Coequalisers under the Lens

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Lenses encode protocols for synchronising systems. We continue the work begun by Chollet et al. at the Applied Category Theory Adjoint School in 2020 to study the properties of the category of small categories and asymmetric delta lenses. The forgetful functor from the category of lenses to the category of functors is already known to reflect monos and epis and preserve epis; we show that it preserves monos, and give a simpler proof that it preserves epis. Together this gives a complete characterisation of the monic and epic lenses in terms of elementary properties of their get functors.

Next, we initiate the study of coequalisers of lenses. We observe that not all parallel pairs of lenses have coequalisers, and that the forgetful functor from the category of lenses to the category of functors neither preserves nor reflects all coequalisers. However, some coequalisers are reflected; we study when this occurs, and then use what we learned to show that every epic lens is regular, and that discrete opfibrations have pushouts along monic lenses. Corollaries include that every monic lens is effective, every monic epic lens is an isomorphism, and the class of all epic lenses and the class of all monic lenses form an orthogonal factorisation system.

1 Introduction

A *bidirectional transformation* between two systems is a specification of when the joint state of the two systems should be regarded as consistent, together with a protocol for updating each system to restore consistency in response to a change in the other [12]. The study of bidirectional transformations goes back to as far as 1981 with Bancilhon and Spyrato's work on the view-update problem for databases [2]. The view-update problem is about *asymmetric* bidirectional transformations; those where the state of one of the systems, called the *view*, is completely determined by that of the other, called the *source*. Bidirectional transformations also arise in many other contexts across computer science, such as when programming with complex data structures and when linking user interfaces to data models.

An *asymmetric state-based lens* is a mathematical encoding of an asymmetric bidirectional transformation in which the consistency restoration updates to the source are assumed to be dependent only on the old source state and the updated view state. If S is the set of source states and V is the set of view states, such a lens consists of a *get function* $S \to V$ and a *put function* $S \times V \to S$ which, ideally, satisfy certain laws. The earliest known account of asymmetric state-based lenses may be found in Oles' PhD thesis [18, Chapter VI], where they are called *extensions* of *store shapes*; they are a key ingredient in Oles' semantics for an imperative stack-based programming language with block-scoped variables because they capture the essential properties of a data store which changes shape as variables come into and go out of scope. All recent notions of lens, including the name *lens*, may be traced back to the work of Pierce et al. [10]; they proposed variants of asymmetric state-based lenses for modelling bidirectional transformations on tree-structured data, and they also introduced the idea of building lenses compositionally with a domain-specific language such as their lens combinators.

Diskin et al. highlighted the inadequacy of state-based lenses as a general mathematical model for bidirectional transformations [8], providing several examples of situations in which consistency restoration would benefit from knowing more about each change to the view than just the view's new state. In

an *asymmetric delta lens*, their proposed alternative, systems are modelled as categories of states and transitions (deltas) rather than simply as sets of states. Also, the put operation takes as input specifically which transition occurred in the view rather than just the end state of that transition.

Application of category theory to the study of lenses has already proved fruitful. Johnson and Rose-brugh's research program [13, 14, 15] has enabled a unified treatment of symmetric and asymmetric delta lenses, with the perspective that a symmetric delta lens is an equivalence class of spans of asymmetric delta lenses. Ahman and Uustalu's observation that asymmetric delta lenses are compatible functor co-functor pairs [1], and Clarke's generalisation of these lenses to the internal category theory setting [6], have enabled an abstract diagrammatic approach to proofs involving these lenses [7], in which we may profit from the already well-developed theory of functors and opfibrations. Yet, until the work of Chollet et al. [5], little was known about the category of asymmetric delta lenses. Building on their work, this paper aims to further our understanding of this category.

Outline

Henceforth, we refer to asymmetric delta lenses simply as *lenses*, which we formally define in Section 2.

In Section 3, we prove the conjecture by Chollet et al. [5] that the forgetful functor from the category of lenses to the category of functors preserves monos. Together with their result that it reflects monos, we deduce that the monic lenses are the unique lenses on cosieves; these are equivalently the out-degree-zero subcategory inclusion functors. We also provide a proof, simpler than the original one sketched by Lack in an unpublished personal communication to Clarke, that the forgetful functor preserves epis.

In Section 4, we initiate the study of coequalisers of lenses. We begin with examples of how they are not as well behaved as one might hope; specifically, not all parallel pairs of lenses have coequalisers, and the forgetful functor neither preserves nor reflects all coequalisers. We then prove our main result, Theorem 4.5, which is about the coequalisers that are actually reflected by the forgetful functor.

In Section 5, we use Theorem 4.5 to show that the category of lenses has pushouts of discrete opfi-brations along monos. We then show that every monic lens is effective. It follows that the classes of all monos, all effective monos, all regular monos, all strong monos and all extremal monos in the category of lenses coincide, and thus also that all lenses which are both monic and epic are isomorphisms.

In Section 6, we use Theorem 4.5 again to show that every epic lens is regular. It follows that the classes of all epis, all regular epis, all strong epis and all extremal epis in the category of lenses coincide. It also follows that the class of all epic lenses is left orthogonal to the class of all monic lenses. Together with other known results, this means that they form an orthogonal factorisation system.

2   Background

Notation

Application of functions (functors, lenses, etc.) is written by juxtaposing the function name with its argument. Application is right associative, so an expression like FGx parses as $F(Gx)$ and not $(FG)x$. Parentheses are only used when needed or for clarity. Binary operators like $\circ$ have lower precedence than application, so an expression like $Fa \circ Fb$ parses as $(Fa) \circ (Fb)$ and not $F((a \circ F)b)$.

Let $\mathcal{C}at$ denote the category whose objects are small categories and whose morphisms are functors. Categories with boldface names $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc. are always small. We write $|\mathbf{C}|$ for the set of objects of a small category $\mathbf{C}$, and, for all $X, Y \in |\mathbf{C}|$, we write $\mathbf{C}(X, Y)$ for the set of morphisms of $\mathbf{C}$ from X to Y. For each $X \in |\mathbf{C}|$, we write $\mathbf{C}(X, *)$ for the set $\bigsqcup_{Y \in |\mathbf{C}|} \mathbf{C}(X, Y)$ of all morphisms in $\mathbf{C}$ out of X. We write

src f and tgt f for, respectively, the source and target of a morphism f. We also write $f\colon X \to Y$ to say that $X, Y \in |\mathbf{C}|$ and $f \in \mathbf{C}(X, Y)$. The composite of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted $g \circ f$.

The category with a single object 0 and no non-identity morphisms, also known as the *terminal category*, is denoted $\mathbf{1}$. The category with two objects 0 and 1 and a single non-identity morphism, namely $u\colon 0 \to 1$, also known as the *interval category*, is denoted $\mathbf{2}$. The category with two objects 0 and 1 and two non-identity morphisms, namely $v\colon 0 \to 1$ and $v^{-1}\colon 1 \to 0$, also known as the *free living isomorphism*, is denoted $\mathbf{I}$. We will identify objects and morphisms of a small category $\mathbf{C}$ with the corresponding functors $\mathbf{1} \to \mathbf{C}$ and $\mathbf{2} \to \mathbf{C}$ respectively.

