced to string diagrams [13, 32]; internal diagrams have served both quantum theory and category theory [3, 24, 28], and can be given semantics into pointed profunctors [40], but again a presentation as string diagrams was missing. A convenient algebra of lenses [39], a particular type of incomplete diagram, has been recently introduced [19], but this is still independent of the semantics of arbitrary internal diagrams.

Finally, the first author has published a blog post that accompanies this manuscript [8].

**Further work.** It should be possible to “destrictify” many of the results of this paper. We have only presented a 1-adjunction between strict *bimodular categories* and bipointed 2-categories, but a higher adjunction would allow us to reuse coherence for bicategories to automatically obtain coherence for *bimodular categories*. We indicated along the paper the conjectures where further work is warranted.

We conjecture that pointed bimodular profunctors form a compact closed tricategory, with the dual of each monoidal category being the *reverse monoidal category*,  $A \otimes_{\text{Rev}} B = B \otimes A$ . Even when it may be conceptually clear what a compact tricategory should be, it is technically challenging to come up with a concrete definition for it in terms of coherence equations.

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## References

- [1] Mario Alvarez-Picallo, Dan R. Ghica, David Sprunger & Fabio Zanasi (2021): *Functorial String Diagrams for Reverse-Mode Automatic Differentiation*. arXiv:2107.13433.
- [2] Bruce Bartlett (2014): *Quasistrict symmetric monoidal 2-categories via wire diagrams*. arXiv:1409.2148.
- [3] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries & Jamie Vicary (2015): *Modular categories as representations of the 3-dimensional bordism 2-category*. arXiv:1509.06811.
- [4] Guillaume Boisseau & Jeremy Gibbons (2018): *What you needa know about Yoneda: profunctor optics and the Yoneda lemma (functional pearl)*. *Proc. ACM Program. Lang.* 2(ICFP), pp. 84:1–84:27, doi:10.1145/3236779.
- [5] Guillaume Boisseau & Pawel Sobocinski (2021): *String Diagrammatic Electrical Circuit Theory*. In Kohei Kishida, editor: *Proceedings of the Fourth International Conference on Applied Category Theory, ACT 2021, Cambridge, United Kingdom, 12-16th July 2021, EPTCS 372*, pp. 178–191, doi:10.4204/EPTCS.372.13.
- [6] Joe Bolt, Jules Hedges & Philipp Zahn (2019): *Bayesian open games*. *CoRR* abs/1910.03656. arXiv:1910.03656.
- [7] Filippo Bonchi, Joshua Holland, Robin Piedeleu, Paweł Sobociński & Fabio Zanasi (2019): *Diagrammatic algebra: from linear to concurrent systems*. *Proc. ACM Program. Lang.* 3(POPL), pp. 25:1–25:28, doi:10.1145/3290338.
- [8] Dylan Braithwaite (2023): *Diagrams for Actegories*. Available at <https://dylanbraithwaite.github.io/2023/01/31/diagrams-for-actegories.html>.
- [9] Matteo Capucci & Bruno Gavranović (2022): *Actegories for the working amthematician*. arXiv preprint arXiv:2203.16351.
- [10] Giulio Chiribella, Giacomo Mauro D’Ariano & Paolo Perinotti (2009): *Theoretical framework for quantum networks*. *Phys. Rev. A* 80, p. 022339, doi:10.1103/PhysRevA.80.022339.
- [11] Bryce Clarke, Derek Elkins, Jeremy Gibbons, Fosco Loregian, Bartosz Milewski, Emily Pillmore & Mario Román (2022): *Profunctor Optics, a Categorical Update*. arXiv:2001.07488.
- [12] Robin B. Cockett & Craig A. Pastro (2009): *The logic of message-passing*. *Science of Computer Programming* 74(8), pp. 498–533, doi:10.1016/j.scico.2007.11.005.
- [13] Robin B. Cockett & Robert A. G. Seely (1999): *Linearly distributive functors*. *Journal of Pure and Applied Algebra* 143(1-3), pp. 155–203, doi:10.1016/S0022-4049(98)00110-8.
- [14] Bob Coecke, Tobias Fritz & Robert W. Spekkens (2016): *A mathematical theory of resources*. *Information and Computation* 250, pp. 59–86, doi:10.1016/j.ic.2016.02.008.
- [15] Elena Di Lovere, Giovanni de Felice & Mario Román (2022): *Monoidal Streams for Dataflow Programming*. In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS ’22*, Association for Computing Machinery, New York, NY, USA, p. 14, doi:10.1145/3531130.3533365.
- [16] Edsger W. Dijkstra (1962): *Over de sequentialiteit van procesbeschrijvingen*. Unpublished. Transcribed by Gerrit Jan Veltink for the E.W. Dijkstra Archive, Center for American History. Available at <https://www.cs.utexas.edu/users/EWD/ewd00xx/EWD35.PDF>.
- [17] Christopher L. Douglas, Christopher Schommer-Pries & Noah Snyder (2019): *The balanced tensor product of module categories*. *Kyoto Journal of Mathematics* 59(1), doi:10.1215/21562261-2018-0006.
- [18] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych & Victor Ostrik (2010): *On braided fusion categories I*. *Selecta Mathematica* 16(1), pp. 1–119, doi:10.1007/s00029-010-0017-z.
- [19] Matt Earnshaw, James Hefford & Mario Román (2023): *The Produoidal Algebra of Process Decomposition*. arXiv:2301.11867.
- [20] Neil Ghani, Jules Hedges, Viktor Winschel & Philipp Zahn (2018): *Compositional Game Theory*. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS ’18*, Association for Computing Machinery, New York, NY, USA, p. 472–481, doi:10.1145/3209108.3209165.

