

# The more legs the merrier: A new composition for symmetric (multi-)lenses

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This paper develops a new composition of symmetric lenses that preserves information which is important for implementing system interoperation. It includes a cut-down but realistic example of a multi-system business supply chain and illustrates the new mathematical content with analysis of the systems, showing how the new composition facilitates the engineering required to implement the interoperations. All of the concepts presented here are based on either pure category theory or on experience in solving business problems using applied category theory.

## 1 Introduction

Lenses are a category theoretic construct and are used in a very wide variety of applications. Lenses come in a wide range of forms, but each kind of lens has a composition (associative, and with identities), and so the various lenses form the morphisms of categories, most often with objects which are themselves categories, usually representing states and transitions of some systems. Among the kinds of lenses we will use here are asymmetric, symmetric, and multiary lenses.

Symmetric lenses compose to, unsurprisingly, form new symmetric lenses. Symmetric lenses are usually represented as spans of asymmetric lenses. Indeed, from the very beginning, symmetric lenses have had various ad-hoc definitions, but in all cases the authors noted that an alternative approach would be to define them as equivalence classes of spans of asymmetric lenses.

In many applications, the fact that a symmetric lens might also be represented as a *cospan* of asymmetric lenses is important, especially for implementation purposes. However, the composition of symmetric lenses does not preserve the property that the lenses can be represented by cospans — two such symmetric lenses may (and frequently do) compose to form a symmetric lens which cannot be represented as a cospan of asymmetric lenses. Thus preserving the factorisation to show how cospans of asymmetric lenses might be used in implementations becomes important.

In 2018, the first work on multilenses was begun. Multilenses can be represented as multi-spans of asymmetric lenses (often called wide spans, these multi-spans are spans with an arbitrary finite number of legs). In this paper we analyse a small but realistic example of a supply chain in which the cospan representations would be ‘composed away’ by ordinary symmetric lens composition, and introduce a new kind of composition which we call *fusion* in which two ordinary symmetric lenses (spans with two legs) fuse to form a multilens with three legs preserving the cospan representations, and more generally, two multilenses, spans with say  $m$  and  $n$  legs, fuse to form a multilens with  $m + n - 1$  legs, again preserving cospan representations.

The plan of the paper is as follows. In Section 2 we present a cut-down example of actual supply chain system interoperations. Although the example is realistic, it has been cut to almost the very minimum required to illustrate the mathematical developments in the rest of the paper. The example is revisited

in several sections as we proceed through the development of the mathematics. In Sections 3 and 4 we review briefly asymmetric and symmetric lenses. In the case of symmetric lenses the approach is the representation of a symmetric lens as a span of asymmetric lenses. In Section 5 we turn to cospans of asymmetric lenses, pointing out the utility of a cospan representation and illustrating this with the example from Section 2.

Section 6 sketches some very new developments in lenses, the multilenses or wide spans of asymmetric lenses. In Section 7 we introduce the new composition, *fusion*, of multilenses, which in particular gives a new composition for symmetric lenses. We still refer to it as a kind of composition because it contains all the information of the composite, but it has more, preserving, unlike normal compositions, some important information about the makeup of the individual lenses that were composed. This extra information is shown in Section 8 to be just what is needed to preserve cospan representations when they exist, and so to facilitate engineering practice.

After a brief interlude in Section 9 to review some other examples of fusion-like compositions, we conclude and outline some future work in Section 10, briefly describing how cospans among the feet of a multispan can be recovered unambiguously from the appropriately fused multispan. This emphasises the engineering importance of using fusion rather than composition.

The paper requires only modest category theoretic background for which we refer readers to any standard text. One particular category theoretic notion we use repeatedly, above and below, is *span*. A *span* in a category is just a pair of arrows with common domain. A *cospan* is, of course, a pair of arrows with common codomain. A *wide span* is a collection of a finite number of arrows with common domain, and similarly a *wide cospan* is a collection of a finite number of arrows with common codomain. We will frequently talk about spans or cospans of asymmetric lenses. Asymmetric lenses are defined in Section 3, but for now note that an asymmetric lens is a functor, normally called the *Get* of the lens, and some further structure. In our categories of lenses the arrows are lenses, and they are oriented in the direction of the lenses' Gets. Thus a span of asymmetric lenses has, inter alia, two Gets (one for each lens) with a common domain.

## 2 An example

We begin with an example which we will use to illustrate the concepts presented in this paper. The example is based on real systems, but they have been cut down to the essential details required to capture the ideas presented here.

Supply chains, especially global supply chains, are very much in the economic news at present because of the disruptions to production and distribution caused by the corona virus crisis. In many cases modern supply chains are managed through system interoperations, with individual organisations owning and operating their own information systems, but sharing enough information for those systems to interoperate automatically. For example, two businesses, a supplier and a customer, might both keep track of the amount of stock available on a customer's premises so that further stock can be supplied in a just in time (JIT) manner. To maintain the consistency of the two representations of the customer's stock levels, information is exchanged between the two systems. This is system interoperation in action, and, as we will see later, it is mathematically captured by lenses — certain kinds of bidirectional transformations.

For our massively simplified businesses, imagine a supplier called ABC Frames. It provides the basic structure, the chassis if you will, for a variety of products that are manufactured by other businesses which make up ABC Frames' customers. One of those customers is XYZ Manufacturing, which is operated as several distinct business entities, XYZ Warehouse, XYZ Logistics and XYZ Production being the

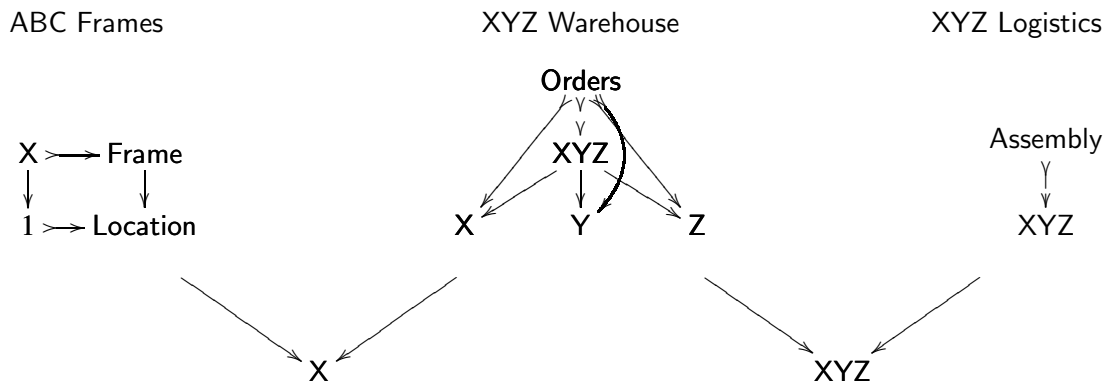


Figure 1: Three (simplified) business entities interoperating via two cospan of lenses

businesses we will consider here.