If the square

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{T} & \mathbf{B} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array} \tag{1}$$

in $\mathcal{C}at$ is a pushout square and $F'\colon \mathbf{A} \to \mathbf{E}$ and $G'\colon \mathbf{B} \to \mathbf{E}$ are functors for which $F' \circ S = G' \circ T$, then we write $[F', G']$ for the functor $\mathbf{C} \to \mathbf{E}$ induced from F' and G' by the universal property of the pushout. Similarly, if the square (1) in $\mathcal{C}at$ is a pullback square and $S'\colon \mathbf{E} \to \mathbf{A}$ and $T'\colon \mathbf{E} \to \mathbf{B}$ are functors for which $F \circ S' = G \circ T'$, then we write $\langle S', T' \rangle$ for the functor $\mathbf{E} \to \mathbf{D}$ induced from S' and T' by the universal property of the pullback. By our identification of objects with functors from $\mathbf{1}$ mentioned above, if $A \in |\mathbf{A}|$ and $B \in |\mathbf{B}|$ are such that $FA = GB$, then $\langle A, B \rangle$ is the object of $\mathbf{D}$ selected by the functor $\mathbf{1} \to \mathbf{D}$ induced by the universal property of the pullback from the functors $\mathbf{1} \to \mathbf{A}$ and $\mathbf{1} \to \mathbf{B}$ that respectively select the objects A and B.

Lenses and discrete opfibrations

First, we recall the definition of a (asymmetric delta) lens [8].

Definition 2.1. Given small categories $\mathbf{A}$ and $\mathbf{B}$, a *lens* $F\colon \mathbf{A} \to \mathbf{B}$ consists of

- a functor $F\colon \mathbf{A} \to \mathbf{B}$, called the *get functor* of F, and

- a function $F^A\colon \mathbf{B}(FA, *) \to \mathbf{A}(A, *)$ for each $A \in |\mathbf{A}|$, collectively known as the *put functions*,

such that

- *PutGet*: $F F^A b = b$ for all $A \in |\mathbf{A}|$ and all $b \in \mathbf{B}(FA, *)$,

- *PutId*: $F^A \mathrm{id}_{FA} = \mathrm{id}_A$ for all $A \in |\mathbf{A}|$, and

- *PutPut*: $F^A(b' \circ b) = F^{A'} b' \circ F^A b$ for all $A \in |\mathbf{A}|$, $b \in \mathbf{B}(FA, *)$, $b' \in \mathbf{B}(FA', *)$, where $A' = \mathrm{tgt}\, F^A b$.

There is a category $\mathcal{L}ens$ whose objects are small categories and whose morphisms are lenses. The composite $G \circ F$ of lenses $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ has get functor which is the composite of the get functors of G and F, and has $(G \circ F)^A c = F^A G^{FA} c$ for all $A \in |\mathbf{A}|$ and all $c \in \mathbf{C}(GFA, *)$. There is also an identity-on-objects forgetful functor $\mathcal{U}\colon \mathcal{L}ens \to \mathcal{C}at$ that sends a lens to its get functor.

Definition 2.2. A functor $F\colon \mathbf{A} \to \mathbf{B}$ is a *discrete opfibration* if, for each $A \in |\mathbf{A}|$ and each $b \in \mathbf{B}(FA, *)$, there is a unique $a \in \mathbf{A}(A, *)$ such that $Fa = b$.

Remark 2.3. If $F\colon \mathbf{A} \to \mathbf{B}$ is a discrete opfibration, then there is a unique lens mapped by $\mathcal{U}$ to F. We will sometimes also use the name F to refer to this unique lens above F.

We also recall Johnson and Roseburgh's "pullback" of a cospan of lenses [13], which we will refer to as their *proxy pullback*, adopting the terminology of Bumpus and Kocsis [17].

Definition 2.4. The *proxy pullback* of a lens cospan $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$ is a lens span $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ where

- the get functors of $\overline{F}$ and $\overline{G}$ form a pullback square

$$
\begin{array}{ccc}
\mathbf{D} & \xrightarrow{\ \mathcal{U}\overline{F}\ } & \mathbf{B} \\
{\scriptstyle \mathcal{U}\overline{G}}\downarrow & & \downarrow{\scriptstyle \mathcal{U}G} \\
\mathbf{A} & \xrightarrow[\ \mathcal{U}F\ ]{} & \mathbf{C}
\end{array}
$$

in $\mathcal{C}at$ (this determines them up to isomorphism), and

- for each $D \in |\mathbf{D}|$, each $a \in \mathbf{A}\big(\overline{G}D, *\big)$, and each $b \in \mathbf{B}\big(\overline{F}D, *\big)$,

$$
\overline{F}^D b = \big\langle F^{\overline{G}D} G b, b \big\rangle \qquad \text{and} \qquad \overline{G}^D a = \big\langle a, G^{\overline{F}D} F a \big\rangle.
$$

When $F = G$, the lenses $\overline{F}, \overline{G} \colon \mathbf{D} \to \mathbf{A}$ are also called the *proxy kernel pair* of F.

3   Characterising monic and epic lenses

Monic lenses

We will study the monos in $\mathcal{L}ens$ via their relation to those in $\mathcal{C}at$, expressed as follows.

Theorem 3.1. *The functor $\mathcal{U}$ preserves and reflects monos.*

Reflection was proved and preservation conjectured by Chollet et al. [5]. Recalling that a morphism is monic if and only if it has a kernel pair with both morphisms equal, we may prove preservation.

Proof that $\mathcal{U}$ preserves monos. Let $M \colon \mathbf{A} \to \mathbf{B}$ be a monic lens, and let $P_1, P_2 \colon \mathbf{Ker}\,\mathcal{U}M \to \mathbf{A}$ be its proxy kernel pair in $\mathcal{L}ens$. As M is monic and $M \circ P_1 = M \circ P_2$, actually $P_1 = P_2$, and so $\mathcal{U}P_1 = \mathcal{U}P_2$. But $\mathcal{U}P_1$ and $\mathcal{U}P_2$ are the (real) kernel pair of $\mathcal{U}M$ in $\mathcal{C}at$. Hence $\mathcal{U}M$ is a monic functor. $\qquad\square$

Chollet et al. [5] also showed that the get functor of a lens is monic if and only if it is a cosieve.

Definition 3.2. A *cosieve* is an injective-on-objects discrete opfibration.

Corollary 3.3. *The functor $\mathcal{U}$ restricts to a bijection between monic lenses and cosieves.*

Proof. A cosieve is a discrete opfibration, so there is a unique lens above it; by reflection, this lens is monic. Conversely, the get functor of a monic lens is, by preservation, monic, and so is a cosieve. $\qquad\square$

The above result says that monic lenses and cosieves are essentially the same. We continue to use the term cosieve for functors when we wish to distinguish these from monic lenses.

Lens images and factorisation

The images of the object and morphism maps of a functor do not always form a subcategory of a functor's target category. The situation is nicer for the get functor of a lens F; in this case, the images actually form an out-degree-zero subcategory $\mathbf{Im}\,F$ of the lens' target category, which we will call the *image* of F. By *out-degree-zero* subcategory, we mean one for which any morphism out of an object in the subcategory belongs to the subcategory. As cosieves are exactly the out-degree-zero subcategory inclusion functors, we obtain the following factorisation result.

Proposition 3.4. *Every lens $F\colon \mathbf{A} \to \mathbf{B}$ has a factorisation*

$$\mathbf{A} \xrightarrow[E]{} \mathbf{Im}\,F \xrightarrow[M]{} \mathbf{B} \qquad (F)$$

in $\mathcal{L}ens$ where M is monic and E is surjective on objects and morphisms.

