

Open Diagrams via Coend Calculus

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Morphisms in a monoidal category are usually interpreted as *processes*, and graphically depicted as square boxes. In practice, we are faced with the problem of interpreting what *non-square boxes* ought to represent in terms of the monoidal category and, more importantly, how should they be composed. Examples of this situation include *lenses* or *learners*. We propose a description of these non-square boxes, which we call *open diagrams*, using the monoidal bicategory of profunctors. A graphical coend calculus can then be used to reason about open diagrams and their compositions.

1 Introduction

1.1 Open Diagrams

Morphisms in monoidal categories are interpreted as processes with inputs and outputs and generally represented by square boxes. This interpretation, however, raises the question of how to represent a process that does not consume all the inputs at the same time or a process that does not produce all the outputs at the same time. For instance, consider a process that consumes an input, produces an output, then consumes a second input and ends producing an output. Graphically, we have a clear idea of how this process should be represented, even if it is not a morphism in the category.

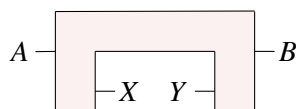


Figure 1: A process with a non-standard shape. The input A is taken at the beginning, then the output X is produced, strictly after that, the input Y is taken; finally, the output B is produced.

Reasoning graphically, it seems clear, for instance, that we should be able to *plug* a morphism connecting the first output to the second input inside this process and get back an actual morphism of the category.

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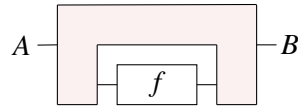


Figure 2: It is possible to plug a morphism $f: X \rightarrow Y$ inside the previous process (Figure 1), and, importantly, get back a morphism $A \rightarrow B$.

The particular shape depicted above has been studied by [29] under the name of (monoidal) *optic*; it can be also called a *monoidal lens*; and it has applications in bidirectional data accessing [28, 5, 21] or compositional game theory [13]. A multi-legged generalization has appeared also in quantum circuit design [8] and quantum causality [20] as a notational convention, see [30]. It can be shown that boxes of that particular shape should correspond to elements of a suitable *coend* (Figure 3, see also §1.2 and [24, 29]). The intuition for this coend representation is to first consider a tuple of morphisms, and then quotient out by the equivalence relation generated by sliding morphisms along connected wires.

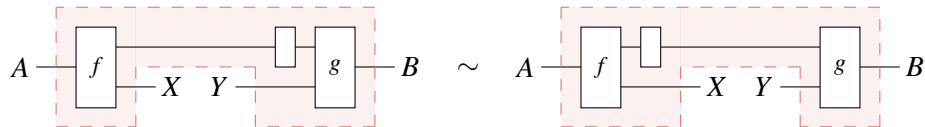


Figure 3: A box of this shape is meant to represent a pair of morphisms in a monoidal category quotiented out by "sliding a morphism" over the upper wire.

It has remained unclear, however, how this process should be carried in full generality and if it was on solid ground. Are we being formal when we use these *open* or *incomplete* diagrams? What happens with all the other possible shapes that one would want to consider in a monoidal category? In general, we cannot assume that they are squares. For instance, the second of the shapes in Figure 4 has three inputs and two outputs, but the first input cannot affect the last output; and the last input cannot affect the first output.¹

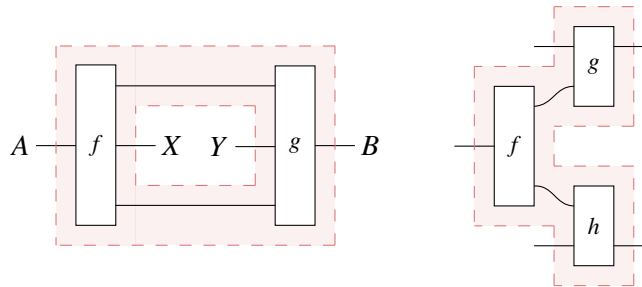


Figure 4: Some other shapes for boxes in a monoidal category.

This article presents the idea that incomplete diagrams should be interpreted as valid diagrams in the monoidal bicategory of profunctors; and that compositions of incomplete diagrams correspond to reductions that employ the monoidal bicategory structure. At the same time, this amounts to a graphical presentation of *coend calculus*.

¹This particular shape comes from a question by Nathaniel Virgo.

1.2 Coend calculus

Coends are particular cases of colimits and *coend calculus* is a practical formalism that uses Yoneda reductions to describe isomorphisms between them. Their dual counterparts are *ends*, and formalisms for both interact nicely in a *(Co)End calculus* [22].

Definition 1.1. The **coend** $\int^{X \in \mathbf{C}} P(X, X)$ of a profunctor $P: \mathbf{C}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$ is the coequalizer of the action of morphisms on both arguments of the profunctor.

$$\int^{X \in \mathbf{C}} P(X, X) \cong \text{coeq} \left(\bigsqcup_{f: B \rightarrow A} P(A, B) \rightrightarrows \bigsqcup_{X \in \mathbf{C}} P(X, X) \right).$$

An element of the coend is an equivalence class of pairs $[X, x \in P(X, X)]$ under the equivalence relation generated by $[X, P(f, \text{id}_X)(p)] \sim [Y, P(\text{id}_Y, f)(p)]$ for each $f: X \rightarrow Y$.