- [21] James Hefford & Cole Comfort (2022): *Coend Optics for Quantum Combs*. arXiv:2205.09027.
- [22] Chris Heunen & Jamie Vicary (2019): *Categories for Quantum Theory: an introduction*. Oxford University Press, doi:10.1093/oso/9780198739623.001.0001.
- [23] Keeley Hoek (2019): *Drinfeld centers for bimodule categories*. Ph.D. thesis, MSc. thesis, The Australian National University.
- [24] Nick Hu & Jamie Vicary (2021): *Traced Monoidal Categories as Algebraic Structures in Prof*. In Ana Sokolova, editor: *Proceedings 37th Conference on Mathematical Foundations of Programming Semantics, MFPS 2021, Hybrid: Salzburg, Austria and Online, 30th August - 2nd September, 2021, EPTCS 351*, pp. 84–97, doi:10.4204/EPTCS.351.6.
- [25] David A Huffman (1954): *The Synthesis of Sequential Switching Circuits*. *Journal of the Franklin Institute* 257(3), pp. 161–190, doi:10.1016/0016-0032(54)90574-8.
- [26] André Joyal & Ross Street (1991): *The geometry of tensor calculus, I*. *Advances in Mathematics* 88(1), pp. 55–112, doi:10.1016/0001-8708(91)90003-P.
- [27] Paul Blain Levy (2022): *Call-by-Push-Value*. *ACM SIGLOG News* 9(2), p. 7–29, doi:10.1145/3537668.3537670.
- [28] Leo Lobski & Fabio Zanasi (2022): *String Diagrams for Layered Explanations*. CoRR abs/2207.03929, doi:10.48550/arXiv.2207.03929.
- [29] Fosco Loregian (2021): *(Co)end Calculus*. London Mathematical Society Lecture Note Series, Cambridge University Press, doi:10.1017/9781108778657.
- [30] Saunders Mac Lane (1971): *Categories for the Working Mathematician*. *Graduate Texts in Mathematics* 5, Springer Verlag, doi:10.1007/978-1-4757-4721-8.
- [31] Paddy McCrudden (2000): *Categories of representations of coalgebroids*. *Advances in Mathematics* 154(2), pp. 299–332, doi:10.1006/aima.2000.1926.
- [32] Paul-André Melliès (2006): *Functorial Boxes in String Diagrams*. In Zoltán Ésik, editor: *Computer Science Logic, 20th International Workshop, CSL 2006, 15th Annual Conference of the EACSL, Szeged, Hungary, September 25-29, 2006, Proceedings, Lecture Notes in Computer Science 4207*, Springer, pp. 1–30, doi:10.1007/11874683\_1.
- [33] Eugenio Moggi (1991): *Notions of Computation and Monads*. *Inf. Comput.* 93(1), pp. 55–92, doi:10.1016/0890-5401(91)90052-4.
- [34] Chad Nester (2020): *A Foundation for Ledger Structures*. In Emmanuelle Anceaume, Christophe Bisière, Matthieu Bouvard, Quentin Bramas & Catherine Casamatta, editors: *2nd International Conference on Blockchain Economics, Security and Protocols, Tokenomics 2020, October 26-27, 2020, Toulouse, France, OASICS 82, Schloss Dagstuhl - Leibniz-Zentrum für Informatik*, pp. 7:1–7:13, doi:10.4230/OASICS.Tokenomics.2020.7.
- [35] Victor Ostrik (2003): *Module categories, weak Hopf algebras and modular invariants*. *Transformation groups* 8, pp. 177–206, doi:10.1007/s00031-003-0515-6.
- [36] Craig Pastro & Ross Street (2007): *Doubles for monoidal categories*. arXiv preprint arXiv:0711.1859.
- [37] Evan Patterson, David I. Spivak & Dmitry Vagner (2021): *Wiring diagrams as normal forms for computing in symmetric monoidal categories*. *Electronic Proceedings in Theoretical Computer Science* 333, pp. 49–64, doi:10.4204/eptcs.333.4.
- [38] Maciej Piróg & Nicolas Wu (2016): *String diagrams for free monads (functional pearl)*. In Jacques Garigue, Gabriele Keller & Eijiro Sumii, editors: *Proceedings of the 21st ACM SIGPLAN International Conference on Functional Programming, ICFP 2016, Nara, Japan, September 18-22, 2016, ACM*, pp. 490–501, doi:10.1145/2951913.2951947.
- [39] Mitchell Riley (2018): *Categories of optics*. arXiv preprint arXiv:1809.00738.
- [40] Mario Román (2020): *Comb Diagrams for Discrete-Time Feedback*. CoRR abs/2003.06214, arXiv:2003.06214.