Recall that a morphism $e\colon A \to B$ is *left orthogonal* to a morphism $m\colon C \to D$ if, for all pairs of morphisms $f\colon A \to C$ and $g\colon B \to D$ such that $g \circ e = m \circ f$, there is a unique morphism $h\colon B \to C$, called the *diagonal filler*, such that $f = h \circ e$ and $g = m \circ h$. Also recall that classes $\mathscr{E}$ and $\mathscr{M}$ of morphisms form an *orthogonal factorisation system* if $\mathscr{E}$ is the class of all morphisms that are left orthogonal to all morphisms in $\mathscr{M}$, and every morphism f factors as $f = m \circ e$ for some $e \in \mathscr{E}$ and some $m \in \mathscr{M}$.

Remark 3.5. The above factorisation is already known to Johnson and Roseburgh, who showed that the surjective-on-objects lenses and the injective-on-objects-and-morphisms lenses form an orthogonal factorisation system on $\mathcal{L}ens$ [16]. Our addition is that this is actually an epi-mono factorisation system; we have already shown that the injective-on-objects-and-morphisms lenses are exactly the monic lenses, and we will show in the next section that the surjective-on-objects lenses are exactly the epic lenses. In Section 6, we will also deduce the orthogonality without explicitly constructing the diagonal fillers.

Epic lenses

We may also study the epis in $\mathcal{L}ens$ via their relation to those in $\mathcal{C}at$.

Theorem 3.6. *The functor $\mathcal{U}$ preserves and reflects epis.*

Again, reflection was proved and preservation conjectured by Chollet et al. [5]. The first proof of preservation was sketched by Lack in an unpublished personal communication to Clarke; we present a new, simpler proof below. First, we recall some preliminary results about epic functors and epic lenses.

Proposition 3.7. *Every epic functor is surjective on objects. Every functor that is surjective both on objects and on morphisms is epic.*

Recall that not all epic functors are surjective on morphisms.

Example 3.8. Let $J\colon \mathbf{2} \to \mathbf{I}$ be the functor that sends the non-identity morphism u of the interval category $\mathbf{2}$ to the morphism v of the free living isomorphism $\mathbf{I}$. Then J is epic because any two functors out of $\mathbf{I}$ which agree on v must also agree on v^{-1}. However, the morphism v^{-1} is not in the image of J.

Proposition 3.9. *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a lens, and let $\bar{J}_1, \bar{J}_2 \colon \mathbf{B} \to \mathbf{C}$ be the cokernel pair of $\mathcal{U}F$. Then $\bar{J}_1$ and $\bar{J}_2$ are cosieves, and the unique lenses J_1 and J_2 above $\bar{J}_1$ and $\bar{J}_2$ satisfy $J_1 \circ F = J_2 \circ F$.*

Proof. Let $F = M \circ E$ be the factorisation of F given in Proposition 3.4. By Proposition 3.7, $\mathcal{U}E$ is an epic functor. As $\bar{J}_1 \circ \mathcal{U}M \circ \mathcal{U}E = \bar{J}_1 \circ \mathcal{U}F = \bar{J}_2 \circ \mathcal{U}F = \bar{J}_2 \circ \mathcal{U}M \circ \mathcal{U}E$, actually $\bar{J}_1 \circ \mathcal{U}M = \bar{J}_2 \circ \mathcal{U}M$. It follows that $\bar{J}_1$ and $\bar{J}_2$ are also the cokernel pair of $\mathcal{U}M$. As cosieves are pushout stable and $\mathcal{U}M$ is a cosieve, so are $\bar{J}_1$ and $\bar{J}_2$. As there is a unique lens above the discrete opfibration $\bar{J}_1 \circ \mathcal{U}M = \bar{J}_2 \circ \mathcal{U}M$, we must have that $J_1 \circ M = J_2 \circ M$. $\qquad\square$

Remark 3.10. Later, we will see that J_1 and J_2 are actually a cokernel pair of F in $\mathcal{L}ens$.

Proof that $\mathcal{U}$ preserves epis. Let $E\colon \mathbf{A} \to \mathbf{B}$ be an epic lens, and J_1 and J_2 the unique lenses above the cokernel pair of $\mathcal{U}E$ from Proposition 3.9. As $J_1 \circ E = J_2 \circ E$ and E is epic, actually $J_1 = J_2$, and so $\mathcal{U}J_1 = \mathcal{U}J_2$. But $\mathcal{U}J_1$ and $\mathcal{U}J_2$ are the cokernel pair of $\mathcal{U}E$, so $\mathcal{U}E$ is also epic. $\qquad\square$

Corollary 3.11. *Let F be a lens. Then the following are equivalent:*

(1) *F is epic,*

(2) $\mathcal{U}F$ *is surjective on objects,*

(3) $\mathcal{U}F$ *is surjective on morphisms.*

Proof. Chollet et al. [5] showed that (2) and (3) are equivalent, and imply (1). Suppose that F is epic. As $\mathcal{U}$ preserves epis (Theorem 3.6), so is $\mathcal{U}F$. By Proposition 3.7, $\mathcal{U}F$ is surjective on objects. □

4   Coequalisers of lenses

Given morphisms $f_1, f_2 \colon A \to B$, we say that a morphism $e \colon B \to C$ *coforks* f_1 and f_2 if $e \circ f_1 = e \circ f_2$. Some authors would use the verb coequalise where we use the verb cofork. Unlike those authors, we say that *e coequalises* f_1 and f_2 only when e is universal among coforks of f_1 and f_2.

Non-existence, non-preservation and non-reflection of coequalisers

Recall that $\mathcal{C}at$ has all coequalisers. Shortly, we will construct several counterexamples to the well-behavedness of coequalisers in $\mathcal{L}ens$, at least with respect to those in $\mathcal{C}at$. To do this, we will use the following proposition, which gives necessary conditions for a cofork of lenses to be a coequaliser.

Proposition 4.1. *Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be lenses with coequaliser $E \colon \mathbf{B} \to \mathbf{C}$ in $\mathcal{L}ens$. Then*

(1) *for each cofork $G \colon \mathbf{B} \to \mathbf{D}$ of F_1 and F_2, $G^B d = E^B E G^B d$ for all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$; and*

(2) *in particular, E is the unique lens above $\mathcal{U}E$ that coforks F_1 and F_2.*

Proof. For (1), if $G \colon \mathbf{B} \to \mathbf{D}$ coforks F_1 and F_2, then there is a lens $H \colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$, and so $G^B d = (H \circ E)^B d = E^B H^{EB} d = E^B E E^B H^{EB} d = E^B E (H \circ E)^B d = E^B E G^B d$. For (2), if $G \colon \mathbf{B} \to \mathbf{C}$ is a lens above $\mathcal{U}E$ that coforks F_1 and F_2, then $G^B c = E^B E G^B c = E^B G G^B c = E^B c$ for each $B \in |\mathbf{B}|$ and each $c \in \mathbf{C}(EB, *)$, and so $G = E$. □

The first example shows that $\mathcal{L}ens$ does not have all coequalisers, nor does $\mathcal{U}$ reflect them.