Our main idea is to use these equivalence relations to deal with the quotienting arising in non-square monoidal boxes.

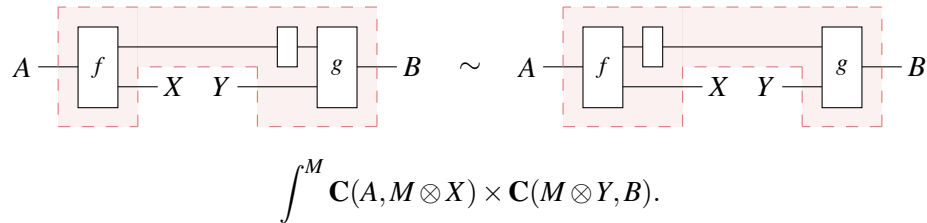


Figure 5: We can go back to Figure 3 to check how it coincides with the quotienting arising from a coend.

1.3 Contributions

Our first contribution is a graphical calculus of *shapes* of open diagrams (§2), with semantics on the monoidal bicategory of profunctors, and with an emphasis on representing monoidal structures. We show how to compose and simplify shapes (§3). Our second contribution is a graphical calculus of *open diagrams*, in terms of the category of pointed profunctors, and hinting at a pseudofunctorial analogue of *functor boxes* [23] (§4).

As examples, we recast the multiple ways of composing *monoidal lenses* and other coend constructions on the literature on optics (§2.3). We also study categories with feedback (§2.4).

2 Shapes of Open Diagrams

In the same sense that morphisms sharing the same domain and codomain are collected into a hom-set; open diagrams sharing the same *shape* will be also collected into a set. Our first step is a graphical language for shapes and a compositional interpretation that assigns a set to each shape (which we anticipate in Figure 6).

$$\begin{aligned}
& \text{Diagram 1: } \text{A} \text{---} \boxed{\text{X} \text{---} \text{Y}} \text{---} \text{B} \cong \int^{M,N} \mathbf{C}(A, M \otimes X \otimes N) \times \mathbf{C}(M \otimes Y \otimes N, B), \\
& \text{Diagram 2: } \begin{array}{c} \text{I}_1 \\ \text{I}_0 \\ \text{I}_2 \end{array} \text{---} \boxed{\phantom{\text{X} \text{---} \text{Y}}} \text{---} \begin{array}{c} \text{O}_1 \\ \text{O}_2 \end{array} \cong \int^{M,N} \mathbf{C}(I_0, M \otimes N) \times \mathbf{C}(I_1 \otimes M, O_1) \times \mathbf{C}(N \otimes I_2, O_2).
\end{aligned}$$

Figure 6: The shapes of Figure 4, abstracted as string diagrams, define sets.

2.1 String Diagrams

Shapes are closed string diagrams in **Prof**, the monoidal bicategory of profunctors [22, §5]. *Wires* represent small categories $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots)$; when unlabelled, they are understood to represent some fixed category. *Diagrams* with input \mathbf{A} and output \mathbf{B} are profunctors $\mathbf{A}^{op} \times \mathbf{B} \rightarrow \mathbf{Set}$. *Deformations* are natural transformations. *Sequential composition* of diagrams with matching wires composes two profunctors $P: \mathbf{A}^{op} \times \mathbf{B} \rightarrow \mathbf{Set}$ and $Q: \mathbf{B}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$ into the profunctor $(P \diamond Q): \mathbf{A}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$ given by

$$(P \diamond Q)(A, C) := \int^{B \in \mathbf{B}} P(A, B) \times Q(B, C).$$

Parallel composition of diagrams uses the cartesian product of categories and the terminal category as unit. Laying two profunctors $P_1: \mathbf{A}_1^{op} \times \mathbf{B}_1 \rightarrow \mathbf{Set}$ and $P_2: \mathbf{A}_2^{op} \times \mathbf{B}_2 \rightarrow \mathbf{Set}$ in parallel yields the profunctor $(P_1 \otimes P_2): (\mathbf{A}_1 \times \mathbf{A}_2)^{op} \times (\mathbf{B}_1 \times \mathbf{B}_2) \rightarrow \mathbf{Set}$ defined by

$$(P_1 \otimes P_2)(A_1, A_2, B_1, B_2) := P_1(A_1, B_1) \times P_2(A_2, B_2).$$

As a consequence, closed string diagrams represent sets, as profunctors $\mathbf{1}^{op} \times \mathbf{1} \rightarrow \mathbf{Set}$.

The string diagrammatic calculus for monoidal bicategories has been studied by Bartlett [2] expanding on a strictification result by Schommer-Pries [31]. It is similar to the graphical calculus of monoidal categories, with the caveat that deformations correspond to invertible 2-cells instead of equalities. Henceforward, the symbols (\rightarrow) and (\cong) between diagrams will denote natural transformations and natural isomorphisms, respectively. It can be also seen as a “sliced” version of *surface diagrams*.

Definition 2.1 (Input and output ports). Every object $A \in \mathbf{C}$ determines two profunctors via its contravariant and covariant Yoneda embeddings, $(\text{A}-) := \mathbf{C}(A, -): \mathbf{1}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$ and $(-\text{A}) := \mathbf{C}(-, A): \mathbf{C}^{op} \times \mathbf{1} \rightarrow \mathbf{Set}$.

