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Table of Contents

Preface

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Non-Classical Logics. Theory and Applications (NCL) is an international conference aimed at presenting novel results and survey works in widely understood non-classical logics and their applications. It was initially held in Łódź, Poland, in September 2008 and 2009. Later on, it was organised alternately in Toruń (2010, 2012, 2015, 2018) and Łódź (2011, 2013, 2016, 2022). The tenth edition of the Conference, organised by the University of Lodz in 2022, was the first one with the Proceedings published in EPTCS. We have a great honour and pleasure to continue this practice and for the second time include all accepted long papers in an EPTCS volume.

This 11th edition is supported by the European Research Council as one of the events organised within the project ExtenDD. In addition to 4 invited talks and 18 contributed talks, we accepted 18 short presentations on the basis of a light reviewing process. This year's edition of NCL was also co-located with the 9th edition of the Workshop on connexive logics and its program included a special session devoted to the presentation of recent results obtained within the project ExtenDD. The conference website can be found at

https://easychair.org/smart-program/NCL'24/.

The Program Committee received about 30 high-quality submissions, which were evaluated on the basis of their significance, novelty and technical correctness. Reviewing was single-blind and each paper was subjected to at least two independent reviews, followed by a thorough discussion within the Program Committee. 18 submissions were selected for presentation on the basis of their quality. This volume contains abstracts of the invited talks and full versions of the accepted submissions.

The Program Committee offered two awards for outstanding submissions. **The Best Paper Award** went to

- Satoru Niki, Hitoshi Omori, *Kamide is in America, Moisil and Leitgeb are in Australia*

The **Best Paper by a Junior Researcher Award** was given to:

- Takahiro Sawasaki, *Semantic Incompleteness of Liberman et al. (2020)'s Hilbert-style System for Term-modal Logic K with Equality and Non-rigid Terms*.

The awards have been financially supported by Springer Nature.

We would like to thank all the people who contributed to the successful performance of NCL'24. In particular, we thank the invited speakers for their talks, the authors for their contributed papers and inspiring presentations, the organisers and participants of the workshop, and all participants for their attendance and discussions. We thank the members of the Program Committee and external reviewers for their careful and competent reviewing.

We also greatly appreciate the financial support of the European Research Council, the University of Lodz, and Springer Nature. Last but not least, one of our invited speakers, professor Hanamantagouda P. Sankappanavar, has covered the costs of travel and accommodation for one of the participants of NCL'24.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics.
Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. iii–iv, doi:10.4204/EPTCS.415.0

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Unorthodox Algebras and Their Associated Logics

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In the first half of my talk, I will introduce five "unorthodox" algebras, four of which have 3-element chain as a lattice-reduct and the fifth one has a 4-element Boolean lattice as a lattice-reduct. These algebras are anti-Boolean and yet have some amazing similarities with Boolean algebras. I develop an algebraic theory of these algebras that leads to an equational axiomatization of the variety $\mathbb{UNO}1$ generated by the five unorthodox algebras. I, then, look at the structure of the lattice of subvarieties of the variety $\mathbb{UNO}1$ and provide bases for all 32 subvarieties of $\mathbb{UNO}1$. I also indicate why these algebras collectively generate a discriminator variety and individually are primal algebras.

In the second half, I will introduce an algebraizable logic (in the sense of Blok and Pigozzi) called "$\mathcal{UNO}1$" whose equivalent algebraic semantics is the variety $\mathbb{UNO}1$. Here I rely on the well-known results of Rasiowa on implicative logics and of Blok and Pigozzi on algebraizability. I will then present axiomatizations for all the axiomatic extensions of $\mathcal{UNO}1$ and discuss decidability of these logics.

References

[1] Juan M. Cornejo & Hanamantagouda P. Sankappanavar (2022): *A logic for dually hemimorphic semi-Heyting algebras and its axiomatic expansions*. Bulletin of the Section of Logic 51(4), pp. 555–645, doi:10.18778/0138-0680.2022.23.

[2] Hanamantagouda P. Sankappanavar (1987): *Semi-De Morgan algebras*. Journal of Symbolic Logic 52(3), pp. 712–724, doi:10.2307/2274359.

[3] Hanamantagouda P. Sankappanavar (2008): *Semi-Heyting algebras: An abstraction from Heyting algebras*. In M. Abad & I. Viglizzo, editors: Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish: Actas del IX Congresso Dr. Antonio A. R. Monteiro, held in Bah à Blanca, May 30–June 1, 2007), pp. 33–66.

[4] Hanamantagouda P. Sankappanavar (2011): *Expansions of Semi-Heyting algebras I: Discriminator varieties*. Studia Logica 98(1–2), pp. 27–81, doi:10.1007/s11225-011-9322-6.

[5] Hanamantagouda P. Sankappanavar (2014): *Dually quasi-De Morgan Stone semi-Heyting algebras I: Regularity*. Categories, General Algebraic Structures and Applications 2(1), pp. 47–64.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics.
Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 1–1, doi:10.4204/EPTCS.415.1

Proof Surgeries in Non-Classical Logics

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This talk explores global proof transformations within sequent and hypersequent calculi. These transformations result in:

(a) restricting cuts to analytic cuts,

(b) replacing hypersequent structures with bounded cuts, and

(c) eliminating the density rule from hypersequent calculi (thus determining whether a given logic qualifies as a fuzzy logic).

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics.
Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 2–2, doi:10.4204/EPTCS.415.2

CEGAR-Tableaux: Improved Modal Satisfiability for Modal and Tense Logics

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CEGAR-tableaux utilise SAT-solvers and modal clause learning to give the current state-of-the-art satisfiability checkers for basic modal logics K, KT and S4. I will start with a brief overview of the basic CEGAR-tableaux method for these logics.

I will show how to extend CEGAR-Tableaux to handle the five basic extensions of K by the axioms D, T, B, 4 and 5, and then indirectly to the whole modal cube. Experiments confirm that the resulting satisfiability-checkers are also the current best ones for these logics.

I will show how to extend CEGAR-tableaux to handle the modal tense logic Kt, which involves modalities for the ”future” and the ”past”. Once again, our experiments show that CEGAR-tableaux are state-of-the-art for these logics.

The talk is intended as an exposition for a broad audience and does not require any knowledge of SAT-solvers or computer science but some knowledge of modal logic would help.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics.
Theory and Applications (NCL’24).
EPTCS 415, 2024, pp. 3–3, doi:10.4204/EPTCS.415.3

Logics for Strategic Reasoning about Socially Interacting Rational Agents

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This work is on using formal logic for capturing reasoning about strategic abilities of rational agents and groups (coalitions) of agents to guarantee achievement of their goals, while acting and interacting within a society of agents. That strategic interaction can be quite complex, as it usually involves various patterns combining cooperation and competition.

The earliest logical systems for formalizing strategic reasoning include Coalition Logic (CL) introduced and studied by Pauly in the early 2000s and the independently introduced and studied at about the same period by Alur, Henzinger and Kupferman Alternating Time Temporal Logic ATL.

Recently more expressive and versatile logical systems, capturing the reasoning about strategic abilities of socially interacting rational agents and coalitions, have been proposed and studied, including:

i. the Socially Friendly Coalition Logic (SFCL), enabling formal reasoning about strategic abilities of individuals and groups to ensure achievement of their private goals while allowing for cooperation with the entire society;

ii. the Logic of Coalitional Goal Assignments (LCGA), capturing reasoning about strategic abilities of the entire society to cooperate in order to ensure achievement of the societal goals, while protecting the abilities of individuals and groups within the society to achieve their individual and group goals;

iii. the Logic for Conditional Strategic Reasoning (ConStR), formalising reasoning about agents' strategic abilities conditional on the goals of the other agents and on the actions that they are expected to take in pursuit of these goals.

Paper [1] provides a recent overview of these.

References

[1] Valentin Goranko (2023): *Logics for Strategic Reasoning of Socially Interacting Rational Agents: An Overview and Perspectives*. *Logics* 1(1), pp. 4–35, doi:10.3390/logics1010003. Available at https://www.mdpi.com/2813-0405/1/1/3.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics.
Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 4–4, doi:10.4204/EPTCS.415.4

A Binary Quantifier for Definite Descriptions in Nelsonian Free Logic

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The method Kürbis used to formalise definite descriptions with a binary quantifier $\mathtt{I}$, such that $\mathtt{I}x[F,G]$ indicates 'the F is G', is examined and improved upon in this work. Kürbis first looked at $\mathtt{I}$ in intuitionistic logic and its negative free form. It is well-known that intuitionistic reasoning approaches truth constructively. We also want to approach falsehood constructively, in Nelson's footsteps. Within the context of Nelson's paraconsistent logic **N4** and its negative free variant, we examine $\mathtt{I}$. We offer an embedding function from Nelson's (free) logic into intuitionistic (free) logic, as well as a natural deduction system for Nelson's (free) logic supplied with $\mathtt{I}$ and Kripke style semantics for it. Our method not only yields constructive falsehood, but also provides an alternate resolution to an issue pertaining to Russell's interpretation of definite descriptions. This comprehension might result in paradoxes. Free logic, which is often used to solve this issue, is insufficiently powerful to produce contradictions. Instead, we employ paraconsistent logic, which is made to function in the presence of contradicting data without devaluing the process of reasoning.

1 Introduction

Kürbis [4] developed a theory of definite descriptions formalised with a binary quantifier $\mathtt{I}$ such that $\mathtt{I}x[F,G]$ means 'the F is G'. This theory is based on intuitionistic first-order logic with identity and its negative free version. Later on, Kürbis presented another version based on intuitionistic positive free logic [6]. The version presented in [4] is a Russellian one; $\mathtt{I}x[F,G]$ is equivalent to Russell's definition of a definite description, that is, $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$. However, Russell does not use a binary quantifier, but a term-forming iota-operator ι: 'the F is G' in Russell's notation is written as $G(\iota x F(x))$. As noticed in [6], one of the problems with this notation is the meaning of $\neg G(\iota x F(x))$: it might be understood as 'the F is not G' or as 'that it is not the case that the F is G'. The use of a binary quantifier allows Kürbis to escape from this ambiguity. So 'the F is not G' is formalised as $\mathtt{I}x[F,\neg G]$ and 'that it is not the case that the F is G' as $\neg\mathtt{I}x[F,G]$.

Generally speaking, the Russellian method might lead to contradiction. There are several ways to deal with that: require G in $G(\iota x F(x))$ to be atomic, introduce scope distinctions, use free logic, use λ-calculus, use paraconsistent logic. In our opinion, the first approach is too restrictive, the second approach might be too clumsy. Free logics lack the deductive strength necessary to deduce a contradiction. Free logic is quite often employed in the study of definite descriptions and is a good solution. The use of λ-calculus works fine as well, although makes the language more complicated. We would like to examine the last option, the use of paraconsistent logic, which is a rather rarely explored option. Contradiction ceases to be an issue in a paraconsistent logic since it prevents us from drawing all the possible conclusions. Therefore, we may answer this problem without employing free logic or λ-calculus by using Nelson's logic **N4** [1] as the foundation for the research of $\mathtt{I}$.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics.
Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 5–15, doi:10.4204/EPTCS.415.5

Intuitionistic logic is known for its constructive view of truth. Nelson's logic **N4** [1] (as well as its non-paraconsistent version **N3** [7]) makes falsity constructive too. One of the aims of this paper is to formulate Kürbis' approach to definite descriptions on the basis of logic with both truth and falsity being constructive. So we study `I` in Nelson's **N4**-first order logic and in its negative free version.

To sum up, our motivation is to avoid negative consequences of contradictions in Russellian theory of definite descriptions by the use of paraconsistent logic and to make this theory constructive, in such a way that both truth and falsity are constructive. The choice of **N4** allows to reach both aims.

Kürbis' [4] approach is proof-theoretic: he uses Tennant's [11] natural deduction system for intuitionistic first-order logic with identity as well as Tennant's natural deduction system for intuitionistic negative free logic with identity and extends them by the rules for `I`.[1] In keeping with this, we also present our results in the form of natural deduction systems. But unlike Kürbis, we also use semantics in our work. Additionally, we establish the following embedding theorems: both syntactically and semantically Nelson's (negative free) logic is embedded into intuitionistic (negative free) logic. As a consequence, we obtain the completeness theorem. Instead of using our embedding processes for `I`, we utilise its definition via quantifiers to derive the sufficient truth and falsity conditions for `I`.

The structure of the paper is as follows. In Section 2, we formulate natural deduction systems for the logics in question. In Section 3, we formulate the semantics for these natural deduction systems. In Section 4, we formulate an embedding function and prove embedding theorems. Section 5 makes concluding remarks.

2 Natural deduction calculi

Let us fix a first-order language $L^{\neg}$ with the following symbols: variables $v_1, v_2, \ldots$; constants: $k, k_1, \ldots$; for every natural number $n > 0$, n-place predicate letters $P_0, P_1, P_2, \ldots$; identity predicate $=$; propositional connectives $\neg, \wedge, \vee, \rightarrow$; quantifiers: $\forall, \exists$; comma, left and right parenthesis. In the case of free logic, we use the symbol $\mathscr{E}$ for the existence predicate. In the metalanguage, we write x, y, z for arbitrary variables, a, b, c for arbitrary constants, $t, t_1, t_2, \ldots$ for terms, A, B, C, F, G for formulas. The notions of a term and a formula of the language $L^{\neg}$ are defined in a standard way. Let $L^{\neg}_{\mathtt{I}}$ be an extension of $L^{\neg}$ by a binary quantifier `I`. Let $L^{\perp}$ ($L^{\perp}_{\mathtt{I}}$) be the language obtained from $L^{\neg}$ ($L^{\neg}_{\mathtt{I}}$) by the replacement $\neg$ with constant falsum $\perp$. Following Kürbis [4], we use the following notation:

> "I will use A^x_t to denote the result of replacing all free occurrences of the variable x in the formula A by the term t or the result of substituting t for the free variable x in A. t is free for x in A means that no (free) occurrences of a variable in t become bound by a quantifier in A after substitution. In using the notation A^x_t I assume that t is free for x in A or that the bound variables of A have been renamed to allow for substitution without 'clashes' of variables, but for clarity I also often mention the condition that t is free for x in A explicitly. I also use the notation Ax to indicate that x is free in A, and At for the result of substituting t for x in A." [4, p. 82]

In what follows, we write **N4** for a first-order version with identity of Nelson's paraconsistent logic from [1], and $\mathbf{N4^{NF}}$ for its negative free version; their extensions by `I` we denote as $\mathbf{N4}_{\mathtt{I}}$ and $\mathbf{N4}^{\mathbf{NF}}_{\mathtt{I}}$. We write **Int** for first-order intuitionsitic logic with identity, and $\mathbf{Int^{NF}}$ for its negative free version; similarly, $\mathbf{Int}_{\mathtt{I}}$ and $\mathbf{Int}^{\mathbf{NF}}_{\mathtt{I}}$ are extensions of **Int** and $\mathbf{Int^{NF}}$ by `I`.

[1] Actually, Tennant has his own approach to definite descriptions [11, 12] and the rules for ι; the paper [5] compares Kürbis' and Tennant's methods.

Based on Prawitz's research [8] as well as Kürbis' investigation [4] of the rules for $\mathtt{I}$, we formulate the following Gentzen-Prawitz-style natural deduction systems for **N4**, $\mathbf{N4^{NF}}$, $\mathbf{N4}_{\mathtt{I}}$, and $\mathbf{N4}_{\mathtt{I}}^{\mathbf{NF}}$. The difference between free and non-free logics lies in the rules for quantifiers, including $\mathtt{I}$, identity (the existence predicate $\mathscr{E}$ is used in the case of free logics), and the usage of special rules for predicates in the case of free logics.

The rules for non-negated propositional connectives are as follows:

$$(\vee I_1)\ \frac{A}{A\vee B} \quad (\vee I_2)\ \frac{B}{A\vee B} \quad (\vee E)^{i,j}\ \frac{A\vee B \quad \begin{matrix}[A]^i\\ \mathfrak{D}_1\\ C\end{matrix} \quad \begin{matrix}[B]^j\\ \mathfrak{D}_2\\ C\end{matrix}}{C} \quad (\to I)^i\ \frac{\begin{matrix}[A]^i\\ \mathfrak{D}\\ B\end{matrix}}{A\to B}$$

$$(\to E)\ \frac{A\to B \quad A}{B} \quad (\wedge I)\ \frac{A \quad B}{A\wedge B} \quad (\wedge E_1)\ \frac{A\wedge B}{A} \quad (\wedge E_2)\ \frac{A\wedge B}{B}$$

The rules for negated propositional connectives as follows:

$$(\neg\neg I)\ \frac{A}{\neg\neg A} \quad (\neg\neg E)\ \frac{\neg\neg A}{A} \quad (\neg{\to}I)\ \frac{A \quad \neg B}{\neg(A\to B)} \quad (\neg{\to}E_1)\ \frac{\neg(A\to B)}{A} \quad (\neg{\to}E_2)\ \frac{\neg(A\to B)}{\neg B}$$

$$(\neg\vee I)\ \frac{\neg A \quad \neg B}{\neg(A\vee B)} \quad (\neg\vee E_1)\ \frac{\neg(A\vee B)}{\neg A} \quad (\neg\vee E_2)\ \frac{\neg(A\vee B)}{\neg B}$$

$$(\neg\wedge I_1)\ \frac{\neg A}{\neg(A\wedge B)} \quad (\neg\wedge I_2)\ \frac{\neg B}{\neg(A\wedge B)} \quad (\neg\wedge E)^{i,j}\ \frac{\neg(A\wedge B) \quad \begin{matrix}[\neg A]^i\\ \mathfrak{D}_1\\ C\end{matrix} \quad \begin{matrix}[\neg B]^j\\ \mathfrak{D}_2\\ C\end{matrix}}{C}$$

The rules for quantifiers are as follows (we give them in both ordinary and free versions (the rules for an ordinary version contain $'$ in their names); the proviso below is given in the form suitable for free version, but can be straightforwardly adapted for the ordinary one):

$$(\forall I)^i\ \frac{\begin{matrix}[\mathscr{E}y]^i\\ \mathfrak{D}\\ A^x_y\end{matrix}}{\forall xA} \quad (\forall E)\ \frac{\forall xA \quad \mathscr{E}t}{A^x_t} \quad (\neg\forall I)\ \frac{\neg A^x_t \quad \mathscr{E}t}{\neg\forall xA} \quad (\neg\forall E)^i\ \frac{\neg\forall xA \quad \begin{matrix}[\neg A^x_y]^i, [\mathscr{E}y]^i\\ \mathfrak{D}\\ C\end{matrix}}{C}$$

$$(\forall I')\ \frac{A^x_y}{\forall xA} \quad (\forall E')\ \frac{\forall xA}{A^x_t} \quad (\neg\forall I')\ \frac{\neg A^x_t}{\neg\forall xA} \quad (\neg\forall E')^i\ \frac{\neg\forall xA \quad \begin{matrix}[\neg A^x_y]^i\\ \mathfrak{D}\\ C\end{matrix}}{C}$$

where in $(\forall I)$, y does not occur free in any undischarged assumptions of $\mathfrak{D}$ except $\mathscr{E}y$, and either y is the same as x or y is not free in A; in $(\forall E)$, t is free for x in A; in $(\neg\forall I)$, t is free for x in A; and in $(\neg\forall E)$, y is not free in C nor any undischarged assumptions of $\mathfrak{D}$, except $\neg A^x_y$ and $\mathscr{E}y$, and either y is the same as x or it is not free in A.

$$(\exists I)\ \frac{A^x_t \quad \mathscr{E}t}{\exists xA} \quad (\exists E)^i\ \frac{\exists xA \quad \begin{matrix}[A^x_y]^i, [\mathscr{E}y]^i\\ \mathfrak{D}\\ C\end{matrix}}{C} \quad (\neg\exists I)^i\ \frac{\begin{matrix}[\mathscr{E}y]^i\\ \mathfrak{D}\\ \neg A^x_y\end{matrix}}{\neg\exists xA} \quad (\neg\exists E)\ \frac{\neg\exists xA \quad \mathscr{E}t}{\neg A^x_t}$$

$$(\exists I')\ \frac{A^x_t}{\exists x A} \quad (\exists E')^i\ \frac{\exists x A \quad \begin{matrix}[A^x_y]^i \\ \mathfrak{D} \\ C\end{matrix}}{C} \quad (\neg\exists I')\ \frac{\neg A^x_y}{\neg\exists x A} \quad (\neg\exists E')\ \frac{\neg\exists x A}{\neg A^x_t}$$

where in $(\exists I)$, t is free for x in A; and in $(\exists E)$, y is not free in C nor any undischarged assumptions of $\mathfrak{D}$, except A^x_y and $\mathscr{E}y$, and either y is the same as x or it is not free in A; $(\neg\exists I)$, y does not occur free in any undischarged assumptions of $\mathfrak{D}$ except $\mathscr{E}y$, and either y is the same as x or y is not free in A; in $(\neg\exists E)$, t is free for x in A.

The rules for identity are given below (both in the ordinary and free versions), where A is an atomic formula or its negation (the rule $(=E)$ is suitable for both ordinary and free versions; while $(=I')$ is used in an ordinary version and $(=I)$ in a free one):

$$(=I)\ \frac{\mathscr{E}t}{t=t} \quad (=I')\ \frac{}{t=t} \quad (=E)\ \frac{t_1=t_2 \quad A^x_{t_1}}{A^x_{t_2}}$$

The special rules for free logic regarding predicates (P stands for an arbitrary predicate, including $=$):

$$(\text{PD})\ \frac{P(t_1,\ldots,t_n)}{\mathscr{E}t_i} \quad (\neg\text{PD})\ \frac{\neg P(t_1,\ldots,t_n)}{\mathscr{E}t_i}$$

The rules for a binary quantifier representation of definite descriptions (both ordinary and free versions):

$$(\mathtt{I}I)^i\ \frac{F^x_t \quad G^x_t \quad \mathscr{E}t \quad \begin{matrix}[F^x_y]^i[\mathscr{E}y]^i \\ \mathfrak{D} \\ y=t\end{matrix}}{\mathtt{I}x[F,G]} \quad (\mathtt{I}I')^i\ \frac{F^x_t \quad G^x_t \quad \begin{matrix}[F^x_y]^i \\ \mathfrak{D} \\ y=t\end{matrix}}{\mathtt{I}x[F,G]}$$

where t is free for x in F and in G, and y is different from x, not free in t and does not occur free in any undischarged assumptions in $\mathfrak{D}$ except F^x_y and $\mathscr{E}y$.

$$(\neg\mathtt{I}E)^{i,j,k}\ \frac{\neg\mathtt{I}x[F,G] \quad \begin{matrix}[\neg F^x_t]^i \\ \mathfrak{D}_1 \\ C\end{matrix} \quad \begin{matrix}[\neg G^x_t]^j \\ \mathfrak{D}_2 \\ C\end{matrix} \quad \begin{matrix}[F^x_y]^k[\mathscr{E}y]^k[\neg y=t]^k \\ \mathfrak{D}_3 \\ C\end{matrix}}{C}$$

$$(\neg\mathtt{I}E')^{i,j,k}\ \frac{\neg\mathtt{I}x[F,G] \quad \begin{matrix}[\neg F^x_t]^i \\ \mathfrak{D}_1 \\ C\end{matrix} \quad \begin{matrix}[\neg G^x_t]^j \\ \mathfrak{D}_2 \\ C\end{matrix} \quad \begin{matrix}[F^x_y]^k[\neg y=t]^k \\ \mathfrak{D}_3 \\ C\end{matrix}}{C}$$

where t is free for x in F and in G, and y is different from x, not free in t and does not occur free in any undischarged assumptions in $\mathfrak{D}_4$ except F^x_y and $\mathscr{E}y$. Free version:

$$(\mathtt{I}E_1)^i\ \frac{\mathtt{I}x[F,G] \quad \begin{matrix}[F^x_y]^i[G^x_y]^i[\mathscr{E}y]^i \\ \mathfrak{D} \\ C\end{matrix}}{C} \quad (\neg\mathtt{I}I_1)\ \frac{\neg F^x_y}{\neg\mathtt{I}x[F,G]} \quad (\neg\mathtt{I}I_2)\ \frac{\neg G^x_y}{\neg\mathtt{I}x[F,G]}$$

Ordinary version:

$$(\mathtt{I}E_1')^i\ \frac{\mathtt{I}x[F,G] \quad \begin{matrix}[F^x_y]^i[G^x_y]^i \\ \mathfrak{D} \\ C\end{matrix}}{C} \quad (\neg\mathtt{I}I_1)\ \frac{\neg F^x_y}{\neg\mathtt{I}x[F,G]} \quad (\neg\mathtt{I}I_2)\ \frac{\neg G^x_y}{\neg\mathtt{I}x[F,G]}$$

where is y not free in C nor any undischarged assumptions it depends on except F^x_y, G^x_y, and $\mathscr{E}y$, and either y is the same as x or it is not free in F nor in G.

$$(\mathtt{I}E_2)\ \dfrac{\mathtt{I}x[F,G] \quad \mathscr{E}t_1 \quad \mathscr{E}t_2 \quad F^x_{t_1} \quad F^x_{t_2}}{t_1 = t_2} \qquad (\neg\mathtt{I}I_3)\ \dfrac{\neg t_1 = t_2 \quad \mathscr{E}t_1 \quad \mathscr{E}t_2 \quad F^x_{t_1} \quad F^x_{t_2}}{\neg\mathtt{I}x[F,G]}$$

$$(\mathtt{I}E_2')\ \dfrac{\mathtt{I}x[F,G] \quad F^x_{t_1} \quad F^x_{t_2}}{t_1 = t_2} \qquad (\neg\mathtt{I}I_3')\ \dfrac{\neg t_1 = t_2 \quad F^x_{t_1} \quad F^x_{t_2}}{\neg\mathtt{I}x[F,G]}$$

where t_1 and t_2 are free for x in F.

Natural deduction systems for **Int**, $\mathbf{Int^{NF}}$, $\mathbf{Int}_{\mathtt{I}}$, and $\mathbf{Int}^{\mathbf{NF}}_{\mathtt{I}}$ can be obtained from natural deduction systems for **N4**, $\mathbf{N4^{NF}}$, $\mathbf{N4}_{\mathtt{I}}$, and $\mathbf{N4}^{\mathbf{NF}}_{\mathtt{I}}$ by implementing the following changes: in the rule $(= E)$, A stands just for atomics formulas (not their negations), all negated rules for connectives, quantifiers, including $\mathtt{I}$, and predicates have to replaced with the following rule

$$(\bot E)\ \dfrac{\bot}{B}$$

As follows from [4, p. 85], $\mathtt{I}x[F,G]$ and $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$ are interderivable in intuitionstic logic. Since in this proof only non-negated rules are used, it is a proof in Nelson logic as well. Thus, $\mathtt{I}x[F,G]$ and $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$ are interderivable in Nelson's logic as well. As follows from [4, p. 90–91], $\mathtt{I}x[F,G]$ and $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$ are interderivable in intuitionstic negative free logic as well. Again, the same proof can be used in the case of Nelson's logic, since only non-negated rules are involved, so we can conclude that $\mathtt{I}x[F,G]$ and $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$ are interderivable in Nelson's free logic.

However, in the case of Nelson's logic a natural question arises: what about negation of $\mathtt{I}x[F,G]$? We can show that $\neg\mathtt{I}x[F,G]$ and $\forall x(\neg F \vee \exists y(F^x_y \wedge \neg y = x) \vee \neg G)$ are interderivable in Nelson's logic. Let us denote $\neg F \vee \exists y(F^x_y \wedge \neg y = x) \vee \neg G$ via $\mathfrak{F}$.

1. $\neg\mathtt{I}x[F,G] \vdash_{\mathbf{N4}} \forall x(\neg F \vee \exists y(F^x_y \wedge \neg y = x) \vee \neg G)$ (where double line means a double application of a disjunction introduction rule):

$$\dfrac{\neg\mathtt{I}x[F,G] \qquad \dfrac{[\neg F]^1}{\mathfrak{F}} \qquad \dfrac{\dfrac{\dfrac{[F^x_y]^2 \qquad [\neg y = x]^3}{F^x_y \wedge \neg y = x}\,(\wedge I)}{\exists y(F^x_y \wedge \neg y = x)}\,(\exists I)}{\mathfrak{F}} \qquad \dfrac{[\neg G]^4}{\mathfrak{F}}}{\dfrac{\mathfrak{F}}{\forall x\mathfrak{F}}\,(\forall I')}\,(\neg\mathtt{I}E)^{1,2,3,4}$$

2. $\forall x(\neg F \vee \exists y(F^x_y \wedge \neg y = x) \vee \neg G) \vdash_{\mathbf{N4}} \neg\mathtt{I}x[F,G]$. Let us denote $F^x_y \wedge \neg y = x$ via $\mathfrak{G}^x_y$.

$$(\vee E)^{1,2,7}\ \dfrac{\dfrac{\forall x\mathfrak{F}}{\mathfrak{F}}\,(\forall E') \qquad \dfrac{[\neg F]^1}{\neg\mathtt{I}x[F,G]} \qquad \dfrac{[\exists y(\mathfrak{G})]^2 \qquad \dfrac{[\exists y(\mathfrak{G})]^2 \qquad \dfrac{\dfrac{[\mathfrak{G}^x_y]^3}{\neg y = x} \quad \dfrac{[\mathfrak{G}^x_x]^4}{F^x_x} \quad \dfrac{[\mathfrak{G}^x_y]^3}{F^x_y}\,(\wedge E)}{\neg\mathtt{I}x[F,G]}\,(\neg\mathtt{I}I_3)}{\neg\mathtt{I}x[F,G]}\,(\exists E')^4}{\neg\mathtt{I}x[F,G]}\,(\exists E')^3 \qquad \dfrac{[\neg G]^7}{\neg\mathtt{I}x[F,G]}}{\neg\mathtt{I}x[F,G]}$$

In the case of Nelson's free logic we have the following deductions.

1. $\neg\mathtt{I}x[F,G] \vdash_{\mathbf{N4}} \forall x(\neg F \vee \exists y(F^x_y \wedge \neg y = x) \vee \neg G)$.

$$\dfrac{\dfrac{\neg \mathtt{I}x[F,G] \qquad \dfrac{[\neg F]^1}{\mathfrak{F}} \qquad \dfrac{\dfrac{\dfrac{[F^x_y]^2 \qquad [\neg y = x]^3}{F^x_y \wedge \neg y = x}(\wedge I) \qquad [\mathscr{E}y]^5}{\exists y(F^x_y \wedge \neg y = x)}(\exists I)}{\mathfrak{F}} \qquad \dfrac{[\neg G]^4}{\mathfrak{F}}}{\mathfrak{F}}(\neg \mathtt{I}E)^{1,2,3,4}}{\forall x \mathfrak{F}}(\forall I)^5$$

2. $\mathscr{E}y, \forall x(\neg F \vee \exists y(F^x_y \wedge \neg y = x) \vee \neg G) \vdash_{\mathbf{N4}} \neg \mathtt{I}x[F,G]$. Let us denote $F^x_y \wedge \neg y = x$ via $\mathfrak{G}^x_y$.

$$\dfrac{\dfrac{\forall x \mathfrak{F} \qquad \mathscr{E}y}{\mathfrak{F}} \qquad \dfrac{[\neg F]^1}{\neg \mathtt{I}x[F,G]} \qquad \dfrac{[\exists y(\mathfrak{G})]^2 \qquad \dfrac{[\exists y(\mathfrak{G})]^2 \qquad \dfrac{\dfrac{[\mathfrak{G}^x_y]^3}{\neg y = x} \qquad [\mathscr{E}x]^4 \qquad [\mathscr{E}y]^5 \qquad \dfrac{[\mathfrak{G}^x_x]^6}{F^x_x} \qquad \dfrac{[\mathfrak{G}^x_y]^3}{F^x_y}}{\neg \mathtt{I}x[F,G]}(\neg \mathtt{I}I_3)}{\neg \mathtt{I}x[F,G]}{\scriptstyle 4,6}}{\neg \mathtt{I}x[F,G]}{\scriptstyle 3,5} \qquad \dfrac{[\neg G]^7}{\neg \mathtt{I}x[F,G]}}{\neg \mathtt{I}x[F,G]}{\scriptstyle 1,2,7}$$

3 Semantics

Let us describe semantics for intuitionistic negative free logic with identity as well as intuitionistic first-order logic with identity. We follow Priest's [9] presentation of semantics for intuitionistic first-order logic with identity.

DEFINITION 3.1 (Intuitionisitic negative free structure). An intuitionistic negative free structure $\mathfrak{I}$ is a seventuple $\langle W, R, H, D, E, J, \varphi \rangle$, where W is the non-empty set of possible worlds, R is a binary reflexive and transitive relation on W, H is a non-empty set of objects, D is the non-empty domain of quantification, which members are functions from W to H such that for any $d \in D$ and $w \in W$ we have $d(w) \in H$ (in what follows, we write $|d|_w$ for $d(w)$), E is the (possibly, empty) set of all existent objects such that $E \subseteq D$, $J = \{|d|_w \in H \mid d \in E\}$, φ is a function such that it maps $w \in W$ to a subset of D, $\varphi(w) \subseteq D$, which we denote as D_w, and satisfies the following conditions, for any $w \in W$:

- $\varphi_w(\mathscr{E}) = J$,
- if c is a constant, then $\varphi(c) \in D_w$,
- if P is an n-place predicate, then $\varphi_w(P) \subseteq J^n$,
- $\varphi_w(=) = \{\langle t, t \rangle \mid t \in J\}$,
- if wRw', then $\varphi_w(P) \subseteq \varphi_{w'}(P)$, for any n-place predicate predicate P, including $=$,
- if wRw', then $D_w \subseteq D_{w'}$.
- if $\langle d_1, \ldots, d_n \rangle \in \varphi_w(P)$, then $d_1 \in \varphi_w(\mathscr{E}), \ldots, d_n \in \varphi_w(\mathscr{E})$.

DEFINITION 3.2 (Intuitionistic structure). An intuitionistic structure is an intuitionistic negative free structure $\mathfrak{I} = \langle W, R, H, D, E, J, \varphi \rangle$ such that $D = E$, and hence $H = J$; and $\varphi_w(\mathscr{E}) = D$.

Following Priest [9], for all $d \in D$, we add a constant to the language, k_d, such that $\varphi(k_d) = d$.

DEFINITION 3.3 (Intuitionistic (negative free) semantics). An intuitionistic (negative free) valuation $\Vdash^I$ on a model $\mathfrak{I} = \langle W, R, H, D, E, J, \varphi \rangle$ is defined as follows, for any $w \in W$:

- $\mathfrak{I}, w \Vdash^I P(t_1, \ldots, t_n)$ iff $\langle |\varphi(t_1)|_w, \ldots, |\varphi(t_n)|_w \rangle \in \varphi_w(P^n)$,
- $\mathfrak{I}, w \nVdash^I \bot$,
- $\mathfrak{I}, w \Vdash^I A \to B$ iff $\forall w' \in W (R(w, w')$ implies $(\mathfrak{I}, w' \Vdash^I A$ implies $\mathfrak{I}, w' \Vdash^I B))$,

- $\mathfrak{I}, w \Vdash^I A \wedge B$ iff $\mathfrak{I}, w \Vdash^I A$ and $\mathfrak{I}, w \Vdash^I B$,
- $\mathfrak{I}, w \Vdash^I A \vee B$ iff $\mathfrak{I}, w \Vdash^I A$ or $\mathfrak{I}, w \Vdash^I B$,
- $\mathfrak{I}, w \Vdash^I \forall x A$ iff $\forall w'(R(w,w')$ implies $\forall d \in E_{w'}, \mathfrak{I}, w' \Vdash^I A^x_{k_d})$
- $\mathfrak{I}, w \Vdash^I \exists x A$ iff $\exists d \in E_w, \mathfrak{I}, w \Vdash^I A^x_{k_d}$.

Using the fact that $\mathtt{I}x[F,G]$ and $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$ are interderivable, we can propose the following semantic condition for $\mathtt{I}x[F,G]$:

- $\mathfrak{I}, w \Vdash^I \mathtt{I}x[F,G]$ iff $\exists d \in E_w, \mathfrak{I}, w \Vdash^I F$ and $\forall w'(R(w,w')$ implies $\forall e \in E_{w'}, \forall w'' \in W(R(w',w'')$ implies $(\mathfrak{I}, w'' \Vdash^I F^{k_d}_{k_e}$ implies $\mathfrak{I}, w'' \Vdash^I k_d = k_e)))$ and $\mathfrak{I}, w \Vdash^I G$.

The semantics for **Int** and $\mathbf{Int}_{\mathtt{I}}$ is based on intuitionistic structures, and for $\mathbf{Int}^{\mathbf{NF}}$ and $\mathbf{Int}^{\mathbf{NF}}_{\mathtt{I}}$ on intuitionistic negative free structures.

DEFINITION 3.4. An inference is valid iff it is truth-preserving in all worlds of all interpretations.

Let us present semantics for Nelson's logics on the basis of Thomason's semantics [13] (see also [9]). However, in contrast to [13, 9], the semantics we use is two-valued with a paradefinite valuation (thus, a formula and its negation can simultaneously be true and false, or simultaneously neither true, nor false).

DEFINITION 3.5 (Nelsonian negative free structure). A Nelsonian negative free structure $\mathfrak{N}$ is an intuitionistic negative free structure $\langle W,R,H,D,E,J,\varphi\rangle$ such that φ is redefined as follows:

- $\varphi_w(\mathscr{E}) = J$, $\varphi_w(\neg\mathscr{E}) = H \setminus J$,
- if c is a constant, then $\varphi(c) \in D$,
- if P is an n-place predicate, then $\varphi_w(P) \subseteq J^n$ and $\varphi_w(\neg P) \subseteq J^n$,
- $\varphi_w(=) = \{\langle t,t\rangle \mid t \in J\}$, $\varphi_w(\neg =) \subseteq J^2$,
- if wRw', then $\varphi_w(P) \subseteq \varphi_{w'}(P)$ and $\varphi_w(\neg P) \subseteq \varphi_{w'}(\neg P)$,
- if wRw', then $D_w \subseteq D_{w'}$,
- if $\langle d_1,\dots,d_n\rangle \in \varphi_w(P)$, then $d_1 \in \varphi_w(\mathscr{E}),\dots,d_n \in \varphi_w(\mathscr{E})$,
- if $\langle d_1,\dots,d_n\rangle \in \varphi_w(\neg P)$, then $d_1 \in \varphi_w(\mathscr{E}),\dots,d_n \in \varphi_w(\mathscr{E})$.

DEFINITION 3.6 (Nelsonian structure). A Nelsonian structure is a Nelsonian negative free structure $\mathfrak{I} = \langle W,R,H,D,E,J,\varphi\rangle$ such that $D = E$, and hence $H = J$; and $\varphi_w(\mathscr{E}) = \varphi_w(\neg\mathscr{E}) = D_w$.

DEFINITION 3.7 (Nelsonian semantics). A Nelsonian paradefinite valuation $\Vdash^N$ on a model $\mathfrak{N} = \langle W,R,H,D,E,J,\varphi\rangle$ is defined as follows, for any $w \in W$:[2]

- $\mathfrak{N}, w \Vdash^N P(t_1,\dots,t_n)$ iff $\langle|\varphi(t_1)|_w,\dots,|\varphi(t_n)|_w\rangle \in \varphi_w(P^n)$,
- $\mathfrak{N}, w \Vdash^N \neg P(t_1,\dots,t_n)$ iff $\langle|\varphi(t_1)|_w,\dots,|\varphi(t_n)|_w\rangle \in \varphi_w(\neg P^n)$,
- $\mathfrak{N}, w \Vdash^N \neg\neg A$ iff $\mathfrak{N}, w \Vdash^N A$,
- $\mathfrak{N}, w \Vdash^N A \to B$ iff $\forall w' \in W(R(w,w')$ implies $(\mathfrak{N}, w' \Vdash^N A$ implies $\mathfrak{N}, w' \Vdash^N B))$,
- $\mathfrak{N}, w \Vdash^N \neg(A \to B)$ iff $\mathfrak{N}, w \Vdash^N A$ and $\mathfrak{N}, w \Vdash^N \neg B$,
- $\mathfrak{N}, w \Vdash^N A \wedge B$ iff $\mathfrak{N}, w \Vdash^N A$ and $\mathfrak{N}, w \Vdash^N B$,
- $\mathfrak{N}, w \Vdash^N \neg(A \wedge B)$ iff $\mathfrak{N}, w \Vdash^N \neg A$ or $\mathfrak{N}, w \Vdash^N \neg B$,
- $\mathfrak{N}, w \Vdash^N A \vee B$ iff $\mathfrak{N}, w \Vdash^N A$ or $\mathfrak{N}, w \Vdash^N B$,

[2]The truth conditions for non-negated formulas, including $\mathtt{I}x[F,G]$, are the same as in the intuitionistic case.

- $\mathfrak{N}, w \Vdash^N \neg(A \vee B)$ iff $\mathfrak{N}, w \Vdash^N \neg A$ and $\mathfrak{N}, w \Vdash^N \neg B$,
- $\mathfrak{N}, w \Vdash^N \forall xA$ iff $\forall w'(R(w,w')$ implies $\forall d \in D_{w'}, \mathfrak{N}, w' \Vdash^N A^x_{k_d})$,
- $\mathfrak{N}, w \Vdash^N \neg\forall xA$ iff $\exists d \in D_{w'}, \mathfrak{N}, w' \Vdash^N \neg A^x_{k_d}$,
- $\mathfrak{N}, w \Vdash^N \exists xA$ iff $\exists d \in D_{w'}, \mathfrak{N}, w' \Vdash^N A^x_{k_d}$,
- $\mathfrak{N}, w \Vdash^N \neg\exists xA$ iff $\forall w'(R(w,w')$ implies $\forall d \in D_{w'}, \mathfrak{N}, w' \Vdash^N \neg A^x_{k_d})$;

Using the fact that $\mathtt{I}x[F,G]$ and $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$ are interderivable as well as $\neg\mathtt{I}x[F,G]$ and $\mathscr{E}y, \forall x(\neg F \vee \exists y(F^x_y \wedge \neg y = x) \vee \neg G)$ are interderivable, we can propose the following semantic condition for $\mathtt{I}x[F,G]$ and $\neg\mathtt{I}x[F,G]$:

- $\mathfrak{N}, w \Vdash^N \mathtt{I}x[F,G]$ iff $\exists d \in E_w, \mathfrak{N}, w \Vdash^I F$ and $\forall w'(R(w,w')$ implies $\forall e \in E_{w'}, \forall w'' \in W(R(w',w'')$ implies $(\mathfrak{N}, w'' \Vdash^N F^{k_d}_{k_e}$ implies $\mathfrak{N}, w'' \Vdash^N k_d = k_e)))$ and $\mathfrak{N}, w \Vdash^N G$,
- $\mathfrak{N}, w \Vdash^N \neg\mathtt{I}x[F,G]$ iff $\langle|\varphi(y)|_w\rangle \in \varphi_w(\mathscr{E})$ and $\forall w'(R(w,w')$ implies $\forall d \in D_{w'}, \mathfrak{N}, w' \Vdash^N \neg F$ or $\exists e \in D_{w'}, (\mathfrak{N}, w' \Vdash^N F^{k_d}_{k_e}$ and $\mathfrak{N}, w' \Vdash^N \neg k_e = k_d)$ or $\mathfrak{N}, w' \Vdash^N \neg G)$.

The semantics for **N4** and $\mathbf{N4}_{\mathtt{I}}$ is based on intuitionistic structures, and for $\mathbf{N4^{NF}}$ and $\mathbf{N4^{NF}_{\mathtt{I}}}$ on intuitionistic negative free structures.

DEFINITION 3.8. An inference is valid iff it is truth-preserving in all worlds of all interpretations.

4 Embedding theorems

We use an embedding function similar to the one used by Gurevich [2], Rautenberg [10], Vorob'ev [14] for **N3** and **Int** as well as Kamide and Shramko [3] for some multilattice logics. One of the specifics this function is the necessity to extend the language of intuitionistic logic with the additional copies of predicate letters. So extend the language $L^{\perp}$ with the set $\{P' \mid P \text{ is a predicate letter}\}$.

DEFINITION 4.1. An embedding function τ from the language $L^{\neg}$ into the language $L^{\perp}$ is inductively defined as follows:

(1) $\tau(P(t_1,\dots,t_n)) = P(t_1,\dots,t_n)$, for any predicate P,
(2) $\tau(\neg P(t_1,\dots,t_n)) = P'(t_1,\dots,t_n)$, for any predicate P,
(3) $\tau(A * B) = \tau(A) * \tau(B)$, where $* \in \{\to, \wedge, \vee\}$
(4) $\tau(\neg\neg A) = \tau(A)$,
(5) $\tau(\neg(A \to B)) = \tau(A) \wedge \tau(\neg B)$,
(6) $\tau(\neg(A \wedge B)) = \tau(\neg A) \vee \tau(\neg B)$,
(7) $\tau(\neg(A \vee B)) = \tau(\neg A) \wedge \tau(\neg B)$,
(8) $\tau(\forall xA) = \forall x\tau(A)$,
(9) $\tau(\exists xA) = \exists x\tau(A)$,
(10) $\tau(\neg\forall xA) = \exists x\tau(\neg A)$,
(11) $\tau(\neg\exists xA) = \forall x\tau(\neg A)$.

Let us prove the following theorem for **N4** and **Int** as well as their negation free versions. A similar theorem has been proven in [2, 10, 14] for **N3** and **Int**.

THEOREM 4.1 (Syntactical embedding). Let τ be a mapping introduced in Definition 4.1. For any formula A, $\vdash_{\mathbf{N4}} A$ iff $\vdash_{\mathbf{Int}} \tau(A)$; $\vdash_{\mathbf{N4^{NF}}} A$ iff $\vdash_{\mathbf{Int^{NF}}} \tau(A)$.

Proof. As an example, we present a proof for the case of negative free logics.

Suppose that $\vdash_{\mathbf{N4^{NF}}} A$. By an induction on the length of the deduction of A. We distinguish cases depending on the last rule applied in the deduction.

Suppose that A is of the form $\mathscr{E}t_i$ and has been obtained by the rule ($\neg$PD) from the formula $\neg P(t_1,\dots,t_n)$. By the induction hypothesis, the translation $\tau(\mathscr{E}t_i)$ is provable in **Int**. Then we can construct a deduction of the translation of $\tau(\mathscr{E}t_i) = \mathscr{E}t_i$ in **Int** using the rule (PD):

$$\frac{\neg P(t_1,\dots,t_n)}{\mathscr{E}t_i}\,(\neg\mathrm{PD}) \;\rightsquigarrow\; \frac{P'(t_1,\dots,t_n)}{\mathscr{E}t_i}\,(\mathrm{PD})$$

Suppose that A is of the form $\neg(B \to C)$ and has been obtained by the rule $(\neg{\to}I)$ from the formulas B and $\neg C$. By the induction hypothesis, the translations $\tau(B)$ and $\tau(\neg C)$ are provable in **Int**. Then we can construct a deduction of the translation of $\tau(\neg(B \to C)) = \tau(B) \wedge \tau(\neg C)$ in **Int** using the rule $(\wedge I)$:

$$\frac{B \qquad \neg C}{\neg(B \to C)}\,(\neg{\to}I) \;\rightsquigarrow\; \frac{\tau(B) \qquad \tau(\neg C)}{\tau(B) \wedge \tau(\neg C)}\,(\wedge I)$$

Suppose that A is of the form $\neg\forall x B$ and has been obtained by the rule $(\neg\forall I)$ from the formulas $\neg B^x_t$ and $\mathscr{E}t$. By the induction hypothesis, the translations $\tau(\neg B^x_t)$ and $\tau(\mathscr{E}t)$ are provable in **Int**. Then we can construct a deduction of the translation of $\tau(\neg\forall x B) = \exists x \tau(\neg B)$ in **Int** using the rule $(\exists I)$:

$$\frac{\neg B^x_t \qquad \mathscr{E}t}{\neg\forall x B}\,(\neg\forall I) \;\rightsquigarrow\; \frac{\tau(\neg B^x_t) \qquad \tau(\mathscr{E}t)}{\exists x \tau(\neg B)}\,(\exists I)$$

The other cases are considered similarly.

Suppose that $\vdash_{\mathbf{Int^{NF}}} \tau(A)$. Similarly to previous cases. $\square$

LEMMA 4.1. Let $\mathfrak{N} = \langle W,R,H,D,E,J,\varphi\rangle$ be a Nelsonian (negative free) structure. Let τ be the mapping defined in Definition 4.1. For any Nelsonian paradefinite valuation $\Vdash^N$ on $\mathfrak{N}$, we can construct an intuitionistic valuation $\Vdash^I$ on an intuitionistic (negative free) structure $\mathfrak{I} = \langle W,R,H,D,E,J,\varphi\rangle$ such that for any formula C, $\mathfrak{N} \Vdash^N C$ iff $\mathfrak{I} \Vdash^I \tau(C)$.

Proof. As an example, we give a proof for the case of non-free logics. Let $\mathscr{P}$ be a set of atomic formulas and let $\mathscr{P}'$ be the set $\{P'(t_1,\dots,t_n) \mid P(t_1,\dots,t_n) \in \mathscr{P}\}$ of atomic formulas. Suppose that $\Vdash^N$ is a Nelsonian paradefinite valuation on $\mathfrak{N}$. Suppose that $\Vdash^I$ is an intuitionistic valuation on $\mathfrak{I}$ such that, for any $w \in W$ and for any atomic formula $P(t_1,\dots,t_n) \in \mathscr{P} \cup \mathscr{P}'$,

(a) $\mathfrak{N},w \Vdash^N P(t_1,\dots,t_n)$ iff $\mathfrak{I},w \Vdash^I P(t_1,\dots,t_n)$,

(b) $\mathfrak{N},w \Vdash^N \neg P(t_1,\dots,t_n)$ iff $\mathfrak{I},w \Vdash^I P'(t_1,\dots,t_n)$.

The lemma is proved by induction on C.

(1) C is an atomic formula $P(t_1,\dots,t_n)$: $\mathfrak{N},w \Vdash^N P(t_1,\dots,t_n)$ iff $\mathfrak{I},w \Vdash^I P(t_1,\dots,t_n)$ (by the assumption) iff $\mathfrak{I},w \Vdash^I \tau(P(t_1,\dots,t_n))$ (by Definition 4.1).

(2) C is a negated atomic formula $\neg P(t_1,\dots,t_n)$: $\mathfrak{N},w \Vdash^N \neg P(t_1,\dots,t_n)$ iff $\mathfrak{I},w \Vdash^I P'(t_1,\dots,t_n)$ (by the assumption) iff $\mathfrak{I},w \Vdash^I \tau(\neg P(t_1,\dots,t_n))$ (by Definition 4.1).

(3) C is $A \to B$: $\mathfrak{N},w \Vdash^N A \to B$ iff $\forall w' \in W(R(w,w')$ implies $(\mathfrak{N},w' \Vdash^N A$ implies $\mathfrak{N},w' \Vdash^N B))$ (by Definition 3.3) iff $\forall w' \in W(R(w,w')$ implies $(\mathfrak{I},w' \Vdash^I \tau(A)$ implies $\mathfrak{I},w' \Vdash^I \tau(B)))$ (by the induction hypothesis) iff $\mathfrak{I},w \Vdash^I \tau(A \to B)$ (by Definition 3.3).

(4) C is $\neg(A \to B)$: $\mathfrak{N},w \Vdash^N \neg(A \to B)$ iff $\mathfrak{N},w \Vdash^N A$ and $\mathfrak{N},w \Vdash^N \neg B$ (be Definition 3.7) iff $\mathfrak{I},w \Vdash^I \tau(A)$ and $\mathfrak{I},w \Vdash^I \tau(\neg B)$ (by the induction hypothesis) iff $\mathfrak{I},w \Vdash^I \tau(A) \wedge \tau(\neg B)$ (by Definition 3.3) iff $\mathfrak{I},w \Vdash^I \tau(\neg(A \to B))$ (by Definition 4.1).

(5) C is $\forall xA$: $\mathfrak{N}, w \Vdash^N \forall xA$ iff $\forall w'(R(w,w')$ implies $\forall d \in D_{w'}, \mathfrak{N}, w' \Vdash^N A^x_{k_d})$ (by Definition 3.7) iff $\forall w'(R(w,w')$ implies $\forall d \in D_{w'}, \mathfrak{I}, w' \Vdash^I \tau(A^x_{k_d}))$ (by the induction hypothesis) iff $\mathfrak{I}, w \Vdash^I \forall xA$ (by Definition 3.3) iff $\mathfrak{I}, w \Vdash^I \tau(\forall xA)$ (by Definition 4.1).

(6) C is $\neg\forall xA$: $\mathfrak{N}, w \Vdash^N \neg\forall xA$ iff $\exists d \in D_{w'}, \mathfrak{N}, w' \Vdash^N \neg A^x_{k_d}$ (by Definition 3.7) iff $\exists d \in D_{w'}, \mathfrak{I}, w' \Vdash^I \tau(\neg A^x_{k_d})$ (by the induction hypothesis) iff $\mathfrak{I}, w \Vdash^I \exists x\tau(\neg A)$ (by Definition 3.3) iff $\mathfrak{I}, w \Vdash^I \tau(\neg\forall xA)$ (by Definition 4.1).

The other cases are considered similarly. □

LEMMA 4.2. Let $\mathfrak{I} = \langle W,R,H,D,E,J,\varphi\rangle$ be an intuitionistic (negative free) structure. Let τ be the mapping defined in Definition 4.1. For any intuitionistic valuation $\Vdash^I$ on $\mathfrak{I}$, we can construct a Nelsonian paraconsistent valuation $\Vdash^N$ on an Nelsonian (negative free) structure $\mathfrak{N} = \langle W,R,H,D,E,J,\varphi\rangle$ such that for any formula C, $\mathfrak{N} \Vdash^N C$ iff $\mathfrak{I} \Vdash^I \tau(C)$.

Proof. Similarly to Lemma 4.1. □

THEOREM 4.2 (Semantic embedding). Let τ be a mapping introduced in Definition 4.1. For any formula C, $\models_{\mathbf{N4}} C$ iff $\models_{\mathbf{Int}} \tau(C)$; $\models_{\mathbf{N4^{NF}}} C$ iff $\models_{\mathbf{Int^{NF}}} \tau(C)$.

Proof. Follows from Lemmas 4.1 and 4.2. □

THEOREM 4.3 (Completeness). For any formula C, $\models_{\mathbf{N4}} C$ iff $\vdash_{\mathbf{N4}} C$; $\models_{\mathbf{N4^{NF}}} C$ iff $\vdash_{\mathbf{N4^{NF}}} C$.

Proof. Follows from Theorems 4.1 and 4.2 as well as completeness of intuitionistic first-order logics with identity and its negative free version. □

LEMMA 4.3. All the rules for $\mathtt{I}$ and $\neg\mathtt{I}$ are sound.

Proof. Left for the reader. □

THEOREM 4.4 (Completeness). For any formula C, it holds that $\models_{\mathbf{N4_I}} C$ iff $\vdash_{\mathbf{N4_I}} C$; $\models_{\mathbf{N4^{NF}_I}} C$ iff $\vdash_{\mathbf{N4^{NF}_I}} C$.

Proof. Follows from Theorem 4.3 and the definition of $\mathtt{I}$ (that is equivalences proved in Section 2) as well as Lemma 4.3. □

5 Conclusion

In this paper, we examined the behaviour of the binary quantifier $\mathtt{I}$ in Nelson's first-order logic with identity and its negative free version, drawing inspiration from Kürbis's method of formalising definite descriptions using $\mathtt{I}$ added to intuitionistic first-order logic with identity and its negative free version. The research described in this article can be continued as follows. As a first task for the future, we leave the problem of an adaptation of the embedding function τ for the case $\mathtt{I}$. As a second task, we can propose to find a proof of the normalisation theorem for the natural deduction systems formulated in this article. As a third task, to conduct a similar study, on the basis of [6], where $\mathtt{I}$ is characterised by different natural deduction rules and is studied on the basis of intuitionistic positive free logic. As a fourth task, carry out comparable research based on **N3** instead of **N4**, or a non-constructive tabular extension of **N4/N3** by Peirce's law (in the latter case, one can think about embedding such logics into classical first-order (free) logic).

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References

[1] Almukdad, A., Nelson, D.: Constructible falsity and inexact predicates. The Journal of Symbolic Logic **49**(01), 231–233 (1984), doi:10.2307/2274105.

[2] Gurevich, Y.: Intuitionistic logic with strong negation. Studia Logica **26**(1-2), 49–59 (1977), doi:10.1007/BF02121114.

[3] Kamide, N., Shramko, Y.: Embedding from multilattice logic into classical logic and vice versa. Journal of Logic and Computation **27**(5), 1549–1575 (2017), doi:10.1093/logcom/exw015.

[4] Kürbis, N.: A binary quantifier for definite descriptions in intuitionist negative free logic: natural deduction and normalisation. Bulletin of the Section of Logic **48**(2), 81–97 (2019), doi:10.18778/0138-0680.48.2.01.

[5] Kürbis, N.: Two Treatments of Definite Descriptions in Intuitionist Negative Free Logic. Bulletin of the Section of Logic **48**(4), 299–317 (2019), doi:10.18778/0138-0680.48.4.04.

[6] Kürbis, N.: Definite Descriptions in Intuitionist Positive Free Logic. Logic and Logical Philosophy **30**, 327–358 (2021), doi:10.12775/LLP.2020.024.

[7] Nelson, D.: Constructible falsity. Journal of Symbolic Logic **14**(1), 16–26 (1949), doi:10.2307/2268973.

[8] Prawitz, D.: Natural Deduction. Stockholm, Göteborg, Uppsala: Almqvist and Wiksell (1965).

[9] Priest, G.: An Introduction to Non-Classical Logic. From If to Is. 2nd ed. Cambridge University Press, Cambridge (2008), doi:10.1017/CBO9780511801174.

[10] Rautenberg, W.: Klassische und nicht-klassische Aussagenlogik. Vieweg, Braunschweig, (1979).

[11] Tennant, N.: Natural Logic. Edinburgh: Edinburgh University Press (1978)

[12] Tennant, N.: A general theory of abstraction operators. The Philosophical Quarterly **54**(214), 105–133 (2004), doi:10.1111/j.0031-8094.2004.00344.x.

[13] Thomason, R. H.: A semantical study of constructible falsity. Zeitschrift Für Mathematische Logik Und Grundlagen Der Mathematik **15**(16-18), 247–257 (1969), doi:10.1002/malq.19690151602.

[14] Vorob'ev, N. N.: A constructive proposition (in Russian). Doklady Akademii Nauk SSSR **85**, 465–468 (1952)

Twist Sequent Calculi for S4 and its Neighbors

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Two Gentzen-style twist sequent calculi for the normal modal logic S4 are introduced and investigated. The proposed calculi, which do not employ the standard logical inference rules for the negation connective, are characterized by several twist logical inference rules for negated logical connectives. Using these calculi, short proofs can be generated for provable negated modal formulas that contain numerous negation connectives. The cut-elimination theorems for the calculi are proved, and the subformula properties for the calculi are also obtained. Additionally, Gentzen-style twist (hyper)sequent calculi for other normal modal logics including S5 are considered.

1 Introduction

Reasoning about negative information or knowledge, especially when involving negations and modalities, holds significant importance in the field of philosophical logic [5, 34, 23, 31, 4]. For instance, Fitch's paradox, a fundamental issue in philosophical logic, has been analyzed through reasoning about negative information within the context of negations and modalities [34]. Effective reasoning in this area requires the development of a robust proof system, such as a Gentzen-style sequent calculus, tailored for standard modal logics like the normal modal logic S4. This Gentzen-style sequent calculus should efficiently manage the interactions between negations and modalities.

The primary objective of this study is to develop an alternative cut-free and analytic Gentzen-style sequent calculus for S4. Specifically, the sequent calculus proposed in this study aims to effectively handle negative information involving negations and modalities. In other words, our focus is on constructing a sequent calculus capable of managing formulas that include both modal operators and multiple negation connectives. The proposed sequent calculi are intended to have the ability to generate relatively short and compact "shortcut (or abbreviated) proofs" for provable negated modal formulas containing numerous negation connectives.

The concept of a "shortcut (or abbreviated) proof" is defined as a proof that incorporates "twist logical inference rules." These twist rules are considered "shortcut (or abbreviated) rules" specifically in relation to negations. To explain these twist rules, we now examine the following twist logical inference rule for negated modal operators, which is included in one of the proposed calculi, gTS4:

$$\frac{\Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \alpha}{\neg\Box\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\neg\Box\text{left}^T).$$

This rule is derivable in a standard sequent calculus as follows:

$$\cfrac{\cfrac{\Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \alpha}{\Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \Box\alpha}\ (\Box\text{right}^k)}{\neg\Box\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ \vdots\ (\neg\text{left}), (\neg\text{right})$$

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where ($\neg$left), ($\neg$right), and ($\Box$right^k) [1] are defined as follows:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{right}) \quad \frac{\Box\Gamma \Rightarrow \Diamond\Delta, \alpha}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\alpha.}\ (\Box\text{right}^k).$$

In this case, we can observe that the applications of the rules ($\neg$left), ($\neg$right), and ($\Box$right^k) are encapsulated within the single rule ($\neg\Box$left^T). Specifically, ($\neg\Box$left^T) serves as a shortcut (or abbreviated) rule for the applications of ($\neg$left), ($\neg$right), and ($\Box$right^k). In other words, many applications of ($\neg$left) and ($\neg$right) in a proof can be abbreviated by a single application of ($\neg\Box$left^T). Therefore, if there are many occurrences of $\neg$ in a given provable sequent, we can obtain a significantly shorter shortcut (or abbreviated) proof for the sequent compared to using the standard calculus. In this sense, gTS4 is effective in proving negated modal formulas containing numerous negation connectives.

In this study, we introduce two cut-free and analytic Gentzen-style twist sequent calculi for the modal logic S4, named lTS4 and gTS4. These calculi handle negation differently: locally in lTS4 and globally in gTS4. Both lTS4 and gTS4 avoid using standard logical inference rules for negation. Instead, they incorporate several twist logical inference rules, which serve as shortcut (or abbreviated) rules specifically designed for handling negated logical connectives. These twist rules are constructed by integrating the standard logical inference rules for the logical connectives $\wedge, \vee, \rightarrow, \neg$ and the modal operators $\Box, \Diamond$ with those for $\neg$.

Due to these twist logical inference rules, lTS4 and gTS4 can generate relatively short and compact shortcut (or abbreviated) proofs for provable negated modal formulas containing multiple negation connectives. This makes lTS4 and gTS4 particularly effective in handling negated modal formulas. Indeed, the proofs produced by lTS4 and gTS4 for the sequents that include negated modal formulas containing numerous negation connective are shorter than those generated by a standard Gentzen-style sequent calculus for S4. Thus, we can understand that lTS4 and gTS4 have the ability to provide effective (shortcut or abbreviated) reasoning in this context.

In this study, we establish the cut-elimination theorems for both lTS4 and gTS4, confirming that they are cut-free. Additionally, we demonstrate the subformula properties for these calculi, ensuring that lTS4 and gTS4 are analytic. Furthermore, we extend similar results to some Gentzen-style twist sequent calculi designed for classical logic and other normal modal logics, including K, KT, and S5. Specifically, a Gentzen-style twist sequent calculus for classical logic, called TCL, is obtained as the common fragment of lTS4 and gTS4 when the modal operators $\Box$ and $\Diamond$ are omitted.

We now examine some closely related traditional and recently proposed Gentzen-style sequent calculi for S4. A cut-free and analytic Gentzen-style sequent calculus for S4 was initially introduced and investigated by Ohnishi and Matsumoto in [24, 25]. Another cut-free and analytic Gentzen-style sequent calculus, referred to here as GS4, was presented by Kripke in [14] (p. 91). Kripke's calculus GS4 was developed by adapting Ohnishi and Matsumoto's calculus to handle the modal operators $\Box$ and $\Diamond$ simultaneously. Grigoriev and Petrukhin introduced and explored some extensions of GS4 in [9], wherein some multilattice extensions of GS4 and its S5 version were studied.

Cut-free (though non-analytic) Gentzen-style sequent calculi NS4, DS4, and SS4 for S4, which are regarded as falsification-aware calculi, have been introduced by Kamide in [12], based on GS4. Furthermore, cut-free (though non-analytic) Gentzen-style sequent calculi $GS4_1$, $GS4_2$, and $GS4_3$ for S4, which are compatible with a Gentzen-style sequent calculus for Avron's self-extensional paradefinite logic, have also recently been introduced by Kamide in [13], based on GS4.

[1] ($\Box$right^k) was originally introduced by Kripke in [14] (p. 91).

The original calculi introduced by Ohnishi and Matsumoto and by Kripke were cut-free and analytic systems, yet they were not effective in proving negated modal formulas containing numerous negation connectives. While NS4, DS4, and SS4 were suitable for falsification-aware reasoning and $GS4_1$, $GS4_2$, and $GS4_3$ were compatible with paraconsistent reasoning, they were not effective for proving negated modal formulas containing numerous negation connectives. Moreover, NS4, DS4, $GS4_1$, $GS4_2$, and $GS4_3$ lacked analyticity (i.e., these calculi lacked the subformula property).

In contrast to these calculi, the proposed twist calculi, lTS4 and gTS4, are cut-free, analytic, and effective in proving negated modal formulas containing numerous negation connectives. For more general information on sequent calculi for modal logics including S4, see, for example, [35, 6, 27, 21, 10, 18, 19, 17] and the references therein. For information on sequent calculi for S5, see, for example, [9, 12, 27, 17, 18, 19, 10] and the references therein. For a very short survey of recent works on sequent calculi for S5, see Section 6 of the present paper.

The structure of this paper is addressed as follows.

In Section 2, we introduce lTS4 and gTS4 and prove some basic propositions for lTS4 and gTS4.

In Section 3, we define Kripke's calculus GS4, establish the equivalence among GS4, lTS4, and gTS4, and observe a comparison among proofs generated by lTS4, gTS4, and GS4.

In Section 4, we prove some basic theorems for lTS4 and gTS4. First, we show the classical-negation-elimination and classical-converse-negation-elimination theorems for lTS4 and gTS4. Second, we prove the cut-elimination theorems for lTS4 and gTS4, relying on key lemmas concerning the cut-free provabilities of lTS4, gTS4, and GS4. Finally, we obtain the subformula properties for lTS4 and gTS4 as a consequence of the cut-elimination theorems.

In Section 5, we introduce Gentzen-style twist sequent calculi for other normal modal logics, including K, KT, and S5. Furthermore, we introduce a twist hyper-sequent calculus for S5. We also show the cut-elimination theorems and subformula properties for these calculi.

In Section 6, we conclude this study, offer some remarks on the potential applications of the proposed calculi to logic programming, and outline prospective future works.

2 Twist sequent calculi for S4

We construct *formulas* of normal modal logic S4 from countably many propositional variables by $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\neg$ (negation), $\Box$ (box), and $\Diamond$ (diamond). We use small letters $p, q, ...$ to denote propositional variables, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to represent finite (possibly empty) sets of formulas. For any set A of symbols (i.e., alphabet), we use the notation $A^\star$ to represent the set of all words of finite length of the alphabet A. For any $\natural \in \{\neg, \Box, \Diamond\}^\star$, we use an expression $\natural\Gamma$ to denote the set $\{\natural\gamma \mid \gamma \in \Gamma\}$. We use the symbol $\equiv$ to denote the equality of symbols. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. We use an expression $\alpha \Leftrightarrow \beta$ to represent the abbreviation of the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. We use an expression $L \vdash S$ to represent the fact that a sequent S is provable in a sequent calculus L. We say that two sequent calculi L_1 and L_2 are *theorem-equivalent* if $\{S \mid L_1 \vdash S\} = \{S \mid L_2 \vdash S\}$. We say that a rule R of inference is *admissible* in a sequent calculus L if the following condition is satisfied: For any instance $\frac{S_1 \cdots S_n}{S}$ of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Furthermore, we say that R is *derivable* in L if there is a derivation from $S_1, \cdots, S_n$ to S in L. We remark the fact that a rule R of inference is admissible in a sequent calculus L if and only if two sequent calculi L and $L + R$ are theorem-equivalent. Since the logics discussed in this study are formulated as Gentzen-style sequent calculi, we will sometimes identify the logic with a Gentzen-style sequent calculus determined by it.

We introduce a Gentzen-style local twist sequent calculus lTS4 for S4.

Definition 2.1 (lTS4) *The initial sequents of* lTS4 *are of the form: For any propositional variable* p*,*

$$p \Rightarrow p \qquad \neg p \Rightarrow \neg p \qquad \neg p, p \Rightarrow \qquad \Rightarrow \neg p, p.$$

The structural inference rules of lTS4 *are of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{cut}) \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The non-twist logical inference rules of lTS4 *are of the form:*

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \to \beta, \Gamma \Rightarrow \Delta}\ (\to\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}\ (\to\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box\text{left}) \quad \frac{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \alpha}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \Box\alpha}\ (\Box\text{right})$$

$$\frac{\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}{\Diamond\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\Diamond\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}\ (\Diamond\text{right}).$$

The (local) twist logical inference rules (or twist rules for short) of lTS4 *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\neg\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\neg\text{left}^t) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\neg\alpha}\ (\neg\neg\text{right}^t)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\neg(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\neg\wedge\text{left}^t) \quad \frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(\alpha \wedge \beta)}\ (\neg\wedge\text{right}^t)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\neg(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\neg\vee\text{left}^t) \quad \frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)}\ (\neg\vee\text{right}^t)$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\neg(\alpha \to \beta), \Gamma \Rightarrow \Delta}\ (\neg\to\text{left}^t) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(\alpha \to \beta)}\ (\neg\to\text{right}^t)$$

$$\frac{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \alpha}{\neg\Box\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\neg\Box\text{left}^t) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\Box\alpha}\ (\neg\Box\text{right}^t)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\Diamond\alpha, \Gamma \Rightarrow \Delta}\ (\neg\Diamond\text{left}^t) \quad \frac{\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha}\ (\neg\Diamond\text{right}^t).$$

Remark 2.2

1. lTS4 *has no standard logical inference rules for* $\neg$ *used in Gentzen's sequent calculus* LK [8]*:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{right}).$$

   Instead, we use the twist logical inference rules in lTS4*.* ($\to$left) *and* ($\neg$right) *are internalized in the twist logical inference rules.*

2. *The twist logical inference rules of* lTS4 *are constructed by integrating the (non-twist or standard) logical inference rules for* $\wedge, \vee, \rightarrow, \neg, \Box$*, and* $\Diamond$ *with the standard logical inference rules for* $\neg$.
3. ($\neg\neg$left^t) *and* ($\neg\neg$right^t) *are also constructed by integrating* ($\neg$left) *with* ($\neg$right). *Thus,* ($\neg\neg$left^t) *and* ($\neg\neg$right^t) *are also said to be twist logical inference rules.*
4. *Let* lTS4* *be the system that is obtained from* lTS4 *by replacing* ($\neg\Box$left^t) *and* ($\neg\Diamond$right^t) *with the simple twist rules of the form:*

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \alpha}{\neg\Box\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}\ (\neg\Box\text{left}^{t\star}) \quad \frac{\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}{\Box\Gamma \Rightarrow \Diamond\Delta, \neg\Diamond\alpha}\ (\neg\Diamond\text{right}^{t\star}).$$

   Then, the sequents of the form $\neg\Box p \Rightarrow \neg\Box p$ *and* $\neg\Diamond p \Rightarrow \neg\Diamond p$ *for any propositional variable* p *cannot be proved in cut-free* lTS4*. *Thus, we adopt* ($\neg\Box$left^t) *and* ($\neg\Diamond$right^t) *in* lTS4.
5. ($\Box$right) *and* ($\Diamond$left) *in* lTS4 *are considered to be compatible with* ($\neg\Diamond$right^t) *and* ($\neg\Box$left^t)*, respectively, in* lTS4. *Actually,* ($\neg\Diamond$right^t) *and* ($\neg\Box$left^t) *are constructed by integrating* ($\Box$right) *and* ($\Diamond$left) *with* ($\neg$left) *and* ($\neg$right). ($\Box$right) *and* ($\Diamond$left) *are required for proving some basic properties. Thus,* ($\Box$right) *and* ($\Diamond$left) *also cannot be replaced with the following simple rules:*

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \alpha}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\alpha}\ (\Box\text{right}^k) \quad \frac{\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}\ (\Diamond\text{left}^k),$$

   which were used in Kripke's Gentzen-style sequent calculus (for S4*) originally introduced in* [14] *(p. 91).*
6. *Let* TCL *be the system that is obtained from* lTS4 *by deleting the logical inference rules concerning* $\Box$ *and* $\Diamond$ *(i.e.,* TCL *is the* $\{\Box, \Diamond\}$*-less fragment of* lTS4*). Then,* TCL *is theorem-equivalent to Gentzen's sequent calculus* LK [8] *for propositional classical logic, and hence* TCL *is a Gentzen-style twist sequent calculus for propositional classical logic.*

Next, we introduce a Gentzen-style global twist sequent calculus gTS4 for S4.