A true representation of ABC Frames will, like most businesses, involve systems which track hundreds, and often many thousands of different types of entities, along in each case with their many attributes (for example, colour, dimensions, location, serial number, base price, etc). For our purposes we focus on a single type of frame, called here just Frames. These frames almost certainly have many attributes including those above, but for now we will just consider location. (Other attributes can be easily managed too, but will just clog up our pictures if we depict them.) So, for our purposes we might think of ABC Frames as having at any instant a set of frames, with each frame having a specified location. For our purposes ABC Frames' information system stores two sets and a function between them

Frame  $\longrightarrow$  Location.

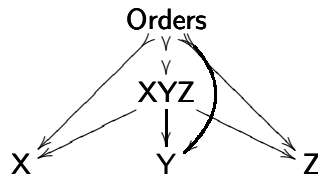
XYZ Manufacturing, in its simplified form, takes frames which they call X, and assembles them with other products called simply Y and Z to produce a consumer product known as XYZ. In particular, XYZ Warehouse keeps stocks of X, Y and Z, along with various attributes of each instance of those stocks (their serial numbers, colours, and so on) which we will not record here. The warehouse information system also calculates from its known stocks the product  $X \times Y \times Z$  which is useful for XYZ's consumer facing operations because it shows all of the possible available combinations of X, Y and Z items that might be assembled and provided to a potential customer. XYZ Warehouse's information system also stores briefly some information about orders, but we will return and fill that in later. So, for now, mathematically speaking, XYZ Warehouse's information systems stores three sets, along with their product and the product projections (all of which can be calculated as required from the three sets).

XYZ Logistics, also known as XYZ Sales because it is the principal interface with XYZ's customers, assembles orders placed by customers after, of course, detailed discussions with those customers about customers' needs and desires, frequently consulting XYZ's current catalogue, which is really just a copy called XYZ of the product  $X \times Y \times Z$  calculated by XYZ Warehouse. So XYZ Logistics' information system stores a set whose value is kept consistent with XYZ Warehouse's calculated product by system interoperations (a lens in fact). In addition, XYZ Logistics keeps track of customer orders, so it maintains a set usually called Assembly Order (or just Assembly for short) whose elements are casually called order-lines or order items, which should be a subset of XYZ (a subset because, of course, we don't want two orders for the same assembled product — each assembly is unique and can only be sold once). As

you can imagine, XYZ Logistics does much more, and has its “Logistics” name, and uses terms like Assembly for orders, because it looks after many logistical issues including the transfer of products from the warehouse to XYZ Production, but we don’t need to discuss these things here and they have been elided. Mathematically, the much simplified information system for XYZ Logistics contains a set XYZ and a subset of that set representing the current order items:

$$\text{Assembly} \twoheadrightarrow \text{XYZ}.$$

The eagle-eyed among readers will have spotted another concern. Since the catalogue, XYZ contains all possible assemblies from products contained in the warehouse, customers might order distinct assemblies which nevertheless contain the same instance of a particular product. For example, two different customers might order two different assemblies both of which are built on the frame with serial number 4097. We need to guard against this as both orders can’t be satisfied — there is only one frame 4097. This is where the “extra information” mentioned above in discussing XYZ Warehouse’s information system comes into play. XYZ Warehouse also keeps a local copy called Orders of XYZ Logistics’ Assembly, maintained again by system interoperations, along with the corresponding subset inclusion of Orders into  $X \times Y \times Z$ . Since that product comes with its product projections in XYZ Warehouse, the information system there can see the composite of the inclusion with each of the product projections, and, as part of its inbuilt constraints, it requires that those compositions are monic (injections). Any attempt to enter an order item that violates that constraint will be rejected, and via the systems interoperations, a customer will be unable to order an assembly containing, for example, frame 4097 if there is already another extant order for an assembly using that frame. Mathematically the full version of the fragment of XYZ Warehouse’s information system that we will be considering is summarised in the diagram below.



That completes our summary of XYZ Manufacturing’s information systems, and we have seen the very simple information systems maintained by ABC Frames, but to complete our automated supply chain we need to see how the two companies’ systems interact, and it is very simple. Among the locations where ABC Frames might keep track of frames is XYZ Warehouse. So in ABC Frames there is an element  $1 \rightarrow \text{Location}$  which picks out XYZ’s warehouse, and the pullback

$$\begin{array}{ccc} X & \twoheadrightarrow & \text{Frame} \\ \downarrow & & \downarrow \\ 1 & \twoheadrightarrow & \text{Location} \end{array}$$

calculates the subset of Frames which are the frames located at XYZ Warehouse. That should of course correspond to X in XYZ’s own systems and system interoperations are used to keep those two sets consistent. This supports for these two companies their version of the JIT supply system described at the beginning of this section.

Although this example is vastly simplified, it does model many interesting aspects of category theoretic information systems interoperation, including, as we will see below, symmetric lenses, multilenses, cospan implementations of interoperations, amendment lenses, and so on. A summary of the three simplified business entities along with rough indications of the lenses between them used for maintaining

interoperations (excluding part of the amendment lens synchronising Assembly and Orders) is shown in Figure 1.

Before saying more about all this we review the relevant concepts from earlier work, and develop the new theory required for this paper.

### 3 Asymmetric lenses

Lenses are used to maintain synchronisation between (or in the case multilenses, among) different systems. In asymmetric lenses, one of the systems (the one with state space  $S$  below) has all the information required to reconstruct the other (the one with state space  $V$  below). An operation, usually called “Get” and frequently denoted  $G$ , gives for any state of the system  $S$ , the corresponding state of the system  $V$ . In the reverse direction, we would not expect a  $V$  state to contain enough information to recreate an entire  $S$  state. Instead, the operation usually called “Put” provides a new state  $s'$  of  $S$  given an old state  $s$  of  $S$  and a change of state in  $V$  from  $Gs$  to some new state  $v'$ , such that the new states of  $S$  and  $V$ ,  $s'$  and  $v'$ , are again synchronised.

Naturally, state spaces will be represented here as categories — a state of a system is an object of the state space, and arrows of the state space are state transitions (and state transitions can be composed associatively, and there are identity transitions corresponding to no-change). Thus the state transition in  $V$  just mentioned is an arrow  $Gs \longrightarrow v'$  in  $V$ .