Definition 2.2 (Junctions and forks). Every monoidal category (\mathbf{C}, \otimes, I) has a canonical pseudomonoid structure on the monoidal bicategory **Prof** given by $(\text{?}-) := \mathbf{C}(- \otimes -, -)$ and $(\text{I}-) := \mathbf{C}(I, -)$, and also a canonical pseudocomonoid structure given by $(-\text{?}) := \mathbf{C}(-, - \otimes -)$ and $(-\text{I}) := \mathbf{C}(-, I)$.

Proposition 2.3. *By definition, $(\text{I}-) \cong (\text{I}-)$ and $(-\text{I}) \cong (-\text{I})$; moreover,*

$$\begin{array}{c} \text{A} \\ \text{B} \end{array} \text{---} \circ \text{---} \cong \boxed{A \otimes B} \text{---} \qquad \text{---} \circ \begin{array}{c} \text{A} \\ \text{B} \end{array} \cong \text{---} \boxed{A \otimes B}$$

In general, Yoneda embeddings are pseudofunctorial.

2.2 Copying and discarding

Shapes define sets in terms of coends, making them less practical for direct manipulation. However, shapes can be reduced to more familiar descriptions in some particular cases. For instance, if \mathbf{C} is cartesian monoidal, the leftmost shape of Figure 7 reduces to a pair of morphisms $\mathbf{C}(I_0 \times I_1, O_1)$ and $\mathbf{C}(I_0 \times I_2, O_2)$. This justifies our previous intuition, back in Figure 4, that the input I_1 should not be able to affect O_2 , while the input I_2 should not be able to affect O_1 .

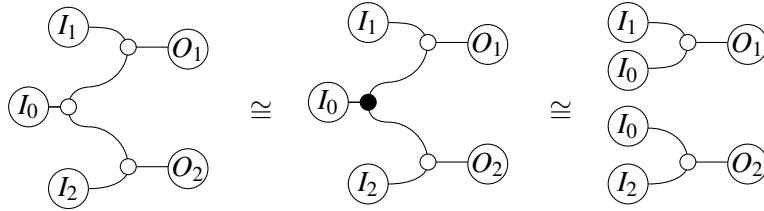


Figure 7: Simplifying a diagram.

Our second step is to justify some reductions like these in the cases of cartesian, cocartesian and symmetric monoidal categories. Every object of the category of profunctors has already a canonical pseudocomonoid structure lifted from \mathbf{Cat} which is given by $(-\lrcorner) := \mathbf{C}(-^0, -^1) \times \mathbf{C}(-^0, -^2)$ and $(-\bullet) := 1$, and also a pseudomonoid structure given by $(\lrcorner-) := \mathbf{C}(-^1, -^0) \times \mathbf{C}(-^2, -^0)$, and $(\bullet-) := 1$. These two structures “copy and discard” representable and corepresentable functors, respectively.

Proposition 2.4 (Cartesian and cocartesian). *A monoidal category is cartesian if and only if $(-\circ) \cong (-\bullet)$ and $(-\lrcorner) \cong (-\lrcorner)$, i.e. the monoidal structure coincides with the canonical one. Dually, a monoidal category is cocartesian if and only if $(\lrcorner-) \cong (\lrcorner-)$ and $(\circ-) \cong (\bullet-)$.*

Proof. The natural isomorphism $\mathbf{C}(X, Y \otimes Z) \cong \mathbf{C}(X, Y) \times \mathbf{C}(X, Z)$ is precisely the universal property of the product; a similar reasoning holds for initial objects, terminal objects and coproducts. \square

Proposition 2.5 (Symmetric monoidal). *If a monoidal category \mathbf{C} is symmetric then its symmetric pseudomonoid structure can be lifted from \mathbf{Cat} to \mathbf{Prof} . The braiding determines $\sigma : (\lrcorner\lrcorner-) \cong (\lrcorner-)$ and $\sigma^* : (-\lrcorner\lrcorner) \cong (-\lrcorner)$, dual 2-cells in the bicategory \mathbf{Prof} that commute with unitors and associators.*

2.3 Example: Lenses

We study lenses using the graphical calculus just described. This presents a new way of describing reductions with coend calculus that also formalizes the intuition of lenses as *diagrams with holes*. Profunctor optics and lenses have been studied in functional programming [21, 24, 28, 5] for bidirectional data accessing. The theory of optics uses coend calculus both to describe how optics compose and how to reduce them in sufficiently well-behaved cases to tuples of morphisms. Categories of monoidal optics and the informal interpretation of optics as *diagrams with holes* are described in [29].

Definition 2.6. A monoidal lens [24, 28, 29, “Optic” in Definition 2.0.1] from $A, B \in \mathbf{C}$ to $X, Y \in \mathbf{C}$ is an element of the following set.

$$\begin{array}{c} \textcircled{A} \\ \textcircled{X} \quad \textcircled{Y} \\ \textcircled{B} \end{array} \cong \int^M \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B)$$

Cartesian lenses are examples of monoidal lenses that are especially important in applications [11, 13].