Example 4.2. Let $\mathbf{A}$ and $\mathbf{B}$ be the preordered sets generated respectively by the following graphs.

$$Y_1 \xleftarrow{\ f_1\ } X \xrightarrow{\ f_2\ } Y_2 \qquad\qquad\qquad Y_1' \xleftarrow{\ f_1'\ } X' \xrightarrow{\ f_2'\ } Y_2'$$
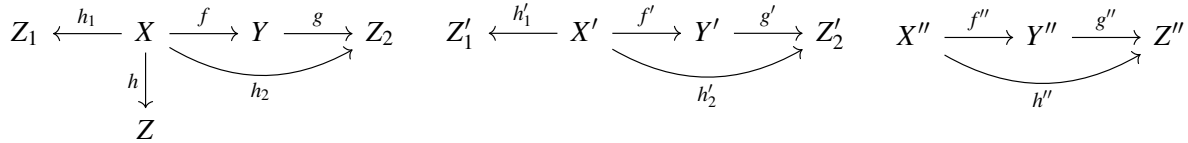$$\Big\downarrow{\scriptstyle f}$$
$$Y$$

Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be the unique lenses that both send X to X', Y_1 to Y_1', Y_2 to Y_2', and such that $F_1 Y = Y_1'$, $F_1{}^X f_1' = f_1$, $F_2 Y = Y_2'$, and $F_2{}^X f_2' = f_2$. Let $G \colon \mathbf{B} \to \mathbf{2}$ be the unique functor that sends X' to 0, and both Y_1' and Y_2' to 1; G coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. There are only two lens structures on G that cofork F_1 and F_2 in $\mathcal{L}ens$; one is determined by $G_1{}^{X'} u = f_1'$ and the other by $G_2{}^{X'} u = f_2'$. By Proposition 4.1, neither G_1 nor G_2 coequalises F_1 and F_2. Thus $\mathcal{U}$ does not reflect the coequaliser G of $\mathcal{U}F_1$ and $\mathcal{U}F_2$.

Actually F_1 and F_2 do not have a coequaliser in $\mathcal{L}ens$. Assume that $E \colon \mathbf{B} \to \mathbf{C}$ is such a coequaliser. Then $E f_1' = E F_1 f = E F_2 f = E f_2'$. As G_1 coforks F_1 and F_2, there is a lens $H \colon \mathbf{C} \to \mathbf{2}$ such that $G_1 = H \circ E$. As $HEX' = G_1 X' \neq G_1 Y_1' = HEY_1'$, we must have $EX' \neq EY_1'$. Hence EX' and EY_1' are distinct

objects of the image of E, and $\mathrm{id}_{EX'}$, Ef_1' and $\mathrm{id}_{EY_1'}$ are distinct morphisms of the image of E. As E is a coequaliser, it is epi, and so, by Corollary 3.11, its image is all of $\mathbf{C}$. Thus $\mathcal{U}H$ is an isomorphism in $\mathcal{C}at$, and so H is an isomorphism in $\mathcal{L}ens$. Hence G_1 also coequalises F_1 and F_2, which is a contradiction.

There are even parallel pairs of lenses for which the coequaliser of their get functors has a unique lens structure that coforks them, and yet does not coequalise them.

Example 4.3. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be the preordered sets generated respectively by the following graphs.

$$Z_1 \xleftarrow{\ h_1\ } X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z_2 \qquad Z_1' \xleftarrow{\ h_1'\ } X' \xrightarrow{\ f'\ } Y' \xrightarrow{\ g'\ } Z_2' \qquad X'' \xrightarrow{\ f''\ } Y'' \xrightarrow{\ g''\ } Z''$$

with $h\colon X \to Z$, $h_2\colon X \to Z_2$, $h_2'\colon X' \to Z_2'$, $h''\colon X'' \to Z''$.

Let $F_1, F_2\colon \mathbf{A} \to \mathbf{B}$ be the unique lenses that both send X to X', Y to Y', Z_1 to Z_1', Z_2 to Z_2', and such that $F_1 Z = Z_1'$, $F_1^X h_1' = h_1$ and $F_2 Z = Z_2'$. Let $E\colon \mathbf{B} \to \mathbf{C}$ be the unique lens that sends X' to X'', Y' to Y'', and both Z_1' and Z_2' to Z''. Then $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$, and E coforks F_1 and F_2 in $\mathcal{L}ens$. However, E does not coequalise F_1 and F_2 in $\mathcal{L}ens$. Indeed, if $G\colon \mathbf{B} \to \mathbf{2}$ is the unique lens that sends X' to 0, all of Y', Z_1' and Z_2' to 1, and for which $G^{X'} u = h_1'$, then $E^{X'} E G^{X'} u = E^{X'} E h_1' = E^{X'} h'' = h_2' \neq h_1' = G^{X'} u$.

The final example shows that $\mathcal{U}$ does not preserve coequalisers. It also shows that there are parallel pairs of lenses for which the coequaliser of their get functors has no lens structure that coforks them.

Example 4.4. Let $\mathbf{A}$ be the preordered set generated by the graph

$$Y_1 \xleftarrow{\ f_1\ } X \xrightarrow{\ f_2\ } Y_2$$
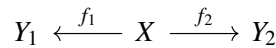
Let $I\colon \mathbf{A} \to \mathbf{A}$ denote the identity lens, and let $S\colon \mathbf{A} \to \mathbf{A}$ be the unique lens that maps X to X, Y_1 to Y_2 and Y_2 to Y_1. The coequaliser of $\mathcal{U}I$ and $\mathcal{U}S$ in $\mathcal{C}at$ is the unique functor $E\colon \mathbf{A} \to \mathbf{2}$ that sends X to 0 and both Y_1 and Y_2 to 1. Recall that $\mathbf{1}$ is terminal in $\mathcal{L}ens$ [5]. We claim that the coequaliser of I and S in $\mathcal{L}ens$ is the unique lens $E\colon \mathbf{A} \to \mathbf{1}$. Let $G\colon \mathbf{A} \to \mathbf{C}$ be a lens that coforks I and S in $\mathcal{L}ens$. Let $f = Gf_1$. Then $f = Gf_1 = GIf_1 = GSf_1 = Gf_2$. As $G^X f \in \mathbf{A}(X, *)$, it is one of f_1, f_2 and id_X. If $G^X f = f_1$, then

$$f_1 = I^X f_1 = I^X G^X f = (G \circ I)^X f = (G \circ S)^X f = S^X G^X f = S^X f_1 = f_2,$$

which is a contradiction. We get a similar contradiction if $G^X f = f_2$. By elimination, $G^X f = \mathrm{id}_X$, and so $f = G G^X f = G\,\mathrm{id}_X = \mathrm{id}_{GX}$. The image of G thus consists of the object GX and the morphism id_{GX}. If $H\colon \mathbf{1} \to \mathbf{C}$ is a lens such that $G = H \circ E$, then H must send 0 to GX, and this uniquely determines H. As the image of any lens, in particular G, is an out-degree-zero subcategory of its target category, this definition of H does indeed give a lens, and $G = H \circ E$. Of course, the factorisation $G = H \circ E$ is really the image factorisation of G from Remark 3.5.

Coequalisers which are reflected

Although the counterexamples above suggest that coequalisers in $\mathcal{L}ens$ have little relation to those in $\mathcal{C}at$, we will see in Theorem 5.4 and Corollary 6.4 two classes of coequalisers in $\mathcal{L}ens$ which do lie over coequalisers in $\mathcal{C}at$. The following theorem, a partial converse to Proposition 4.1, reduces checking the coequaliser property in these cases to checking that Equation (2) below always holds.