Database view updating [2] provides a typical (and longstanding) example: Suppose that  $S$  is the state space of the information system of ABC Frames, one of the organisations discussed in the previous section. The object  $X$  of the previous section is the state space of a view of  $S$ . A state of  $X$  is just a set, the current set of frames located at the XYZ Warehouse according to the current state of the information system  $S$ . And the pullback above shows how to calculate  $X$  from a current state of ABC Frame’s information system (as discussed in Section 2, a state of ABC Frame’s information system is just a function  $\text{Frame} \longrightarrow \text{Location}$ ). The Get of this view is calculated by the pullback, which in database terms is simply the query “select Frame where Location equals XYZ Warehouse”.

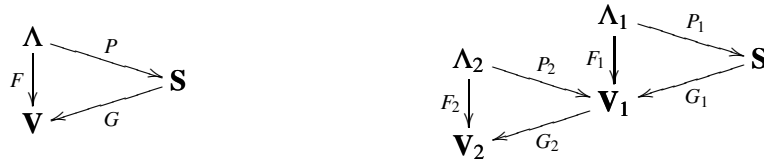
If the view  $X$  is changed, then the Put needs to construct sets  $\text{Frame}$  and  $\text{Location}$  and a function  $\text{Frame} \longrightarrow \text{Location}$ . The most natural choice of Put in this case starts from the old function  $\text{Frame} \longrightarrow \text{Location}$ , leaves  $\text{Location}$  unchanged, changes  $\text{Frame}$  to correspond to the new  $X$  by adding or deleting elements as required, retains the values of the function for all those elements of  $\text{Frame}$  that remain in the new  $X$ , and assigns any extra elements of  $X$  to have location XYZ Warehouse (since if the set  $X$  is intended to be the result of the above query, any extra (new) elements in  $X$  can be assumed to located at XYZ Warehouse).

All of this can be formalised easily using the theory of database modelling via EA-sketches [12] in which diagrams like those from the previous section are the base graphs of sketches, limits are used to ensure that things like products and monics are appropriately realised, colimits are used to define attributes, and the state spaces just described are models of the sketches, that is, full subcategories of finite set-valued functor categories for which the functors preserve finite limits and finite coproducts.

We turn now to the formal definition of asymmetric lenses. For readers who are most familiar with early work on lenses [21, 20] or with lenses as implemented in Haskell, this definition might come as a surprise, but it elegantly captures the generality required in clear category theoretic terms. Asymmetric lenses, as defined here, are sometimes called d-lenses or delta lenses [7], and unify a wide range of different types of lenses [13].

**Definition 1** (Clarke [3]): *An asymmetric lens is a commutative triangle of functors, as depicted below*

left, in which  $F$  is a discrete opfibration,  $P$  is bijective on objects, and  $G$  and  $P$  are called the Get and the Put respectively.



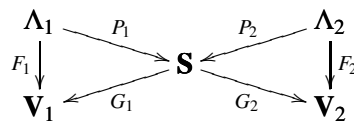
The category  $\Lambda$  is, up to equivalence, a category with the same objects as  $\mathbf{S}$  and with an arrow  $\alpha$  from  $s$  to  $s'$  if and only if  $\alpha : Gs \rightarrow Gs'$  in  $\mathbf{V}$  (using the notation  $s'$  from the beginning of this section). The span  $(\Lambda, F, P)$  is a co-functor [1] from  $\mathbf{V}$  to  $\mathbf{S}$ . For further motivation and details we refer the reader to [3]. As noted there, composition of asymmetric lenses is defined by simply composing the Gets and pulling back the second Put,  $P_2$ , along the first lens's discrete opfibration,  $F_1$  (see the above right diagram). Thus, there is a category  $\mathbf{ALens}$  whose objects are categories and whose arrows are asymmetric lenses, oriented in the direction of the Get (so the above left triangle is an arrow of  $\mathbf{ALens}$  from  $\mathbf{S}$  to  $\mathbf{V}$ ).

### 4 Symmetric lenses as spans of asymmetric lenses

Lenses, as just defined, are examples of bidirectional transformations [10]. To reiterate, a bidirectional transformation maintains consistency between two systems as one or the other changes, and the functor part and the cofunctor part of an asymmetric lens embody the two updates required, one in each direction, to restore consistency after a change of state of one system or the other. As we've noted, such lenses are often called *asymmetric* lenses to emphasise the asymmetry noted at the beginning of Section 3: A state of one system,  $\mathbf{S}$ , has all the information required to construct a state of the other system,  $\mathbf{V}$ , and this is reflected in the fact that one of the updates,  $G$ , is simply a functor.

While asymmetric lenses do arise in real world applications of bidirectional transformations, there are many important cases where neither system has the information to reconstruct the other completely. Instead, each system “knows” things that the other system does not. What's required is a *symmetric* lens [11, 8]. As was conjectured in both the papers just cited, and in [9], a symmetric lens can be defined as an equivalence class of spans of asymmetric lenses [14]. In this paper we will elide the details about the equivalence (full details are available in [14]) and work with representatives of equivalence classes. Again, the best available modern treatment is due to Bryce Clarke.

**Definition 2** This formulation is due to Clarke [4]: A (*representative for a*) symmetric lens is a span of asymmetric lenses as shown,



in which the objects are categories, the arrows are functors, the vertical arrows  $F_1$  and  $F_2$  are discrete opfibrations, and the functors  $P_1$  and  $P_2$  are bijective on objects.

This “bowtie” representation of symmetric lenses turns out to be particularly convenient. For example, as a bidirectional transformation, a symmetric lens should show how to restore consistency if a state of either  $\mathbf{V}_1$  or  $\mathbf{V}_2$  is changed. These two operations have variously been called the Rightward and Leftwards [11] and Forwards and Backwards [8] propagations. Each propagation is easily visible in the

bowtie, with, for example, the Forwards propagation given by the span  $(\Lambda_1, F_1, G_2P_1)$ , or in short, the South-East diagonal  $G_2P_1$ . In more detail: The systems with state spaces  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are synchronised when there is an  $s$  in  $\mathbf{S}$  with the current states of  $\mathbf{V}_1$  and  $\mathbf{V}_2$  equal to  $G_1s$  and  $G_2s$  respectively. If  $\mathbf{V}_1$  then changes state via say an  $\alpha : G_1s \longrightarrow v'_1$ , then that determines a unique arrow of  $\Lambda_1$ ,  $\hat{\alpha} : s \longrightarrow s'$  and  $G_2P_1 \hat{\alpha}$  is an arrow of  $\mathbf{V}_2$ , of the form  $G_2s \longrightarrow v'_2$ . Furthermore, the new states  $v'_1$  and  $v'_2$  are synchronised by  $s'$ . Synchronisation has been restored by the Forward propagation  $G_2P_1$ .