Proposition 2.7. *In a cartesian category \mathbf{C} , a lens $(A, B) \rightarrow (X, Y)$ is given by a pair of morphisms $\mathbf{C}(A, X)$ and $\mathbf{C}(A \times Y, B)$. In a cocomonoidal category, lenses are called prisms [21] and they are given by a pair of morphisms $\mathbf{C}(S, A + T)$ and $\mathbf{C}(B, T)$.*

Proof. We write the proof for lenses, the proof for prisms is dual and can be obtained by mirroring the diagrams. The coend derivation can be found, for instance, in [24].

$$\begin{array}{l}
 \begin{array}{c} \textcircled{A} \\ \textcircled{X} \quad \textcircled{Y} \\ \textcircled{B} \end{array} \\
 \cong \{(-\curvearrowright) \cong (-\curvearrowleft)\} \\
 \begin{array}{c} \textcircled{A} \\ \bullet \\ \textcircled{X} \quad \textcircled{Y} \\ \textcircled{B} \end{array} \\
 \cong \{\text{Copy}\} \\
 \begin{array}{c} \textcircled{A} \\ \textcircled{X} \\ \textcircled{A} \\ \textcircled{Y} \\ \textcircled{B} \end{array}
 \end{array}
 \quad
 \begin{array}{l}
 \int^M \mathbf{C}(A, M \times X) \times \mathbf{C}(M \times Y, B) \\
 \cong \{\text{Universal property of the product}\} \\
 \int^M \mathbf{C}(A, M) \times \mathbf{C}(A, X) \times \mathbf{C}(M \times Y, B) \\
 \cong \{\text{Yoneda lemma}\} \\
 \mathbf{C}(A, X) \times \mathbf{C}(A \times Y, B) \quad \square
 \end{array}$$

2.4 Example: Feedback

Shapes do not need to be limited to a single category. For instance, we can make use of the opposite category to introduce feedback, in the sense of the *categories with feedback* of [19]. Wires in the opposite category will be marked with an arrow to distinguish them.

$$\begin{array}{c} \leftarrow \\ \textcircled{X} \quad \textcircled{Y} \end{array} = \int^{M \in \mathbf{C}} \mathbf{C}(M \otimes X, M \otimes Y).$$

Figure 8: A shape with feedback, interpreted as a set.

Proposition 2.8 (see [33]). *Profunctors form a compact closed bicategory. The dual of a category is its opposite category.*

3 Composing and Reducing Shapes

We have been focusing on the invertible transformations between shapes, but arguably the most interesting case is that of non-invertible transformations. Our next step is to describe rules for composing and reducing diagrams that translate to valid coend calculus reductions.

Definition 3.1 (Joining and splitting wires). Identities and composition define natural transformations $\eta_A: () \rightarrow (A-A)$ and $\varepsilon_A: (-A A-) \rightarrow (—)$. They determine an adjunction, as the following transformations are identities.

$$(-A) \xrightarrow{\eta} (-A A-A) \xrightarrow{\varepsilon} (-A); \quad (A-) \xrightarrow{\varepsilon} (A-A A-) \xrightarrow{\eta} (A-).$$

In the same vein, junctions and forks have natural transformations $\varepsilon_{\otimes}: (-\circlearrowleft-\circlearrowright-) \rightarrow (—)$ and $\eta_{\otimes}: (—) \rightarrow (\circlearrowleft-\circlearrowright-)$. They determine an adjunction, as the following transformations are identities.

$$(\circlearrowleft-) \xrightarrow{\eta} (\circlearrowleft-\circlearrowright-) \xrightarrow{\varepsilon} (\circlearrowleft-); \quad (-\circlearrowright-) \xrightarrow{\eta} (-\circlearrowleft-\circlearrowright-) \xrightarrow{\varepsilon} (-\circlearrowright-).$$

For instance, as we saw in the introduction (Figure 2), a lens $(A, B) \rightarrow (X, Y)$ can be composed with a morphism $X \rightarrow Y$ to obtain a morphism $A \rightarrow B$.