- [41] Mario Román (2020): *Open Diagrams via Coend Calculus*. In David I. Spivak & Jamie Vicary, editors: *Proceedings of the 3rd Annual International Applied Category Theory Conference 2020, ACT 2020, Cambridge, USA, 6-10th July 2020, EPTCS 333*, pp. 65–78, doi:[10.4204/EPTCS.333.5](https://doi.org/10.4204/EPTCS.333.5).
- [42] Mario Román (2022): *Promonads and String Diagrams for Effectful Categories*. CoRR abs/2205.07664, doi:[10.48550/arXiv.2205.07664](https://doi.org/10.48550/arXiv.2205.07664). arXiv:[2205.07664](https://arxiv.org/abs/2205.07664).
- [43] Michael Shulman (2016): *Categorical logic from a categorical point of view*. Available on the web. Available at <https://mikeschulman.github.io/catlog/catlog.pdf>.
- [44] Abraham Silberschatz, Peter Baer Galvin & Greg Gagne (2018): *Operating System Concepts, 10th Edition*. Wiley. Available at <http://os-book.com/OS10/index.html>.
- [45] David I. Spivak (2022): *Generalized Lens Categories via functors  $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}$* . arXiv:[1908.02202](https://arxiv.org/abs/1908.02202).
- [46] Sam Staton & Paul Blain Levy (2013): *Universal properties of impure programming languages*. In Roberto Giacobazzi & Radhia Cousot, editors: *The 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '13, Rome, Italy - January 23 - 25, 2013*, ACM, pp. 179–192, doi:[10.1145/2429069.2429091](https://doi.org/10.1145/2429069.2429091).
- [47] Ross Street (1981): *Cauchy characterization of enriched categories*. *Rendiconti del Seminario Matematico e Fisico di Milano* 51(1), pp. 217–233, doi:[10.1007/BF02924823](https://doi.org/10.1007/BF02924823).
- [48] Zoran Škoda (2009): *Some equivariant constructions in noncommutative algebraic geometry*. *Georgian Mathematical Journal* 16(1), pp. 183–202, doi:[10.1515/GMJ.2009.183](https://doi.org/10.1515/GMJ.2009.183). arXiv:[0811.4770](https://arxiv.org/abs/0811.4770).