Theorem 4.5. *Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be lenses. Let $E \colon \mathbf{B} \to \mathbf{C}$ be a cofork of F_1 and F_2 in $\mathcal{L}ens$, and suppose that $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. Then E coequalises F_1 and F_2 in $\mathcal{L}ens$ if and only if for all lenses $G \colon \mathbf{B} \to \mathbf{D}$ that cofork F_1 and F_2 in $\mathcal{L}ens$, all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$, we have*

$$G^B d = E^B E G^B d. \tag{2}$$

In the proof of the following lemma and again, later, in the proof of Lemma 5.3, we use the induction principle for the equivalence relation $\simeq$ on a set S generated by a binary relation R on S, that is,

$$\forall P\, x_0\, y_0. \begin{bmatrix} x_0 \simeq y_0 \\ \quad \wedge \quad \forall x\, y.\ x\, R\, y \implies P(x,y) \\ \quad \wedge \quad \forall x.\ P(x,x) \\ \quad \wedge \quad \forall x\, y.\ [x \simeq y \wedge P(x,y)] \implies P(y,x) \\ \quad \wedge \quad \forall x\, y\, z.\ [x \simeq y \wedge P(x,y) \wedge y \simeq z \wedge P(y,z)] \implies P(x,z) \end{bmatrix} \implies P(x_0, y_0). \tag{3}$$

Lemma 4.6. *Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be lenses. Let $E \colon \mathbf{B} \to \mathbf{C}$ be a cofork of F_1 and F_2 in $\mathcal{L}ens$, and suppose that $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. Let $G \colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2 in $\mathcal{L}ens$, and let $H \colon \mathbf{C} \to \mathbf{D}$ be the unique functor such that $\mathcal{U}G = H \circ \mathcal{U}E$. Then there is a unique lens structure on H that, for all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$, satisfies the equation*

$$H^{EB} d = E G^B d. \tag{4}$$

Proof. For each $C \in |\mathbf{C}|$, as $\mathcal{U}E$ is epic, there is a $B \in |\mathbf{B}|$ such that $EB = C$. Hence, we may define H^C using Equation (4), so long as, for all $B_1, B_2 \in |\mathbf{B}|$, if $EB_1 = EB_2$ then, for all $d \in \mathbf{D}(EB_1, *)$, we have $E G^{B_1} d = E G^{B_2} d$. Let $\simeq$ be the smallest equivalence relation on $|\mathbf{B}|$ such that $F_1 A \simeq F_2 A$ for all $A \in |\mathbf{A}|$. As $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$, we have [3, Proposition 4.1], for all $B_1, B_2 \in |\mathbf{B}|$, that $EB_1 = EB_2$ if and only if $B_1 \simeq B_2$. We proceed using the induction principle in Equation (3). The proof obligations from the reflexivity, symmetry and transitivity axioms for $\simeq$ hold as $=$ is an equivalence relation. For the remaining one, for all $A \in |\mathbf{A}|$ and all $d \in \mathbf{D}(F_1 A, *)$, we have

$$E G^{F_1 A} d = E F_1 F_1^A G^{F_1 A} d = (E \circ F_1)(G \circ F_1)^A d = (E \circ F_2)(G \circ F_2)^A d = E F_2 F_2^A G^{F_2 A} d = E G^{F_2 A} d.$$

Define H^C using Equation (4). It remains to check that the lens laws hold for H. For all $C \in |\mathbf{C}|$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $H^C \mathrm{id}_{HC} = E G^B \mathrm{id}_{GB} = E \mathrm{id}_B = \mathrm{id}_C$; hence PutId holds. For all $C \in |\mathbf{C}|$, all $d \in \mathbf{D}(HC, *)$ and all $d' \in \mathbf{D}(\mathrm{tgt}\, d, *)$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and

$$H^C(d' \circ d) = E G^B(d' \circ d) = E\big(G^{B'} d' \circ G^B d\big) = E G^{B'} d' \circ E G^B d = H^{C'} d' \circ H^C d,$$

where $B' = \mathrm{tgt}\, G^B d$ and $C' = EB'$; hence PutPut holds. Finally, for all $C \in |\mathbf{C}|$ and all $d \in \mathbf{D}(HC, *)$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $HH^C d = HEG^B d = GG^B d = d$; hence PutGet holds. $\square$

Proof of Theorem 4.5. We proved the *only if* direction in Proposition 4.1. For the *if* direction, suppose, for all lenses $G \colon \mathbf{B} \to \mathbf{D}$ that cofork F_1 and F_2, that Equation (2) always holds. We must show that E is the universal cofork of F_1 and F_2 in $\mathcal{L}ens$. Let $G \colon \mathbf{B} \to \mathbf{D}$ be another cofork of F_1 and F_2 in $\mathcal{L}ens$. Suppose that there is a lens $H \colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$. Then $\mathcal{U}G = \mathcal{U}H \circ \mathcal{U}E$, and so $\mathcal{U}H$ is the unique functor that composes with $\mathcal{U}E$ to give $\mathcal{U}G$. Let $C \in |\mathbf{C}|$ and $d \in \mathbf{D}(HC, *)$. As $\mathcal{U}E$ is epic, there is a $B \in |\mathbf{B}|$ such that $EB = C$. Then $H^C d = E E^B H^C d = E(H \circ E)^B d = E G^B d$. Hence H is uniquely determined. Now let $H \colon \mathbf{C} \to \mathbf{D}$ be the lens defined as in Lemma 4.6. For all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$, we have $G^B d = E^B E G^B d = E^B H^{EB} d = (H \circ E)^B d$, and so $G = H \circ E$. $\square$

Corollary 4.7. *Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be lenses. Let $E \colon \mathbf{B} \to \mathbf{C}$ be a cofork of F_1 and F_2 in $\mathcal{L}ens$, and suppose that $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$. If $\mathcal{U}E$ is a discrete opfibration then E coequalises F_1 and F_2.*

Proof. Let $G \colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2, let $B \in |\mathbf{B}|$ and let $d \in \mathbf{D}(GB, *)$. Then $G^B d$ and $E^B E G^B d$ are both elements of $\mathbf{B}(B, *)$ which are sent by E to the same morphism $EG^B d$ of $\mathbf{C}$. If $\mathcal{U}E$ is a discrete opfibration, then $EG^B d$ has a unique lift to $\mathbf{B}(B, *)$, and so $G^B d$ and $E^B E G^B d$ must be equal. □

5 Pushouts of discrete opfibrations along monos

In the proof that $\mathcal{U}$ preserves epis (Theorem 3.6), we used the well-known result that cosieves are pushout stable to explain why the pushout in $\mathcal{C}at$ of the get functors of a span of monic lenses lifts uniquely to a commutative square in $\mathcal{L}ens$; this lifted square is actually a pushout square in $\mathcal{L}ens$. In this section, we will show, more generally, that $\mathcal{L}ens$ has pushouts of discrete opfibrations along monics, and that $\mathcal{U}$ creates these pushouts. In what follows, we use square brackets for equivalence classes of elements.

Fritsch and Latch [11, Proposition 5.2] explicitly construct the pushout in $\mathcal{C}at$ of a functor along a full monic functor. Specialising to when the full monic functor is a cosieve, and recalling that the image of a cosieve is out-degree-zero, we obtain the following simplification of Fritsch and Latch's construction.