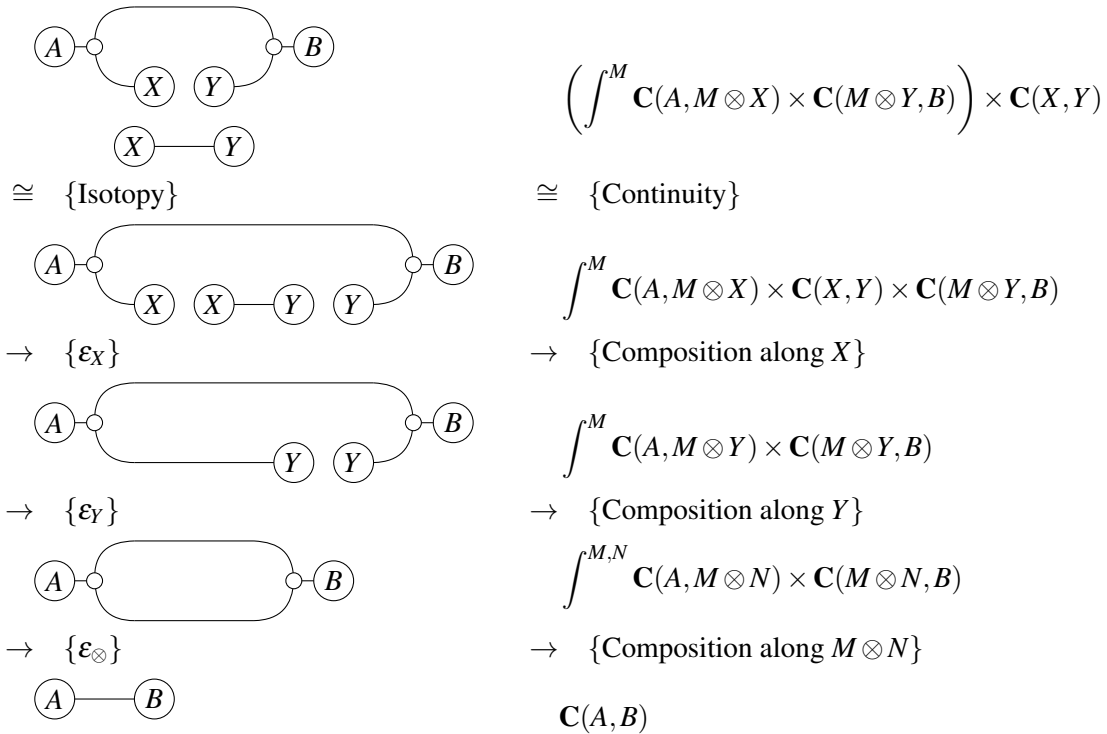


Figure 9: Composing a lens with a morphism, formalizing Figure 2.

3.1 Example: Categories of Optics

Two lenses of types $(A, B) \rightarrow (X, Y)$ and $(X, Y) \rightarrow (U, V)$ can be composed with each other to form a *category of optics* [29]. There is, however, another way of composing two lenses. When the base

category is symmetric, a lens $(A, Y) \rightarrow (X, V)$ can be composed with a lens $(X, B) \rightarrow (U, Y)$ into a lens $(A, B) \rightarrow (U, Y)$. We will observe that, even if **Prof** is symmetric, the reduction explicitly uses symmetry on the base category **C**.