This might be a convenient moment to say a few more words about the equivalence relation that we are mostly suppressing in this short version to ease the reader's burden. A symmetric lens is a bidirectional transformation between  $\mathbf{V}_1$  and  $\mathbf{V}_2$ , while  $\mathbf{S}$  is generally considered to be hidden coordination information. If two bowtie representations between  $\mathbf{V}_1$  and  $\mathbf{V}_2$  have the same Forwards and Backwards propagations then they should be considered to be representatives of the same abstract symmetric lens even if they happen to manage their coordination via different categories  $\mathbf{S}_1$  and  $\mathbf{S}_2$ . Again, we refer readers interested in full details about the required equivalence relation to [14].

Another convenience of the bowtie representation is that it shows immediately how symmetric lenses compose. In fact, it shows that in two distinct, but equivalent ways. First, operationally, the Forward propagation described two paragraphs ago results in an arrow of  $\mathbf{V}_2$ , which can be in exactly the same manner Forward propagated along a symmetric lens from  $\mathbf{V}_2$  to  $\mathbf{V}_3$  defining a composite Forward propagation. Similarly, Backward propagations can be iterated thus determining, operationally, a composite symmetric lens from  $\mathbf{V}_1$  to  $\mathbf{V}_3$ . This can be shown to be equivalent to the composite span of asymmetric lenses described in Remark 4 below using lens structures on the pullback in **cat**. First we describe these “pulled back” lenses and their basic properties.

**Proposition 3** *Given a cospan of asymmetric lenses as shown*

$$\begin{array}{ccccc}
 & & \Lambda_2 & & \Lambda'_1 \\
 & \swarrow P_2 & \downarrow F_2 & & \swarrow P'_1 \\
 \mathbf{S} & & & & \mathbf{S}' \\
 & \searrow G_2 & & & \swarrow G'_1 \\
 & & \mathbf{V}_2 = \mathbf{V}'_1 & & 
 \end{array}$$

1. Each lens pulls back along the other lens's Get to give a lens
2. The resulting square of asymmetric lenses commutes in **ALens**, and will be referred to as the “pullback” of the cospan
3. The cospan itself determines operationally Forwards and Backwards propagations
4. And the propagations determined by the cospan and by its “pulled back” span coincide.

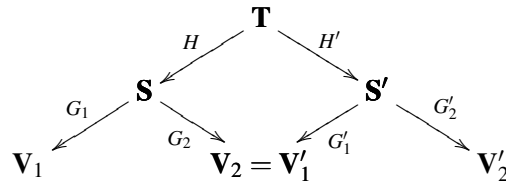
**Proof.** We give brief proof outlines:

1. This is proved by explicitly constructing the Puts in [14], but here we just outline the simple proof due to Clarke [3]: Part 1 follows immediately from the pullback pasting lemma and the facts that discrete opfibrations pull back along functors to give discrete opfibrations and bijective on objects functors pullback along functors to give bijective on object functors.
2. The Gets of the two sides of the square commute by construction (the Gets form a pullback square in **cat**) and it's easy to see from the explicit construction of the Puts along the two sides of the square that the compositions of the cofunctors coincide too.

3. For a cospan labelled as above, we call objects  $s$  of  $\mathbf{S}$  and  $s'$  of  $\mathbf{S}'$  *synchronised* when  $G_2s = G'_1s'$ . If  $s$  and  $s'$  are synchronised, and  $\alpha : s \rightarrow r$  is a change of state of the system with state space  $\mathbf{S}$ , then the Forward propagation of  $\alpha$  is the Put,  $P'_1$ , of  $G_2\alpha : G'_1s' \rightarrow G_2r$ , which will be an arrow of  $\mathbf{S}'$  with domain  $s'$  and whose codomain is then synchronised with  $r$ . The Backward propagation is defined similarly.
4. Finally, by the constructions of pullbacks in **cat**,  $s$  and  $s'$  are synchronised by the pulled back span if and only if they are synchronised by the given cospan. Furthermore, by inspection, the two Forward propagations coincide, and by symmetry the same inspections shows that the two Backward propagations coincide. ■

The inverted commas around “pullback” are to remind us that while the pullback in **cat**, along with the lenses constructed on its pullback projections, might look like a pullback diagram in **ALens**, it is not necessarily a pullback in that category (the universally determined mediating functors do not in general have canonical lens structures on them).

**Remark 4** Given composable representatives of symmetric lenses, that is, spans in **ALens** which agree on one of their feet (as in  $\mathbf{V}_2 = \mathbf{V}'_1$  in the proposition), they can be composed using the “pullback” of the cospan exactly as one does for span composition in **cat**. In more detail: Imagine that the two triangles in the proposition are the right side of the bow-tie displayed in Definition 2, and the left side of a similar bowtie in which all the labels have added primes, then the pullback of the proposition gives a new span of asymmetric lenses with peak  $\mathbf{T}$  say, between  $\mathbf{S}$  and  $\mathbf{S}'$ . As in ordinary span composition these asymmetric lenses can be composed with the asymmetric lenses  $\mathbf{S} \rightarrow \mathbf{V}_1$  and  $\mathbf{S}' \rightarrow \mathbf{V}'_2$  to yield the composite span of asymmetric lenses with peak  $\mathbf{T}$  and feet  $\mathbf{V}_1$  and  $\mathbf{V}'_2$ . Here is the picture, labelling the various lenses with their Gets, and in which  $H$  and  $H'$  are the pullback projections in **cat**.



Furthermore, using (4) from the proposition, the operational propagations of the composite span agree with the composite of the operational propagations as described just before the proposition.

## 5 Symmetric lenses as cospans of asymmetric lenses

In the previous section we saw that every cospan of asymmetric lenses yields, by pullback, a span of asymmetric lenses, that is, a representative for a symmetric lens. In fact, the cospan presentation of an asymmetric lens is especially valuable and is the main way system interoperations are actually built.