Proposition 5.1. *Let $F \colon \mathbf{A} \to \mathbf{C}$ be a functor and $J \colon \mathbf{A} \to \mathbf{B}$ be a cosieve. Then*

$$
\begin{array}{ccc}
\mathbf{A} & \overset{J}{\rightarrowtail} & \mathbf{B} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \overline{F}} \\
\mathbf{C} & \underset{\overline{J}}{\rightarrowtail} & \mathbf{D}
\end{array}
$$

is a pushout square in $\mathcal{C}at$ and $\overline{J}$ is a cosieve, where $\mathbf{D}$, $\overline{F}$ and $\overline{J}$ are defined as follows:

- Object set:

$$
|\mathbf{D}| = |\mathbf{C}| \sqcup \big(|\mathbf{B}| \setminus |\mathbf{A}| \big)
$$

- Hom-sets: *for all $C_1, C_2 \in |\mathbf{C}|$ and all $B_1, B_2 \in |\mathbf{B}| \setminus |\mathbf{A}|$,*

$$
\begin{aligned}
\mathbf{D}(C_1, C_2) &= \mathbf{C}(C_1, C_2) & \mathbf{D}(C_1, B_2) &= \emptyset \\
\mathbf{D}(B_1, B_2) &= \mathbf{B}(B_1, B_2) & \mathbf{D}(B_1, C_2) &= \Big(\coprod_{A \in |\mathbf{A}|} \mathbf{C}(FA, C_2) \times \mathbf{B}(B_1, A) \Big) \big/ {\sim}
\end{aligned}
$$

  where $\sim$ is the equivalence relation on $\coprod_{A \in |\mathbf{A}|} \mathbf{C}(FA, C_2) \times \mathbf{B}(B_1, A)$ generated by $(c, a \circ b) \sim (c \circ Fa, b)$ for all $A_1, A_2 \in |\mathbf{A}|$, all $b \in \mathbf{B}(B_1, A_1)$, all $a \in \mathbf{A}(A_1, A_2)$ and all $c \in \mathbf{C}(FA_2, C_2)$.

- Composition: *for all $B_1, B_2, B_3 \in |\mathbf{B}| \setminus |\mathbf{A}|$, all $A \in |\mathbf{A}|$, all $C_1, C_2, C_3 \in |\mathbf{C}|$, all $b_1 \in \mathbf{D}(B_1, B_2)$, all $b_2 \in \mathbf{D}(B_2, B_3)$, all $a \in \mathbf{D}(B_2, A)$, all $c \in \mathbf{D}(FA, C_2)$, all $c_1 \in \mathbf{D}(C_1, C_2)$ and all $c_2 \in \mathbf{D}(C_2, C_3)$,*

$$
\begin{aligned}
b_2 \circ_\mathbf{D} b_1 &= b_2 \circ_\mathbf{B} b_1 & [(c, a)] \circ_\mathbf{D} b_1 &= [(c, a \circ_\mathbf{B} b_1)] \\
c_2 \circ_\mathbf{D} c_1 &= c_2 \circ_\mathbf{C} c_1 & c_2 \circ_\mathbf{D} [(c, a)] &= [(c_2 \circ_\mathbf{C} c, a)]
\end{aligned}
$$

- Identity morphisms: *same as in $\mathbf{B}$ and $\mathbf{C}$.*

- Injections: *the functor $\bar{J}\colon \mathbf{C} \to \mathbf{D}$ is the obvious inclusion of* $\mathbf{C}$ *as a full subcategory of* $\mathbf{D}$; *the functor* $\overline{F}\colon \mathbf{B} \to \mathbf{D}$ *is defined, for all* $B, B' \in |\mathbf{B}| \setminus |\mathbf{A}|$, *all* $A, A' \in |\mathbf{A}|$, *all* $b \in \mathbf{B}(B, B')$, *all* $b' \in \mathbf{B}(B, A)$ *and all* $a \in \mathbf{B}(A, A')$, *as follows:*

$$\overline{F}B = B \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{F}A = FA$$
$$\overline{F}b = b \qquad\qquad \overline{F}b' = [(\mathrm{id}_{FA}, b')] \qquad\qquad \overline{F}a = Fa$$

Theorem 5.2. *The pushout in $\mathcal{C}at$ of a discrete opfibration along a cosieve is a discrete opfibration.*

Lemma 5.3. *Let $F\colon \mathbf{A} \to \mathbf{C}$ be a discrete opfibration, let $J\colon \mathbf{A} \to \mathbf{B}$ be a cosieve, let $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and let $C \in |\mathbf{C}|$. Then, for all $A_1, A_2 \in \mathbf{A}$, all $b_1 \in \mathbf{B}(B, A_1)$, all $b_2 \in \mathbf{B}(B, A_2)$, all $c_1 \in \mathbf{C}(FA_1, C)$ and all $c_2 \in \mathbf{C}(FA_2, C)$, if $(c_1, b_1) \sim (c_2, b_2)$ then $F^{A_1} c_1 \circ b_1 = F^{A_2} c_2 \circ b_2$.*

Proof. We proceed by induction, using the induction principle for $\sim$ in Equation (3). The proof obligations from the reflexivity, symmetry and transitivity axioms for $\sim$ hold because $=$ is an equivalence relation. For the remaining proof obligation, for all $A_1, A_2 \in |\mathbf{A}|$, all $b \in \mathbf{B}(B, A_1)$, all $a \in \mathbf{A}(A_1, A_2)$ and all $c \in \mathbf{C}(FA_2, C)$, we have $F^{A_1} Fa = a$ as F is a discrete opfibration, and so

$$F^{A_2} c \circ (a \circ b) = F^{A_2} c \circ F^{A_1} Fa \circ b = F^{A_1}(c \circ Fa) \circ b. \qquad \square$$

Proof of Theorem 5.2. Using the notation of Proposition 5.1, suppose that F is a discrete opfibration. We must show that $\overline{F}$ is also a discrete opfibration. Let $B \in |\mathbf{B}|$ and $d \in \mathbf{D}(\overline{F}B, *)$.

Suppose that $B \in |\mathbf{A}|$. Then $\overline{F}B = FB$, and $d \in \mathbf{C}(FB, *)$. As F is a discrete opfibration, there is a unique $a \in \mathbf{A}(B, *)$ such that $d = Fa$. But $\mathbf{A}(B, *) = \mathbf{B}(B, *)$ as $\mathbf{A}$ is out-degree-zero in $\mathbf{B}$; also $\overline{F}a = Fa$ for each $a \in \mathbf{B}(B, *)$. Hence there is a unique $a \in \mathbf{B}(B, *)$ such that $d = \overline{F}a$.

Suppose that $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and $\mathrm{tgt}\, d \in |\mathbf{B}| \setminus |\mathbf{A}|$. Then $\overline{F}B = B$, $d \in \mathbf{B}(B, *)$ and $\overline{F}d = d$. As $\overline{F}$ is injective on the morphisms of $\mathbf{B}$ not in $\mathbf{A}$, d is the unique morphism in $\mathbf{B}(B, *)$ mapped by $\overline{F}$ to d.