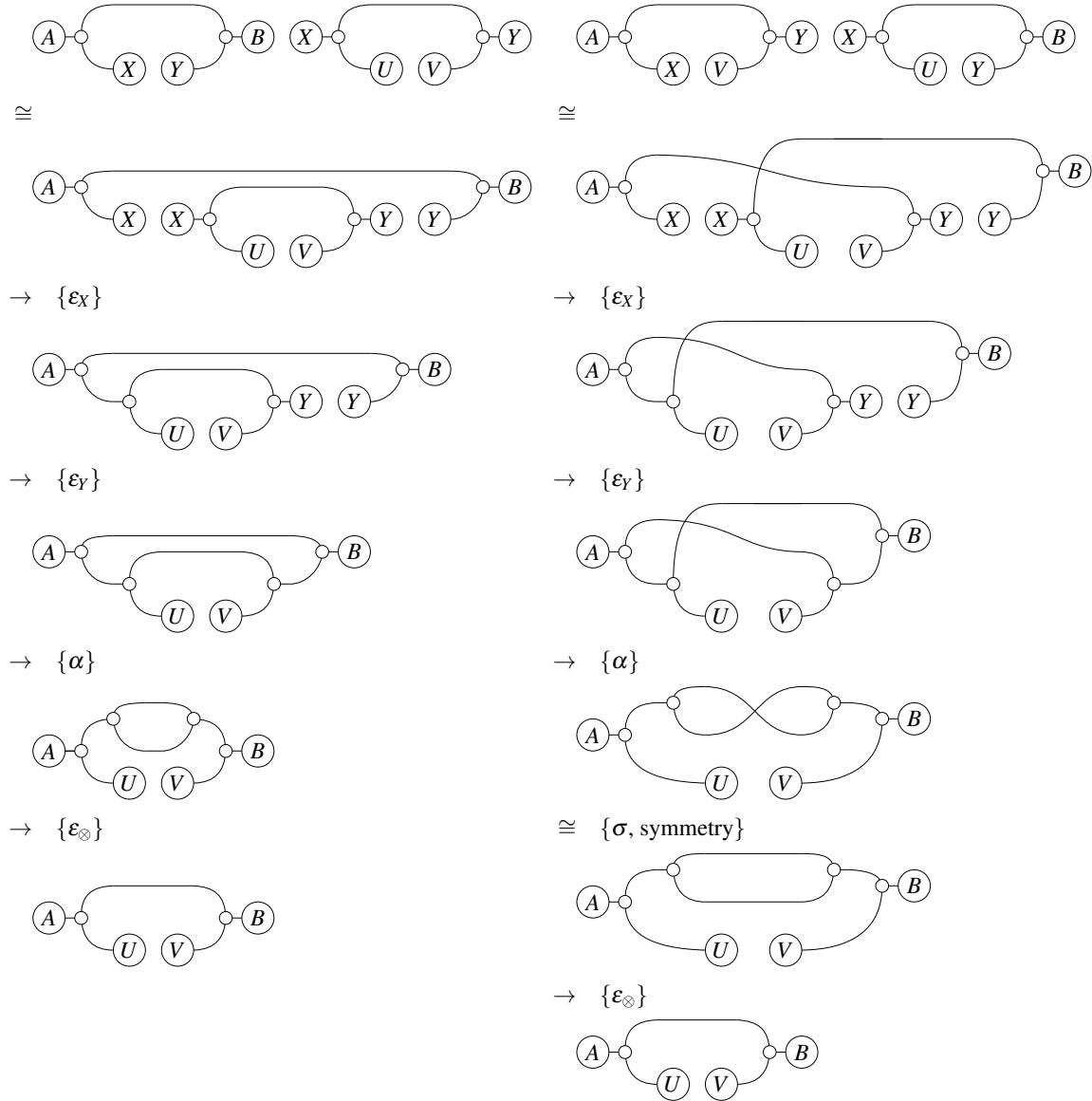


Figure 10: In parallel, two possible compositions of optics.

3.2 Example: from Lenses to Dynamical Systems

In [32, Definition 2.3.1], a discrete dynamical system, a Moore machine, is characterized to have the same data as a lens $(A, A) \rightarrow (X, Y)$. The following derivation is a conceptual justification of this coincidence: a lens with suitable types can be made into a morphism of the free category with feedback [19], subsuming

particular cases such as *Moore machines*.

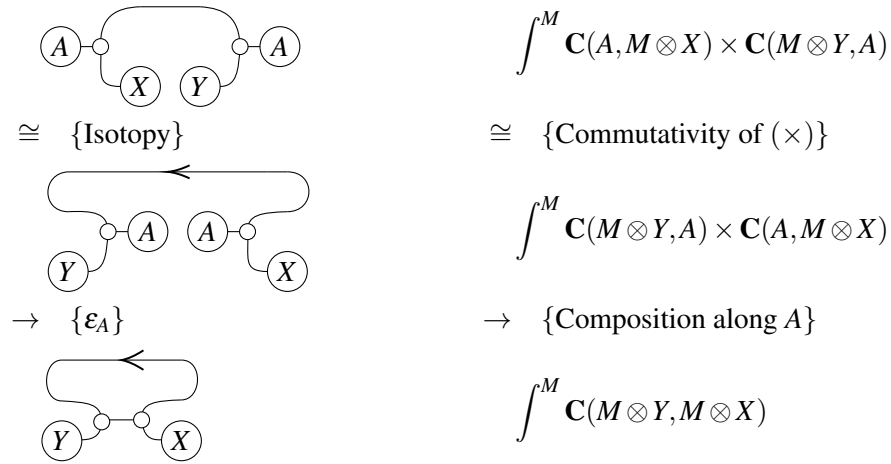


Figure 11: From lenses to dynamical systems.

4 Open Diagrams

Our final contribution is to justify how to obtain the diagrams that originally motivated this article (*open diagrams*) by “looking inside” the shapes. So far, the element of a set described by a shape could be only expressed as a derivation of the shape from the empty diagram. In this section, we show diagrams that summarize these derivations and that represent specific elements of the shape.

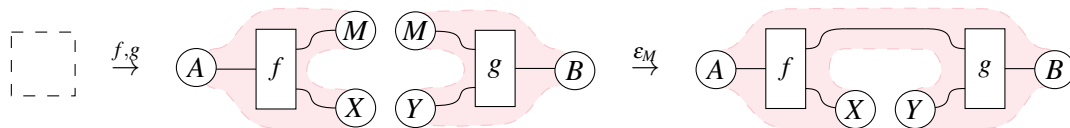


Figure 12: Open diagrams represent specific elements.

4.1 Open Diagrams

Open diagrams will be interpreted in \mathbf{Prof}_* , the symmetric monoidal bicategory of pointed profunctors. Its 0-cells are categories with a chosen object; its 1-cells from (\mathbf{A}, X) to (\mathbf{B}, Y) are profunctors $P: \mathbf{A}^{op} \times \mathbf{B} \rightarrow \mathbf{Set}$ with a chosen point $p \in P(X, Y)$; and its 2-cells are natural transformations preserving that chosen point. The point will keep track of a specific element of the shape.

Proposition 4.1. *Reductions on shapes can be lifted to reductions on open diagrams.*

Proof. There exists a pseudofunctor $\mathcal{U}: \mathbf{Prof}_* \rightarrow \mathbf{Prof}$ that forgets about the specific point. It holds that $a \in A$ for every element $(A, a) \in \mathbf{Prof}_*((\mathbf{1}, 1), (\mathbf{1}, 1))$. Natural transformations $\alpha: P \rightarrow Q$ can be lifted

to $\alpha_*: (P, p) \rightarrow (Q, \alpha(p))$ in a unique way, determining a discrete opfibration $\mathbf{Prof}_*((\mathbf{A}, A), (\mathbf{B}, B)) \rightarrow \mathbf{Prof}(\mathbf{A}, \mathbf{B})$ for every pair of pointed categories (\mathbf{A}, A) and (\mathbf{B}, B) . \square