To revisit our example from Section 2, we have already seen that there is an asymmetric lens between ABC Frames and the (possibly imaginary, but frequently built) system X. Furthermore, there is an asymmetric lens from XYZ Warehouse to X. The Get of that lens is just a projection from among all the data stored at XYZ Warehouse, and returns simply the current state of the set X in XYZ Warehouse. The Put starts from a known state of XYZ Warehouse and a new state of X, and constructs a new state of XYZ Warehouse by changing its set X to match, leaving the sets Y and Z unchanged, recalculating the product  $X \times Y \times Z$ , and usually leaving the set of Orders unchanged, but if some elements of X have



been deleted, and if there are orders depending on those elements of  $X$ , then those orders are also deleted (called by database people a “cascading delete”). Notice that if instead new elements of  $X$  had been inserted, then Orders would not change, but the injection into  $X \times Y \times Z$  would be adjusted to account for its new larger codomain.

Thus we have a cospan of asymmetric lenses between ABC Frames and XYZ Warehouse. We could say now, following the previous section, that we “pullback” the span to obtain a representative for a symmetric lens, thus providing interoperations between ABC Frames’ and XYZ Warehouse’s systems. That is indeed theoretically true. The resulting system at the peak of the span  $\mathbf{T}$  is sometimes called the *federated information system* because it is the state space for the system that combines all of the information held at ABC Frames with all of the information held at XYZ Warehouse, subject only to ensuring that those two subsystems remain consistent via the same  $X$  state. Such symmetric lenses are theoretically important because we can reason with them and prove properties of the combined system (for example, that certain things remain consistent or that certain operations avoid deadlock or . . .). But these systems are hardly ever built. To begin with, ABC Frames and XYZ are separate companies, and are unlikely to want to, or indeed be able to, break commercial-in-confidence agreements and share all data that they might hold. There are commercial, privacy, and cyber security [17], reasons, to name just a few, for not building the system  $\mathbf{T}$ .

Instead, the system  $X$  might be built, along with the two asymmetric lenses to it described above (one from ABC Frames and one from XYZ Warehouse). Or, alternatively, the Forward and Backward propagations from such a cospan can be implemented as message passing and through Applications Programmer Interfaces (APIs) the messages can keep the two systems of the two companies synchronised. These options limit the exposure of each of the companies and their systems to the minimum required for the system interoperations [17], and those system interoperations are in the interests of the efficiencies of both organisations (after all, we only build such systems if there is a commercial imperative).

Of course, as noted in Proposition 3 part 4, the propagations determined by the cospan through  $X$  or by the span through the federated information system are the same. But the former is a minor piece of engineering work, which can even be separated into three tasks: Implementing the small common system  $X$ , and the two asymmetric lenses from ABC Frames to it (which can be done exclusively by ABC Frames engineers) and from XYZ Warehouse to it (which can be done exclusively by XYZ Warehouse engineers). On the other hand, working with the federated system, either by constructing it or by simulating propagations through it, is a major piece of work that is generally hard to partition into secure and independent tasks.

The message of this section is that cospan representations of symmetric lenses are very much preferred for engineering purposes.

It is worth noting however, that not all symmetric lenses have cospan representations. The paper [15] establishes necessary and sufficient conditions for the existence of cospan representations. For now, suffice it to say, having a cospan representation is something that one wants to keep.

And so there is another important point to note: The composition of symmetric lenses does not preserve cospan representability. Two cospan representable symmetric lenses may compose to give a symmetric lens which is not in itself cospan representable.

Again, the example from Section 2 provides an illustration.

We will not work through the details here, but there is a cospan representation for the symmetric lens between XYZ Warehouse and XYZ Logistics. To make the example more realistic we have included in this interoperation an example of *half-duplex interoperation* (see [5]). In short, XYZ Logistics is not permitted to change the state of the catalogue XYZ — it is read-only. There is also an opportunity here to introduce a non-trivial amendment lens (see [6]) between XYZ Warehouse and XYZ Logistics, but

to keep things simple let's assume that the company XYZ enters, processes and fills single orders at a time (otherwise orders in XYZ Logistics might have to be reversed (amended) by XYZ Warehouse if the monic constraints in XYZ Warehouse were violated).

The cospan of symmetric lenses between XYZ Warehouse and XYZ Logistics determines by “pull-back” a representative for a symmetric lens. The two symmetric lenses (between ABC Frames and XYZ Warehouse, and between XYZ Warehouse and XYZ Logistics) can be composed, either by “pulling back” (creating an even larger federated system  $\mathbf{T}''$ ), or by composing the propagations, and it may be that for whatever reason the composite symmetric lens is the subject of our interest. But note well: The composite symmetric lens is not cospan representable. Presented merely with the composite symmetric lens (and so, no information about how XYZ Warehouse mediates the information between ABC Frames and XYZ Logistics) there is no simple shared data that the two organisations can synchronise upon. The super federated system could be used in theory to build interoperations, but the information about the engineering appropriate cospans is gone.

Perhaps it would be better if the example from Section 2 were treated as a multilens, since then all three organisations, and their interactions, could be captured in a single mathematical entity.

## 6 Multilenses

**Definition 5** For  $n$  a positive integer, an  $n$ -lens consists of  $n$  asymmetric lenses  $f_i$  with common domain  $\mathbf{S}$ , such that  $f_i : \mathbf{S} \longrightarrow A_i$ .

A 1-lens is an asymmetric lens  $f_1 : \mathbf{S} \longrightarrow A_1$ . A 2-lens is a representative for a symmetric lens — a span in  $\mathbf{ALens}$  as in Definition 2.

For an  $n$ -lens  $L = (f_i : \mathbf{S} \longrightarrow A_i)$ , since it is in general an  $n$ -wide span in  $\mathbf{ALens}$ , we adopt, and adapt, the terminology usually used for parts of wide spans (including ordinary spans). Thus the  $f_i$  are called the *legs* of  $L$ , and the  $A_i$  are called the *feet* of  $L$ . The category  $\mathbf{S}$  is called the *peak* of  $L$ . We call  $f_1$  the *leftmost* leg of  $L$  and  $f_n$  the *rightmost* leg of  $L$ , and  $A_1$  the *leftmost* foot of  $L$  and  $A_n$  the *rightmost* foot of  $L$ . Of course, for  $n = 1$  the leftmost and rightmost legs of  $L$  coincide, and are both  $f_1$ , and likewise  $A_1$  is both the leftmost and the rightmost foot of  $L$ . When  $n \leq 2$ , the “most” of leftmost or rightmost is superfluous in normal usage, and it is common to say just “the left leg” or “the right foot” etc, and even when  $n > 2$ , if there is little chance of confusion, we may still say “left” for “leftmost” and “right” for “rightmost”.