Definition 2.3 (gTS4) gTS4 *is obtained from* lTS4 *by replacing* ($\Box$right), ($\Diamond$left), ($\neg\Box$left^t)*, and* ($\neg\Diamond$left^t) *with the (global) twist logical inference rules of the form:*

$$\frac{\Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \alpha}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \Box\alpha}\ (\Box\text{right}^T) \quad \frac{\alpha, \Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2}{\Diamond\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\Diamond\text{left}^T)$$

$$\frac{\Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \alpha}{\neg\Box\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\neg\Box\text{left}^T) \quad \frac{\alpha, \Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha}\ (\neg\Diamond\text{right}^T).$$

Remark 2.4 *We now address a comparison between* lTS4 *and* gTS4. *In a sense,* lTS4 *is a local calculus for handling* $\neg$ *and* gTS4 *is a global calculus for handling* $\neg$. *On the one hand, the twist logical inference rules for* $\neg\Box$ *and* $\neg\Diamond$ *in* lTS4 *are applied only for the principal formulas* $\neg\Box\alpha$ *and* $\neg\Diamond\alpha$ *of the twist rules. Namely, the occurrences of* $\neg$ *in the non-principal contexts of the lower sequents of the twist rules are retained in the upper sequents (i.e.,* $\neg$ *is handled locally). On the other hand, the upper sequents of the twist rules for* $\neg\Box$ *and* $\neg\Diamond$ *in* gTS4 *have no* $\neg$. *Namely, all the occurrences of* $\neg$ *in the contexts of the lower sequents of the twist rules are deleted in the upper sequents (i.e.,* $\neg$ *is handled globally). Thus, we call* lTS4 *and* gTS4 *local and global twist calculi, respectively.*

Proposition 2.5 *Let L be* lTS4 *or* gTS4. *The following sequents are provable in cut-free L: For any formula* α*,* *1.* $\alpha \Rightarrow \alpha$,

2. $\alpha, \neg\alpha \Rightarrow$,

3. $\Rightarrow \alpha, \neg\alpha$.

Proof. We only prove the proposition for lTS4, because the proposition for gTS4 can be proved similarly. We now prove the statements 1 and 2 for lTS4. The statement 3 for lTS4 can be proved in a similar way as that for 2. Thus, the proof of the statement 3 for lTS4 is omitted.

1. We prove the statement 1 by induction on α. We distinguish the cases according to the form of α and show only the case $\alpha \equiv \neg\beta$. In this case, we distinguish the cases according to the form of β and show some cases.

   (a) Case $\beta \equiv \beta_1 \rightarrow \beta_2$: We obtain the required proof:

$$\dfrac{\dfrac{\dfrac{\vdots\ Ind.\ hyp.}{\beta_1 \Rightarrow \beta_1}}{\beta_1 \Rightarrow \beta_2, \beta_1}\text{(we-right)} \quad \dfrac{\dfrac{\vdots\ Ind.\ hyp.}{\beta_2 \Rightarrow \beta_2}}{\beta_2, \beta_1 \Rightarrow \beta_2}\text{(we-left)}}{\dfrac{\beta_1 \Rightarrow \neg(\beta_1 \rightarrow \beta_2), \beta_2}{\neg(\beta_1 \rightarrow \beta_2) \Rightarrow \neg(\beta_1 \rightarrow \beta_2)}(\neg{\rightarrow}\text{left}^t).}(\neg{\rightarrow}\text{right}^t)$$

   (b) Case $\beta \equiv \Box\beta_1$: We can obtain the required proof:

$$\dfrac{\dfrac{\dfrac{\vdots\ Ind.\ hyp.}{\beta_1 \Rightarrow \beta_1}}{\Rightarrow \neg\Box\beta_1, \beta_1}(\neg\Box\text{right}^t)}{\neg\Box\beta_1 \Rightarrow \neg\Box\beta_1}(\neg\Box\text{left}^t).$$

   We remark that we cannot prove this case using the simple rule ($\neg\Box\text{left}^{t\star}$) considered in Remark 2.2.

2. We prove the statement 2 by induction on α. We distinguish the cases according to the form of α and show only the following cases. We have to prove some cases by using the statement 1.

   (a) Case $\alpha \equiv \beta_1 \rightarrow \beta_2$: We obtain the required proof:

$$\dfrac{\dfrac{\dfrac{\vdots\ Prop.\ 2.5(1)}{\beta_1 \Rightarrow \beta_1}}{\beta_1 \Rightarrow \beta_1, \beta_2}\text{(we-right)} \quad \dfrac{\dfrac{\vdots\ Prop.\ 2.5(1)}{\beta_2 \Rightarrow \beta_2}}{\beta_1, \beta_2 \Rightarrow \beta_2}\text{(we-left)}}{\dfrac{\beta_1, \beta_1 \rightarrow \beta_2 \Rightarrow \beta_2}{\beta_1 \rightarrow \beta_2, \neg(\beta_1 \rightarrow \beta_2) \Rightarrow}(\neg{\rightarrow}\text{left}^t).}({\rightarrow}\text{left})$$

   (b) Case $\alpha \equiv \Box\beta$: We obtain the required proof:

$$\dfrac{\dfrac{\dfrac{\vdots\ Prop.\ 2.5(1)}{\beta \Rightarrow \beta}}{\Box\beta \Rightarrow \beta}(\Box\text{left})}{\Box\beta, \neg\Box\beta \Rightarrow}(\neg\Box\text{left}^t).$$

■

3 Equivalence and comparison among calculi

In this section, we define Kripke's Gentzen-style sequent calculus GS4 for S4 and show the theorem-equivalence among GS4, lTS4, and gTS4.

Definition 3.1 (GS4) GS4 *is obtained from* lTS4 *by replacing* ($\Box$right)*,* ($\Diamond$left)*, all the twist logical inference rules, and the negated initial sequents of the form* ($\neg p \Rightarrow \neg p$)*,* ($\neg p, p \Rightarrow$)*, and* ($\Rightarrow \neg p, p$) *with the logical inference rules of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{right}) \quad \frac{\Box\Gamma \Rightarrow \Diamond\Delta, \alpha}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\alpha}\ (\Box\text{right}^k) \quad \frac{\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond\alpha, \Box\Gamma \Rightarrow \Diamond\Delta}\ (\Diamond\text{left}^k).$$

Remark 3.2

1. *Strictly speaking,* GS4 *is regarded as a non-essential and small modification of Kripke's original Gentzen-style sequent calculus (for* S4*) introduced in* [14] *(p. 91) to deal with* $\Box$ *and* $\Diamond$ *simultaneously. The original system by Kripke has the formula-based initial sequents of the form* $\alpha \Rightarrow \alpha$ *for any formula* α *instead of the propositional-variable-based initial sequents. This original system was introduced by modifying Ohnishi and Matsumoto's Gentzen-style sequent calculus (for* S4*) introduced in* [24, 25]*. Some extensions and modifications of the system of this type have been recently introduced and studied by Grigoriev and Petrukhin in* [9] *and by Kamide in* [12].
2. *The difference between Kripke's system (and its small modification* GS4*) and Ohnishi and Matsumoto's system is the form of* ($\Box$right^k) *and* ($\Diamond$left^k)*. Ohnishi and Matsumoto's system has no* $\Diamond\Delta$ *in* ($\Box$right^k) *and* $\Box\Gamma$ *in* ($\Diamond$left^k)*. Using the rules of* GS4*, we can show that the sequents of the form* $\Box\alpha \Leftrightarrow \neg\Diamond\neg\alpha$ *and* $\Diamond\alpha \Leftrightarrow \neg\Box\neg\alpha$ *for any formula* α *are provable in cut-free* GS4*. These sequents cannot be proved in Ohnishi and Matsumoto's system. For more information on these characteristic rules, see* [14, 9, 12].
3. *The sequents of the form* $\alpha \Rightarrow \alpha$ *for any formula* α *are provable in cut-free* GS4*. This fact can be shown by induction on* α*. Thus, we can take the sequents of the form* $\alpha \Rightarrow \alpha$ *for any formula* α *as initial sequents of* GS4.
4. *The following rules are derivable in* GS4 *using* (cut)*:*

$$\frac{\Gamma \Rightarrow \Delta, \neg\alpha}{\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}^{-1}) \quad \frac{\neg\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\neg\text{right}^{-1}).$$

5. *The cut-elimination and Kripke-completeness theorems hold for Kripke's original system. Thus, the same theorems also hold for* GS4*. For more information on these theorems, see* [14, 9].

Theorem 3.3 (Equivalence among lTS4**,** gTS4**, and** GS4**)** *Let L be* lTS4 *or* gTS4*. The systems L and* GS4 *are theorem-equivalent.*

Proof. We only prove the theorem for lTS4, because the proof of the theorem for gTS4 can be obtained similarly. Obviously, the negated initial sequents of lTS4 are provable in cut-free GS4, and the negated logical inference rules of lTS4 are derivable in GS4. For example, the derivability of ($\neg\Box$left^t) in GS4 is shown as follows.

$$\dfrac{\dfrac{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \alpha}{\Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \alpha}\ \vdots\ (\neg\text{left}^{-1}), (\neg\text{right}^{-1})}{\dfrac{\Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \Box\alpha}{\neg\Box\alpha, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ \vdots\ (\neg\text{left}), (\neg\text{right})}\ (\Box\text{right})$$

where ($\neg$left^{-1}) and ($\neg$right^{-1}) are derivable in GS4 using (cut). Conversely, ($\neg$left) and ($\neg$right) in GS4 are derivable in lTS4 using (cut) by:

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha \quad \overset{\vdots\ \text{Prop. 2.5 (2)}}{\alpha, \neg\alpha \Rightarrow}}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\text{cut}) \qquad \dfrac{\overset{\vdots\ \text{Prop. 2.5 (3)}}{\Rightarrow \alpha, \neg\alpha} \quad \alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\text{cut}).$$

Therefore, lTS4 and GS4 are theorem-equivalent. ■

Remark 3.4 *The proofs generated by* lTS4 *and* gTS4 *are shorter than those of* GS4*. Furthermore, both the proofs generated by* lTS4 *and* gTS4 *are composed of subformulas of the formulas included in the last sequent. If* $\neg$ *appears many times in a given provable sequent, then the generated proofs by* lTS4 *or* gTS4 *are quite shorter than those generated by* GS4*. Thus,* lTS4 *and* gTS4 *are regarded as effective systems for proving negated modal formulas containing numerous negation connectives. We will illustrate a comparison among proofs generated by* lTS4*,* gTS4*, and* GS4*.*

Example 3.5 *We consider the provable sequent* $\neg\neg\neg\Diamond\neg p \Rightarrow \neg\Diamond\neg\neg\Diamond\neg\neg\neg p$ *with a propositional variable* p*. The proofs of this sequent in* lTS4*,* gTS4*, and* GS4 *are addressed as follows. First, we show the short proof generated by* lTS4 *using the twist rules* ($\neg\neg$left^t)*,* ($\neg\Diamond$left^t)*, and* ($\neg\Diamond$left^t) *and the negated initial sequent* $\neg p \Rightarrow \neg p$*.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg p \Rightarrow \neg p}{\neg p, \neg\Diamond\neg p \Rightarrow}\ (\neg\Diamond\text{left}^t)}{\neg\neg\neg p, \neg\Diamond\neg p \Rightarrow}\ (\neg\neg\text{left}^t)}{\Diamond\neg\neg\neg p, \neg\Diamond\neg p \Rightarrow}\ (\Diamond\text{left})}{\neg\neg\Diamond\neg\neg\neg p, \neg\Diamond\neg p \Rightarrow}\ (\neg\neg\text{left}^t)}{\neg\Diamond\neg p \Rightarrow \neg\Diamond\neg\neg\Diamond\neg\neg\neg p}\ (\neg\Diamond\text{right}^t)}{\neg\neg\neg\Diamond\neg p \Rightarrow \neg\Diamond\neg\neg\Diamond\neg\neg\neg p}\ (\neg\neg\text{left}^t).$$

Next, we show the short proof generated by gTS4 *using the twist rules* ($\neg\neg$left^t)*,* ($\neg\Diamond$right^T)*, and* ($\Diamond$left^T) *and the negated initial sequent* $\neg p \Rightarrow \neg p$*.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg p \Rightarrow \neg p}{\neg p \Rightarrow \Diamond\neg p}\ (\Diamond\text{right})}{\neg\neg\neg p \Rightarrow \Diamond\neg p}\ (\neg\neg\text{left}^t)}{\Diamond\neg\neg\neg p \Rightarrow \Diamond\neg p}\ (\Diamond\text{left}^T)}{\neg\neg\Diamond\neg\neg\neg p \Rightarrow \Diamond\neg p}\ (\neg\neg\text{left}^t)}{\neg\Diamond\neg p \Rightarrow \neg\Diamond\neg\neg\Diamond\neg\neg\neg p}\ (\neg\Diamond\text{right}^T)}{\neg\neg\neg\Diamond\neg p \Rightarrow \neg\Diamond\neg\neg\Diamond\neg\neg\neg p}\ (\neg\neg\text{left}^t).$$

Finally, we show the usual (long) proof generated by GS4 *using the standard logical inference rules*

(¬left) *and* (¬right).

$$
\begin{array}{cl}
p \Rightarrow p & \\ \hline
\Rightarrow \neg p, p & (\neg\text{right}) \\ \hline
\neg p \Rightarrow \neg p & (\neg\text{left}) \\ \hline
\Rightarrow \neg p, \neg\neg p & (\neg\text{right}) \\ \hline
\neg\neg\neg p \Rightarrow \neg p & (\neg\text{left}) \\ \hline
\neg\neg\neg p \Rightarrow \Diamond\neg p & (\Diamond\text{right}) \\ \hline
\Diamond\neg\neg\neg p \Rightarrow \Diamond\neg p & (\Diamond\text{left}^k) \\ \hline
\Rightarrow \Diamond\neg p, \neg\Diamond\neg\neg\neg p & (\neg\text{right}) \\ \hline
\neg\neg\Diamond\neg\neg\neg p \Rightarrow \Diamond\neg p & (\neg\text{left}) \\ \hline
\Diamond\neg\neg\Diamond\neg\neg\neg p \Rightarrow \Diamond\neg p & (\Diamond\text{left}^k) \\ \hline
\neg\Diamond\neg p, \Diamond\neg\neg\Diamond\neg\neg\neg p \Rightarrow & (\neg\text{left}) \\ \hline
\Diamond\neg\neg\Diamond\neg\neg\neg p \Rightarrow \neg\neg\Diamond\neg p & (\neg\text{right}) \\ \hline
\neg\neg\neg\Diamond\neg p, \Diamond\neg\neg\Diamond\neg\neg\neg p \Rightarrow & (\neg\text{left}) \\ \hline
\neg\neg\neg\Diamond\neg p \Rightarrow \neg\Diamond\neg\neg\Diamond\neg\neg\neg p & (\neg\text{right}).
\end{array}
$$

4 Cut-elimination and subformula property

In this section, we prove some basic theorems for lTS4 and gTS4.

Theorem 4.1 (Classical-negation-elimination for lTS4 and gTS4) *Let L be* lTS4 *or* gTS4. *The rules* (¬left) *and* (¬right) *are admissible in cut-free L.*

Proof. We show only the admissibility of (¬left), because the admissibility of (¬right) can be shown similarly. We consider the proof of the form:

$$
\dfrac{\begin{array}{c}\vdots\, P \\ \Gamma \Rightarrow \Delta, \alpha\end{array}}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}).
$$

Then, we prove the theorem by induction on P. We distinguish the cases according to the last inference of P and show some cases.

1. Case (→right): The last inference of P is of the form:

$$
\dfrac{\begin{array}{c}\vdots \\ \alpha_1, \Gamma \Rightarrow \Delta, \alpha_2\end{array}}{\Gamma \Rightarrow \Delta, \alpha_1 \to \alpha_2}\ (\to\text{right})
$$

   where $\alpha \equiv \alpha_1 \to \alpha_2$. We then obtain the required fact:

$$
\dfrac{\begin{array}{c}\vdots \\ \alpha_1, \Gamma \Rightarrow \Delta, \alpha_2\end{array}}{\neg(\alpha_1 \to \alpha_2), \Gamma \Rightarrow \Delta}\ (\neg\to\text{left}^t).
$$

2. Case (¬→right^t): The last inference of P is of the form:

$$
\dfrac{\begin{array}{c}\vdots \\ \Gamma \Rightarrow \Delta, \alpha_1\end{array} \quad \begin{array}{c}\vdots \\ \alpha_2, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta, \neg(\alpha_1 \to \alpha_2)}\ (\neg\to\text{right}^t)
$$

where $\alpha \equiv \neg(\alpha_1 \to \alpha_2)$. We then obtain the required fact:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha_1\end{array} \quad \begin{array}{c}\vdots\\ \alpha_2, \Gamma \Rightarrow \Delta\end{array}}{\alpha_1 \to \alpha_2, \Gamma \Rightarrow \Delta}\ (\to\text{left})}{\neg\neg(\alpha_1 \to \alpha_2), \Gamma \Rightarrow \Delta}\ (\neg\neg\text{left}^t).$$

3. Case ($\Box$right) for lTS4: The last inference of P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \alpha_1\end{array}}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \Box\alpha_1}\ (\Box\text{right})$$

where $\Gamma \Rightarrow \Delta, \alpha$ is $\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \Box\alpha_1$ and $\alpha \equiv \Box\alpha_1$. We then obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \alpha_1\end{array}}{\neg\Box\alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\neg\Box\text{left}^t).$$

4. Case ($\neg\Diamond$right^t) for lTS4: The last inference of P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2\end{array}}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha_1}\ (\neg\Diamond\text{right}^t)$$

where $\Gamma \Rightarrow \Delta, \alpha$ is $\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha_1$ and $\alpha \equiv \neg\Diamond\alpha_1$. We then obtain the required fact:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2\end{array}}{\Diamond\alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\Diamond\text{left})}{\neg\neg\Diamond\alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\neg\neg\text{left}^t).$$

5. Case ($\Box$right^T) for gTS4: The last inference of P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \alpha_1\end{array}}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \Box\alpha_1}\ (\Box\text{right}^T)$$

where $\Gamma \Rightarrow \Delta, \alpha$ is $\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \Box\alpha_1$ and $\alpha \equiv \Box\alpha_1$. We then obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2, \alpha_1\end{array}}{\neg\Box\alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}\ (\neg\Box\text{left}^T).$$

6. Case ($\neg\Diamond$right^T) for gTS4: The last inference of P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha_1, \Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2\end{array}}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha_1}\ (\neg\Diamond\text{right}^T)$$

where $\Gamma \Rightarrow \Delta, \alpha$ is $\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha_1$ and $\alpha \equiv \neg\Diamond\alpha_1$. We then obtain the required fact:

$$\dfrac{\dfrac{\vdots}{\alpha_1, \Box\Gamma_1, \Box\Delta_2 \Rightarrow \Diamond\Delta_1, \Diamond\Gamma_2} \quad}{\dfrac{\Diamond\alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2}{\neg\neg\Diamond\alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2} \ (\neg\neg\text{left}^t)} \ (\Diamond\text{left}^T).$$

■

Next, we show the following theorem using Theorem 4.1.

Theorem 4.2 (Classical-converse-negation-elimination for lTS4 and gTS4) *Let L be* lTS4 *or* gTS4. *The following rules are admissible in cut-free L:*

$$\dfrac{\Gamma \Rightarrow \Delta, \neg\alpha}{\alpha, \Gamma \Rightarrow \Delta} \ (\neg\text{left}^{-1}) \qquad \dfrac{\neg\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \ (\neg\text{right}^{-1}).$$

Proof. We only prove the theorem for lTS4. We show only the admissibility of ($\neg$left^{-1}). The admissibility of ($\neg$right^{-1}) can be shown similarly. We consider the proof of the form:

$$\dfrac{\vdots\, P \atop \Gamma \Rightarrow \Delta, \neg\alpha}{\alpha, \Gamma \Rightarrow \Delta} \ (\neg\text{left}^{-1}).$$

Then, we prove the theorem by induction on P. We distinguish the cases according to the last inference of P and show some cases.

1. Case ($\neg\neg$right^t): The last inference of P is of the form:

$$\dfrac{\vdots \atop \Gamma \Rightarrow \Delta, \alpha_1}{\Gamma \Rightarrow \Delta, \neg\neg\alpha_1} \ (\neg\neg\text{right}^t)$$

   where $\alpha \equiv \neg\alpha_1$. We then obtain the required fact:

$$\dfrac{\vdots \atop \Gamma \Rightarrow \Delta, \alpha_1}{\neg\alpha_1, \Gamma \Rightarrow \Delta} \ (\neg\text{left})$$

   where ($\neg$left) is admissible in cut-free lTS4 by Theorem 4.1.

2. Case ($\neg\!\rightarrow$right^t): The last inference of P is of the form:

$$\dfrac{\vdots \atop \Gamma \Rightarrow \Delta, \alpha_1 \quad \vdots \atop \alpha_2, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(\alpha_1 \rightarrow \alpha_2)} \ (\neg\!\rightarrow\text{right}^t)$$

   where $\alpha \equiv \alpha_1 \rightarrow \alpha_2$. We then obtain the required fact:

$$\dfrac{\vdots \atop \Gamma \Rightarrow \Delta, \alpha_1 \quad \vdots \atop \alpha_2, \Gamma \Rightarrow \Delta}{\alpha_1 \rightarrow \alpha_2, \Gamma \Rightarrow \Delta} \ (\rightarrow\text{left}).$$

3. Case ($\neg\Diamond$right^t): The last inference of P is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2\end{array}}{\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha_1} \ (\neg\Diamond\text{right}^t)$$

where $\Gamma \Rightarrow \Delta, \alpha$ is $\Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2, \neg\Diamond\alpha_1$ and $\alpha \equiv \Diamond\alpha_1$. We then obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2\end{array}}{\Diamond\alpha_1, \Box\Gamma_1, \neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1, \neg\Box\Delta_2} \ (\Diamond\text{left}).$$

In this case, we note that ($\Diamond$left) in lTS4 cannot be replaced with ($\Diamond$left^k) in GS4.

■

Next, we show the following lemma using Theorem 4.1.

Lemma 4.3 *Let L be* lTS4 *or* gTS4*. For any sequent S, if S is provable in cut-free* GS4*, then S is provable in cut-free L.*

Proof. We only prove the theorem for lTS4. Suppose that a sequent $\Gamma \Rightarrow \Delta$ is provable in cut-free GS4. Then, we show this lemma by induction on the cut-free proofs P of $\Gamma \Rightarrow \Delta$. We distinguish the cases according to the last inference of P and show only the cases for ($\neg$left) and ($\neg$right). The proofs of these cases can be obtained using ($\neg$left) and ($\neg$right), which are admissible in cut-free lTS4 by Theorem 4.1. ■

We show the following cut-elimination theorem using Lemma 4.3.

Theorem 4.4 (Cut-elimination for lTS4 and gTS4) *Let L be* lTS4 *or* gTS4*. The rule* (cut) *is admissible in cut-free L.*

Proof. We only prove the theorem for lTS4. Suppose that a sequent S is provable in lTS4. Then, S is provable in GS4 by Theorem 3.3. Thus, S is provable in cut-free GS4 by the cut-elimination theorem for GS4. Thus, S is provable in cut-free lTS4 by Lemma 4.3. ■

Theorem 4.5 (Subformula property for lTS4 **and** gTS4**)** *Let L be* lTS4 *or* gTS4*. The system L has the subformula property. Namely, if a sequent S is provable in L, then there is a proof P of S such that all formulas appear in P are subformulas of some formula in S.*

Proof. By a consequence of Theorem 4.4. ■

Remark 4.6 lTS4 *and* gTS4 *are conservative extensions of the Gentzen-style twist sequent calculus* TCL *for propositional classical logic, which was considered in Remark 2.2. This fact is obtained by Theorem 4.4. The cut-elimination theorem and subformula property also hold for* TCL.

5 Twist sequent calculi for K, KT, and S5

First, we introduce Gentzen-style global twist sequent calculi gTK, gTKT, and gTS5 for K, KT, and S5, respectively.

Definition 5.1 (gTK, gTKT, and gTS5)

1. gTK *is obtained from* gTS4 *by replacing* ($\Box$left), ($\Box$right^T), ($\Diamond$left^T), ($\Diamond$right), ($\neg\Box$left^T), ($\neg\Box$right^t), ($\neg\Diamond$left^t), *and* ($\neg\Diamond$right^T) *with the following global twist logical inference rules:*

$$\frac{\Gamma_1,\Delta_2 \Rightarrow \Delta_1,\Gamma_2,\alpha}{\Box\Gamma_1,\neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1,\neg\Box\Delta_2,\Box\alpha}\ (\Box\text{K-right}^T)$$

$$\frac{\alpha,\Gamma_1,\Delta_2 \Rightarrow \Delta_1,\Gamma_2}{\Diamond\alpha,\Box\Gamma_1,\neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1,\neg\Box\Delta_2}\ (\Diamond\text{K-left}^T)$$

$$\frac{\Gamma_1,\Delta_2 \Rightarrow \Delta_1,\Gamma_2,\alpha}{\neg\Box\alpha,\Box\Gamma_1,\neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1,\neg\Box\Delta_2}\ (\neg\Box\text{K-left}^T)$$

$$\frac{\alpha,\Gamma_1,\Delta_2 \Rightarrow \Delta_1,\Gamma_2}{\Box\Gamma_1,\neg\Diamond\Gamma_2 \Rightarrow \Diamond\Delta_1,\neg\Box\Delta_2,\neg\Diamond\alpha}\ (\neg\Diamond\text{K-right}^T).$$

2. gTKT *is obtained from* gTK *by adding* ($\Box$left), ($\Diamond$right), *and the logical inference rules* ($\neg\Box$right^T) *and* ($\neg\Diamond$left^T).

3. gTS5 *is obtained from* gTS4 *by replacing* ($\Box$right^T), ($\Diamond$left^T), ($\neg\Box$left^T), ($\neg\Diamond$right^T) *with the following global twist logical inference rules:*

$$\frac{\Box\Gamma_1,\Diamond\Delta_2,\Box\Lambda_2 \Rightarrow \Box\Delta_1,\Diamond\Lambda_1,\Diamond\Gamma_2,\alpha}{\Box\Gamma_1,\neg\Diamond\Gamma_2 \Rightarrow \Box\Delta_1,\neg\Diamond\Delta_2,\Diamond\Lambda_1,\neg\Box\Lambda_2,\Box\alpha}\ (\Box\text{S5-right}^T)$$

$$\frac{\alpha,\Box\Gamma_1,\Diamond\Sigma_1,\Box\Delta_2 \Rightarrow \Diamond\Delta_1,\Diamond\Gamma_2,\Box\Sigma_2}{\Diamond\alpha,\Box\Gamma_1,\neg\Diamond\Gamma_2,\Diamond\Sigma_1,\neg\Box\Sigma_2 \Rightarrow \Diamond\Delta_1,\neg\Box\Delta_2}\ (\Diamond\text{S5-left}^T)$$

$$\frac{\Box\Gamma_1,\Diamond\Sigma_1,\Box\Delta_2 \Rightarrow \Diamond\Delta_1,\Diamond\Gamma_2,\Box\Sigma_2,\alpha}{\neg\Box\alpha,\Box\Gamma_1,\neg\Diamond\Gamma_2,\Diamond\Sigma_1,\neg\Box\Sigma_2 \Rightarrow \Diamond\Delta_1,\neg\Box\Delta_2}\ (\neg\Box\text{S5-left}^T)$$

$$\frac{\alpha,\Box\Gamma_1,\Diamond\Delta_2,\Box\Lambda_2 \Rightarrow \Box\Delta_1,\Diamond\Lambda_1,\Diamond\Gamma_2}{\Box\Gamma_1,\neg\Diamond\Gamma_2 \Rightarrow \Box\Delta_1,\neg\Diamond\Delta_2,\Diamond\Lambda_1,\neg\Box\Lambda_2,\neg\Diamond\alpha}\ (\neg\Diamond\text{S5-right}^T).$$

Remark 5.2 *We can also consider the local-type twist sequent calculi* lTKT *and* lTS5*. However, we cannot consider the local-type twist sequent calculus* lTK*. The Kripke-style non-twist sequent calculi for* K*,* KT*, and* S5 *were introduced and studied in* [12]*. On the one hand, the cut-elimination theorems for the Gentzen-style twist sequent calculi* lTS5 *and* gTS5 *do not hold. A counter example sequent for this fact is* $p \Rightarrow \Box\neg\Box\neg p$ *where* p *is a propositional variable. This counterexample sequent was given by Takano in* [33] *for the cut-elimination theorem for a standard Gentzen-style sequent calculus for* S5*, introduced by Ohnishi and Matsumoto. On the other hand, we can show the cut-elimination theorem for a twist hypersequent calculus,* HTS5*, for* S5*. The cut-elimination theorem for* HTS5 *will be shown. In* HTS5*, there is no distinction between local and global. For more information on hypersequent calculi for* S5*, see e.g.,* [28, 1, 30, 26, 15, 16, 3, 9, 12] *and the references therein.*

Next, we introduce a twist hypersequent calculus HTS5 for S5. We call an expression of the form $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ *hypersequent*. We define the hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ as a finite multiset of sequents $\Gamma_k \Rightarrow \Delta_k$ $(1 \leq k \leq n)$. We use capital letters H, G, ... to represent hypersequents.

Definition 5.3 (HTS5) *The initial hypersequents of* HTS5 *are of the form: For any propositional variable p,*

$$p \Rightarrow p \qquad \neg p \Rightarrow \neg p \qquad p, \neg p \Rightarrow \qquad \Rightarrow p, \neg p.$$

The structural inference rules of HTS5 *are of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha \mid H \quad \alpha, \Sigma \Rightarrow \Pi \mid G}{\Gamma, \Sigma \Rightarrow \Delta, \Pi \mid H \mid G}\ \text{(cut)} \qquad \frac{\Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid H}{\Gamma, \Sigma \Rightarrow \Delta, \Pi \mid H}\ \text{(merge)}$$

$$\frac{\Gamma \Rightarrow \Delta \mid H}{\alpha, \Gamma \Rightarrow \Delta \mid H}\ \text{(in-we-left)} \qquad \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \alpha \mid H}\ \text{(in-we-right)}$$

$$\frac{H}{\alpha \Rightarrow\ \mid H}\ \text{(ex-we-left)} \qquad \frac{H}{\Rightarrow \alpha \mid H}\ \text{(ex-we-right)}.$$

The non-twist logical inference rules of HTS5 *are of the form:*

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta \mid H}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta \mid H}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \quad \Gamma \Rightarrow \Delta, \beta \mid G}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta \mid H \mid G}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \mid H \quad \beta, \Gamma \Rightarrow \Delta \mid G}{\alpha \vee \beta, \Gamma \Rightarrow \Delta \mid H \mid G}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta \mid H}{\Gamma \Rightarrow \Delta, \alpha \vee \beta \mid H}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \mid H \quad \beta, \Gamma \Rightarrow \Delta \mid G}{\alpha \to \beta, \Gamma \Rightarrow \Delta \mid H \mid G}\ (\to\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta \mid H}{\Gamma \Rightarrow \Delta, \alpha \to \beta \mid H}\ (\to\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Box\alpha \Rightarrow\ \mid \Gamma \Rightarrow \Delta \mid H}\ (\Box\text{left}) \qquad \frac{\Rightarrow \alpha \mid H}{\Rightarrow \Box\alpha \mid H}\ (\Box\text{right})$$

$$\frac{\alpha \Rightarrow\ \mid H}{\Diamond\alpha \Rightarrow\ \mid H}\ (\Diamond\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\Gamma \Rightarrow \Delta \mid\ \Rightarrow \Diamond\alpha \mid H}\ (\Diamond\text{right}).$$

The twist logical inference rules of HTS5 *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\neg\neg\alpha, \Gamma \Rightarrow \Delta \mid H}\ (\neg\neg\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\Gamma \Rightarrow \Delta, \neg\neg\alpha \mid H}\ (\neg\neg\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \mid H \quad \Gamma \Rightarrow \Delta, \beta \mid G}{\neg(\alpha \wedge \beta), \Gamma \Rightarrow \Delta \mid H \mid G}\ (\neg\wedge\text{left}) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg(\alpha \wedge \beta) \mid H}\ (\neg\wedge\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha, \beta \mid H}{\neg(\alpha \vee \beta), \Gamma \Rightarrow \Delta \mid H}\ (\neg\vee\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta \mid H \quad \beta, \Gamma \Rightarrow \Delta \mid G}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta) \mid H \mid G}\ (\neg\vee\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta \mid H}{\neg(\alpha \to \beta), \Gamma \Rightarrow \Delta \mid H}\ (\neg\to\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \quad \beta, \Gamma \Rightarrow \Delta \mid G}{\Gamma \Rightarrow \Delta, \neg(\alpha \to \beta) \mid H \mid G}\ (\neg\to\text{right})$$

$$\frac{\Rightarrow \alpha \mid H}{\neg\Box\alpha \Rightarrow\ \mid H}\ (\neg\Box\text{S5-left}^h) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid\ \Rightarrow \neg\Box\alpha \mid H}\ (\neg\Box\text{S5-right}^h)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\neg\Diamond\alpha \Rightarrow\ \mid \Gamma \Rightarrow \Delta \mid H}\ (\neg\Diamond\text{S5-left}^h) \qquad \frac{\alpha \Rightarrow\ \mid H}{\Rightarrow \neg\Diamond\alpha \mid H}\ (\neg\Diamond\text{S5-right}^h).$$

Theorem 5.4 (Cut-elimination for gTK**,** gTKT**, and** HTS5**)** *Let L be* gTK*,* gTKT*, or* HTS5*. The rule* (cut) *is admissible in cut-free L.*

Proof. Similar to the proof of Theorem 4.4. For the case of HTS5, we use a cut-free (non-twist) hypersequent calculus for S5, that includes the following standard logical inference rules for $\neg$:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\neg\alpha, \Gamma \Rightarrow \Delta \mid H}\ (\neg\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg\alpha \mid H}\ (\neg\text{right}).$$

For more information on this standard hypersequent calculus, see [30, 9, 12]. ■

Theorem 5.5 (Subformula property for gTK**,** gTKT**, and** HTS5**)** *Let L be* gTK*,* gTKT*, or* HTS5*. The system L has the subformula property.*

Proof. By a consequence of Theorem 5.4. ■

6 Concluding remarks

In this study, we introduced and investigated the cut-free and analytic Gentzen-style local and global twist sequent calculi, lTS4 and gTS4, for the normal modal logic S4. In these calculi, negations are handled locally in lTS4 and globally in gTS4. Unlike standard calculi, lTS4 and gTS4 do not include standard logical inference rules for negation. Instead, they employ several twist logical inference rules, which serve as “shortcut (or abbreviated)” rules specifically for negated logical connectives. As a result, lTS4 and gTS4 can generate relatively short “shortcut (or abbreviated)” proofs for provable modal formulas containing numerous negation connectives.

We proved the cut-elimination theorems for lTS4 and gTS4 and obtained the subformula properties for them. Additionally, we observed that if a given provable modal formula contains numerous negation connectives, the lengths of the proofs generated by lTS4 and gTS4 are shorter than those generated by the standard Gentzen-style sequent calculus GS4. Thus, we have identified a method for generating short proofs for modal formulas containing numerous negation connectives. We also obtained similar results for the Gentzen-style twist sequent calculi, gTK and gTKT, for the normal modal logics K and KT, respectively. Additionally, we obtained a similar result for the twist hypersequent calculus, HTS5, for the normal modal logic S5.

On the one hand, as mentioned in Section 5, we could construct the cut-free twist hypersequent calculus HTS5 for S5, in a similar way to those in [9, 12]. On the other hand, we have not yet considered other types of twist sequent calculi for S5 based on *tree-hypersequent calculi* studied by Poggiolesi and Lellmann [27, 17], *2-sequent calculi* studied by Martini, Masini, and Zorzi [18, 19], or *bisequent calculi* studied by Indrzejczak [10]. Additionally, in this study, we have not yet considered twist-style calculi in the usual sequent, hypersequent, tree-hypersequent, 2-sequent, or bisequent formats for non-normal modal logics. These issues are left as future work.

As mentioned in Section 1, reasoning about negative information or knowledge involving both negations and modalities holds significant importance in the field of philosophical logic. This type of reasoning is also crucial in computer science, particularly in logic programming and knowledge representation. Modal logic programming and knowledge representation involving modalities and negations have been extensively studied [29, 2, 22, 32, 7]. In these areas, an effective proof system that can efficiently handle both modalities and negations simultaneously is required.

We believe that the proposed Gentzen-style twisted sequent calculi are useful for implementing a sequent calculus-based goal-directed logic programming language, known as a uniform proof-based abstract logic programming language, which was originally developed by Miller, Nadathur, Pfenning, and Scedrov [20]. In relation to this, abstract paraconsistent logic programming with uniform proof was studied by Kamide in [11], where a uniform proof-theoretic foundation for that programming language, along with its applications, was proposed. Therefore, a promising future direction is to develop a uniform proof-theoretic abstract modal logic programming framework based on the proposed twisted sequent calculi, focusing on negations and modalities.

We also believe that shortcut (or abbreviated) reasoning, based on the proposed twist calculi, plays a crucial role in logic programming involving modalities and negations. This is because true negative information (or knowledge) in logic programming, represented by provable negated modal formulas containing modal operators and multiple negation connectives, often arises in real-world situations [2, 22, 32, 7]. In such cases, the proofs, which are often lengthy, are regarded as evidence. This evidence should be concise and ideally represented by short and compact shortcut (or abbreviated) proofs. In this context, short proofs are valuable and necessary for explaining evidence concisely.

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References

[1] Arnon Avron (1996): *The method of hypersequents in the proof theory of propositional non-classical logic*. In: Logic: from Foundations to Applications, pp. 1-32, doi:10.1093/oso/9780198538622.003.0001.

[2] Chitta Baral & Michael Gelfond (1994): *Logic programming and knowledge representation*. Journal of Logic Programing 19/20, pp. 73-148, doi:10.1016/0743-1066(94)90025-6.

[3] Kaja Bednarska & Andrzej Indrzejczak (2015): *Hypersequent calculi for S5: The methods of cut elimination*. Logic and Logical Philosophy 24 (3), pp. 277-311, doi:10.12775/LLP.2015.018.

[4] Sergey Drobyshevich & Heinrich Wansing (2020): *Proof systems for various FDE-based modal logics*. Review of Symbolic Logic 13(4), pp. 720-747, doi:10.1017/S1755020319000261.

[5] Melvin Fitting (1991): *Many-valued modal logics*. Fundamenta Informaticae 15, pp. 235-254, doi:10.3233/FI-1991-153-404.

[6] Melvin Fitting (2007): *Modal proof theory*. In: Handbook of Modal Logic (Studies in Logic and Practical Reasoning 3), pp. 85-138, doi:10.1016/S1570-2464(07)80005-X.

[7] Michael Gelfond, Jorge Fandinno & Evgenii Balai (2023): *Embracing background knowledge in the analysis of actual causality: An answer set programming approach*. Theory and Practice of Logic Programing 23 (4), pp. 715-729, doi:10.1017/S1471068423000248.

[8] Gerhard Gentzen (1969): *Collected papers of Gerhard Gentzen*. M.E. Szabo (ed.), Studies in logic and the foundations of mathematics, North-Holland (English translation), doi:10.2307/2272429.

[9] Oleg Grigoriev & Yaroslav Petrukhin (2019): *On a multilattice analogue of a hypersequent S5 calculus*. Logic and Logical Philosophy 28, pp. 683-730, doi:10.12775/LLP.2019.031.

[10] Andrzej Indrzejczak (2019): *Two is enough - Bisequent calculus for S5*. Proceedings of the 12th International Symposium on Frontiers of Combining Systems (FroCoS 2019), pp. 277-294, doi:10.1007/978-3-030-29007-8_16.

[11] Norihiro Kamide (2007): *A uniform proof-theoretic foundation for abstract paraconsistent logic programming*. Journal of Functional and Logic Programming 2007, pp. 1-36.

[12] Norihiro Kamide (2023): *Falsification-aware calculi and semantics for normal modal logics including S4 and S5*. Journal of Logic, Language, and Information 32 (3), pp. 395-440, doi:10.1007/S10849-022-09386-7.

[13] Norihiro Kamide (2024): *Alternative cut-free sequent calculi for S4 that are compatible with paradefinite four-valued logic*. Journal of Applied Logics, to appear.

[14] Saul Aaron Kripke (1963): *Semantical analysis of modal logic I Normal modal propositional calculi*. Zeitschr. math. Logik und Grundlagen d. Math. Bd. 9, S. pp. 67-96, doi:10.1002/malq.19630090502.

[15] Hidenori Kurokawa (2013): *Hypersequent calculi for modal logics extending S4*. In: New Frontiers in Artificial Intelligence, Lecture Notes in Computer Science 8417, pp. 51-68, doi:10.1007/978-3-319-10061-6_4.

[16] Ori Lahav (2013): *From frame properties to hypersequent rules in modal logics*. Proceedings of the 28th Annual ACM/IEEE Symposium on Logic in Computer Science, pp. 408-417, doi:10.1109/LICS.2013.47.

[17] Björn Lellmann & Francesca Poggiolesi (2024): *Nested sequent or tree-hypersequents: A survey*. Saul Kripke on Modal logic (Outstanding Contributions to Logic 30), R. Padro and Y. Weiss (eds.), Springer.

[18] Simone Martini, Andrea Masini & Margherita Zorzi (2021): *From 2-sequents and linear nested sequents to natural deduction for normal modal logics*. ACM Transaction on Computational Logic 22 (3), pp. 19:1-19:29, doi:10.1145/3461661.

[19] Simone Martini, Andrea Masini & Margherita Zorzi (2023): *Cut-elimination for extended sequent calculi*. Bulletin of the Section of Logic 52 (4), pp. 459-495, doi:10.18778/0138-0680.2023.22.

[20] Dale Miller, Gopalan Nadathur, Frank Pfenning & Andre Scedrov (1991): *Uniform proofs as a foundation for logic programming*. Annals of Pure and Applied Logic 51, pp. 125-157, doi:10.1016/0168-0072(91)90068-W.

[21] Sara Negri (2011): *Proof theory for modal logic*. Philosophy Compass 6 (8), pp. 523-538, doi:10.1111/j.1747-9991.2011.00418.x.

[22] Linh Anh Nguyen (2009): *Modal logic programming revisited*. Journal of Applied Non-Classical Logics 19 (2), pp.167-181, doi:10.3166/jancl.19.167-181.

[23] Sergei P. Odintsov & Heinrich Wansing (2010): *Modal logics with Belnapian truth values*. Journal of Applied Non-Classical Logics 20, pp. 279-301, doi:10.3166/JANCL.20.279-301.

[24] Masao Ohnishi & Kazuo Matsumoto (1957): *Gentzen method in modal calculi*. Osaka Mathematical Journal 9, pp. 113-130.

[25] Masao Ohnishi & Kazuo Matsumoto (1959): *Gentzen method in modal calculi II*. Osaka Mathematical Journal 11, pp. 115-120.

[26] Francesca Poggiolesi (2008): *A cut-free simple sequent calculus for modal logic S5*. Review of Symbolic Logic 1 (1), pp. 3-15, doi:10.1017/S1755020308080040.

[27] Francesca Poggiolesi (2010): *Gentzen Calculi for Modal Propositional Logic*. Trends in Logic 32, Springer.

[28] Garrel Pottinger (1983): *Uniform cut-free formulations of T, S4 and S5 (abstract)*. Journal of Symbolic Logic 48, p. 900.

[29] Vaughan R. Pratt (1980): *Application of modal logic to programming*. Studia Logica 39 (2/3), pp. 257-274, doi:10.1007/BF00370324.

[30] Greg Restall (2007): *Proofnets for S5: Sequents and circuits for modal logic*. Proceedings of Logic Colloquium 2005, Lecture Notes in Logic 28, pp. 151-172, Cambridge University Press.

[31] Umberto Rivieccio, Achim Jung & Ramon Jansana (2017): *Four-valued modal logic: Kripke semantics and duality*. Journal of Logic and Computation 27, pp. 155-199, doi:10.1093/LOGCOM/EXV038.

[32] Yi-Dong Shen & Thomas Eiter (2016): *Evaluating epistemic negation in answer set programming*. Artificial Intelligence 237, pp. 115-135, doi:10.1016/J.ARTINT.2016.04.004.

[33] Mitio Takano (1992): *Subformula property as a substitute for cut-elimination in modal propositional logics*. Mathematica japonica 37, pp.1129-1145.

[34] Heinrich Wansing (2002): *Diamonds are a philosopher's best friends*. Journal of Philosophical Logic 31(6), pp. 591-612, doi:10.1023/A:1021256513220.

[35] Heinrich Wansing (2002): *Sequent systems for modal logics*. In: Gabbay, D.M., Guenthner, F. (eds), Handbook of Philosophical Logic 8, pp 61-145.

Nested-sequent Calculus for Modal Logic MB

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Quantum logic (**QL**) is a non-classical logic for analyzing the propositions of quantum physics. Modal logic **MB**, which is a logic that handles the value of the inner product that appears in quantum mechanics, was constructed with the development of **QL**. Although the basic properties of this logic have already been analyzed in a previous study, some essential parts still need to be completed. They are concerned with the completeness theorem and the decidability of the validity problem of this logic. This study solves those problems by constructing a nested-sequent calculus for **MB**. In addition, new logic **MB+** with the addition of new modal symbols is discussed.

1 Introduction

Quantum logic (**QL**) has developed from both quantum physics and mathematical logic aspects since [4]. *Modular lattices* and *orthomodular lattices* have been analyzed as algebraic semantics of **QL**. These lattices are based on a *Hilbert space*, which is the state space of a particle. In quantum mechanics, the value of a physical quantity can only be predicted probabilistically. The absolute value of the *inner product* of two states (two unit vectors in a Hilbert space) is intrinsically related to the probability distribution of the physical quantity.

As counterparts of orthomodular lattice, some *Kripke frames* (binary relation frames) have also been analyzed. In the simplest Kripke frame of **QL**, possible worlds represent states, and the binary relation abstractly represents the *orthogonal relation* between states. Intuitively, on this frame, we can only deal with the binary concept of whether a proposition is 100 % true or not because the orthogonal relation expresses that the inner product between states is zero. Although such logic has developed as an essential foundation for **QL**, developing logic that can handle detailed probability values is also desirable. Because the absolute value of the inner product is independent of the order of the elements, the binary relation is constructed to satisfy *symmetry* in these frames.

Extended quantum logic (**EQL**) [23] has been developed to handle some properties of the absolute value of the inner product. In [23], two logics, **EQL** and **MB**, are constructed. The truth values of the formulas of **EQL** range over the unit interval $I = [0,1]$, which is related to the absolute value of the inner product. **MB** (multi-modal extension of **B**) is the modal logic counterpart of **EQL**. This relation could be regarded as the well-known McKinsey–Tarski translation. In **MB**, the truth value is binary, but the concept of the inner product can be expressed using a modal symbol containing numerical values. This study focuses on **MB**.

Technically, as a relation between states, we can also consider frames that introduce not the absolute value of the inner product but the inner product itself. However, when analyzing the critical factor of probability, a frame that introduces the inner product itself becomes somewhat unnecessarily complex. Therefore, the study of **MB** deals with frames that introduce only absolute values [23]. Other studies have introduced the *transitions* between two states in Hilbert space as a binary relation of the frame. For example, the frame of *dynamic quantum logic* introduces the concepts of *unitary transformations* and *projections* [2]. Each of these has its logical characteristics and has been studied separately.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics. Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 33–47, doi:10.4204/EPTCS.415.7

Although the basic concept of **MB** has already been analyzed in [23], there is room for analysis of the following concepts:

1. In [23], only the Hilbert-style deduction system has been analyzed.

2. There is a mistake in the proof of the completeness theorem in [23] originating from symmetry frames. Furthermore, in [23], the proof of decidability of the validity problem of **MB** is based on the finite model property, which is related to the proof of the completeness theorem. Therefore, it is important to reestablish decidability.

   Here, an overview of the error is provided. In proving the completeness theorem for a Hilbert-style deduction system for modal logic with symmetry frames, the following problem arises. To construct a finite canonical model for modal logic from an unprovable formula A, a set Γ_A consisting of all subformulas of A (and all their negative forms in some cases) is usually constructed. In a canonical model, *consistent* subsets of Γ_A are defined as possible worlds. The binary relation R of a canonical model is defined as follows: $(\Gamma', \Gamma'') \in R$ if for all $\Box B \in \Gamma'$, $B \in \Gamma''$. To show symmetry, we must prove that $(\Gamma'', \Gamma') \in R$ also holds on this definition. The following types of methods are generally used to prove this relation. Suppose $\Box B \in \Gamma''$. From $(\Gamma', \Gamma'') \in R$, $\Box\neg\Box B \notin \Gamma'$. Because $\neg B \to \Box\neg\Box B$ is provable, $\neg B \notin \Gamma'$. Therefore, $B \in \Gamma'$. However, this proof fails as follows. Even if $\Box B \in \Gamma_A$, there is no guarantee of $\Box\neg\Box B \in \Gamma_A$ because $\Box\neg\Box B$ is not a subformula of $\Box B$. This mistake is on page 562, line 12 of [23]. This method works if an infinite set of all formulas, not just subformulas of A, is adopted as Γ_A. (If completeness is all needed, we can change to this infinite model and use the method described in [23] to prove it.) However, that method would make the canonical model infinite, and we could not prove the decidability.

3. **MB** has only the modal comparison symbols. Leaving room for analysis of the modal symbols corresponding to each number. (Details are provided in Section 5.)

To solve these problems, in this study, *nested-sequent calculus* for **MB** that satisfies the *cut-elimination theorem* is constructed, and the cut-free completeness theorem is proved. The decidability of the validity problem of **MB** is shown by using this new calculus. In addition, a nested-sequent calculus for new logic **MB+** (**MB** with new modal symbols) is also constructed.

The concept of nested-sequent were introduced independently in [6] [7] [13] [21]. For logic that satisfies specific properties, using ordinary sequent may be inconvenient. It is well known that in logics involving symmetry frames as semantics (e.g., **S5** and **B**), it is complex to construct the usual sequent calculus that satisfies the cut-elimination theorem. Various developed sequent systems have been proposed to overcome this problem, including nested-sequent (also known as *tree-hypersequent*) and others such as *hypersequent*, and *labelled sequent*. These developed sequents are structures constructed by combining multiple sequents. In many cases, These developmental sequents contain semantic elements. Intuitively, each sequent in nested-sequent or labelled-sequent corresponds to each possible world of a Kripke frame. The nested-sequent have a tree-like structure with the sequents as nodes, which intuitively corresponds to the tree-like part of the Kripke frame. One of the characteristics of tree-like sequents is that it is easy to translate the entire tree-like structure into a single formula by translating sequents into formulas, starting from the leaf sequents in turn. A labelled-sequent uses specific labels to represent each possible world in the Kripke frame. In these developed sequent calculi, when constructing a canonical model, transforming just one sequent ensures that the canonical model does not become an infinite model while preserving conditions such as symmetry. In this study, we employ a nested-sequent, which exhibits relatively manageable properties among these candidates. Studies about these developed sequents are discussed, for example, in [1] [12] [18] [19] [21] [22]. A comparison and summary of these developed sequents are discussed in [17].

In this study, we adopt a development of the usual nested-sequent. In the nested-sequent of standard modal logic, brackets [ ] represent modal concepts of $\Box$. In other words, intuitively, [ ] expresses the difference between possible worlds. This part needs to be developed in nested-sequents for logics that use more complex notions of modality. Because **MB** includes the modal symbol $\Box^d_\alpha$ to concretely express the number α of the absolute value of the inner product, in this study, we use the bracket $[\ ]^d_\alpha$. Except for this difference, almost the same concept as the standard nested-sequent is employed.

In section 2, the basics of **MB** are reviewed. In section 3, the basics of nested-sequent for **MB** are defined. In section 4, a nested-sequent calculus for **MB** is defined, and some theorems are established. In section 5, a nested-sequent calculus for **MB+** is discussed.

Because this study is entirely the result of mathematical logic, a more detailed explanation of the quantum mechanical background of **MB** is omitted. For such an explanation, see [23]. For more detailed explanations of the quantum mechanical background of **QL**, see [2] [3] [8] [9] [10]. For more details about recent studies of sequent calculi and developed sequent systems for **QL**, see, for example, [11] [14] [15] [16] [20].

2 Modal logic MB

This section reviews **MB** defined in [23]. The language of **MB** consists of the following vocabulary:

propositional variables: $p, q, \ldots$

propositional constants: $\top, \bot$

logical connectives: $\neg, \wedge, \Box^c_\alpha, \Box^o_\alpha$ $(\alpha \in J)$

where J is a finite subset of the unit interval $I = [0,1]$ that includes 0 and 1. As in [23], in this study, we assume that J is fixed to one particular set. c stands for "closed", and o stands for "open". These meanings can be seen in the definition of the valuation of formulas in a frame, which will be discussed later.

The formulas of **MB** are defined as follows:

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \Box^c_\alpha A \mid \Box^o_\alpha A \quad (\alpha \in J)$$

Formulas are denoted $A, B, \ldots$, and finite sets of formulas are denoted $\Gamma, \Delta, \Sigma, \ldots$. Elements of $\{c, o\}$ are denoted $d, d', \ldots$. We use the following abbreviations. $A \vee B = \neg(\neg A \wedge \neg B)$, $A \to B = \neg A \vee B$, $\Diamond^c_\alpha A = \neg \Box^c_\alpha \neg A$, $\Diamond^o_\alpha A = \neg \Box^o_\alpha \neg A$.

An *EQL-frame* (S, R) is defined as follows:

S: a non-empty set, an element referred to as a possible world (or physically, a pure quantum state).

R: an I-valued accessibility relation on S, i.e., $R : S \times S \to I$, satisfying the following conditions: $R(s,t) = 1$ iff $s = t$ (reflexivity), $R(s,t) = R(t,s) (\forall s, t \in S)$ (symmetry). (This R represents the absolute value of the inner product between states.)

We write $s(\alpha)t$ for $R(s,t) = \alpha$.

An *MB-realization* is a structure $M = (S, R, P, V)$, where

(S, R) is an EQL-frame.

P is a set of subsets of S, including S and $\emptyset$, being closed under set-theoretic finite intersection, set-theoretic complement relative to S, and the two series of operations $\Box^c_\alpha$, $\Box^o_\alpha$ on a set for each $\alpha \in J$ that are defined as follows:

$\Box^c_\alpha S' \stackrel{\text{def}}{=} \{s \in S | \forall t \in S\ (\alpha \leqq R(s,t) \text{ implies } t \in S')\}$.

$\Box^o_\alpha S' \stackrel{\text{def}}{=} \{s \in S | \forall t \in S\ (\alpha < R(s,t) \text{ implies } t \in S')\}$.

(Although the modal symbols used here as operations on sets are the same as those in the language of **MB**, these are defined independently of the language of **MB**. This concept is introduced to ensure that when dealing with V, the sets of possible worlds are closed in P in the operation of logical connective $\Box^d_\alpha$ [23].)

Valuation V is a map from propositional variables to P.

V is extended inductively as follows:

$V(\top) = S$,

$V(\bot) = \emptyset$,

$V(A \wedge B) = V(A) \cap V(B)$,

$V(\neg A) = V(A)^c$,

$V(\Box^c_\alpha A) = \{s \in S|$ for all $t \in S$, if $\alpha \leqq R(s,t)$, then $t \in V(A)\ \}$,

$V(\Box^o_\alpha A) = \{s \in S|$ for all $t \in S$, if $\alpha < R(s,t)$, then $t \in V(A)\ \}$.

Formula A is *true* at $s \in S$ if $s \in V(A)$ and we write $s \models A$. A is *valid in an* **MB**-*realization* (S,R,P,V) if for all $s \in S$, A is true at s. A is *valid in an EQL-frame* (S,R) if for all P and V, A is valid in (S,R,P,V). A is *valid* if A is valid in all EQL-frames.

3 Nested-sequent

This section defines the basics of the nested-sequent for **MB**.

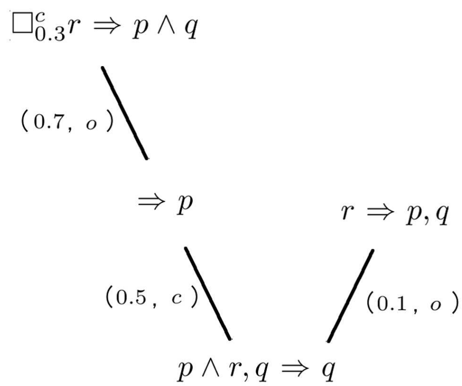
A *sequent* is a structure $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas. A *nested-sequent* is defined inductively as follows:

1. A sequent is a nested-sequent (a tree with only a root).
2. $\Gamma \Rightarrow \Delta, \mathscr{T}$ is a nested-sequent where $\Gamma \Rightarrow \Delta$ is a sequent and $\mathscr{T}$ is a finite set of nested-sequents enclosed in each *modal brackets* $[\ ]^d_\alpha$ where $d \in \{c,o\}$ and $\alpha \in J - \{1\}$.

For example, $p \wedge r, q \Rightarrow q, [\Rightarrow p, [\Box^c_{0.3} r \Rightarrow p \wedge q]^o_{0.7}]^c_{0.5}, [r \Rightarrow p, q]^o_{0.1}$ is a nested-sequent. A nested-sequent can be considered a *tree* structure if the leftmost sequent is regarded as the root, each internal sequent is considered a *node*, and each modal bracket is regarded as an *edge* labelled with (α, d).

A number α *appears* in a nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$ if $\Box^c_\alpha A$ or $\Box^o_\alpha A$ appear in it for some A, or some brackets $[\ ]^d_\alpha$ appear in it. The set $(\Gamma \Rightarrow \Delta, \mathscr{T})_N$ is defined as the set of all nodes of $\Gamma \Rightarrow \Delta, \mathscr{T}$. If the same sequent appears multiple times, they are treated as separate nodes. For example, the first $p \Rightarrow q$ and the last $p \Rightarrow q$ in $p \Rightarrow q, [r \Rightarrow s, [p \Rightarrow q]^o_{0.7}]^c_{0.5}$ are different nodes. The ordered set $(\Gamma \Rightarrow \Delta, \mathscr{T})_J$ is defined as the set of all $\alpha \in J$ that appear in $\Gamma \Rightarrow \Delta, \mathscr{T}$ with 0 and 1. For example, $(p \wedge r, q \Rightarrow q, [\Rightarrow p, [\Box^c_{0.2} r \Rightarrow p \wedge q]^o_{0.7}]^c_{0.5}, [r \Rightarrow p, q]^o_{0.1})_J = \{0, 0.1, 0.2, 0.5, 0.7, 1\}$.

We write $\|\Gamma \Rightarrow \Delta, \mathscr{T}\|$ for the abbreviated nested-sequent in which $\Gamma \Rightarrow \Delta, \mathscr{T}$ appears as a subtree. This expression is used when focusing only on a specific part, $\Gamma \Rightarrow \Delta, \mathscr{T}$, of a nested-sequent. Note that



Example: Tree representation of $p \wedge r, q \Rightarrow q, [\Rightarrow p, [\Box^c_{0.3} r \Rightarrow p \wedge q]^o_{0.7}]^c_{0.5}, [r \Rightarrow p, q]^o_{0.1}$.

even if $\Gamma \Rightarrow \Delta, \mathscr{T}$ appears multiple times in a nested-sequent, when this notation is used, we are focusing on one particular subtree. In a situation in which we focus on a specific $\Gamma \Rightarrow \Delta, \mathscr{T}$ in a nested-sequent $\Gamma' \Rightarrow \Delta', \mathscr{T}'$, we write $\|\Gamma \Rightarrow \Delta, \mathscr{T}\| = \Gamma' \Rightarrow \Delta', \mathscr{T}'$. After writing such an abbreviation, the discussion will proceed, assuming that the abbreviation is fixed. For example, after writing $\|p \Rightarrow q\| = p \Rightarrow q, [r \Rightarrow s]^o_{0.5}, [p \Rightarrow q]^c_{0.3}$ (and if it is determined from the context that $p \Rightarrow q$ refers to the first one), $\|p \Rightarrow q, r\|$ means $p \Rightarrow q, r, [r \Rightarrow s]^o_{0.5}, [p \Rightarrow q]^c_{0.3}$.

For convenience, in the following, we will equate the sequent $\Gamma \Rightarrow \Delta$ with the nested-sequent $\Gamma \Rightarrow \Delta, \emptyset$ that has the empty set of trees. Therefore, if $\|\Gamma \Rightarrow \Delta, \mathscr{T}\|$ is written, $\Gamma \Rightarrow \Delta$ may be a leaf of the tree.

The order $\prec$ on $I \times \{c, o\}$ is defined as follows:

In case of $d = d'$: $(\alpha, d) \prec (\beta, d')$ if $\alpha < \beta$.

In case of $d \neq d'$: $(\alpha, c) \prec (\beta, o)$ if $\alpha \leqq \beta$. $(\beta, o) \prec (\alpha, c)$ if $\alpha > \beta$.

Intuitively, this order represents the inverse of the inclusion relation of the upper closed subsets of I. It is easy to see that this order is total.

We write $(\Gamma \Rightarrow \Delta, \mathscr{T}) \lhd (\Gamma' \Rightarrow \Delta', \mathscr{T}')$ if $\Gamma \Rightarrow \Delta, \mathscr{T}$ is a subtree of $\Gamma' \Rightarrow \Delta', \mathscr{T}'$. In particular, if $\Gamma \Rightarrow \Delta$ is a node of $\Gamma' \Rightarrow \Delta', \mathscr{T}'$, we write $(\Gamma \Rightarrow \Delta) \lhd (\Gamma' \Rightarrow \Delta', \mathscr{T}')$.

An *embedding* of a nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$ in an **MB**-realization (S, R, P, V) is a function $\mathscr{E}$ from $(\Gamma \Rightarrow \Delta, \mathscr{T})_N$ to S that satisfies the following conditions:

If $(\Gamma_1 \Rightarrow \Delta_1, [\Gamma_2 \Rightarrow \Delta_2, \mathscr{T}']^c_\alpha) \lhd (\Gamma \Rightarrow \Delta, \mathscr{T})$ and $R((\mathscr{E}(\Gamma_1 \Rightarrow \Delta_1), (\mathscr{E}(\Gamma_2 \Rightarrow \Delta_2)) = \beta$, then $\alpha \leqq \beta$.

If $(\Gamma_1 \Rightarrow \Delta_1, [\Gamma_2 \Rightarrow \Delta_2, \mathscr{T}']^o_\alpha) \lhd (\Gamma \Rightarrow \Delta, \mathscr{T})$ and $R((\mathscr{E}(\Gamma_1 \Rightarrow \Delta_1), (\mathscr{E}(\Gamma_2 \Rightarrow \Delta_2)) = \beta$, then $\alpha < \beta$.

A nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$ is *false* in an **MB**-realization (S, R, P, V) under $\mathscr{E}$ if for all sequents $\Gamma' \Rightarrow \Delta'$ in $\Gamma \Rightarrow \Delta, \mathscr{T}$, all $A \in \Gamma'$ are true at $\mathscr{E}(\Gamma' \Rightarrow \Delta')$ and all $A \in \Delta'$ are false at $\mathscr{E}(\Gamma' \Rightarrow \Delta')$. A nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$ is *true* in (S, R, P, V) under $\mathscr{E}$ if $\Gamma \Rightarrow \Delta, \mathscr{T}$ is not false in (S, R, P, V) under $\mathscr{E}$. A nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$ is *valid in* (S, R, P, V) if for all $\mathscr{E}$, $\Gamma \Rightarrow \Delta, \mathscr{T}$ is true under $\mathscr{E}$. A nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$ is *valid* if it is valid in all (S, R, P, V).

The *interpretation* τ of a nested-sequent to a formula is defined inductively as follows:

$\tau(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta$.

$$\tau(\Gamma \Rightarrow \Delta, [\Gamma_1 \Rightarrow \Delta_1, \mathscr{T}_1]^{d_1}_{\alpha_1}, \ldots, [\Gamma_n \Rightarrow \Delta_n, \mathscr{T}_n]^{d_n}_{\alpha_n})$$
$$= \tau(\Gamma \to \Delta) \vee \Box^{d_1}_{\alpha_1} \tau(\Gamma_1 \Rightarrow \Delta_1, \mathscr{T}_1) \vee \ldots \vee \Box^{d_n}_{\alpha_n} \tau(\Gamma_n \Rightarrow \Delta_n, \mathscr{T}_n).$$

where $\bigwedge \Gamma$ denotes a formula connecting all the formulas in Γ with $\wedge$, and $\bigvee \Delta$ denotes a formula connecting all the formulas in Δ with $\vee$.

As in the case of other studies of nested-sequent, the following theorem holds.

Theorem 3.1. $\Gamma \Rightarrow \Delta, \mathscr{T}$ is valid iff $\tau(\Gamma \Rightarrow \Delta, \mathscr{T})$ is valid.

Proof. $\tau(\Gamma \Rightarrow \Delta, \mathscr{T})$ generally has the following form:

$$(\bigwedge \Gamma \to \bigvee \Delta) \vee \Box^{d_1}_{\alpha_1}((\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee T^1_1 \vee \ldots \vee T^1_m) \vee \ldots \vee \Box^{d_n}_{\alpha_n}((\bigwedge \Gamma_n \to \bigvee \Delta_n) \vee T^n_1 \vee \ldots \vee T^n_l).$$

Suppose $\Gamma \Rightarrow \Delta, \mathscr{T}$ is false under $\mathscr{E}$. Then, $\bigwedge \Gamma \to \bigvee \Delta$ is false at $\mathscr{E}(\Gamma \Rightarrow \Delta)$. Furthermore, for all $i \in \{1, \ldots, n\}$, $\Gamma_i \Rightarrow \Delta_i$ is false at $\mathscr{E}(\Gamma_i \Rightarrow \Delta_i)$ and $\alpha_i \leqq R(\mathscr{E}(\Gamma \Rightarrow \Delta), \mathscr{E}(\Gamma_i \Rightarrow \Delta_i))$ (if $d_i = c$) or $\alpha_i < R(\mathscr{E}(\Gamma \Rightarrow \Delta), \mathscr{E}(\Gamma_i \Rightarrow \Delta_i))$ (if $d_i = o$). Continuing this procedure up to all leaves of the tree confirms that for all $i \in \{1, \ldots, n\}$ and for each j, $\Box^{d_i}_{\alpha_i}((\bigwedge \Gamma_i \to \bigvee \Delta_i) \vee T^i_1 \vee \ldots \vee T^i_j)$ is false at $\mathscr{E}(\Gamma \Rightarrow \Delta)$. Then, $\tau(\Gamma \Rightarrow \Delta, \mathscr{T})$ is false at $\mathscr{E}(\Gamma \Rightarrow \Delta)$.

Suppose $\tau(\Gamma \Rightarrow \Delta, \mathscr{T})$ is false at $x \in S$. Then $\Gamma \Rightarrow \Delta$ is false at x. Furthermore, for all $i \in \{1, \ldots, n\}$, there exists $x_i \in S$ such that $\Gamma_i \Rightarrow \Delta_i$ is false at x_i and $\alpha_i \leqq R(x, x_i)$ (if $d_i = c$) or $\alpha_i < R(x, x_i)$ (if $d_i = o$). This notion applies inductively to each T^i_j until it reaches the leaves. $\mathscr{E}$ is defined as a function that transfers each sequent to each element that makes it false. That is, $\mathscr{E}(\Gamma \Rightarrow \Delta) = x$, $\mathscr{E}(\Gamma_1 \Rightarrow \Delta_1) = x_1, \ldots$. Then, $\Gamma \Rightarrow \Delta, \mathscr{T}$ is false under $\mathscr{E}$.