Proposition 4.2. *Diagrams on the base category can be lifted to open diagrams.*

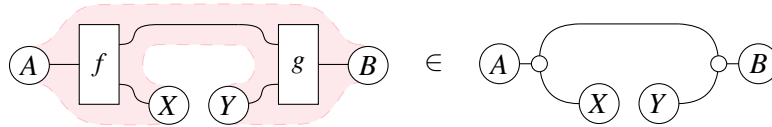
Proof. Let \mathbf{C} be a small category. There exists a pseudofunctor $\mathbf{C} \rightarrow \mathbf{Prof}_*$ sending every object $A \in \mathbf{C}$ to the 0-cell pair (\mathbf{C}, A) and every morphism $f \in \mathbf{C}(A, B)$ to the 1-cell pair $(\text{hom}_{\mathbf{C}}, f)$. Moreover, when (\mathbf{C}, \otimes, I) is monoidal, the pseudofunctor is lax and oplax monoidal (weak pseudofunctor in [25]), with oplaxators being left adjoint to laxators.

This can be called an *op-ajax monoidal pseudofunctor*, following the notion of *ajax monoidal functor* from [12]. \square

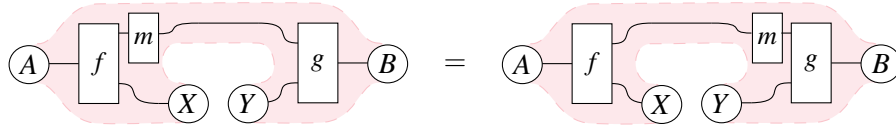
The graphical calculus for open diagrams can then be interpreted as the graphical calculus of pointed profunctors enhanced with a pseudofunctorial box, in the same vein as the functor boxes of [23]. Similar “*internal diagrams*” have been described before by [3] as a “graphical mnemonic notation”.

4.2 Example: Categories of Optics

The lens $\langle g, f \rangle: (A, B) \rightarrow (X, Y)$ is depicted as the following open diagram.

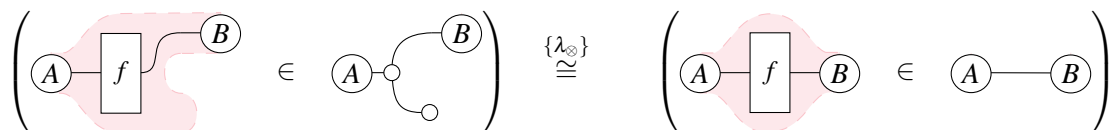


The quotienting that makes $\langle g, (m \otimes \text{id}_X) \circ f \rangle = \langle g \circ (m \otimes \text{id}_Y), f \rangle$ is explicit in this graphical calculus. The following two diagrams are equal in the category \mathbf{Prof}_* : they represent the same set and the same element within it.

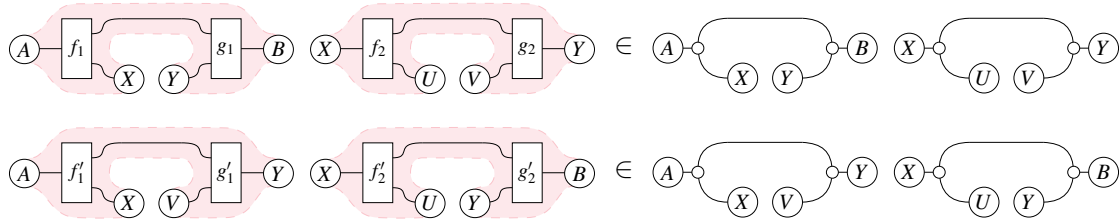


Let us repeat an important caveat: the same diagram, after a deformation, describes a different, although isomorphic, set. A diagram describes a set only up to isomorphism. This raises a subtlety: we cannot speak of equality between open diagrams with different shapes, for they belong to different sets. We could however speak of equality between two open diagrams such that the shape of the first can be deformed into the shape of the second. The deformation determines a particular isomorphism between the sets defined by the shapes. Equality of elements on isomorphic sets is understood to be equality after applying that isomorphism.

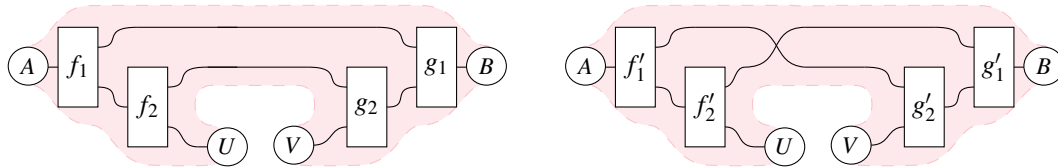
For instance, the following two elements, $(\lambda \circ f) \in \mathbf{C}(A, B \otimes I)$ and $f \in \mathbf{C}(A, B)$, are equal after the deformation given by counitality of the pseudocomonoid structure.



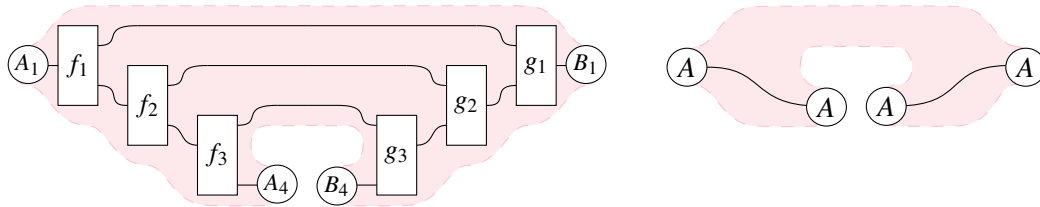
We will now use open diagrams to justify that both compositions from Example 3.1 determine a category. Consider two pairs of lenses of suitable types.