For  $n > 1$  an  $n$ -lens is a wide span in the category  $\mathbf{ALens}$ . Relating this to previous work, an  $n$ -lens for  $n > 1$  is a multiary lens [16] in which every asymmetric amendment lens is closed, that is, all amendments are trivial. Such  $n$ -lenses form the “special case” (wide spans of  $d$ -lenses) referred to in the final paragraph of [16]. Thus, for  $n > 1$ ,  $n$ -lenses are a specialisation of multiary lenses — the special case in which all amendments are trivial. It may be worth emphasising that this “special case” is what the authors see as the main case. There are occasionally circumstances in which non-trivial amendments are useful, and the paper [16] dealt with nontrivial amendments to have the broadest possible generality and to link directly with the extant work of Diskin et al [6], but in this paper we restrict our attention to multilenses: wide spans of asymmetric lenses without amendments.

**Definition 6** A multilens  $L$  is an  $n$ -lens for some (positive integer)  $n$ . If  $n > 1$  the multilens  $L$  is said to be non-trivial.

## 7 The fusion of multilenses

The multiary lenses of [16] compose, as shown there, with a multicategory [19] structure. In the terminology of this paper, using the composite defined in [16], an  $m$ -lens and an  $n$ -lens compose to give an  $(m+n-2)$ -lens (think for example of “plugging” the left leg of one lens into the right leg of the other with those two legs “disappearing”). That composition is a generalisation of the usual composition of symmetric lenses, or indeed of spans or relations — a 2-lens composes with a 2-lens if the leftmost foot of one equals the rightmost foot of the other, and the result is a  $(2+2-2)$ -lens with peak a pullback calculated over the common foot (see Remark 4). Notice that in this familiar composition the common foot and the two legs to that foot all disappear (hence the subtraction of two in the count of legs).

The simple, but important, change in this paper is the introduction of a new composition called fusion which retains the foot that has been composed over.

**Definition 7** *Suppose  $L = (f_i : \mathbf{S} \longrightarrow A_i)$  is an  $m$ -lens and  $L' = (f'_i : \mathbf{S} \longrightarrow A'_i)$  is an  $n$ -lens with the rightmost foot of  $L$  being equal to the leftmost foot of  $L'$ . Then the fusion of  $L$  and  $L'$ , denoted here simply by the juxtaposition  $LL'$ , is the lens  $LL' = (g_i : \mathbf{S} \longrightarrow B_i)$  given as follows:  $LL'$  is an  $(m+n-1)$ -lens with feet  $B_i = A_i$  for  $i \leq m$ , and  $B_i = A'_{i-m+1}$  for  $i \geq m$ . Let  $\mathbf{T}$  be the pullback of  $f_m$  along  $g_1$  with projections  $H$  and  $H'$ , then  $g_i = f_i H$  if  $i \leq m$ , and  $g_i = f'_{i-m+1} H'$  if  $i \geq m$ .*

**Remark 8** We record here a few basic results about the fusion operation.

1. Well-definedness: The use of both  $i \leq m$  and  $i \geq m$  in the definition is deliberate, and is intended to reinforce the sense of fusion. If  $i = m$  then  $B_i = A_i = A'_{i-m+1}$  by assumption, and  $g_i = f_i H = f'_{i-m+1} H'$  by Proposition 3 part 2, so the fusion is well-defined.
2. Identities: Identity 1-lenses are, up to equivalence, left and right identities for fusion.
3. Associativity: Up to span isomorphism in **cat**, the fusion operation is associative. The equivalence relation presented in [14] (and mostly avoided here) is coarser than span isomorphism in **cat**, and is a congruence for fusion (and for the composition of [16]), so fusion is also associative for equivalence classes of multilenses. (The calculations are tedious, but routine, and follow the path traced in [14], so they have been suppressed here.)

We would like to emphasise that fusion is a minor change from multilens composition. Non-trivial multilenses are fusable if and only if they are composable — fusion simply keeps the foot that one composes over along with the single (by Proposition 3 part 2) leg to that foot. It is still an operation which combines composable multilenses to get multilenses. But fusion often feels like a significant change for people who are used to composing symmetric lenses because the fusion of two symmetric lenses is not a symmetric lens but rather a 3-multilens. This difference is exactly what we need for our applications. We will return to this in Section 9 where we illustrate a few other well-known fusion operators for comparison purposes and to set readers’ minds at ease.

## 8 Sometimes lens fusions are better than lens compositions

We turn now to some basic examples of fusion, and then revisit the example of Section 2.

What happens when we fuse 1-lenses, recalling that 1-lenses are themselves simply asymmetric lenses?

If  $L$  and  $L'$  are 1-lenses, then  $LL'$  is also a  $(1+1-1=1)$ -lens, so fusion is an operation on asymmetric lenses. But it is not the usual composition of asymmetric lenses because fusable 1-lenses have common

codomains. That is, they form a cospan of asymmetric lenses  $L : \mathbf{S} \longrightarrow A_1 = A'_1 \longleftarrow \mathbf{S}' : L'$ . So, how do we fuse a cospan? Definition 7 tells us that we pull the two asymmetric lenses back along each other, and the resulting asymmetric lens is the diagonal of the “pullback” square  $\mathbf{T} \longrightarrow A_1$ . This is sometimes known as the *consistency lens*. In database terms, if the trough of the cospan  $A_1$  is the system of states of common data, then the peak of the “pullback” is, up to isomorphism, the category whose objects are consistent pairs of states of the systems  $\mathbf{S}$  and  $\mathbf{S}'$ , consistent in as much as they share the same common data state, and whose arrows are pairs of transitions, one from  $\mathbf{S}$  and one from  $\mathbf{S}'$  which are consistent in as much as they involve the same transition in  $A_1$  for the shared data. The 1-lens  $LL'$  is an asymmetric lens. The Get, the functor part of the diagonal, tells us how the shared data changes when a  $\mathbf{T}$  transition takes place, and the Put tells us how to change the consistent states in  $\mathbf{T}$  when the shared data is changed.

But there is yet another way that we might fuse the asymmetric lenses  $L$  and  $L'$ . It is well-known that an asymmetric lens can be represented as a symmetric lens in two ways: For the asymmetric lens  $L$ , form a span of asymmetric lenses (Definition 2) by pairing  $L$  with the identity on  $\mathbf{S}$  on either the left or the right. If we do that on the left for  $L$ , and on the right for  $L'$  (using of course the identity on  $\mathbf{S}'$ ) we obtain two 2-lenses which we know are symmetric forms of the asymmetric lenses  $L$  and  $L'$ , and these two 2-lenses are again fusible. This time Definition 7 tells us that the resulting lens will be a  $(2 + 2 - 1 = 3)$ -lens: It is the three legged “pullback” cone over the cospan  $\mathbf{S} \longrightarrow A_1 = A'_1 \longleftarrow \mathbf{S}'$  consisting of the consistency lens in the middle, and the two “pullback” projection lenses  $H : \mathbf{T} \longrightarrow \mathbf{S}$  and  $H' : \mathbf{T} \longrightarrow \mathbf{S}'$  on the left and right.