□

4 Nested-sequent calculus NSMB

This section discusses the nested-sequent calculus for **MB** that satisfies the cut-elimination theorem. The nested-sequent calculus **NSMB** is defined as follows:

Axioms:

$$\|A \Rightarrow A, \mathscr{T}\| \quad \| \Rightarrow \top, \mathscr{T}\| \quad \|\bot \Rightarrow \mathscr{T}\| \quad \| \Rightarrow \Box^o_1 A, \mathscr{T}\|$$

Rules:

$$\frac{\|\Gamma \Rightarrow \Delta, A, \mathscr{T}\| \quad \|A, \Gamma \Rightarrow \Delta, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta, \mathscr{T}\|} \text{ (cut)}$$

$$\frac{\|\Gamma \Rightarrow \Delta, \mathscr{T}\|}{\|A, \Gamma \Rightarrow \Delta, \mathscr{T}\|} \text{ (wL)} \quad \frac{\|\Gamma \Rightarrow \Delta, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta, A, \mathscr{T}\|} \text{ (wR)} \quad \frac{\|\Gamma \Rightarrow \Delta, A, \mathscr{T}\|}{\|\neg A, \Gamma \Rightarrow \Delta, \mathscr{T}\|} \text{ (}\neg\text{L)} \quad \frac{\|A, \Gamma \Rightarrow \Delta, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta, \neg A, \mathscr{T}\|} \text{ (}\neg\text{R)}$$

$$\frac{\|A, B, \Gamma \Rightarrow \Delta, \mathscr{T}\|}{\|A \wedge B, \Gamma \Rightarrow \Delta, \mathscr{T}\|} \text{ (}\wedge\text{L)} \quad \frac{\|\Gamma \Rightarrow \Delta, A, \mathscr{T}\| \quad \|\Gamma \Rightarrow \Delta, B, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta, A \wedge B, \mathscr{T}\|} \text{ (}\wedge\text{R)}$$

$$\frac{\|\Gamma \Rightarrow \Delta, [A, \Gamma' \Rightarrow \Delta', \mathscr{T}']^{d'}_{\beta}, \mathscr{T}\|}{\|\Box^d_\alpha A, \Gamma \Rightarrow \Delta,\ [\Gamma' \Rightarrow \Delta', \mathscr{T}']^{d'}_{\beta}, \mathscr{T}\|} \text{ (}\Box\text{ L)}^{(1)} \quad \frac{\|A, \Gamma \Rightarrow \Delta, [\Gamma' \Rightarrow \Delta', \mathscr{T}']^{d'}_{\beta}, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta,\ [\Box^d_\alpha A, \Gamma' \Rightarrow \Delta', \mathscr{T}']^{d'}_{\beta}, \mathscr{T}\|} \text{ (}\Box\text{ L sym)}^{(1)}$$

$$\frac{\|A, \Gamma \Rightarrow \Delta, \mathscr{T}\|}{\|\Box^d_\alpha A, \Gamma \Rightarrow \Delta, \mathscr{T}\|} \text{ (}\Box\text{ L self)}^{(2)} \quad \frac{\|A, \Gamma \Rightarrow \Delta, \mathscr{T}\|}{\|\Box^c_0 A, \Gamma' \Rightarrow \Delta', \mathscr{T}'\|} (\Box^c_0)^{(3)}$$

$$\frac{\|\Gamma \Rightarrow \Delta, [\Rightarrow A]^d_\alpha, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta, \Box^d_\alpha A, \mathscr{T}\|} \text{ (}\Box\text{ R)} \quad \frac{\|\Gamma \Rightarrow \Delta, A, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta, \Box^c_1 A, \mathscr{T}\|} \text{ (}\Box\text{ R self)}$$

* In all rules except ($\Box^c_0$), the parts other than those specified parts must be the same at the top and bottom. For example, in ($\neg$ L), the only difference between the upper and lower nested-sequents is the change from $\Gamma \Rightarrow \Delta, A$ to $\neg A, \Gamma \Rightarrow \Delta$ in the stated node $\Gamma \Rightarrow \Delta, A$. In the case of (cut) and ($\wedge$R), this condition is also imposed on the top two sequents. In the case of (cut) and ($\wedge$R), the top two and the bottom one nested-sequents must be the same for all three except for the stated parts.

(1) $(\alpha, d) \preceq (\beta, d')$.

(2) $(\alpha, d) \neq (1, o)$.

(3) This rule erases A from the left of one node in the tree and adds $\Box^c_0 A$ to the left of another arbitrary node of the same tree.

The following deduction is an example of a proof of $A \Rightarrow \Box^c_{0.5} \Diamond^o_{0.3} A$ in **NSMB**.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A, [\Rightarrow]^c_{0.5}}{\neg A, A \Rightarrow [\Rightarrow]^c_{0.5}}\ (\neg\ \mathrm{L})}{A \Rightarrow [\Box^o_{0.3} \neg A \Rightarrow]^c_{0.5}}\ (\Box\ \mathrm{L\ sym})}{A \Rightarrow [\Rightarrow \neg \Box^o_{0.3} \neg A]^c_{0.5}}\ (\neg\ \mathrm{R})}{A \Rightarrow \Box^c_{0.5} \Diamond^o_{0.3} A}\ (\Box\ \mathrm{R})$$

Theorem 4.1 (Soundness theorem for **NSMB**). If $\Gamma \Rightarrow \Delta, \mathscr{T}$ is provable in **NSMB**, then $\Gamma \Rightarrow \Delta, \mathscr{T}$ is valid.

Proof. It is proved by induction on the construction of the proof of nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$. We only show the cases in which the last rule used in the proof is ($\Box$ L) or ($\Box$ R). The proofs for the other cases are simpler. First, we show the case in which the last rule is ($\Box$ L).

Suppose that $\|\Box^d_\alpha A, \Gamma \Rightarrow \Delta,\ [\Gamma' \Rightarrow \Delta', \mathscr{T}']^{d'}_\beta, \mathscr{T}\|$ is false in (S, R, P, V) under embedding $\mathscr{E}$. Then, $\Box^d_\alpha A$ is true at $\mathscr{E}(\Box^d_\alpha A, \Gamma \Rightarrow \Delta)$. From the condition of the rule, $(\alpha, d) \preceq (\beta, d')$.

In the case of $d = d' = c$, from the definition of embedding, $\beta \leqq R((\mathscr{E}(\Box^c_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$. Therefore, $\alpha \leqq R((\mathscr{E}(\Box^c_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$.

In the case of $d = c$ and $d' = o$, from the definition of embedding, $\beta < R((\mathscr{E}(\Box^c_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$. Therefore, $\alpha < R((\mathscr{E}(\Box^c_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$.

In the case of $d = o$ and $d' = c$, from the definition of $\prec$, $\alpha < \beta$. From the definition of embedding, $\beta \leqq R((\mathscr{E}(\Box^o_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$. Therefore, $\alpha < R((\mathscr{E}(\Box^o_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$.

In the case of $d = d' = o$, from the definition of $\prec$, $\alpha < \beta$. From the definition of embedding, $\beta < R((\mathscr{E}(\Box^o_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$. Therefore, $\alpha < R((\mathscr{E}(\Box^o_\alpha A, \Gamma \Rightarrow \Delta), (\mathscr{E}(\Gamma' \Rightarrow \Delta'))$.

Therefore, in any case, A is true at $\mathscr{E}(\Gamma' \Rightarrow \Delta')$, and $\|\Gamma \Rightarrow \Delta,\ [A, \Gamma' \Rightarrow \Delta', \mathscr{T}']^{d'}_\beta, \mathscr{T}\|$ is false under $\mathscr{E}'$ where $\mathscr{E}'$ is exactly the same as $\mathscr{E}$ except that $\mathscr{E}'(\Gamma \Rightarrow \Delta) = \mathscr{E}(\Box^d_\alpha A, \Gamma \Rightarrow \Delta)$ and $\mathscr{E}'(A, \Gamma' \Rightarrow \Delta') = \mathscr{E}(\Gamma' \Rightarrow \Delta')$.

Next, we show the case where the last rule is ($\Box$ R). Suppose that $\|\Gamma \Rightarrow \Delta, \Box^d_\alpha A, \mathscr{T}\|$ is false in (S, R, P, V) under $\mathscr{E}$. Then there exists $s \in S$ such that $\mathscr{E}(\Gamma \Rightarrow \Delta, \Box^d_\alpha A)(\beta)s$, $\alpha \leqq \beta$ (if $d = c$), $\alpha < \beta$ (if $d = o$), and A is false at s. Let $\mathscr{E}'$ be the embedding from $\|\Gamma \Rightarrow \Delta, [\Rightarrow A]^d_\alpha\|$ to (S, R, P, V) such that $\mathscr{E}'(\Rightarrow A) = s$, $\mathscr{E}'(\Gamma \Rightarrow \Delta) = \mathscr{E}(\Gamma \Rightarrow \Delta, \Box^d_\alpha A)$ and $\mathscr{E}' = \mathscr{E}$ for the other sequents. Then, $\|\Gamma \Rightarrow \Delta, [\Rightarrow A]^d_\alpha\|$ is false in (S, R, P, V) under $\mathscr{E}'$. ☐

For the completeness theorem, the contraposition of the theorem is proved. In other words, we show that if a nested-sequent $\Gamma \Rightarrow \Delta, \mathscr{T}$ is not provable in **NSMB**, then an **MB**-realization (S,R,P,V) exists with an embedding $\mathscr{E}$ of $\Gamma \Rightarrow \Delta, \mathscr{T}$ to (S,R,P,V) such that $\Gamma \Rightarrow \Delta, \mathscr{T}$ is false in (S,R,P,V) under $\mathscr{E}$.

Suppose $\Gamma \Rightarrow \Delta, \mathscr{T}$ is not provable. (We assume that $\Gamma \Rightarrow \Delta, \mathscr{T}$ is fixed to one particular nested-sequent to the end of this section.) To construct a model in which $\Gamma \Rightarrow \Delta, \mathscr{T}$ is false, a new nested-sequent $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$ is formed from $\Gamma \Rightarrow \Delta, \mathscr{T}$ by the following iterative procedure. This procedure is continued until the nested-sequent is no longer changed by applying any of the following steps. Changes in the sequent are denoted by $\Gamma_0 \Rightarrow \Delta_0, \mathscr{T}_0 (= \Gamma \Rightarrow \Delta, \mathscr{T}), \Gamma_1 \Rightarrow \Delta_1, \mathscr{T}_1, ..., \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i, \Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} ....$

1. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$ and $A \wedge B \in \Gamma'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1}$ by adding A and B to Γ' of $\Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$. That is, $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|A, B, \Gamma' \Rightarrow \Delta', \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\wedge$L).

2. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$ and $A \wedge B \in \Delta'$, at least one of $\|\Gamma' \Rightarrow \Delta', A, \mathscr{T}'\|$ and $\|\Gamma' \Rightarrow \Delta', B, \mathscr{T}'\|$ is not provable because of the rule ($\wedge$R). Of these, the unprovable one is adopted as $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1}$.

3. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$ and $\neg A \in \Gamma'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|\Gamma' \Rightarrow \Delta', A, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\neg$L).

4. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$ and $\neg A \in \Delta'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|A, \Gamma' \Rightarrow \Delta', \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\neg$R).

5. If $\|\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, $(\alpha, d) \preceq (\beta, d')$, and $\Box^d_\alpha A \in \Gamma'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|\Gamma' \Rightarrow \Delta', [A, \Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\Box$L).

6. If $\|\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, $(\alpha, d) \preceq (\beta, d')$, and $\Box^d_\alpha A \in \Gamma''$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|A, \Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\Box$L sym).

7. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, and $\Box^d_\alpha A \in \Gamma'$ ($(\alpha, d) \neq (1, o)$), then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|A, \Gamma' \Rightarrow \Delta', \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\Box$L self).

8. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, and $\Box^d_\alpha A \in \Delta'$ ($\alpha \neq 1$), then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|\Gamma' \Rightarrow \Delta', [\Rightarrow A]^d_\alpha, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\Box$R). This step is performed once per occurrence of $\Box^d_\alpha A$.

9. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, and $\Box^c_1 A \in \Delta'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|\Gamma' \Rightarrow \Delta', A, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule ($\Box$R self).

10. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \|\Gamma'' \Rightarrow \Delta'', \mathscr{T}''\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, that is, $\Gamma' \Rightarrow \Delta'$ and $\Gamma'' \Rightarrow \Delta''$ are (could be the same) nodes of $\Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, and if $\Box^c_0 A \in \Gamma'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|A, \Gamma'' \Rightarrow \Delta'' \mathscr{T}''\|$. This new nested-sequent is also not provable because of the rule ($\Box^c_0$).

This procedure stops within a finite number of steps for the following reasons:

- The number of nodes and formulas appearing in $\Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$ is always finite.
- All of the procedures decrease the complexity of the formulas.

– Step 8 increases the number of nodes, but it is applied only once at most for one formula. In this procedure, only subformulas of the formulas in the first nested-sequent appear. Therefore, the number of nodes can only increase by a finite amount from the initial nested-sequent.

Let $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$ be the nested-sequent obtained at the end of this procedure, that is not provable. A *canonical model* is constructed from $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$ with the following notion.

We say a ordered set $\mathbb{U}$ is an *interpolated set* of $(\Gamma \Rightarrow \Delta, \mathscr{T})_J$ if it satisfies the following conditions:

1. $(\Gamma \Rightarrow \Delta, \mathscr{T})_J \subset \mathbb{U}$
2. If $\alpha \in (\Gamma \Rightarrow \Delta, \mathscr{T})_J$, $\beta \in (\Gamma \Rightarrow \Delta, \mathscr{T})_J$, $\alpha \neq \beta$, and there is no $\gamma \in (\Gamma \Rightarrow \Delta, \mathscr{T})_J$ that satisfies $\alpha < \gamma < \beta$, then there exists exactly one $\delta \in I$ in $\mathbb{U}$ that satisfies $\alpha < \delta < \beta$.

For example, $\{0, 0.05, 0.1, 0.15, 0.2, 0.4, 0.7, 0.9, 1\}$ is an interpolated set of $\{0, 0.1, 0.2, 0.7, 1\}$. This set is necessary to ensure that all modalities do not affect each other when constructing a canonical model. We write $Suc(\alpha)$ for the successor of element α in an interpolated set with $Suc(1) = 1$.

Let $\mathbb{U}_C$ be a certain interpolated set of $(\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C)_J$. A canonical model (S_C, R_C, P_C, V_C) of $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$ (with $\mathbb{U}_C$) is defined as follows:

$S_C \stackrel{\text{def}}{=} (\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C)_N$

R_C: Defined in the following cases:

(I) If $\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^c_\beta, \mathscr{T}' \lhd \Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$, then $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) \stackrel{\text{def}}{=} \beta$.

(II) If $\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^o_\beta, \mathscr{T}' \lhd \Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$, then $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) \stackrel{\text{def}}{=} Suc(\beta)$.

(III) $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma' \Rightarrow \Delta')) \stackrel{\text{def}}{=} 1$. (Same nodes)

(IV) In all other cases, $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) \stackrel{\text{def}}{=} 0$.

$P_C \stackrel{\text{def}}{=} \{S' \subseteq S \mid \exists A\ V_C(A) = S'\}$

$V_C(p) \stackrel{\text{def}}{=} \{\Gamma' \Rightarrow \Delta' \mid p \in \Gamma'\}$

Lemma 4.2. (S_C, R_C, P_C, V_C) is an **MB**-realization.

Proof. By the definition of R_C, every pair of nodes is associated with a single number. Furthermore, it is only in the case of $s = t$ that $R_C(s,t) = 1$ for the following reasons. From the definition of the bracket in a nested-sequent, if $\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^c_\beta, \mathscr{T}' \lhd \Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$, then $\beta \neq 1$, and if $\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^o_\beta, \mathscr{T}' \lhd \Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$, then $Suc(\beta) \neq 1$, because of the definition of $\mathbb{U}_C$ and $\beta \in (\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C)_J$.

The definition of V for compound formulas corresponds to each condition of P. For example, $V(A \wedge B) = V(A) \cap V(B)$ corresponds to the condition that P is closed under a set-theoretic finite intersection. Therefore, P_C meets the conditions of P. □

The embedding $\mathscr{E}_C$ form $\Gamma \Rightarrow \Delta, \mathscr{T}$ to (S_C, R_C, P_C, V_C) is defined as follows. From the configuration of $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$, all the nodes that existed in $\Gamma \Rightarrow \Delta, \mathscr{T} (= \Gamma_0 \Rightarrow \Delta_0, \mathscr{T}_0)$ also exist in $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$ (but with the added formulas). $\mathscr{E}_C$ is defined as a function that transfers to that "same" node. It can be proved from the composition of $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$ and the definition of R_C that $\mathscr{E}_C$ satisfies the embedding conditions.

Lemma 4.3. If $\mathscr{E}_C(\Gamma' \Rightarrow \Delta') = \Gamma'' \Rightarrow \Delta''$ and $A \in \Gamma' (A \in \Delta')$, then $A \in \Gamma'' (A \in \Delta'')$.

Proof. All steps do not remove formulas in the composition of $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$. Therefore, all formulas present in $\Gamma_0 \Rightarrow \Delta_0, \mathscr{T}_0$ remain in $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$. □

Lemma 4.4. For all $(\Gamma' \Rightarrow \Delta') \in S_C$, if $A \in \Gamma'$, then A is true at $\Gamma' \Rightarrow \Delta' \in S_C$. If $A \in \Delta'$, then A is false at $\Gamma' \Rightarrow \Delta' \in S_C$.

Proof. It is proved by induction on the construction of the formulas in Γ' and Δ'.

- From the definition of V_C, the axiom $\|A \Rightarrow A, \mathscr{T}\|$, and the unprovability of $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$, $(\Gamma' \Rightarrow \Delta') \models p$ if $p \in \Gamma'$ and $(\Gamma' \Rightarrow \Delta') \not\models p$ if $p \in \Delta'$.
- Suppose $A \wedge B \in \Gamma'$. From Step 1, $A \in \Gamma'$ and $B \in \Gamma'$. From the inductive hypothesis, $(\Gamma' \Rightarrow \Delta') \models A$ and $(\Gamma' \Rightarrow \Delta') \models B$. Therefore, $(\Gamma' \Rightarrow \Delta') \models A \wedge B$.
- Suppose $A \wedge B \in \Delta'$. From Step 2, at least one of $A \in \Delta'$ or $B \in \Delta'$ is established. From the inductive hypothesis, $(\Gamma' \Rightarrow \Delta') \not\models A$ or $(\Gamma' \Rightarrow \Delta') \not\models B$. Therefore, $(\Gamma' \Rightarrow \Delta') \not\models A \wedge B$.
- Suppose $\neg A \in \Gamma'$. From Step 3, $A \in \Delta'$. From the inductive hypothesis, $(\Gamma' \Rightarrow \Delta') \not\models A$. Therefore, $(\Gamma' \Rightarrow \Delta') \models \neg A$.
- Suppose $\neg A \in \Delta'$. From Step 4, $A \in \Gamma'$. From the inductive hypothesis, $(\Gamma' \Rightarrow \Delta') \models A$. Therefore, $(\Gamma' \Rightarrow \Delta') \not\models \neg A$.
- Suppose $\Box^c_\alpha A \in \Gamma'$ and $\alpha \neq 0$.

  Suppose $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) = \beta$, and $\alpha \leqq \beta$. If the reason for β is (I), from $(\alpha, c) \preceq (\beta, c)$ and Step 5 or 6, $A \in \Gamma''$. If the reason for β is (II), suppose $\beta = Suc(\beta')$. Then, $(\alpha, c) \preceq (\beta', o)$ is established for the following reason. If $(\beta', o) \prec (\alpha, c)$, then $Suc(\beta') < \alpha$ because $\alpha, \beta' \in (\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C)_J$ and from the definitions of $\prec$ and $\mathbb{U}_C$, $\beta' < Suc(\beta') < \alpha$. In this case, $\beta < \alpha$, which is contrary to the assumption. Therefore, from Step 5 or 6, $A \in \Gamma''$. If the reason for β is (III), $\beta = 1$. From Step 7, $A \in \Gamma''$. From the inductive hypothesis, $(\Gamma'' \Rightarrow \Delta'') \models A$ holds in all cases. Therefore, $(\Gamma' \Rightarrow \Delta') \models \Box^c_\alpha A$.
- Suppose $\Box^c_0 A \in \Gamma'$.

  From Step 10, $A \in \Gamma''$ for all $(\Gamma'' \Rightarrow \Delta'') \in S_C$. From the inductive hypothesis, $(\Gamma'' \Rightarrow \Delta') \models A$ for all $(\Gamma'' \Rightarrow \Delta'') \in S_C$. Therefore, $(\Gamma' \Rightarrow \Delta') \models \Box^c_0 A$.
- Suppose $\Box^o_\alpha A \in \Gamma'$.

  $\Box^o_1 A$ is always true because there is no relation greater than 1.

  Suppose $\alpha \neq 1$, $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) = \beta$, $\alpha < \beta$. If the reason for β is (I), from $(\alpha, o) \preceq (\beta, c)$ and Step 5 or 6, $A \in \Gamma''$. If the reason for β is (II), suppose $\beta = Suc(\beta')$. From $\alpha, \beta' \in (\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C)_J$, $\alpha < Suc(\beta')$, and the definitions of $\mathbb{U}_C$, $\alpha \leqq \beta'$. From $(\alpha, o) \preceq (\beta', o)$ and Step 5 or 6, $A \in \Gamma''$. If the reason for β is (III), $\beta = 1$. From Step 7, $A \in \Gamma''$.

  From the inductive hypothesis, $(\Gamma'' \Rightarrow \Delta'') \models A$ holds in all cases. Therefore, $(\Gamma' \Rightarrow \Delta') \models \Box^o_\alpha A$.
- Suppose $\Box^c_\alpha A \in \Delta'$.

  If $\alpha = 1$, from Step 9, $A \in \Delta'$. From the inductive hypothesis, $(\Gamma' \Rightarrow \Delta') \not\models A$. If $\alpha \neq 1$, from Step 8 and the definition of R_C, there exists $(\Gamma'' \Rightarrow \Delta'') \in S_C$ such that $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) = \alpha$ and $A \in \Delta''$. From the inductive hypothesis, $(\Gamma'' \Rightarrow \Delta'') \not\models A$. Therefore, $(\Gamma' \Rightarrow \Delta') \not\models \Box^c_\alpha A$.
- Suppose $\Box^o_\alpha A \in \Delta'$. $\alpha \neq 1$ because of the axiom, (wL), and (wR). From Step 8 and the definition of R_C, there exists $(\Gamma'' \Rightarrow \Delta'') \in S_C$ such that $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) = Suc(\alpha)$ and $A \in \Delta''$. From the inductive hypothesis, $(\Gamma'' \Rightarrow \Delta'') \not\models A$. Therefore, $(\Gamma' \Rightarrow \Delta') \not\models \Box^o_\alpha A$.

□

Lemma 4.5. $\Gamma \Rightarrow \Delta, \mathscr{T}$ is false in (S_C, R_C, P_C, V_C) under $\mathscr{E}_C$.

Proof. The corollary of Lemma 4.3 and Lemma 4.4. □

Theorem 4.6 (Completeness theorem for **NSMB**)**.** If $\Gamma \Rightarrow \Delta, \mathscr{T}$ is valid, then $\Gamma \Rightarrow \Delta, \mathscr{T}$ is provable in **NSMB**.

Proof. From Lemma 4.5, if $\Gamma \Rightarrow \Delta, \mathscr{T}$ is not provable in **NSMB**, there exists an **MB**-realization (S_C, R_C, P_C, V_C) and an embedding $\mathscr{E}_C$ such that $\Gamma \Rightarrow \Delta, \mathscr{T}$ is false under $\mathscr{E}_C$. □

Theorem 4.7 (Cut-elimination theorem for **NSMB**)**.** If $\Gamma \Rightarrow \Delta, \mathscr{T}$ is provable in **NSMB**, there exists a proof of $\Gamma \Rightarrow \Delta, \mathscr{T}$ that does not include the rule (cut).

Proof. The completeness theorem is proved without the rule (cut). Therefore, the provability of a nested-sequent in **NSMB** does not depend on whether **NSMB** contains (cut). □

The construction of a canonical model stops within a finite number of steps. The discussion does not change in essence if J (and $\mathbb{U}$) is replaced by a suitable total ordered finite set instead of a set of real numbers. Therefore, comparing (α, d) and (β, d') can also be completed in a finite number of steps.

Theorem 4.8 (Finite model property for **MB**)**.** If A is not valid, there exists an **MB**-realization (S, R, P, V) such that S is a finite set and A is not valid in it.

Proof. If A is not valid, the above method could construct a finite canonical model of nested-sequent $\Rightarrow A$. □

Theorem 4.9. The validity problem for **MB** is decidable.

Proof. The corollary of Theorem 4.8. □

5 Nested-sequent calculus NSMB+

From a multi-relational frame point of view, R in a **MB**-realization is regarded as a set of binary relations with the *conditions* such as "If there is a relation α from s to t, then there is no relation β ($\beta \neq \alpha$) from s to t." In general, those binary relations are defined independently. Some ingenuity is required to handle these conditions using formulas. For example, the condition "If there is a relation R' from s to t, then there is no other relation R'' from s to t" cannot be defined as a formula in standard modal logic. (Here, "define" has the same meaning as, for example, $\Box p \to \Box\Box p$ defines the transitivity of a binary relation in a frame of modal logic.) If the conditions of a frame cannot be defined as a formula, some problems may occur when proving the completeness theorem in a Hilbert-style system or a standard sequent system (see [5] for these problems).This problem does not occur in **MB** because it only handles relational operators $\Box^c_\alpha$ and $\Box^o_\alpha$. That is, the following "normal" modal symbols that correspond to only one modality are not included in **MB** (other than $\Box^c_1$).

$V(\Box^{=}_\alpha A) = \{s \in S |$ for all $t \in S$, if $\alpha = R(s,t)$, then $t \in V(A)$ $\}$.

Relational operators make it simple to construct the canonical model. By employing only the maximum value among the numbers that satisfy a specific condition as a binary relation, we can have only one binary relation between any two possible worlds in the canonical model. (See [23] for concrete definitions. As mentioned briefly in the introduction, the completeness theorem of the Hilbert style system in [23] can be proved with this method if the infinite canonical model is acceptable.) However, the above issue arises in a Hilbert-style system or a standard sequent system if $\Box^{=}_{\alpha}$ is added to the language. Therefore, developed sequent becomes intrinsically important to adding $\Box^{=}_{\alpha}$.

Adding $\Box^{=}_{\alpha}A$ to the language of **MB** and constructing a new logic is essential from both a physics and mathematical logic point of view since it broadens the range of expression. Because $V(\Box^{c}_{\alpha}A) = V(\Box^{o}_{\alpha}A \wedge \Box^{=}_{\alpha}A)$ holds, $\Box^{c}_{\alpha}$ can be represented by $\Box^{o}_{\alpha}$ and $\Box^{=}_{\alpha}$, but $\Box^{c}_{\alpha}$ and $\Box^{o}_{\alpha}$ cannot represent $\Box^{=}_{\alpha}$. Therefore, it is desirable to define $\Box^{c}_{\alpha}A$ as an abbreviation of $\Box^{o}_{\alpha}A \wedge \Box^{=}_{\alpha}A$ rather than a primitive formula.

Because $\Box^{c}_{0}$ is a universal modality, it is not directly related to 0-relation, but 0-relation is relevant to $\Box^{=}_{0}$. The definition (IV) of R_C is inappropriate for $\Box^{=}_{0}$ because (IV) is defined independently of occurrence of $\Box^{=}_{0}A$ in the nested-sequent. Therefore, the truth of $\Box^{=}_{0}A$ in the canonical model changes from intention, and the proof of the completeness theorem fails. (Even if we add the concept of 0-relation to embedding, the soundness of (□R) will not be satisfied this time. It is currently unclear how this problem can be resolved if $\Box^{=}_{0}$ is added.) Therefore, we define the formulas of new logic **MB+** by removing all $\Box^{c}_{\alpha}A$ $(\alpha \neq 0)$ from the formulas of **MB** and adding all $\Box^{=}_{\alpha}A$ $(0 < \alpha \leqq 1)$.

Basic definitions for **MB+** are constructed as follows (but we only briefly describe the differences from the **MB** case). The relational symbols ${}^{d}_{\alpha}$ used in the modal symbols and the brackets in nested-sequent are ${}^{=}_{\alpha}(0 < \alpha \leqq 1)$, ${}^{o}_{\alpha}(0 \leqq \alpha \leqq 1)$, and ${}^{c}_{0}$. The definition of embedding is changed by adding the following condition:

If $(\Gamma_1 \Rightarrow \Delta_1, [\Gamma_2 \Rightarrow \Delta_2, \mathscr{T}']^{=}_{\alpha}) \lhd (\Gamma \Rightarrow \Delta, \mathscr{T})$ and $R((\mathscr{E}(\Gamma_1 \Rightarrow \Delta_1), (\mathscr{E}(\Gamma_2 \Rightarrow \Delta_2)) = \beta$, then $\alpha = \beta$.

NSMB+ is defined by changing **NSMB** as follows:

1. (□ R self) is removed, and the following rule is added.

$$\frac{\|\Gamma \Rightarrow \Delta, A, \mathscr{T}\|}{\|\Gamma \Rightarrow \Delta, \Box^{=}_{1}A, \mathscr{T}\|} \ (= \text{R self})$$

2. The conditions (1) and (2) in the annotation of **NSMB** are changed as follows:

   (1) d and d' are $=$, and $\alpha = \beta$, or
   d and d' are o, and $\alpha < \beta$, or
   d is o, d' is $=$, and $\alpha < \beta$.

   (2) d' is $=$ and $\alpha = 1$, or
   d is o and $\alpha \neq 1$.

Theorem 5.1 (Soundness theorem for **NSMB+**). If $\Gamma \Rightarrow \Delta, \mathscr{T}$ is provable in **NSMB+**, then $\Gamma \Rightarrow \Delta, \mathscr{T}$ is valid.

Proof. Almost the same as Theorem 4.1. □

Some procedure for the composition of $\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$ is modified as follows:

5. If $\|\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i, \alpha, \beta, d$ and d' satisfy condition (1) of **NSMB+**, and $\Box^{d}_{\alpha}A \in \Gamma'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|\Gamma' \Rightarrow \Delta', [A, \Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule (□L).

6. If $\|\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, α, β, d and d' satisfy condition (1) of **NSMB+**, and $\Box^d_\alpha A \in \Gamma''$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|A, \Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{d'}_{\beta}, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule (□ L sym).
7. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, α and d satisfy condition (2) of **NSMB+**, and $\Box^d_\alpha A \in \Gamma'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|A, \Gamma' \Rightarrow \Delta', \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule (□ L self).
9. If $\|\Gamma' \Rightarrow \Delta', \mathscr{T}'\| = \Gamma_i \Rightarrow \Delta_i, \mathscr{T}_i$, and $\Box^=_1 A \in \Delta'$, then we construct $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \mathscr{T}_{i+1} = \|\Gamma' \Rightarrow \Delta', A, \mathscr{T}'\|$. This new nested-sequent is also not provable because of the rule (=R self).

For the definition of R_C of the canonical model, the following (I)' is added.

(I)' If $\Gamma' \Rightarrow \Delta', [\Gamma'' \Rightarrow \Delta'', \mathscr{T}'']^{=}_{\beta}, \mathscr{T}' \lhd \Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C$, then $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) \overset{\text{def}}{=} \beta$.

Theorem 5.2 (Completeness theorem for **NSMB+**)**.** If $\Gamma \Rightarrow \Delta, \mathscr{T}$ is valid, then $\Gamma \Rightarrow \Delta, \mathscr{T}$ is provable in **NSMB+**.

Proof. We change some parts of the proof of Lemma 4.4 as follows:

- Suppose $\Box^=_\alpha A \in \Gamma'$ and $\alpha \neq 0$.
  Suppose $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) = \beta$, and $\alpha = \beta$. If the reason for β is (I)', from Step 5 or 6, $A \in \Gamma''$. From the nature of $\mathbb{U}$ and $\alpha = \beta$, there is no case where (II) is the reason for β.
- Suppose $\Box^o_\alpha A \in \Gamma'$.
  Suppose $\alpha \neq 1$, $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) = \beta$, $\alpha < \beta$.
  If the reason for β is (I)', from Step 5 or 6, $A \in \Gamma''$. If the reason for β is (II), suppose $\beta = Suc(\beta')$. From $\alpha, \beta' \in (\Gamma_C \Rightarrow \Delta_C, \mathscr{T}_C)_J$, $\alpha < Suc(\beta')$, and the definitions of $\mathbb{U}_C$, $\alpha \leqq \beta'$. From $\alpha < \beta'$ and Step 5 or 6, $A \in \Gamma''$. If the reason for β is (III), $\beta = 1$. From Step 7, $A \in \Gamma''$.
- Suppose $\Box^=_\alpha A \in \Delta'$.
  If $\alpha = 1$, from Step 9, $A \in \Delta'$. From the inductive hypothesis, $(\Gamma' \Rightarrow \Delta') \not\models A$. If $\alpha \neq 1$, from Step 8 and the definition of R_C, there exists $(\Gamma'' \Rightarrow \Delta'') \in S_C$ such that $R_C((\Gamma' \Rightarrow \Delta'), (\Gamma'' \Rightarrow \Delta'')) = \alpha$ and $A \in \Delta''$.

□

The following theorems can also be proved in the same way as the **NSMB** case.

Theorem 5.3 (Cut-elimination theorem for **NSMB+**)**.** If $\Gamma \Rightarrow \Delta, \mathscr{T}$ is provable in **NSMB+**, there exists a proof of $\Gamma \Rightarrow \Delta, \mathscr{T}$ that does note include the rule (cut).

Theorem 5.4 (Finite model property for **MB+**)**.** If A is not a valid formula of **MB+**, there exists an **MB**-realization (S, R, P, V) such that S is a finite set and A is not valid in it.

Theorem 5.5. The validity problem for **MB+** is decidable.

The definition of interpretation τ is the same as for **MB** (except that d could be $=$).

Theorem 5.6. $\Gamma \Rightarrow \Delta, \mathscr{T}$ is valid iff $\tau(\Gamma \Rightarrow \Delta, \mathscr{T})$ is valid.

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References

[1] A. Avron (1996): *The Method of Hypersequents in the Proof Theory of Propositional Non-classical Logics*. In W. Hodges, M. Hyland, C. Steinhorn & J. Truss, editors: *Logic: from Foundations to Applications: European logic colloquium*, chapter I, Oxford University Press, pp. 1–32, doi:10.1093/oso/9780198538622.003.0001.

[2] A. Baltag & S. Smets (2004): *The Logic of Quantum Programs*. *QPL 2004*, pp. 39–56, doi:10.48550/arXiv.2109.06792.

[3] E. Beltrametti, M. L. Dalla Chiara, R. Giuntini, R. Leporini & G. Sergioli (2014): *A Quantum Computational Semantics for Epistemic Logical Operators. Part I: Epistemic Structures*. *International Journal of Theoretical Physics* 53(10), pp. 3279–3292, doi:10.1007/s10773-013-1642-z.

[4] G. Birkhoff & J. Von Neumann (1936): *The Logic of Quantum Mechanics*. *The Annals of Mathematics* 37(4), pp. 823–843, doi:10.2307/1968621.

[5] P. Blackburn, M. de Rijke & Y. Venema (2001): *Modal Logic*. Cambridge university press, doi:10.1017/CBO9781107050884.

[6] K. Brünnler (2006): *Deep sequent systems for modal logic*. In G. Governatori, I. Hodkinson & Y. Venema, editors: *Advances in Modal Logic*, 6, College Publications, p. 107–119, doi:10.1007/s00153-009-0137-3.

[7] R. A. Bull (1992): *Cut elimination for propositional dynamic logic without **. *Mathematical Logic Quarterly* 38, p. 85–100, doi:10.1002/malq.19920380107.

[8] G. Cattaneo, M. L. Dalla Chiara, R. Giuntini & F. Paoli (2007): *Quantum Logic and Nonclassical Logics*. In Kurt Engesser, Dov M. Gabbay & Daniel Lehmann, editors: *Handbook of Quantum Logic and Quantum Structures: Quantum Structures*, Elsevier Science, doi:10.1016/B978-0-444-52870-4.X5020-1.

[9] M. L. Dalla Chiara (1977): *Quantum Logic and Physical Modalities*. *Journal of Philosophical Logic* 6, pp. 391–404, doi:10.1007/BF00262076.

[10] M. L. Dalla Chiara & R. Giuntini (2002): *Quantum Logics*. In D. M. Gabbay & F. Guenthner, editors: *Handbook Of Philosophical Logic 2nd Edition*, 6, Springer, pp. 129–228, doi:10.1007/978-94-017-0460-1_2.

[11] D Fazio, A Ledda, F Paoli & G St. John. (2023): *A Substructural Gentzen Calculus for Orthomodular Quantum Logic*. *Review of Symbolic Logic* 16(4), pp. 1177–1198, doi:10.1017/S1755020322000016.

[12] D.M. Gabbay (1996): *Labelled Deductive Systems*. Oxford University Press, doi:10.1093/oso/9780198538332.001.0001.

[13] R. Kashima (1994): *Cut-free sequent calculi for some tense logics*. *Studia Logica* 53, p. 119–135, doi:10.1007/BF01053026.

[14] T. Kawano (2018): *Labeled Sequent Calculus for Orthologic*. *Bulletin of the Section of Logic* 47(4), pp. 217–232, doi:10.18778/0138-0680.47.4.01.

[15] T. Kawano (2022): *Sequent Calculi for Orthologic with Strict Implication*. *Bulletin of the Section of Logic* 51(1), pp. 73–89, doi:10.18778/0138-0680.2021.22.

[16] A. Kornell (2023): *A Natural Deduction System for Orthomodular Logic*. *The Review of Symbolic Logic*, p. 1–40, doi:10.1017/S1755020323000229.

[17] B. Lellmann & F. Poggiolesi (2023): *Nested sequent or Tree-hypersequents: A survey*. In R. Padro & Y. Weiss, editors: *Saul Kripke on Modal logic*.

[18] S. Negri (2005): *Proof Analysis in Modal Logic*. *Journal of Philosophical Logic* 34, pp. 507–544, doi:10.1007/s10992-005-2267-3.

[19] S. Negri (2011): *Proof Theory for Modal Logic*. *Philosophy Compass* 6(8), pp. 523–538, doi:10.1111/j.1747-9991.2011.00418.x.

[20] H. Nishimura (1980): *Sequential Method in Quantum Logic*. *The Journal of Symbolic Logic* 45(2), pp. 339–352, doi:10.2307/2273194.

[21] F. Poggiolesi (2009): *The method of tree-hypersequents for modal propositional logic*. In D. Makinson, J. Malinowski & H. Wansing, editors: *Trends in Logic: Towards Mathematical Philsophy*, Springer, pp. 31–51, doi:10.1007/978-1-4020-9084-4_3.

[22] F. Poggiolesi (2010): *Gentzen Calculi for Modal Propositional Logic*. Springer, doi:10.1007/978-90-481-9670-8.

[23] K. Tokuo (2003): *Extended Quantum Logic*. *Journal of Philosophical Logic* 32, pp. 549–563, doi:10.1023/A:1025693128153.

Two Cases of Deduction with Non-referring Descriptions

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Formal reasoning with non-denoting terms, esp. non-referring descriptions such as "the King of France", is still an under-investigated area. The recent exception being a series of papers e.g. by Indrzejczak and Zawidzki. The present paper offers an alternative to their approach since instead of free logic and sequent calculus, it's framed in partial type theory with natural deduction in sequent style. Using a Montague- and Tichý-style formalization of natural language, the paper successfully handles deduction with intensional transitives whose complements are non-referring descriptions, and derives Strawsonian rules for existential presuppositions of sentences with such descriptions.

1 Introduction

In his groundbreaking 1905 paper "On Denoting", Russell [29] offered a widely adopted theory of *(definite) descriptions*, i.e. the *singular terms* of the form "*the F*", the most famous example being "*the King of France*". Russell rightly indicated that

1. Each (definite) description is satisfied by at most one entity. (*Uniqueness*)
2. Descriptions typically involve predicative (some say: descriptive) content. (*Predicativity*)

Which has been generally accepted, cf. e.g. Ludlow [19]. But the true brilliance of Russell's theory lies in its capability to handle even the fact that

3. Some descriptions (e.g. "*the King of France*") are *non-referring*. (*Non-Referring Descriptions*)

However, Russell's own elaboration of *formal semantics* of descriptions became divisive. On one side, many theoreticians praised Russell for paradigmatic philosophical analysis – which states that

(r1) Descriptions have no meaning in isolation, so "*the King of France*" is meaningless *per se*.

(r2) Descriptions only contribute to sentence's meaning by scattered bits such as the meaning of "*F*".

(r3) The sentential meaning of e.g. "*The King of France is bald*" is to be reconstructed in terms of first-order logic with identity as an *existential statement* of the form $\exists x(F(x) \wedge G(x) \wedge \forall y(F(y) \to y = x))$.

While (r2) has rarely been challenged since it obviously matches Point 2, (r3)'s consequence that sentences with descriptions in 'referential positions' (cf. e.g. "*The King of France is an F*.") are definitely true or false (which was seen as an advantage by Russell and some his allies) has been persistently criticized by Strawson [31] and his numerous supporters.

But the clash between Strawson and Russell as regards (r3) overshadows the fact that both Russell and Strawson were followed by many writers (e.g. Tichý [34], Farmer [6], Feferman [7], Indrzejczak and Zawidzki [16]) who *did* adopt Point 3 (neglecting here Strawson's stress on *use* of descriptions). The corresponding area of research is now known as the *logic of non-denoting terms*, or, more generally, as *partial logic*. For an introduction, see e.g. Farmer [6], Feferman [7], or the present author's [17].

Non-denoting terms are in fact ubiquitous in

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Theory and Applications (NCL'24).
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a. (formalized) mathematics, cf. e.g. "$3 \div 0$", "$\sqrt{x}$" (for negative x), "$\lim_{x \mapsto a} f(x)$" (for some values);

b. natural language, cf. e.g. "*the greatest prime*", "*the King of France*";

c. computer science, cf. e.g. abortive halting programs, unsuccessful database searches, etc.

Yet in (philosophical) logic such *partiality phenomena* have been largely abandoned. In particular, many logical textbooks and related writings offer no sufficient discussion of descriptions and simply reiterate Russell's controversial points (r1) and (r3). But once we overview further literature, we find various broadly Strawsonian approaches; they roughly fit the following quadruple of views:

(s1) Descriptions D do have a self-sustaining meaning: either (s1.a) D's meaning is identical with D's reference/denotation, or (s1.b) D's meaning determines D's reference/denotation.

(s2) Sentences with descriptions D in 'referential position' are either (s2.a) implicitly existential claims, or (s2.b) are in no sense existential claims.

Free logic (FL) seems to provide the largest platform for positions revolving mainly on (s2)-topics, cf. e.g. Bencivenga [2]. As repeatedly argued by its proponents, FL delivers desired truth conditions for sentences with descriptions and other singular terms in 'referential position' regardless their actual reference. Some writers, e.g. Farmer [6], Fitting and Mendelssohn [8], follow Frege [9] and Scott [30] and maintain that D refers to a *dummy value* (sometimes denoted $\bot_\tau$ or $*_\tau$), an artificially chosen object either from the 'domain we live in' (sometimes identified with *inner domain*), or some *outer domain*. Some writers at least briefly discuss so induced *existential commitments* (i.e. s2.a), but many (e.g. Blamey [3]) consider dummy values being mere technical devices. On the other hand, some theoreticians, e.g. Lehmann [18], Tichý [35] and also the present writer, rather favour the view that

4. Non-referring descriptions refer to nothing whatsoever (i.e. not to dummy entities). (*Genuine Partiality*)

Whereas Occam's Principle of Parsimony provides a potent argument in favour of such a position.

Another assumption of the present paper, which is now widely adopted in literature, is an overt dismissal of Russell's (r1):

5. Descriptions have meaning even in isolation. (*Descriptions' Meaning*)

As argued on numerous places in literature, in particular by Tichý [32, 35], Montague [20], Fitting and Mendelsohn [8], Indrzejczak and Zawidzki [13, 16], Orlandelli [22], and even the present author [26],

6. The reference of 'empirical' descriptions such as "*the King of France*" is a contingent affair, i.e. the reference of expressions depends on possible worlds and time instants. (*Modality, Temporality*)

Moreover, the present paper relies on arguments developed by Tichý (e.g. [35], Moschovakis [21] and others (incl. the present author's [26, 17]) in favour of the view that

7. Meanings of descriptions are algorithmic computations that determine possible-worlds intensions. (*Algorithmic Meanings*)

Note that Point 7 sustains the *Principle of Compositionality*: the meaning of a compound expression E depends on the meaning of E's parts – regardless their contingent reference (if any).

1.1 Problems addressed in the present paper

So far we have sketched an overall background of our investigation; now it's time for a brief and informal discussion of problems addressed in this paper, indicating also their solution elaborated below.

Problem 1. In his [5], Church published a decisive counter-argument against Russell's theory of descriptions. It employs so-called *intensional transitive verbs* (ITVs) such as "*seek*", cf. e.g. the sentence

"*Ponce de León searched for the Fountain of Youth*".

As correctly observed by Church, and emphasised by Quine in his seminal paper [23], such sentences *lack existential commitment* as regards complements of ITVs. The sought object need not to exist, so we are not allowed to derive that (say) the Fountain of Youth exists. Yet such a fallacious inference is not prevented by Russell's theory (since no discrimination between primary/secondary occurrence of a description can be employed here as in case of propositional attitudes). Which thus presents its fatal flaw.

Church [5] noted that Frege's theory of singular terms is therefore superior to Russell's, since it can reject undesired inferences by pointing out the confusion of reference (*Bedeutung*) and sense (*Sinn*). The distinction was elaborated by Carnap [4] and other adherents of *possible-worlds semantics* (*PWS*) in terms of extensions and possible-worlds *intensions* (i.e. certain functions to extensions). Intensions such as the *individual concept* of the Fountain of Youth figure as complement objects of the relations(-in-intensions) which are meanings of ITVs, cf. Tichý [32, 35], Montague [20], or e.g. [26].

The widely adopted solution, and even the problem itself, is surprisingly entirely missing in recent studies on reasoning with descriptions (cf. e.g. [8, 16]). One of the aims of the present paper is to suggest (on a particular example of a chosen deduction system) that any logical framework adopting *PWS* can successfully cope with Problem 1. Of course, a FL restricted to first-order quantification is not useful here, since adoption of PWS-intensions typically amounts to adoption of *quantification over functions* and so *higher-order logic* HOL – e.g. the *type theory* TT^* [26, 28, 17] utilised below.

Problem 2. For investigation of Problem 1, the logical system TT^* deployed below might be perhaps seen as over-dimensioned. But it's deduction system – a natural deduction in sequent style $\mathsf{ND}_{\mathsf{TT}^*}$, [26, 28, 17, 33] – is a *labelled calculus*, for which esp. Gabbay [10] provided an extensive argumentation. In particular, a part of the present paper shows how labelled (or 'signed') formulas allow to control inference even in cases the formulas being non-denoting expressions. (To avoid misunderstanding: according to the present approach all well-formed expressions always have certain meaning, viz. an algorithmic computation, yet they may lack a reference/denotation.)

Being so equipped, a formal reconstruction of Strawson's [31] 'logic' of *existential presupposition* is possible. We will, for example, derive an exact logical rule of $\mathsf{ND}_{\mathsf{TT}^*}$ that corresponds to Strawson's claim (p. 330) that

> If the sentence "*The King of France doesn't exist*" is false, then the sentence "*The King of France is (not) bald*" is without a truth value.

Albeit such Strawsonian reasoning is considered sound by many linguists and some philosophers of language, its formal reconstruction seems to be entirely missing in logical literature.

Structure of the paper. In Secs. 2 and 3, we expose the *partial type theory* TT^* and a natural deduction system for it, $\mathsf{ND}_{\mathsf{TT}^*}$. In Sec. 4, we first show how to formalize meanings of descriptions and expressions involving them, and how to formally check natural language arguments. We test the proposal against two groups of frequently neglected inferences, namely (a) inferences with intensional transitives (well understood in formal semantics), (b) Strawsonian inferences (rarely reflected in formal logic). Note: though the paper utilises many Tichý's ideas, it also employs numerous ideas developed by the present author, some of them being alien or even contradictory to Tichý's.

2 Partial type theory TT^*

We adopt here Tichý's [33, 35] (see also Moschovakis [21]) idea that expressions of language *express* (or: depict) abstract, structured, not necessarily effective, acyclic *algorithmic computations*, called by Tichý *constructions*. In our construal [26, 17], constructions construct objects – each from a particular *domain* $\mathscr{D}_{\tau^n}$ that interprets the *type* τ^n (see below) – that are different from them. For an illustrative example, "3×1" and "$5 - 2$" express two different (but congruent) constructions, namely $\times(\mathbf{3},\mathbf{1})$ and $-(\mathbf{5},\mathbf{2})$, of the number $\mathtt{3}$. Constructions may aptly serve as *fine-grained meanings* of expressions, while the objects constructed by them serve as their *denotata* (the double-layered semantics is *neo-Fregean* in its spirit):

$$expressions/terms \underset{express}{\longrightarrow} constructions \underset{construct}{\longrightarrow} objects(denotata)$$

Constructing is dependent on *assignment* v and *model* $\mathscr{M}$ (see below), so constructions are said to *v-construct* objects *in* $\mathscr{M}$. Constructions v-constructing other constructions in $\mathscr{M}$ are also allowed. Each *assignment* v (into *frame* $\mathscr{F} \in \mathscr{M}$, see below) is the union of all *total* functions $v_i^{\tau^n}$, one for each τ^n, such that each variable(-as-construction) x_i *ranging over* type τ^n is assigned a τ^n-object $\mathtt{X}_i \in \mathscr{D}_{\tau^n}$. Notation: $v(\vec{\mathtt{X}}/\vec{x})$ abbreviates $v(\mathtt{X}_1/x_1;...;\mathtt{X}_m/x_m)$, which stands for v's $\vec{x}$-*modification* v' such that for each $1 \leq i \leq m$, it assigns a τ_i^n-object $\mathtt{X}_i$ to x_i/τ_i^n.

Some constructions, e.g. $\div(\mathbf{3},\mathbf{0})$, v-construct nothing at all in $\mathscr{M}$, they are called *v-improper in* $\mathscr{M}$; they serve as meanings of *non-denoting expressions*. Two constructions are called *v-congruent in* $\mathscr{M}$, $\cong$, iff they v-construct the same object in $\mathscr{M}$ (examples above), or they are both v-improper in $\mathscr{M}$.

By *functions* we mean here set-theoretical *functions-as-mappings* (graphs, ...), not *functions-as-computations*. Each function $\mathtt{f}$ has a certain domain $\mathscr{D}_{\mathtt{x}}$ of $\mathtt{f}$'s arguments and a (co-)domain $\mathscr{D}_{\mathtt{y}}$ of $\mathtt{f}$'s values; a function $\mathtt{f}$ is called *total* / *partial* iff all / some-but-not-all members of its $\mathscr{D}_{\mathtt{x}}$ are mapped to some members of its $\mathscr{D}_{\mathtt{y}}$. Unlike any total function, each partial function thus *lacks a value* – i.e. it's *undefined* – for at least one of its arguments. Functions-as-computations may be identified with some constructions; some of them are *strict*, so the applications involving them are v-improper in $\mathscr{M}$.

2.1 Language $\mathscr{L}_{\mathsf{TT}^*}$

Constructions are best recorded using λ-notation. Let for any E_i (construction/object/type), $1 \leq i \leq m$, $\vec{E}$ be short for $E_1,...,E_m$, while "$\lambda\vec{x}.$" rather unpacks to "$\lambda x_1...x_m.$". Whenever possible, we employ two languages: (i) an *object language* whose part is e.g. "$\mathtt{X}$", which stands for the object $\mathtt{X}$ (which is often an object that isn't a construction) and (ii) a *meta-language* whose part is e.g. "X", which stands for the construction X of $\mathtt{X}$ (if any). Let $X := Y$ mean that X is defined (takes the form, ...) as Y.

Each construction of TT^* (and so each $\mathscr{L}_{\mathsf{TT}^*}$'s proper expression) is always typed:

Definition 1 (Forms of constructions (and of terms of the language $\mathscr{L}_{\mathsf{TT}^*}$))**.**

	Form of X:	X*'s name:*	X*'s typing rule* X/τ^n*:*
i.	x	*variable*	x/τ^n
ii.	$\ulcorner\mathtt{X}\urcorner$	*acquisition*	$\ulcorner\mathtt{X}\urcorner/\tau^n$; if $\ulcorner\mathtt{X}\urcorner/\tau^n :\neq *^n$, one writes $\mathbf{X}$
iii.	$F(\vec{X})$	*application*	$F(\vec{X})/\tau$, where $X_1/\tau_1^n;...;X_m/\tau_m^n; F/\langle\vec{\tau}^n\rangle\to\tau^n$
iv.	$\lambda\vec{x}.Y$	λ*-abstraction*	$\lambda\vec{x}.Y/\langle\vec{\tau}^n\rangle\to\tau^n$, where $Y/\tau^n; x_1/\tau_1^n;...;x_m/\tau_m^n$

Notes. Auxiliary expressions (note that we are not pedantic as regards quotation marks): $(,),\lambda x.$ and $\ulcorner,\urcorner$; auxiliary brackets: $[,]$. *Acquisitions* $\ulcorner\mathtt{X}\urcorner$ are *primitive* constructions, they are not applications of a certain function to X. Each acquisition $\ulcorner \mathtt{X} \urcorner$ v-constructs X in just one direct construction step of 'delivering' X and leaving it as it is. Acquisitions can be thus seen as 'procedural constants'; *variables* are 'procedural', too. *Applications* $F(\vec{X})$ are 'juxtapositions' of constructions such that if F v-constructs a function f in $\mathscr{M}$ whose argument $\langle \vec{\mathtt{x}} \rangle$ consists of entities v-constructed by $\vec{X}$ in $\mathscr{M}$, and f is defined for $\langle \vec{\mathtt{x}} \rangle$, then the whole application v-constructs $\mathtt{y} := \mathtt{f}(\vec{\mathtt{x}})$ in $\mathscr{M}$. (Irreducibility of m-ary partial functions to unary ones, proved in [33], necessitates $F(\vec{X})$ instead of $F'(X_1...(X_{m-1}(X_m)))$; similarly for types.) Each *abstraction* $\lambda \vec{x}.Y$ v-constructs a function f in $\mathscr{M}$ from m-tuples v-constructed by $\vec{X}$ in $\mathscr{M}$ even on $\vec{x}$-modifications of v, i.e. v', to values that are $v^{(')}$-constructed in $\mathscr{M}$ by abstraction's body Y. See our [17] for an *exact* description of $\mathscr{L}_{\mathsf{TT}^*}$'s semantics.

2.2 Types, orders, frames, models

Typing. Let $\tau^n, \tau_0^n, \vec{\tau}^n$ be *type variables* (in the following sections, "^{n}" will be suppressed) and $o, \iota, *^1, ..., *^n$ be *type constants*. Expressions of $\mathscr{L}_{\mathsf{TT}^*}$, but primarily TT^*'s constructions, are typed via *typing statements* of the form X/τ^n, saying that for any v, the construction X should v-construct an object of type τ^n; X/τ^n is often called a τ^n-*construction*. Notation: $X, Y/\tau^n$ is short for $X/\tau^n; Y/\tau^n$. Examples: x/τ^n; $\div(\mathbf{3},\mathbf{1}), \div(\mathbf{3},\mathbf{0})/\iota$, where ι is interpreted as $\mathbb{R}$; $\mathbf{0},\mathbf{1},\mathbf{3}/\iota$; $\div/\langle \iota,\iota \rangle \to \iota$ (cf. below).

Interpretation of types. Types τ^n are interpreted by sets of objects called *domains* $\mathscr{D}_{\tau^n}$. Members of $\mathscr{D}_{\tau^n}$ are called τ^n-*objects*. Let $\mathscr{T}$ be a set of types for $\mathscr{L}_{\mathsf{TT}^*}$. A *frame* $\mathscr{F} = \{\mathscr{D}_{\tau^n} \mid \tau^n \in \mathscr{T}\}$ consists of all domains that interpret all types in $\mathscr{T}$; each $\mathscr{D}_{\tau^n} \in \mathscr{F}$ contains the equality relation $=^{\tau^n}$ and Σ^{τ^n} (below). A *model* $\mathscr{M}$ is an *interpretation* for $\mathscr{L}_{\mathsf{TT}^*}$, i.e. a couple $\langle \mathscr{F}, \mathscr{I} \rangle$ such that the *interpretation mapping* $\mathscr{I}$ maps acquisitions expressed by $\mathscr{L}_{\mathsf{TT}^*}$'s constants (e.g. "$=^{\tau^n}$") to objects of $\mathscr{F}$ ([17]).

Definition 2 (Types τ^n). Let $1 \leq n \in \mathbb{N}$.

- $\mathscr{B}$ Let $\mathscr{B} = \{o, \iota\}$ be a *type base* for $\mathscr{L}_{\mathsf{TT}^*}$ such that $\mathscr{D}_o = \{\mathtt{T}, \mathtt{F}\}$ (*truth values*; $\mathtt{T} \neq \mathtt{F}$) and $\mathscr{D}_\iota$ are '*entities*' (e.g. $\mathscr{D}_\iota = \mathbb{R}$).
- τ^1 1*st-order types*: (a) each type $\tau^{\mathscr{B}} \in \mathscr{B}$ is a 1*st-order type*, and (b) if $\vec{\tau}^1$ and τ_0^1 are 1st-order types, $\langle \vec{\tau}^1 \rangle \to \tau_0^1$ is also a 1*st-order type*; $\mathscr{D}_{\langle \vec{\tau}^1 \rangle \to \tau_0^1}$ consists of total and partial functions $\mathscr{D}_{\tau_1^1} \times ... \times \mathscr{D}_{\tau_m^1} \to \mathscr{D}_{\tau_0^1}$.
- $*^n$ Let $*^n$ be type such that $\mathscr{D}_{*^n}$ consists of all *nth-order constructions*, i.e. constructions whose subconstructions v-construct (if v-proper) objects in $\mathscr{M}$ of nth-order types.
- τ^{n+1} $(n+1)$*st-order types*: (a) each nth-order type τ^n is an $(n+1)$*st-order type*; (b) the type $*^n$ is an $(n+1)$*st-order type*, and, (c) if $\vec{\tau}^{n+1}$ and τ_0^{n+1} are $(n+1)$st-order types, then $\langle \vec{\tau}^{n+1} \rangle \to \tau_0^{n+1}$ is also an $(n+1)$*st-order type*; $\mathscr{D}_{\langle \vec{\tau}^{n+1} \rangle \to \tau_0^{n+1}}$ consists of total and partial functions $\mathscr{D}_{\tau_1^{n+1}} \times ... \times \mathscr{D}_{\tau_m^{n+1}} \to \mathscr{D}_{\tau_0^{n+1}}$.

Notes. Auxiliary brackets: (,). Types defined in steps (τ^1.b) and (τ^{n+1}.c) are called *function types*, for they're interpreted by domains consisting of m-ary functions. *Sets* of τ-objects, i.e. of $\mathscr{D}_\tau$'s members, are identified with characteristic functions in $\mathscr{D}_{\tau \to o}$; similarly for m-ary *relations*. Domains are pairwise disjoint, except $\mathscr{D}_{*^1} \subset \mathscr{D}_{*^2} \subset ... \subset \mathscr{D}_{*^n}$ (*cumulativity*); there is no greatest order $n \in \mathbb{N}$. Neither *Russell's paradox*, nor e.g. *Russell-Myhill's paradox* about propositions (as identified with o-constructions) is possible in TT^* (cf. [26]). We cannot enjoy the higher orders in this short paper. If X v-constructs X (if any) in $\mathscr{M}$: X/τ^n indicates $\mathtt{X} \in \mathscr{D}_{\tau^n}$ and $X \in \mathscr{D}_{*^n}$.

3 Natural deduction in sequent style, $\mathsf{ND}_{\mathsf{TT}^*}$

$\mathsf{ND}_{\mathsf{TT}^*}$, which we borrow and slightly adjust from [17, 28], stems from Tichý's systems [33, 34] for his partial TT. It's essentially an ND *in sequent style*, but with '*signed formulas*', so it's a kind of *labelled calculi*, cf. Gabbay [10]. In Kuchyňka and Raclavský [17], *Henkin-completeness* of $\mathsf{ND}_{\mathsf{TT}^*}$, and thus the *higher-order logic* (HOL) we apply here, w.r.t. an exact semantics of $\mathscr{L}_{\mathsf{TT}^*}$ is proved in details.

3.1 Matches, sequents and derivation rules

$\mathsf{ND}_{\mathsf{TT}*}$'s *rules* $\mathtt{R}$ are made from sequents, while *sequents* $\mathtt{S}$ are made from $\mathsf{ND}_{\mathsf{TT}*}$'s *statements* called *matches* $\mathtt{M}$. Here are three motivations a.–c. for introducing matches.

a. Each $\mathtt{M}$ states *v-congruence* in $\mathscr{M}$ of a certain (typically compound) construction X with a (typically simple) variable or acquisition $\mathbf{x}$. So the best notation for $\mathtt{M}$ would be $X \cong \mathbf{x}$, where $\cong$ is the *strong equality* operator (it holds even if X and $\mathbf{x}$ are both v-improper $\mathscr{M}$), which indicates the 'equational character' of the system. We rather write $X :^{\tau} \mathbf{x}$, which displays the type τ of each of X and $\mathbf{x}$ and underlines that matches present *signed formulas*. As signed formulas, matches obviously increase the deduction power of $\mathsf{ND}_{\mathsf{TT}^*}$; to illustrate, from $\supset(\varphi,\psi) :^{o} \mathbf{F}$ one deduces e.g. $\psi :^{o} \mathbf{F}$. The term "match" is of course auxiliary and our above explanation admittedly specific: "φ true" or "T : φ" (both saying 'the formula φ has the value True', which is encoded even by our $\varphi :^{o} \mathbf{T}$) are a familiar and ubiquitous concept in most (if not all) computer-science-related writings on natural deduction and was first employed in semantic tableaux method.

b. The use of signed formulas is especially fruitful when dealing with partiality. Let $\bot^{\tau}/\tau$ be any v-improper τ-construction; "$^{\tau}$" will usually be suppressed. $\bot^{\tau}$ may perhaps seem to play a role of so-called *dummy value* (or *null value*) known from algebraic approaches of e.g. FL by Scott [30]. But there is a crucial difference: Scott and many others use *denotational semantics* in which something (namely the dummy value) must interpret a non-denoting expression, otherwise it's meaningless (just as non-well-formed expressions); in the *procedural semantics* followed in this paper, however, a non-denoting (well-formed) expression lacks denotation (reference), but expresses as its *meaning* a specific improper construction $\bot^{\tau}$. To illustrate such matches, let $\mathbf{3},\mathbf{0}/\iota$ (the numbers-as-objects $3,0$), $\div/\langle\iota,\iota\rangle\!\rightarrow\!\iota$ (the familiar division mapping): the match $\div(\mathbf{3},\mathbf{0}) :^{\iota} \bot$ says that the two constructions flanking $:^{\iota}$ are v-congruent in $\mathscr{M}$ (for they are both v-improper); note that we do not postulate a 'dummy number' in our ontology that is allegedly computed by $\div(\mathbf{3},\mathbf{0})$.

c. Last but not least, the *monotonicity* of $\vDash$ is preserved, for each $\mathtt{M}$ definitely either holds, or not. Then the following situation of common partial logics, criticised by Blamey [3], is excluded: let $\sim$ be the familiar function of negation; if φ and so even $\sim\varphi$ have the value $\bot$, and $\varphi \vDash \psi$, then $\sim\psi \nvDash \sim\varphi$.

i. Matches. Let $X, x, \mathbf{X}, \ulcorner \mathtt{X} \urcorner/\tau$. Matches split into two types, a. and b. Each of three a.-type matches

$$\text{a.} \qquad \mathtt{M} := X :^{\tau} \mathbf{X} \mid X :^{\tau} \ulcorner \mathtt{X} \urcorner \mid X :^{\tau} x$$

says that X is v-proper in $\mathscr{M}$. Notation: $X :^{\tau} \mathbf{x}$ represents any a.-type matches. Each b.-type match

$$\text{b.} \qquad \mathtt{M} := X :^{\tau} \bot$$

says that X is v-improper in $\mathscr{M}$. Notation: $X :^{\tau} \underline{\mathbf{x}}$ covers variants $X :^{\tau} \mathbf{x}$ and $X :^{\tau} \bot$. An assignment v *satisfies* $X :^{\tau} \underline{\mathbf{x}}$ in $\mathscr{M}$ iff $X \cong \underline{\mathbf{x}}$ in $\mathscr{M}$.

ii. Sequents. A *sequent*

$$\mathtt{S} := \Gamma \longrightarrow \mathtt{M}$$

may be seen as a couple consisting of a finite *set* (not multiset) Γ of matches and a match $\mathtt{M}$ that follows from Γ. $\mathtt{S}$ is *valid* in $\mathscr{M}$ iff every v that satisfies all members of Γ in $\mathscr{M}$ also satisfies $\mathtt{M}$ in $\mathscr{M}$. Notation: where Δ is a set of matches, $\Gamma,\Delta \longrightarrow \mathtt{M}$ abbreviates $\Gamma \cup \Delta \longrightarrow \mathtt{M}$; $\Gamma,\mathtt{M} \longrightarrow \mathtt{M}$ abbreviates $\Gamma \cup \{\mathtt{M}\} \longrightarrow \mathtt{M}$.

iii. Rules. A *(derivation) rule* $\vec{\mathtt{S}} \vdash \mathtt{S}$, is a validity-preserving operation on sequents, usually written

$$\mathtt{R} := \frac{\vec{\mathtt{S}}}{\mathtt{S}},$$

where $\vec{\mathtt{S}}$ are its *premisses*, $\mathtt{S}$ its *conclusion*. Each $\mathtt{R}$ says that $\mathtt{S}$ is valid in all models in which $\vec{\mathtt{S}}$ are valid.

Let H be an arbitrary *set of sequents*. A finite *sequence S of sequents*, each member of which being either a member of H, or the result of the application of a rule from a *set of rules R* to some preceding members of S or members of H is called a *derivation* $\mathtt{D}$ *of S*'s last sequent $\mathtt{S}$ from H. $\mathtt{D}$ is also called in brief *proof* and (numbered) members of S are called *steps*. $H \vdash \mathtt{S}$ presents a *derived rule*.

3.2 $\mathsf{ND}_{\mathsf{TT}^*}$'s derivation rules

The rules of $\mathsf{ND}_{\mathsf{TT}^*}$ may be divided into four groups: i. *structural rules*, ii. *form rules*, iii. *operational rules* and iv. *rules for extralogical constants*. The i.-type rules present general properties of validity, the ii.-type rules present properties of validity w.r.t. forms of constructions. The iii.-type rules make TT^* a HOL.

Even a cursory inspection of the i.- and ii.-type rules reveals that they rather resemble rules familiar from ND for modern STT, compare e.g. Hindley and Seldin [11] and $\mathsf{ND}_{\mathsf{TT}^*}$'s rules (AX), (WR), (CUT) (see Def. 3 below). Those NDs usually utilise sequents of the form $\Gamma \longrightarrow t : \tau$, in which term t is typed by τ, while we use $\Gamma \longrightarrow X :^{\tau} \underline{\mathbf{x}}$ to the same effect. Nevertheless, labelling X by $\underline{\mathbf{x}}$ (cf. below) for the reasons stated above gives rise to a few new rules; in Def. 3, see esp. (EXH). Deduction systems STT by Beeson [1], Feferman [7] and Farmer [6] are not sequent-style ones as $\mathsf{ND}_{\mathsf{TT}^*}$ is, so their encoding mechanisms differ. To illustrate, the fact that both variables and constants always denote is expressed by their axioms $x_\tau \downarrow$ (where $\downarrow$ reads 'is denoting') and $c_\tau \downarrow$, while $\mathsf{ND}_{\mathsf{TT}^*}$ uses (TM) (cf. Def. 4) for both; similarly for $\lambda x_\tau.t \downarrow$ and our (λ-INST) (cf. Def. 4).

Notational agreement (holding unless stated otherwise). Let the following symbols be any: $\mathtt{M}_{(i)}$ – match; $\mathtt{S}_{(i)}$ – sequent; Γ (or Δ) – set of matches; $x_{(i)}, y, f, g$ – variables; $\mathbf{x}_{(i)}, \mathbf{y}, \mathbf{f}, \mathbf{g}$ – acquisitions/variables; $X_{(i)}, Y, F$ – constructions. The constructions fit types as follows: $X, Y, \mathbf{x}, \mathbf{y}/\tau; \mathbf{x}_1, X_1/\tau; ...; \mathbf{x}_m, X_m/\tau; F, \mathbf{f}, \mathbf{g}/\langle\vec{\tau}\rangle \rightarrow \tau$; let ϕ abbreviate $\langle\vec{\tau}\rangle \rightarrow \tau$. *Conditions* of each relevant $\mathtt{R}$ typically include: (i) the variables occurring within $\mathtt{R}$ are pairwise distinct and (ii) they are not free in $\Gamma, \mathtt{M}$ and other constructions occurring in $\mathtt{R}$.[1] Let $Y_{[X/x]}$ stand for the construction Y in which *free* occurrences of x are *substituted* by X, as defined in [17]; steps differing by rewriting terms on the basis of substitution are suppressed.

Definition 3 (Structural rules)**.** For informal description of (nearly all) our rules, see [26]. (AX) is the *axiom rule*; (WR) is the *weakening rule*; (CUT) is the *deletional cut rule* (cf. [10]); (EFQ) is the *ex falso/contradictione quodlibet rule*. (EXH) is the *exhaustation rule* – it says that if the assumptions that X is / is not v-proper are needed for $\mathtt{M}$'s following from Γ, then $\mathtt{M}$ follows from Γ independently of the assumptions.

$$\frac{}{\Gamma, \mathtt{M} \longrightarrow \mathtt{M}} \text{(AX)} \qquad \frac{\Gamma \longrightarrow \mathtt{M}_1 \qquad \Gamma, \mathtt{M}_1 \longrightarrow \mathtt{M}_2}{\Gamma \longrightarrow \mathtt{M}_2} \text{(CUT)} \qquad \frac{\Gamma \longrightarrow \mathtt{M}}{\Gamma, \Delta \longrightarrow \mathtt{M}} \text{(WR)}$$

$$\frac{\Gamma \longrightarrow \mathtt{M}_1 \qquad \Gamma \longrightarrow \mathtt{M}_2}{\Gamma \longrightarrow \mathtt{M}} \text{(EFQ)} \qquad \frac{\Gamma, X :^{\tau} \bot \longrightarrow \mathtt{M} \qquad \Gamma, X :^{\tau} x \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}} \text{(EXH)}$$

Condition (EFQ): $\mathtt{M}_1$ and $\mathtt{M}_2$ are *patently incompatible* – they are either of the forms $X :^{\tau} \mathbf{x}$ and $X :^{\tau} \bot$, or of the forms $X :^{\tau} \mathbf{x}_1$ and $X :^{\tau} \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ acquire distinct objects $\mathtt{X}_1$ and $\mathtt{X}_2$. Patently incompatible matches are never satisfied (in $\mathscr{M}$) by the same v.

[1] x is called *free in* $\mathtt{M}$ of the form $X :^{\tau} \underline{\mathbf{x}}$ iff it's free in X or $\mathbf{x}$; x is called *free in* Γ iff it's free at least in one $\mathtt{M}_i \in \Gamma$.