Otherwise, $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and $\mathrm{tgt}\, d \in |\mathbf{C}|$. Then $\overline{F}B = B$, and $d = [(c_1, b_1)]$ for some $A_1 \in |\mathbf{A}|$, some $b_1 \in \mathbf{B}(B, A_1)$ and some $c_1 \in \mathbf{C}(FA_1, C)$, where $C = \mathrm{tgt}\, d$. For uniqueness of lifts, suppose that $b_2 \in \mathbf{B}(B, *)$ is such that $d = \overline{F}b_2$. Let $A_2 = \mathrm{tgt}\, b_2$. Then $A_2 \in |\mathbf{A}|$ as $\overline{F}A_2 = \mathrm{tgt}\, d = C$, and so $\overline{F}b_2 = [(\mathrm{id}_C, b_2)]$. As $d = \overline{F}b_2$, we have $(\mathrm{id}_C, b_2) \sim (c_1, b_1)$. By Lemma 5.3, $b_2 = F^{A_2} \mathrm{id}_C \circ b_2 = F^{A_1} c_1 \circ b_1$; this determines b_2. For existence of lifts, note that $\overline{F}(F^{A_1} c_1 \circ b_1) = [(\mathrm{id}_C, F^{A_1} c_1 \circ b_1)] = [(FF^{A_1} c_1, b_1)] = [(c_1, b_1)] = d.$ $\square$

Theorem 5.4. *The functor $\mathcal{U}$ creates pushouts of monic lenses with discrete opfibrations.*

Proof. Using the notation of Proposition 5.1, suppose that F is a discrete opfibration. Then $\overline{F}$ is also a discrete opfibration (Theorem 5.2). Let $J_{\mathbf{B}}\colon \mathbf{B} \to \mathbf{B} \sqcup \mathbf{C}$ and $J_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{B} \sqcup \mathbf{C}$ be the coproduct injection functors. Coproduct injections in $\mathcal{C}at$ are always discrete opfibrations, as is the coproduct copairing of any two discrete opfibrations. Hence $J_{\mathbf{B}}$, $J_{\mathbf{C}}$ and $[\bar{J}, \overline{F}]$ are all discrete opfibrations. As the composite of two discrete opfibrations is a discrete opfibration, so are $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$. So far, we know that $[\bar{J}, \overline{F}]$ is the coequaliser in $\mathcal{C}at$ of $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$, all of these functors have canonical lens structures as they are discrete opfibrations, and $[\bar{J}, \overline{F}]$ coforks $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$ in $\mathcal{L}ens$. As $[\bar{J}, \overline{F}]$ is a discrete opfibration, the conditions of Theorem 4.5 are satisfied, and so $[\bar{J}, \overline{F}]$ coequalises $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$ in $\mathcal{L}ens$. As $\mathcal{U}$ creates coproducts [5], it follows that $\bar{J}$ and $\overline{F}$ exhibit $\mathbf{D}$ as the pushout of J and F in $\mathcal{L}ens$. $\square$

One might hope that the above result generalises to pushouts of two discrete opfibrations, or of arbitrary lenses along monics; this is not the case. The following is an example of two discrete opfibrations whose pushout injection functors have no lens structures that give a commutative square of lenses.

Example 5.5. Let **A** and **B** be the preordered sets generated respectively by the following graphs.

$$Y_1' \xleftarrow{\;f_1'\;} X' \xrightarrow{\;f_2'\;} Y_2' \qquad\qquad Y_1 \xleftarrow{\;f_1\;} X \xrightarrow{\;f_2\;} Y_2$$
$$Y_1'' \xleftarrow{\;f_1''\;} X'' \xrightarrow{\;f_2''\;} Y_2''$$

Let $F\colon \mathbf{A} \to \mathbf{B}$ be the unique functor that sends both X' and X'' to X, both Y_1' and Y_1'' to Y_1, and both Y_2' and Y_2'' to Y_2. Let $G\colon \mathbf{A} \to \mathbf{B}$ be the unique functor that sends both X' and X'' to X, both Y_1' and Y_2'' to Y_1, and both Y_2' and Y_1'' to Y_2. Both F and G are discrete opfibrations. Their pushout in $\mathcal{C}at$ is **2**; the pushout injections $\overline{F}, \overline{G}\colon \mathbf{B} \to \mathbf{2}$ are both the unique functor that sends X to 0, and both Y_1 and Y_2 to 1. There are two different lens structures on this functor; one lifts the unique morphism u of **2** to f_1, the other lifts it to f_2. This gives four different combinations of lens structures on $\overline{F}$ and $\overline{G}$. Assume, for a contradiction, that one of these combinations satisfies $\overline{F}\,\overline{G} = \overline{G}\,\overline{F}$ in $\mathcal{L}ens$. As $G^{X'}\overline{F}^X u = F^{X'}\overline{G}^X u$, we must have $\overline{F}^X u = \overline{G}^X u$. If $\overline{F}^X u = f_1$, then $G^{X''}\overline{F}^X u = G^{X''}f_1 = f_2'$ and $F^{X''}\overline{G}^X u = F^{X''}f_1 = f_1' \neq f_2'$, which is a contradiction. If $\overline{F}^X u = f_2$, we obtain a similar contradiction.

Next is an example of a lens and a cosieve where the pushout of the get functor of the lens along the cosieve does not have a lens structure (incidentally this lens and cosieve do not have a pushout in $\mathcal{L}ens$).

Example 5.6. Let **B** and **D** be the preordered sets generated respectively by the following graphs.

$$
\begin{array}{ccc}
X & \xleftarrow{\;f\;} W \xrightarrow{\;g\;} Y \\
\downarrow{\scriptstyle s} & \downarrow & \downarrow{\scriptstyle t} \\
Z_2 & \longleftarrow Z_1 \longrightarrow Z_3
\end{array}
\qquad\qquad
\begin{array}{ccc}
X' \xleftarrow{\;f'\;} W' \xrightarrow{\;g'\;} Y' \\
{\scriptstyle s'}\searrow \quad \downarrow \quad \swarrow{\scriptstyle t'} \\
Z'
\end{array}
$$

Let **A** be the out-degree-zero subcategory of **B** on the objects Z_1, Z_2 and Z_3, and let $J\colon \mathbf{A} \rightarrowtail \mathbf{B}$ be the inclusion lens. As **1** is terminal in $\mathcal{L}ens$ [5], there is a unique lens $F\colon \mathbf{A} \to \mathbf{1}$. By Proposition 5.1, the pushout of $\mathcal{U}F$ along $\mathcal{U}J$ in $\mathcal{C}at$ is the unique functor $\overline{F}\colon \mathbf{B} \to \mathbf{D}$ that maps W to W', X to X', Y to Y', and all of Z_1, Z_2 and Z_3 to Z'. The functor $\overline{F}$ has no lens structure, otherwise we could derive the contradiction

$$s \circ f = \overline{F}^X s' \circ \overline{F}^W f' = \overline{F}^W (s' \circ f') = \overline{F}^W (t' \circ g') = \overline{F}^Y t' \circ \overline{F}^W g' = t \circ g.$$

From Theorem 5.4, every monic lens has a cokernel pair. Actually, using the epi-mono factorisation, every lens has a cokernel pair, namely, the cokernel pair of its mono factor.

Proposition 5.7. *Every monic lens is effective (i.e. equalises its cokernel pair).*

Proof. Let $M\colon \mathbf{A} \to \mathbf{B}$ be a monic lens, and let $J_1, J_2\colon \mathbf{B} \to \mathbf{Coker}\,M$ be its cokernel pair. Based on Proposition 5.1, if $B \in |\mathbf{B}|$ is such that $J_1 B = J_2 B$, then $B \in |\mathbf{A}|$; and similarly for morphisms of **B**. In particular, the image of any lens which forks J_1 and J_2 is contained in **A**, and thus its corestriction to **A** is the unique comparison lens. $\square$

Corollary 5.8. *In $\mathcal{L}ens$, the classes of all monos, effective monos, regular monos, strong monos and extremal monos coincide.*

Corollary 5.9. *Every lens that is both epic and monic is an isomorphism.*

6 Regular epic lenses

In this section, we show that all epis in $\mathcal{L}ens$ are regular. This gives us another class of coequalisers in $\mathcal{L}ens$, namely, the epic lenses. For contrast, recall that not all epis in $\mathcal{C}at$ are regular.