The last paragraph describes a special case of the fusion of two symmetric lenses: Two 2-lenses fuse to form a 3-lens. The three legs are again the consistency lens in the middle, and the outer two legs are, together, the usual symmetric lens composite. The fusion “remembers” the foot  $A_1$  that the symmetric lenses have been composed over, and its relationship to the peak  $\mathbf{T}$  via the consistency lens. This is a small, but important difference, as we will soon see. The fusion remembers the way the composed up symmetric lens factors into two symmetric lenses. And why is this important? It is because, as noted in Section 5, the fact that there might be a cospan representation of each symmetric lens is important in engineering, but the composed symmetric lens might have no cospan representation so the factorisation is vital for actually building system interoperations. Let’s look again at the example presented in Section 2 and further developed in Sections 3 and 5.

Recall that there are symmetric lenses, 2-lenses, between ABC Frames and XYZ Warehouse, and between XYZ Warehouse and XYZ Logistics, and that those two symmetric lenses are cospan representable, but that the composite 2-lens between ABC Frames and XYZ Logistics is not cospan representable.

If we are concerned, as was hypothesised in Section 5, with the interoperations between ABC Frames and XYZ Logistics, we might have composed the two symmetric lenses and lost the information that the composite factors into two symmetric lenses that are cospan representable. But alternatively, we could have fused the two symmetric lenses, the two 2-lenses, to obtain a 3-lens. That 3-lens does indeed contain all the information required to study and prove properties about the interactions between ABC Frames and XYZ Logistics. But it also includes the base information to see how to factorise that interaction through XYZ Warehouse, and through that factorisation and the cospan representations, the interoperations are then easy to engineer as well.

Sometimes, lens fusions are better than lens compositions.

Thus, the more legs the merrier: In our example in fact, if we have the cospans the best fusion is five legged! See below. But first, a little more about fusions.

## 9 Other fusions

At first sight, some people find the fusion operator confronting because it works like a composition (at least, one can only fuse composable lenses), but  $n$ -lenses, and especially concerning the familiar 2-lenses normally called symmetric lenses, aren't closed under fusion. In general, fusion takes things of certain types and produces things of different types.

So it seems worthwhile to point out that operators like fusion are common, and have a long history in mathematics and software engineering. To offer just a few examples:

1. Path categories: Paths in a topological space  $X$  are normally defined to be continuous functions from the unit interval  $[0, 1]$  to  $X$ . Two paths with common end and start points can be composed by reparameterization to get a new path — the first path is traversed “twice as fast” so that the second path can be traversed in the second half of the unit interval. Probably this is all familiar. Of course, such composition is not associative because three paths will be traversed in either the first two quarters and the last half of the unit interval or the first half and the last two quarters of the unit interval depending on which composition is taken first. So, we introduce an equivalence relation, homotopy, which allows reparameterized paths to be treated as equivalent.

Alternatively, one might approach the problem in the style of fusion. Let paths be continuous functions from an interval of length  $n$ , say for non-negative integers  $n$ , and paths are fused (but usually still called “composed”) by having a path from  $[0, m]$  and a path from  $[0, n]$  form a path from  $[0, m + n]$ . This fusion is associative and has paths of length zero, paths from  $[0, 0]$ , also known as points, as identities. Such paths form a category, the Moore Path Category, with no need for any equivalence relation.

2. Free monoid construction (including list concatenation): The discrete form of the example just given is familiar to computer scientists. Lists, or indeed arrays, might be seen as functions from sets  $[n]$  of  $n$  elements into a data type  $X$ . The concatenation of lists of length  $m$  and length  $n$  gives a list of length  $m + n$  by fusing the functions. In mathematics of course, this is the construction of the free monoid on a set  $X$  via words in the alphabet  $X$ . Again associativity is immediate, and identity comes from the empty domain  $[0]$ .
3. Other free constructions from  $F1$ : In both the fusion examples just given, the collection of domains has the form of a coalgebra family [18] and can be calculated from the free algebra on the terminal. There are more examples of the same kind.
4. Composition of (lax) natural transformations: To offer an example of a rather different kind, natural transformations are, in chain complex terms, degree 1 maps. Given a natural transformation  $\eta : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$  an *object* of  $\mathcal{A}$  is sent to an *arrow*  $\eta_A$  of  $\mathcal{B}$ , and an *arrow* of  $\mathcal{A}$  is sent to a commutative *square* in  $\mathcal{B}$ , or if  $\mathcal{B}$  is a 2-category and  $\eta$  a lax natural transformation, to a *2-cell* of  $\mathcal{B}$ , etc. The horizontal composite of natural transformations is normally said to yield a natural transformation because a commuting square is indeed a 1-cell. But a very natural composition, in analogy with the examples above, one might say a fusion, of lax natural transformations, yields a modification (a degree 2 map), and indeed the fusion of three lax natural transformations yields a perturbation (a degree 3 map). And more generally degree  $m$  and degree  $n$  maps in higher categories can fuse to give degree  $m + n$  maps. An extensive study of algebras with these kinds of compositions was undertaken by the Dutch mathematician Sjoerd Crans under the name *teisi* (singular *tas*).

We end this section with a note about counting. Recall that the fusion of an  $m$ -lens and an  $n$ -lens is an  $(m + n - 1)$ -lens. But the examples just given “fuse”  $m$ -structures and  $n$ -structures to get  $(m + n)$ -

structures. In fact there is no substantive difference, and the apparent difference arises just from how we count. The natural way of counting, and hence of labelling, multilenses is by counting the legs. But the examples above are counted and labelled instead by the equivalent of the *spaces between* the legs (the interval  $[0, n]$  for example is of length  $n$  because there are  $n$  unit intervals in the spaces between the “feet”  $\{0, 1, \dots, n\}$ ). The formulas coincide exactly if we use the same counting paradigm in each case.

A further example, and one in which the natural count again corresponds to the way we count legs in  $n$ -lenses, is the natural join operation for databases (this example was suggested to us by an anonymous referee). To take it in its simple form, two tables in a database consisting of say  $m$  and  $n$  columns respectively, are joined on some common data (often a key attribute) to yield a table with  $m + n - 1$  columns.