Definition 4 (Form rules)**.** The sense of (TM), the *trivial match rule*, and (λ-INST), the *λ-instantiation rule*, was indicated above. The rest of the form rules govern applications that are *v*-proper. The *β-conversion rules* (β-CON) (*contraction r.*) and (β-EXP) (*expansion r.*) are very important, while the rules for *substitution in applications* (a-SUB) are very useful, too; (EXT) is the *extensionality rule*.

$$\frac{}{\Gamma \longrightarrow \mathbf{x} :^{\tau} \mathbf{x}}\ \text{(TM)} \qquad \frac{\Gamma, \lambda\vec{x}.Y :^{\phi} f \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ (\lambda\text{-INST}) \qquad \frac{\Gamma \longrightarrow [\lambda\vec{x}.Y](\vec{X}) :^{\tau} \mathbf{y}}{\Gamma \longrightarrow Y_{(\vec{X}/\vec{x})} :^{\tau} \mathbf{y}}\ (\beta\text{-CON})$$

$$\frac{\Gamma \longrightarrow Y_{(\vec{X}/\vec{x})} :^{\tau} \mathbf{y} \qquad \Gamma \longrightarrow X_1 :^{\tau_1} \mathbf{x}_1 \qquad \ldots \qquad \Gamma \longrightarrow X_m :^{\tau} \mathbf{x}_m}{\Gamma \longrightarrow [\lambda\vec{x}.Y](\vec{X}) :^{\tau} \mathbf{y}}\ (\beta\text{-EXP})$$

$$\frac{\Gamma \longrightarrow F(\vec{X}) :^{\tau} \mathbf{y} \qquad \Gamma \longrightarrow X_1 :^{\tau_1} \mathbf{x}_1 \qquad \ldots \qquad \Gamma \longrightarrow X_m :^{\tau_m} \mathbf{x}_m}{\Gamma \longrightarrow F(\vec{\mathbf{x}}) :^{\tau} \mathbf{y}}\ \text{(a-SUB.i)}$$

$$\frac{\Gamma \longrightarrow F(\vec{\mathbf{x}}) :^{\tau} \mathbf{y} \qquad \Gamma \longrightarrow X_1 :^{\tau_1} \mathbf{x}_1 \qquad \ldots \qquad \Gamma \longrightarrow X_m :^{\tau_m} \mathbf{x}_m}{\Gamma \longrightarrow F(\vec{X}) :^{\tau} \mathbf{y}}\ \text{(a-SUB.ii)}$$

$$\frac{\Gamma \longrightarrow F(\vec{X}) :^{\tau} \mathbf{y} \qquad \Gamma, F :^{\phi} f, X_1 :^{\tau_1} x_1, \ldots, X_m :^{\tau_m} x_m \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ \text{(a-INST)}$$

$$\frac{\Gamma, \mathbf{f}(\vec{x}) :^{\tau} y \longrightarrow \mathbf{g}(\vec{x}) :^{\tau} y \qquad \Gamma, \mathbf{g}(\vec{x}) :^{\tau} y \longrightarrow \mathbf{f}(\vec{x}) :^{\tau} y}{\Gamma \longrightarrow \mathbf{g} :^{\tau} \mathbf{f}}\ \text{(EXT)}$$

$$\frac{\Gamma \longrightarrow F :^{\phi} f \qquad \Gamma, X_1 :^{\tau_1} x_1; \ldots; X_m :^{\tau_m} x_m \longrightarrow \mathtt{M}}{\Gamma \longrightarrow F(\vec{X}) :^{\tau} \bot}\ (\text{a-IMP}^{\bot}) \qquad \text{Condition: except } \tau,\ \phi :\neq \langle\vec{\tau}\rangle \to \tau.$$

TT^* employs the following familiar functions-as-mappings: the *negation* $\sim$ maps $\mathtt{T}$ to $\mathtt{F}$ and vice versa; the *material conditional* $\supset$ maps $\langle\mathtt{T},\mathtt{F}\rangle$ to $\mathtt{F}$ but $\langle\mathtt{T},\mathtt{T}\rangle$, $\langle\mathtt{F},\mathtt{T}\rangle$, $\langle\mathtt{F},\mathtt{F}\rangle$ to $\mathtt{F}$; the *universal quantifier* Π^{τ} maps the function $\mathscr{D}_{\tau} \to \mathscr{D}_{o}$ that assigns $\mathtt{T}$ to all τ-objects to $\mathtt{T}$, but all other functions $\mathscr{D}_{\tau} \to \mathscr{D}_{o}$ to $\mathtt{F}$; the *existential quantifier* Σ^{τ} (irreducible to Π^{τ}, [28]) maps each function $\mathscr{D}_{\tau} \to \mathscr{D}_{o}$ that assigns $\mathtt{T}$ to at least one τ-object to $\mathtt{T}$, but all other functions $\mathscr{D}_{\tau} \to \mathscr{D}_{o}$ to $\mathtt{F}$; the *identity relation* $=^{\tau}$ maps each couple pairing the same τ-object to $\mathtt{T}$, but couples pairing different τ-objects to $\mathtt{F}$; the *singularization* (or *iota*) *function* $\imath^{\tau}$ maps each function $\mathscr{D}_{\tau} \to \mathscr{D}_{o}$ that assigns $\mathtt{T}$ to just one τ-object to that τ-object, and is undefined for all other functions $\mathscr{D}_{\tau} \to \mathscr{D}_{o}$. Their acquisitions $\boldsymbol{\sim}, \boldsymbol{\supset}, \boldsymbol{\Pi}^{\tau}, \boldsymbol{\Sigma}^{\tau}, \boldsymbol{=}^{\tau}, \boldsymbol{\imath}^{\tau}$ are governed by the following rules.

Definition 5 (Operational rules)**.** Specifying Def. 1 (point iii): $\mathbf{T}, \mathbf{F}, \mathbf{o}, \mathbf{o}', O, O'/o; \boldsymbol{\sim}/o \to o; \boldsymbol{\supset}/\langle o,o\rangle \to o$; $\boldsymbol{\Pi}^{\tau}, \boldsymbol{\Sigma}^{\tau}/(\tau \to o) \to o; \boldsymbol{=}^{\tau}/\langle\tau,\tau\rangle \to o; \boldsymbol{\imath}^{\tau}/(\tau \to o) \to \tau; \bot^{\tau}/\tau; \mathbf{c}, C/\tau \to o$ ('class').

$$\frac{\Gamma, \mathbf{o} :^{o} \mathbf{o}' \longrightarrow \mathtt{M}_1 \qquad \Gamma, \mathbf{o} :^{o} \mathbf{o}' \longrightarrow \mathtt{M}_2}{\Gamma \longrightarrow \boldsymbol{\sim}(\mathbf{o}) :^{o} \mathbf{o}'}\ (\sim\text{-I}) \qquad \textit{Condition } (\sim\text{-I}): \mathtt{M}_1 \text{ and } \mathtt{M}_2 \text{ are patently incompatible.}$$

$$\frac{\Gamma, \mathbf{o} :^{o} \mathbf{T} \longrightarrow \mathtt{M} \qquad \Gamma, \mathbf{o} :^{o} \mathbf{F} \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ \text{(RA)} \qquad \frac{\Gamma, \boldsymbol{\sim}(\mathbf{o}) :^{o} o \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ (\sim\text{-INST})$$

$$\frac{\Gamma, \mathbf{o} :^{o} \mathbf{T} \longrightarrow \mathbf{o}' :^{o} \mathbf{T}}{\Gamma \longrightarrow \boldsymbol{\supset}(\mathbf{o}, \mathbf{o}') :^{o} \mathbf{T}}\ (\supset\text{-I}) \quad \frac{\Gamma \longrightarrow \boldsymbol{\supset}(O, O') :^{o} \mathbf{T} \quad \Gamma \longrightarrow O :^{o} \mathbf{T}}{\Gamma \longrightarrow O' :^{o} \mathbf{T}}\ (\supset\text{-E}) \quad \frac{\Gamma, \boldsymbol{\supset}(\mathbf{o}, \mathbf{o}') :^{o} o \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ (\supset\text{-INST})$$

$$\frac{\Gamma \longrightarrow C(X) :^{o} \mathbf{T}}{\Gamma \longrightarrow \boldsymbol{\Sigma}^{\tau}(C) :^{o} \mathbf{T}}\ (\Sigma\text{-I}) \quad \frac{\Gamma \longrightarrow \boldsymbol{\Sigma}^{\tau}(C) :^{o} \mathbf{T} \quad \Gamma, C(\mathbf{x}) :^{o} \mathbf{T} \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ (\Sigma\text{-E}) \quad \frac{\Gamma, \boldsymbol{\Sigma}^{\tau}(\mathbf{c}) :^{o} o \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ (\Sigma\text{-INST})$$

$$\frac{\Gamma \longrightarrow C(x){:}^{o}\,\mathbf{T}}{\Gamma \longrightarrow \mathbf{\Pi}^{\tau}(C){:}^{o}\,\mathbf{T}}\ (\Pi\text{-I}) \qquad \frac{\Gamma \longrightarrow \mathbf{\Pi}^{\tau}(C){:}^{o}\,\mathbf{T}}{\Gamma \longrightarrow C(\mathbf{x}){:}^{o}\,\mathbf{T}}\ (\Pi\text{-E}) \qquad \frac{\Gamma, \mathbf{\Pi}^{\tau}(\mathbf{c}){:}^{o}\,o \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ (\Pi\text{-INST})$$

$$\frac{\Gamma \longrightarrow X{:}^{\tau}\,\mathbf{x}}{\Gamma \longrightarrow =^{\tau}(X,\mathbf{x}){:}^{o}\,\mathbf{T}}\ (=\text{-I}) \quad \frac{\Gamma \longrightarrow =^{\tau}(X,\mathbf{x}){:}^{o}\,\mathbf{T}}{\Gamma \longrightarrow X{:}^{\tau}\,\mathbf{x}}\ (=\text{-E}) \quad \frac{\Gamma, =^{\tau}(\mathbf{x},\mathbf{y}){:}^{o}\,o \longrightarrow \mathtt{M}}{\Gamma \longrightarrow \mathtt{M}}\ (=\text{-INST})$$

$$\frac{\Gamma \longrightarrow C(\mathbf{x}){:}^{o}\,\mathbf{T} \qquad \Gamma, C(y){:}^{o}\,\mathbf{T} \longrightarrow y{:}^{\tau}\,\mathbf{x}}{\Gamma \longrightarrow \boldsymbol{\imath}^{\tau}(C){:}^{\tau}\,\mathbf{x}}\ (\imath\text{-I}) \qquad \frac{\Gamma \longrightarrow \boldsymbol{\imath}^{\tau}(C){:}^{\tau}\,\mathbf{x}}{\Gamma \longrightarrow C(\mathbf{x}){:}^{o}\,\mathbf{T}}\ (\imath\text{-E})$$

$$\frac{\Gamma, \boldsymbol{\imath}^{\tau}(C){:}^{\tau}\,\mathbf{x} \longrightarrow \mathtt{M} \qquad \Gamma \longrightarrow C(\mathbf{x}){:}^{o}\,o}{\Gamma \longrightarrow \mathtt{M}}\ (\imath\text{-INST})$$

Notes on operational rules. There is a difference between (i) e.g. $\mathbf{o}$, which is an acquisition or variable – in both cases an always v-proper o-construction, and (ii) O, which is any form of o-constructions – which needn't be v-proper if an application occurs in the place of O. Note then that all INST-rules require $\mathbf{o}$ (etc.) being a v-proper construction (in systems that do not employ partial functions, or, more precisely, improper constructions, INST-rules are not needed). Rules such as ($\supset$-E) or (Π-E) omit the condition only seemingly: the condition is imposed on O, O' or C through the fact that $\supset(O, O')$ or $\mathbf{\Pi}^{\tau}(C)$ are v-proper (they v-construct $\mathtt{T}$), hence their subconstructions O, O' and C must be v-proper, too. Most of the operational rules have a straightforward reading; for example, ($\supset$-E) says that if both an implication and its antecedent are true, then we may conclude that its consequent is also true; (Π-E) says that if the 'higher-order concept' `All` applies to a set (some say: class) `C` of τ-objects, then we may conclude that *any* τ-object `x` falls in `C`. All these rules, incl. (RA), *the redundant assumption rule*, occur in Tichý's [34] but without any comment or explanation. But we add here $\imath$-rules (commented below), i.e. the rules for the *iota operator* $\boldsymbol{\imath}^{\tau}$ first proposed by the present author in [28].

Notes on $\boldsymbol{\imath}^{\tau}$ *rules*. The $\imath$-operator is *prima facie* not 'defined' in terms of $\exists, \forall, =$ as in standard approaches, cf. e.g. Russell's contextual introduction of $\imath$-operator in [37], $G(\imath x.F(x)) := \exists x((\forall y(F(y) \leftrightarrow y = x) \wedge G(x))$ (notation adjusted). But a conscientious eye quickly reveals that Russell's $y = x$ is encoded by our $y{:}^{\tau}x$ (reread our informal description of matches in 3.1.a). A version of the rule ($\imath$-I) with the match $=^{\tau}(y,\mathbf{x}){:}^{o}\,\mathbf{T}$ instead of $y{:}^{\tau}x$ is easily derivable using the (=-I) rule. Russell's $\forall y$ is encoded by our 'any' y (again, deploy (Π-I) to obtain a version of the rule in which Π^{τ}, corresponding to $\forall$, is explicit). Only Russell's $\exists$, the operator of 'ontological existence', is not immediately recoverable, ($\imath$-I) thus retains the well-known oscillation between generic/maximality and particular/existential readings of descriptions, cf. e.g. [19]. But if certain conditions related to $\mathbf{x}$ are met, (Π-E) and (Σ-I) allow us to derive the existential reading. ($\imath$-E) captures the well-known idea that the only F is an F, which many writers state as an axiom but in our rule-based approach the idea is naturally presented as a rule. ($\imath$-INST) differs from the other INST-rules because the function $\imath$ is partial, not total, so the second premiss had to be added. Finally, let us stress at least one consequence of the above indicated fact that any application consisting of $\sim, \supset, =^{\tau}, \boldsymbol{\imath}^{\tau}$ or $\supset$ and X (and Y) that is v-improper in $\mathscr{M}$, e.g. $=^{\tau}(X,Y)$, is v-improper in $\mathscr{M}$ – 'error' is thus 'propagated up', 'functions' are *strict*. An application $\boldsymbol{\imath}^{\tau}(C)$ is v-improper if C is v-improper: in such a case, $G(\boldsymbol{\imath}^{\tau}(C))$ and even $=^{\iota}(\boldsymbol{\imath}^{\tau}(C), \boldsymbol{\imath}^{\tau}(C))$ are also v-improper. Contra negative/positive FLs, cf. e.g. Scott [30], Feferman [7], Farmer [6], Bencivenga [2], Lehmann [18], Indrzejczak [15], and even Russell [29, 37].

Numerous rules are *derivable* in $\mathsf{ND}_{\mathsf{TT}^*}$. For proof and discussion of the *Rule of Existential Generalisation* (EG), see [28, 27]; for proofs of (L-$\sim$.iii) and (L-APP.$\perp$), see [17]. Let $\perp^{\tau_{(i)}}/\tau_{(i)}$.

$$\frac{\Gamma \longrightarrow C_{(X/x)}{:}^{o}\,\mathbf{T}}{\Gamma \longrightarrow \mathbf{\Sigma}^{\tau}(\lambda x.(C(x))){:}^{o}\,\mathbf{T}}\ (\text{EG}) \qquad \frac{\Gamma \longrightarrow \sim(o)\,{:}^{o}\,\mathbf{T}}{\Gamma \longrightarrow o\,{:}^{o}\,\mathbf{F}}\ (\text{L-}\sim\text{.iii}) \qquad \frac{\Gamma \longrightarrow X_i\,{:}^{\tau_i}\,\perp}{\Gamma \longrightarrow Y(\vec{X})\,{:}^{\tau}\,\perp}\ (\text{L-APP.}\perp)$$

4 Applications to reasoning framed within natural language NL

The above TT^* can be extended to endorse various methods of *natural language processing* (NLP), e.g. Tichý's *transparent intensional logic* (*TIL*) (e.g. [35]), or its more effective variant *transparent hyperintensional logic* (*THL*) proposed by Kuchyňka (p.c.) and developed in Raclavský [26]. For simplicity reasons we use a simplified TIL here (with only one, alethic *modality*) which is rather close to THL. For that sake let $\mathscr{T}$ be extended by the atomic type ω such that $\mathscr{D}_\omega$ consists of (primitive) entities $\mathtt{w}_1, \mathtt{w}_2, \ldots$, called *possible worlds*. ($\mathbf{\Pi}^\omega$ and $\mathbf{\Sigma}^\omega$ may serve as modal operators.) The meanings of NL expressions are constructions of TT^*. In case of expressions whose *reference* varies dependently *on* $\mathtt{w}$, the meanings in question are constructions of *possible-worlds intensions*, i.e. total or partial functions $\mathscr{D}_\omega \to \mathscr{D}_\tau$.

Propositions are intensions with $\tau := o$; *properties* (or *m-ary relations-in-intensions*) of τ_1-objects are intensions with $\tau := (\tau_1 \to o)$ (or $\tau := \langle \vec{\tau} \rangle \to o$); *individual offices* are intensions with $\tau := \iota$; *offices of individual offices* are intensions with $\tau := \omega \to \iota$, etc. The well-known PWS-style notion of *individual concepts* was adjusted by Tichý to his notion of individual offices as total/partial functions from $\langle$possible world, time instant$\rangle$ couples. We simplify the concept here due to the omission of time-instants parameter.[2]

To simplify things, (declarative) sentences are assumed to express *o-constructions*, i.e. constructions of truth values, not propositions; they typically contain a free *possible world variable* w, i.e. w/ω. For further simplification, instead of constructions of properties, we will often deploy F such that $F/\iota \to o$.

With Tichý we maintain that (typical empirical, definite) *descriptions* "D" *of individuals* (such as e.g. "the King of England/France") express constructions of individual offices; i.e. $D/\omega \to \iota$. In many cases, D is a complex construction, often involving the iota operator $\boldsymbol{\imath}^\tau$; for examples, see below. (Analogously for other types of definite descriptions.) Note carefully that the meaning of description "D" is the construction D, not an office $\mathtt{D}$. On the other hand, the denotation of an *empirical description* "D" is an office $\mathtt{D}$ and the value of $\mathtt{D}$ in $\mathtt{w}$ is called the *reference* of "D" *in* $\mathtt{w}$ – while Tichý used an apt term *occupant* of $\mathtt{D}$ in w. In case of *non-empirical descriptions* "D" such as e.g. "the only number n such that $n = 3 \div 1$" we usually got rid of dull functional dependence on $\mathtt{w}$, "D"'s denotation is thus not an office, but simply its constant value, which is thus not distinguished from "D"'s reference. Recall that each (well-formed) description always has a meaning: in case of non-empirical descriptions it is a construction of a τ-object (if any), in case of empirical descriptions it is a construction of a τ-office whose value in given $\mathtt{w}$ is a τ-object (if any).

Sentences such as

"The D is an F." ("The D is in R with D'" etc.)

have often two readings, called *extensional and intensional reading* (it is surprising that such useful distinction evaporated from recent philosophical logic)

(i) In their *extensional reading*, such sentences are aptly paraphrased as

"The occupant of the office D is an F".

In such a reading they express an o-construction S in which the construction D of the office occurs as applied to w, i.e. $D(w)$, which is abbreviated to D_w (D_w/ι if $D/\omega \to \iota$). (Similarly for other types of expressions denoting intensions.) If there is *no reference* of "D" in $\mathtt{w}$, as in the case of "the King of France", sentences involving them typically *lack a truth value*.

(ii) In their *intensional reading*, such sentences are aptly paraphrased as

[2] Unlike the original notion of individual concepts, Tichý repeatedly attempted to provide philosophical elucidations of offices, see esp. his papers "Individuals and their Roles" and "Existence and God" in [36] and, of course, his [35].

"*The office D is an F*".

In such a reading they express an o-construction S in which the construction D is not so applied. I.e., the subject of such an assertion is the individual office *per se*, not its occupant in w (as in the extensional case). (Similarly for other types of expressions denoting intensions.) The type of reading is often indicated by the predicate; to illustrate, let "D" be "*the US president*": if "F" is "*to be blue-eyed*", i.e. a predicate applicable to individuals, not offices, one naturally renders "*The D is F*" in the extensional sense; if "F" is "*to be one of the highest offices*", i.e. a predicate applicable to offices, not individuals, one naturally renders "*The D is F*" in the intensional sense.

Examples. Recapitulation of some type annotations added or changed in this section: $x, y/\iota$ ($\mathscr{D}_\iota$ consists of individuals); $D/\omega{\to}\iota; F/\iota{\to}o; =^\iota/\langle\iota,\iota,\rangle{\to}o; \boldsymbol{\imath}^\iota/(\iota{\to}o){\to}\iota; \mathbf{T}, \mathbf{F}, \bot^o/o; w/\omega$.

Expression for extension	*its meaning/type*	*expression for intension*	*its meaning/type*
"*be self-identical*"	$\lambda x.{=}^\iota(x,x)/\iota{\to}o$	"*be bald*"	$\mathbf{B}/\omega{\to}(\iota{\to}o)$
"*be identical with*"	$=^\iota/\langle\iota,\iota\rangle{\to}o$	"*be the King of sth.*"	$\mathbf{K}/\omega{\to}(\langle\iota,\iota\rangle{\to}o)$
"*France*"	$\mathbf{Fr}/\iota$	"*the King of France*"	$\lambda w.\boldsymbol{\imath}^\iota(\lambda x.\mathbf{K}_w(x,\mathbf{Fr}))/\omega{\to}\iota$

The sentence "*The King of France is bald.*" expresses the o-construction $\mathbf{B}_w(\boldsymbol{\imath}^\iota(\lambda x.\mathbf{K}_w(x,\mathbf{Fr})))$.

Validity of NL *arguments* A such as

$$\frac{\text{“The King of France is identical with Louis.”}}{\text{“Louis is a King of France.”}}$$

is proof-theoretically *justified* by showing a (derived) rule R of $\mathsf{ND}_{\mathsf{TT}^*}$ (where $\mathbf{L}/\iota$):

$$\frac{\Gamma \longrightarrow =^\iota(\boldsymbol{\imath}^\iota(\lambda x.\mathbf{K}_w(x,\mathbf{Fr})),\mathbf{L}) :^o \mathbf{T}}{\Gamma \longrightarrow \mathbf{K}_w(\mathbf{L},\mathbf{Fr}) :^o \mathbf{T}} \text{ (L.=.Desc-E) (an instance of)}$$

such that (i) each formalisation (meaning) $P_1,...,P_n$ of A's premisses is matched with $\mathbf{T}$ (i.e. $P_i :^o \mathbf{T}$, for each $1 \leq i \leq m$), forming thus the succedents of R's premisses, while (ii) the formalization of A's conclusion is matched with $\mathbf{T}$, too, forming thus succedent of R's conclusion. (Equivalently, the set of R's premisses is empty, but all matches $P_i :^o \mathbf{T}$ occur on the left of $\longrightarrow$ as antecedents in R's conclusion.)

Proof of (the instance of) (L.=.Desc-E).

$$\cfrac{\cfrac{\cfrac{\Gamma \longrightarrow =^\iota(\boldsymbol{\imath}^\iota(\lambda x.\mathbf{K}_w(x,\mathbf{Fr})),\mathbf{L}) :^o \mathbf{T}}{\Gamma \longrightarrow \boldsymbol{\imath}^\iota(\lambda x.\mathbf{K}_w(x,\mathbf{Fr})) :^\iota \mathbf{L}}\,(=\text{-E})}{\Gamma \longrightarrow [\lambda x.\mathbf{K}_w(x,\mathbf{Fr})](\mathbf{L}) :^o \mathbf{T}}\,(\imath\text{-E})}{\Gamma \longrightarrow \mathbf{K}_w(\mathbf{L},\mathbf{Fr}) :^o \mathbf{T}}\,(\beta\text{-CON})$$

□

4.1 Case: Intensional Transitives

Intensional transitive (verbs) (ITV) are verbs such as "*seek*", "*looking for*", "*wish [being something]*"; they attribute a connection to *agents* and *objects* of *intentional attitudes*. In this paper, we will put aside all ITVs such as "*believe*", "*know*", "*wish [that]*" whose sentential complements are sentences, forming thus sentences called *reports* of *propositional attitudes* (for their investigation, see e.g. our [26]).

Since Church [5] and Quine [23], who discussed examples such as "*Ponce de León searched for the Fountain of Youth*", it's widely held that *object terms* complementing ITVs only serve to indicate to which *notion* (not material object) an agent is intentionally related to. For not only that there's no point in e.g. looking for an object to which an agent is already consciously related to: sometimes the sought object under the description needn't to exist.

This gives rise to two widely accepted observations, (1) and (2).

(1) *Sentences with object terms in the scope of ITVs lack existential import as regards them.*

For example, the following type of arguments is obviously invalid (as indicated by – – – –):[3]

Ponce de León searched for the Fountain of Youth.
– –
The Fountain of Youth exists.

(2) *Substitution for object terms in the scope of ITVs fails.*

For an example, consider a so-called *hidden description* "*Endora*" and:

Ponce de León seeks the Fountain of Youth.
Endora is the Fountain of Youth.
– – – – – – – – – – – – – – – – – –
Ponce de León seeks Endora.

A natural choice for fulfilment of the requirements (1) – (2) is to employ Fregean *modes of presentations* (*senses*), explained in the Carnapian [4] spirit as possible-worlds intensions called (say) *individual concepts*. Explaining thus ITVs as denoting relations(-in-intension) between agents and the individual concepts. Montague (e.g. [20]) is famous for this, but Tichý's proposal (cf. e.g. [36, 35]) is more elaborated: his offices (i) can be partial functions (such offices are *unoccupied* in the respective worlds $\mathtt{w}$), (ii) they are functions from $\langle$possible world, time instant$\rangle$ couples (which we simplify in this paper), (iii) and systematically occur even in extensional contexts (via constructions D applied to w, i.e. D_w).

It remains to explain why the above two arguments fail. Let $\mathbf{S}/\omega{\rightarrow}(\omega{\rightarrow}\iota)$ (searched for); $\mathbf{FY}/\omega{\rightarrow}\iota$ (for simplicity); $\mathbf{L}/\iota$ (León). The (major) premiss of the arguments illustrating (1) and (2) expresses

$$P := \mathbf{S}_w(\mathbf{L}, \mathbf{FY}).$$

To P, one cannot apply the type-theoretical version of (EG) that targets ι-constructions such as $\mathbf{FY}_w$, since they're missing in P. The only applicable version of (EG) (as regards object terms) targets constructions of individual offices, here $\mathbf{FY}$. Then, one may only infer the uninformative $\mathbf{\Sigma}^{\omega\rightarrow\iota}(=^{\omega\rightarrow\iota}(d, \mathbf{FY}))$, where $d/\omega{\rightarrow}\iota$, expressed by "*There is an individual office of the Fountain of Youth*".

Similarly for the argument illustrating point (2). Let us adjust (SI) (proved in [34]) to two versions:

$$\frac{\Gamma \longrightarrow S_{[D_w/x]} :^o \mathbf{T} \quad \Gamma \longrightarrow =^\iota(D_w, D'_w) :^o \mathbf{T}}{\Gamma \longrightarrow S_{[D'_w/x]} :^o \mathbf{T}}\ (\mathrm{SI}_1) \qquad \frac{\Gamma \longrightarrow S_{[D/d]} :^o \mathbf{T} \quad \Gamma \longrightarrow =^{\omega\rightarrow\iota}(D, D') :^o \mathbf{T}}{\Gamma \longrightarrow S_{[D'/d]} :^o \mathbf{T}}\ (\mathrm{SI}_2)$$

The rule (SI$_1$), which uses a non-trivial *co-reference* identity statement, cannot be applied in our case (for P doesn't contain $\mathbf{FY}_w$, but mere $\mathbf{FY}$). Only (SI$_2$) is applicable. But since according to (SI$_2$)'s second premiss "D" is *co-denotative* with "D'", one only changes the names of one and the same office $\mathtt{D}$ that is reportedly the object of the agent's attitude.

[3] As noted by Church [5], Russell's theory of descriptions blatantly fails here, since (unlike in the case of propositional-attitudes reports), only primary occurrence elimination of the description is possible here, so the unwelcome conclusion is derivable.

4.2 Case: Strawsonian Reasoning about Existential Presuppositions

By its design, $\mathsf{ND}_{\mathsf{TT}^*}$ is powerful in capturing reasoning about partiality. It is then no surprise that it allows formalization of Strawson's famous views concerning *existential presuppositions* (as indicated in [24]). Recall that these are sentences "E" ascribing existence to some object, if any, fitting the description "D" that must be true in order the sentences "S" in which "D" is in '*referential position*' be either true, or false – not without a truth value. If, on the other hand, "E" is false, the corresponding "S" is without a true value (being *gappy*). We'll consider three arguments concerning "E"s.

($\mathtt{A}_1$) On p. 330 of Strawson's [31], we find two formulations of the following argument (let "*the KF*" abbreviate "*the King of France*"):

$$\frac{\textit{The sentence } \text{“The KF is (not) bald”} \textit{ has a truth value (true or false).}}{\textit{The sentence } \text{“The KF exists”} \textit{ is true.}}$$

Setting aside its meta-linguistic mode, assume the argument as an inference is captured by

$$\frac{\Gamma \longrightarrow [\sim]F(D_w) :^o o}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(D_w),x) :^o \mathbf{T}}$$

The following derived rule of $\mathsf{ND}_{\mathsf{TT}^*}$, which we will call the *Strawsonian Presupposition Rule* 1 (SPR1), covers it (recall that $\mathbf{o}$ is either $o, \mathbf{T}$, or $\mathbf{F}$).

Theorem 1. The following is a *derived rule* of $\mathsf{ND}_{\mathsf{TT}^*}$:

$$\frac{\Gamma \longrightarrow F(D_w) :^o \mathbf{o}}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(D_w,x) :^o \mathbf{T}}\ \text{(SPR1)}$$

Proof. We begin with an assumption introduced by (AX) that fits the premiss that $F(D_w)$ is v-proper:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, D_w :^\iota x \longrightarrow D_w :^\iota x}\ \text{(AX)}}{\Gamma, D_w :^\iota x \longrightarrow =^\iota(D_w,x) :^o \mathbf{T}}\ \text{(=-I)} \qquad \dfrac{}{\Gamma \longrightarrow x :^\iota x}\ \text{(TM)}}{\Gamma, D_w :^\iota x \longrightarrow [\lambda x.=^\iota(D_w,x)](x) :^o \mathbf{T}}\ (\beta\text{-EXP})}{\Gamma, D_w :^\iota x \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(D_w,x)) :^o \mathbf{T}}\ (\Sigma\text{-I})}{\Gamma, D_w :^\iota x, F :^{\iota\to o} f \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(D_w,x)) :^o \mathbf{T}}\ \text{(WR)} \qquad \Gamma \longrightarrow F(D_w) :^o \mathbf{o}}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(D_w,x)) :^o \mathbf{T}}\ \text{(a-INST)}$$

□

($\mathtt{A}_2$) On p. 330 of Strawson's [31], one also finds an argument resembling to:

$$\frac{\textit{The sentence } \text{“The KF doesn't exist”} \textit{ is true.}}{\textit{The sentence } \text{“The KF is bald”} \textit{ is without a truth value.}}$$

The argument can be seen as justified by (what we call) *Strawsonian Presupposition Rule* 2 (SPR2).[4]

Theorem 2. The following is a *derived rule* of $\mathsf{ND}_{\mathsf{TT}^*}$:

$$\frac{\Gamma \longrightarrow \sim(\mathbf{\Sigma}^\iota(\lambda x.=^\iota(D_w,x))) :^o \mathbf{T}}{\Gamma \longrightarrow F(D_w) :^o \bot}\ \text{(SPR2)}$$

Proof. To simplify the proof presentation, let's first state auxiliary matches $\mathtt{M}_1, \mathtt{M}_1$ and derivation $\mathtt{D}_1$:

[4] To really justify the above argument, one should derive the conclusion $\Gamma \longrightarrow \sim\mathbf{\Sigma}^\iota(\lambda o.=^\iota(F(D_w),o)) :^o \mathbf{T}$, using (SPR3) and (L-$\sim$.iii) on (SPR2)'s actual conclusion.

$$\mathtt{M}_1 := \mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x)) :^o o \qquad \mathtt{M}_2 := D_w :^\iota x \qquad \mathtt{D}_1 := \dfrac{}{\Gamma, \mathtt{M}_1 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x)) :^o o}\,(\text{AX})$$

Derivation $\mathtt{D}$. Now we develop the left branch $\mathtt{D}$ of the whole proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \longrightarrow {\sim}(\mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x))) :^o \mathbf{T}}{\Gamma,\mathtt{M}_1 \longrightarrow {\sim}(\mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x))) :^o \mathbf{T}}\,(\text{WR}) \quad \mathtt{D}_1}{\Gamma,\mathtt{M}_1 \longrightarrow {\sim}(o) :^o \mathbf{T}}\,(\text{a-SUB})}{\Gamma,\mathtt{M}_1 \longrightarrow o :^o \mathbf{F}}\,(\text{L-}{\sim}\text{.iii}) \quad \dfrac{\dfrac{}{\Gamma \longrightarrow o :^o o}\,(\text{TM})}{\Gamma,\mathtt{M}_1 \longrightarrow o :^o o}\,(\text{WR})}{\Gamma,\mathtt{M}_1 \longrightarrow [\lambda o.o](o) :^o \mathbf{F}}\,(\beta\text{-EXP}) \quad \mathtt{D}_1}{\dfrac{\dfrac{\dfrac{\Gamma,\mathtt{M}_1 \longrightarrow [\lambda o.o](\mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x))) :^o \mathbf{F}}{\Gamma,\mathtt{M}_1 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x)) :^o \mathbf{F}}\,(\beta\text{-CON})}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x)) :^o \mathbf{F}}\,(\Sigma\text{-INST})}{\Gamma,\mathtt{M}_2 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x)) :^o \mathbf{F}}\,(\text{WR})}\,(\text{a-SUB})$$

In the middle branch, an assumption *per absurdum* that "D" is referring in $\mathtt{w}$ is introduced by (AX):

$$\dfrac{\dfrac{\mathtt{D} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\mathtt{M}_2 \longrightarrow D_w :^\iota x}\,(\text{AX})}{\Gamma,\mathtt{M}_2 \longrightarrow {=^\iota}(D_w,x) :^o \mathbf{T}}\,(=\text{-I}) \quad \dfrac{\dfrac{}{\Gamma \longrightarrow x :^\iota x}\,(\text{TM})}{\Gamma,\mathtt{M}_2 \longrightarrow x :^\iota x}\,(\text{WR})}{\Gamma,\mathtt{M}_2 \longrightarrow [\lambda x.{=^\iota}(D_w,x)](x) :^o \mathbf{T}}\,(\beta\text{-EXP})}{\Gamma,\mathtt{M}_2 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(D_w,x)) :^o \mathbf{T}}\,(\Sigma\text{-I})}{}}{\Gamma,\mathtt{M}_2 \longrightarrow D_w :^\iota \bot}\,(\text{EFQ}) \quad \dfrac{}{\Gamma, D_w :^\iota \bot \longrightarrow D_w :^\iota \bot}\,(\text{AX})}{\dfrac{\Gamma \longrightarrow D_w :^\iota \bot}{\Gamma \longrightarrow F(D_w) :^o \bot}\,(\text{L-APP}^\bot\text{.ii})}\,(\text{EXH})$$

□

($\mathtt{A}_3$) On p. 331 of Strawson's [31], we find an argument quite fitting the rule (L-APP$^\bot$.ii). Let us rather study a justification of an argument which looks like an inverse of $\mathtt{A}_2$.

$$\dfrac{\textit{The sentence } \text{“The KF is (not) bald”} \textit{ is without a truth value.}}{\textit{The sentence } \text{“The KF exists”} \textit{ is false.}}$$

It can be seen as justified by (what we call) the *Strawsonian Presupposition Rule* 3 (SPR3).

Theorem 3. The following is a *derived rule* of $\mathsf{ND}_{\mathsf{TT}^*}$:

$$\dfrac{\Gamma \longrightarrow F(D_w) :^o \bot \qquad \Gamma \longrightarrow F(y) :^o o}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(x,D_w)) :^o \mathbf{F}}\,(\text{SPR3})$$

Remark. In (SPR3)'s second premiss, we require F *v*-constructs a *total* characteristic function (in $\mathscr{M}$). For in cases when F *v*-constructed a partial characteristic function (in $\mathscr{M}$), the whole application $F(D_w)$ would also be *v*-improper (in $\mathscr{M}$), so we couldn't derive (SPR)'s conclusion for sure.[5]

(SPR3)'s proof (occurring in the end of this section) becomes simple, once two derived rules are established.

Lemma 1. The following is a *derived rule* of $\mathsf{ND}_{\mathsf{TT}^*}$:

$$\dfrac{\Gamma \longrightarrow F(D_w) :^o \bot \qquad \Gamma \longrightarrow F(y) :^o o}{\Gamma \longrightarrow D_w :^\iota \bot}\,(\text{L-Desc}^\bot\text{-APP})$$

[5] To justify the above argument, the first premiss of the rule should be converted to $\Gamma \longrightarrow {\sim}\mathbf{\Sigma}^\iota(\lambda x.{=^\iota}(x,D_w)) :^o \mathbf{T}$.

Proof. An assumption *per absurdum* is introduced by (AX) in the right middle branch. First, auxiliary derivations $\mathtt{D}_1$ and $\mathtt{D}_2$ are stated:

$$\mathtt{D}_1 := \frac{\Gamma \longrightarrow F(D_w) :^o \bot}{\Gamma, D_w :^\iota y \longrightarrow F(D_w) :^o \bot}\,(\text{WR}) \qquad \mathtt{D}_2 := \frac{}{\Gamma, D_w :^\iota \bot \longrightarrow D_w :^\iota \bot}\,(\text{AX})$$

$$\dfrac{\dfrac{\mathtt{D}_1 \qquad \dfrac{\dfrac{\Gamma \longrightarrow F(y) :^o o}{\Gamma, D_w :^\iota y \longrightarrow F(y) :^o o}\,(\text{WR}) \qquad \dfrac{}{\Gamma, D_w :^\iota y \longrightarrow D_w :^\iota y}\,(\text{AX})}{\Gamma, D_w :^\iota y \longrightarrow F(D_w) :^\iota o}\,(\text{a-SUB})}{\Gamma, D_w :^\iota y \longrightarrow D_w :^\iota \bot}\,(\text{EFQ}) \qquad \mathtt{D}_2}{\Gamma \longrightarrow D_w :^\iota \bot}\,(\text{EXH})$$

□

Lemma 2. The following is a *derived rule* of $\mathsf{ND}_{\mathsf{TT}^*}$:

$$\frac{\Gamma \longrightarrow D_w :^\iota \bot}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=}^\iota(x, D_w)) :^o \mathbf{F}}\,(\text{L-}\Sigma.\text{Desc}^\bot\text{-APP})$$

Proof. The presentation of the proof is split in three pieces. First, auxiliary matches are stated:

$$\mathtt{M}_1 := \mathbf{\Sigma}^\tau(\lambda x.{=}^\iota(D_w, x)) :^o o \qquad \mathtt{M}_2 := o :^o \mathbf{T} \qquad \mathtt{M}_3 := [\lambda x.{=}^\iota(D_w, x)](x) :^o \mathbf{T}$$

Derivation $\mathtt{D}_1$. We begin with the assumption *per absurdum* that it is true that an individual $\mathtt{y}$ belongs to the (one-membered) set of individuals who are the reference of "D" in $\mathtt{w}$ (cf. $\mathtt{M}_3$). This will suggest that the truth value of the relevant existence ascription (cf. $\mathtt{M}_1$) is $\mathtt{F}$.

$$\dfrac{\dfrac{\dfrac{\Gamma \longrightarrow D_w :^\iota \bot}{\Gamma, \mathtt{M}_3 \longrightarrow D_w :^\iota \bot}\,(\text{WR}) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \mathtt{M}_3 \longrightarrow [\lambda x.{=}^\iota(D_w, x)](x) :^o \mathbf{T}}\,(\text{AX})}{\Gamma, \mathtt{M}_3 \longrightarrow {=}^\iota(D_w, x) :^o \mathbf{T}}\,(\beta\text{-CON})}{\Gamma, \mathtt{M}_3 \longrightarrow D_w :^\iota x}\,(\text{=-E})}{\Gamma, \mathtt{M}_3 \longrightarrow o :^o \mathbf{F}}\,(\text{EFQ})}{\Gamma, \mathtt{M}_1, \mathtt{M}_2, \mathtt{M}_3 \longrightarrow o :^o \mathbf{F}}\,(\text{WR})$$

Derivation $\mathtt{D}_2$. Now we elaborate the redundant assumption (below, we'll therefore use (RA)) that the truth value of the relevant existence ascription is $\mathtt{T}$ (cf. $\mathtt{M}_2$ and the left middle branch).

$$\dfrac{\mathtt{D}_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \mathtt{M}_2 \longrightarrow o :^o \mathbf{T}}\,(\text{AX}) \qquad \dfrac{\dfrac{}{\Gamma \longrightarrow o :^o o}\,(\text{TM})}{\Gamma, \mathtt{M}_2 \longrightarrow o :^o o}\,(\text{WR})}{\Gamma, \mathtt{M}_2 \longrightarrow [\lambda o.o](o) :^o \mathbf{T}}\,(\beta\text{-EXP})}{\Gamma, \mathtt{M}_1, \mathtt{M}_2 \longrightarrow [\lambda o.o](o) :^o \mathbf{T}}\,(\text{WR}) \qquad \dfrac{\dfrac{}{\Gamma, \mathtt{M}_1 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=}^\iota(D_w, x)) :^o o}\,(\text{AX})}{\Gamma, \mathtt{M}_1, \mathtt{M}_2 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=}^\iota(D_w, x)) :^o o}\,(\text{WR})}{\Gamma, \mathtt{M}_1, \mathtt{M}_2 \longrightarrow [\lambda o.o](\mathbf{\Sigma}^\iota(\lambda x.{=}^\iota(D_w, x))) :^o \mathbf{T}}\,(\text{a-SUB})}{\Gamma, \mathtt{M}_1, \mathtt{M}_2 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.{=}^\iota(D_w, x)) :^o \mathbf{T}}\,(\beta\text{-CON})}{\Gamma, \mathtt{M}_1, \mathtt{M}_2 \longrightarrow o :^o \mathbf{F}}\,(\Sigma\text{-E})$$

Finally, we put the truth value $\mathtt{F}$ with the existence ascription together (first, we eliminate $\mathtt{M}_2$, cf. left).

$$\dfrac{\dfrac{\dfrac{\mathtt{D}_2 \quad \dfrac{\dfrac{}{\Gamma, o :^o \mathbf{F} \longrightarrow o :^o \mathbf{F}}\,(\text{AX})}{\Gamma, \mathtt{M}_1, o :^o \mathbf{F} \longrightarrow o :^o \mathbf{F}}\,(\text{WR})}{\Gamma, \mathtt{M}_1 \longrightarrow o :^o \mathbf{F}}\,(\text{RA}) \quad \dfrac{\dfrac{}{\Gamma \longrightarrow o :^o o}\,(\text{TM})}{\Gamma, \mathtt{M}_1 \longrightarrow o :^o o}\,(\text{WR})}{\Gamma, \mathtt{M}_1 \longrightarrow [\lambda o.o](o) :^o \mathbf{F}}\,(\beta\text{-EXP}) \quad \dfrac{}{\Gamma, \mathtt{M}_1 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(D_w, x)) :^o o}\,(\text{AX})}{\dfrac{\Gamma, \mathtt{M}_1 \longrightarrow [\lambda o.o](\mathbf{\Sigma}^\iota(\lambda x.=^\iota(x, D_w))) :^o \mathbf{F}}{\dfrac{\Gamma, \mathtt{M}_1 \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(x, D_w)) :^o \mathbf{F}}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(x, D_w)) :^o \mathbf{F}}\,(\Sigma\text{-INST})}\,(\beta\text{-CON})}\,(\text{a-SUB})$$

□

Proof of (SPR3).

$$\dfrac{\dfrac{\Gamma \longrightarrow F(D_w) :^o \bot \qquad \Gamma \longrightarrow F(x) :^o o}{\Gamma \longrightarrow D_w :^\iota \bot}\,(\text{L-Desc}^\bot\text{-APP})}{\Gamma \longrightarrow \mathbf{\Sigma}^\iota(\lambda x.=^\iota(x, D_w)) :^o \mathbf{F}}\,(\text{L-}\Sigma.\text{Desc}^\bot\text{-APP})$$

□

5 Conclusion

We exposed a specific theory of definite descriptions in Tichýan spirit whose essential features were listed in Sec. 1. The derivation rules of the partial type theory TT^* that govern the $\imath$-operator were exposed and briefly discussed in Sec. 3. In Sec. 4, we showed its application in natural language processing, in particular to two famous cases of reasoning: (a) the case with intensional transitives whose complements are non-referring descriptions, and (b) the case of Strawsonian rules for existential presuppositions concerning non-referring descriptions – which have not been studied in a formal way in literature. Future work should focus more on (i) proof-theoretic properties of the above $\imath$-rules and (ii) comparison with rival logical approaches both in free and modal logic (cf. [15, 16, 22]).

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References

[1] Michael J. Beeson (1985): *Foundations of Constructive Mathematics: Metamathematical Studies*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 6, Springer, doi:10.1007/978-3-642-68952-9.

[2] Ermanno Bencivenga (2019): *Free Logics*. In Dov Gabbay & Franz Guenthner, editors: *Handbook of Philosophical Logic*, 5, Springer, Dordrecht, pp. 147–196, doi:10.1007/978-94-017-0458-8_3.

[3] Stephen Blamey (2002): *Partial Logic*. In Dov Gabbay & Franz Guenthner, editors: *Handbook of Philosophical Logic*, 5, Springer, pp. 261–353, doi:10.1007/978-94-017-0458-8_5.

[4] Rudolf Carnap (1947): *Meaning and Necessity*. The University of Chicago Press.

[5] Alonzo Church (1951): *The Need for Abstract Entities*. *American Academy of Arts and Sciences Proceedings* 80, pp. 100–113, doi:10.2307/20023640.

[6] William M. Farmer (1990): *A Partial Functions Version of Church's Simple Theory of Types*. *Journal of Symbolic Logic* 55(3), pp. 1269–1291, doi:10.2307/2274487.

[7] Solomon Feferman (1995): *Definedness*. *Erkenntnis* 43(3), pp. 295–320, doi:10.1007/BF01135376.

[8] Melvin Fitting & Richard L. Mendelsohn (1998): *First-Order Modal Logic*. Kluwer, doi:10.1007/978-94-011-5292-1.

[9] Gottlob Frege (1892): *Über Sinn und Bedeutung*. *Zeitschrift für Philosophie und philosophishe Kritik* 100, pp. 25–50.

[10] Dov M. Gabbay (1996): *Labelled Deductive Systems*. Oxford University Press, doi:10.1093/oso/9780198538332.001.0001.

[11] James R. Hindley & Jonathan P. Seldin (2008): *Lambda-Calculus and Combinators, an Introduction*. Cambridge University Press, doi:10.1017/CBO9780511809835.

[12] Andrzej Indrzejczak (2019): *Fregean Description Theory in Proof-Theoretical Setting*. *Logic and Logical Philosophy* 28(1), p. 137–155, doi:10.12775/LLP.2018.008.

[13] Andrzej Indrzejczak (2020): *Existence, Definedness and Definite Descriptions in Hybrid Modal Logic*. In Nicola Olivetti nad Rineke Verbrugge, Sara Negri & Gabriel Sandu, editors: *Advances in Modal Logic*, 13, College Publications, pp. 349–368. Available at http://www.aiml.net/volumes/volume13/Indrzejczak.pdf.

[14] Andrzej Indrzejczak (2020): *Free Definite Description Theory - Sequent Calculi and Cut Elimination*. *Logic and Logical Philosophy* 29(4), pp. 505–539, doi:10.12775/LLP.2019.020.

[15] Andrzej Indrzejczak (2023): *Russellian Definite Description Theory – A Proof-theoretic Approach*. *The Review of Symbolic Logic* 16(2), pp. 624–649, doi:10.1017/s1755020321000289.

[16] Andrzej Indrzejczak & Michał Zawidzki (2023): *When Iota Meets Lambda*. *Synthèse* 201(72), doi:10.1007/s11229-023-04048-y.

[17] Petr Kuchyňka & Jiří Raclavský (2024): *Completeness in Partial Type Theory*. *Journal of Logic and Computation* 34(1), pp. 1–32, doi:10.1093/logcom/exac089.

[18] Scott Lehmann (2019): *More Free Logic*. In Dov Gabbay & Franz Guenthner, editors: *Handbook of Philosophical Logic*, 5, Springer, Dordrecht, pp. 197–259, doi:10.1007/978-94-017-0458-8_4.

[19] Peter Ludlow (2019): *Descriptions*. In Edward N. Zalta & Uri Nodelman, editors: *The Stanford Encyclopedia of Philosophy*, winter 2023 edition. Available at https://plato.stanford.edu/archives/win2023/entries/descriptions/.

[20] Richard Montague (1973): *The Proper Treatment of Quantification in Ordinary English*. In K.J.J. Hintikka, J.M.E. Moravcsik & P. Suppes, editors: *Approaches to Natural Language. Proceedings of the 1970 Stanford Workshop on Grammar and Semantics*, D. Reidel, pp. 221–242, doi:10.1007/978-94-010-2506-5_10.

[21] Yiannis N. Moschovakis (2005): *A Logical Calculus of Meaning and Synonymy*. *Linguistics and Philosophy* 29(1), pp. 27–89, doi:10.1007/s10988-005-6920-7.

[22] Eugenio Orlandelli (2021): *Labelled Calculi for Quantified Modal Logics with Definite Descriptions*. *Journal of Logic and Computation* 31(3), p. 923–946, doi:10.1093/logcom/exab018.

[23] Willard Van Orman Quine (1956): *Quantifiers and Propositional Attitudes*. *The Journal of Philosophy* 53(5), pp. 177–187, doi:10.2307/2022451.

[24] Jiří Raclavský (2011): *Semantic Concept of Existential Presupposition*. *Human Affairs* 21(3), pp. 249–261, doi:10.2478/s13374-011-0026-4.

[25] Jiří Raclavský (2014): *Explicating Truth in Transparent Intensional Logic*. In Roberto Ciuni, Heinrich Wansing & Caroline Willkommen, editors: *Recent Trends in Philosophical Logic*, Springer Verlag, pp. 167–177, doi:10.1007/978-3-319-06080-4_12.

[26] Jiří Raclavský (2020): *Belief Attitudes, Fine-Grained Hyperintensionality and Type-Theoretic Logic*. Studies in Logic 88, College Publications.

[27] JiříRaclavský (2022): *Puzzles of Existential Generalisation from Type-theoretic Perspective*. *Electronic Proceedings in Theoretical Computer Science* 358, pp. 68–83, doi:10.4204/eptcs.358.6.

[28] Jiří Raclavský (2022): *The Rule of Existential Generalisation and Explicit Substitution*. *Logic and Logical Philosophy* 31(1), pp. 105–141, doi:10.12775/LLP.2021.011.

[29] Bertrand Russell (1905): *On Denoting*. *Mind* 14(56), pp. 479–493, doi:10.1093/mind/XIV.4.479.

[30] Dana Scott (1979): *Identity and Existence in Intuitionistic Logic*. In M.P. Fourman, C.J. Mulvey & D.S. Scott, editors: *Applications of Sheaves. Lecture Notes in Mathematics, vol 753*, Springer, pp. 660–696, doi:10.1007/BFb0061839.

[31] Peter Frederick Strawson (1950): *On Referring*. *Mind* 59(235), pp. 320–344, doi:10.1093/mind/LIX.235.320.

[32] Pavel Tichý (1971): *An Approach to Intensional Analysis*. *Noûs* 5(3), pp. 273–297, doi:10.2307/2214668.

[33] Pavel Tichý (1982): *Foundations of Partial Type Theory*. *Reports on Mathematical Logic* 14, pp. 57–72.

[34] Pavel Tichý (1986): *Indiscernibility of Identicals*. *Studia Logica* 45(3), pp. 251–273, doi:10.1007/BF00375897.

[35] Pavel Tichý (1988): *The Foundations of Frege's Logic*. Walter de Gruyter, doi:10.1515/9783110849264.

[36] Pavel Tichý (2004): *Pavel Tichý's Collected Papers in Logic and Philosophy*. The University of Otago Press and Filosofia.

[37] Alfred N. Whitehead & Bertrand Russell (1910–1913): *Principia Mathematica*. Cambridge University Press.

Semantic Incompleteness of Liberman et al. (2020)'s Hilbert-style System for Term-modal Logic K with Equality and Non-rigid Terms

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In this paper, we prove the semantic incompleteness of the Hilbert-style system for the minimal normal term-modal logic with equality and non-rigid terms that was proposed in Liberman et al. (2020) "Dynamic Term-modal Logics for First-order Epistemic Planning." Term-modal logic is a family of first-order modal logics having term-modal operators indexed with terms in the first-order language. While some first-order formula is valid over the class of all frames in the Kripke semantics for the term-modal logic proposed there, it is not derivable in Liberman et al. (2020)'s Hilbert-style system. We show this fact by introducing a non-standard Kripke semantics which makes the meanings of constants and function symbols relative to the meanings of relation symbols combined with them.

1 Introduction

In this paper, we prove the semantic incompleteness of the Hilbert-style system H**K** for the minimal normal term-modal logic **K** with equality and non-rigid terms that was proposed in Liberman et al. [11]. Term-modal logic, developed by Thalmann [22] and Fitting et al. [4], is a family of first-order modal logics having term-modal operators $[t]$ indexed with terms t in the first-order language. In the language of term-modal logic, for example, $[x]P(x)$, $[f(x)]P(x)$ and $\forall x[f(x)]P(x)$ are formulas. Term-modal logic is more expressive than multi-modal propositional logic and has been applied to epistemic logic [10, 17, 18, 21, 1, 23, 24, 16, 15, 13, 11, 14] and deontic logic [20, 19, 8, 9, 7, 5, 6]. Some other developments of term-modal logic have been overviewed e.g. in [11, pp. 22-4] and [5, pp. 48-50].

The logic developed in Liberman et al. [11] is a first-order dynamic epistemic logic for epistemic planning, and term-modal logic is invoked as its underlying logic. Technically speaking, their term-modal logic is a two-sorted normal term-modal logic of the constant domain with equality and non-rigid terms. They make their logic two-sorted because, whereas letting the domain of a model include both agents and objects, they read an epistemically interpreted term-modal operator K_t as "*agent t* knows." The language defined in [11] allows $K_t\varphi$ to be a formula only if t is a term for an agent, and thereby excludes the possibility that terms denoting objects appear in the argument of the term-modal operator. The Hilbert-style system H**K** found in [11, p. 17] was originally presented in [1] which is probably based on [17, 18]. Later, two issues on action model and reduction axiom were fixed in the erratum [12] of [11].

Unfortunately, H**K** is semantically incomplete due to the unprovability of a first-order formula $x = c \to (P(x) \to P(c))$. In Section 3, we show that it is valid over the class of all frames whereas it is unprovable in H**K**. To this end, we there introduce a non-standard Kripke semantics which makes the

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meanings of constants and function symbols relative to the meanings of relation symbols combined with them.

It is worth noting here that, as the above first-order formula suggests, the semantic incompleteness of H**K** is irrelevant to its term-modal aspects. To make the point clear, let $\mathscr{L}$ be a first-order modal language having equality, constants and only the ordinary non-indexed modal operator as its modal operators, and say the semantics for first-order modal logic (the FOML-semantics for short) to refer to the Kripke semantics of the constant domain given to $\mathscr{L}$ in which the accessibility relation is just a binary relation on worlds and constants are interpreted relative to worlds. Using a semantics similar to the non-standard semantics introduced in Section 3, we can in fact prove that the Hilbert-style system naturally obtained from H**K** by changing from the two-sorted term-modal language to $\mathscr{L}$ becomes semantically incomplete with respect to the FOML-semantics similarly due to the unprovability of $x = c \to (P(x) \to P(c))$. The question to be asked here is what is the formulation of a sound and complete Hilbert-style system to the FOML-semantics. To the best of our knowledge, this is still an open question. Such a Hilbert-style system seems to have never been provided together with a detailed proof in the literature.[1]

This paper will proceed as follows. In Section 2 we first introduce the syntax in [11]. Since there are some minor defects on the definitions for type, we do this with some modifications. Then we introduce the Kripke semantics and the Hilbert-style system H**K** given in [11]. In Section 3 we prove the semantic incompleteness of H**K** by introducing a non-standard Kripke semantics for which H**K** is sound but in which $x = c \to (P(x) \to P(c))$ is not valid.

2 Syntax, Semantics and the Hilbert-style System HK

We will first introduce the syntax presented in [11, pp. 3-4] with some modifications. The idea there is to define the notions of term and formula while assigning (sequences of) types "`agt`", "`obj`" or "`agt_or_obj`" to all symbols like variables or relation symbols. It is basically the same idea as in Enderton [2, Section 4.3], but there is an important difference. In the syntax of [11], not only `agt` or `obj` but also `agt_or_obj` may be assigned to the arguments of function symbols and relation symbols, so that $P(x)$ seems to be intended to become a formula even when x has type `agt` and P takes type `agt_or_obj`.

However, the original definitions 1–3 for the syntax seem to have two minor defects. First, the original definition 1 for type assignment and the original definition 2 for term are dependent upon one another, thus they are circular definitions. Second, whereas $P(x)$ seems to be intended to become a formula when x has type `agt` and P takes type `agt_or_obj`, it does not actually become a formula since the original definition 3 for formula requires that the type of x and the type of the argument of P must be the same. Accordingly, for example, $x = x$ cannot be a formula in any signature since the type of x is either `agt` or `obj` but the type of the arguments of $=$ is always `agt_or_obj`.

To amend the above two defects, we redefine the syntax in [11, pp. 3-4] as follows.

Definition 1 (Signature)**.** Let `Var` be a countably infinite set of *variables*, `Cn` a countable set of *constants*, `Fn` a countable set of *function symbols*, and `Rel` a countable set of *relation symbols* containing the *equality symbol* $=$. Let $\langle \mathtt{TYPE}, \preccurlyeq \rangle$ be also the ordered set of *types* where $\mathtt{TYPE} = \{\mathtt{agt}, \mathtt{obj}, \mathtt{agt\_or\_obj}\}$ and $\preccurlyeq$ is the reflexive ordering on `TYPE` with $\mathtt{agt} \preccurlyeq \mathtt{agt\_or\_obj}$ and $\mathtt{obj} \preccurlyeq \mathtt{agt\_or\_obj}$, i.e.,

$$\preccurlyeq \;:=\; \{\langle \tau, \tau \rangle \mid \tau \in \mathtt{TYPE}\} \cup \{\langle \mathtt{agt}, \mathtt{agt\_or\_obj} \rangle, \langle \mathtt{obj}, \mathtt{agt\_or\_obj} \rangle\}.$$

[1] As a sound and complete proof system with respect to the multi-modal FOML-semantics with the epistemic accessibility relation for each agent, Fagin et al. [3, p. 90] offered a Hilbert-style system having two first-order principles $A(t/x) \to \exists x A$ and $t = s \to (A(t/z) \leftrightarrow A(s/z))$ as axioms with a restriction that t, s must be variables if A has any occurrence of an (not term-modal) epistemic operator K_a. However, the proof of this system's completeness is omitted there.

A *type assignment* $\mathtt{t}\colon \mathtt{Var}\cup\mathtt{Cn}\cup\mathtt{Fn}\cup\mathtt{Rel}\to\bigcup_{n\in\mathbb{N}}\mathtt{TYPE}^n$ is an assignment mapping

1. a variable x to a type $\mathtt{t}(x)\in\{\,\mathtt{agt},\mathtt{obj}\,\}$ such that both $\mathtt{Var}\cap\mathtt{t}^{-1}[\{\,\mathtt{agt}\,\}]$ and $\mathtt{Var}\cap\mathtt{t}^{-1}[\{\,\mathtt{obj}\,\}]$ are countably infinite, where $\mathtt{t}^{-1}[X]$ is the inverse image of a set X;
2. a constant c to a type $\mathtt{t}(c)\in\{\,\mathtt{agt},\mathtt{obj}\,\}$;
3. a function symbol f to a sequence of types $\mathtt{t}(f)\in\mathtt{TYPE}^n\times\{\,\mathtt{agt},\mathtt{obj}\,\}$ for some $n\in\mathbb{N}$;
4. the equality symbol $=$ to the sequence of types $\mathtt{t}(=)=\langle\mathtt{agt\_or\_obj},\mathtt{agt\_or\_obj}\rangle$;
5. a relation symbol P distinct from $=$ to a sequence of types $\mathtt{t}(P)\in\mathtt{TYPE}^n$ for some $n\in\mathbb{N}$.

The tuple $\langle\mathtt{Var},\mathtt{Cn},\mathtt{Fn},\mathtt{Rel},\mathtt{t}\rangle$ is called a *signature*.

Definition 2 (Term of Type)**.** Let $\langle\mathtt{Var},\mathtt{Cn},\mathtt{Fn},\mathtt{Rel},\mathtt{t}\rangle$ be a signature. The set of *terms of types* is defined as follows.

1. any variable $x\in\mathtt{Var}$ is a term of type $\mathtt{t}(x)$.
2. any constant $c\in\mathtt{Cn}$ is a term of type $\mathtt{t}(c)$.
3. If $t_1,\dots,t_n$ are terms of types $\tau_1,\dots,\tau_n$ and f is a function symbol in $\mathtt{Fn}$ such that $\mathtt{t}(f)=\langle\tau'_1,\dots,\tau'_n,\tau'_{n+1}\rangle$ and $\tau_i\preccurlyeq\tau'_i$, then $f(t_1,\dots,t_n)$ is a term of type τ'_{n+1}.

For convenience, henceforth we use a type assignment $\mathtt{t}$ to mean its uniquely extended assignment by letting $\mathtt{t}(f(t_1,\dots,t_n))=\tau$ for each term of the form $f(t_1,\dots,t_n)$ of type τ.

Definition 3 (Language)**.** Let $\langle\mathtt{Var},\mathtt{Cn},\mathtt{Fn},\mathtt{Rel},\mathtt{t}\rangle$ be a signature. The *language* is the set of formulas φ defined in the following BNF.

$$\varphi ::= P(t_1,\dots,t_n)\mid\neg\varphi\mid\varphi\wedge\varphi\mid K_s\varphi\mid\forall x\varphi,$$

where $t_1,\dots,t_n,s$ are terms with $\mathtt{t}(s)=\mathtt{agt}$ and $P\in\mathtt{Rel}$ such that $\mathtt{t}(P)=\langle\tau_1,\dots,\tau_n\rangle$ and $\mathtt{t}(t_i)\preccurlyeq\tau_i$. Note here that P can be $=$.

As usual, we use the notations $t\neq s:=\neg(t=s)$, $\varphi\to\psi:=\neg(\varphi\wedge\neg\psi)$ and $\exists x\varphi:=\neg\forall x\neg\varphi$.

We believe that our definitions successfully capture what was intended in the original definitions 1–3. On top of these definitions, we will follow [11, p. 4] to define the notions of *free variable* and *bound variable* in a formula as usual, where the set of free variables in $K_t\varphi$ is defined as the union of the set of variables in t and the set of free variables in φ. For a variable x, terms t,s and a formula φ such that $\mathtt{t}(x)=\mathtt{t}(s)$ and no variables in s are bound variables in φ, we also define *substitutions* $t(s/x)$ and $\varphi(s/x)$ of s for x in t and φ in a usual manner, except that $(K_t\varphi)(s/x)=K_{t(s/x)}\varphi(s/x)$. Whenever we write $t(s/x)$ or $\varphi(s/x)$, we tacitly assume that $\mathtt{t}(x)=\mathtt{t}(s)$ and no variables in s are bound variables in φ. We also define the lengths of term and formula as usual.

Let us now introduce the Kripke semantics presented in [11, pp. 5-6].

Definition 4 (Frame, [11, Def. 4])**.** A *frame* is a tuple $F=\langle D,W,R\rangle$ where

1. $D:=D_{\mathtt{agt\_or\_obj}}:=D_{\mathtt{agt}}\sqcup D_{\mathtt{obj}}$ is the disjoint union of a non-empty set $D_{\mathtt{agt}}$ of *agents* and a non-empty set $D_{\mathtt{obj}}$ of *objects*;
2. W is a non-empty set of *worlds*;
3. R is a mapping that assigns to each agent $i\in D_{\mathtt{agt}}$ a binary relation R_i on W, i.e., $R\colon D_{\mathtt{agt}}\to\mathscr{P}(W\times W)$.

Definition 5 (Model, [11, Def. 5])**.** Let $\langle\mathtt{Var},\mathtt{Cn},\mathtt{Fn},\mathtt{Rel},\mathtt{t}\rangle$ be a signature. A *model* is a tuple $M=\langle D,W,R,I\rangle$ where $\langle D,W,R\rangle$ is a frame and I is an *interpretation* that maps

1. a pair $\langle c, w\rangle$ of some $c \in \mathtt{Cn}$ and some $w \in W$ to an element $I(c,w) \in D_{\mathtt{t}(c)}$;
2. a pair $\langle f, w\rangle$ of some $f \in \mathtt{Fn}$ and some $w \in W$ to a function $I(f,w)\colon (D_{\tau_1} \times \cdots \times D_{\tau_n}) \to D_{\tau_{n+1}}$, where $\mathtt{t}(f) = \langle \tau_1, \dots, \tau_n, \tau_{n+1}\rangle$;
3. a pair $\langle =, w\rangle$ of the equality symbol $=$ and some $w \in W$ to the set $I(=,w) = \{\langle d,d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}$;
4. a pair $\langle P, w\rangle$ of some $P \in \mathtt{Rel} \setminus \{=\}$ and some $w \in W$ to a subset $I(P,w)$ of $D_{\tau_1} \times \cdots \times D_{\tau_n}$, where $\mathtt{t}(P) = \langle \tau_1, \dots, \tau_n\rangle$.

Definition 6 (Valuation, [11, Def. 6, 7])**.** A *valuation* is a mapping $v\colon \mathtt{Var} \to D$ such that $v(x) \in D_{\mathtt{t}(x)}$ and the valuation $v[x \mapsto d]$ is the same valuation as v except for assigning to a variable x an element $d \in D_{\mathtt{t}(x)}$. Given a valuation v, a world w and an interpretation I in a model, the extension $[\![t]\!]^{I,v}_w$ of a term t is defined by $[\![x]\!]^{I,v}_w = v(x)$, $[\![c]\!]^{I,v}_w = I(c,w)$, and $[\![f(t_1,\dots,f_n)]\!]^{I,v}_w = I(f,w)([\![t_1]\!]^{I,v}_w, \dots, [\![t_n]\!]^{I,v}_w)$.

Definition 7 (Satisfaction, [11, Def. 8])**.** The *satisfaction* $M,w \models_v \varphi$ of a formula φ at a world w in a model M under a valuation v is defined as follows.

$$
\begin{array}{lll}
M,w \models_v P(t_1,\dots,t_n) & \text{iff} & \langle [\![t_1]\!]^{I,v}_w, \dots, [\![t_n]\!]^{I,v}_w\rangle \in I(P,w) \qquad (P \text{ can be } =)\\
M,w \models_v \neg\varphi & \text{iff} & M,w \not\models_v \varphi\\
M,w \models_v \varphi \wedge \psi & \text{iff} & M,w \models_v \varphi \quad \text{and} \quad M,w \models_v \psi\\
M,w \models_v \forall x\varphi & \text{iff} & M,w \models_{v[x\mapsto d]} \varphi \quad \text{for all } d \in D_{\mathtt{t}(x)}\\
M,w \models_v K_t\varphi & \text{iff} & M,w' \models_v \varphi \quad \text{for all } w' \in W \text{ such that } \langle w,w'\rangle \in R_{[\![t]\!]^{I,v}_w}
\end{array}
$$

Definition 8 (Validity, [11, p. 25])**.** A formula φ is *valid* if for all models M, all worlds $w \in W$ and all valuations v, it holds that $M,w \models_v \varphi$.

Remark 1. Instead of the x-variant of a valuation v used in [11], we adopted the valuation $v[x \mapsto d]$ to give the satisfaction for $\forall x\varphi$. This change is just for the clarity of our proof and does not affect the satisfiability of formulas. As for validity, because unlike [11] we are only interested here in the validity of formula over the class of all frames, for the sake of brevity we defined the validity of formula independently of any class of frames.

For ease of reference, henceforth we call this semantics *TML-semantics*.

Finally, we will introduce by Table 1 the Hilbert-style system H**K** for the minimal normal term-modal logic **K** presented in Liberman et al. [11, p. 17]. The notion of provability is defined as usual.

What is involving the semantic incompleteness of H**K** here is UE and PS. As remarked in Fagin et al. [3, pp. 88-9], the ordinary first-order axioms $\forall x\varphi \to \varphi(t/x)$ and $t = s \to (\varphi(t/z) \to \varphi(s/z))$ are not valid in Kripke semantics for first-order modal logic where constants or function symbols are interpreted as non-rigid. In order to avoid making invalid formulas provable, Liberman et al. [11] adopted the variable-restricted versions UE and PS of these two axioms. The problem is that PS or its combinations with UE or ∃Id are not sufficient to derive a valid formula $x = c \to (P(x) \to P(c))$.

3 Semantic Incompleteness of the Hilbert-style System H**K**

In this section, we prove the semantic incompleteness of H**K** by showing that $x = c \to (P(x) \to P(c))$ is valid in the TML-semantics but not provable in H**K**. As expected, there is no difficulty to show the former.

Axiom				
	all propositional tautologies			
UE	$\forall x\varphi \to \varphi(y/x)$	K	$K_t(\varphi \to \psi) \to (K_t\varphi \to K_t\psi)$	
Id	$t = t$	BF	$\forall x K_t\varphi \to K_t\forall x\varphi$	for x not occurring in t
PS	$x = y \to (\varphi(x/z) \to \varphi(y/z))$	KNI	$x \neq y \to K_t x \neq y$	
∃Id	$c = c \to \exists x(x = c)$			
DD	$x \neq y$ if $\mathtt{t}(x) \neq \mathtt{t}(y)$			
Inference rules				
MP	From φ and $\varphi \to \psi$, infer ψ			
KG	From φ, infer $K_t\varphi$			
UG	From $\varphi \to \psi$, infer $\varphi \to \forall x\psi$ for x not free in φ			

Table 1: The Hilbert-style system H**K** for the minimal normal term-modal logic **K**

Proposition 1. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature, $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ and $P \in \mathtt{Rel}$ with $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj}\rangle$. A formula $x = c \to (P(x) \to P(c))$ is valid in the TML-semantics.