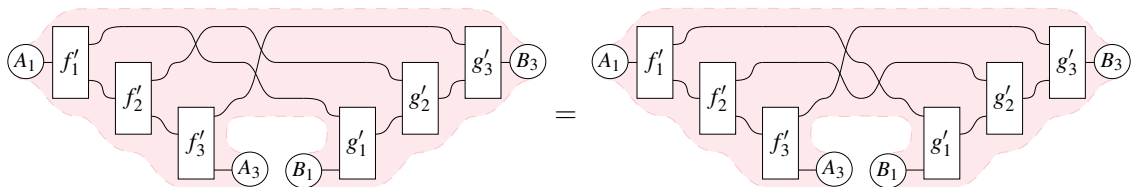
We can use Proposition 4.1 to lift the two compositions in Example 3.1 to two deformations of open diagrams that send the two pairs of lenses to the following two open diagrams, respectively.



Let us show that a category can be defined from the first composition. Consider three lenses o_i for $i = 1, 2, 3$. We have two ways of composing them, as $o_1 \circ (o_2 \circ o_3)$ or $(o_1 \circ o_2) \circ o_3$, but they both give rise to the same final diagram, thanks to associativity of the base monoidal category. The identity is the diagram on the right.



For the second composition, checking associativity amounts to the following equality. The identity is the same as in the previous case.



The graphical calculus is hiding at the same time the details of two structures. The first is the quotient relation given by the coend in the monoidal bicategory of profunctors; the second is the coherence of the base monoidal category inside the pseudofunctorial box.

5 Related and Further Work

The graphical calculus for profunctors can be seen as a direction in which the graphical calculus for the cartesian bicategory of relations [6, 12] can be categorified. A notion of *cartesian bicategory* generalizing

relations is discussed in [7]. For a slightly different future direction, we could try to relate this work to many of the interesting applications of *compact closed bicategories* (see [33]); such as *resistor networks*, *double-entry bookkeeping* [18] or *higher linear algebra* [17].

Certain shapes open diagrams have been described in the literature. Specifically, finite *combs* were used as notation by [8, 20, 29]; the relation with lenses is described in [30]. Previous graphical calculi for lenses and optics [16, 4] have elegantly captured some aspects of optics by working on the Kleisli or Eilenberg-Moore categories of the Pastro-Street monoidal monad [27]. The present approach diverges from previous formalisms on optics by focusing on the *monoidal* structure of the bicategory of profunctors, which seems to be crucial for the case of optics while not considered by previous work (neither for arbitrary profunctors nor for Tambara modules). It is more general than considering combs, as it can express arbitrary shapes in non-symmetric monoidal categories. In any case, it enables us to reason about categories of optics themselves; the results on optics of [9] can be greatly simplified in this calculus. We believe that it is closer to, and it provides a formal explanation to the *diagrams with holes* of [29, Definition 2.0.1], which were missing from previous approaches.

Most of our first part can be repeated for arbitrary monoidal bicategories such as enriched profunctors or spans. Multiple approaches to open systems (decorated cospans [10], structured cospans [1]) could be related in this way to open diagrams, but we have not explored this possibility yet. Another potential direction is to repeat this reasoning for the case of double categories and obtain a “tile” version of these diagrams (see [26, 14]).

6 Conclusions

We have presented a way to study and compose *processes* in monoidal categories that do not necessarily have the usual shape of a square box without losing the benefits of the usual language of monoidal categories. Direct applications seem to be circuit design, see [8], or the theory of optics [9]. This technique is justified by the formalism of coend calculus [22] and string diagrams for monoidal bicategories [2]. We also argue that the graphical representation of coend calculus is helpful to its understanding: contrasting with usual presentations of coends that are usually centered around the Yoneda reductions, the graphical approach seems to put more weight in the non-reversible transformations while making most applications of Yoneda lemma transparent. Regarding open diagrams, we can think of many other applications that have not been described in this article: we could speak of multiple categories at the same time and combine open diagrams of any of them using functors and adjunctions. This work has opened many paths that we aim to further explore.

We have been working in the symmetric monoidal bicategory of profunctors for simplicity, but similar results extend to the symmetric monoidal bicategory of \mathcal{V} -profunctors for \mathcal{V} a Bénabou cosmos [22, §5]. We can also consider arbitrary monoidal bicategories and drop the requirements for symmetry, copying or discarding. Finally, there is an important shortcoming to this approach that we leave as further work: the present graphical calculus is an extremely good tool for *coend calculus*, but it remains to see if it is so for *(co)end calculus*. In other words, *ends* “enter the picture” only as natural transformations (see [34]), and this can feel limiting even if, after applying Yoneda embeddings, it usually suffices for most applications. As it happens with diagrammatic presentations of regular logic [6, 12], the existential quantifier plays a more prominent role. Diagrammatic approaches to obtaining the universal quantifier in a situation like this go back to Peirce and are described by [15].

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