Example 6.1. In Example 3.8, we saw that the functor $J\colon \mathbf{2} \to \mathbf{I}$ is epic. It is, however, not a regular epi. Indeed, if J coforks $F_1, F_2\colon \mathbf{A} \to \mathbf{2}$, then $F_1 = F_2$ as J is monic, and so $\mathrm{id}_{\mathbf{2}}$ is the coequaliser of F_1 and F_2, but $\mathbf{2}$ and $\mathbf{I}$ are not isomorphic.

Proposition 6.2. *The get functor of every epic lens is an effective epi in $\mathcal{C}at$.*

A functor $E\colon \mathbf{B} \to \mathbf{C}$ is *surjective on composable pairs* if for each composable pair (c, c') of $\mathbf{C}$, there is a composable pair (b, b') of $\mathbf{B}$ such that $Eb = c$ and $Eb' = c'$; such functors are necessarily also surjective on objects and morphisms. If $E\colon \mathbf{B} \to \mathbf{C}$ is an epic lens, then $\mathcal{U}E$ is surjective on composable pairs; indeed, if (c, c') is a composable pair of $\mathbf{C}$, then there is a $B \in |\mathbf{B}|$ such that $EB = \mathrm{src}\, c$, and $(E^B c, E^{\mathrm{tgt}\, E^B c} c')$ is a composable pair above (c, c'). Hence it suffices to prove the following lemma.

Lemma 6.3. *All functors that are surjective on composable pairs are effective epis in $\mathcal{C}at$.*

Proof. Let $E\colon \mathbf{B} \to \mathbf{C}$ be a functor that is surjective on composable pairs, and let its kernel pair be $F_1, F_2\colon \mathbf{Ker}\, E \to \mathbf{B}$. We must show that E coequalises F_1 and F_2. Let $G\colon \mathbf{B} \to \mathbf{D}$ cofork F_1 and F_2.

Suppose that there is a functor $H\colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$. As E is surjective on objects, for all $C \in |\mathbf{C}|$ there is a $B \in |\mathbf{B}|$ such that $EB = C$, and so $HC = HEB = GB$; this equation determines H on objects. As E is surjective on morphisms, a similar equation determines H on morphisms.

To define $H\colon \mathbf{C} \to \mathbf{D}$ with these equations, the values of GB and Gb should be independent of the choice of B above C and b above c. For all $C \in |\mathbf{C}|$ and all $B, B' \in |\mathbf{B}|$ such that $EB = EB' = C$, we have $GB = GF_1\langle B, B'\rangle = GF_2\langle B, B'\rangle = GB'$, where $\langle B, B'\rangle \in |\mathbf{Ker}\, E|$ comes from the pullback property; hence the object map of H is well defined. Its morphism map is similarly also well defined.

Define H with the above equations. By construction, $G = H \circ E$. We must show that H is a functor. For all $C \in |\mathbf{C}|$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $H\,\mathrm{id}_C = G\,\mathrm{id}_B = \mathrm{id}_{GB} = \mathrm{id}_{HC}$; thus H preserves identities. For all composable pairs c and c' of $\mathbf{C}$, there is a composable pair b and b' of $\mathbf{B}$ such that $Eb = c$ and $Eb' = c'$, and $H(c' \circ c) = G(b' \circ b) = Gb' \circ Gb = Hc' \circ Hc$; thus H preserves composites. $\qquad\square$

Corollary 6.4. *Every epic lens coequalises its proxy kernel pair, and so is regular.*

Proof. Let $E\colon \mathbf{B} \to \mathbf{C}$ be an epic lens. Let $F_1, F_2\colon \mathbf{Ker}\,\mathcal{U}E \to \mathbf{B}$ be the proxy kernel pair of E in $\mathcal{L}ens$. By Proposition 6.2, $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. Let $G\colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2, let $B \in |\mathbf{B}|$, let $d \in \mathbf{D}(GB, *)$, and let $C = EB$. Then $(G \circ F_1)^{\langle B, B\rangle} d = F_1^{\langle B, B\rangle} G^B d = \langle G^B d, E^B E G^B d\rangle$, and similarly $(G \circ F_2)^{\langle B, B\rangle} d = \langle E^B E G^B d, G^B d\rangle$. As G coforks F_1 and F_2, we have $G^B d = E^B E G^B d$. By Theorem 4.5, E coequalises F_1 and F_2 in $\mathcal{L}ens$. $\qquad\square$

Corollary 6.5. *In $\mathcal{L}ens$, the classes of all epis, regular epis, strong epis and extremal epis coincide.*

Corollary 6.6. *In $\mathcal{L}ens$, the class of all morphisms that are left orthogonal to the class of all monos is the class of all epis.*

Proof. As $\mathcal{L}ens$ has equalisers [5], every morphism that is left orthogonal to the class of all monos is an epi. Conversely, we have already shown that every epi is a strong epi. $\qquad\square$

Remark 6.7. As every lens factors as an epi followed by a mono (Remark 3.5), it follows that the class of all epis and the class of all monos together form an orthogonal factorisation system on $\mathcal{L}ens$.

7   Conclusion

In this article, we have seen a number of results which advance our understanding of the category $\mathcal{L}ens$ of (asymmetric delta) lenses. We now have a complete elementary characterisation of the monos and epis in $\mathcal{L}ens$, the monos being the unique lenses on cosieves and the epis being the surjective-on-objects lenses; from this, we see that Johnson and Roseburgh's factorisation system on $\mathcal{L}ens$ [16] is actually an epi-mono factorisation system. We have also initiated a study of the coequalisers in $\mathcal{L}ens$. Despite $\mathcal{L}ens$ not having all coequalisers, nor the forgetful functor from $\mathcal{L}ens$ to $\mathcal{C}at$ preserving or reflecting them, we have two interesting positive results. First, every epic lens coequalises its proxy kernel pair. Second, $\mathcal{L}ens$ has pushouts of discrete opfibrations along cosieves. Our characterisation of the epic lenses played a central role in the proof of both of these results, and hopefully will enable future work to completely characterise the coequalisers in $\mathcal{L}ens$.

That every epic lens coequalises its proxy kernel pair is yet another result that emphasises the parallels between proxy pullbacks in $\mathcal{L}ens$ and real pullbacks in other categories. An interesting question for future work is whether there is an axiomatisation of the notion of proxy pullback from which one may prove interesting general results which also apply to other categories. Existing work in this direction include Bumpus and Kocsis' *proxy pushout* [17], which inspired our use of the name proxy pullback, as well as Böhm's *relative pullbacks* [4] and Simpson's *local independent products* [19]. One potential use for such an axiomatised proxy pullback would be to give a generalised notion of regular category; the category $\mathcal{L}ens$ is an obvious candidate example from which to draw inspiration. This notion of a proxy regular category may even be helpful for understanding symmetric lenses, which are known to be equivalence classes of spans of asymmetric ones, as some kind of relations in $\mathcal{L}ens$, although this is as yet merely speculation.

Acknowledgements

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