Proof. Suppose $M, w \models_v x = c$ and $M, w \models_v P(x)$. Since $[\![x]\!]_w^{I,v} = [\![c]\!]_w^{I,v}$ and $[\![x]\!]_w^{I,v} \in I(P,w)$, we have $[\![c]\!]_w^{I,v} \in I(P,w)$. Thus $M, w \models_v P(c)$. □

To establish the unprovability of $x = c \to (P(x) \to P(c))$, it is sufficient to find a new semantics to which H**K** is sound but in which this formula is not valid. To this end, we will first introduce the notion of non-standard model as follows.

Definition 9 (Non-standard Model)**.** Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature. A *non-standard model* is a tuple $N = \langle D, W, R, J\rangle$ where $\langle D, W, R\rangle$ is a frame in the sense of Definition 4 and J is an interpretation that maps

1. a triple $\langle c, w, X\rangle$ of some $c \in \mathtt{Cn}$, some $w \in W$ and some $X \subseteq D^n$ for some $n \in \mathbb{N}$ to an element $J(c, w, X) \in D_{\mathtt{t}(c)}$;
2. a triple $\langle f, w, X\rangle$ of some $f \in \mathtt{Fn}$, some $w \in W$ and some $X \subseteq D^n$ for some $n \in \mathbb{N}$ to a function $J(f, w, X)\colon (D_{\tau_1} \times \cdots \times D_{\tau_n}) \to D_{\tau_{n+1}}$, where $\mathtt{t}(f) = \langle \tau_1, \ldots, \tau_{n+1}\rangle$;
3. a pair $\langle =, w\rangle$ of the equality symbol $=$ and some $w \in W$ to the set $J(=, w) = \{\langle d, d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}$;
4. a pair $\langle P, w\rangle$ of some $P \in \mathtt{Rel} \setminus \{=\}$ and some $w \in W$ to a subset $J(P, w)$ of $D_{\tau_1} \times \cdots \times D_{\tau_n}$, where $\mathtt{t}(P) = \langle \tau_1, \ldots, \tau_n\rangle$.

Here is the intuition. A subset X of D^n is a set of sequences consisting of either/both of agents and objects. Thus, the set X mentioned in the meanings $J(c, w, X)$ and $J(f, w, X)$ of a constant c and a function symbol f can serve as the meaning of a relation symbol. This trick enables us to make the meanings of constants and function symbols relative to the meanings of relation symbols combined with them.

We then define the notion of satisfaction of formula in non-standard model. In what follows, we use the same notion of valuation as in the TML-semantics and define the extension $[\![t]\!]_{w,X}^{J,v}$ of a term t in a given non-standard model similarly by letting $[\![x]\!]_{w,X}^{J,v} = v(x)$, $[\![c]\!]_{w,X}^{J,v} = J(c, w, X)$ and $[\![f(t_1, \ldots, t_n)]\!]_{w,X}^{J,v} = J(f, w, X)([\![t_1]\!]_{w,X}^{J,v}, \ldots, [\![t_n]\!]_{w,X}^{J,v})$.

Definition 10 (Satisfaction in Non-standard Model)**.** The *satisfaction* $N,w \models_v \varphi$ of a formula φ at a world w in a non-standard model N under a valuation v is defined as follows.

$$
\begin{array}{lll}
N,w \models_v P(t_1,\dots,t_n) & \text{iff} & \langle [\![t_1]\!]^{J,v}_{w,J(P,w)},\dots,[\![t_n]\!]^{J,v}_{w,J(P,w)}\rangle \in J(P,w) \qquad (P \text{ can be } =) \\
N,w \models_v \neg\varphi & \text{iff} & N,w \not\models_v \varphi \\
N,w \models_v \varphi \wedge \psi & \text{iff} & N,w \models_v \varphi \quad \text{and} \quad N,w \models_v \psi \\
N,w \models_v \forall x\varphi & \text{iff} & N,w \models_{v[x\mapsto d]} \varphi \quad \text{for all } d \in D_{\mathtt{t}(x)} \\
N,w \models_v K_t\varphi & \text{iff} & N,w' \models_v \varphi \quad \text{for all } w' \in W \text{ such that } \langle w,w'\rangle \in R_{[\![t]\!]^{J,v}_{w,\emptyset}}
\end{array}
$$

What we should pay attention here is the satisfactions of atomic formula $P(t_1,\dots,t_n)$ and term-modal formula $K_t\varphi$. In the satisfaction of $P(t_1,\dots,t_n)$ in non-standard model, the meaning $[\![t_i]\!]^{J,v}_{w,J(P,w)}$ of each t_i in $P(t_1,\dots,t_n)$ is determined by the interpretation J, the valuation v, the world w and *the meaning* $J(P,w)$ *of the relation symbol P combined with terms* $t_1,\dots,t_n$. Thus, as explained in the following Example 1, the meaning of a constant c occurring in $P(c)$ could be different from that of c occurring in $Q(c)$.

Example 1. Let $lewis \in \mathtt{Cn}$ with $\mathtt{t}(lewis) = \mathtt{agt}$ and $SL, CF \in \mathtt{Rel}$ with $\mathtt{t}(SL) = \mathtt{t}(CF) = \langle\mathtt{agt}\rangle$, and consider a non-standard model such that

$$
\begin{aligned}
J(SL,w) &= \{i \in D_{\mathtt{agt}} \mid i \text{ is one of the authors of } \textit{Symbolic Logic}\}, \\
J(CF,w) &= \left\{ i \in D_{\mathtt{agt}} \mid i \text{ is the author of } \textit{Counterfactuals} \right\},
\end{aligned}
$$

$J(lewis,w,J(SL,w))$ is C. I. Lewis and $J(lewis,w,J(CF,w))$ is D. Lewis. The meaning $J(lewis,w,J(SL,w))$ of *lewis* occurring in $SL(lewis)$ is then different from the meaning $J(lewis,w,J(CF,w))$ of *lewis* occurring in $CF(lewis)$. Note that, although $J(lewis,w,J(SL,w)) \in J(SL,w)$ holds in the above non-standard model, we can technically have a non-standard model such that $J(lewis,w,J(SL,w)) \notin J(SL,w)$ holds by assigning D. Lewis to $J(lewis,w,J(SL,w))$.

On the other hand, because the meaning $[\![t]\!]^{J,v}_{w,\emptyset}$ of t in K_t is determined independently of the meaning of any relation symbol, the satisfaction of $K_t\varphi$ in non-standard model is in effect the same as the satisfaction of $K_t\varphi$ in model of the TML-semantics. By this fact we can validate axioms K and BF in this semantics.

The notion of validity is defined as in the TML-semantics. For ease of reference, henceforth we call this semantics *non-standard semantics*.

Now it is easy to see the invalidity of $x = c \to (P(x) \to P(c))$ in the non-standard semantics.

Proposition 2. Let $\langle\mathtt{Var},\mathtt{Cn},\mathtt{Fn},\mathtt{Rel},\mathtt{t}\rangle$ be a signature, $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ with $\mathtt{t}(x) = \mathtt{t}(c)$ and $P \in \mathtt{Rel}$ with $\mathtt{t}(P) = \langle\mathtt{agt\_or\_obj}\rangle$. A formula $x = c \to (P(x) \to P(c))$ is not valid in the non-standard semantics.

Proof. We may assume $\mathtt{t}(x) = \mathtt{t}(c) = \mathtt{agt}$ without loss of generality. Let $N = \langle D,W,R,J\rangle$ be a non-standard model such that $w \in W$, $D_{\mathtt{agt}} = \{\alpha,\beta\}$, $J(c,w,\{\langle d,d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}) = \alpha$, $J(c,w,\{\alpha\}) = \beta$ and $J(P,w) = \{\alpha\}$. Let v be also a valuation such that $v(x) = \alpha$. Since

$$[\![x]\!]^{J,v}_{w,J(=,w)} = v(x) = \alpha = J(c,w,\{\langle d,d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}) = J(c,w,J(=,w)) = [\![c]\!]^{J,v}_{w,J(=,w)},$$

we have $N,w \models_v x = c$. It is also easy to see $N,w \models_v P(x)$. However, since

$$[\![c]\!]^{J,v}_{w,J(P,w)} = J(c,w,J(P,w)) = J(c,w,\{\alpha\}) = \beta,$$

it fails that $N,w \models_v P(c)$. Therefore $x = c \to (P(x) \to P(c))$ is not valid in the non-standard semantics. □

On top of this, we can prove as below that H**K** is sound with respect to the non-standard semantics.

Proposition 3. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature and $x, y \in \mathtt{Var}$ with $\mathtt{t}(x) = \mathtt{t}(y)$. Let $N = \langle D, W, R, J\rangle$ be also a non-standard model, w a world, X a subset of D^n for some $n \in \mathbb{N}$ and v a valuation. For all terms t,

$$[\![t(y/x)]\!]^{J,v}_{w,X} \quad = \quad [\![t]\!]^{J,v[x\mapsto v(y)]}_{w,X}.$$

Proof. By induction on the length of terms.

- For t being the variable x, $[\![x(y/x)]\!]^{J,v}_{w,X} = v(y) = v[x \mapsto v(y)](x) = [\![x]\!]^{J,v[x\mapsto v(y)]}_{w,X}$.
- For t being a variable z distinct from x, $[\![z(y/x)]\!]^{J,v}_{w,X} = v(z) = v[x \mapsto v(y)](z) = [\![z]\!]^{J,v[x\mapsto v(y)]}_{w,X}$.
- For t being a constant c, $[\![c(y/x)]\!]^{J,v}_{w,X} = J(c, w, X) = [\![c]\!]^{J,v[x\mapsto v(y)]}_{w,X}$.
- For t being of the form $f(t_1, \dots, t_n)$,

$$\begin{aligned}
[\![f(t_1,\dots,t_n)(y/x)]\!]^{J,v}_{w,X} &= [\![f(t_1(y/x),\dots,t_n(y/x))]\!]^{J,v}_{w,X} \\
&= J(f,w,X)([\![t_1(y/x)]\!]^{J,v}_{w,X},\dots,[\![t_n(y/x)]\!]^{J,v}_{w,X}) \\
&= J(f,w,X)([\![t_1]\!]^{J,v[x\mapsto v(y)]}_{w,X},\dots,[\![t_n]\!]^{J,v[x\mapsto v(y)]}_{w,X}) \quad \text{(inductive hypothesis)} \\
&= [\![f(t_1,\dots,t_n)]\!]^{J,v[x\mapsto v(y)]}_{w,X}.
\end{aligned}$$

□

Proposition 4. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature, $x, y \in \mathtt{Var}$ with $\mathtt{t}(x) = \mathtt{t}(y)$ and $N = \langle D, W, R, J\rangle$ a non-standard model. For all worlds w, all valuations v and all formulas φ,

$$N, w \models_v \varphi(y/x) \quad \text{iff} \quad N, w \models_{v[x\mapsto v(y)]} \varphi.$$

Proof. By induction on the length of formulas. Since the proof of the cases for $\neg\psi$ and $\psi \wedge \gamma$ are straightforward, we see only the cases for $P(t_1, \dots, t_n)$, $\forall z\psi$ and $K_t\psi$.

- For φ being of the form $P(t_1, \dots, t_n)$,

$$\begin{aligned}
N,w \models_v P(t_1,\dots,t_n)(y/x) \quad &\text{iff} \quad \langle [\![t_1(y/x)]\!]^{J,v}_{w,J(P,w)},\dots,[\![t_n(y/x)]\!]^{J,v}_{w,J(P,w)}\rangle \in J(P,w) \\
&\text{iff} \quad \langle [\![t_1]\!]^{J,v[x\mapsto v(y)]}_{w,J(P,w)},\dots,[\![t_n]\!]^{J,v[x\mapsto v(y)]}_{w,J(P,w)}\rangle \in J(P,w) \quad \text{(Proposition 3)} \\
&\text{iff} \quad N,w \models_{v[x\mapsto v(y)]} P(t_1,\dots,t_n)
\end{aligned}$$

- For φ being of the form $\forall z\psi$, if $z = x$, then $N, w \models_v (\forall x\psi)(y/x)$ iff $N, w \models_v \forall x\psi$ iff $N, w \models_{v[x\mapsto v(y)]}$ $\forall x\psi$. So suppose $z \neq x$. Then

$$\begin{aligned}
N,w \models_v (\forall z\psi)(y/x) \quad &\text{iff} \quad N,w \models_v \forall z\psi(y/x) \\
&\text{iff} \quad N,w \models_{v[z\mapsto d]} \psi(y/x) \quad \text{for all } d \in D_{\mathtt{t}(z)} \\
&\text{iff} \quad N,w \models_{v[z\mapsto d][x\mapsto v[z\mapsto d](y)]} \psi \quad \text{for all } d \in D_{\mathtt{t}(z)} \quad \text{(inductive hypothesis)} \\
&\text{iff} \quad N,w \models_{v[x\mapsto v(y)][z\mapsto d]} \psi \quad \text{for all } d \in D_{\mathtt{t}(z)} \quad (z \neq x \text{ and } z \neq y) \\
&\text{iff} \quad N,w \models_{v[x\mapsto v(y)]} \forall z\psi
\end{aligned}$$

- For φ being of the form $K_t\psi$,

$$
\begin{array}{lll}
N,w \models_v (K_t\psi)(y/x) & \text{iff} & N,w \models_v K_{t(y/x)}\psi(y/x) \\
 & \text{iff} & N,w' \models_v \psi(y/x) \quad \text{all } w' \in W \text{ such that } \langle w,w'\rangle \in R_{[\![t(y/x)]\!]^{J,v}_{w,\emptyset}} \\
 & \text{iff} & N,w' \models_v \psi(y/x) \quad \text{all } w' \in W \text{ such that } \langle w,w'\rangle \in R_{[\![t]\!]^{J,v[x\mapsto v(y)]}_{w,\emptyset}} \\
 & & \text{(Proposition 3)} \\
 & \text{iff} & N,w' \models_{v[x\mapsto v(y)]} \psi \quad \text{all } w' \in W \text{ such that } \langle w,w'\rangle \in R_{[\![t]\!]^{J,v[x\mapsto v(y)]}_{w,\emptyset}} \\
 & & \text{(inductive hypothesis)} \\
 & \text{iff} & N,w \models_{v[x\mapsto v(y)]} K_t\psi.
\end{array}
$$

□

Theorem 1 (Soundness)**.** If φ is provable in H**K**, then φ is valid in the non-standard semantics.

Proof. It is sufficient to prove that all axioms are valid and that all inference rules preserve validity. Since the proof of the latter is done as usual, we see only the former.

- For any propositional tautology, its validity is obvious since the non-standard semantics gives the ordinary satisfactions for $\neg$ and $\wedge$.
- For UE, i.e., $\forall x\varphi \to \varphi(y/x)$, suppose $N,w \models_v \forall x\varphi$. Then $N,w \models_{v[x\mapsto v(y)]} \varphi$. Thus by Proposition 4 $N,w \models_v \varphi(y/x)$ holds, as required.
- For Id, i.e., $t = t$, its validity is obvious.
- For PS, i.e., $x = y \to (\varphi(x/z) \to \varphi(y/z))$, its validity is shown by induction on φ.
  - For φ being of the form $P(t_1,\dots,t_n)$, suppose $N,w \models_v x = y$ and $N,w \models_v P(t_1,\dots,t_n)(x/z)$. Since
  $$\langle [\![t_1(x/z)]\!]^{J,v}_{w,J(P,w)},\dots,[\![t_n(x/z)]\!]^{J,v}_{w,J(P,w)}\rangle \in J(P,w),$$
  we can use $v(x) = v(y)$ and Proposition 3 to obtain
  $$\langle [\![t_1(y/z)]\!]^{J,v}_{w,J(P,w)},\dots,[\![t_n(y/z)]\!]^{J,v}_{w,J(P,w)}\rangle \in J(P,w).$$
  Thus $N,w \models_v P(t_1,\dots,t_n)(y/z)$.
  - For φ being of the forms $\neg\psi$ or $\psi \wedge \gamma$, the proof is straightforward.
  - For φ being of the form $\forall z'\psi$, suppose $N,w \models_v x = y$ and $N,w \models_v (\forall z'\psi)(x/z)$. If $z' = z$, obviously $N,w \models_v (\forall z'\psi)(y/z)$. If $z' \neq z$, then we have $N,w \models_v \forall z'\psi(x/z)$ thus $N,w \models_{v[z'\mapsto d]} \psi(x/z)$ for all $d \in D_{\mathtt{t}(z')}$. Since we have $N,w \models_{v[z'\mapsto d]} x = y$ for all $d \in D_{\mathtt{t}(z')}$, by inductive hypothesis we obtain $N,w \models_{v[z'\mapsto d]} \psi(y/z)$ for all $d \in D_{\mathtt{t}(z')}$. Therefore, $N,w \models_v (\forall z'\psi)(y/z)$.
  - For φ being of the form $K_t\psi$, suppose $N,w \models_v x = y$ and $N,w \models_v (K_t\psi)(x/z)$. Then $N,w' \models_v \psi(x/z)$ for all $w' \in W$ such that $\langle w,w'\rangle \in R_{[\![t(x/z)]\!]^{J,v}_{w,\emptyset}}$. Now, we have $N,w' \models_v x = y$ for all $w' \in W$, as well as $[\![t(x/z)]\!]^{J,v}_{w,\emptyset} = [\![t(y/z)]\!]^{J,v}_{w,\emptyset}$ by $v(x) = v(y)$ and Proposition 3. So by inductive hypothesis we obtain $N,w' \models_v \psi(y/z)$ for all $w' \in W$ such that $\langle w,w'\rangle \in R_{[\![t(y/z)]\!]^{J,v}_{w,\emptyset}}$. Thus, $N,w \models_v (K_t\psi)(y/z)$.

- For $\exists$Id, i.e., $c = c \to \exists x(x = c)$, suppose $N, w \models_v c = c$. Since $N, w \models_{v[x \mapsto J(c,w,J(=,w))]} x = c$, we have $N, w \models_v \exists x(x = c)$, as required.
- For DD, i.e., $x \neq y$ if $\mathtt{t}(x) \neq \mathtt{t}(y)$, suppose $\mathtt{t}(x) \neq \mathtt{t}(y)$ and let N, w and v be arbitrary. By the definition of valuation, each of $v(x)$ and $v(y)$ is in $D_{\mathtt{t}(x)}$ and $D_{\mathtt{t}(y)}$, respectively. Since $\mathtt{t}(x) \neq \mathtt{t}(y)$, $D_{\mathtt{t}(x)}$ and $D_{\mathtt{t}(y)}$ must be disjoint. Thus $N, w \models_v x \neq y$, as required.
- For K, i.e., $K_t(\varphi \to \psi) \to (K_t\varphi \to K_t\psi)$, suppose $N, w \models_v K_t(\varphi \to \psi)$ and $N, w \models_v K_t\varphi$. Let w' be any world such that $\langle w, w' \rangle \in R_{[\![t]\!]^{J,v}_{w,\emptyset}}$. Then we have $N, w' \models_v \varphi \to \psi$ and $N, w' \models_v \varphi$. Thus $N, w' \models_v \psi$, as required.
- For BF, i.e., $\forall x K_t\varphi \to K_t\forall x\varphi$ for x not occurring in t, suppose $N, w \models_v \forall x K_t\varphi$. To show $N, w \models_v K_t\forall x\varphi$, let w' be any world such that $\langle w, w' \rangle \in R_{[\![t]\!]^{J,v}_{w,\emptyset}}$ and take any $d \in D_{\mathtt{t}(x)}$. By our supposition, we have $N, w \models_{v[x \mapsto d]} K_t\varphi$. Now $[\![t]\!]^{J,v}_{w,\emptyset} = [\![t]\!]^{J,v[x \mapsto d]}_{w,\emptyset}$ holds since x does not occur in t. Thus $N, w' \models_{v[x \mapsto d]} \varphi$, as required.
- For KNI, i.e., $x \neq y \to K_t x \neq y$, suppose $N, w \models_v x \neq y$. By definition, obviously $N, w' \models_v x \neq y$ for all worlds w'. Thus $N, w \models K_t x \neq y$, as required.

By the above argument the proof has completed. □

We can now prove the semantic incompleteness of H**K** as follows.

Theorem 2. Let $\Sigma = \langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature, $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ with $\mathtt{t}(x) = \mathtt{t}(c)$ and $P \in \mathtt{Rel}$ with $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj} \rangle$. A formula $x = c \to (P(x) \to P(c))$ is not provable in H**K**.

Proof. If $x = c \to (P(x) \to P(c))$ is provable in H**K**, then by the soundness (Theorem 1) it must be valid in the non-standard semantics, which contradicts Proposition 2. □

Corollary 1 (Semantic Incompleteness of H**K**)**.** The Hilbert-style system H**K** is semantically incomplete with respect to the TML-semantics, i.e., there exists some formula φ such that φ is valid in the TML-semantics but not provable in H**K**.

Proof. By Proposition 1 and Theorem 2. □

4 Conclusion

In this paper, we proved that Liberman et al.[11]'s Hilbert-style system H**K** for the term-modal logic **K** with equality and non-rigid terms is semantically incomplete by introducing the non-standard semantics for which H**K** is sound but in which $x = c \to (P(x) \to P(c))$ is not valid.

A further direction to be pursued is to give sound and complete Hilbert-style systems for term-modal logics including **K** with equality and non-rigid terms. Such systems, for example, might be obtained as slight modifications of the system given in Fagin et al. [3, p. 90]. Another further direction that might be worth studying is to apply the non-standard semantics to the analysis of natural language. As Example 1 suggests, it is reasonable to see $J(P)$ in $J(c, w, J(P))$ as a kind of *context* uniquely determining the denotation of a constant c at a world w. Thus, the non-standard semantics might be seen as a semantics capturing the context-dependency of the denotations of nouns in natural language.

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References

[1] A. Achen (2017): *Putting the Agents Back in the Domain: A Two-Sorted Term-Modal Logic*. Master's thesis, University of Copenhagen, doi:10.48550/arXiv.1811.01590. Bachelor thesis.

[2] Herbert Enderton (2001): *A Mathematical Introduction to Logic*, second edition. Academic Press, doi:10.1016/C2009-0-22107-6.

[3] Ronald Fagin, Yoram Moses, Joseph Y. Halpern & Moshe Y. Vardi (2003): *Reasoning about Knowledge*. MIT press.

[4] Melvin Fitting, Lars Thalmann & Andrei Voronkov (2001): *Term-Modal Logics*. *Studia Logica* 69, pp. 133–169, doi:10.1023/A:1013842612702.

[5] Stef Frijters (2021): *All Doctors Have an Obligation to Care for Their Patients: Term-modal Logics for Ethical Reasoning with Quantified Deontic Statements*. Ph.D. thesis, Ghent University.

[6] Stef Frijters (2023): *An Andersonian-Kangerian Reduction of Term-Modal Deontic Logics*. In: *Deontic Logic and Normative Systems 16th International Conference, DEON 2023*, College Publications, pp. 159–176.

[7] Stef Frijters & Thijs De Coninck (2021): *The Manchester Twins : Conflicts between Directed Obligations*. In Fenrong Liu, Alessandra Marra, Paul Portner & Frederik Van De Putte, editors: *Deontic Logic and Normative Systems: 15th International Conference, Proceedings*, College Publications.

[8] Stef Frijters, Joke Meheus & Frederik Van De Putte (2019): *Quantifing over the Indexes of Obligation Operators*. In: *Formal Ethics 2019*, Ghent University, Gent.

[9] Stef Frijters & Frederik Van De Putte (2020): *Classical Term-modal Logics*. *Journal of Logic and Computation*, doi:10.1093/logcom/exaa053.

[10] B. Kooi (2008): *Dynamic Term-modal Logic*. In J. van Benthem, S. Ju & F. Veltman, editors: *A Meeting of the Minds. Proceedings of the Workshop on Logic, Rationality and Interaction, Beijing, 2007, Texts in Computing Computer Science 8*, College Publications, pp. 173–185.

[11] Andrés Occhipinti Liberman, Andreas Achen & Rasmus Kræmmer Rendsvig (2020): *Dynamic Term-modal Logics for First-order Epistemic Planning*. *Artificial Intelligence* 286, p. 103305, doi:10.1016/j.artint.2020.103305.

[12] Andrés Occhipinti Liberman, Andreas Achen & Rasmus Kræmmer Rendsvig (2023): *Erratum to "Dynamic Term-Modal Logics for First-Order Epistemic Planning" [artif. Intell. 286 (2020) 103305]*. *Artificial Intelligence* 323, p. 103969, doi:10.1016/j.artint.2023.103969.

[13] Andrés Occhipinti Liberman & Rasmus K. Rendsvig (2019): *Dynamic Term-Modal Logic for Epistemic Social Network Dynamics*. In: *Logic, Rationality, and Interaction*, Springer Berlin Heidelberg, pp. 168–182, doi:10.1007/978-3-662-60292-8_13.

[14] Andrés Occhipinti Liberman & Rasmus Kræmmer Rendsvig (2020): *Decidability Results in First-Order Epistemic Planning*. In: *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence*, International Joint Conferences on Artificial Intelligence Organization, pp. 4161–4167, doi:10.24963/ijcai.2020/575.

[15] Pavel Naumov & Anna Ovchinnikova (2023): *De Re and De Dicto Knowledge in Egocentric Setting*, doi:10.48550/ARXIV.2308.00001.

[16] Pavel Naumov & Jia Tao (2019): *Everyone Knows that Someone Knows: Quantifiers over Epistemic Agents*. *The Review of Symbolic Logic* 12(2), pp. 255–270, doi:10.1017/s1755020318000497.

[17] Rasmus K. Rendsvig (2010): *Epistemic Term-Modal Logic*. In Marija Slavkovik, editor: *Proceedings of the 15th Student Session of the European Summer School in Logic, Language and Information*, pp. 37–46.

[18] Rasmus K. Rendsvig (2011): *Towards a Theory of Semantic Competence*. Master's thesis, Roskilde University.

[19] Takahiro Sawasaki & Katsuhiko Sano (2021): *Term-Sequence-Dyadic Deontic Logic*. In Fenrong Liu, Alessandra Marra, Paul Portner & Frederik Van De Putte, editors: *Deontic Logic and Normative Systems: 15th International Conference, DEON 2020/2021*, College Publication, pp. 376–393.

[20] Takahiro Sawasaki, Katsuhiko Sano & Tomoyuki Yamada (2019): *Term-Sequence-Modal Logics*. In Patrick Blackburn, Emiliano Lorini & Meiyun Guo, editors: *Logic, Rationality, and Interaction: 7th International Workshop, LORI 2019, Chongqing, China, October 18–21, 2019, Proceedings*, 11813, Springer-Verlag Berlin Heidelberg, pp. 244–258, doi:10.1007/978-3-662-60292-8_18.

[21] Igor Sedlár (2014): *Term-Modal Logics of Evidence*. *Epistemic Logic for Individual, Social, and Interactive Epistemology (ESSLLI 2014), Tübingen, Germany*.

[22] Lars Thalmann (2000): *Term-Modal Logic and Quantifier-free Dynamic Assignment Logic*. Ph.D. thesis, Uppsala University.

[23] Yanjing Wang & Jeremy Seligman (2018): *When Names are not Commonly Known: Epistemic Logic with Assignments*. In Guram Bezhanishvili, Giovanna D'Agostino, George Metcalfe & Thomas Studer, editors: *Advances in Modal Logic, Proceedings of the12th Conference on Advances in Modal Logic, Held in Bern, Switzerland, August 27-31, 2018* 12, College Publications, pp. 611–628, doi:10.48550/arXiv.1805.03852.

[24] Yanjing Wang, Yu Wei & Jeremy Seligman (2022): *Quantifier-free Epistemic Term-modal Logic with Assignment Operator*. *Annals of Pure and Applied Logic* 173(3), p. 103071, doi:10.1016/j.apal.2021.103071.

Agent-Knowledge Logic for Alternative Epistemic Logic

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Epistemic logic is known as a logic that captures the knowledge and beliefs of agents and has undergone various developments since Hintikka (1962). In this paper, we propose a new logic called agent-knowledge logic by taking the product of individual knowledge structures and the set of relationships among agents. This logic is based on the Facebook logic proposed by Seligman et al. (2011) and the Logic of Hide and Seek Game proposed by Li et al. (2021). We show two main results; one is that this logic can embed the standard epistemic logic, and the other is that there is a proof system of tableau calculus that works in finite time. We also discuss various sentences and inferences that this logic can express.

1 Introduction

Investigations into knowledge and beliefs form part of philosophy, which is now called epistemology. This area has been the subject of various studies from the standpoint of logic. One of these was conducted by applying modal logic, which is nowadays called epistemic logic. The operator K_i, which is the key element of this logic, has form $K_i\varphi$, which expresses that "agent i knows that φ." On this basis, it is possible to represent various concepts related to knowledge and belief in formal language. As far as I know, the pioneering work on epistemic logic was done by Hintikka in 1962 [10], and there is a wide range of research being done today; see Fagin et al. [7] and van Benthem [2].

A more recent logic for human knowledge is Facebook logic, developed by Seligman et al. in 2011 [17]. This logic was invented to describe personal knowledge plus the friendships of agents in two-dimensional hybrid logic. For instance, consider this sentence: "I am Andy's friend, and Andy knows he has pollen allergy. Then, one of my friends knows that they have pollen allergy." This inference can be written using the language of Facebook logic as follows:

$$\langle\texttt{Friend}\rangle i \wedge @_i[\texttt{Know}]p \to \langle\texttt{Friend}\rangle[\texttt{Know}]p$$

where $p =$ "they have pollen allergy." and $i =$ "This is Andy." The at sign @ in the logical formula is the operator of hybrid logic, where $@_i p$ can be read as "p holds at point i." Facebook logic uses nominals, a tool of hybrid logic, to make reference to individual agents. For a thorough introduction into hybrid logics, we refer the reader to Blackburn & ten Cate [3], Indrzejczak [11], and Braüner [5]. Sano [15] provides further details on two-dimensional hybrid logic.

In fact, Facebook logic treats propositional variables differently from epistemic logic. The truth of a propositional variable p depends not only on the epistemic alternative but also on the agent under consideration. Therefore, the propositions represented by the propositional variables here are personal properties, such as, "I have a pollen allergy."

The new logic proposed in this paper — we will call it *agent-knowledge logic* — is a modification of the aforementioned Facebook logic. One feature of this logic is that the fragment of it is compatible

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EPTCS 415, 2024, pp. 77–92, doi:10.4204/EPTCS.415.10

with epistemic logic. This property allows us to use agent-knowledge logic as an alternative to epistemic logic. Indeed, this paper shows how to embed epistemic logic into our new logic. Furthermore, agent-knowledge logic is able to formalize a variety of sentences that cannot be represented by traditional epistemic logic, such as "one of my friends knows p." Some of the examples given in this paper may be only part of the possibilities our new logic opens up.

In this paper, we also introduce a proof system, by constructing a tableau calculus. The tableau calculus is not only a proof system but also a system for discovering a counterexample model in which the formula is not valid. In particular, by constructing a tableau calculus with the termination property — in short, that the proof ends in finite time — we can show that the logic is decidable.

This logic has two parents: one is Facebook logic, and the other is, which seems to have nothing to do with epistemic logic, the Logic of Hide and Seek Game (LHS, in short) created by Li et al. in 2021 [12, 13]. This logic was originally invented to illustrate the hide and seek game (also known as cops and robbers). In LHS, propositional variables are split into two sets, which are related to hider and seeker, respectively. We borrow this idea to express the *agent-free* propositions ("the sun rises in the east," for example.)

We proceed as follows: Section 2 reviews the well-known epistemic logic and explains the parents of agent-knowledge logic, Facebook logic, and LHS, briefly. In Section 3, we introduce our new logic, that is, agent-knowledge logic. Section 4 shows how we embed epistemic logic into our new logic. In Section 5, we construct a tableau calculus with the termination property and completeness. Finally, in Section 6, we write about some future prospects.

2 Preliminary

2.1 Epistemic Logic

This section is mostly based on the work of Fagin et al. [7, Chapter 2].

In epistemic logic, we have another set $\mathbf{A}$ of agents besides a usual set $\mathbf{Prop}$ of propositional variables. The elements of $\mathbf{A}$ occur in a new operator K_i. The intuitive meaning of $K_i\varphi$ is that "agent i knows φ."

Definition 2.1. We have two disjoint sets, $\mathbf{Prop}$ and $\mathbf{A}$. A formula φ of the *epistemic logic* $\mathscr{L}_{\mathrm{EL}}$ is defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi$$

where $p \in \mathbf{Prop}$ and $i \in \mathbf{A}$.

We only use $\neg$ and $\wedge$ as primitives since other Boolean operators, such as $\vee$ and $\rightarrow$, can be defined as compounds of the first two operators.

Definition 2.2. A *Kripke model for epistemic logic* (we call it *EL model*) $\mathscr{M}_{\mathrm{EL}}$ is a tuple $(W, (R_i)_{i\in\mathbf{A}}, V)$ where

- W is a non-empty set,
- For each $i \in \mathbf{A}$, R_i is a binary relation on W,
- $V : \mathbf{Prop} \rightarrow \mathscr{P}(W)$.

Definition 2.3. Given an EL model $\mathscr{M}_{\text{EL}}$, its point w, and a formula $\varphi \in \mathscr{L}_{\text{EL}}$, the *satisfaction relation* $\mathscr{M}_{\text{EL}}, w \models \varphi$ is defined inductively as follows:

$$\begin{aligned}
\mathscr{M}_{\text{EL}}, w \models p &\iff w \in V(p) \text{ where } p \in \mathbf{Prop},\\
\mathscr{M}_{\text{EL}}, w \models \neg\varphi &\iff \text{Not } \mathscr{M}_{\text{EL}}, w \models \varphi \ (\mathscr{M}_{\text{EL}}, w \not\models \varphi),\\
\mathscr{M}_{\text{EL}}, w \models \varphi \wedge \psi &\iff \mathscr{M}_{\text{EL}}, w \models \varphi \text{ and } \mathscr{M}_{\text{EL}}, w \models \psi,\\
\mathscr{M}_{\text{EL}}, w \models K_i\varphi &\iff \text{For all } v \in W, wR_iv \text{ implies } \mathscr{M}_{\text{EL}}, v \models \varphi.
\end{aligned}$$

As for epistemic logic, we define the validity of a formula. Later we discuss embedding epistemic logic into our new logic, so the formal definition is needed.

Definition 2.4. A formula φ is *valid* with respect to the class of EL models (written as $\models_{\text{EL}} \varphi$) if $\mathscr{M}_{\text{EL}}, w \models \varphi$ for every model $\mathscr{M}_{\text{EL}}$ and its every world w.

2.2 Facebook Logic

Facebook logic, firstly invented by Seligman et al. [17], has two characteristics compared to classical modal logic.

First, we have two modal operators, K and F. These modal operators correspond to knowledge and friendship, respectively. Correspondingly, a possible world is decomposed into two components: one representing an agent and the other representing an epistemic alternative of an individual.

Another addition is the introduction of special propositional variables called *nominals*. A nominal n is a proposition corresponding to only one agent, which is a proposition for the *name* of the agent. In addition, we introduce the satisfaction operator @ used in hybrid logic. The intuitive meaning of $@_n p$ is that "p holds for agent n."

Let us introduce a formal definition. We have two disjoint infinite sets, **Prop** of propositional variables and **Nom** of nominals. A formula φ of the *Facebook logic* is defined as follows:

$$\varphi ::= p \mid n \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid F\varphi \mid @_n\varphi$$

where $p \in \mathbf{Prop}$ and $n \in \mathbf{Nom}$. If needed, we can define the dual $\langle K \rangle$ and $\langle F \rangle$ of each modal operators as $\langle K \rangle\varphi := \neg K \neg\varphi$ and $\langle F \rangle\varphi := \neg F \neg\varphi$.

The semantics of Facebook logic is based on *epistemic social network models*. An epistemic social network model is a tuple $(W, A, (\sim_a)_{a \in A}, (\asymp_w)_{w \in W}, V)$, where

- W is a set of epistemic alternatives,
- A is a set of agents,
- For each $a \in A$, $\sim_a$ is an equivalence relation on W,
- For each $w \in W$, $\asymp_w$ is an irreflexive and symmetric relation of friendship on A, and
- V is a valuation function, which assigns a propositional variable p to a subset of $W \times A$ and a nominal n to a set $W \times \{a\}$ for some $a \in A$.

The reason for a relation $\asymp_w$ over A being irreflexive and symmetric can be understood when we assume it as a friendship; no one is a friend to oneself, and if a person is your friend, then you are a friend of them.

Then, the truth of formulas in Facebook logic is defined inductively. The Boolean cases are omitted since they are the same as those in classical modal logic. Also, the element $a \in A$ such that $V(n) = W \times \{a\}$ holds is abbreviated as n^V.

$$\begin{aligned}
\mathcal{M}, w, a \models p &\iff (w,a) \in V(p) \text{ where } p \in \mathbf{Prop}, \\
\mathcal{M}, w, a \models n &\iff n^V = a, \text{ where } n \in \mathbf{Nom} \\
\mathcal{M}, w, a \models K\varphi &\iff \mathcal{M}, v, a \models \varphi \text{ for every } v \sim_a w, \\
\mathcal{M}, w, a \models F\varphi &\iff \mathcal{M}, w, b \models \varphi \text{ for every } b \asymp_w a, \\
\mathcal{M}, w, a \models @_n\varphi &\iff \mathcal{M}, w, n^V \models \varphi.
\end{aligned}$$

As mentioned in the Introduction, the truth of a propositional variable depends on both an epistemic alternative and an agent.

Example 2.5. The following formulas of Facebook logic can be translated into natural language as follows.

- Kp: I know that I am p.
- KFp: I know that all of my friends are p.
- FKp: Each of my friends knows that they are p.
- $\langle F \rangle n$: I have a friend n.
- $@_nKp$: An agent n knows that they are p.

For the readers who would like to study it deeper, Seligman et al. [17] and its sequel, Seligman et al. [18], should be of help.

2.3 Logic of Hide and Seek Game

The logic of hide and seek game (LHS), as the name implies, is a logic for describing a hide and seek game. There are two players, a hider and a seeker, and a set of propositional variables $\mathbf{Prop}_H$ and $\mathbf{Prop}_S$ for each player to describe their state. Moreover, there is a special propositional variable I. This is a proposition to describe that the hider and seeker are in the same place, i.e., expressing "I find you!"

The main difference from Facebook logic is that we use the same structure (W, R, V) as in usual modal logic, which is appropriate considering that the hide and seek game is played by two players on the same board.

Here is a definition of a formula of LHS φ, where $p_H \in \mathbf{Prop}_H$ and $p_S \in \mathbf{Prop}_S$.

$$\varphi ::= p_H \mid p_S \mid I \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_H \varphi \mid \Diamond_S \varphi$$

The truth value of LHS formulas is defined inductively as follows. Note that both x and y are elements of W.

$$\begin{aligned}
\mathcal{M}, x, y \models p_H &\iff x \in V(p_H) \text{ where } p_H \in \mathbf{Prop}_H, \\
\mathcal{M}, x, y \models p_S &\iff y \in V(p_S) \text{ where } p_S \in \mathbf{Prop}_S, \\
\mathcal{M}, x, y \models I &\iff x = y, \\
\mathcal{M}, x, y \models \Diamond_H\varphi &\iff \text{there is some } x' \text{ such that } xRx' \text{ and } \mathcal{M}, x', y \models \varphi, \\
\mathcal{M}, x, y \models \Diamond_S\varphi &\iff \text{there is some } y' \text{ such that } yRy' \text{ and } \mathcal{M}, x, y' \models \varphi.
\end{aligned}$$

Using this language, we can describe the hide and seek game. For example, $\Box_H \Diamond_S I$ means that no matter how the hider moves, the seeker has a one-step move to catch the hider. This expression shows the existence of a winning strategy for the seeker.

In addition to the already mentioned Li et al. [12], Li et al. [13] may also help readers who want to know more about LHS.

3 Agent-Knowledge Logic

Here, we introduce a new logic, called *agent-knowledge logic*. As you read in Section 1, this logic is a mixture of Facebook logic and LHS. We have two dimensions, which correspond to agents and their knowledge, respectively. This structure and the intention behind it are very similar to that of Facebook logic. On the other hand, the idea that we use both $\mathbf{Prop}_A$ and $\mathbf{Prop}_K$ is unique for LHS.

3.1 Agent-Knowledge Model

Definition 3.1. We have four disjoint sets $\mathbf{Prop}_A$, $\mathbf{Prop}_K$, $\mathbf{Nom}_A$, and $\mathbf{Nom}_K$. A formula φ of the *agent-knowledge logic* $\mathscr{L}_{\mathrm{AK}}$ is defined as follows:

$$\varphi ::= p_A \mid p_K \mid a \mid k \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_A \varphi \mid \Box_K \varphi \mid @_a \varphi \mid @_k \varphi$$

where $p_A \in \mathbf{Prop}_A$, $p_K \in \mathbf{Prop}_K$, $a \in \mathbf{Nom}_A$, and $k \in \mathbf{Nom}_K$.

We call an element of both $\mathbf{Nom}_A$ or $\mathbf{Nom}_K$ a *nominal*. As we mentioned in Section 2.2, they point to a specific agent and a specific epistemic alternative, respectively. As well as $\vee$ and $\rightarrow$, if we need, we can define $\Diamond_A$ and $\Diamond_K$ in the usual way.

Definition 3.2. A *agent-knowledge model (AK model)* $\mathscr{M}_{\mathrm{AK}}$ is a tuple $(W_A, W_K, (R_y)_{y \in W_K}, (S_x)_{x \in W_A}, V_A, V_K)$ where

- W_A, W_K are non-empty sets,
- For each $y \in W_K$, R_y is a binary relation on W_A,
- For each $x \in W_A$, S_x is a binary relation on W_K,
- $V_A : \mathbf{Prop}_A \cup \mathbf{Nom}_A \rightarrow \mathscr{P}(W_A)$ where if $a \in \mathbf{Nom}_A$, then $V_A(a) = \{x\}$ for some $x \in W_A$,
- $V_K : \mathbf{Prop}_K \cup \mathbf{Nom}_K \rightarrow \mathscr{P}(W_K)$ where if $k \in \mathbf{Nom}_K$, then $V_K(k) = \{y\}$ for some $y \in W_K$.

Note that the image of a nominal $\mathbf{Nom}_A$ by V_A is a singleton (the same fact holds for $\mathbf{Nom}_K$ and V_K.) Owing to this definition, nominal behaves as a *name* for each possible world.

We can illustrate an agent-knowledge model as if we write Cartesian coordinates in Figure 1. In this circumstance, a nominal is represented as a horizontal or vertical line. Likely, a propositional variable is depicted as a set of parallel lines.

We write V to express $V_A \cup V_K$. For instance, $V(p_A) = V_A(p_A)$. Moreover, we abbreviate $x \in W_A$ such that $V_A(a) = \{x\}$ by a^V. We do the same for k^V.

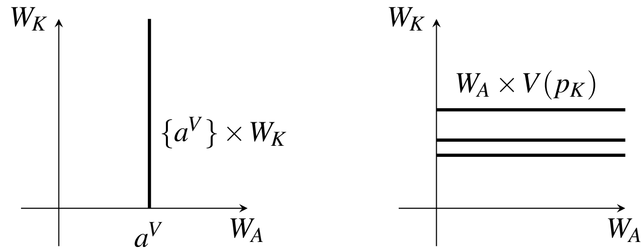

Figure 1: An agent-knowledge model.

Definition 3.3. Given a model $\mathscr{M}_{\text{AK}}$, its points $(x,y) \in W_A \times W_K$, and a formula $\varphi \in \mathscr{L}_{\text{AK}}$, the *satisfaction relation* $\mathscr{M}_{\text{AK}},(x,y) \models \varphi$ is defined inductively as follows:

$$\begin{aligned}
\mathscr{M}_{\text{AK}},(x,y) \models p_A &\iff x \in V(p_A) \text{ where } p_A \in \mathbf{Prop}_A,\\
\mathscr{M}_{\text{AK}},(x,y) \models p_K &\iff y \in V(p_K) \text{ where } p_K \in \mathbf{Prop}_K,\\
\mathscr{M}_{\text{AK}},(x,y) \models a &\iff x = a^V \text{ where } a \in \mathbf{Nom}_A,\\
\mathscr{M}_{\text{AK}},(x,y) \models k &\iff y = k^V \text{ where } k \in \mathbf{Nom}_K,\\
\mathscr{M}_{\text{AK}},(x,y) \models \neg\varphi &\iff \text{Not } \mathscr{M}_{\text{AK}},(x,y) \models \varphi \ (\mathscr{M}_{\text{AK}},(x,y) \not\models \varphi),\\
\mathscr{M}_{\text{AK}},(x,y) \models \varphi \wedge \psi &\iff \mathscr{M}_{\text{AK}},(x,y) \models \varphi \text{ and } \mathscr{M}_{\text{AK}},(x,y) \models \psi,\\
\mathscr{M}_{\text{AK}},(x,y) \models \Box_A\varphi &\iff \text{For all } x' \in W_A, xR_yx' \text{ implies } \mathscr{M}_{\text{AK}},(x',y) \models \varphi,\\
\mathscr{M}_{\text{AK}},(x,y) \models \Box_K\varphi &\iff \text{For all } y' \in W_K, yS_xy' \text{ implies } \mathscr{M}_{\text{AK}},(x,y') \models \varphi,\\
\mathscr{M}_{\text{AK}},(x,y) \models @_a\varphi &\iff \mathscr{M}_{\text{AK}},(a^V,y) \models \varphi,\\
\mathscr{M}_{\text{AK}},(x,y) \models @_k\varphi &\iff \mathscr{M}_{\text{AK}},(x,k^V) \models \varphi.
\end{aligned}$$

The truth of each propositional variable is determined by either $x \in W_A$ or $y \in W_K$. Especially whether p_K is true or false is independent of the element of W_A, so p_K can be assumed as an agent-free proposition.

The usage of the satisfaction operator @ should also be mentioned. It refers to a specific agent or epistemic alternative while ignoring the current one. For example, the meaning of $@_a\varphi$ is "for an agent whose name is a, φ holds." The current element of W_A is no longer necessary information to determine the truth of that formula.

Definition 3.4. A formula φ is *valid* with respect to the class of $\mathscr{M}_{\text{AK}}$ (written as $\models_{\text{AK}} \varphi$) if $\mathscr{M}_{\text{AK}},(x,y) \models \varphi$ for every model $\mathscr{M}_{\text{AK}}$ and its every pair (x,y).

3.2 Examples

As we do in Facebook logic, we can compound friendship and knowledge in agent-knowledge logic. We read $\Box_K\varphi$ as "I know φ," and $\Box_A\varphi$ as "All of my friend are φ." For example, we can write some sentences as follows.

- $\Box_A\Box_K p_K$: All of my friends know p_K.
- $\Diamond_A\Box_K p_K$: Some of my friends know p_K.
- $\Box_K\Diamond_A\Box_K$: I know that some of my friends know p_K.

Moreover, we can designate an individual by calling their name owing to nominals. Consider this sentence:

I am Andy's friend, and Andy knows he has that the Earth goes around the Sun Then, one of my friends knows the heliocentric theory.

This inference can be symbolized in the agent-knowledge logic as follows:

$$\Diamond_A a \wedge @_a \Box_K p_K \to \Diamond_A \Box_K p_K,$$

where p_K shows "the Earth goes around the Sun" and a shows "This is Andy."

The difference between agent-knowledge logic and Facebook logic becomes more pronounced when we assume that the binary relations over epistemic alternatives are equivalence relations. For example, in Facebook logic, the formula $@_n Kp \to p$ is not valid even if $\asymp_w$ is an equivalence relation. Define $\mathscr{M} = (W, A, (\sim_a)_{a \in A}, (\asymp_w)_{w \in W}, V)$ as follows:

$$\begin{aligned} W &= \{w, v\}, \\ A &= \{a, b\}, \\ \sim_a &= \sim_b = W \times W, \\ \asymp_w &= \asymp_v = \emptyset, \\ V(p) &= \{(w,b), (v,b)\}, \\ V(n) &= W \times \{b\}. \end{aligned}$$

Then, $\mathscr{M}, (w,a) \models @_n Kp$ holds but we have $\mathscr{M}, (w,a) \not\models p$. However, in agent-knowledge logic, the situation changes.

Proposition 3.5. The formula $@_a \Box_K p_K \to p_K$ is valid with respect to the class of $\mathscr{M}_{\mathrm{AK}}$ where all of S_x are equivalence relations.

Proof. Suppose that $\mathscr{M}_{\mathrm{AK}}, (x,y) \models @_a \Box_K p_K$. Then, we have $\mathscr{M}_{\mathrm{AK}}, (a^V, y) \models \Box_K p_K$. By the reflexivity of S_y, especially we have $\mathscr{M}_{\mathrm{AK}}, (a^V, y) \models p_K$. Since the truth value of p_K is determined only by an element of W_K, we have $\mathscr{M}_{\mathrm{AK}}, (x,y) \models p_K$. ■

This fact may be better understood if we interpret those formulas into natural language. Even though Andy knows he has pollen allergy, it does not mean so does I. However, if he knows that the Earth goes around the Sun, then it is true; the Earth really goes around the Sun.

In addition to the relationships between epistemic alternatives, we can also impose restrictions on the relationships between agents as needed. For example, in Facebook logic, the relationship between agents should be irreflexive and symmetric. Also, we have another way to capture relationships between agents, for example, to read $xR_y x'$ as "in the situation y, the agent x can see the post of x'" in X[1]. Then, we can read $\Box_A \Box_K p_K$ as "all the people know p_K, *as far as I know*."

4 Embedding Epistemic Logic into Agent-Knowledge Logic

One of the aims of our new logic is to make it an alternative to Facebook logic. In fact, any sentence we can express in basic epistemic logic can be rewritten in this agent-knowledge logic. In this section, we show that we can embed epistemic logic into agent-knowledge logic.

First of all, we identify the theorem we wish to prove. A proper translation T exists, and the following theorem holds.

[1]Most of the readers are familiar with the name once it had; twitter.

Theorem 4.1. For all $\varphi \in \mathscr{L}_{\mathrm{EL}}$,

$$\models_{\mathrm{EL}} \varphi \iff \models_{\mathrm{AK}} T(\varphi).$$

To prove it, let us define how to translate a formula of epistemic logic.

Definition 4.2. We define a *translation* $T : \mathscr{L}_{\mathrm{EL}} \to \mathscr{L}_{\mathrm{AK}}$ as follows:

$$\begin{aligned} T : \mathbf{Prop} \ni p \mapsto p_K \in \mathbf{Prop}_K &\text{ is a bijection,} \\ T : \mathbf{A} \ni i \mapsto a \in \mathbf{Nom}_A &\text{ is a bijection,} \\ T(\neg\varphi) &= \neg T(\varphi), \\ T(\varphi \wedge \psi) &= T(\varphi) \wedge T(\psi), \\ T(K_i\varphi) &= @_{T(i)} \Box_K T(\varphi). \end{aligned}$$

Example 4.3. Here is one example of translation.

$$T(K_i(p \wedge K_j \neg q)) = @_{a_i} \Box_K (p_K \wedge @_{a_j} \Box_K \neg q_K).$$

We write a_i to abbreviate $T(i)$ $(i \in \mathbf{A})$.

In fact, the idea of rewriting $K_i\varphi$ as $@_{T(i)}\Box_K T(\varphi)$ was presented in Sano's review in 2011 [16] for Japanese, which introduces Seligman et al. [17]. Unfortunately, this translation does not work for Facebook logic, but it does work when the target logic is agent-knowledge logic.

Definition 4.4. Given an EL model $\mathscr{M}_{\mathrm{EL}} = (W, (R_i)_{i \in \mathbf{A}}, V)$, the induced AK model $\mathscr{M}^{\alpha}_{\mathrm{AK}}$ is defined as follows:

$\mathscr{M}^{\alpha}_{\mathrm{AK}} = (\mathbf{A}, W, \emptyset, (R_i)_{i \in \mathbf{A}}, V^{\alpha})$, where

- For any $p_A \in \mathbf{Prop}_A$, $V^{\alpha}(p_A) = \emptyset$,
- For any $p_K \in \mathbf{Prop}_K$, $V^{\alpha}(p_K) = V(T^{-1}(p_K))$,
- For any $a \in \mathbf{Nom}_A$, $V^{\alpha}(a) = V(T^{-1}(a))$,
- Take one $y_0 \in W$, and for any $k \in \mathbf{Nom}_K$, $V^{\alpha}(k) = \{y_0\}$.

Note that we do not care about the definitions of $(R_y)_{y \in W_K}, V^{\alpha}(p_A)$, and $V^{\alpha}(k)$. It is because the formula translated by T requires only $\mathbf{Prop}_K, \mathbf{Nom}_A$, Boolean operators, $\Box_K$, and $@_a$.

Lemma 4.5. For any $\varphi \in \mathscr{L}_{\mathrm{EL}}$ and for any $i \in \mathbf{A}$, we have:

$$\mathscr{M}_{\mathrm{EL}}, w \models \varphi \iff \mathscr{M}^{\alpha}_{\mathrm{AK}}, (i, w) \models T(\varphi).$$

Proof. By induction on the complexity of φ.

$(\varphi = p)$ For all $i \in \mathbf{A}$,

$$\begin{aligned} \mathscr{M}_{\mathrm{EL}}, w \models p &\iff w \in V(p) \\ &\iff w \in V^{\alpha}(T(p)) \\ &\iff \mathscr{M}^{\alpha}_{\mathrm{AK}}, (i, w) \models T(p). \end{aligned}$$

$(\varphi = \neg\psi, \psi \wedge \chi)$ Straightforward.

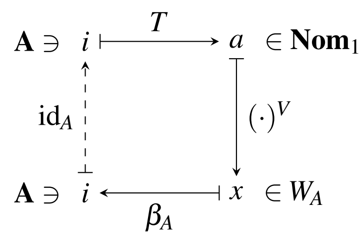

Figure 2: The condition β_A satisfies (id_A is the identity on $\mathbf{A}$).

$(\varphi = K_j\psi)$ First, we prove the left-to-right direction.

Suppose that $\mathscr{M}_{\mathrm{EL}}, w \models K_j\psi$. Then, for all v such that wR_jv, we have $\mathscr{M}_{\mathrm{EL}}, v \models \psi$. We divide the proof into two cases depending on whether such a world $v \in W$ exists.

(i) If there is some $v \in W$, take arbitrary one. Then, we have $\mathscr{M}_{\mathrm{EL}}, v \models \varphi$. By the induction hypothesis, especially $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (j,v) \models T(\varphi)$. Since we took v arbitrarily, it follows that $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (j,w) \models \Box_K T(\varphi)$. By the definition of V^{α}, we finally get that $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (i,w) \models @_{T(j)}\Box_K T(\varphi)$ for all $i \in A$.

(ii) If there is no $v \in W$ such that wR_jv, we straightforwardly get that $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (j,w) \models \Box_K T(\varphi)$. In the same way as in the former case, we have $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (i,w) \models @_{T(j)}\Box_K T(\varphi)$ for all $i \in \mathbf{A}$.

In both cases, we can reach the result that $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (i,w) \models @_{T(j)}\Box_K T(\varphi)$ for all $i \in \mathbf{A}$. Therefore, we have $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (i,w) \models T(K_j\varphi)$.

Next, we prove the other direction. Take one $i \in A$ and suppose that $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (i,w) \models T(K_j\psi)$. It means that for all v such that wR_jv, $\mathscr{M}^{\alpha}_{\mathrm{AK}}, (j,v) \models T(\psi)$ holds. Take one v such that wR_jv (if we cannot, then $\mathscr{M}_{\mathrm{EL}}, w \models K_j\psi$ is straightforward.) By the induction hypothesis, we have $\mathscr{M}_{\mathrm{EL}}, v \models \psi$. Since we took v arbitrarily, it follows that $\mathscr{M}_{\mathrm{EL}}, w \models K_j\psi$.

■

Definition 4.6. Given an AK model $\mathscr{M}_{\mathrm{AK}} = (W_A, W_K, (R_y)_{y \in W_K}, (S_x)_{x \in W_A}, V)$, the induced EL model $\mathscr{M}^{\beta}_{\mathrm{EL}}$ is defined as follows:

$\mathscr{M}^{\beta}_{\mathrm{EL}} = (W_K, (S^{\beta}_i)_{i \in \mathbf{A}}, V^{\beta})$, where

- $\mathbf{A}$ is the set used in Definition 2.1,
- $yS^{\beta}_i z$ in $\mathscr{M}^{\beta}_{\mathrm{EL}}$ iff $yS_{T(i)^V}z$ in $\mathscr{M}_{\mathrm{AK}}$,
- $V^{\beta}(p) = V(T(p))$.

Let us consider a function $\beta_A : W_A \to \mathbf{A}$ such that $\beta(T(i)^V) = i$ for all $i \in \mathbf{A}$. It expresses the correspondence between an agent in W_A and an agent in $\mathbf{A}$. The illustration of this condition in Figure 2 may help your understanding.

Lemma 4.7. For any $\varphi \in \mathscr{L}_{\mathrm{EL}}$ and for any $x \in W_A$,

$$\mathscr{M}_{\mathrm{AK}}, (x,y) \models T(\varphi) \iff \mathscr{M}^{\beta}_{\mathrm{EL}}, y \models \varphi.$$

Proof. By induction on the complexity of φ.

$(\varphi = p)$ For all $x \in W^A$,

$$\begin{aligned}\mathscr{M}_{\mathrm{AK}},(x,y) \models T(p) &\iff y \in V(T(p)) \\ &\iff y \in V^{\beta}(p) \\ &\iff \mathscr{M}^{\beta}_{\mathrm{EL}}, y \models p.\end{aligned}$$

$(\varphi = \neg\psi, \psi \wedge \chi)$ Straightforward.

$(\varphi = K_j\psi)$ First, we prove the left-to-right direction.

Suppose that $\mathscr{M}_{\mathrm{AK}},(x,y) \models T(K_j\psi)$. That is, we assume that $\mathscr{M}_{\mathrm{AK}},(x,y) \models @_{T(j)}\Box_K T(\psi)$. Then, for all z such that $yS_{T(j)^V}z$, we have $\mathscr{M}_{\mathrm{AK}},(T(j)^V,z) \models T(\psi)$. Bearing the definition of S^{β}_i, it suffices to pick up one $z \in W_K$ such that $yS^{\beta}_j z$ (if we cannot, it is straightforward that $\mathscr{M}^{\beta}_{\mathrm{EL}}, y \models K_j\psi$ holds.) By the assumption, we have $\mathscr{M}_{\mathrm{AK}},(T(j)^V,z) \models T(\psi)$. By the induction hypothesis, $\mathscr{M}^{\beta}_{\mathrm{EL}}, z \models \psi$. Since we picked up z arbitrarily, we have $\mathscr{M}^{\beta}_{\mathrm{EL}}, y \models K_j\psi$.

Next, we prove the other direction. Suppose that $\mathscr{M}^{\beta}_{\mathrm{EL}}, y \models K_j\psi$. It means that for all $z \in W_K$ such that $yS^{\beta}_j z$, $\mathscr{M}^{\beta}_{\mathrm{EL}}, z \models \psi$ holds. Now, pick $z \in W_A$ such that $yS_{T(j)^V}z$ arbitrarily (if we cannot, we have $\mathscr{M}_{\mathrm{AK}},(x,y) \models T(K_j\psi)$ for all $x \in W_A$.), and we have $yS^{\beta}_j z$. Then, we have $\mathscr{M}^{\beta}_{\mathrm{AK}}, z \models \psi$. By the induction hypothesis, $\mathscr{M}_{\mathrm{AK}},(T(j)^V,z) \models T(\psi)$. Since we pick up z arbitrarily, it follows that $\mathscr{M}_{\mathrm{AK}},(x,y) \models @_{T(j)}\Box_K T(\psi)$ for any $x \in W_A$, which means $\mathscr{M}_{\mathrm{AK}},(x,y) \models T(K_j\psi)$.

■

Now, we are ready to prove the main theorem, Theorem 4.1. Here is the proof.

Proof. We prove it by showing the contraposition. To prove the left-to-right direction, suppose that we have some φ such that $\not\models_{\mathrm{AK}} T(\varphi)$. Then, there is a model $\mathscr{M}_{\mathrm{AK}}$ and its pair of points (x,y) such that $\mathscr{M}_{\mathrm{AK}},(x,y) \models \neg T(\varphi)$, which means that $\mathscr{M}_{\mathrm{AK}},(x,y) \models T(\neg\varphi)$. Then, by Lemma 4.7, we have $\mathscr{M}^{\beta}_{\mathrm{EL}}, y \models \neg\varphi$, which leads us to the conclusion that $\not\models_{\mathrm{EL}} \varphi$. The case of the other direction can be done by using Lemma 4.5. ■

We usually treat binary relations of EL models as equivalence relations. Moreover, once we want to deal with beliefs by means of a modal operator, we impose yet another condition on accessibility relations. The following corollary shows how embedding can reflect these restrictions.

Proposition 4.8. We have the following properties:

(i) For every $i \in \mathbf{A}$, if R_i in $\mathscr{M}_{\mathrm{EL}}$ is reflexive (or serial, symmetric, transitive, euclidian), then so is R_i in $\mathscr{M}^{\alpha}_{\mathrm{AK}}$.

(ii) For every $x \in W_A$, if S_x in $\mathscr{M}_{\mathrm{AK}}$ is reflexive (or serial, symmetric, transitive, euclidian), then so is S^{β}_i in $\mathscr{M}^{\beta}_{\mathrm{EL}}$.

Proof. The former is obvious, and the latter is straightforward from the definition of S^{β}_i. ■

5 Proof System

In this section, we introduce a tableau calculus as a proof system.

In constructing a tableau calculus for agent-knowledge logic, we have made significant references to that for hybrid logic. The primary reference is the work of Bolander and Blackburn [4]. We also refer to Nishimura [14], which studies tableau calculi for some two-dimensional hybrid logics.

For simplicity, this section deals only with the negation normal form (NNF, in short) of formulas. For the satisfaction operators, a formula $\neg @_a \varphi$ is equivalent to $@_a \neg \varphi$. That is, for any model and its possible world (x, y), a formula φ, and a nominal $a \in \mathbf{Nom}_A$, we have

$$\mathscr{M}_{\mathrm{AK}}, (x,y) \models @_a \neg \varphi \iff \mathscr{M}_{\mathrm{AK}}, (x,y) \models \neg @_a \varphi.$$

The same equivalence holds for the case of $k \in \mathbf{Nom}_K$. Transformations to the NNF involving Boolean and modal operators can be done in the usual way.

5.1 Tableau Calculus

Here we provide a tableau calculus of agent-knowledge logic, denoted by $\mathbf{T}_{\mathrm{AK}}$.

Definition 5.1. A *tableau* is a well-founded tree constructed in the following way:

- Start with a formula of the form $@_a @_k \varphi$ (called the *root formula*), where φ is a formula of agent-knowledge logic and $a \in \mathbf{Nom}_A, k \in \mathbf{Nom}_K$ does not occur in φ.
- For each branch, extend it by applying rules (see Definition 5.3) to all nodes as often as possible. However, we can no longer add any formula in a branch if at least one of the following conditions is satisfied:
  - (i) Every new formula generated by applying any rule already exists in the branch.
  - (ii) The branch is closed (see Definition 5.2.)

Here, a *branch* means a maximal path of a tableau. If a formula φ occurs in a branch Θ, we write $\varphi \in \Theta$.

Definition 5.2. A branch of a tableau Θ is *closed* if one of the following condition holds.

(i) There are $a \in \mathbf{Nom}_A$, $k, l \in \mathbf{Nom}_K$, and $p_A \in \mathbf{Prop}_A$ such that $@_a @_k p_A, @_a @_l \neg p_A \in \Theta$.

(ii) There are $a, b \in \mathbf{Nom}_A$, $k \in \mathbf{Nom}_K$, and $p_K \in \mathbf{Prop}_K$ such that $@_a @_k p_K, @_b @_k \neg p_K \in \Theta$.

(iii) There are $a, b \in \mathbf{Nom}_A$ and $k, l \in \mathbf{Nom}_K$ such that $@_a @_k b, @_a @_l \neg b \in \Theta$.

(iv) There are $a, b \in \mathbf{Nom}_A$ and $k, l \in \mathbf{Nom}_K$ such that $@_a @_k l, @_b @_k \neg l \in \Theta$.

We say that Θ is *open* if it is not closed. A tableau is called *closed* if all branches in the tableau are closed.

Definition 5.3. We provide the rules of $\mathbf{T}_{\mathrm{AK}}$ in Figure 3.

Definition 5.4 (provability)**.** Given a formula φ, we say that φ is *provable* in $\mathbf{T}_{\mathrm{AK}}$ if there is a closed tableau whose root formula is $@_a @_k \varphi'$, where $a \in \mathbf{Nom}_A$ and $k \in \mathbf{Nom}_K$ does not occur in φ, and φ' is an NNF of $\neg \varphi$.

$$\frac{}{@_a@_k a}\ [Ref_A]^{*1} \qquad \frac{}{@_a@_k k}\ [Ref_K]^{*1}$$

$$\frac{@_a@_k\neg\neg\varphi}{@_a@_k\varphi}\ [\neg\neg] \qquad \frac{@_a@_k(\varphi\wedge\psi)}{\begin{array}{c}@_a@_k\varphi\\ @_a@_k\psi\end{array}}\ [\wedge] \qquad \frac{@_a@_k(\varphi\vee\psi)}{@_a@_k\varphi \mid @_a@_k\psi}\ [\vee]$$

$$\frac{@_a@_k\Diamond_A\varphi}{\begin{array}{c}@_a@_k\Diamond_A b\\ @_b@_k\varphi\end{array}}\ [\Diamond_A]^{*2,*3,*4} \qquad \frac{@_a@_k\Diamond_K\varphi}{\begin{array}{c}@_a@_k\Diamond_K l\\ @_a@_l\varphi\end{array}}\ [\Diamond_K]^{*2,*3,*5} \qquad \frac{\begin{array}{c}@_a@_k\Box_A\varphi\\ @_a@_k\Diamond_A b\end{array}}{@_b@_k\varphi}\ [\Box_A]^{*6} \qquad \frac{\begin{array}{c}@_a@_k\Box_K\varphi\\ @_a@_k\Diamond_K l\end{array}}{@_a@_l\varphi}\ [\Box_K]^{*6}$$

$$\frac{@_a@_k@_b\varphi}{@_b@_k\varphi}\ [@_A] \qquad \frac{@_a@_k@_l\varphi}{@_a@_l\varphi}\ [@_K] \qquad \frac{\begin{array}{c}@_a@_k\varphi\\ @_a@_k b\end{array}}{@_b@_k\varphi}\ [Id_A]^{*3} \qquad \frac{\begin{array}{c}@_a@_k\varphi\\ @_a@_k l\end{array}}{@_a@_l\varphi}\ [Id_K]^{*3}$$

*1: $a \in \mathbf{Nom}_A$ and $k \in \mathbf{Nom}_K$ have already occurred in the branch.
*2: This rule can be applied only one time per formula.
*3: The formula above the line is not an accessibility formula. Here, an *accessibility formula* is the formula of the form $@_a@_k\Diamond_A b$ ($@_a@_k\Diamond_K l$) generated by $[\Diamond_A]$ ($[\Diamond_K]$), where b (l) is a new nominal.
*4: $b \in \mathbf{Nom}_A$ does not occur in the branch.
*5: $l \in \mathbf{Nom}_K$ does not occur in the branch.
*6: The second formula above the line is an accessibility formula.

In these rules, the formulas above the line show the formulas that have already occurred in the branch, and the formulas below the line show the formulas that will be added to the branch. The vertical line in the $[\vee]$ means that the branch splits to the left and right.

Figure 3: The rules of $\mathbf{T}_{\mathrm{AK}}$

5.2 Termination and Completeness

A tableau calculus has *the termination property* if, for any tableau constructed in the system, all branches have finite length. We firstly prove that the tableau calculus $\mathbf{T}_{\mathrm{AK}}$ introduced above has the termination property. Due to the limited space of the paper, we provide a brief outline of the proof.

Definition 5.5. Let Θ be a branch of a tableau, and let $a,b \in \mathbf{Nom}_A$ and $k,l \in \mathbf{Nom}_K$ be nominals occurring in Θ. A pair (b,l) of nominals is *generated* by (a,k) in Θ (written: $(a,k) \prec_\Theta (b,l)$) if one of the following conditions holds.

(i) $k = l$ and b is introduced by applying $[\Diamond_A]$ to $@_a@_k\Diamond_A\varphi$.

(ii) $a = b$ and l is introduced by applying $[\Diamond_K]$ to $@_a@_k\Diamond_K\varphi$.

Lemma 5.6. Let Θ be a branch of a tableau. The length of Θ is infinite if and only if there is an infinite sequence

$$(a_0,k_0) \prec_\Theta (a_1,k_1) \prec_\Theta \cdots .$$

Definition 5.7. Let Θ be a branch of a tableau, and let $a \in \mathbf{Nom}_A$ and $k \in \mathbf{Nom}_K$ be nominals occurring in Θ. We define a function $m_\Theta : \mathbf{Nom}_A \times \mathbf{Nom}_K \to \mathbb{N}$ as follows:

$$m_\Theta((a,k)) = \max\{|\varphi| \mid @_a @_k \varphi \in \Theta\}.$$

Lemma 5.8. Let Θ be a branch of a tableau. If $(a,k) \prec_\Theta (b,l)$, then $m_\Theta((a,k)) > m_\Theta((b,l))$.

Theorem 5.9. The tableau calculus $\mathbf{T}_{\mathrm{AK}}$ has the termination property.

Proof. By *reductio ad absurdum*. Suppose there is a branch Θ of a tableau that is infinite. Then, by Lemma 5.6, we have an infinite sequence

$$(a_0,k_0) \prec_\Theta (a_1,k_1) \prec_\Theta \cdots .$$

Applying Lemma 5.8, we have an infinite decreasing sequence

$$m_\Theta((a_0,k_0)) > m_\Theta((a_1,k_1)) > \cdots ,$$

which contradict to the definition of m_Θ. ■

The soundness of $\mathbf{T}_{\mathrm{AK}}$ can be proved in a similar way introduced in [14]. Then, we move on to prove the completeness of $\mathbf{T}_{\mathrm{AK}}$. In preparation, we define some terms.

First, we use the term *subformula* with an expanded meaning. Given two formulas $@_a@_k\varphi$ and $@_b@_l\psi$, the formula $@_a@_k\varphi$ is a *subformula* of the other formula $@_b@_l\psi$ if φ is a subformula (in the usual way) of ψ. Second, we say a branch Θ *saturated* if every new formula generated by applying some rules already exists in Θ.