## 10 Conclusion and further work

We have seen that the *fusion* operation is sometimes better than the normal composition of, for example, symmetric lenses, because it preserves information about what structures have been composed over, and this factorisation information may be very valuable in knowing where to find cospan representations. And those cospan representations may be very useful in the implementation of systems interoperations.

But, the reader might ask, if we have those cospan representations, why would we even do fusion? We don't want to lose the cospan representations if they're so useful for implementations, and the cospan representations are *among* the feet of the spans. The fusion preserves legs and feet, but nothing between the feet. The results of fusions are wide spans: A peak, some legs, and some (bare) feet.

In further work we have shown that this loss of the cospans need not be an issue at all.

If we have the cospan representations, then we begin with a “zig-zag“ of asymmetric lenses among the feet. As discussed in Section 8 we can treat each of those asymmetric lenses as a symmetric lens, a 2-lens, (by pairing it, on the *outside* of the cospan, with identities) and then fuse the 2-lenses so that each cospan becomes a 3-lens. We can even do all this in one go — and this is a general result — and the fusion will be a wide span of asymmetric lenses canonically built on the limit cone of the zig-zag as calculated in **cat**.

In the case of our supply chain example, the previous paragraph means that we end up with a 5-lens. The five legs are asymmetric lenses with codomains the three business systems (ABC Frames, XYZ Warehouse and XYZ Logistics) and the two common data subsystems (X and Orders  $\longrightarrow$  XYX). As noted above, the cospans among the feet are indeed gone, but their objects remain, and, remarkably, the cospans can be *uniquely* recovered whenever required. This follows from a particularly nice orthogonal factorisation system on **ALens** using image factorisations. And image factorisations are much simpler for lenses than for arbitrary functors, and are frequently used, often tacitly, by engineers who cut lenses down to their image factorisations routinely. The unique fillers for the orthogonal factorisation system restore the asymmetric lenses making up the cospans whenever they are required. The fusion contains not just the factorisation information required to search for the cospan representations, but in fact all the information needed to fully determine the cospan representations, and so all the information needed for an effective implementation.

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## References

- [1] Aguiar, M. (1997) *Internal Categories and Quantum Groups*. Ph.D. thesis, Cornell University.
- [2] Bancilhon, F. and Spyratos, N. (1981) Update semantics of relational views. *ACM Transactions on Database Systems* **6**, 557–575. doi:10.1145/319628.319634
- [3] Clarke, B. (2019) Internal lenses as functors and cofunctors. *Electronic Proceedings in Theoretical Computer Science*, **323**, 183–195. doi:10.4204/EPTCS.323.13
- [4] Clarke, B. (2020) A diagrammatic approach to symmetric lenses. *Electronic Proceedings in Theoretical Computer Science*, to appear, 13pp.
- [5] Dampney, C.N.G and Johnson, M. (2001) Half-duplex interoperations for cooperating information systems. *Advances in Concurrent Engineering*, 565–571. IN-1, (International Institute of Concurrent Engineering, ISBN 09710461-0-7, 2000).
- [6] Diskin, Z., König, H., and Lawford, M. (2018) Multiple model synchronization with multiary delta lenses, *Lecture Notes in Computer Science* **10802**, 21–37. doi:10.1007/978-3-319-89363-1\_2
- [7] Diskin, Z., Xiong, Y., and Czarnecki, K. (2011) From State- to Delta-Based Bidirectional Model Transformations: the Asymmetric Case. *Journal of Object Technology* **10**, 1–25. doi:10.5381/jot.2011.10.1.a6
- [8] Diskin, Z., Xiong, Y., Czarnecki, K., Ehrig, H., Hermann, F., and Orejas, F. (2011) From State- to Delta-Based Bidirectional Model Transformations: the Symmetric Case. *Lecture Notes in Computer Science* **6981**, 304–318. doi:10.1007/978-3-642-24485-8\_22
- [9] Diskin, Z. and Maibaum, T. (2012) Category Theory and Model-Driven Engineering: From Formal Semantics to Design Patterns and Beyond, 7th ACCAT Workshop on Applied and Computational Category Theory, *Electronic Proceedings in Theoretical Computer Science*, **93**, 1–21. doi:10.4204/EPTCS.93.1
- [10] Gibbons, J. and Stevens, P. (2016) Bidirectional Transformations. *Lecture Notes in Computer Science* **9715**. doi:10.1007/978-3-319-79108-1
- [11] Hofmann, M., Pierce, B., and Wagner, D. (2011) Symmetric Lenses. In ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL), *ACM SIGPLAN Notices* **46**, 371–384. doi:10.1145/1925844.1926428
- [12] Johnson, M. and Rosebrugh, R. (2007) Fibrations and universal view updatability. *Theoretical Computer Science* **388**, 109–129. doi:10.1016/j.tcs.2007.06.004
- [13] Johnson, M. and Rosebrugh, R. (2016) Unifying set-based, delta-based and edit-based lenses. Proceedings of the 5th International Workshop on Bidirectional Transformations, Eindhoven *CEUR Proceedings* **1571**, 1–13.
- [14] Johnson, M. and Rosebrugh, R. (2017) Symmetric delta lenses and spans of asymmetric delta lenses. *Journal of Object Technology*, **16**, 2:1–32. doi:10.5381/jot.2017.16.1.a2
- [15] Johnson, M. and Rosebrugh, R. (2018) Cospans and symmetric lenses. *Programming 18 Companion*, ACM, 21–29. doi:10.1145/3191697.3191717
- [16] Johnson, M. and Rosebrugh, R. (2019) Multicategories of multiary lenses. *CEUR Proceedings* **2355**, 30–44.
- [17] Johnson, M. and Stevens, P. (2018) Confidentiality in the process of (model driven) software development. *Programming 18 Companion*, ACM, 1–8. doi:10.1145/3191697.3191714
- [18] Johnson, M. and Walters, R.F.C. (1992) Algebra objects and algebra families for finite limit theories. *Journal of Pure and Applied Algebra*, **83**, 283–293.
- [19] Leinster, T. (2004) *Higher Operads, Higher Categories*. Cambridge University Press. arXiv:math/0305049 doi:10.1017/CBO9780511525896

- [20] Oles, F. J. (1982) *A category-theoretic approach to the semantics of programming languages*. PhD Thesis, Syracuse University.
- [21] Pierce, B. and Schmitt, A. (2003) Lenses and view update translation. Preprint.