Definition 5.10. Given a branch Θ of a tableau, we define $\sim^A_\Theta \subset \mathbf{Nom}_A \times \mathbf{Nom}_A$ and $\sim^K_\Theta \subset \mathbf{Nom}_K \times \mathbf{Nom}_K$ as follows.

- $a \sim^A_\Theta b$ if there is a nominal $k \in \mathbf{Nom}_K$ such that $@_a@_k b \in \Theta$.
- $k \sim^K_\Theta l$ if there is a nominal $a \in \mathbf{Nom}_A$ such that $@_a@_k l \in \Theta$.

We can show that if Θ is saturated, then both $\sim^A_\Theta$ and $\sim^K_\Theta$ are equivalence relations. They enable us to take a representative of nominals.

Definition 5.11. Let Θ be a tableau branch and $a \in \mathbf{Nom}_A$ a nominal occurring in Θ. The *urfather* of a on Θ (written: $u_\Theta(a)$) is the earliest introduced nominal b such that $a \sim^A_\Theta b$. For $k \in \mathbf{Nom}_K$, $u_\Theta(k)$ is defined in the same way.

Definition 5.12. Given an open saturated branch Θ, a model $\mathscr{M}^\Theta_{\mathrm{AK}} = (W^\Theta_A, W^\Theta_K, (R^\Theta_y)_{y\in W^\Theta_K}, (S^\Theta_x)_{x\in W^\Theta_A}, V^\Theta)$ generated from Θ is defined as follows:

$$\begin{aligned}
W^\Theta_A &= \{u_\Theta(a) \mid a \in \mathbf{Nom}_A \text{ occurs in } \Theta\},\\
W^\Theta_K &= \{u_\Theta(k) \mid k \in \mathbf{Nom}_K \text{ occurs in } \Theta\},\\
R^\Theta_{u_\Theta(k)} &= \{(u_\Theta(a), u_\Theta(b)) \mid \text{accessibility formula } @_a@_k\Diamond_A b \in \Theta\},\\
S^\Theta_{u_\Theta(a)} &= \{(u_\Theta(k), u_\Theta(l)) \mid \text{accessibility formula } @_a@_k\Diamond_K l \in \Theta\},\\
V^\Theta(p_A) &= \{u_\Theta(a) \mid \text{there is } k \in \mathbf{Nom}_K \text{ such that } @_a@_k p_A \in \Theta\} \text{ where } p_A \in \mathbf{Prop}_A,\\
V^\Theta(p_K) &= \{u_\Theta(k) \mid \text{there is } a \in \mathbf{Nom}_A \text{ such that } @_a@_k p_K \in \Theta\} \text{ where } p_K \in \mathbf{Prop}_K,\\
V^\Theta(a) &= \{u_\Theta(a)\} \text{ where } a \in \mathbf{Nom}_A,\\
V^\Theta(k) &= \{u_\Theta(k)\} \text{ where } k \in \mathbf{Nom}_K.
\end{aligned}$$

Lemma 5.13. Let Θ be an open saturated branch and let $@_a @_k \varphi$ be a subformula of the root formula of Θ. Then, we have:

$$\text{if } @_a @_k \varphi \in \Theta, \text{ then } \mathcal{M}^{\Theta}_{\mathrm{AK}}, (u_\Theta(a), u_\Theta(k)) \models \varphi.$$

This lemma is called *model existence lemma*. Note that by combining it with the termination property of $\mathbf{T}_{\mathrm{AK}}$, we can show the finite model property of agent-knowledge logic as well as the completeness.

Theorem 5.14. The tableau calculus $\mathbf{T}_{\mathrm{AK}}$ is complete for the class of all AK models.

Proof. We show the contraposition.

Suppose that φ is not provable in $\mathbf{T}_{\mathrm{AK}}$. Then, we can find an open and saturated branch Θ with the root formula $@_a @_k \varphi'$, where $a \in \mathbf{Nom}_A$ and $k \in \mathbf{Nom}_K$ does not occur in φ, and φ' is an NNF of $\neg\varphi$. Then, by Lemma 5.13, we have $\mathcal{M}^{\Theta}_{\mathrm{AK}}, (u_\Theta(a), u_\Theta(k)) \models \varphi'$. It means that there is an AK model and its possible world which falsify φ. ■

The termination property and completeness of the tableau calculus tell us about the decidability of logic. If φ is provable, then it is provable in finite time. By contrast, if φ is unprovable, we can make a finite counterexample model. From them, the following corollary holds.

Corollary 5.15. The agent-knwoledge logic is decidable.

6 Future Work and Perspective

6.1 Seeking More Usage

As one of the expected future research endeavours involving agent-knowledge logic I plan to examine a greater variety of representations. The use of the tools presented in this paper would be just the tip of the iceberg. For example, $\mathbf{Nom}_A$ is the set of agents, and $\mathbf{Prop}_K$ is an agent-independent proposition. However, it is difficult to say that sufficient utilization has been found for $\mathbf{Prop}_A$ and $\mathbf{Nom}_K$.

It is also fruitful to imitate various operators of epistemic logic. For example, given a group $G \subseteq \mathbf{A}$ of agents, the *everybody knows operator* E_G is defined as follows:

$$\mathcal{M}_{\mathrm{EL}}, w \models E_G \varphi \iff \mathcal{M}, w \models K_i \varphi \text{ for all } i \in G.$$

Intuitively, this formula says that everyone in the group G knows φ. In agent-knowledge logic, $E_G \varphi$ can be expressed by the following formula:

$$\bigwedge_{i \in G} @_{T(i)} \Box_K T(\varphi).$$

Also, we may mimic other operators used in epistemic logic, such as the operator for common knowledge C_G and the operator for distributed knowledge D_G. Research in this direction may be able to reflect various results in epistemic logic in agent-knowledge logic as well.

Additionally, there is another direction to research about agent-knowledge logic, to introduce a universal operator used in hybrid logic, which may enable us to symbolize more expressions in natural language. The definition of the universal operators A_A and E_A are as follows:

$$\mathcal{M}_{\mathrm{AK}}, (x, y) \models \mathsf{A}_A \varphi \iff \mathcal{M}_{\mathrm{AK}}, (z, y) \models \varphi \text{ for all } z \in W_A,$$
$$\mathcal{M}_{\mathrm{AK}}, (x, y) \models \mathsf{E}_A \varphi \iff \text{there is some } z \in W_A \text{ such that } \mathcal{M}_{\mathrm{AK}}, (z, y) \models \varphi.$$

Owing to these operators, we can write some expressions as follows:

- $\mathsf{E}_A \Box_K p_K$: Someone knows p_K.
- $\Box_K \mathsf{A}_A \Box_K p_K$: I know that all the people know p_K.
- $\mathsf{E}_A \Box_A \Box_K p_K$: There is a person all of whose friends know p_K.

6.2 Hilbert-Style Axiomatization

In this paper, we have given the tableau calculus for agent-knowledge logic as a proof system. We can give another proof system, for example, Hilbert-style axiomatization.

Fortunately, there is already abundant previous research in the surrounding fields. In addition to the aforementioned Sano's work [15], Balbiani and Fernández González [1] has shown the Hilbert-style axiomatization of Facebook logic. For LHS, recent research by Chen and Li [6] gives the axiomatization.

6.3 Complexity

In this paper, we have shown the decidability of the agent-knowledge logic using tableau calculus. But what about its computational complexity? As already known, the satisfiability problem for epistemic logic is PSPACE-complete [9]. If agent-knowledge logic is used as an alternative to epistemic logic, it must also be PSPACE-complete.

The analysis of computational complexity for a fusion in modal logic may provide a clue to solving this problem. An explanation for a fusion is in [8, p. 111]:

> Let L_1 and L_2 be two multimodal logics formulated in languages $\mathscr{L}_1$ and $\mathscr{L}_2$, both containing the language $\mathscr{L}$ of classical propositional logic, but having disjoint sets of modal operators. Denote by $\mathscr{L}_1 \otimes \mathscr{L}_2$ the union of $\mathscr{L}_1$ and $\mathscr{L}_2$. Then the *fusion* $L_1 \otimes L_2$ of L_1 and L_2 is the smallest multimodal logic L in the language $\mathscr{L}_1 \otimes \mathscr{L}_2$ containing $L_1 \cup L_2$.

From the results of Halpern and Moses [9], we can obtain that the satisfiability problem for $\mathbf{K} \otimes \mathbf{K}$ is PSPACE-complete. Since agent-knowledge logic is based on $\mathbf{K} \otimes \mathbf{K}$, we may be able to answer the question with reference to this proof.

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References

[1] Philippe Balbiani & Saúl Fernández González (2020): *Indexed Frames and Hybrid Logics*. In: *Advances in Modal Logic*.

[2] Johan van Benthem (2006): *Epistemic Logic and Epistemology: The State of their Affairs*. *Philosophical Studies* 128, pp. 249–76, doi:10.1007/s11098-005-4052-0.

[3] Patrick Blackburn & Balder ten Cate (2006): *Pure Extensions, Proof Rules, and Hybrid Axiomatics*. *Studia Logica* 84(2), pp. 277–322, doi:10.1007/s11225-006-9009-6.

[4] Thomas Bolander & Patrick Blackburn (2007): *Termination for Hybrid Tableaus*. *Journal of Logic and Computation* 17(3), pp. 517–554, doi:10.1093/logcom/exm014.

[5] Torben Braüner (2011): *Hybrid Logic and its Proof-Theory*. 37, Springer Science & Business Media, doi:10.1007/978-94-007-0002-4.

[6] Qian Chen & Dazhu Li (2024): *Logic of the Hide and Seek Game: Characterization, Axiomatization, Decidability*. In Nina Gierasimczuk & Fernando R. Velázquez-Quesada, editors: *Dynamic Logic. New Trends and Applications*, Springer Nature Switzerland, pp. 20–34, doi:10.1007/978-3-031-51777-8_2.

[7] Ronald Fagin, Joseph Y. Halpern, Yoram Moses & Moshe Y. Vardi (1995): *Reasoning about knowledge*. MIT press, doi:10.7551/mitpress/5803.001.0001.

[8] Dov M. Gabbay (2003): *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier North Holland.

[9] Joseph Y. Halpern & Yoram Moses (1992): *A Guide to Completeness and Complexity for Modal Logics of Knowledge and Belief*. *Artificial Intelligence* 54(3), pp. 319–379, doi:10.1016/0004-3702(92)90049-4.

[10] Jaakko Hintikka (1962): *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, second edition. Cornell University Press.

[11] Andrzej Indrzejczak (2007): *Modal Hybrid Logic*. *Logic and Logical Philosophy* 16(2-3), pp. 147–257, doi:10.12775/LLP.2007.006.

[12] Dazhu Li, Sujata Ghosh, Fenrong Liu & Yaxin Tu (2021): *On The Subtle Nature of a Simple Logic of The Hide and Seek Game*. In: *Logic, Language, Information, and Computation: 27th International Workshop, WoLLIC 2021, Virtual Event, October 5–8, 2021, Proceedings 27*, Springer, pp. 201–218, doi:10.1007/978-3-030-88853-4_13.

[13] Dazhu Li, Sujata Ghosh, Fenrong Liu & Yaxin Tu (2023): *A Simple Logic of The Hide and Seek Game*. *Studia Logica* 111(5), pp. 821–853, doi:10.1007/s11225-023-10039-4.

[14] Yuki Nishimura (2024): *Completeness of Tableau Calculi for Two-dimensional Hybrid Logics*. *Journal of Logic and Computation*, p. exae018, doi:10.1093/logcom/exae018.

[15] Katsuhiko Sano (2010): *Axiomatizing Hybrid Products: How Can We Reason Many-Dimensionally in Hybrid Logic?* *Journal of Applied Logic* 8(4), pp. 459–474, doi:10.1016/j.jal.2010.08.006.

[16] Katsuhiko Sano (2011): *Paper Review: Logic in The Community*. *Journals of The Japanese Society for Artificial Intelligence* 26(6), pp. 703–707, doi:10.11517/jjsai.26.6_703. (written in Japanese).

[17] Jeremy Seligman, Fenrong Liu & Patrick Girard (2011): *Logic in the Community*. In: *Proceedings of the 4th Indian Conference on Logic and Its Applications*, Springer Berlin Heidelberg, pp. 178–188, doi:10.1007/978-3-642-18026-2_15.

[18] Jeremy Seligman, Fenrong Liu & Patrick Girard (2013): *Facebook and the Epistemic Logic of Friendship*. In: *Proceedings of the 14th Conference on Theoretical Aspects of Rationality and Knowledge*, pp. 229–238.

Syntactic Cut-Elimination for Provability Logic GL via Nested Sequents

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The cut-elimination procedure for the provability logic is known to be problematic: a Löb-like rule keeps cut-formulae intact on reduction, even in the principal case, thereby complicating the proof of termination. In this paper, we present a syntactic cut-elimination proof based on nested sequents, a generalization of sequents that allows a sequent to contain other sequents as single elements. A similar calculus was developed by Poggiolesi (2009), but there are certain ambiguities in the proof. Adopting the idea of Kushida (2020) into nested sequents, our proof does not require an extra measure on cuts or error-prone, intricate rewriting on derivations, but only straightforward inductions, thus leading to less ambiguity and confusion.

Keywords: Cut-elimination · Provability logic · Nested sequents · Proof theory

1 Introduction

The provability logic **GL**, named after Gödel and Löb, is a modal logic extending **K** with the Löb axiom $\Box(\Box A \to A) \to \Box A$, where $\Box A$ can be roughly read as "A is provable in Peano arithmetic" (see, e.g., Boolos [2] for more details). Computationally, the Löb axiom represents a kind of recursion (e.g., [8, 12]), and indeed in Kripke semantics the axiom is interpreted as just an induction on its model. Therefore, **GL** exhibits a certain "recursiveness" as its nature.

From a proof-theoretical viewpoint, a sequent calculus for **GL** is obtained by the following single modal rule [10]:

$$\frac{\Gamma, \Box\Gamma, \Box A \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta}\,(\mathrm{GLR})$$

where $\Box A$ is called *diagonal formula*. It is not difficult to prove the cut-elimination theorem using semantical arguments [16, 1], but syntactically, the diagonal formula is quite problematic: it appears in both the premise and the conclusion of (GLR), so the standard double induction on the cut-formula and height fails.

Valentini [18] proposed a proof using a third induction parameter, called the *width* of a cut, to justify the reduction involving the rule (GLR). Nevertheless, Valentini's proof is very brief and only describes the principal cases for (GLR), which raised a question about its termination (see [6, 11]). In response, Goré and Ramanayake [6] confirmed the validity of Valentini's arguments by carefully analyzing the notion of width, but also pointed out, overlooked by Valentini, that the width can be increased by reduction in some cases [6, Remark 21]. Although such an increase is certainly acceptable [6, Lemma 19], it makes their proof more complicated and non-trivial.

In this paper, we propose a more clarifying approach to syntactic cut-elimination for **GL**; unlike Valentini's, our calculus is based on *nested sequents* [7, 4, 14]. A similar calculus was developed by

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics. Theory and Applications (NCL'24).
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Poggiolesi [15], along with a syntactic cut-elimination proof using a third induction parameter specific to its sequent structure. This proof is rather simple and seemingly sound, but there are still certain ambiguities around the third parameter (see Section 3), and thus the termination is again imprecise. These matters suggest that while an additional measure on cuts could indeed resolve the problem, it would not necessarily lead to a straightforward triple-induction proof, but might require more careful checks, even where there seems less troublesome.

Instead of following the triple-induction approach, we adopt the idea presented in Kushida [9], also in Borga [3], of introducing a subprocedure, called *diagonal-formula-elimination* in this paper, that removes the diagonal formula in the premise prior to the reduction in question. This helps us avoid the problematic cut-reduction and recover the standard double induction proof of cut-elimination. The advantage of nested sequents in employing this method is that, thanks to their sequent structure, it is easier to talk about where the diagonal formula is used in a derivation. We take this advantage further by introducing *annotations*, demonstrating that the nested-sequent basis allows for much more concise and clear arguments, based solely on a series of straightforward inductions.

The rest of the paper is organized as follows: Section 2 defines a nested sequent calculus for **GL**, and Section 3 illustrates the problem on the cut-reduction method and gives an overview of our approach. Section 4 introduces an auxiliary calculus with additional information on the use of the diagonal formula. Section 5 demonstrates the procedure for eliminating diagonal formulae, which leads to the cut-elimination theorem in Section 6. Finally, we conclude with some discussions in Section 7.

2 The Calculus

In this section, we introduce a nested sequent calculus for **GL** with a one-sided formulation. Our system is not very special as a nested sequent calculus, so we only give a brief description here. For a more detailed and general introduction to nested sequent calculus itself, see, e.g., Brünnler [4].

A *formula* is defined by the following grammar:

$$A, B ::= \alpha \mid \alpha^{\perp} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A,$$

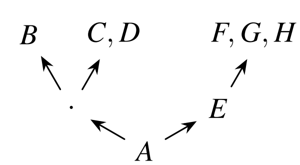
where α and $\alpha^{\perp}$ denote positive and negative atoms respectively, both taken from a certain countable set. The *negation* $A^{\perp}$ of a formula A is defined inductively in the usual way of extending duality on atoms using De Morgan's laws. We may use $A \to B$ as an abbreviation for $A^{\perp} \vee B$.

A *(nested) sequent* is defined by the following grammar:

$$\Gamma, \Delta ::= \cdot \mid \Gamma, A \mid \Gamma, [\Delta],$$

where "$\cdot$" denotes an empty sequent, and the notation $[\Delta]$ indicates that the sequent Δ is being placed as an element in another sequent (i.e., *nested*). We may apply exchange implicitly as usual, so for example, we identify $A, B, [C], \big[D, [E]\big]$ with $B, [C], A, \big[[E], D\big]$. The juxtaposition of two sequents Γ and Δ is written simply with a comma "," as Γ, Δ as usual.

Intuitively, a nested sequent represents a tree consisting of ordinary sequents (i.e., multisets of formulae) by means of the bracket $[-]$ nesting; for example, the sequent $A, \big[[B], [C, D]\big], \big[E, [F, G, H]\big]$ corresponds to the following tree structure:



$$\frac{}{\Gamma\{\alpha^{\perp},\alpha\}}\,(\mathrm{id}) \qquad \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}}\,(\wedge) \qquad \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}}\,(\vee)$$

$$\frac{\Gamma\big\{[\Diamond A^{\perp},A]\big\}}{\Gamma\{\Box A\}}\,(\Box) \qquad \frac{\Gamma\big\{\Delta\{A\},\Diamond A\big\}}{\Gamma\big\{\Delta\{\},\Diamond A\big\}}\,(\Diamond)\ \text{ if } \mathrm{depth}(\Delta\{-\})>0$$

Figure 1: Inference rules for **GL**.

By considering such a tree as a shape within a Kripke model, we can obtain modal rules directly from Kripke semantics. From the perspective of structural proof theory, on the other hand, $[-]$ is a structure corresponding to $\Box$, just as "," is to $\vee$, allowing for modal reasoning in an analytic way.

Before getting into our proof system, we need to introduce the notion of *context*. A *unary context* is informally a sequent with a single *hole* $\{-\}$ as a placeholder, formally defined by the following grammar:

$$\Gamma\{-\} ::= \Delta,\{-\} \;\big|\; \Delta,\big[\Gamma\{-\}\big].$$

Given a unary context $\Gamma\{-\}$ and a sequent Δ, we write $\Gamma\{\Delta\}$ for the sequent obtained by replacing $\{-\}$ with Δ in $\Gamma\{-\}$. For instance, $\Gamma\{-\}\equiv A,[B,C],\big[D,\{-\}\big]$ is a unary context, and then $\Gamma\big\{E,[F,G]\big\}$ represents the sequent $A,[B,C],\big[D,E,[F,G]\big]$. When filling an empty sequent into a context, we omit its symbol "$\cdot$" from the result; that is, $\Gamma\{\}$ means the sequent $\Gamma\{\cdot\}$, which is of course also distinguished from the context $\Gamma\{-\}$. A *binary context* $\Gamma\{-_1\}\{-_2\}$, a sequent with two distinct holes of $\{-_1\}$ and $\{-_2\}$, is formally defined and used in a similar way.

Definition 2.1 (Depth)**.** The *depth* of a unary context is defined inductively as follows:

$$\begin{aligned}\mathrm{depth}\big(\Delta,\{-\}\big) &= 0;\\ \mathrm{depth}\big(\Delta,[\Gamma\{-\}]\big) &= \mathrm{depth}\big(\Gamma\{-\}\big)+1.\end{aligned}$$

It is, in short, the nesting depth of the bracket $[-]$ at the hole $\{-\}$ position.

Figure 1 shows the inference rules of our system. The non-modal rule are fairly standard, except for the form of sequents. The rule $(\Box)$ is a kind of the Löb rule, as is the rule (GLR), and is the only rule in the system that consumes a $[-]$. Reflecting the transitivity of **GL**-models, we can deduce $\Diamond A$ by the rule $(\Diamond)$ from A at a deeper location within several $[-]$'s. A contraction for $\Diamond A$ is incorporated into the rule $(\Diamond)$ to ensure the admissibility of contraction (Lemma 2.6).

This is indeed a complete proof system for **GL** in the following sense, but we omit the proof here.

Theorem 2.2 (Completeness)**.** *A formula A is a theorem of* **GL** *if and only if the sequent A is provable in the calculus.*

Example (The Löb axiom). A proof of $\Box(\Box\alpha \to \alpha) \to \Box\alpha \equiv \Diamond(\Box\alpha \wedge \alpha^\perp) \vee \Box\alpha$ is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Diamond(\Box\alpha\wedge\alpha^\perp),\,[\Diamond\alpha^\perp,\alpha,[\Diamond\alpha^\perp,\alpha,\alpha^\perp]]}\,(\mathrm{id})}{\Diamond(\Box\alpha\wedge\alpha^\perp),\,[\Diamond\alpha^\perp,\alpha,[\Diamond\alpha^\perp,\alpha]]}\,(\Diamond)}{\Diamond(\Box\alpha\wedge\alpha^\perp),\,[\Diamond\alpha^\perp,\alpha,\Box\alpha]}\,(\Box) \qquad \dfrac{}{\Diamond(\Box\alpha\wedge\alpha^\perp),\,[\Diamond\alpha^\perp,\alpha,\alpha^\perp]}\,(\mathrm{id})}{\Diamond(\Box\alpha\wedge\alpha^\perp),\,[\Diamond\alpha^\perp,\alpha,\Box\alpha\wedge\alpha^\perp]}\,(\wedge)}{\dfrac{\Diamond(\Box\alpha\wedge\alpha^\perp),\,[\Diamond\alpha^\perp,\alpha]}{\Diamond(\Box\alpha\wedge\alpha^\perp),\,\Box\alpha}\,(\Box)}\,(\Diamond)}{\Diamond(\Box\alpha\wedge\alpha^\perp)\vee\Box\alpha}\,(\vee)$$

⌟

Remark. Poggiolesi [15] also developed a nested sequent calculus for **GL**, but under the name *tree-hypersequents*, with a two-sided representation. The main difference[1] is the rules for $\Diamond$ (or, the left rules for $\Box$). Poggiolesi instead employed the following two rules (but in our notation):

$$\frac{\Gamma\{[\Delta,\Box A^\perp],\Diamond A\} \qquad \Gamma\{[\Delta,A],\Diamond A\}}{\Gamma\{[\Delta],\Diamond A\}} \qquad\qquad \frac{\Gamma\{[\Delta,\Diamond A],\Diamond A\}}{\Gamma\{[\Delta],\Diamond A\}}$$

Nevertheless, there is no essential difference, especially as for provability. We shall discuss Poggiolesi's cut-elimination proof in Section 3. ⌟

Definition 2.3 (Cut)**.** A *cut* in our calculus has the following form:

$$\frac{\Gamma\{A\} \qquad \Gamma\{A^\perp\}}{\Gamma\{\}}\,(\mathrm{cut})$$

The *height* of a derivation is defined in the standard manner, i.e., the maximum length of consecutive applications of inference rules in that derivation. Several common rules required for the cut-elimination procedure are shown to be (height-preserving) admissible:

Lemma 2.4 (Inversion)**.** *The following rules are height-preserving admissible:*

$$\frac{\Gamma\{A\wedge B\}}{\Gamma\{A\}}\,(\wedge)^{-1} \qquad \frac{\Gamma\{A\wedge B\}}{\Gamma\{B\}}\,(\wedge)^{-1} \qquad \frac{\Gamma\{A\vee B\}}{\Gamma\{A,B\}}\,(\vee)^{-1} \qquad \frac{\Gamma\{\Box A\}}{\Gamma\{[\Diamond A^\perp, A]\}}\,(\Box)^{-1}$$

Proof. By induction on derivation. □

Lemma 2.5 (Identity)**.** *The following rule is admissible:*

$$\frac{}{\Gamma\{A^\perp, A\}}\,(\mathrm{id})$$

Proof. By induction on A. □

Lemma 2.6 (Structural rules)**.** *The following rules are height-preserving admissible:*

$$\frac{\Gamma\{\}}{\Gamma\{\Delta\}}\,(\mathrm{weak}) \qquad \frac{\Gamma\{A,A\}}{\Gamma\{A\}}\,(\mathrm{contract}) \qquad \frac{\Gamma\{\Delta\{\},[\Delta']\}}{\Gamma\{\Delta\{\Delta'\}\}}\,(\mathrm{rebase}) \ \text{ if } \mathrm{depth}(\Delta\{-\}) > 0$$

Proof. By induction on derivation, along with Lemma 2.4. □

Semantically, the rebasing rule is to instantiate an arbitrary transition denoted by $[-]$ into a more concrete one described by $\Delta\{-\}$, and its side-condition corresponds to the transitivity, as in the rule $(\Diamond)$.

[1] Another difference is a form of cut, which shall be discussed in Section 7.

3 Problem on Cut-Reduction

In this section, we explain why the standard cut-reduction method does not work as expected for **GL**, even in nested sequents, together with the problem with Poggiolesi's proof. We also give an overview of our approach with an informal description of our rewriting procedure.

The standard double induction fails in the principal case of $\Diamond$ and $\Box$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \Gamma\{\Delta\{A^\perp\},\Diamond A^\perp\}\end{matrix}}{\Gamma\{\Delta\{\},\Diamond A^\perp\}}(\Diamond) \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_2\\ \Gamma\{\Delta\{\},[\Diamond A^\perp,A]\}\end{matrix}}{\Gamma\{\Delta\{\},\Box A\}}(\Box)}{\Gamma\{\Delta\{\}\}}(\text{cut}) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \Gamma\{\Delta\{A^\perp\},\Diamond A^\perp\}\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_2\\ \Gamma\{\Delta\{\},[\Diamond A^\perp,A]\}\end{matrix}}{\Gamma\{\Delta\{\},\Box A\}}(\Box)}{\Gamma\{\Delta\{A^\perp\},\Box A\}}(\text{weak})}{\Gamma\{\Delta\{A^\perp\}\}}(\text{cut})^{*1}}{\Gamma\{\Delta\{\Diamond A^\perp,A^\perp\}\}}(\text{weak}) \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_2\\ \Gamma\{\Delta\{\},[\Diamond A^\perp,A]\}\end{matrix}}{\Gamma\{\Delta\{\Diamond A^\perp,A\}\}}(\text{rebase})}{\Gamma\{\Delta\{\Diamond A^\perp\}\}}(\text{cut})^{*2} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_2\\ \Gamma\{\Delta\{\},[\Diamond A^\perp,A]\}\end{matrix}}{\Gamma\{\Delta\{[]\},[\Diamond A^\perp,A]\}}(\text{weak})}{\Gamma\{\Delta\{[\Diamond A^\perp,A]\}\}}(\text{rebase})}{\Gamma\{\Delta\{\Box A\}\}}(\Box)}{\Gamma\{\Delta\{\}\}}(\text{cut})^{*3} \tag{3.1}$$

The first cut[*1] is admissible because of the smaller derivation of the left premise, and so is the second cut[*2] because of the smaller size of the cut-formula, but neither is small for the third cut.[*3]

Naïve Attempt. Although the cut-formula stays the same, it can be seen that on the third cut, compared to the original, the cut-formula $\Diamond A^\perp$ has moved by the depth of $\Delta\{-\}$ toward the leaves of the tree represented by the sequent $\Gamma\{\Delta\{\}\}$. So one might think that the reduction could be justified by appealing to the remaining distance to the leaves, namely, by induction on the following lexicographic ordering:

(i) The size of the cut-formula;

(ii) The maximum number of steps required for the cut-formula to reach a leaf; and

(iii) The total height of the premise derivations.

This approach, unfortunately, does not work as expected: the well-foundedness of the ordering requires that a tree not grow its branches due to the reductions admitted by condition (iii), which is in fact not true in the following case:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \Gamma\{[\Diamond A^\perp,A]\}\{B\}\end{matrix}}{\Gamma\{\Box A\}\{B\}}(\Box) \qquad \begin{matrix}\vdots\,\mathcal{D}_2\\ \Gamma\{\Box A\}\{B^\perp\}\end{matrix}}{\Gamma\{\Box A\}\{\}}(\text{cut}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \Gamma\{[\Diamond A^\perp,A]\}\{B\}\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_2\\ \Gamma\{\Box A\}\{B^\perp\}\end{matrix}}{\Gamma\{[\Diamond A^\perp,A]\}\{B^\perp\}}(\Box)^{-1}}{\Gamma\{[\Diamond A^\perp,A]\}\{\}}(\text{cut})}{\Gamma\{\Box A\}\{\}}(\Box) \tag{3.2}$$

This permutation exposes the $[-]$ previously discharged by $(\Box)$ in the left premise, potentially increasing the measure (ii).[2] Even were we to address this case by considering the number of $\Box$'s as well as of $[-]$'s in (ii), it would impose a strong restriction on weakening, thus breaking the argument in other cases.

Poggiolesi's Approach. Poggiolesi [15] proposed a similar triple-induction proof using the notion of *position* instead of the measure (ii). The position is, in brief, a variant of (ii) that estimates the maximum number of steps by considering not just the end-sequent but also all sequents appearing in a derivation, whereby the reduction (3.2) is no longer a problem. However, this rather causes trouble with the third cut mentioned above. More specifically, Poggiolesi's admissible rule $(\tilde{4})$ [15, Lemma 4.10], analogous to our the rule (rebase), is used to move a subtree upwards on reduction [15, Lemma 4.26-Case 3.2-4(a)], which can cause an increase in position since rewriting subderivations affects the position on the end-sequent. Such an operation is essential for the interaction between $\Diamond$ and $\Box$, and the transitive property makes the trouble unavoidable.

Poggiolesi's approach basically follows the work by Negri [13], presented a proof based on *labeled sequents* with an additional parameter called *range*, similar to the position but defined in terms of labels. Both position and range attempt to capture the well-foundedness of **GL**-models by means of their sequent structures, but there are crucial differences. Negri used *label substitution* [13, Lemma 4.3] to achieve the required transformation without increasing range, which makes the triple-induction proof effective. Here, it takes advantage of the fact that a substitution yields a *graph* structure rather than a tree, and precisely for this reason, such an operation cannot be fully reproduced in nested sequents. Poggiolesi seems to have overlooked this point, and consequently, without filling this gap, the argument would be inadequate to simulate Negri's method.

Our Approach. As in the case of Valentini, a more detailed analysis might make up for this piece, but contrary to Poggiolesi's expectation, even in nested sequents it is not so obvious how to make triple induction work. In addition, adding a third induction parameter is annoying since it has a relatively broad impact on the overall induction, which induces oversights in some boring cases such as permutations. The triple-induction approach is not so ideal for these reasons, and a more reliable method based on more intuitive and purely syntactic concept is desirable.

The reason why the problematic third cut is necessary is to eliminate the diagonal formula $\Diamond A^{\perp}$ in the premise of $(\Box)$; if we could do that in any other way, then the problem should be resolved. Kushida [9] showed that it is indeed possible by relying on cuts only on A (notice, not on $\Diamond A^{\perp}$), motivated by a syntactic cut-elimination proof for provability logic **S** [17] in ordinary sequents.

Let us review the basic idea of Kushida [9], but in the form of nested sequents. Suppose $\Gamma\{[\Diamond A^{\perp}, A]\}$ is cut-free provable with a derivation $\mathcal{D}$, and consider dropping $\Diamond A^{\perp}$ to obtain $\Gamma\{[A]\}$. If $\Diamond A^{\perp}$ is not used in $\mathcal{D}$ at all, then we can remove it from all initial sequents of $\mathcal{D}$ to obtain a derivation of $\Gamma\{[A]\}$. Otherwise, there must be a pair of relevant applications of $(\Diamond)$ and $(\Box)$ in $\mathcal{D}$, as shown in Figure 2a. For simplicity, assume that the rule $(\Diamond)$ is not applied with the $\Diamond A^{\perp}$ in $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$. Then, to obtain $\Gamma\{[A]\}$, we need to erase the assumption $A^{\perp}$ in the premise of $(\Diamond)^{*4}$ without using $(\Diamond)$.

Next, we consider how to erase $A^{\perp}$. This is done in the following steps. First, truncate $\mathcal{D}$ above the application of $(\Box)$,^{*5} including the use of $(\Diamond)$, by adding $\Diamond B^{\perp}$ as an assumption (Figure 2b). Then return to the original $\mathcal{D}$ and erase $A^{\perp}$ by a cut (Figure 2c), where many applications of admissible rules are required to adjust the shape of sequents. The added $\Diamond B^{\perp}$ is dealt with as a diagonal formula and does

[2] For instance, take $\Gamma\{-_1\}\{-_2\} \equiv \{-_1\}, \{-_2\}$. Then, on the left-hand side the cut-formula B is already on the leaf, but on the right-hand side it can go one step ahead.

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_3\\ \Gamma''\big\{\big[\Diamond A^{\perp},\Delta'\big\{[\Diamond B^{\perp},\Theta,A^{\perp}]\big\}\big]\big\}\end{array}}{\Gamma''\big\{\big[\Diamond A^{\perp},\Delta'\big\{[\Diamond B^{\perp},\Theta]\big\}\big]\big\}}\,(\Diamond)^{*4}\quad\vdots\ \mathcal{D}_2}{}
$$

$$
\begin{array}{c}
\vdots\ \mathcal{D}_3\\
\dfrac{\Gamma''\big\{\big[\Diamond A^{\perp},\Delta'\big\{[\Diamond B^{\perp},\Theta,A^{\perp}]\big\}\big]\big\}}{\Gamma''\big\{\big[\Diamond A^{\perp},\Delta'\big\{[\Diamond B^{\perp},\Theta]\big\}\big]\big\}}\,(\Diamond)^{*4}\\
\vdots\ \mathcal{D}_2\\
\dfrac{\Gamma'\big\{\big[\Diamond A^{\perp},\Delta\{[\Diamond B^{\perp},B]\}\big]\big\}}{\Gamma'\big\{[\Diamond A^{\perp},\Delta\{\Box B\}]\big\}}\,(\Box)^{*5}\\
\vdots\ \mathcal{D}_1\\
\Gamma\big\{[\Diamond A^{\perp},A]\big\}
\end{array}
$$

(a) Original derivation $\mathcal{D}$.

$$
\begin{array}{c}
\overline{\Gamma'\big\{\big[\Diamond B^{\perp},\Delta\{[\Diamond B^{\perp},B,B^{\perp}]\}\big]\big\}}\ (\mathrm{id})\\
\dfrac{\Gamma'\big\{\big[\Diamond B^{\perp},\Delta\{[\Diamond B^{\perp},B,B^{\perp}]\}\big]\big\}}{\dfrac{\Gamma'\big\{\big[\Diamond B^{\perp},\Delta\{[\Diamond B^{\perp},B]\}\big]\big\}}{\Gamma'\big\{\big[\Diamond B^{\perp},\Delta\{\Box B\}\big]\big\}}\,(\Box)}\,(\Diamond)\\
\vdots\ \mathcal{D}'_1\\
\Gamma\big\{[\Diamond B^{\perp},A]\big\}
\end{array}
$$

(b) First step: Truncate the derivation above the application of $(\Box)$ by adding $\Diamond B^{\perp}$ as an assumption.

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_3\\ \Gamma''\big\{\big[\Diamond A^{\perp},\Delta'\big\{[\Diamond B^{\perp},\Theta,A^{\perp}]\big\}\big]\big\}\end{array}\qquad \begin{array}{c}\vdots\ \mathrm{b}\\ \Gamma\big\{[\Diamond B^{\perp},A]\big\}\\ \overline{\Gamma\big\{[\Diamond B^{\perp},A],\big[\Diamond A^{\perp},\Delta'\big\{[\Theta]\big\}\big]\big\}}\ (\mathrm{weak})\\ \overline{\Gamma\big\{\big[\Diamond A^{\perp},\Delta'\big\{[\Diamond B^{\perp},\Theta,A]\big\}\big]\big\}}\ (\mathrm{rebase})\end{array}}{\Gamma'''\big\{\big[\Diamond A^{\perp},\Delta'\big\{[\Diamond B^{\perp},\Theta]\big\}\big]\big\}}\,(\mathrm{cut})\\
\vdots\ \mathcal{D}'_2+(\Box)+\mathcal{D}''_1\\
\Gamma\big\{[\Diamond A^{\perp},A]\big\}
\end{array}
$$

(c) Second step: Displace the application of $(\Diamond)$ with a cut on A.

Here $\mathcal{D}'_1$, $\mathcal{D}''_1$, and $\mathcal{D}'_2$ denote minor modifications of $\mathcal{D}_1$, $\mathcal{D}_1$, and $\mathcal{D}_2$ with admissible rules applied several times, respectively.

Figure 2: Overview of the diagonal-formula-elimination subprocedure.

not remain in the conclusion. The resulting derivation no longer requires the $\Diamond A^{\perp}$, allowing us to obtain $\Gamma\{[A]\}$.

This is the base case of our rewriting process, and in general, it can be done by repeating this as many times as necessary. However, it requires global manipulation of the derivation and several tweaks of sub-derivations by admissible rules, making precise discussion difficult. In addition, it may seem a bit counterintuitive that even if only *two* instances of the rule $(\Diamond)$ are involved, at most *three* cuts are needed. To avoid pitfalls, we introduce *annotations* in the next section to allow for more precise arguments.

4 Annotated System

In this section, we introduce an auxiliary calculus with tiny annotations, which keep track of the use of diagonal formulae in a derivation.

An *annotated sequent* is defined by the following grammar:

$$\Gamma, \Delta ::= \cdot \mid \Gamma, C \mid \Gamma, \Diamond A_\Sigma \mid \Gamma, [\Delta]_B,$$

where C is a formula not of the form $\Diamond A$, and B and Σ are a formula and a set of formulae, respectively. Accordingly, there are two sorts of annotations, each with the following roles:

- $[-]_B$ indicates that the $[-]$ is to be discharged by applying the rule $(\Box)$ to B:

$$\frac{\Gamma\{[\Diamond B^\perp, B]_B\}}{\Gamma\{\Box B\}}\,(\Box)$$

- The set Σ of $\Diamond A_\Sigma$ records the provenances of A. That is, $B \in \Sigma$ implies that we have used the rule $(\Diamond)$ to A directly contained inside some $[-]_B$, absorbing it into the $\Diamond A$:

$$\frac{\Gamma\{\Delta\{[\Delta', A]_B\}, \Diamond A_\Sigma\}}{\Gamma\{\Delta\{[\Delta']_B\}, \Diamond A_{\Sigma \cup \{B\}}\}}\,(\Diamond)$$

This is all our annotations do, and we shall see in the next section that they do indeed provide sufficient information for induction.

To put it a little more strictly, for annotations to make sense, we require the following conditions be placed on the inference rules:

- An initial sequent shall contain only emptysets as an annotation to $\Diamond$-formulae, since the rule $(\Diamond)$ has not yet been applied here.

✔ $\dfrac{}{\Diamond A_\emptyset, [\alpha, \alpha^\perp, [\Diamond B \wedge C]_\alpha]_\beta}\,(\mathrm{id})$ ✘ $\dfrac{}{\Diamond A_{\{\alpha\}}, [\alpha, \alpha^\perp, [A]_\alpha]_\beta}\,(\mathrm{id})$

- Even if A comes out of multiple $[-]$'s by the rule $(\Diamond)$, only the innermost one is essential.

✔ $\dfrac{[[\Delta, A]_\alpha]_\beta, \Diamond A_\Sigma}{[[\Delta]_\alpha]_\beta, \Diamond A_{\Sigma\cup\{\alpha\}}}\,(\Diamond)$ ✘ $\dfrac{[[\Delta, A]_\alpha]_\beta, \Diamond A_\Sigma}{[[\Delta]_\alpha]_\beta, \Diamond A_{\Sigma\cup\{\alpha,\beta\}}}\,(\Diamond)$ ✘ $\dfrac{[[\Delta, A]_\alpha]_\beta, \Diamond A_\Sigma}{[[\Delta]_\alpha]_\beta, \Diamond A_{\Sigma\cup\{\beta\}}}\,(\Diamond)$

- We never use annotations for subformulae,[3] so whenever logical rules are applied to $\Diamond$-formulae, their annotations are simply discarded.

✔ $\dfrac{\Diamond A_\Sigma, \Diamond B_\Pi}{\Diamond A \vee \Diamond B}\,(\vee)$ ✘ $\dfrac{\Diamond A_\Sigma, \Diamond B_\Pi}{\Diamond A_\Sigma \vee \Diamond B_\Pi}\,(\vee)$ ✘ $\dfrac{\Diamond A_\Sigma \quad \Diamond B_\Pi}{(\Diamond A \wedge \Diamond B)_{\Sigma\cup\Pi}}\,(\wedge)$

✔ $\dfrac{[\Delta, \Diamond A_\Sigma]_\alpha, \Diamond\Diamond A_\Pi}{[\Delta]_\alpha, \Diamond\Diamond A_{\Pi\cup\{\alpha\}}}\,(\Diamond)$ ✘ $\dfrac{[\Delta, \Diamond A_\Sigma]_\alpha, \Diamond\Diamond A_\Pi}{[\Delta]_\alpha, \Diamond\Diamond A_{\Sigma\cup\Pi\cup\{\alpha\}}}\,(\Diamond)$

- For the two-premise rules (i.e., $(\wedge)$ and (cut)):

[3] This is because in those cases the induction works just fine due to the smaller formula sizes.

$$\dfrac{}{\Gamma\{\alpha^\perp,\alpha\}}\,(\mathrm{id})^{*\mathrm{a}} \qquad \dfrac{\Gamma\{A\} \qquad \Gamma\{B\}}{\Gamma\{A\wedge B\}}\,(\wedge)^{*\mathrm{b},*\mathrm{c}} \qquad \dfrac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}}\,(\vee)^{*\mathrm{b}}$$

$$\dfrac{\Gamma\big\{[\Diamond A^\perp{}_\Sigma, A]_A\big\}}{\Gamma\{\Box A\}}\,(\Box)^{*\mathrm{b}} \qquad \dfrac{\Gamma\big\{\Delta\big\{[\Delta', A]_B\big\}, \Diamond A_\Sigma\big\}}{\Gamma\big\{\Delta\big\{[\Delta']_B\big\}, \Diamond A_{\Sigma\cup\{B\}}\big\}}\,(\Diamond)^{*\mathrm{b}} \qquad \dfrac{\Gamma\{A\} \qquad \Gamma\{A^\perp\}}{\Gamma\{\}}\,(\mathrm{cut})^{*\mathrm{c}}$$

a) $\Gamma\{-\}$ must contain only $\emptyset$ as annotation sets. b) Discard the annotation(s) of A (and B) if exist(s).
c) Each $[-]$ must have the same annotation for premises. For each $\Diamond$-formula, annotation sets are merged in the conclusion.

Figure 3: Inference rules with annotations.

- For each $[-]$, its annotation formula shall be *shared* by both premises; whereas
- For each $\Diamond$-formula, its annotation set may be *independent* on two premises, and in the conclusion two sets are to be merged by set union $\cup$.

$$\checkmark\ \dfrac{A, \Diamond C_{\{E,F\}}, \Diamond D^\perp{}_\emptyset \qquad B, \Diamond C_{\{E\}}, \Diamond D^\perp{}_{\{G\}}}{A\wedge B, \Diamond C_{\{E,F\}}, \Diamond D^\perp{}_{\{G\}}}\,(\wedge)$$

Since exchange may be used implicitly, it is not always possible to uniquely determined which formula of each premise is paired, but in that case anything is ok.

We may omit annotations if not important, and note that a mere $\Diamond A^\perp$ does not imply $\Diamond A^\perp{}_\emptyset$, but rather $\Diamond A^\perp{}_\Sigma$ for some (possibly empty) Σ.

Example (The axiom (K)). A proof of $\Box(\alpha\to\beta)\to\Box\alpha\to\Box\beta \equiv \Diamond(\alpha\wedge\beta^\perp)\vee(\Diamond\alpha^\perp\vee\Box\beta)$ is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Diamond(\alpha\wedge\beta^\perp)_\emptyset, \Diamond\alpha^\perp{}_\emptyset, \big[\Diamond\beta^\perp{}_\emptyset, \beta, \alpha^\perp, \alpha\big]_\beta}\,(\mathrm{id})}{\Diamond(\alpha\wedge\beta^\perp)_\emptyset, \Diamond\alpha^\perp{}_{\{\beta\}}, \big[\Diamond\beta^\perp{}_\emptyset, \beta, \alpha\big]_\beta}\,(\Diamond) \qquad \dfrac{}{\Diamond(\alpha\wedge\beta^\perp)_\emptyset, \Diamond\alpha^\perp{}_\emptyset, \big[\Diamond\beta^\perp{}_\emptyset, \beta, \beta^\perp\big]_\beta}\,(\mathrm{id})}{\Diamond(\alpha\wedge\beta^\perp)_\emptyset, \Diamond\alpha^\perp{}_{\{\beta\}}, \big[\Diamond\beta^\perp{}_\emptyset, \beta, \alpha\wedge\beta^\perp\big]_\beta}\,(\wedge)}{\Diamond(\alpha\wedge\beta^\perp)_{\{\beta\}}, \Diamond\alpha^\perp{}_{\{\beta\}}, \big[\Diamond\beta^\perp{}_\emptyset, \beta\big]_\beta}\,(\Diamond)}{\Diamond(\alpha\wedge\beta^\perp)_{\{\beta\}}, \Diamond\alpha^\perp{}_{\{\beta\}}, \Box\beta}\,(\Box)}{\Diamond(\alpha\wedge\beta^\perp)_{\{\beta\}}, \Diamond\alpha^\perp\vee\Box\beta}\,(\vee)}{\Diamond(\alpha\wedge\beta^\perp)\vee(\Diamond\alpha^\perp\vee\Box\beta)}\,(\vee)}{}$$

⌟

We summarize the annotated system in Figure 3. Observe that the rules in Figure 3, with all annotations dropped, are exactly the same as those in Figure 1 and Definition 2.3, including in particular the rule $(\Diamond)$. Also note that annotations are not a restriction, but merely clues, because, given an unannotated derivation, we can always lift it straightforwardly to an annotated one; more precisely, this is done by the following two steps:

1. First, we need to annotate all $[-]$'s appropriately. Look at a given derivation from bottom to top, and if we find that the rule $(\Box)$ is applied to some A, then annotate $[-]$ there with A. Taking the

derivation of the Löb axiom (p. 95) as an example, there are two instances, annotated as follows:

$$\dfrac{\begin{array}{c}\vdots\\ \Diamond(\Box\alpha\wedge\alpha^{\perp}),[\Diamond\alpha^{\perp},\alpha]\end{array}}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp}),\Box\alpha\\ \vdots\end{array}}\,(\Box)\quad\rightsquigarrow\quad\dfrac{\begin{array}{c}\vdots\\ \Diamond(\Box\alpha\wedge\alpha^{\perp}),[\Diamond\alpha^{\perp},\alpha]_{\boldsymbol{\alpha}}\end{array}}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp}),\Box\alpha\\ \vdots\end{array}}\,(\Box)$$

$$\dfrac{\begin{array}{c}\vdots\\ \Diamond(\Box\alpha\wedge\alpha^{\perp}),\big[\Diamond\alpha^{\perp},\alpha,[\Diamond\alpha^{\perp},\alpha]\big]_{\alpha}\end{array}}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp}),[\Diamond\alpha^{\perp},\alpha,\Box\alpha]_{\alpha}\\ \vdots\end{array}}\,(\Box)\quad\rightsquigarrow\quad\dfrac{\begin{array}{c}\vdots\\ \Diamond(\Box\alpha\wedge\alpha^{\perp}),\big[\Diamond\alpha^{\perp},\alpha,[\Diamond\alpha^{\perp},\alpha]_{\boldsymbol{\alpha}}\big]_{\alpha}\end{array}}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp}),[\Diamond\alpha^{\perp},\alpha,\Box\alpha]_{\alpha}\\ \vdots\end{array}}\,(\Box)$$

If the end-sequent contains $[-]$'s, their annotations are not important and may be annotated in any way.

2. Then, annotate all $\Diamond$-formulae in an initial sequent with an emptyset and, from top to bottom, collect their usage. In the case of the example (p. 95):

$$\dfrac{\dfrac{}{\Diamond(\Box\alpha\wedge\alpha^{\perp}),\big[\Diamond\alpha^{\perp},\alpha,\big[\Diamond\alpha^{\perp},\alpha,\alpha^{\perp}\big]_{\alpha}\big]_{\alpha}}\,(\mathrm{id})}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp}),\big[\Diamond\alpha^{\perp},\alpha,\big[\Diamond\alpha^{\perp},\alpha\big]_{\alpha}\big]_{\alpha}\\ \vdots\end{array}}\,(\Diamond)\quad\rightsquigarrow\quad\dfrac{\dfrac{}{\Diamond(\Box\alpha\wedge\alpha^{\perp})_{\boldsymbol{\emptyset}},\big[\Diamond\alpha^{\perp}{}_{\boldsymbol{\emptyset}},\alpha,\big[\Diamond\alpha^{\perp}{}_{\boldsymbol{\emptyset}},\alpha,\alpha^{\perp}\big]_{\alpha}\big]_{\alpha}}\,(\mathrm{id})}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp})_{\boldsymbol{\emptyset}},\big[\Diamond\alpha^{\perp}{}_{\boldsymbol{\{\alpha\}}},\alpha,\big[\Diamond\alpha^{\perp}{}_{\boldsymbol{\emptyset}},\alpha\big]_{\alpha}\big]_{\alpha}\\ \vdots\end{array}}\,(\Diamond)$$

$$\dfrac{\begin{array}{c}\vdots\\ \Diamond(\Box\alpha\wedge\alpha^{\perp})_{\emptyset},\big[\Diamond\alpha^{\perp}{}_{\{\alpha\}},\alpha,\Box\alpha\wedge\alpha^{\perp}\big]_{\alpha}\end{array}}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp}),[\Diamond\alpha^{\perp},\alpha]_{\alpha}\\ \vdots\end{array}}\,(\Diamond)\quad\rightsquigarrow\quad\dfrac{\begin{array}{c}\vdots\\ \Diamond(\Box\alpha\wedge\alpha^{\perp})_{\emptyset},\big[\Diamond\alpha^{\perp}{}_{\{\alpha\}},\alpha,\Box\alpha\wedge\alpha^{\perp}\big]_{\alpha}\end{array}}{\begin{array}{c}\Diamond(\Box\alpha\wedge\alpha^{\perp})_{\boldsymbol{\{\alpha\}}},\big[\Diamond\alpha^{\perp}{}_{\boldsymbol{\{\alpha\}}},\alpha\big]_{\alpha}\\ \vdots\end{array}}\,(\Diamond)$$

Remark. The use of *sets* to annotate $\Diamond$-formulae is not essential in our proof. In fact, it is possible to use *multisets* instead, but using sets makes the proof a bit simpler since we can deal with identical formulae at once (see Lemmas 5.2 and 5.4). ⌟

Hereafter, we shall focus on a fixed $\Diamond A^{\perp}$ and treat a cut on A as a rather first-class inference rule. We write $\vdash\Gamma$ if the sequent Γ is cut-free provable, and $\vdash^{A}\Gamma$ if provable with cuts on A admitted. It is easily checked that the admissible rules in Lemmas 2.5 and 2.6 are still admissible under proper annotations, with or without cuts; in particular, the following forms are admissible:

$$\dfrac{}{\vdash^{A}\Gamma\{\Diamond A^{\perp}{}_{\emptyset}\}\{B^{\perp},B\}}\,(\mathrm{id})\qquad\dfrac{\vdash^{A}\Gamma\{\}}{\vdash^{A}\Gamma\{\Diamond A^{\perp}{}_{\emptyset}\}}\,(\mathrm{weak})\qquad\dfrac{\vdash^{A}\Gamma\{\Diamond A^{\perp}{}_{\Sigma},\Diamond A^{\perp}{}_{\Sigma'}\}}{\vdash^{A}\Gamma\{\Diamond A^{\perp}{}_{\Sigma\cup\Sigma'}\}}\,(\mathrm{contract})$$

Lemma 4.1. *The following rule is admissible:*

$$\dfrac{\vdash^{A}\Gamma\big\{\Delta\{\Diamond A^{\perp}{}_{\Sigma}\}\big\}}{\vdash^{A}\Gamma\big\{\Delta\{\},\Diamond A^{\perp}{}_{\Sigma}\big\}}\,(\text{cherry-pick})$$

Proof. By induction on derivation. □

5 Diagonal-Formula-Elimination

In this section, we prove the key lemma of this paper by fully using annotations. Our goal is to show the following:

Lemma 5.1 (Diagonal-formula-elimination)**.** *If* $\vdash \Gamma\{[\Diamond A^{\perp}, A]\}$*, then* $\vdash^{A} \Gamma\{[A]\}$.

We first show that in exchange for adding an extra assumption, we can reduce annotated formulae from $\Diamond A^{\perp}$. This is similar to the step shown in Figure 2b, but we process all $[-]$'s annotated with the same formula at once.

Lemma 5.2. *Suppose* $\vdash^{A} \Gamma\{[\Diamond A^{\perp}{}_{\Sigma}, \Delta]\}$*, and let* $B \in \Sigma$*. If* Δ *contains no* $[-]$ *annotated with* B*, then* $\vdash^{A} \Gamma\{[\Diamond A^{\perp}{}_{\Sigma'}, \Diamond B^{\perp}, \Delta]\}$ *for some* $\Sigma' \subseteq \Sigma \setminus \{B\}$.

Proof. By induction on derivation. Here we show only two cases.

Case 1. The most important case is where the rule ($\Box$) is applied to B within Δ:

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \Gamma\big\{\big[\Diamond A^{\perp}{}_{\Sigma}, \Delta'\{[\Diamond B^{\perp}, B]_B\}\big]\big\}\end{array}}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Sigma}, \Delta'\{\Box B\}\big]\big\}}\,(\Box)$$

We cannot then use the induction hypothesis because the newly appering $[-]_B$ in the premise breaks the requirement of the claim. So instead, we derive the desired sequent without using $\mathcal{D}$ as follows:

$$\dfrac{\dfrac{\overline{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\emptyset}, \Diamond B^{\perp}, \Delta'\{[\Diamond B^{\perp}, B, B^{\perp}]_B\}\big]\big\}}\,(\mathrm{id})}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\emptyset}, \Diamond B^{\perp}, \Delta'\{[\Diamond B^{\perp}, B]_B\}\big]\big\}}\,(\Diamond)}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\emptyset}, \Diamond B^{\perp}, \Delta'\{\Box B\}\big]\big\}}\,(\Box)$$

Case 2. Suppose

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \Gamma\big\{\big[\Diamond A^{\perp}{}_{\Sigma}, \Delta'\{A^{\perp}\}\big]\big\}\end{array}}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Sigma\cup\{C\}}, \Delta'\{\}\big]\big\}}\,(\Diamond)$$

By assumption, we have $C \not\equiv B$; otherwise, there exists a $[-]_B$ within $\Delta'\{-\}$, a contradiction. Hence $B \in \Sigma$, and using the induction hypothesis we have the following:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \Gamma\big\{\big[\Diamond A^{\perp}{}_{\Sigma}, \Delta'\{A^{\perp}\}\big]\big\}\end{array}}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Sigma'}, \Diamond B^{\perp}, \Delta'\{A^{\perp}\}\big]\big\}}\,(\mathrm{IH})}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Sigma'\cup\{C\}}, \Diamond B^{\perp}, \Delta'\{\}\big]\big\}}\,(\Diamond)$$

where $\Sigma' \subseteq \Sigma \setminus \{B\}$, and so $\Sigma' \cup \{C\} \subseteq (\Sigma \cup \{C\}) \setminus \{B\}$ as required.

The other cases are straightforward. $\Box$

Definition 5.3. Let Δ be an annotated sequent. Then, the sequent Δ^+ is defined inductively as follows:

$$(\cdot)^+ \equiv \cdot\,;$$
$$\big(\Gamma, C\big)^+ \equiv \Gamma^+, C;$$
$$\big(\Gamma, [\Delta]_C\big)^+ \equiv \Gamma^+, [\Delta^+, \Diamond C^\perp]_C.$$

The weakened context $\Delta^+\{-\}$ for $\Delta\{-\}$ is defined in a similar way.

In short, Δ^+ is the sequent in which all subsequents of the form $[\Delta']_C$ within Δ are weakened to $[\Delta', \Diamond C^\perp]_C$ with the diagonal formula $\Diamond C^\perp$. For example, $\big(A, \big[[B]_B, [C,D]_E\big]_C\big)^+$ represents the sequent

$$A, \big[[B, \Diamond B^\perp]_B, [C, D, \Diamond E^\perp]_E, \Diamond C^\perp\big]_C.$$

We next show that the extra assumption $\Diamond B^\perp$ added in Lemma 5.2 can be moved to its "proper place," namely, inside some $[-]_B$, where it should eventually be contracted with the diagonal formula. This corresponds to the step of Figure 2c, but again we handle multiple instances simultaneously.

Lemma 5.4. *Suppose*

$$\vdash^A \Gamma\big\{[\Diamond A^\perp{}_\Sigma, \Diamond B^\perp, A]\big\}. \tag{†1}$$

If

$$\vdash^A \Gamma\{[\Diamond A^\perp{}_\Pi, \Delta]\}, \tag{†2}$$

then $\vdash^A \Gamma\{[\Diamond A^\perp{}_{\Pi'}, \Delta^+]\}$ *for some* $\Pi' \subseteq \Sigma \cup (\Pi \setminus \{B\})$.

Proof. By induction on the derivation of (†2). Here we show only two important cases.

Case 1. Suppose

$$\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ \Gamma\big\{\big[\Diamond A^\perp{}_\Pi, \Delta'\big\{[\Delta'', A^\perp]_B\big\}\big]\big\}\end{matrix}}{\Gamma\big\{\big[\Diamond A^\perp{}_{\Pi\cup\{B\}}, \Delta'\big\{[\Delta'']_B\big\}\big]\big\}}\ (\Diamond)$$

To avoid B appended to $\Diamond A^\perp$ again, we dispense with $(\Diamond)$ using a cut as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ \Gamma\big\{\big[\Diamond A^\perp{}_\Pi, \Delta'\big\{[\Delta'', A^\perp]_B\big\}\big]\big\}\end{matrix}}{\Gamma\big\{\big[\Diamond A^\perp{}_{\Pi'}, \Delta'^+\big\{[\Delta''^+, \Diamond B^\perp, A^\perp]_B\big\}\big]\big\}}\ (\text{IH}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\big\{[\Diamond A^\perp{}_\Sigma, \Diamond B^\perp, A]\big\}}\ (†1)}{\Gamma\big\{\big[\Delta'^+\{[\Delta''^+]_B\}\big], [\Diamond A^\perp{}_\Sigma, \Diamond B^\perp, A]\big\}}\ (\text{weak})}{\Gamma\big\{\big[\Delta'^+\big\{[\Delta''^+, \Diamond A^\perp{}_\Sigma, \Diamond B^\perp, A]_B\big\}\big]\big\}}\ (\text{rebase})}{\Gamma\big\{\big[\Diamond A^\perp{}_\Sigma, \Delta'^+\big\{[\Delta''^+, \Diamond B^\perp, A]_B\big\}\big]\big\}}\ (\text{cherry-pick})}{\Gamma\big\{\big[\Diamond A^\perp{}_{\Sigma\cup\Pi'}, \Delta'^+\big\{[\Delta''^+, \Diamond B^\perp]_B\big\}\big]\big\}}\ (\text{cut})$$

where $\Pi' \subseteq \Sigma \cup (\Pi \setminus \{B\})$, and so $\Sigma \cup \Pi' \subseteq \Sigma \cup (\Pi \setminus \{B\})$ as required. Observe that the $\Diamond B^\perp$ in the left premise is due to the operation $(-)^+$ of the induction hypothesis, whereas that in the right comes from (†1).

Case 2. Suppose

$$\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ \Gamma\big\{\big[\Diamond A^\perp{}_\Pi, \Delta'\{[\Diamond C^\perp, C]_C\}\big]\big\}\end{matrix}}{\Gamma\big\{\big[\Diamond A^\perp{}_\Pi, \Delta'\{\Box C\}\big]\big\}}\ (\Box)$$

Then we have

$$\dfrac{\dfrac{\dfrac{\vdots\,\mathcal{D}}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Pi}, \Delta'\{[\Diamond C^{\perp}, C]_C\}\big]\big\}}}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Pi'}, \Delta'^{+}\{[\Diamond C^{\perp}, \Diamond C^{\perp}, C]_C\}\big]\big\}}\,(\mathrm{IH})}{\dfrac{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Pi'}, \Delta'^{+}\{[\Diamond C^{\perp}, C]_C\}\big]\big\}}{\Gamma\big\{\big[\Diamond A^{\perp}{}_{\Pi'}, \Delta'^{+}\{\Box C\}\big]\big\}}\,(\Box)}\,(\text{contract})$$

for some $\Pi' \subseteq \Sigma \cup (\Pi \setminus \{B\})$. We notice that when $C \equiv B$ here, we have successfully canceled out that we added $\Diamond B^{\perp}$ in Lemma 5.2.

The other cases are straightforward. □

Repeated application of these two lemmas sweeps all the annotations away from $\Diamond A^{\perp}$:

Lemma 5.5. *If*

$$\vdash^{A} \Gamma\{[\Diamond A^{\perp}{}_{\Sigma}, A]\}, \tag{‡}$$

then $\vdash^{A} \Gamma\{[\Diamond A^{\perp}{}_{\emptyset}, A]\}$.

Proof. By induction on the size of Σ. If $\Sigma = \emptyset$, the proof is complete; otherwise, let $B \in \Sigma$. From (‡), Lemma 5.2 yields $\vdash^{A} \Gamma\{[\Diamond A^{\perp}{}_{\Sigma'}, \Diamond B^{\perp}, A]\}$ for some $\Sigma' \subseteq \Sigma \setminus \{B\}$. Then putting this into (†1) and (‡) into (†2) in Lemma 5.4, we have $\vdash^{A} \Gamma\{[\Diamond A^{\perp}{}_{\Sigma''}, A]\}$ for some $\Sigma'' \subseteq \Sigma' \cup (\Sigma \setminus \{B\}) \subsetneq \Sigma$. Applying the induction hypothesis to Σ'', we obtain the conclusion. □

Now we may drop unused assumptions from a derived sequent, which can be restated formally in a specific form we need as follows:

Lemma 5.6 (Thinning)**.** *If* $\vdash^{A} \Gamma\{\Diamond A^{\perp}{}_{\emptyset}\}$*, then* $\vdash^{A} \Gamma\{\}$.

Proof. By induction on derivation. □

Lemma 5.1 follows immediately from Lemmas 5.5 and 5.6.

6 Syntactic Cut-Elimination

We are now ready to prove the cut-elimination theorem. We use the standard double induction to show the reduction lemma:

Lemma 6.1 (Reduction)**.** *Suppose*

$$\dfrac{\dfrac{\vdots\,\mathcal{D}_1}{\Gamma\{A\}} \qquad \dfrac{\vdots\,\mathcal{D}_2}{\Gamma\{A^{\perp}\}}}{\Gamma\{\}}\,(\text{cut})$$

If $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are both cut-free derivations, then we have* $\vdash \Gamma\{\}$.

Proof. We proceed by induction on the following lexicographic ordering:

(i) The size of A; and

(ii) The sum of the heights of $\mathcal{D}_1$ and $\mathcal{D}_2$

and reduce the cut to lower ones.

We here show only the case in question:

$$\dfrac{\dfrac{\dfrac{\vdots\,\mathcal{D}_1}{\Gamma\{\Delta\{A^\perp\},\Diamond A^\perp\}}}{\Gamma\{\Delta\{\},\Diamond A^\perp\}}(\Diamond)\quad \dfrac{\dfrac{\vdots\,\mathcal{D}_2}{\Gamma\{\Delta\{\},[\Diamond A^\perp, A]\}}}{\Gamma\{\Delta\{\},\Box A\}}(\Box)}{\Gamma\{\Delta\{\}\}}(\mathrm{cut})$$

By Lemma 5.1 we have $\vdash^A \Gamma\{\Delta\{\},[A]\}$, where all cuts are admissible by the induction hypothesis, and hence $\vdash \Gamma\{\Delta\{\},[A]\}$. We can now reduce the cut above as follows:

$$\dfrac{\dfrac{\dfrac{\vdots\,\mathcal{D}_1}{\Gamma\{\Delta\{A^\perp\},\Diamond A^\perp\}}\quad \dfrac{\dfrac{\dfrac{\vdots\,\mathcal{D}_2}{\Gamma\{\Delta\{\},[\Diamond A^\perp, A]\}}}{\Gamma\{\Delta\{\},\Box A\}}(\Box)}{\Gamma\{\Delta\{A^\perp\},\Box A\}}(\mathrm{weak})}{\Gamma\{\Delta\{A^\perp\}\}}(\mathrm{cut})\quad \dfrac{\dfrac{\vdots}{\Gamma\{\Delta\{\},[A]\}}}{\Gamma\{\Delta\{A\}\}}(\mathrm{rebase})}{\Gamma\{\Delta\{\}\}}(\mathrm{cut})$$

Both of the cuts here are admissible by the induction hypothesis and, unlike the reduction (3.1), a third cut is no longer required thanks to the diagonal-formula-elimination subprocedure.

The other cases are standard and almost the same as for the **K** case of Brünnler [4], where no special consideration regarding well-foundedness is needed since we add no new induction parameters. □

Theorem 6.2 (Cut-elimination)**.** *The cut-rule is admissible.*

Proof. Immediate from Lemma 6.1. □

7 Conclusion

We have presented a syntactic cut-elimination proof for **GL** by combining the following two ideas:

- The diagonal-formula-elimination subprocedure splits off the difficult part of **GL**'s cut-elimination, thereby allowing for the proof in a more modular way without any trouble on the termination of the entire procedure.
- The nested-sequent approach enables straightforward induction proofs, where we rely only on local assumptions and can perform rewriting without having to grasp the entire derivation with the help of annotations.

This allows for a more concise and clear proof than previous methods in a composable way.

We employed a context-sharing form of the cut-rule in this paper, but other forms can be considered. For example, Poggiolesi [15] adopted a context-independent one, and Brünnler [4] considered a special form of multicut called *Y-cut* for modal logics with the axiom (4). These variants have some impact on cut-elimination, but do not seem to provide a fundamental solution in the **GL** case. Another possibility is to extend cut to be applicable to subsequents. Such a generalization has been developed for basic modal logics such as **K** and **S4** by Chaudhuri, Marin, and Straßburger [5], and seems to fit also well with **GL**, which is future work.

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References

[1] Arnon Avron (1984): *On modal systems having arithmetical interpretations*. The Journal of Symbolic Logic 49(3), pp. 935–942, doi:10.2307/2274147.

[2] George Boolos (1994): *The Logic of Provability*. Cambridge University Press, doi:10.1017/CBO9780511625183.

[3] Marco Borga (1983): *On some proof theoretical properties of the modal logic* **GL**. Studia Logica 42(4), pp. 453–459, doi:10.1007/BF01371633.

[4] Kai Brünnler (2009): *Deep sequent systems for modal logic*. Archive for Mathematical Logic 48(6), pp. 551–577, doi:10.1007/s00153-009-0137-3.

[5] Kaustuv Chaudhuri, Sonia Marin & Lutz Straßburger (2016): *Focused and synthetic nested sequents*. In Bart Jacobs & Christof Löding, editors: Foundations of Software Science and Computation Structures, Springer, pp. 390–407, doi:10.1007/978-3-662-49630-5_23.

[6] Rajeev Goré & Revantha Ramanayake (2012): *Valentini's cut-elimination for provability logic resolved*. The Review of Symbolic Logic 5(2), pp. 212–238, doi:10.1017/S1755020311000323.

[7] Ryo Kashima (1994): *Cut-free sequent calculi for some tense logics*. Studia Logica 53(1), pp. 119–135, doi:10.1007/BF01053026.

[8] G Alex Kavvos (2021): *Intensionality, intensional recursion, and the Gödel–Löb axiom*. IfCoLoG Journal of Logics and their Applications 8(8), pp. 2287–2312. Available at `https://hdl.handle.net/1983/6a9e73f9-e55f-4b0c-925b-0a9099e30214`.

[9] Hirohiko Kushida (2020): *A proof theory for the logic of provability in true arithmetic*. Studia Logica 108(4), pp. 857–875, doi:10.1007/s11225-019-09891-0.

[10] Daniel Leivant (1981): *On the proof theory of the modal logic for arithmetic provability*. Journal of Symbolic Logic 46(3), p. 531–538, doi:10.2307/2273755.

[11] Anders Moen (2001): *The proposed algorithms for eliminating cuts in provability calculus* **GLS** *do not terminate*. The 13th Nordic Workshop in Programming Theory (NWPT'01). Available at `https://nr.no/en/publication/146681/`.

[12] Hiroshi Nakano (2001): *Fixed-point logic with the approximation modality and its Kripke completeness*. In: International Symposium on Theoretical Aspects of Computer Software, Springer, pp. 165–182, doi:10.1007/3-540-45500-0_8.

[13] Sara Negri (2005): *Proof analysis in modal logic*. Journal of Philosophical Logic 34(5), pp. 507–544, doi:10.1007/s10992-005-2267-3.

[14] Francesca Poggiolesi (2009): *The method of tree-hypersequents for modal propositional logic*. In: Towards Mathematical Philosophy: Papers from the Studia Logica conference Trends in Logic IV, Springer, pp. 31–51, doi:10.1007/978-1-4020-9084-4_3.

[15] Francesca Poggiolesi (2009): *A purely syntactic and cut-free sequent calculus for the modal logic of provability*. The Review of Symbolic Logic 2(4), pp. 593–611, doi:10.1017/S1755020309990244.

[16] Giovanni Sambin & Silvio Valentini (1982): *The modal logic of provability. The sequential approach*. Journal of Philosophical Logic 11(3), pp. 311–342, doi:10.1007/BF00293433.

[17] Robert M Solovay (1976): *Provability interpretations of modal logic*. Israel Journal of Mathematics 25(3), pp. 287–304, doi:10.1007/BF02757006.

[18] Silvio Valentini (1983): *The modal logic of provability: Cut-elimination*. Journal of Philosophical logic 12(4), pp. 471–476, doi:10.1007/BF00249262.

Incomplete Descriptions and Qualified Definiteness

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According to Russell, strict uses of the definite article 'the' in a definite description 'the F' involve uniqueness; in case there is more than one F, 'the F' is used somewhat loosely, and an indefinite description 'an F' should be preferred. We give an account of constructions of the form 'the F is G' in which the definite article is used loosely (and in which 'the F' is, therefore, incomplete), essentially by replacing the usual notion of identity in Russell's uniqueness clause with the notion of qualified identity, i.e., 'a is the same as b in all $\mathcal{Q}$-respects', where $\mathcal{Q}$ is a subset of the set of predicates $\mathcal{P}$. This modification gives us qualified notions of uniqueness and definiteness. A qualified definiteness statement 'the $\mathcal{Q}$-unique F is G' is strict in case $\mathcal{Q} = \mathcal{P}$ and loose in case $\mathcal{Q}$ is a proper subset of $\mathcal{P}$. The account is made formally precise in terms of proof theory and proof-theoretic semantics.

Keywords: definiteness, incomplete descriptions, proof-theoretic semantics, uniqueness

1 Introduction

Sometimes we use the definite description 'the F' in cases in which there is a unique F. According to Russell ([10]: 481), the definite article 'the' is used strictly in such cases. For example, speaking about Francis, we use 'the pope' in (1.1) in this way.

(1.1) The pope is bald.

Sometimes, as Russell notes, we use 'the F' also in cases, in which there is more than one F. For example, 'the bishop' in (1.2) is used in this loose way (as would be 'the pope' during a schism).

(1.2) The pope blesses the bishop.

According to Russell, such loose uses of 'the F' should be avoided in favour of the indefinite description 'an F'.

In this paper, we propose a formal account of both uses of 'the F' in terms of qualified definiteness. On a Russellian analysis, a construction of the form 'the F is G' is explained in terms of an existence, a uniqueness, and a predication clause:

(E) There is at least one F.

(U) There is at most one F.

(P) Every F is G.

We modify this analysis mainly by replacing the usual notion of identity in the definition of uniqueness with the notion of *qualified identity* proposed in [16], i.e., 'a is the same as b in all $\mathcal{Q}$-respects', where $\mathcal{Q}$ is a subset of the set of predicates $\mathcal{P}$. The notion of *qualified uniqueness* that results from this replacement says:

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(QU) For every x and y, if they are F, then they are identical with respect to every predicate in $\mathcal{Q}$.

Finally, a statement of *qualified definiteness* says, combining the three Russellian components:

(QD) The $\mathcal{Q}$-unique x which is F is G.

Qualified definiteness, unlike standard definiteness, allows for fine-tuning. Let $\mathcal{Q}'$ be a proper subset of $\mathcal{P}$ (i.e., $\mathcal{Q}' \subset \mathcal{P}$). If $\mathcal{Q} = \mathcal{P}$ in (QD), then we get the reading 'the *only* x which is F is G'. We may use this reading only in case there is a single x that is F. This is definiteness proper. If, on the other hand, we put $\mathcal{Q} = \mathcal{Q}'$, then we get: 'the x which is F is G'. We may use this reading only in case there are at least two things which are F that are indiscernible with respect to $\mathcal{Q}'$, but discernible with respect to $\mathcal{P} \setminus \mathcal{Q}'$. This is restricted definiteness. What is subject to restriction, on this account, is thus the set of $\mathcal{Q}$-respects (rather than, e.g., a domain of quantifiers [12]).

Below, we provide the details of this proposal. It will differ from competing semantic analyses of incomplete descriptions also in that it will be couched in a framework of proof-theoretic semantics (see [11] for an overview) rather than in some version of model-theoretic semantics. (For an overview of the literature on incomplete descriptions see, e.g., [8]: sect. 5.3. An elaborate model-theoretic account is [1].)

Sect. 2 defines the formal language. Sect. 3 recapitulates the relevant fragment of the intuitionistic bipredicational natural deduction systems defined in [16] and combines it with the rules for definiteness proposed in [2], [3] into proof systems for qualified definiteness, establishing normalization and the subexpression (and subformula) property for them. Sect. 4 defines a proof-theoretic semantics for qualified definiteness, and Sect. 5 applies this semantics to incomplete descriptions in the manner suggested above. The paper ends with a brief outlook in Sect. 6.

2 The language

We extend the bipredicational language $\mathcal{L}$ motivated and defined in [16] with contextually defined operators for qualified definiteness and call the extended language $\mathcal{L}\iota$.

$\mathcal{L}$ is a first-order language. It is bipredicational, since it allows for both predication and predication failure. We first recapitulate those parts of its definition which are relevant for present purposes.

Definition 2.1. $\mathcal{C}$ is the set of individual (or nominal) constants (form: α_i) and $\mathcal{P}$ is the set of n-ary predicate constants (form: φ_i^n) of $\mathcal{L}$. Moreover, Atm is the set of atomic sentences (form: $\varphi^n\alpha_1...\alpha_n$) of $\mathcal{L}$. $Atm(\alpha) =_{def} \{A \in Atm : A$ contains at least one occurrence of $\alpha \in \mathcal{C}\}$ and $Atm(\varphi^n) =_{def} \{A \in Atm : A$ contains an occurrence of $\varphi^n \in \mathcal{P}\}$. A nominal term o_i is either a nominal constant or a nominal variable x_i. Atomic formulae have the form $\varphi^n o_1...o_n$ and are used for predication. Negative predications (or predication failures) take the form $-\varphi^n o_1...o_n$ (reading: 'the ascriptive combination of φ^n with $o_1,...,o_n$ fails').

Definition 2.2. Defined symbols of $\mathcal{L}$:

1. $\neg A =_{def} A \supset \bot$ (negation)
2. $A \leftrightarrow B =_{def} (A \supset B)\&(B \supset A)$ (equivalence)
3. Let φ^n be an n-ary predicate constant.

$$\begin{aligned} &P^n_{\varphi^n}(o_1,o_2) =_{def} \\ &\quad \forall z_1...\forall z_{n-1}\forall z_n\ ((\varphi^n o_1 z_2...z_n \leftrightarrow \varphi^n o_2 z_2...z_n) \\ &\quad \&\ (\varphi^n z_1 o_1...z_n \leftrightarrow \varphi^n z_1 o_2...z_n) \\ &\quad \&\ ...\ \&\ (\varphi^n z_1...z_{n-1} o_1 \leftrightarrow \varphi^n z_1...z_{n-1} o_2)) \end{aligned}$$

$$N^n_{\varphi^n}(o_1, o_2) =_{def}$$
$$\forall z_1 ... \forall z_{n-1} \forall z_n \ ((-\varphi^n o_1 z_2 ... z_n \leftrightarrow -\varphi^n o_2 z_2 ... z_n)$$
$$\& \ (-\varphi^n z_1 o_1 ... z_n \leftrightarrow -\varphi^n z_1 o_2 ... z_n)$$
$$\& \ ... \ \& \ (-\varphi^n z_1 ... z_{n-1} o_1 \leftrightarrow -\varphi^n z_1 ... z_{n-1} o_2))$$

Let $\varphi_1^{k_1}, ..., \varphi_m^{k_m}$ be all the predicate constants in $\mathcal{Q}$, where φ_i is k_i-ary and $\mathcal{Q} \subseteq \mathcal{P}$.

Positive qualified identity:

$o_1 \overset{+}{=}_{\mathcal{Q}} o_2 =_{def} P^{k_1}_{\varphi_1}(o_1, o_2) \ \& \ ... \ \& \ P^{k_m}_{\varphi_m}(o_1, o_2)$

('o_1 is the same as o_2 in all $\mathcal{Q}$-respects')

Negative qualified identity:

$o_1 \overset{-}{=}_{\mathcal{Q}} o_2 =_{def} N^{k_1}_{\varphi_1}(o_1, o_2) \ \& \ ... \ \& \ N^{k_m}_{\varphi_m}(o_1, o_2)$

('o_1 is the same as o_2 in no $\mathcal{Q}$-respect')

Remark 2.1. Note that, in contrast to $\neg$, the operator for predication failure $-$ is primitive. Moreover, unlike the former, it is sensitive to the internal structure of the formula to which it is prefixed.

$\mathcal{L}\iota$ extends $\mathcal{L}$ with operators for qualified definiteness by adapting the definitions from [2], [3].

Definition 2.3. We write $\varphi(x)$, suppressing the arity of φ, for atomic formulae $\varphi^n o_1 ... o_n$ containing (possibly multiple occurrences of) x. Let $\mathcal{Q} \subseteq \mathcal{P}$.

1. *Positive qualified definiteness*:

$$\psi(\iota_{\mathcal{Q}} x \varphi(x)) =_{def} \exists x \varphi(x) \ \& \ \underbrace{\forall u \forall v((\varphi(u) \ \& \ \varphi(v)) \supset u \overset{+}{=}_{\mathcal{Q}} v)}_{\text{Positive qualified uniqueness}} \ \& \ \forall w(\varphi(w) \supset \psi(w))$$

('the $\mathcal{Q}$-unique x which is φ is ψ'; simpler: 'the $\mathcal{Q}$-unique φ is ψ')

2. *Negative qualified definiteness*:

$$\psi(\iota_{\mathcal{Q}} x - \varphi(x)) =_{def} \exists x - \varphi(x) \ \& \ \underbrace{\forall u \forall v((-\varphi(u) \ \& \ -\varphi(v)) \supset u \overset{-}{=}_{\mathcal{Q}} v)}_{\text{Negative qualified uniqueness}} \ \& \ \forall w(-\varphi(w) \supset \psi(w))$$

('the $\mathcal{Q}$-unique x which fails to be φ is ψ'; simpler: 'the $\mathcal{Q}$-unique $-\varphi$ is ψ')

Remark 2.2. The definition of positive qualified definiteness differs from the definition of definiteness proposed in [2], [3], in that it does not make use of the familiar primitive notion of identity in the uniqueness part. In this respect, it significantly departs also from the tradition.

Qualified definiteness allows for degrees.

Definition 2.4. Let $\mathcal{Q}' \subset \mathcal{P}$. It has (i) the highest degree of definiteness in case $\mathcal{Q} = \mathcal{P}$ and (ii) a lower degree, in case $\mathcal{Q} = \mathcal{Q}'$. Given $\mathcal{Q}' \subset \mathcal{P}$, we can make the following distinction:

1. *Maximal definiteness*:
   (a) $\psi(\iota_{\mathcal{P}} x \varphi(x))$: 'the only x which is φ is ψ';
   (b) $\psi(\iota_{\mathcal{P}} x - \varphi(x))$: 'the only x which fails to be φ is ψ'.
2. *Restricted definiteness*:
   (a) $\psi(\iota_{\mathcal{Q}'} x \varphi(x))$: 'the x which is φ is ψ';
   (b) $\psi(\iota_{\mathcal{Q}'} x - \varphi(x))$: 'the x which fails to be φ is ψ'.

A loosely used definite description 'the F' is, thus, construed as a restriction of a strictly used 'the F' (i.e., the maximally definite description 'the only F').

Definition 2.5. Negative predications with qualified definite descriptions take the following forms:

1. $-\psi(\iota_{\mathcal{Q}} x \varphi(x))$: 'the $\mathcal{Q}$-unique x which is φ fails to be ψ';
2. $-\psi(\iota_{\mathcal{Q}} x - \varphi(x))$: 'the $\mathcal{Q}$-unique x which fails to be φ fails to be ψ'.

3 Proof systems

In order to obtain a proof system for reasoning with qualified definiteness, we enrich the intuitionistic bipredicational $\mathbf{I0}(\mathcal{S}_b^{=})$-systems defined in [16] with rules for qualified definiteness, by adapting the rules for definiteness presented in [2], [3]. We call the resulting systems $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-systems.

3.1 Bipredicational natural deduction

We first repeat the parts of the definition of $\mathbf{I0}(\mathcal{S}_b^{=})$-systems from [16] which are relevant for present purposes.

3.1.1 Bipredicational subatomic systems

Definition 3.1. A *bipredicational subatomic system* $\mathcal{S}_b$ is a pair $\langle \mathcal{I}, \mathcal{R}_b \rangle$, where $\mathcal{I}$ is a *subatomic base* and $\mathcal{R}_b$ is a set of *introduction and elimination rules for atomic sentences and negative predications*. $\mathcal{I}$ is a 3-tuple $\langle \mathcal{C}, \mathcal{P}, v \rangle$, where v is such that:

1. For any $\alpha \in \mathcal{C}$, $v : \mathcal{C} \to \wp(Atm)$, where $v(\alpha) \subseteq Atm(\alpha)$.
2. For any $\varphi^n \in \mathcal{P}$, $v : \mathcal{P} \to \wp(Atm)$, where $v(\varphi^n) \subseteq Atm(\varphi^n)$.

We let $\tau\Gamma =_{def} v(\tau)$ for any $\tau \in \mathcal{C} \cup \mathcal{P}$, and call $\tau\Gamma$ the set of *term assumptions* for τ. $\mathcal{R}_b$ contains I/E-rules of the following form:

$$\frac{\begin{matrix}\mathcal{D}_0 \\ \varphi_0^n\Gamma\end{matrix} \quad \begin{matrix}\mathcal{D}_1 \\ \alpha_1\Gamma\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{D}_n \\ \alpha_n\Gamma\end{matrix}}{\varphi_0^n\alpha_1\dots\alpha_n}\ (as\mathrm{I}) \qquad \frac{\begin{matrix}\mathcal{D}_1 \\ \varphi_0^n\alpha_1\dots\alpha_n\end{matrix}}{\tau_i\Gamma}\ (as\mathrm{E}_i)$$

$$\frac{\begin{matrix}\mathcal{D}_0 \\ \varphi_0^n\Gamma\end{matrix} \quad \begin{matrix}\mathcal{D}_1 \\ \alpha_1\Gamma\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{D}_n \\ \alpha_n\Gamma\end{matrix}}{-\varphi_0^n\alpha_1\dots\alpha_n}\ (-as\mathrm{I}) \qquad \frac{\begin{matrix}\mathcal{D}_1 \\ -\varphi_0^n\alpha_1\dots\alpha_n\end{matrix}}{\tau_i\Gamma}\ (-as\mathrm{E}_i)$$

Side conditions:

1. asI: $\varphi_0^n\alpha_1\dots\alpha_n \in \varphi_0^n\Gamma \cap \alpha_1\Gamma \cap \dots \cap \alpha_n\Gamma$.
2. $-as$I: $\varphi_0^n\alpha_1\dots\alpha_n \notin \varphi_0^n\Gamma \cap \alpha_1\Gamma \cap \dots \cap \alpha_n\Gamma$.
3. $as\mathrm{E}_i$ and $-as\mathrm{E}_i$: $i \in \{0,\dots,n\}$ and $\tau_i \in \{\varphi_0^n, \alpha_1, \dots, \alpha_n\}$.

Terminology: We say that $-\varphi_0^n\alpha_1\dots\alpha_n$ is *negatively contained* in $\varphi_0^n\Gamma \cap \alpha_1\Gamma \cap \dots \cap \alpha_n\Gamma$, in case the side condition on $-as$I is satisfied.

Definition 3.2. *Derivations in $\mathcal{S}_b$-systems.*

Basic step. Any term assumption $\tau\Gamma$, any atomic sentence (resp. negative predication), i.e., a derivation from the open assumption of $\varphi_0^n\alpha_1\dots\alpha_n$ (resp. $-\varphi_0^n\alpha_1\dots\alpha_n$) is an $\mathcal{S}_b$-derivation.

Induction step. If $\mathcal{D}_i$, for $i \in \{0,\dots,n\}$, are $\mathcal{S}_b$-derivations, then an $\mathcal{S}_b$-derivation can be constructed by means of the I/E-rules for as and $-as$ displayed above.

Remark 3.1. The term assumptions are, so to speak, proof-theoretic semantic values of the non-logical constants. Applications of the subatomic introduction rules asI and $-as$I serve to establish, on the basis of these values, the truth of atomic sentences and negative predications, respectively.

3.1.2 Bipredicational subatomic identity systems

Definition 3.3. Atomic sentences $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are *mirror atomic sentences* if and only if they are exactly alike except that the former contains occurrences of α_1 at all the places at which the latter contains occurrences of α_2, and vice versa.

Definition 3.4. A *bipredicational subatomic identity system* $\mathcal{S}_b^{=}$ is a 3-tuple $\langle \mathcal{I}, \mathcal{R}_b, \mathcal{R}_b^{=} \rangle$, which extends a bipredicational subatomic system with a set $\mathcal{R}_b^{=}$ of *I/E-rules for (positive/negative) qualified identity sentences*, where $\mathcal{Q} \subseteq \mathcal{P}$.

1. $\overset{+}{=}_{\mathcal{Q}}$:

$$\dfrac{\begin{matrix} [\varphi_1(\alpha_1)]^{(1_1)} & [\varphi_1(\alpha_2)]^{(1_2)} & & [\varphi_k(\alpha_1)]^{(k_1)} & [\varphi_k(\alpha_2)]^{(k_2)} \\ \mathcal{D}_{1_1} & \mathcal{D}_{1_2} & & \mathcal{D}_{k_1} & \mathcal{D}_{k_2} \\ \varphi_1(\alpha_2) & \varphi_1(\alpha_1) & \dots & \varphi_k(\alpha_2) & \varphi_k(\alpha_1) \end{matrix}}{\alpha_1 \overset{+}{=}_{\mathcal{Q}} \alpha_2} \, (\overset{+}{=}_{\mathcal{Q}}\mathrm{I}), 1_1, \dots, k_2$$

$$\dfrac{\begin{matrix} \mathcal{D}_1 & \mathcal{D}_{i_1} \\ \alpha_1 \overset{+}{=}_{\mathcal{Q}} \alpha_2 & \varphi_i(\alpha_1) \end{matrix}}{\varphi_i(\alpha_2)} \, (\overset{+}{=}_{\mathcal{Q}}\mathrm{E}_i1) \qquad \dfrac{\begin{matrix} \mathcal{D}_1 & \mathcal{D}_{i_2} \\ \alpha_1 \overset{+}{=}_{\mathcal{Q}} \alpha_2 & \varphi_i(\alpha_2) \end{matrix}}{\varphi_i(\alpha_1)} \, (\overset{+}{=}_{\mathcal{Q}}\mathrm{E}_i2)$$

where $\varphi_i \in \mathcal{Q}$, $i \in \{1, \dots, k\}$, and $\varphi_i(\alpha_1)$ and $\varphi_i(\alpha_2)$ are mirror atomic sentences.

2. $\overset{-}{=}_{\mathcal{Q}}$:

$$\dfrac{\begin{matrix} [-\varphi_1(\alpha_1)]^{(1_1)} & [-\varphi_1(\alpha_2)]^{(1_2)} & & [-\varphi_k(\alpha_1)]^{(k_1)} & [-\varphi_k(\alpha_2)]^{(k_2)} \\ \mathcal{D}_{1_1} & \mathcal{D}_{1_2} & & \mathcal{D}_{k_1} & \mathcal{D}_{k_2} \\ -\varphi_1(\alpha_2) & -\varphi_1(\alpha_1) & \dots & -\varphi_k(\alpha_2) & -\varphi_k(\alpha_1) \end{matrix}}{\alpha_1 \overset{-}{=}_{\mathcal{Q}} \alpha_2} \, (\overset{-}{=}_{\mathcal{Q}}\mathrm{I}), 1_1, \dots, k_2$$

$$\dfrac{\begin{matrix} \mathcal{D}_1 & \mathcal{D}_{i_1} \\ \alpha_1 \overset{-}{=}_{\mathcal{Q}} \alpha_2 & -\varphi_i(\alpha_1) \end{matrix}}{-\varphi_i(\alpha_2)} \, (\overset{-}{=}_{\mathcal{Q}}\mathrm{E}_i1) \qquad \dfrac{\begin{matrix} \mathcal{D}_1 & \mathcal{D}_{i_2} \\ \alpha_1 \overset{-}{=}_{\mathcal{Q}} \alpha_2 & -\varphi_i(\alpha_2) \end{matrix}}{-\varphi_i(\alpha_1)} \, (\overset{-}{=}_{\mathcal{Q}}\mathrm{E}_i2)$$

where $\varphi_i \in \mathcal{Q}$, $i \in \{1, \dots, k\}$, and $\varphi_i(\alpha_1)$ and $\varphi_i(\alpha_2)$ are mirror atomic sentences.

Remark 3.2. In contrast to the standard I-rules for identity, the I-rules for qualified identity allow one to introduce formulae in which the identity predicate is not necessarily flanked by two occurrences of the same constant. Note that these rules reflect the definitions of the qualified identity predicates.

Definition 3.5. It will sometimes be convenient to use the notation $\{\mathcal{D}\}$ for the set of the subderivations $\mathcal{D}_{2_1}, \mathcal{D}_{2_2}, \dots, \mathcal{D}_{k_1}, \mathcal{D}_{k_2}$ in applications of I-rules for qualified identity.

3.1.3 Bipredicational subatomic natural deduction systems

Definition 3.6. *Derivations in* $\mathbf{I0}(\mathcal{S}_b^{=})$*-systems.*

Basic step. Any derivation in an $\mathcal{S}_b^{=}$-system and any formula A (i.e., a derivation from the open assumption of A) is a derivation in an $\mathbf{I0}(\mathcal{S}_b^{=})$-system.

Induction step. If $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are derivations in an $\mathbf{I0}(\mathcal{S}_b^{=})$-system, and C possibly a term assumption, then a derivation in an $\mathbf{I0}(\mathcal{S}_b^{=})$-system can be constructed by means of the rules:

$$\dfrac{\begin{matrix} \mathcal{D}_1 & \mathcal{D}_2 \\ A & B \end{matrix}}{A \& B} \, (\&\mathrm{I}) \qquad \dfrac{\begin{matrix} \mathcal{D}_1 \\ A \& B \end{matrix}}{A} \, (\&\mathrm{E1}) \qquad \dfrac{\begin{matrix} \mathcal{D}_1 \\ A \& B \end{matrix}}{B} \, (\&\mathrm{E2}) \qquad \dfrac{\begin{matrix} \mathcal{D}_1 \\ A \end{matrix}}{A \vee B} \, (\vee\mathrm{I1}) \qquad \dfrac{\begin{matrix} \mathcal{D}_1 \\ B \end{matrix}}{A \vee B} \, (\vee\mathrm{I2})$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A \vee B\end{array} \quad \begin{array}{c}[A]^{(u)}\\ \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[B]^{(v)}\\ \mathcal{D}_3\\ C\end{array}}{C}\,(\vee\text{E}), u, v \qquad \dfrac{\begin{array}{c}[A]^{(u)}\\ \mathcal{D}_1\\ B\end{array}}{A \supset B}\,(\supset\text{I}), u \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \supset B\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}\,(\supset\text{E})$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A(x/o)\end{array}}{\forall xA}\,(\forall\text{I}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \forall xA\end{array}}{A(x/o)}\,(\forall\text{E}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A(x/o)\end{array}}{\exists xA}\,(\exists\text{I}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists xA\end{array} \quad \begin{array}{c}[A(x/o)]^{(u)}\\ \mathcal{D}_2\\ C\end{array}}{C}\,(\exists\text{E}), u$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \bot\end{array}}{A}\,(\bot\text{i})$$

Side conditions:

1. In $\forall$I: (i) if o is a proper variable y, then $o \equiv x$ or o is not free in A, and o is not free in any assumption of a formula which is open in the derivation of $A(x/o)$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\forall xA$, nor in a term assumption leaf $o\Gamma$; (iii) o is nominal constant and $\begin{array}{c}\mathcal{D}_1\\ A(x/o)\end{array}$ for all $o \in \mathcal{C}$.
2. In $\forall$E: o is free for x in A.
3. In $\exists$E: (i) if o is a proper variable y, then $o \equiv x$ or o is not free in A, and o is not free in C nor in any assumption of a formula which is open in the derivation of the upper occurrence of C other than $[A(x/o)]^{(u)}$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\exists xA$, nor in C, nor in a term assumption leaf $o\Gamma$.
4. In $\exists$I: o is free for x in A.

Minimal bipredicational subatomic natural deduction systems, $\mathbf{M0}(\mathcal{S}_b^=)$-systems, result from $\mathbf{I0}(\mathcal{S}_b^=)$-systems, in case $\bot$i is removed.

In case we employ the $\forall$I-rule according to the provisos for it given in (i) [(ii), (iii)], we use the labels $\forall$I.i [$\forall$I.ii, $\forall$I.iii]. Similarly, for the $\exists$E-rule and the labels $\exists$E.i and $\exists$E.ii.

3.2 Bipredicational natural deduction for qualified definiteness

We now add rules for the introduction and elimination of qualified definiteness to $\mathbf{I0}(\mathcal{S}_b^=)$-systems in order to obtain $\mathbf{I0}(\mathcal{S}_b^=)\iota$-systems which are sufficient to define a proof-theoretic semantics for the simplest possible constructions involving definite descriptions.

Definition 3.7. Let $\mathcal{Q} \subseteq \mathcal{P}$. In the $\iota_\mathcal{Q}$I-rule below, the conclusion of $\mathcal{D}_1$ [$\mathcal{D}_2$, $\mathcal{D}_3$] corresponds to the (E)- [(QU)-, (P)-] clause. Likewise for $\iota_\mathcal{Q}$–I.

1. *Rules for positive qualified definiteness*:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists x\varphi(x)\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \forall u\forall v((\varphi(u)\ \&\ \varphi(v)) \supset u \overset{\pm}{=}_\mathcal{Q} v)\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ \forall w(\varphi(w) \supset \psi(w))\end{array}}{\psi(\iota_\mathcal{Q} x\varphi(x))}\,(\iota_\mathcal{Q}\text{I})$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi(\iota_\mathcal{Q} x\varphi(x))\end{array}}{\exists x\varphi(x)}\,(\iota_\mathcal{Q}\text{E1}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi(\iota_\mathcal{Q} x\varphi(x))\end{array}}{\forall u\forall v((\varphi(u)\ \&\ \varphi(v)) \supset u \overset{\pm}{=}_\mathcal{Q} v)}\,(\iota_\mathcal{Q}\text{E2}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi(\iota_\mathcal{Q} x\varphi(x))\end{array}}{\forall w(\varphi(w) \supset \psi(w))}\,(\iota_\mathcal{Q}\text{E3})$$

The $\iota_{\mathcal{Q}}$I/E-rules for $-\psi(\iota_{\mathcal{Q}}x\varphi(x))$ are analogous.

2. *Rules for negative qualified definiteness*:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \exists x-\varphi(x)\end{matrix}\qquad \begin{matrix}\mathcal{D}_2\\ \forall u\forall v((-\varphi(u)\ \&\ -\varphi(v))\supset u\overset{-}{=}_{\mathcal{Q}} v)\end{matrix}\qquad \begin{matrix}\mathcal{D}_3\\ \forall w(-\varphi(w)\supset\psi(w))\end{matrix}}{\psi(\iota_{\mathcal{Q}}x-\varphi(x))}\ (\iota_{\mathcal{Q}}-\text{I})$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \psi(\iota_{\mathcal{Q}}x-\varphi(x))\end{matrix}}{\exists x-\varphi(x)}\ (\iota_{\mathcal{Q}}-\text{E1})\qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \psi(\iota_{\mathcal{Q}}x-\varphi(x))\end{matrix}}{\forall u\forall v((-\varphi(u)\ \&\ -\varphi(v))\supset u\overset{-}{=}_{\mathcal{Q}} v)}\ (\iota_{\mathcal{Q}}-\text{E2})\qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \psi(\iota_{\mathcal{Q}}x-\varphi(x))\end{matrix}}{\forall w(-\varphi(w)\supset\psi(w))}\ (\iota_{\mathcal{Q}}-\text{E3})$$

The $\iota_{\mathcal{Q}}-$I/E-rules for $-\psi(\iota_{\mathcal{Q}}x-\varphi(x))$ are analogous.

Example 3.1. Let $\mathcal{Q}=\{\varphi_1,\ldots,\varphi_k\}$, $\mathcal{Q}\subseteq\mathcal{P}$, and $\varphi_i,\varphi_j\in\mathcal{Q}$, where $i,j\in\{1,\ldots,k\}$ and $i\neq j$.

$$\mathcal{D}_1=\dfrac{\dfrac{\varphi_i\Gamma\quad\ldots\quad\alpha\Gamma}{\varphi_i(\alpha)}}{\exists x\varphi_i(x)}\tag{1}$$

$$\mathcal{D}_2=\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi_1(\alpha)]^{(1_1)}}{\varphi_1\Gamma}\quad\ldots\quad\dfrac{\dfrac{[\varphi_i(\alpha)\&\varphi_i(\beta)]^{(1)}}{\varphi_i(\beta)}}{\beta\Gamma}}{\varphi_1(\beta)}\qquad \dfrac{\dfrac{[\varphi_1(\beta)]^{(1_2)}}{\varphi_1\Gamma}\quad\ldots\quad\dfrac{\dfrac{[\varphi_i(\alpha)\&\varphi_i(\beta)]^{(1)}}{\varphi_i(\alpha)}}{\alpha\Gamma}}{\varphi_1(\alpha)}\qquad\{\mathcal{D}\}}{\alpha\overset{\pm}{=}_{\mathcal{Q}}\beta}\ 1_1,\ldots,k_2}{(\varphi_i(\alpha)\&\varphi_i(\beta))\supset\alpha\overset{\pm}{=}_{\mathcal{Q}}\beta}\ 1}{\forall v((\varphi_i(\alpha)\&\varphi_i(v))\supset\alpha\overset{\pm}{=}_{\mathcal{Q}} v)}\ \text{iii}}{\forall u\forall v((\varphi_i(u)\&\varphi_i(v))\supset u\overset{\pm}{=}_{\mathcal{Q}} v)}\ \text{iii}\tag{2}$$

$$\mathcal{D}_3=\dfrac{\dfrac{\dfrac{\varphi_j\Gamma\quad\ldots\quad\dfrac{[\varphi_i(\alpha)]^{(2)}}{\alpha\Gamma}}{\varphi_j(\alpha)}}{\varphi_i(\alpha)\supset\varphi_j(\alpha)}\ 2}{\forall w(\varphi_i(w)\supset\varphi_j(w))}\ \text{iii}\tag{3}$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \exists x\varphi_i(x)\end{matrix}\qquad \begin{matrix}\mathcal{D}_2\\ \forall u\forall v((\varphi_i(u)\&\varphi_i(v))\supset u\overset{\pm}{=}_{\mathcal{Q}} v)\end{matrix}\qquad \begin{matrix}\mathcal{D}_3\\ \forall w(\varphi_i(w)\supset\varphi_j(w))\end{matrix}}{\varphi_j(\iota_{\mathcal{Q}}x\varphi_i(x))}\ (\iota_{\mathcal{Q}}\text{I})\tag{4}$$

3.3 Normalization and the subformula property

Normalization and the subformula property for $\mathbf{I0}(\mathcal{S}_b^{=})$-systems have been established in [16] making use of the methods developed in [9]; see also [14]. These results guarantee, e.g., the consistency of the systems and simplify proof search in them.

In order to prove normalization for $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-systems, we make use of the following conversions.

Definition 3.8. The *conversions (detour, permutation, simplification)* for $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-systems comprise those for $\mathbf{I0}(\mathcal{S}_b^{=})$-systems (see [16]) and the following detour conversions:

1. ι_Q-*Conversions*:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists x\varphi(x)\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \forall u\forall v((\varphi(u)\ \&\ \varphi(v))\supset u \stackrel{\pm}{=}_Q v)\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ \forall w(\varphi(w)\supset\psi(w))\end{array}}{\psi(\iota_Q x\varphi(x))}\,(\iota_Q\mathrm{I})}{\exists x\varphi(x)}\,(\iota_Q\mathrm{E1}) \quad \text{conv} \quad \begin{array}{c}\mathcal{D}_1\\ \exists x\varphi(x)\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists x\varphi(x)\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \forall u\forall v((\varphi(u)\ \&\ \varphi(v))\supset u \stackrel{\pm}{=}_Q v)\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ \forall w(\varphi(w)\supset\psi(w))\end{array}}{\psi(\iota_Q x\varphi(x))}\,(\iota_Q\mathrm{I})}{\forall u\forall v((\varphi(u)\ \&\ \varphi(v))\supset u \stackrel{\pm}{=}_Q v)}\,(\iota_Q\mathrm{E2})$$

$$\text{conv}$$

$$\begin{array}{c}\mathcal{D}_2\\ \forall u\forall v((\varphi(u)\ \&\ \varphi(v))\supset u \stackrel{\pm}{=}_Q v)\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists x\varphi(x)\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \forall u\forall v((\varphi(u)\ \&\ \varphi(v))\supset u \stackrel{\pm}{=}_Q v)\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ \forall w(\varphi(w)\supset\psi(w))\end{array}}{\psi(\iota_Q x\varphi(x))}\,(\iota_Q\mathrm{I})}{\forall w(\varphi(w)\supset\psi(w))}\,(\iota_Q\mathrm{E3})$$

$$\text{conv}$$

$$\begin{array}{c}\mathcal{D}_3\\ \forall w(\varphi(w)\supset\psi(w))\end{array}$$

2. ι_Q--*Conversions*: analogous.

Remark 3.3. Unlike the ιE2-rules in [2], [3], the above E2-rules have a single premiss and invert directly.

Theorem 3.1. *Normalization*: Any derivation $\mathcal{D}$ in an $\mathbf{I0}(\mathcal{S}_b^=)\iota$-system can be transformed into a normal $\mathbf{I0}(\mathcal{S}_b^=)\iota$-derivation.

Proof. We repeat the corresponding proof for $\mathbf{I0}(\mathcal{S}_b^=)$-systems in [16], taking also the detour conversions for qualified definiteness into account. As a result, all detours can be eliminated from derivations in these systems. □

Importantly, $\mathbf{I0}(\mathcal{S}_b^=)\iota$-systems enjoy the subformula property as a special case of the subexpression property. The latter property deals with units and expressions. Roughly, a unit is either a formula or a term assumption $\tau\Gamma$, and an expression is either a formula or the non-logical constant τ of $\tau\Gamma$.

Theorem 3.2. *Subexpression property*: If $\mathcal{D}$ is a normal derivation of a unit U from a set of units Γ in an $\mathbf{I0}(\mathcal{S}_b^=)\iota$-system, then each unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma\cup\{U\}$.

Proof. We proceed like in the corresponding proof for $\mathbf{I0}(\mathcal{S}_b^=)$-systems in [16]. As a result, all expressions in $\mathcal{D}$ are subexpressions of either the root or the leaves of $\mathcal{D}$. □

Corollary 3.1. *Subformula property*: If $\mathcal{D}$ is a normal $\mathbf{I0}(\mathcal{S}_b^=)\iota$-derivation of formula A from a set of formulae Γ, then each formula in $\mathcal{D}$ is a subformula of a formula in $\Gamma\cup\{A\}$.

Remark 3.4. Since the identity predicates used in the proof systems [2], [3], are primitive, such a subformula result is not available for these systems. This remark also applies to other available intuitionistic natural deduction systems for definiteness (e.g., [7], [13]).

Corollary 3.2. *Internal completeness*. Internal completeness in the sense of [4] (pp. 139–140) is given by Corollary 3.1. To establish internal completeness for $\mathbf{I0}(\mathcal{S}_b^=)\iota$-systems in the sense of [16] (p. 127), we proceed like described therein.

4 A proof-theoretic semantics

On the basis of the results obtained, we may formulate a subatomic proof-theoretic semantics for qualified definiteness. For this purpose, we adjust the corresponding definitions form [16] to the present systems.

Definition 4.1. 1. A derivation $\mathcal{D}$ of a formula A in an $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-system is a *canonical derivation* iff it derives A by means of an application of an I-rule (in the last step of $\mathcal{D}$).

2. A canonical derivation $\mathcal{D}$ of A in an $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-system is a *canonical proof* of A in that system iff there are no applications of *as*-rules or $-$*as*-rules in $\mathcal{D}$ and all assumptions of $\mathcal{D}$ have been discharged.

3. The conclusions of canonical $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-derivations are $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-*theses* and the conclusions of $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-derivations which are also proofs are $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-*theorems*.

Definition 4.2. *Meaning*: Let I be an $\mathbf{I0}(\mathcal{S}_b^{=})\iota$-system.

1. The meaning of a *non-logical constant* τ is given by the term assumptions $\tau\Gamma$ for τ which are determined by the subatomic base of the $\mathcal{S}_b^{=}$-system of I.

2. The meaning of a *formula* A of $\mathcal{L}\iota$ is given by the set of canonical derivations of A in I.

Remark 4.1. The rules for qualified identity defined in [16] allow not only for reductions in terms of conversions, but also for expansions (cf. [15]: 256). This is a further point, in which they differ from the standard natural deduction rules for identity (cf. [16]: 104). For an overview of the structural proof theory of identity see [5].

Remark 4.2. Note that this formal account of meaning does not make use of a semantic ontology (e.g., individuals, possible worlds), something essential to model-theoretic semantics. Specifically, the meaning of $\exists$-formulae does not presuppose a domain of individuals. Strictly speaking, $\exists xA$ reads: 'For at least one x, A', where x is a nominal variable ranging over $\mathcal{C}$. This feature of the present semantics makes it particularly natural for the analysis of constructions which involve non-denoting (or empty) terms (e.g., 'Pegasus', 'the captive unicorn').

5 On incomplete descriptions

Qualified uniqueness allows for fine-tuning.

Remark 5.1. Let $\{\varphi_i\} \subset \mathcal{Q}' \subset \mathcal{P}$ and $\varphi_i \in \mathcal{P}$, where $i \in \{1,\ldots,k\}$. We consider the following cases: (i) $\mathcal{Q} = \mathcal{P}$, (ii) $\mathcal{Q} = \mathcal{Q}'$, and (iii) $\mathcal{Q} = \{\varphi_i\}$.

Case (i): Like (2), but with $\mathcal{Q}$ replaced by $\mathcal{P}$. This case gives us the maximal degree of qualified uniqueness. For every x and y, if they are φ_i, then they are identical with respect to every predicate (i.e., they are indiscernible in every respect).

Case (ii): Like case (i), but with $\mathcal{P}$ replaced by $\mathcal{Q}'$ and with $\{\mathcal{D}\}$ replaced by $\{\mathcal{D}\}'$, where $\{\mathcal{D}\}' \subset \{\mathcal{D}\}$. This case gives us an intermediate degree of qualified uniqueness. For every x and y, if they are φ_i, then they are identical with respect to every predicate in $\mathcal{Q}'$ (i.e., they are indiscernible with respect to $\mathcal{Q}'$, but discernible with respect to $\mathcal{P} \setminus \mathcal{Q}'$).

Case (iii):

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi_i(\alpha)]^{(1_1)}}{\varphi_i\Gamma} \quad \dots \quad \dfrac{\dfrac{[\varphi_i(\alpha)\&\varphi_i(\beta)]^{(1)}}{\varphi_i(\beta)}}{\beta\Gamma}}{\varphi_i(\beta)} \qquad \dfrac{\dfrac{[\varphi_i(\beta)]^{(1_2)}}{\varphi_i\Gamma} \quad \dots \quad \dfrac{\dfrac{[\varphi_i(\alpha)\&\varphi_i(\beta)]^{(1)}}{\varphi_i(\alpha)}}{\alpha\Gamma}}{\varphi_i(\alpha)}}{\dfrac{\alpha \overset{\pm}{=}_{\{\varphi_i\}} \beta}{(\varphi_i(\alpha)\&\varphi_i(\beta)) \supset \alpha \overset{\pm}{=}_{\{\varphi_i\}} \beta}\,1}\,1_1,1_2}{\dfrac{\forall y((\varphi_i(\alpha)\&\varphi_i(y)) \supset \alpha \overset{\pm}{=}_{\{\varphi_i\}} y)}{\forall x\forall y((\varphi_i(x)\&\varphi_i(y)) \supset x \overset{\pm}{=}_{\{\varphi_i\}} y)}\,\text{iii}}\,\text{iii} \tag{5}$$

This case gives us the minimal degree of qualified uniqueness. For every x and y, if they are φ_i, then they are identical with respect to every predicate in the singleton $\{\varphi_i\}$ (i.e., they are indiscernible with respect to the predicate φ_i, but discernible with respect to any other predicate in $\mathcal{P} \smallsetminus \{\varphi_i\}$). (Likewise for negative qualified uniqueness.)

Qualified definiteness allows for fine-tuning, since it involves qualified uniqueness.

Remark 5.2. Let $\{\varphi_i\} \subset \mathcal{Q}' \subset \mathcal{P}$, let $P = \varphi_i$, and $B = \varphi_j$ for $\varphi_i, \varphi_j \in \mathcal{Q}'$, where $i, j \in \{1, \dots, k\}$ and $i \neq j$. P: '... is a pope'; B: '... is bald'. And let $\mathcal{D}_2(i)$ [$\mathcal{D}_2(ii)$, $\mathcal{D}_2(iii)$] refer to the derivation for case (i) [(ii), (iii)] mentioned in the previous remark. We may, then, distinguish three general cases of qualified definiteness.

Case (i). Maximal qualified definiteness:

$$\dfrac{\overset{\mathcal{D}_1}{\exists x\varphi_i(x)} \qquad \overset{\mathcal{D}_{2(i)}}{\forall u\forall v((\varphi_i(u)\&\varphi_i(v)) \supset u \overset{\pm}{=}_{\mathcal{P}} v)} \qquad \overset{\mathcal{D}_3}{\forall w(\varphi_i(w) \supset \varphi_j(w))}}{\varphi_j(\iota_{\mathcal{P}}x\varphi_i(x))}\,(\iota_{\mathcal{P}}\text{I}) \tag{6}$$

The premisses of the $\iota_{\mathcal{P}}$I-application say that there is at least one thing which is φ_i, that any two things which are φ_i are the same in any respect, and that everything that is φ_i is φ_j. The conclusion $\varphi_j(\iota_{\mathcal{P}}x\varphi_i(x))$ can be read: 'the $\mathcal{P}$-unique x which is φ_i is φ_j', or, simplifying the reading of Definition 2.4(1) further, 'the only φ_i is φ_j'. We may use these readings only in case there is a single x that is φ_i. This is definiteness proper. We use it for the analysis of (1.1), in case there is no schism.

Case (ii). Intermediate qualified definiteness:

$$\dfrac{\overset{\mathcal{D}_1}{\exists x\varphi_i(x)} \qquad \overset{\mathcal{D}_{2(ii)}}{\forall u\forall v((\varphi_i(u)\&\varphi_i(v)) \supset u \overset{\pm}{=}_{\mathcal{Q}'} v)} \qquad \overset{\mathcal{D}_3}{\forall w(\varphi_i(w) \supset \varphi_j(w))}}{\varphi_j(\iota_{\mathcal{Q}'}x\varphi_i(x))}\,(\iota_{\mathcal{Q}'}\text{I}) \tag{7}$$

The premisses of the $\iota_{\mathcal{Q}'}$I-application say that there is at least one thing which is φ_i, that any two things which are φ_i are the same (only) in any $\mathcal{Q}'$-respect, and that everything that is φ_i is φ_j. The conclusion $\varphi_j(\iota_{\mathcal{Q}'}x\varphi_i(x))$ can be read: 'the $\mathcal{Q}'$-unique x which is φ_i is φ_j', or simply 'the φ_i is φ_j'. We may use these readings only in case there are at least two things that are φ_i which are discernible with respect to $\mathcal{P} \smallsetminus \mathcal{Q}'$. It will be natural to use this restricted kind of definiteness for the analysis of (1.1) in times of schism.

Case (iii). Minimal qualified definiteness:

$$\dfrac{\overset{\mathcal{D}_1}{\exists x\varphi_i(x)} \qquad \overset{\mathcal{D}_{2(iii)}}{\forall u\forall v((\varphi_i(u)\&\varphi_i(v)) \supset u \overset{\pm}{=}_{\{\varphi_i\}} v)} \qquad \overset{\mathcal{D}_3}{\forall w(\varphi_i(w) \supset \varphi_j(w))}}{\varphi_j(\iota_{\{\varphi_i\}}x\varphi_i(x))}\,(\iota_{\{\varphi_i\}}\text{I}) \tag{8}$$

The premisses of the $\iota_{\{\varphi_i\}}$I-application say that there is at least one thing which is φ_i, that any two things which are φ_i are the same only with respect to $\{\varphi_i\}$, and that everything that is φ_i is φ_j. The conclusion $\varphi_j(\iota_{\{\varphi_i\}}x\varphi_i(x))$ can be read: 'the $\{\varphi_i\}$-unique x which is φ_i is φ_j'. We may use this reading only in case there are at least two things that are φ_i which are discernible with respect to $\mathcal{P} \setminus \{\varphi_i\}$. In a sense, this minimal degree of definiteness comes close to generic definiteness: 'the generic φ_i is φ_j' (e.g., 'The Englishman is brave'). Similarly for negative qualified definiteness.

Remark 5.3. A negative predication with a definite description:

(1.3) The king of France is not real.
$-Real(\iota_{\mathcal{P}}x(King\text{-}of(x, France)))$

Cf. Remark 4.2.

6 Outlook

Adapting the resources of [2], [3] to the present framework, we may use it also for the analysis of constructions such as, e.g., (1.2), (1.4)-(1.6), and further challenging cases discussed in the literature.

(1.4) The dog descends from the wolf. (Cf. [8]: (33).)
$Descends\text{-}from(\iota_{\{Dog\}}x(Dog(x)), \iota_{\{Wolf\}}y(Wolf(y)))$

(1.5) The pope put the zucchetto on the zucchetto. (Cf. [8]: (38).)
$Put\text{-}on(\iota_{\mathcal{P}}x(Pope(x)), \iota_{\mathcal{Q}'}y(Zucchetto(y)), \iota_{\mathcal{Q}''}z(Zucchetto(z)))$

(1.6) The man wearing the beret with the button is French. ([6]: 450.)
$French(\iota_{\mathcal{Q}}x(Man(x)\ \&\ Wears(x, \iota_{\mathcal{Q}}y(Beret(y)\ \&\ Has(y, \iota_{\mathcal{Q}}z(Button(z)))))))$

References

[1] Paul Elbourne (2016): *Incomplete descriptions and indistinguishable participants*. Natural Language Semantics 24(1), pp. 1–43, doi:10.1007/s11050-015-9118-8.

[2] Nissim Francez & Bartosz Więckowski (2014): *A proof-theoretic semantics for contextual definiteness*. In Enrico Moriconi & Laura Tesconi, editors: Second Pisa Colloquium in Logic, Language and Epistemology, Edizioni ETS, Pisa, pp. 181–212.

[3] Nissim Francez & Bartosz Więckowski (2017): *A proof-theory for first-order logic with definiteness*. The IfCoLog Journal of Logics and their Applications 4(2), pp. 313–331. Available at http://www.collegepublications.co.uk/downloads/ifcolog00011.pdf. Special issue on Hilbert's Epsilon and Tau in Logic, Informatics and Linguistics edited by Stergios Chatzikyriakidis, Fabio Pasquali and Christian Retoré.

[4] Jean-Yves Girard (2003): *From foundations to ludics*. The Bulletin of Symbolic Logic 9(2), pp. 131–168, doi:10.2178/bsl/1052669286.

[5] Andrzej Indrzejczak (2024): *The logicality of equality*. In Thomas Piecha & Kai F. Wehmeier, editors: Peter Schroeder-Heister on Proof-Theoretic Semantics, Outstanding Contributions to Logic 29, Springer, Cham, Switzerland, pp. 211–238, doi:10.1007/978-3-031-50981-0_7.

[6] Steven T. Kuhn (2000): *Embedded definite descriptions: Russellian analysis and semantic puzzles*. Mind 109(435), pp. 443–454, doi:10.1093/mind/109.435.443.

[7] Nils Kürbis (2019): *A binary quantifier for definite descriptions in intuitionist negative free logic: Natural deduction and normalisation*. Bulletin of the Section of Logic 48(2), pp. 81–97, doi:10.18778/0138-0680.48.2.01.

[8] Peter Ludlow (2023): *Descriptions*. In Edward N. Zalta & Uri Nodelman, editors: *The Stanford Encyclopedia of Philosophy (Winter 2023 Edition)*. Available at https://plato.stanford.edu/archives/win2023/entries/descriptions/.

[9] Dag Prawitz (1965): *Natural Deduction. A Proof-Theoretical Study*. Almqvist and Wiksell, Stockholm. (Reprint: Mineola/NY, Dover Publications, 2006.).

[10] Bertrand Russell (1905): *On denoting*. *Mind* 14(56), pp. 479–493, doi:10.1093/mind/XIV.4.479.

[11] Peter Schroeder-Heister (2023): *Proof-theoretic semantics*. In Edward N. Zalta & Uri Nodelman, editors: *The Stanford Encyclopedia of Philosophy (Fall 2023 Edition)*. Available at https://plato.stanford.edu/archives/fall2023/entries/proof-theoretic-semantics.

[12] Jason Stanley & Zoltán G. Szabó (2000): *On quantifier domain restriction*. *Mind and Language* 15(2-3), pp. 219–261, doi:10.1111/1468-0017.00130.

[13] Sören Stenlund (1975): *Descriptions in intuitionistic logic*. In Stig Kanger, editor: *Proceedings of the Third Scandinavian Logic Symposium*, *Studies in Logic and the Foundations of Mathematics* 82, North-Holland, Amsterdam, pp. 197–212, doi:10.1016/S0049-237X(08)70732-8.

[14] Anne S. Troelstra & Helmut Schwichtenberg (2000): *Basic Proof Theory*, 2nd edition. *Cambridge Tracts in Theoretical Computer Science* 43, Cambridge University Press, Cambridge, doi:10.1017/CBO9781139168717.

[15] Bartosz Więckowski (2016): *Subatomic natural deduction for a naturalistic first-order language with non-primitive identity*. *Journal of Logic, Language and Information* 25(2), pp. 215–268, doi:10.1007/s10849-016-9238-7.

[16] Bartosz Więckowski (2023): *Negative predication and distinctness*. *Logica Universalis* 17(1), pp. 103–138, doi:10.1007/s11787-022-00321-9.

Many-Valued Modal Logic

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We combine the concepts of modal logics and many-valued logics in a general and comprehensive way. Namely, given any finite linearly ordered set of truth values and any set of propositional connectives defined by truth tables, we define the many-valued minimal normal modal logic, presented as a Gentzen-like sequent calculus, and prove its soundness and strong completeness with respect to many-valued Kripke models. The logic treats necessitation and possibility independently, i.e., they are not defined by each other, so that the duality between them is reflected in the proof system itself. We also prove the finite model property (that implies strong decidability) of this logic and consider some of its extensions. Moreover, we show that there is exactly one way to define negation such that De Morgan's duality between necessitation and possibility holds. In addition, we embed many-valued intuitionistic logic into one of the extensions of our many-valued modal logic.

1 Introduction

The (two-valued) logic K is the minimal normal modal logic. It extends classical propositional calculus with the modal connective $\Box$, the rule of inference

$$\frac{\varphi}{\Box\varphi} \tag{1}$$

and the axiom scheme

$$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi) \tag{2}$$

Semantically, K is characterized by Kripke models [11, 12].

In this paper we define a many-valued counterpart ***mv-K*** of K, in which the necessity connective is interpreted as the infimum of all relevant values and the possibility connective is interpreted as their supremum, and nothing is assumed about the underlying propositional connectives. Syntactically, our proof system is an extension of that in [8] to the modal case. The possibility connective $\Diamond$ is treated explicitly. The reason for such a treatment is that, in ***mv-K***, $\Box$ and $\Diamond$ are not necessarily interdefinable. This is because our set of connectives does not necessarily contain negation, and even if it does, nothing is assumed about its truth table. We also show extensions of ***mv-K***, which are counterparts of some well-known extensions of K. We establish the finite model property of ***mv-K*** and its extensions. We then show the unique definition of negation such that De Morgan's duality between $\Box$ and $\Diamond$ holds. Finally, we prove that many-valued intuitionistic logic is a fragment of one of the extensions of ***mv-K***.

A number of many-valued normal modal logics is known from the literature. In [17], an n-valued modal logic is based on the Łukasiewicz classical n-valued connectives. The paper contains Hilbert-style calculi for the generalizations of the two-valued normal modal logics T,S4, and S5, and the author notes that other generalizations are also possible. This work seems to generalize [19], in which three-valued modal Łukasiewicz logics are considered. Three-valued modal logics with different connectives are considered in [20].

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics. Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 121–137, doi:10.4204/EPTCS.415.13

In [24] and [15], general notions of many-valued modal logics are suggested, using designated values and a rather general interpretation of the modal connective.

In [16], proof systems relying on matrices are presented for normal three-valued modal logics based on any arbitrary set of propositional connectives. Among other logics, the three-valued counterparts of the two-valued T,S4, and S5 are presented.

In [4], the author presents a sequent calculus for modal logics based on any finite lattice of truth values. These logics, in addition to all propositional constants, have all classical propositional connectives. The semantics relies on a many-valued accessibility relation, that is further discussed in [5]. This paper also addresses the possibility connective $\Diamond$ that is treated explicitly, because, in general, $\Diamond$ and $\Box$ are not interdefinable.[1] In addition, some extensions of the many-valued modal logics are mentioned at the end of [5].

The most general approach (for our purposes) was, probably, taken in [23], where a proof system, relying on matrices of labelled formulas, is presented for many-valued normal modal logics with an arbitrary set of propositional connectives. The system is appropriate for any semantic interpretation of the necessity connective satisfying certain conditions – not only for its interpretation as the infimum. The system is weakly complete and possesses the subformula property (that implies weak decidability). However, the possibility connective is not addressed in [23] at all, and extensions of the logic are not presented there.

Another general approach is taken in [3], where proof systems using tableaux are suggested for a variety of finite-valued modal logics with generalized modalities.

In [2], counterparts of K, using Hilbert-style proof systems, are presented for any finite residuated lattice of truth values, allowing many-valued accessibility relations. The logics address only the necessity operator (it is only mentioned that the possibility operator should, in general, be addressed separately and not as an abbreviation of $\neg\Box\neg$), and they are based on a fixed set of propositional connectives. The semantics use designated values to interpret validity of formulas.

In [14, Chapter 9.1], many-valued modal logics are discussed, referring also to Gentzen systems, logic extensions and logic embeddings. Again, a fixed set of connectives is assumed and the semantic interpretation is algebraic.

Our research introduces a novel and comprehensive framework for many-valued modal logics that stands out by integrating several key features simultaneously: the use of an arbitrary "base logic", the use of Gentzen-like sequents of labelled formulas, the independent treatment of both necessity and possibility modalities, the demonstration of strong completeness and strong decidability, and addressing all basic logic extensions. While each of these elements has been explored individually in previous studies, our work combines them into a single coherent system. This combination allows for a more robust and flexible logical framework that can handle a wider variety of logical scenarios and applications. By employing labeled formulas (discussed, e.g., in [1] and [8]), we can address any truth value rather than being limited to designated ones, providing a significant advantage in terms of expressive power. The motivation behind this research lies in the importance of many-valued modal logics in contexts with inherent uncertainty or gradations of truth, such as fuzzy logic systems and multi-agent systems. Additionally, extending these logics to include features like transitive accessibility relations is crucial for modeling more complex systems. Our results include the finite model property, ensuring the logics' strong decidability, and embedding many-valued intuitionistic logic within our framework, thus offering a comprehensive and robust tool for logical analysis.

The paper is organized as follows. In Section 2, we introduce a many-valued modal logic ***mv-K*** and

[1] As noted above, these connectives are not interdefinable in our paper either, but for a different reason.

present a sound and strongly complete[2] proof system for it. Section 3 deals with the *canonical model* theorem and the proof of the strong completeness of ***mv-K***. Section 4 contains some extensions of ***mv-K*** and their soundness and completeness with respect to certain classes of Kripke models and, in Section 5, we explain why ***mv-K*** and its extensions from Section 4 possess the finite model property[3]. Then, in Section 6, we present the appropriate definition of negation so that $\Box$ and $\Diamond$ are interdefinable. Finally, in Section 7, we embed many-valued intuitionistic logic in our many-valued counterpart of S4.

We conclude this section with the note that, because of the limitation on the publications length, a number of proofs is omitted.

2 Many-valued modal logic

In this section we define a many-valued logic, ***mv-K***, assuming a linear order on the set of truth values.

In what follows, $V = \{v_1, \dots, v_n\}$, $n \geq 2$, is a set of truth values ordered by

$$v_1 < v_2 < \cdots < v_n$$

Formulas of ***mv-K*** are built from propositional variables by means of propositional connectives (of arbitrary arities) and the modal connectives $\Box$ and $\Diamond$. The set of all ***mv-K*** formulas will be denoted by $\mathscr{F}$. The semantics of propositional connectives is given by truth tables, where, as usual, the truth table of an ℓ-ary propositional connective $*$ is a function $* : V^\ell \to V$ and the semantics of the modal connectives is given below.

A *labelled* formula is a pair (φ, k), where φ is a formula and $k = 1, \dots, n$. The intended meaning of such a labelled formula is that v_k is the truth value associated with φ.

Sequents are expressions of the form $\Gamma \to \Delta$, where Γ and Δ are finite (possibly empty) sets of labelled formulas and $\to$ is not a symbol of the underlying language.

The ***mv-K*** semantics is as follows.

A *many-valued Kripke model* (or *many-valued K-model* or just Kripke model) is a triple $M = \langle W, R, I \rangle$, where

- W is a nonempty set (of possible worlds),
- R is a binary (accessibility) relation on W, and
- $I : W \times \mathscr{P} \to V$, where $\mathscr{P}$ is the set of propositional variables, is a (valuation) function.

For a world $u \in W$, we define the set of *successors* of u, denoted by $S(u)$, as

$$S(u) = \{v \in W : uRv\}$$

and extend I to $W \times \mathscr{F}$, recursively, as follows.

- $I(u, *(\varphi_1, \dots, \varphi_\ell)) = *(I(u, \varphi_1), \dots, I(u, \varphi_\ell))$,
- $I(u, \Box\varphi) = \inf(\{I(v, \varphi) : v \in S(u)\})$, where $\inf(\emptyset)$ is v_n, and
- $I(u, \Diamond\varphi) = \sup(\{I(v, \varphi) : v \in S(u)\})$, where $\sup(\emptyset)$ is v_1.

[2]That is, complete with respect to the consequence relation.
[3]Thus, they are strongly decidable.

Note that, if $S(u) \neq \emptyset$, then, since V is finite and linearly ordered, $\inf(\{I(v,\varphi) : v \in S(u)\})$ and $\sup(\{I(v,\varphi) : v \in S(u)\})$ are, actually, $\min(\{I(v,\varphi) : v \in S(u)\})$ and $\max(\{I(v,\varphi) : v \in S(u)\})$, respectively.

We also write $M,u \models (\varphi,k)$, if $I(u,\varphi) = v_k$.

The satisfiability relation $\models$ between worlds of W and sequents of labelled formulas is defined as follows.

A world u satisfies a sequent $\Gamma \to \Delta$, denoted $M,u \models \Gamma \to \Delta$, if the following holds.

- If for each $(\varphi,k) \in \Gamma$, $I(u,\varphi) = v_k$, then for some $(\varphi,k) \in \Delta$, $I(u,\varphi) = v_k$.[4]

A Kripke model M satisfies a sequent $\Gamma \to \Delta$, if each world in W satisfies $\Gamma \to \Delta$ and M satisfies a set of sequents Σ, if it satisfies each sequent in Σ. Finally, a set of sequents Σ *semantically entails* a sequent $\Gamma \to \Delta$, denoted $\Sigma \models \Gamma \to \Delta$, if each many-valued Kripke model satisfying Σ also satisfies $\Gamma \to \Delta$.

Let i and j be nonnegative integers. We denote the set of integers between i and j by $[i,j]$. That is

$$[i,j] = \{k : i \leq k \leq j\}$$

In particular, if $i > j$, $[i,j]$ is empty.

By definition,

$$\overline{[i,j]} = \{1,\ldots,n\} \setminus [i,j] = [1,i-1] \cup [j+1,n]$$

For convenience, we define

$$(\varphi,k)^+ = \{\varphi\} \times [k,n]$$

and

$$(\varphi,k)^- = \{\varphi\} \times [1,k]$$

Definition 1 For a set of labelled formulas Γ, the set of labelled formulas $\Gamma^\times$ is defined as follows.

$$\Gamma^\times = \bigcup \{\{\psi\} \times \overline{[i_\psi, j_\psi]} : (\Box\psi, i_\psi), (\Diamond\psi, j_\psi) \in \Gamma\}$$

That is, for all ψ such that $(\Box\psi, i_\psi), (\Diamond\psi, j_\psi) \in \Gamma$, $\Gamma^\times$ includes the set $\{\psi\} \times \overline{[i_\psi, j_\psi]}$ and nothing more.

The idea lying behind the definition of $\Gamma^\times$ is, that in a Kripke model M, if uRv and u satisfies *every* element of Γ, then v satisfies *no* element of $\Gamma^\times$.

We define next the proof system of ***mv-K***.

The axioms are:

$$(\varphi,k) \to (\varphi,k) \tag{3}$$

and

$$(\varphi_1,k_1),\ldots,(\varphi_\ell,k_\ell) \to (*(\varphi_1,\ldots,\varphi_\ell),k) \tag{4}$$

for each table entry $v_{k_1},\ldots,v_{k_\ell}$ such that $*(v_{k_1},\ldots,v_{k_\ell}) = v_k$, and the logical rules are

$$\frac{(\varphi,k) \to \Gamma^\times}{(\Box\varphi,k),\Gamma \to} \quad k \neq n \tag{5}$$

$$\frac{(\varphi,k) \to \Gamma^\times}{(\Diamond\varphi,k),\Gamma \to} \quad k \neq 1 \tag{6}$$

[4] In other words, v satisfies a sequent $\Gamma \to \Delta$, if the metavalue of the classical metasequent $\{I(u,\varphi) = v_k : (\varphi,k) \in \Gamma\} \to \{I(u,\varphi) = v_k : (\varphi,k) \in \Delta\}$ is "true."

and the structural rules below.
k-left-shift:

$$\frac{\Gamma, (\varphi, k) \to \Delta}{\Gamma \to \Delta, \{\varphi\} \times \overline{\{k\}}} \tag{7}$$

k', k''-right-shift:

$$\frac{\Gamma \to \Delta, (\varphi, k')}{\Gamma, (\varphi, k'') \to \Delta} \quad k' \neq k'' \tag{8}$$

k-left-weakening:

$$\frac{\Gamma \to \Delta}{\Gamma, (\varphi, k) \to \Delta} \tag{9}$$

k-right-weakening:

$$\frac{\Gamma \to \Delta}{\Gamma \to \Delta, (\varphi, k)} \tag{10}$$

k-cut:

$$\frac{\Gamma' \to \Delta', (\varphi, k) \qquad \Gamma'', (\varphi, k) \to \Delta''}{\Gamma', \Gamma'' \to \Delta', \Delta''} \tag{11}$$

k', k''-resolution:

$$\frac{\Gamma' \to \Delta', (\varphi, k') \qquad \Gamma'' \to \Delta'', (\varphi, k'')}{\Gamma', \Gamma'' \to \Delta', \Delta''} \quad k' \neq k'' \tag{12}$$

In fact, cut and resolution are derivable from each other, see [8, Proposition 3.3].

We shall also need the two following derivable rules. One is "multi-shift"

$$\frac{\{\Gamma_k, (\varphi, k) \to \Delta_k : k \in K\}}{\bigcup_{k \in K} \Gamma_k \to \bigcup_{k \in K} \Delta_k, \{\varphi\} \times \overline{K}} \tag{13}$$

for $K \subset \{1, \ldots, n\}$ and $\overline{K} = \{1, \ldots, n\} \setminus K$, see [8, Remark 3.5], and the other is its generalization

$$\frac{\{\Gamma_{k_1,\ldots,k_\ell}, (\varphi_1, k_1), \ldots, (\varphi_\ell, k_\ell) \to \Delta_{k_1,\ldots,k_\ell} : k_1 \in K_1, \ldots, k_\ell \in K_\ell\}}{\bigcup_{k_1 \in K_1,\ldots,k_\ell \in K_\ell} \Gamma_{k_1,\ldots,k_\ell} \to \bigcup_{k_1 \in K_1,\ldots,k_\ell \in K_\ell} \Delta_{k_1,\ldots,k_\ell}, \{\varphi_1\} \times \overline{K_1}, \ldots, \{\varphi_\ell\} \times \overline{K_\ell}} \tag{14}$$

The derivation of (14) is rather long and is omitted.[5]

We precede the statement of the soundness and completeness theorem for ***mv-K*** with a number of examples.

Example 2 Sequents

$$(\Box\varphi, k) \to (\Diamond\varphi, k)^+ \quad k \neq n \tag{15}$$

and

$$(\Diamond\varphi, k) \to (\Box\varphi, k)^- \quad k \neq 1 \tag{16}$$

are ***mv-K*** derivable.

The derivation of (15) is as follows, where, in steps 2_j and 3_j, $j < k$.

1.	$(\varphi, k) \to (\varphi, k)$	axiom (3)
2.	$(\varphi, k) \to \{\varphi\} \times \overline{[1, n]}$	follows from 1 by $n-1$ right weakenings (10)
2_j.	$(\varphi, k) \to \{\varphi\} \times \overline{[k, j]}$	follows from 2, because, for $j < k$, $[k, j] = \emptyset$
3_j.	$(\Box\varphi, k), (\Diamond\varphi, j) \to$	follows from 2_j by (5) with Γ being $\{(\Box\varphi, k), (\Diamond\varphi, j)\}$
4.	$(\Box\varphi, k) \to (\Diamond\varphi, k)^+$	follows from all 3_j, $j < k$, by multi-shift (13)

[5]A skeptical reader can easily verify that this rule is valid and then add it to ***mv-K***.

The derivation of (16) is dual to that of (15) and is omitted.

Example 3 Sequent

$$(\Box\varphi, n) \to (\Diamond\varphi, 1), (\Diamond\varphi, n) \tag{17}$$

is ***mv-K*** derivable.

Indeed, for $k \neq 1, n$, by k n, j-right-shifts (8), $j \leq k$, on (16), we obtain

$$(\Diamond\varphi, k), (\Box\varphi, n) \to \quad k = 2, 3, \ldots, n-1$$

from which (17) follows by multi-shift (13).

Example 4 Sequents

$$(\Box\varphi, n), (\Diamond\varphi, 1) \to (\Box\psi, n) \tag{18}$$

and

$$(\Box\varphi, n), (\Diamond\varphi, 1) \to (\Diamond\psi, 1) \tag{19}$$

are ***mv-K*** derivable.

The derivation of (18) is as follows, where, in steps 2_i and 3_i, $i \neq n$.

1.	$\to \{\varphi\} \times [1, n]$	derivable sequent, see [8, Proposition 3.4]
2_i.	$(\psi, i) \to \{\varphi\} \times [1, n]$	follows from 1 by i-left-weakening (9)
3_i.	$(\Box\psi, i), (\Box\varphi, n), (\Diamond\varphi, 1) \to$	follows from 2_i by (5), with Γ being $\{(\Box\varphi, n), (\Diamond\varphi, 1)\}$, because $\overline{[n, 1]} = [1, n]$
4.	$(\Box\varphi, n), (\Diamond\varphi, 1) \to (\Box\psi, n)$	follows from all 3_i, $i \neq n$, by multi-shift (13)

The derivation of (19) is dual to that of (18) and is omitted.

Example 5 In this example we show that the sequent

$$(\Box(p \supset q), 3), (\Box p, 3) \to (\Box q, 3)$$

is derivable in the modal extension of the Łukasiewicz three-valued logic. That is, $n = 3$ and the truth table of implication $\supset$ is as follows.

$\supset$	1	2	3
1	3	3	3
2	2	3	3
3	1	2	3

In steps $3_k, 4_k, 5_k, 7_k$ and 9_k of the proof below, $k \in \{1,2\}$.

1.	$(p,3),(q,1) \to (p \supset q,1)$	axiom (4)
2.	$(p,3),(q,2) \to (p \supset q,2)$	axiom (4)
3_k.	$(p,3),(q,k),(p \supset q,3) \to$	follows from either 1 or 2 by right-shift (8)
4_k.	$(q,k) \to \{p\} \times [1,2], \{p \supset q\} \times [1,2]$	follows from 3_k by left-shifts (7)
5_k.	$(\Box q,k),(\Box p,3),(\Diamond p,3),(\Box(p \supset q),3),(\Diamond(p \supset q),3) \to$	follows from 4_k by (5), because $[1,2] = \overline{[3,3]}$
6.	$(\Box p,3) \to (\Diamond p,3),(\Diamond p,1)$	(17)
7_k.	$(\Box q,k),(\Box p,3),(\Box(p \supset q),3),(\Diamond(p \supset q),3) \to (\Diamond p,1)$	follows from 5_k and 6 by cut (11)
8.	$(\Box(p \supset q),3) \to (\Diamond(p \supset q),3),(\Diamond(p \supset q),1)$	(17)
9_k.	$(\Box q,k),(\Box p,3),(\Box p \supset q,3) \to (\Diamond p,1),(\Diamond(p \supset q),1)$	follows from 7_k and 8 by cut (11)
10.	$(\Box p,3),(\Box p \supset q,3) \to (\Diamond p,1),(\Diamond(p \supset q),1),(\Box q,3)$	follows from all 9_k by multi-shift (13)
11.	$(\Box p,3),(\Diamond p,1) \to (\Box q,3)$	(18)
12.	$(\Box p,3),(\Box p \supset q,3) \to (\Diamond(p \supset q),1),(\Box q,3)$	follows from 10 and 11 by cut (11)
13.	$(\Box(p \supset q),3),(\Diamond(p \supset q),1) \to (\Box q,3)$	(18)
14.	$(\Box p,3),(\Box p \supset q,3) \to (\Box q,3)$	follows from 12 and 13 by cut (11)

Theorem 6 *Let Σ and $\Gamma \to \Delta$ be a set of sequents and a sequent, respectively. Then $\Sigma \vdash \Gamma \to \Delta$ if and only if $\Sigma \models \Gamma \to \Delta$.*

The proof of the "only if" part of theorem (soundness) is by induction on the derivation length, and the proof of the "if" part of theorem (strong completeness) is rather involved and follows from the canonical model theorem in the next section.

3 The canonical model theorem and the proof of the "if" part of Theorem 6

For the proof of the strong completeness of ***mv-K***, we extend the definition of provability to infinite sets of labelled formulas.

For a set of sequents Σ, a (not necessarily finite) set of labelled formulas Γ, and a finite set of labelled formulas Δ, we write $\Sigma \vdash \Gamma \to \Delta$, if there exists a finite subset Γ' of Γ such that $\Sigma \vdash \Gamma' \to \Delta$.

A set of labelled formulas Γ is called Σ-*consistent*, if $\Sigma \not\vdash \Gamma \to$.

A set of sequents Σ is called *consistent*, if $\Sigma \not\vdash \to$.[6]

Lemma 7 *If $\Sigma \not\vdash \Gamma \to \Delta$, then there exists a maximal (with respect to inclusion) Σ-consistent set Γ' including Γ such that $\Sigma \not\vdash \Gamma' \to \Delta$.*

The proof is straightforward, by Zorn's lemma, and is omitted.

Lemma 8 ([8, Lemma 3.12 and the following observation]) *If Γ is a maximal set for which $\Sigma \not\vdash \Gamma \to \Delta$, then for every formula φ there exists a unique $k \in \{1,\dots,n\}$ such that $(\varphi,k) \in \Gamma$.*

From now on, we enumerate the set of all formulas $\mathscr{F}$ as $\psi_1, \psi_2, \dots$.

For a consistent set of sequents Σ, the Σ-*canonical model* $M_\Sigma = \langle W_\Sigma, R_\Sigma, I_\Sigma \rangle$ is defined as follows.

- W_Σ is the set of all maximal Σ-consistent sets. Since Σ is consistent, by Lemma 7, W_Σ is nonempty.

[6] Equivalently, Σ is consistent, if there exists a Σ-consistent set of formulas Γ.

- For worlds $\Gamma', \Gamma'' \in W_\Sigma$, $\Gamma' R_\Sigma \Gamma''$ if and only if for the unique $i_1, i_2, \ldots$, $j_1, j_2, \ldots$, and $k_1, k_2, \ldots$ such that
$$(\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \ldots \in \Gamma'$$
and
$$(\psi_1, k_1), (\psi_2, k_2), \ldots \in \Gamma''$$
provided by Lemma 8, $i_m \leq k_m \leq j_m$ for all $m = 1, 2, \ldots$[7]
- For $u \in W_\Sigma$ and $p \in \mathscr{P}$, $I_\Sigma(u, p)$ is the unique value v_k such that $(p, k) \in u$.

Theorem 9 (The canonical model theorem) *For all labelled formulas (φ, k) and all $u \in W_\Sigma$, $(\varphi, k) \in u$ if and only if $M_\Sigma, u \models (\varphi, k)$.*

For the proof of Theorem 9 we need the lemma below.

Lemma 10 *Let Γ be a Σ-consistent set of formulas and let*
$$(\Box\varphi', k)(\Box\psi, i), (\Diamond\psi, j) \in \Gamma \tag{20}$$
where $k \neq n$. Then $i \leq j$.

Proof Assume to the contrary that $i > j$. We distinguish among the cases of, $i \neq n$, $j \neq 1$, and $i = n$ and $j = 1$.

If $i \neq n$, then

1.	$\Gamma \to (\Box\psi, i)$	follows from axiom (3), with φ being $\Box\psi$ and k being i, and (20)
2.	$\Gamma \to (\Diamond\psi, j)$	follows from axiom (3), with φ being $\Diamond\psi$ and k being j, and (20)
3.	$(\Box\psi, i) \to (\Diamond\psi, i)^+$	(15) with φ being ψ and k being i
4.	$\Gamma \to (\Diamond\psi, i)^+$	follows from 1 and 3 by cut
5.	$\Gamma \to$	follows from 2 and 4 by $n - i$ resolutions (12)

However, $\Sigma \vdash \Gamma \to$ contradicts the Σ-consistency of Γ.

The case of $j \neq 1$ is dual to that of $i \neq n$ and is omitted.

Let $i = n$ and $j = 1$. Then

1.	$\Gamma \to (\Box\varphi', k)$	follows from axiom (3), with φ being $\Box\varphi'$
2.	$\Gamma \to (\Box\psi, n)$	follows from axiom (3), with φ being $\Box\psi$ and k being n, and (20)
3.	$\Gamma \to (\Diamond\psi, 1)$	follows from axiom (3), with φ being $\Diamond\psi$ and k being 1, and (20)
4.	$(\Box\psi, n), (\Diamond\psi, 1) \to (\Box\varphi', n)$	(18) with φ being ψ and ψ being φ'
5.	$\Gamma \to (\Box\varphi', n)$	follows from 2, 3, and 4 by two cuts
6.	$\Gamma \to$	follows from 1 and 5 by resolution (12)

Again, $\Sigma \vdash \Gamma \to$ contradicts the Σ-consistency of Γ. ∎

[7] Equivalently, $\Gamma' R_\Sigma \Gamma''$ if and only if $(\Gamma')^\times \cap \Gamma'' = \emptyset$.

Proof of Theorem 9 It is sufficient to prove the "only if" part of the theorem, i.e., that $(\varphi,k) \in u$ implies $M_\Sigma, u \models (\varphi,k)$. This is because, if $(\varphi,k) \notin u$, then, by Lemma 8, $(\varphi,k') \in u$ for $k' \neq k$. Therefore by the "only if" part of the theorem, $M_\Sigma, u \models (\varphi,k')$, implying $M_\Sigma, u \not\models (\varphi,k)$.

The proof is by induction on the complexity of φ. For the cases of an atomic formula and a propositional principal connective, see [8, Proposition 3.13].

Let φ be of the form $\Box\varphi'$ and assume that for some $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$,

$$(\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \ldots \in u$$

We distinguish between the cases of $k \neq n$ and $k = n$.

• Let $k \neq n$. By the induction hypothesis and the definition of R_Σ, for each world $v \in S(u)$, $v_k \leq I_\Sigma(v, \varphi')$.[8] Therefore, for the proof of

$$v_k = \min(\{I_\Sigma(v,\varphi') : v \in S(u)\}) = I_\Sigma(u,\varphi)$$

it suffices to show that

$$v_k \in \{I_\Sigma(v,\varphi') : v \in S(u)\}$$

i.e., that there exist a world $v \in W_\Sigma$ and $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \ldots$ such that

$$(\varphi',k), (\psi_1,k_1), (\psi_2,k_2), \ldots \in v$$

This is because, by definition of R_Σ, $uR_\Sigma v$ and, by the induction hypothesis, $I_\Sigma(v,\varphi') = v_k$.

By Lemma 7, for existence of such $k_1, k_2, \ldots$ and v, it suffices to show that there exist $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \ldots$ such that the set of labelled formulas

$$\{(\varphi',k), (\psi_1,k_1), (\psi_2,k_2), \ldots\} \tag{21}$$

is Σ-consistent.

For the proof, assume to the contrary that for all $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \ldots$, (21) is Σ-inconsistent. That is,

$$\Sigma \vdash (\varphi',k), (\psi_1,k_1), (\psi_2,k_2), \ldots \to \quad k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \ldots \tag{22}$$

Note that, by Lemma 10, $i_m \leq j_m$, for all $m = 1, 2, \ldots$. Thus, the set of sequents in (22) is nonempty.

We contend that there exists a non-negative integer L such that

$$\begin{aligned} \Sigma \vdash (\varphi',k), (\psi_1,k_1), (\psi_2,k_2), \ldots, (\psi_L,k_L) \to \\ k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \ldots, k_L \in [i_L, j_L] \end{aligned} \tag{23}$$

Then we shall apply rules (14) and (5) to the set of sequents in (23).

For the proof of our contention, we consider a tree T whose nodes are sets of labelled formulas of the form

$$\begin{aligned} \{(\psi_1,k_1), (\psi_2,k_2), \ldots, (\psi_m,k_m)\} \\ k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \ldots, k_m \in [i_m, j_m], \end{aligned} \tag{24}$$

$m = 0, 1, \ldots$, such that each node (24) is Σ-consistent when (φ',k) is added to it as an element, and the successors of a node (24) are nodes of the form

$$\{(\psi_1,k_1), (\psi_2,k_2), \ldots, (\psi_m,k_m), (\psi_{m+1},k_{m+1})\}$$

[8] This is because, for some $m = 1, 2, \ldots,$, φ' is ψ_m.

where $k_{m+1} \in [i_{m+1}, j_{m+1}]$.

Thus, nodes (24) are of height m. In particular, the root of T is $\emptyset$, if $\{(\varphi', k)\}$ is Σ-consistent. Otherwise, T is empty.

This tree T is of a finite branching degree, because a node of height m has at most $j_{m+1} - i_{m+1} + 1$ successors. Also, T has no infinite paths. Indeed, an infinite path would correspond to a choice of $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \ldots$. However, the set of labelled formulas $\{(\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \ldots\}$ is Σ-inconsistent. Thus, the path contains a node that becomes Σ-inconsistent, when (φ', k) is added to it, in contradiction with the definition of T. Therefore, by the contraposition of the König infinite lemma [10], T is finite.

Let H be the height of T (H is defined as -1, if T is empty). Then, for $L = H + 1$, we have (23), which proves our contention.

Now, from (23), by (14) we obtain

$$\Sigma \vdash (\varphi', k) \to \{\psi_1\} \times \overline{[i_1, j_1]}, \{\psi_2\} \times \overline{[i_2, j_2]}, \ldots, \{\psi_L\} \times \overline{[i_L, j_L]}$$

from which, by (5) we obtain

$$\Sigma \vdash (\Box\varphi', k), (\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \ldots, (\Box\psi_L, i_L), (\Diamond\psi_L, j_L) \to$$

that contradicts the Σ-consistency of u.

• Let $k = n$. If $S(u) = \emptyset$, then, trivially, $M_\Sigma, u \models (\Box\varphi, k)$. Otherwise, by the induction hypothesis and the definition of R_Σ, for all worlds $v \in S(u)$, we have $v_n \leq I_\Sigma(v, \varphi')$, implying $I_\Sigma(v, \varphi') = v_n$. Thus,

$$\min(\{I_\Sigma(v, \varphi') : v \in S(u)\}) = \min(\{v_n\}) = v_n$$

and $M_\Sigma, u \models (\varphi, k)$ follows.

The case of $\Diamond$ is dual to that of $\Box$. We just replace $\Box$ with $\Diamond$, $\Diamond$ with $\Box$, min with max, n with 1, and $\leq$ with $\geq$. We leave the details to the reader. ∎

Corollary 11 *We have $M_\Sigma \models \Sigma$.*

Proof Let $u \in W_\Sigma$ and let $\Gamma \to \Delta \in \Sigma$. Assume that u satisfies all labelled formulas in Γ and assume to the contrary that u satisfies no labelled formula in Δ. By Theorem 9, $\Gamma \subseteq u$, and for all $(\varphi, k_\varphi) \in \Delta$ there is $k'_\varphi \neq k_\varphi$ such that $(\varphi, k'_\varphi) \in u$. By definition, $\Sigma \vdash \Gamma \to \Delta$. Therefore, by k_φ, k'_φ-right-shifts (8),

$$\Sigma \vdash \Gamma, \{(\varphi, k'_\varphi) : (\varphi, k_\varphi) \in \Delta\} \to$$

implying $\Sigma \vdash u \to$, because $\Gamma, \{(\varphi, k'_\varphi) : (\varphi, k_\varphi) \in \Delta\} \subseteq u$. This, however, contradicts Σ-consistency of u. ∎

Proof of the "if" part of Theorem 6 Assume $\Sigma \not\vdash \Gamma \to \Delta$. By Lemma 7, there exists a maximal set Γ' including Γ such that $\Sigma \not\vdash \Gamma' \to \Delta$. By the definition of M_Σ, $\Gamma' \in W_\Sigma$. We contend that $M_\Sigma \not\models \Gamma \to \Delta$. Namely, $M_\Sigma, \Gamma' \not\models \Gamma \to \Delta$.

Since $\Gamma \subseteq \Gamma'$, by Theorem 9, Γ' satisfies all labelled formulas in Γ. However, it satisfies no labelled formula in Δ, because, otherwise, by Theorem 9, such a formula would belong to Γ', implying $\Sigma \vdash \Gamma' \to \Delta$, in contradiction with the definition of Γ'. Thus, $M_\Sigma \not\models \Gamma \to \Delta$, which completes the proof of our contention and, together with Corollary 11, completes the proof of the "if" part of the theorem. ∎

4 Extensions of *mv-K*

In this section, L is an extension of ***mv-K*** with additional axioms.

We write $\Sigma \vdash_L \Gamma \to \Delta$, if $\Gamma \to \Delta$ is derivable from Σ in L (and we keep writing $\Sigma \vdash \Gamma \to \Delta$, if L is ***mv-K*** itself). We generalize this notation to sequents $\Gamma \to \Delta$ with an infinite antecedent Γ, like in the previous section.

Clearly, the results of the previous section apply also to any extension L. Below, we just rewrite them with respect to L.

Definition 12 A set of labeled formulas Γ is called *L-Σ-consistent*, if $\Sigma \not\vdash_L \Gamma \to$.

Definition 13 A set of sequents Σ is called *L-consistent*, if $\Sigma \not\vdash_L \to$, or, equivalently, if there exists an L-Σ-consistent set, cf. footnote 6.

Lemma 14 (Cf. Lemma 7.) *If $\Sigma \not\vdash_L \Gamma \to \Delta$, then there exists a maximal L-Σ-consistent set Γ' including Γ such that $\Sigma \not\vdash_L \Gamma' \to \Delta$.*

Lemma 15 (Cf. Lemma 8.) *If Γ is a maximal set for which $\Sigma \not\vdash_L \Gamma \to \Delta$, then for every formula $\varphi \in \mathscr{F}$ there exists a unique $k \in \{1,\dots,n\}$ such that $(\varphi,k) \in \Gamma$.*

For an L-consistent set of sequents Σ, we define the *L-Σ-canonical model* $M_{L,\Sigma} = \langle W_{L,\Sigma}, R_{L,\Sigma}, I_{L,\Sigma}\rangle$ just like the Σ-canonical model M_Σ in Section 3, except that $W_{L,\Sigma}$ is the set of all maximal L-Σ-consistent sets. Note that $W_{L,\Sigma}$ is nonempty, because Σ is L-consistent.

Corollary 16 *For an L-consistent set of sequents Σ the following holds.*

- (i) (Cf. Theorem 9.) *For all labelled formulas (φ,k) and all $u \in W_{L,\Sigma}$, $(\varphi,k) \in u$ if and only if $M_{L,\Sigma}, u \models (\varphi,k)$.*
- (ii) (Cf. Corollary 11.) $M_{L,\Sigma} \models \Sigma$.
- (iii) (Cf. the "if" part of Theorem 6.) *If $\Sigma \not\vdash_L \Gamma \to \Delta$, then $M_{L,\Sigma} \not\models \Gamma \to \Delta$.*

We proceed with some extensions of ***mv-K*** which are sound and strongly complete for the many-valued Kripke models defined below.

Definition 17 A binary relation $R \subseteq W \times W$ is called *serial* (or *with no dead-ends*), if for all $u \in W$, $S(u) \neq \emptyset$, and is called *Eucledian*, if for all $u,v,w \in W$, uRv and uRw imply vRw.

Definition 18 A many-valued Kripke model $M = \langle W,R,I\rangle$ is called *serial/ reflexive/ transitive/ symmetric/ Euclidean*, if the accessibility relation R is serial/ reflexive/ transitive/ symmetric/ Euclidean, respectively.

In this section, the many-valued modal logics, we shall deal with, result from ***mv-K*** by adding some subsets of the following axioms.

$$(\Box\varphi,n) \to (\Diamond\varphi,n) \tag{25}$$

$$(\Box\varphi,k) \to (\varphi,k)^+ \tag{26}$$

$$(\varphi,k) \to (\Diamond\varphi,k)^+ \tag{27}$$

$$(\Box\varphi,k) \to (\Box\Box\varphi,k)^+ \tag{28}$$

$$(\Diamond\Diamond\varphi, k) \to (\Diamond\varphi, k)^+ \tag{29}$$

$$(\varphi, k) \to (\Box\Diamond\varphi, k)^+ \tag{30}$$

$$(\Diamond\Box\varphi, k) \to (\varphi, k)^+ \tag{31}$$

$$(\Diamond\varphi, k) \to (\Box\Diamond\varphi, k)^+ \tag{32}$$

$$(\Diamond\Box\varphi, k) \to (\Box\varphi, k)^+ \tag{33}$$

Theorem 19 *Let L be an extension of* ***mv-K*** *and let* Σ *be an L-consistent set of sequents.*

(i) *If* (25) *is an axiom of L, then* $M_{L,\Sigma}$ *is serial.*

(ii) *If* (26) *and* (27) *are axioms of L, then* $M_{L,\Sigma}$ *is reflexive.*

(iii) *If* (28) *and* (29) *are axioms of L, then* $M_{L,\Sigma}$ *is transitive.*

(iv) *If* (30) *and* (31) *are axioms of L, then* $M_{L,\Sigma}$ *is symmetric.*

(v) *If* (32) *and* (33) *are axioms of L, then* $M_{L,\Sigma}$ *is Euclidean.*

Next we define the the many-valued counterparts of the two-valued modal logics D, T, K4, S4, B, and S5.

Definition 20

- The many-valued modal logic ***mv-D*** is obtained from ***mv-K*** by adding to it (25).
- The many-valued modal logic ***mv-T*** is obtained from ***mv-K*** by adding to it (26) and (27).
- The many-valued modal logic ***mv-K4*** is obtained from ***mv-K*** by adding to it (28) and (29).
- The many-valued modal logic ***mv-S4*** is obtained from ***mv-T*** by adding to it (28) and (29).
- The many-valued modal logic ***mv-B*** is obtained from ***mv-K*** by adding to it (30) and (31).
- The many-valued modal logic ***mv-S5*** is obtained from ***mv-T*** by adding to it (32) and (33).

The above many-valued logics, but ***mv-D*** are defined by pairs of axioms - the many valued counterpart of the two-valued one and its dual, because the logics under consideration do not necessarily have negation. Thus, unlike in the two-valued case, $\Box$ and $\Diamond$ are not interdefinable. We address the extension of these logics with negation in Section 6.

Note that the above axioms are many-valued counterparts of axioms D, see [7, p. 29], $T, 4, B$, see [7, p. 10], and E, see [7, p. 11].

Theorem 21

(i) ***mv-D*** *is sound and (strongly) complete with respect to serial Kripke models.*

(ii) ***mv-T*** *is sound and (strongly) complete with respect to reflexive Kripke models.*

(iii) ***mv-K4*** *is sound and (strongly) complete with respect to transitive Kripke models.*

(iv) ***mv-S4*** *is sound and (strongly) complete with respect to reflexive and transitive (preordered) Kripke models.*

(v) ***mv-B*** *is sound and (strongly) complete with respect to symmetric Kripke models.*

(vi) ***mv-S5*** *is sound and (strongly) complete with respect to reflexive and Euclidean Kripke models.*[9]

[9] This is the class of all Kripke models whose accessibility relation is an equivalence relation.

5 Decidability of *mv-K* and its extensions

In what follows, L can be any of the logics ***mv-K***,***mv-D***,***mv-T***,***mv-K4***,***mv-S4***,***mv-B*** or ***mv-S5*** and C_L is the class of the respective Kripke models, see Theorem 21.

We show that L possesses the finite model property. The proof is based on the filtration technique, cf. [21, Chapter I, Section 7], where this technique is applied to some two-valued modal logics.

Let Φ be a subformula-closed set of formulas[10] and let $M = \langle W, R, I\rangle$ be a Kripke model. The equivalence relation $\equiv_\Phi$ on W is defined as follows.

$$u \equiv_\Phi v \text{ if and only if } I(u,\varphi) = I(v,\varphi) \text{ for all } \varphi \in \Phi.$$

The *L-filtration of M through* Φ is the Kripke model $M^\star_{L,\Phi} = \langle W^\star_{L,\Phi}, R^\star_{L,\Phi}, I^\star_{L,\Phi}\rangle$, where

- $W^\star_{L,\Phi}$ is the set of all equivalence classes of $\equiv_\Phi$. That is, $W^\star_{L,\Phi} = \{[u] : u \in W\}$ where $[u]$ is the $\equiv_\Phi$ equivalence class of u.
- For $[u] \in W^\star$ and a propositional variable $p \in \Phi$, $I^\star_{L,\Phi}([u], p) = I(u, p)$. By the definition of $\equiv_\Phi$, $I^\star_{L,\Phi}$ is well defined and the value of $I^\star_{L,\Phi}$ for $p \notin \Phi$ does not matter for our purposes.
- The definition of $R^\star_{L,\Phi}$ depends on L.
  - For ***mv-K***,***mv-D*** and ***mv-T***, $[u]R^\star_{L,\Phi}[v]$ if and only if there exist $u' \in [u]$ and $v' \in [v]$ such that $u'Rv'$.
  - For ***mv-K4***, $[u]R^\star_{L,\Phi}[v]$ if and only if
    * for all $\Box\varphi' \in \Phi$, $I(u,\Box\varphi') \le I(v,\Box\varphi')$ and $I(u,\Box\varphi') \le I(v,\varphi')$; and
    * for all $\Diamond\varphi' \in \Phi$, $I(u,\Diamond\varphi') \ge I(v,\Diamond\varphi')$ and $I(u,\Diamond\varphi') \ge I(v,\varphi')$.
  - For ***mv-S4***, $[u]R^\star_{L,\Phi}[v]$ if and only if
    * for all $\Box\varphi' \in \Phi$, $I(u,\Box\varphi') \le I(v,\Box\varphi')$; and
    * for all $\Diamond\varphi' \in \Phi$, $I(u,\Diamond\varphi') \ge I(v,\Diamond\varphi')$.
  - For ***mv-B***, $[u]R^\star_{L,\Phi}[v]$ if and only if
    * for all $\Box\varphi' \in \Phi$, $I(u,\Box\varphi') \le I(v,\varphi')$ and $I(v,\Box\varphi') \le I(u,\varphi')$; and
    * for all $\Diamond\varphi' \in \Phi$, $I(u,\Diamond\varphi') \ge I(v,\varphi')$ and $I(v,\Diamond\varphi') \ge I(u,\varphi')$.
  - For ***mv-S5***, $[u]R^\star_{L,\Phi}[v]$ if and only if
    * for all $\Box\varphi' \in \Phi$, $I(u,\Box\varphi') = I(v,\Box\varphi')$ and
    * for all $\Diamond\varphi' \in \Phi$, $I(u,\Diamond\varphi') = I(v,\Diamond\varphi')$.

Theorem 22 *Let M be in* C_L *and let* $M^\star_{L,\Phi}$ *be its L-filtration through* Φ. *Then*

- *For all* $\varphi \in \Phi$ *and* $u \in W$, $I(u,\varphi) = I^\star_{L,\Phi}([u],\varphi)$ *and*
- $M^\star_{L,\Phi}$ *is in* C_L.

Definition 23 A logic L possesses the *finite model property*, if for each finite set of sequents Σ and each sequent $\Gamma \to \Delta$ such that $\Sigma \not\vdash_L \Gamma \to \Delta$, there exists a finite Kripke model $M \in C_L$ (i.e. the set of worlds of M is finite) such that $M \models \Sigma$, but $M \not\models \Gamma \to \Delta$.

Theorem 24 *Each of the logics considered above possesses the finite model property.*

Corollary 25 *Each of the logics considered above is strongly decidable.*

Proof The decision procedure is standard. We, in parallel, search for a proof of $\Gamma \to \Delta$ from Σ and for a finite Kripke model provided by Theorem 24 that satisfies Σ, but does not satisfy $\Gamma \to \Delta$. ∎

[10] That is, if $\varphi \in \Phi$, then each subformula of φ also belongs to Φ.

6 Duality of □ and ◇ via negation

In ***mv-K***, the existence of any specific connective is not assumed and ◇ is not defined as the De Morgan dual ¬□¬ of □, but is defined independently, both semantically and syntactically via the proof system.

In this section we define the truth table for negation ¬ in such a way that □ and ◇ become the De Morgan dual. That is, the sequents

$$(\Diamond\varphi,k) \to (\neg\Box\neg\varphi,k) \tag{34}$$

and

$$(\Box\varphi,k) \to (\neg\Diamond\neg\varphi,k) \tag{35}$$

are provable in ***mv-K***.[11] We shall show that this is the only appropriate definition of negation, for which (34) and (35) are derivable in ***mv-K***.

The truth table of ¬ is

$$\neg(v_k) = v_{n-k+1} \quad k = 1,2,\ldots,n \tag{36}$$

That is,

$$\neg(v_1) = v_n,\ \neg(v_2) = v_{n-1},\ \ldots,\ \neg(v_{n-1}) = v_2, \text{ and } \neg(v_n) = v_1$$

Therefore, axioms (4) for ¬ are

$$(\varphi,k) \to (\neg\varphi,n-k+1)$$

Example 26 Sequents

$$(\neg\varphi,n-k+1) \to (\varphi,k) \tag{37}$$

are ***mv-K*** derivable.

The derivation is as follows.

$1_{j\neq k}$.	$(\varphi,j) \to (\neg\varphi,n-j+1)$, $j \neq k$	axiom (4)
$2_{j\neq k}$.	$(\varphi,j),(\neg\varphi,n-k+1) \to$	follows from 1_j by $n-j+1,n-k+1$-right-shift (8)
3.	$(\neg\varphi,n-k+1) \to (\varphi,k)$	follows from $2_{j\neq k}$ by multi-shift (13)

Remark 27 Sequents (34) and (35) immediately imply their reversals. For (34), since each sequent in the set

$$\{(\Diamond\varphi,k') \to (\neg\Box\neg\varphi,k') : k' \neq k\}$$

is derivable, by right shifts, we derive

$$\{(\Diamond\varphi,k'),(\neg\Box\neg\varphi,k) \to : k' \neq k\}$$

from which, by multi-shift, we obtain

$$(\neg\Box\neg\varphi,k) \to (\Diamond\varphi,k)$$

and, dually, for (35).

Theorem 28 *Let ¬ be a unary connective. Then, sequents* (34) *and* (35) *are derivable in **mv-K** if and only if, for all* $k = 1,2,\ldots,n$, $\neg(v_k) = v_{n-k+1}$.

[11] In particular, in the three-valued logics of Łukasiewicz [13] and Kleene [9], these connectives are interdefinable.

Remark 29 If we define negation as above, then rule (6) becomes redundant, which can be shown as follows.

1.	$(\varphi,k) \to \Gamma^{\times},\ k \neq 1$	assumption of (6)
2.	$(\neg\varphi, n-k+1) \to (\varphi,k)$	(37)
3.	$(\neg\varphi, n-k+1) \to \Gamma^{\times},\ n-k+1 \neq n$	follows from 1 and 2 by cut
4.	$(\Box\neg\varphi, n-k+1), \Gamma \to,\ n-k+1 \neq n$	follows from 3 by (5)
5.	$(\neg\Box\neg\varphi, k) \to (\Box\neg\varphi, n-k+1)$	(37)
6.	$(\neg\Box\neg\varphi, k), \Gamma \to,\ k \neq 1$	follows from 4 and 5 by cut
7.	(6)	because, by (34), $\neg\Box\neg$ is $\Diamond$

Also, it can be shown that (27), (29), (31), and (33) follow from (26), (28), (30), and (32), respectively, and vice-versa.

7 Embedding many-valued intuitionistic logic into *mv-S4*

In [22], following [18], Takano defined a quite general notion of many-valued intuitionistic logic, that we shall denote by ***mvIL***. We focus on the semantics, because we embed ***mvIL*** into ***mv-S4*** semantically. Also, we restrict ourselves to the case of linearly ordered set of truth values V in which ***mvIL***-interpretations may be defined, recursively, as follows.

The language of ***mvIL*** is that of many-valued propositional logic, i.e., it does not contain the modal connectives $\Box$ or $\Diamond$.

An ***mvIL***-interpretation $M = \langle W,R,I\rangle$ is a preordered (reflexive and transitive) many-valued Kripke model satisfying the (monotonic valuation) requirement below.

For all propositional variables $p \in \mathscr{P}$ and for all $u, v \in W$ such that uRv,

$$I(u,p) \leq I(v,p)$$

The definition of I extends to formulas of the form $*(\varphi_1,\ldots,\varphi_\ell)$ as

$$I(u,*(\varphi_1,\ldots,\varphi_\ell)) = \inf\{*(I(v,\varphi_1),\ldots,I(v,\varphi_\ell)) : v \in S(u)\} \tag{38}$$

A straightforward induction on the formula complexity shows that I is monotonic not only on $W \times \mathscr{P}$, but on the whole $W \times \mathscr{F}$.

We write $M,u \models_{\boldsymbol{mvIL}} (\varphi,k)$, if $I(u,\varphi) = v_k$. For a sequent $\Gamma \to \Delta$ and a set of sequents Σ, we define the relations $M,u \models_{\boldsymbol{mvIL}} \Gamma \to \Delta$, $M \models_{\boldsymbol{mvIL}} \Gamma \to \Delta$, $M \models_{\boldsymbol{mvIL}} \Sigma$, and $\Sigma \models_{\boldsymbol{mvIL}} \Gamma \to \Delta$ like in the beginning of Section 2.

Our translation of ***mvIL*** to ***mv-S4***, is a generalization of the two-valued case (first suggested in [6]).

Definition 30 Let φ be a formula in the language of ***mvIL***. The translation φ^t of an ***mvIL*** formula φ is obtained from φ by inserting $\Box$ before every its subformula. That is, φ^t is defined recursively as follows.

- For a propositional variable p, p^t is $\Box p$, and
- if φ is of the form $*(\varphi_1,\ldots,\varphi_\ell)$, then φ^t is $\Box * (\varphi_1^t,\ldots,\varphi_\ell^t)$.

Lemma 31 *Let $M = \langle W,R,I\rangle$ be a preordered Kripke model and let $\widehat{M} = \langle W,R,\widehat{I}\rangle$ be such that, for all $u \in W$ and all $p \in \mathscr{P}$, $\widehat{I}(u,p) = I(u,\Box p)$. Then $\widehat{M}$ is an **mvIL**-interpretation, and, for all $u \in W$ and all formulas φ in the language of **mvIL**,*

$$\widehat{I}(u,\varphi) = I(u,\varphi^t) \tag{39}$$

Proof To show $\widehat{M}$ is an $\boldsymbol{mvIL}$-interpretation, we need to show that, for all $u,v \in W$ such that uRv and for all $p \in \mathscr{P}$, $\widehat{I}(u,p) \leq \widehat{I}(v,p)$, i.e., by the definition of $\widehat{I}$, we need to show $I(u,\Box p) \leq I(v,\Box p)$, which is clear, because M is transitive.

The proof of (39) is by induction on the complexity of φ (extending $\widehat{I}$ to an intuitionistic valuation).

The basis, i.e., the case of φ being a propositional variable, is by the definition of $\widehat{I}$, and, for the induction step, if φ is of the form $*(\varphi_1,\dots,\varphi_\ell)$, then

$$\begin{aligned}\widehat{I}(u,\varphi) &= \inf\{*(\widehat{I}(v,\varphi_1),\dots,\widehat{I}(v,\varphi_\ell)) : v \in S(u)\}\\ &= \inf\{*(I(v,\varphi_1^t),\dots,I(v,\varphi_\ell^t)) : v \in S(u)\}\\ &= \inf\{I(v,*(\varphi_1^t,\dots,\varphi_\ell^t) : v \in S(u)\}\\ &= I(u,\Box *(\varphi_1^t,\dots,\varphi_\ell^t))\\ &= I(u,\varphi^t)\end{aligned}$$

where the first equality is by (38), the second equality is by the induction hypothesis, the third and the fourth equalities are by the definition of the extension of I onto $W \times \mathscr{F}$, and the last equality is by the definition of translation ^{t}. ∎

It follows from (39) that $\widehat{M} \models_{\boldsymbol{mvIL}} \Gamma \to \Delta$ if and only if $M \models \Gamma^t \to \Delta^t$, where Γ^t and Δ^t are obtained from Γ and Δ, respectively, by translating every formula appearing in them. Similarly, $\widehat{M} \models_{\boldsymbol{mvIL}} \Sigma$ if and only if $M \models \Sigma^t$ where Σ^t is obtained from Σ by translating every sequent appearing in it.

Theorem 32 $\Sigma \models_{\boldsymbol{mvIL}} \Gamma \to \Delta$ *if and only if* $\Sigma^t \models_{\mathbf{C}} \Gamma^t \to \Delta^t$*, where* $\mathbf{C}$ *is the class of preordered Kripke models.*

Proof If $\Sigma^t \not\models_{\mathbf{C}} \Gamma^t \to \Delta^t$, there exists a preordered Kripke model M such that $M \models \Sigma^t$, but $M \not\models \Gamma^t \to \Delta^t$. By Lemma 31, $\widehat{M} \models_{\boldsymbol{mvIL}} \Sigma$, but $\widehat{M} \not\models_{\boldsymbol{mvIL}} \Gamma \to \Delta$. Thus, $\Sigma \not\models_{\boldsymbol{mvIL}} \Gamma \to \Delta$.

Conversely, if $\Sigma \not\models_{\boldsymbol{mvIL}} \Gamma \to \Delta$, there exists an $\boldsymbol{mvIL}$ interpretation M such that $M \models \Sigma$, but $M \not\models \Gamma \to \Delta$. By definition, M is also a preordered Kripke model and $\widehat{M}$ defined in Lemma 31 is M itself, because by the definition of an intuitionistic valuation, the value of a propositional variable p in a world u is already the minimum of the values of p in $S(u)$. Therefore M, as an ***mv-S4*** model, satisfies Σ^t but not $\Gamma^t \to \Delta^t$. ∎

It follows that strong decidability (and completeness) of ***mv-S4*** implies strong decidability of $\boldsymbol{mvIL}$.

Remark 33 If the principal connective $*$ of a formula is monotonic,[12] then there is no need to insert $\Box$ before $*$ in the translation. This is because $\widehat{I}$ is "local" on this connective, like in modal logic.

References

[1] Matthias Baaz, Christian G. Fermüller, Gernot Salzer & Richard Zach (1998): *Labeled Calculi and Finite-valued Logics*. *Studia Logica* 61, pp. 7–33, doi:`10.1023/A:1005022012721`.

[2] Félix Bou, Francesc Esteva, Lluís Godo & Ricardo Oscar Rodríguez (2011): *On the Minimum Many-Valued Modal Logic over a Finite Residuated Lattice*. *Journal of Logic and Computation* 21, p. 739–790, doi:`10.1093/logcom/exp062`.

[12] That is, if $v_{k_1} \leq v_{k_1'},\dots,v_{k_\ell} \leq v_{k_\ell'}$, then $*(v_{k_1},\dots,v_{k_\ell}) \leq *(v_{k_1'},\dots,v_{k_\ell'})$. For example, in the three-valued logics of Łukasiewicz [13] and Kleene [9], disjunction $\vee$ and conjunction $\wedge$ are monotonic.

[3] Christian G. Fermüller & Herbert Langsteiner (1998): *Tableaux for Finite-Valued Logics with Arbitrary Distribution Modalities*. In: Automated Reasoning with Analytic Tableaux and Related Methods, Springer, Berlin, Heidelberg, pp. 156–171, doi:10.1007/3-540-69778-0_20.

[4] Melvin. C. Fitting (1991): *Many-valued modal logics*. Fundamenta informaticae 15, pp. 235–254, doi:10.3233/FI-1991-153-404.

[5] Melvin. C. Fitting (1992): *Many-valued modal logics II*. Fundamenta informaticae 17, pp. 55–73, doi:10.3233/FI-1992-171-205.

[6] Kurt Gödel (1933): *Eine Interpretation des intuitionistischen Aussagenkalküls*. Ergebnisse eines mathematischen Kolloquiums 4, pp. 39–40.

[7] George Edward Hughes & Maxwell John Cresswell (1984): *A Companion to Modal Logic*. Methuen & Co., London.

[8] Michael Kaminski & Nissim Francez (2021): *Calculi for Many-Valued Logics*. Logica Universalis 15, pp. 193–226, doi:10.1007/s11787-021-00274-5.

[9] Stephen Cole Kleene (1938): *On a notation for ordinal numbers*. The Journal of Symbolic Logic 3, pp. 150–155, doi:10.2307/2267778.

[10] Dénes König (1926): *Sur les correspondences multivoques des ensembles*. Fundamenta Mathematicae 8, pp. 114–134, doi:10.4064/fm-8-1-114-134.

[11] Saul A. Kripke (1959): *A completeness theorem in modal logic*. The Journal of Symbolic Logic 24, pp. 1–14, doi:10.2307/2964568.

[12] Saul A. Kripke (1963): *Semantical analysis of modal logic, I. Normal propositional calculi*. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 9, pp. 67–96, doi:10.1002/malq.19630090502.

[13] Jan Łukasiewicz (1920): *O logice trójwartosściowej*. Ruch Filozoficzny 5, pp. 169–171.

[14] George Metcalfe, Nicola Olivetti & Dov M. Gabbay (2008): *Proof theory for fuzzy logics*. Applied Logic Series 36, Springer Science & Business Media, doi:10.1007/978-1-4020-9409-5.

[15] Charles G. Morgan (1979): *Local and global operators and many-valued modal logics*. Notre Dame Journal of Formal Logic 20, pp. 401–411, doi:10.1305/ndjfl/1093882548.

[16] Osamu Morikawa (1989): *Some modal logics based on a three-valued logic*. Notre Dame Journal of Formal Logic 30, pp. 130–137, doi:10.1305/ndjfl/1093635000.

[17] Pascal Ostermann (1988): *Many-valued modal propositional calculi*. Mathematical Logic Quarterly 34, pp. 343–354, doi:10.1002/malq.19880340411.

[18] George Rousseau (1970): *Sequents in many valued logic II*. Fundamenta Mathematicae 67, pp. 125–131, doi:10.4064/fm-67-1-125-131.

[19] Peter K. Schotch, Jorgen B. Jensen, Peter F. Larsen & Edwin J. MacLellan (1978): *A note on three-valued modal logic*. Notre Dame Journal of Formal Logic 19, pp. 63–68, doi:10.1305/ndjfl/1093888207.

[20] Krister Segerberg (1967): *Some modal logics based on a three-valued logic*. Theoria 33, pp. 53–71, doi:10.1111/j.1755-2567.1967.tb00610.x.

[21] Krister Segerberg (1971): *An essay in classical modal logic*. Filosofiska Studier 13.

[22] Mitio Takano (1986): *Extending the Family of Intuitionistic Many-Valued Logics Introduced by Rousseau*. Annals of the Japan Association for Philosophy of Science 7, pp. 47–56, doi:10.4288/jafpos1956.7.47.

[23] Mitio Takano (1994): *Subformula Property in Many-Valued Modal Logics*. The Journal of Symbolic Logic 59, pp. 1263–1273, doi:10.2307/2275704.

[24] Steven K. Thomason (1978): *Possible Worlds and Many Truth Values*. Studia Logica: An International Journal for Symbolic Logic 37, pp. 195–204, doi:10.1007/bf02124804.

Modal Logics – RNmatrices vs. Nmatrices

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In this short paper we will discuss the similarities and differences between two semantic approaches to modal logics – non-deterministic semantics and *restricted* non-deterministic semantics. Generally speaking, both kinds of semantics are similar in the sense that they employ non-deterministic matrices as a starting point but differ significantly in the way extensions of the minimal modal logic **M** are constructed.

Both kinds of semantics are many-valued and truth-values are typically expressed in terms of tuples of 0s and 1s, where each dimension of the tuple represents either truth/falsity, possibility/non-possibility, necessity/non-necessity etc. And while non-deterministic semantics for modal logic offers an intuitive interpretation of the truth-values and the concept of modality, with restricted non-deterministic semantics are more general in terms of providing extensions of **M**, including normal ones, in an uniform way.

On the example of three modal logics, **MK**, **MKT** and **MKT4**, we will show the differences and similarities of those two approaches. Additionally, we will briefly discuss (current) restrictions of both approaches.

1 Introduction

We begin our study with the weakest system of modal logic – **M**. This system is an expansion of classical propositional logic with a unary operator $\ominus$ and is characterized as follows:

- **M** contains all (classical) tautologies
- **M** is closed under uniform substitution
- **M** is closed under Modus Ponens

This starting point for investigating modal logics is not new. Logicians like Krister Segerberg [30], David Makinson [21], Heinrich Wansing [31] and Lloyd Humberstone [14] started their studies of modal logics with a similar weak system of modal logic, as well.[1]

In the presentations for the smallest modal system by Segerberg, Makinson, Wansing or Humberstone the meaning of the modal operator $\ominus$ is not kept for all extensions of the smallest modal system. Since in practice, what happens is a shift of meaning for the operator $\ominus$. From no meaning in **M** to a meaning constituted in possible worlds, where in all extensions, e.g. $\ominus A$ is true in a world iff A is true in all accessible worlds.[2] Similar things can be said about neighborhood frames or some versions of truth-maker semantics. These shifts are made rather abruptly in order to generate the needed behavior of the modal operator. In our approach we keep the meaning of $\ominus$ the same, and thus establish a uniform theory of modal operators.

[1] They either call it $\mathbf{L}_0$, **PC** or **S**. In the more recent [13] this system is called **0**. The main difference between their and our starting point is that they interpret $\ominus$ from the beginning as a necessity operator.

[2] If we interpret $\ominus$ as necessity.

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Theory and Applications (NCL'24).
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This way of aiming at a uniform theory of modal operators is not new. In recent publications, cf. [9, 25], **M** and some of its normal extensions were investigated as part of a larger discussion concerning non-deterministic semantics for non-normal modal logics, in the sense that the rule of necessitation is absent, and normal modal logics. There, the authors build upon the framework of non-deterministic semantics, which was systematically introduced by Arnon Avron and his collaborators, cf. [4], but already used in the context of modal logics by Yuri Ivlev and John Kearns, cf. [15, 16], [17, 18], and further developed more recently for example in [10, 13, 22, 23, 24, 26, 27, 28].

In this paper, we have a humble objective. We will present two different strategies of constructing semantics for modal logics via Nmatrices and via RNmatrices. In Section 2 we will introduce the minimal modal logic **M** and show how it can be extended by either eliminating truth-values or non-determinacy in Section 3 or by restricting the set of acceptable valuations in Section 4[3]. This is then followed by Section 5, where we will briefly compare both strategies, discuss some open problems and hint a future research of both semantics approaches.

2 The minimal modal logic M

To start with, consider a modal propositional signature Σ with unary connectives $\neg$ and $\ominus$ (classical negation and modality, respectively) and a binary connective $\to$ (material implication). Let $\mathscr{V}$ be a denumerable set of propositional variables $\mathscr{V} = \{p_0, p_1, \ldots\}$ and let $For(\Sigma)$ be the algebra of formulas over Σ freely generated by $\mathscr{V}$. As usual, conjunction $\wedge$, disjunction $\vee$ and bi-implication $\leftrightarrow$ are defined from $\neg$ and $\to$ as follows: $A \wedge B := \neg(A \to \neg B)$, $A \vee B ;= \neg A \to B$ and $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. Note that we could also take $\wedge$, $\vee$ and $\leftrightarrow$ as primitive rather than defined connectives. However, due to the non-truth-functional nature of our semantics, presented below, this would require more care wrt the truth-tables and the formulation of later results. Hence, in order to keep our approach accessible to a broader audience, we decided to take smaller set of connectives as primitive.

In this section we consider a set of four-valued[4] non-deterministic matrices (Nmatrices, for short) defined from swap structures (see for instance [7, Ch. 6] and [11]) in which each truth-value is an ordered pair (or *snapshot*) $z = (z_1, z_2)$ in $\mathbf{2}^2$, for $\mathbf{2} = \{0, 1\}$. Here, z_1 and z_2 represent, respectively, the truth value of A and of $\ominus A$ for a given formula A over Σ. This produces four truth-values $(1,0)$, $(1,1)$, $(0,1)$ and $(0,0)$. Let V_4 be the set of such truth-values. Accordingly, the set of designated values will be $D_4 = \{z \in V_4 \ : \ z_1 = 1\} = \{(1,0),(1,1)\} = (1,*)$. On the other hand, the set of non-designated values is given as $ND_4 = \{(0,0),(0,1)\} = (0,*)$.[5]

Because of the intended meaning of the snapshots, i.e. classical operators should behave classically, negation and implication between snapshots are computed over **2** in the first coordinate, while the second one can takes an arbitrary value. That is:

$$\begin{array}{rcl} \tilde{\neg} z & := & (\sim z_1, *); \\ z \tilde{\to} w & := & (z_1 \Rightarrow w_1, *) \end{array}$$

Here, $\sim$ and $\Rightarrow$ denote the Boolean negation and the implication in **2**. Observe that the second coordinate is arbitrary since at this moment $\ominus$ remains uninterpreted, i.e. there are no axioms ruling the value of $\ominus\neg A$ and the value of $\ominus(A \to B)$.

[3] A largely extended version of that section is currently under review at another venue.

[4] The number of truth-values is not arbitrary but depends on the number of independent modal operators. In case we would like to consider two, three etc. independent modal operators, the number of truth-values would increase to eight, sixteen etc. values.

[5] Note that by $(1,*)$ and $(0,*)$ we mean sets of values rather than undefined values.

The interpretation of $\ominus$ is a multioperator which simply 'reads' the second coordinate, while the second coordinate (corresponding to $\ominus\ominus A$) will be arbitrary at this point, as well:

$$\tilde{\ominus} z \quad := \quad (z_2, *).$$

Let $\mathscr{M} = \langle V_4, D_4, \mathscr{O} \rangle$ be the obtained 4-valued Nmatrix, where $\mathscr{O}(\#) = \tilde{\#}$ for every connective # in Σ, with $\tilde{\#} : V_4 \longrightarrow \mathscr{P}(V_4)$.[6] The truth-tables for $\mathscr{M}$ can be displayed as follows:[7]

$\tilde{\to}$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$
$(1,1)$	$(1,*)$	$(1,*)$	$(0,*)$	$(0,*)$
$(1,0)$	$(1,*)$	$(1,*)$	$(0,*)$	$(0,*)$
$(0,1)$	$(1,*)$	$(1,*)$	$(1,*)$	$(1,*)$
$(0,0)$	$(1,*)$	$(1,*)$	$(1,*)$	$(1,*)$

A	$\tilde{\neg} A$	$\tilde{\ominus} A$
$(1,1)$	$(0,*)$	$(1,*)$
$(1,0)$	$(0,*)$	$(0,*)$
$(0,1)$	$(1,*)$	$(1,*)$
$(0,0)$	$(1,*)$	$(0,*)$

Now, let $\mathscr{F}$ be the set of all the valuations over the Nmatrix $\mathscr{M}$, such that $v \in \mathscr{F}$ iff $v : For(\Sigma) \to V_4$ is a function satisfying the following properties:

- $v(\#A) \in \tilde{\#}\, v(A)$ for $\# \in \{\neg, \ominus\}$;
- $v(A \to B) \in v(A) \,\tilde{\to}\, v(B)$.

The logic **M** generated by the Nmatrix $\mathscr{M}$ is then defined as follows: $\Gamma \vDash_{\mathscr{M}} A$ iff, for every $v \in \mathscr{F}$: if $v(B) \in D_4$ for every $B \in \Gamma$ then $v(A) \in D_4$.

Alternatively, any valuation $v \in \mathscr{F}$ can be written as $v = (v_1, v_2)$ such that $v_1, v_2 : For(\Sigma) \to \mathbf{2}$. Hence, $v(A) = (v_1(A), v_2(A))$ for every formula A. This means that, for all formulas A and B:

- $v(A) \in D_4$ iff $v_1(A) = 1$;
- $v_1(\neg A) = \sim v_1(A)$;
- $v_1(\ominus A) = v_2(A)$;
- $v_1(A \to B) = v_1(A) \Rightarrow v_1(B)$.

The Hilbert calculus $\mathscr{H}$ for **M** consists of the following axioms and a rule of inference.[8]

$$A \to (B \to A) \qquad \textsf{(Ax1)}$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \qquad \textsf{(Ax2)}$$
$$(\neg B \to \neg A) \to (A \to B) \qquad \textsf{(Ax3)}$$
$$\frac{A \quad A \to B}{B} \qquad \textsf{(MP)}$$

We write $\Gamma \vdash_{\mathscr{H}} A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence either (i) belongs to Γ; (ii) is an axiom of $\mathscr{H}$; (iii) is obtained by (MP) from formulas preceding it in sequence.

The following result is then easy to prove:

Theorem 1 (Soundness and completeness of $\mathscr{H}$ w.r.t. $\mathscr{M}$)**.** *For every $\Gamma \cup \{A\} \subseteq For(\Sigma)$ it holds: $\Gamma \vdash_{\mathscr{H}} A$ iff $\Gamma \vDash_{\mathscr{M}} A$.*

[6] I.e., truth-function for the connective assign non-empty sets of designated or non-designated values.

[7] The Nmatrix semantics $\mathscr{M}$ for **M** was already introduced by H. Omori and D. Skurt in [25], but with a slight different interpretation of the truth-values.

[8] Note that no axioms nor rules for $\ominus$ are given.

Being the minimal modal logic, at first glance, **M** seems to be nothing else than **CPL** presented in a language with a modal operator $\ominus$ without an interpretation. For instance, it does neither satisfy the axiom $(\mathsf{K}) : \ominus(A \to B) \to (\ominus A \to \ominus B)$ nor the rule of necessitation. However, **M** can not be characterized by a finite deterministic matrix, since any such characterization would designate a formula similar to the well-known Dugundji construction, which is of course not derivable in **M**, cf. [13].

Since $\ominus$ is supposed to represent any given modal operator (for instance, a possibility operator $\Diamond$) it should be expected that $\ominus$ has no fixed interpretation yet. But it can be shown that the nature of the modality, whether $\ominus$ can be interpreted as necessity, possibility, knowledge, obligation etc., will strongly depend on our choice of axioms we want to be valid. However, $\ominus$ is not meaningless, since $\ominus A$ will be designated, iff $v_2(A) = 1$.

In the next sections we will extend the minimal modal logic **M** with two modal axioms, $\ominus(A \to B) \to (\ominus A \to \ominus B)$ (K), $\ominus A \to A$ (T) and $\ominus A \to \ominus\ominus A$ $(\mathsf{4})$, respectively, and later the rule of necessitation and thus show differences and similarities between two approaches for constructing non-deterministic semantics for modal logics. To this end, let $\mathscr{H}_{\mathsf{K}}$ be the Hilbert calculus over Σ obtained from $\mathscr{H}$ by adding axiom schema (K). And let $\mathscr{H}_{\mathsf{KT}}$ be the Hilbert calculus over Σ obtained from $\mathscr{H}_{\mathsf{K}}$ by adding axiom schema (T). Furthermore, let $\mathscr{H}_{\mathsf{KT4}}$ be the Hilbert calculus over Σ obtained from $\mathscr{H}_{\mathsf{KT}}$ by adding the axiom schema $(\mathsf{4})$. The corresponding consequence relations $\vdash_{\mathscr{H}_{\mathsf{K}}}$, $\vdash_{\mathscr{H}_{\mathsf{KT}}}$ and $\vdash_{\mathscr{H}_{\mathsf{KT4}}}$ are defined in similar manner than $\vdash_{\mathscr{H}}$

Before we continue, however, we will quickly show that these three axioms, (K), (T), $(\mathsf{4})$, are not valid in **M**. For (K) consider a valuation such that $v(A) = (1,1)$ and $v(B) = (1,0)$, hence $v(\ominus A)$ is designated and $v(B)$ is non-designated, i.e. $\ominus A \to \ominus B$ is non-designated. Because $A \to B$ is designated and there is valuation such that $\ominus(A \to B)$ is designated, namely $v(A \to B) = (1,1)$, $\ominus(A \to B) \to (\ominus A \to \ominus B)$ is not valid in **M**. As for (T), consider a valuation such that $v(A) = (0,1)$. Then $v(\ominus A) = (1,1)$ or $(1,0)$. Hence, $\ominus A \to A$ will get a non-designated value. For $(\mathsf{4})$ just take a valuation that assigns $\ominus A$ the value $(1,0)$.

3 Nmatrices for MK, MKT and MKT4

In this section, we will quickly recapitulate results from previous works, e.g. [8, 11] or [22, 25] and show how to systematically develop non-deterministic semantics for certain extensions of **M** by either eliminating some non-determinacy from the truth-tables or eliminating truth-values.

MK Let $\mathscr{M}_{\mathsf{K}} = \langle V_4, D_4, \mathscr{O} \rangle$. The truth-tables for $\mathscr{M}_{\mathsf{K}}$ can be displayed as follows:[9]

$\tilde{\to}$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$
$(1,1)$	$(1,*)$	$(1,0)$	$(0,*)$	$(0,0)$
$(1,0)$	$(1,*)$	$(1,*)$	$(0,*)$	$(0,*)$
$(0,1)$	$(1,*)$	$(1,0)$	$(1,*)$	$(1,0)$
$(0,0)$	$(1,*)$	$(1,*)$	$(1,*)$	$(1,*)$

A	$\tilde{\neg} A$	$\tilde{\ominus} A$
$(1,1)$	$(0,*)$	$(1,*)$
$(1,0)$	$(0,*)$	$(0,*)$
$(0,1)$	$(1,*)$	$(1,*)$
$(0,0)$	$(1,*)$	$(0,*)$

Alternatively, we can calculate the value of $\tilde{\to}$ by adding the following conditions to $\mathscr{M}$:

$$z \tilde{\to} w \quad := \quad (x_1, x_2) = (x_1 = (z_1 \Rightarrow w_1)\ ,\ x_2 \leq z_2 \Rightarrow w_2) \qquad \text{or} \qquad v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B)$$

[9] We omit brackets for sets.

MKT Now, let $V_3 = \{(z_1,z_2) \in V_4 \ : \ z_1 \geq z_2\}$, i.e. $V_3 = V_4 \backslash \{(0,1\}$. Accordingly, the set of designated values will be $D_3 = \{z \in V_3 \ : \ z_1 = 1\} = D_4$. Then we can define $\mathscr{M}_{\mathsf{KT}} = \langle V_3, D_3, \mathscr{O} \rangle$. The truth-tables for $\mathscr{M}_{\mathsf{KT}}$ can be displayed as follows:

$\tilde{\to}$	$(1,1)$	$(1,0)$	$(0,0)$
$(1,1)$	$(1,*)$	$(1,0)$	$(0,0)$
$(1,0)$	$(1,*)$	$(1,*)$	$(0,*)$
$(0,0)$	$(1,*)$	$(1,*)$	$(1,*)$

A	$\tilde{\neg} A$	$\tilde{\ominus} A$
$(1,1)$	$(0,*)$	$(1,*)$
$(1,0)$	$(0,*)$	$(0,*)$
$(0,0)$	$(1,*)$	$(0,*)$

In this case, we do not need to change the definitions of the operations.

MKT4 Let $\mathscr{M}_{\mathsf{KT4}} = \langle V_3, D_3, \mathscr{O} \rangle$. The truth-tables for $\mathscr{M}_{\mathsf{KT4}}$ can be displayed as follows:

$\tilde{\to}$	$(1,1)$	$(1,0)$	$(0,0)$
$(1,1)$	$(1,*)$	$(1,0)$	$(0,0)$
$(1,0)$	$(1,*)$	$(1,*)$	$(0,*)$
$(0,0)$	$(1,*)$	$(1,*)$	$(1,*)$

A	$\tilde{\neg} A$	$\tilde{\ominus} A$
$(1,1)$	$(0,*)$	$(1,1)$
$(1,0)$	$(0,*)$	$(0,*)$
$(0,0)$	$(1,*)$	$(0,*)$

Alternatively, we can calculate the value of $\tilde{\ominus}$ by adding the following conditions to $\mathscr{M}$:

$$\tilde{\ominus} z \quad := \quad (x_1, x_2) = (x_1 = z_2 \ , \ x_2 \geq z_2) \qquad \text{or} \qquad v_2(\ominus A) \geq v_2(A)$$

Let $\mathsf{Ax} \in \{\mathsf{K}, \mathsf{KT}, \mathsf{KT4}\}$, then we have the following results:

Theorem 2 (Soundness and completeness of $\mathscr{H}_{\mathsf{Ax}}$ w.r.t. the Nmatrix $\mathscr{M}_{\mathsf{Ax}}$)**.** *Let* $\Gamma \cup \{A\} \subseteq For(\Sigma)$. *Then:* $\Gamma \vdash_{\mathscr{H}_{\mathsf{Ax}}} A$ *iff* $\Gamma \vDash_{\mathscr{M}_{\mathsf{Ax}}} A$.

We omit the proofs, as they can be found in detail in previous publications.

4 RNmatrices for MK, MKT and MKT4

In this section, we will go into a little more detail, since we do not expect readers to be familiar with what in [12] was called *restricted Nmatrices* (RNmatrices). In short, the set $\mathscr{F}$ of valuations over the Nmatrix $\mathscr{M}$ will be restricted to specific subsets $\mathscr{F}' \subseteq \mathscr{F}$ with the aim of satisfying certain modal axiom(s). In particular, we will consider RNmatrices of the form $\mathscr{RM} = \langle \mathscr{M}, \mathscr{F}' \rangle$ such that $\mathscr{F}' \subseteq \mathscr{F}$. I.e., each axiom that extends **M** will restrict the set of acceptable valuation.

MK Consider

$$(\mathsf{K}) : \ominus(A \to B) \to (\ominus A \to \ominus B)$$

Then, a valuation $v \in \mathscr{F}$ satisfies (K) iff, $v_1(\ominus(A \to B) \to (\ominus A \to \ominus B)) = 1$ for every A, B iff $v_1(\ominus(A \to B)) \Rightarrow (v_1(\ominus A) \Rightarrow v_1(\ominus B)) = 1$ for every A, B iff $v_1(\ominus(A \to B)) \leq v_1(\ominus A) \Rightarrow v_1(\ominus B)$ for every A, B iff $v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B)$ for every A, B. Hence, the logic **MK** satisfying axiom (K) is characterized by the RNmatrix $\mathscr{RM}_{\mathsf{K}} = \langle \mathscr{M}, \mathscr{F}_{\mathsf{K}} \rangle$ such that

$$\mathscr{F}_{\mathsf{K}} = \{v \in \mathscr{F} \ : \ v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B) \ \text{ for every } A, B\}.$$

MKT Consider

$$(\mathsf{T}) : \ominus A \to A$$

Then, a valuation $v \in \mathscr{F}$ satisfies (T) iff, $v_1(\ominus A \to A) = 1$ for every A iff $v_1(\ominus A) \Rightarrow v_1(A) = 1$ for every A iff $v_1(\ominus A) \leq v_1(A)$ for every A iff $v_2(A) \leq v_1(A)$ for every A. Hence, the logic **MKT** satisfying axioms $(\mathsf{K})$and (T) is characterized by the RNmatrix $\mathscr{RM}_{\mathsf{KT}} = \langle \mathscr{M}, \mathscr{F}_{\mathsf{KT}} \rangle$ such that

$$\mathscr{F}_{\mathsf{KT}} = \{v \in \mathscr{F} \ : \ v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B) \text{ and } v_2(A) \leq v_1(A) \text{ for every } A, B\}.$$

MKT4 Consider

$$(4) : \ominus A \to \ominus\ominus A$$

Then, a valuation $v \in \mathscr{F}$ satisfies (4) iff, $v_1(\ominus A \to \ominus\ominus A) = 1$ for every A iff $v_1(\ominus A) \Rightarrow v_1(\ominus\ominus A) = 1$ for every A iff $v_1(\ominus A) \leq v_1(\ominus\ominus A)$ for every A iff $v_2(A) \leq v_2(\ominus A)$ for every A. Hence, the logic **MKT4** satisfying axioms (K), $(\mathsf{T})$and (4) is characterized by the RNmatrix $\mathscr{RM}_{\mathsf{KT4}} = \langle \mathscr{M}, \mathscr{F}_{\mathsf{KT4}} \rangle$ such that

$$\mathscr{F}_{\mathsf{KT4}} = \{v \in \mathscr{F} \ : \ v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B) \text{ and } v_2(A) \leq v_1(A) \text{ and } v_2(A) \leq v_2(\ominus A) \text{ for every } A, B\}.$$

Each of the generated RNmatrices will be *structural*, that is, the set $\mathscr{F}'$ is required to be closed under substitutions: if $v \in \mathscr{F}'$ and ρ is a substitution over Σ then $v \circ \rho \in \mathscr{F}'$. As proved in [12], any structural RNmatrix generates a Tarskian and structural consequence relation defined as expected: $\Gamma \vDash_{\mathscr{RM}} A$ iff, for every $v \in \mathscr{F}'$: if $v(B) \in D_4$ for every $B \in \Gamma$ then $v(A) \in D_4$.

Recall, a *substitution* over the signature Σ is a function $\sigma : \mathscr{V} \to For(\Sigma)$. Since $For(\Sigma)$ is an absolutely free algebra, each σ can be extended to a unique endomorphism in $For(\Sigma)$ (which will be also denoted by σ). That is, $\sigma : For(\Sigma) \to For(\Sigma)$ is such that $\sigma(\#A) = \#\sigma(A)$ for $\# \in \{\neg, \ominus\}$, and $\sigma(A \to B) = \sigma(A) \to \sigma(B)$. The set of substitutions over σ (seen as endomorphisms in $For(\Sigma)$) will be denoted by $Subs(\Sigma)$.

Clearly, $\mathscr{RM}_{\mathsf{K}}$, $\mathscr{RM}_{\mathsf{KT}}$ and $\mathscr{RM}_{\mathsf{KT4}}$ are structural, hence generate a Tarskian and structural consequence relation $\vDash_{\mathscr{RM}_{\mathsf{K}}}, \vDash_{\mathscr{RM}_{\mathsf{KT}}}, \vDash_{\mathscr{RM}_{\mathsf{KT4}}}$.

E.g., let ρ be a substitution and let $v \in \mathscr{F}_{\mathsf{K}}$. Observe that $v \circ \rho = (v_1 \circ \rho, v_2 \circ \rho)$. Then, for every A, B: $v_2 \circ \rho(A \to B) = v_2(\rho(A \to B)) = v_2(\rho(A) \to \rho(B)) \leq v_2(\rho(A)) \Rightarrow v_2((\rho(B)) = v_2 \circ \rho(A) \Rightarrow v_2 \circ \rho(B)$. Hence $v \circ \rho \in \mathscr{F}_{\mathsf{K}}$.

We will now sketch soundness and completeness results for $\mathscr{H}_{\mathsf{K}}$, the proofs for $\mathscr{H}_{\mathsf{KT}}$ and $\mathscr{H}_{\mathsf{KT4}}$ follow the same structure. More details for soundness and completeness of Hilbert calculi wrt RNmatrices can be found for example [12]. To this end, we will make use of well-known definitions.

Recall that, given a Tarskian and finitary logic $\mathbf{L}$, a set of formulas Δ is said to be A-saturated (where A is a formula) if $\Delta \nvdash_{\mathbf{L}} A$ but $\Delta, B \vdash_{\mathbf{L}} A$ for every formula B such that $B \notin \Delta$. If Δ is A-saturated then it is a closed theory, that is: $\Delta \vdash_{\mathbf{L}} B$ iff $B \in \Delta$. It is well-known that, in any Tarskian and finitary logic $\mathbf{L}$, if $\Gamma \nvdash_{\mathbf{L}} A$ then there exists an A-saturated set Δ such that $\Gamma \subseteq \Delta$. Since the logic generated by $\mathscr{H}_{\mathsf{K}}$ is Tarskian and finitary, it has this property.

Proposition 1. *Let Δ be an A-saturated set in $\mathscr{H}_{\mathsf{K}}$. Then, for every formulas A, B:*
(1) $\neg A \in \Delta$ iff $A \notin \Delta$;
(2) $A \to B \in \Delta$ iff either $A \notin \Delta$ or $B \in \Delta$;
(3) if $\ominus(A \to B) \in \Delta$ and $\ominus A \in \Delta$ then $\ominus B \in \Delta$.

Proof. Immediate, by definition of $\mathscr{H}_{\mathsf{K}}$ and the fact that Δ is a closed theory. □

Corollary 1. *Let Δ be an A-saturated set in $\mathscr{H}_{\mathsf{K}}$. Then, $B \in \Delta$ iff $v(B) \in D_4$ for some $v \in \mathscr{F}_{\mathsf{K}}$.*

Proof. It is immediate from Proposition 1.[10] □

Theorem 3 (Soundness and completeness of $\mathscr{H}_{\mathsf{K}}$). *Let $\Gamma \cup \{A\} \subseteq For(\Sigma)$. Then: $\Gamma \vdash_{\mathscr{H}_{\mathsf{K}}} A$ iff $\Gamma \vDash_{\mathscr{RM}_{\mathsf{K}}} A$.*

Proof.
(Soundness): It is an easy exercise to show that every axiom of $\mathscr{H}_{\mathsf{K}}$ is valid w.r.t. $\mathscr{RM}_{\mathsf{K}}$ and that MP preserves the designated values. Hence, by induction on the length of a derivation in $\mathscr{H}_{\mathsf{K}}$ of A from Γ, it is easy to see the following: $\Gamma \vdash_{\mathscr{H}_{\mathsf{K}}} A$ implies that $\Gamma \vDash_{\mathscr{RM}_{\mathsf{K}}} A$.
(Completeness): Suppose that $\Gamma \nvdash_{\mathscr{H}_{\mathsf{K}}} A$. As observed above, there exists an A-saturated set Δ such that $\Gamma \subseteq \Delta$. By Corollary 1 there is a valuation $v \in \mathscr{F}_{\mathsf{K}}$ such that $v(B) = 1$ for every $B \in \Gamma$ but $v(A) = 0$. This shows that $\Gamma \nvDash_{\mathscr{RM}_{\mathsf{K}}} A$. □

5 Nmatrices vs. RNmatrices

In this short article we sketched out two general semantical framework for modal logics that do not rely on the notion of possible worlds. Both frameworks adopt non-deterministic semantics in a creative way in order to construct alternative semantics for systems with modalities, but have different strategies for constructing extensions of the minimal modal logic **M**. While the status of **M** as a modal logic itself, can be discussed, in this section we will briefly compare both approaches and consider some similarities and differences of them.

We will start by showing how to extend the systems discussed in this article with the rule of necessitation in order to construct normal modal logics. This can be done for Nmatrices and RNmatrices in a similar manner. Then, we will show some limitations of both approaches. Finally, we will conclude with remarks on some philosophical issues concerning both approaches.

5.1 Normal modal logics

The systems presented before, even though we call them modal logics, are generally not received as such, with the reason being that rules for the modal operators are not present. And while we will not in full detail discuss our rationale behind our terminology, we can certainly provide the technical means such that the systems can be extended by (any) modal rules.

As an example, we will present the changes to the semantics needed for validating the rule of necessitation (N).

Let $\mathsf{Ax} \in \{\mathsf{K}, \mathsf{KT}, \mathsf{KT4}\}$. Then $\mathscr{H}^{\mathsf{N}}_{\mathsf{Ax}}$ is the Hilbert calculus obtained from $\mathscr{H}_{\mathsf{Ax}}$ by adding the necessitation inference rule (where A is a propositional variable):

$$\frac{A}{\ominus A} \quad (\mathsf{N})$$

A formula B is derivable in $\mathscr{H}^{\mathsf{N}}_{\mathsf{Ax}}$ if there exists a finite sequence of formulas $A_1 \ldots A_m$ such that $A_m = B$ and, for every $1 \leq i \leq m$, either A_i is an instance of an axiom, or it follows from $A_j = A_k \to A_i$ and A_k by MP), for some $j,k < i$, or $A_i = \ominus A_j$ follows from A_j (for some $j < i$) by the N-rule. Finally, A is derivable from Γ in $\mathscr{H}^{\mathsf{N}}_{\mathsf{Ax}}$ if either A is derivable in $\mathscr{H}^{\mathsf{N}}_{\mathsf{Ax}}$, or $B_1 \to (B_2 \to (\ldots \to (B_k \to A) \ldots)$ is derivable in $\mathscr{H}^{\mathsf{N}}_{\mathsf{Ax}}$ for some nonempty finite set $\{B_1, \ldots, B_k\} \subseteq \Gamma$.

[10] More detailed proofs regarding soundness and completeness for RNmatrices can be found in [12]

First observe that neither Nmatrices nor RNmatrices will capture the behavior of the N-rule. To see this, take for example any classical tautology, e.g. $A \to A$. Due to both kinds of semantics, this formula will receive the value $(1,0)$, for some valuation v. But then for this valuation we have $v(\ominus(A \to A)) \notin D_4$. Hence the N-rule is not valid.

The key idea for the validity of that rule was given by John Kearns in [17]. There he restricted the set of acceptable valuations by a simple strategy. If a formula A receives a designated value wrt all possible valuation, than $\ominus A$ will also receive a designated value. I.e., for the formula A in question, all valuations v, such that $v(A) = (1,0)$, will be eliminated from the set of acceptable valuations. The original idea of Kearns was later put in the modern context of non-deterministic semantics, see for example [8, 22]. Following the results from [8, 17, 22], we define the set of valuations over the RNmatrix $\mathscr{RM}_{\mathsf{Ax}} = \langle \mathscr{M}, \mathscr{F}_{\mathsf{Ax}} \rangle$ as follows:

- $\mathscr{F}^0_{\mathsf{Ax}} = \mathscr{F}_{\mathsf{Ax}}$
- $\mathscr{F}^{m+1}_{\mathsf{Ax}} = \big\{ v \in \mathscr{F}^m_{\mathsf{Ax}} \ : \ \forall B \in For(\Sigma), \text{ if } \forall w \in \mathscr{F}^m_{\mathsf{Ax}}(w_1(B) = 1) \text{ then } v_2(B) = 1 \big\}$
- $\mathscr{F}^{\mathsf{N}}_{\mathsf{Ax}} = \bigcap\limits_{m=0}^{\infty} \mathscr{F}^m_{\mathsf{Ax}}$

Observe that $\mathscr{F}^{\mathsf{N}}_{\mathsf{Ax}}$ coincides with the original definition of level valuations introduced by Kearns and therefore can also applied to Nmatrices by using instead of the restricted set of valuation $\mathscr{F}_{\mathsf{Ax}}$, the set of all valuation $\mathscr{F}$.

We then define a new semantical consequence as follows:

1. A is valid, denoted by $\models_{\mathscr{RM}_{\mathsf{Ax}}} A$, if $v_1(A) = 1$ for every $v \in \mathscr{F}^N_{\mathsf{Ax}}$.
2. A is a semantical consequence of Γ, denoted by $\Gamma \models_{\mathscr{RM}_{\mathsf{Ax}}} A$, if either A is valid, or $B_1 \to (B_2 \to (\ldots \to (B_k \to A)\ldots)$ is valid for some nonempty finite set $\{B_1, \ldots, B_k\} \subseteq \Gamma$.

For both, Nmatrices and RNmatrices, soundness and completeness results can be obtained in a straight forward manner, by adapting the results from [8, 17, 22].

Recent unpublished results by the authors also show that this technique can be adapted for any (global) modal rule, and thus offering semantics for well-known non-normal modal logics as well. In this regard, both kinds of semantics offer a unifying framework for a various range of (non)-normal modal logics. But while the technique of level valuations applied to Nmatrices might seem ad hoc, it is a generalization of restricting the valuations for RNmatrices. In fact, one could interpret the restriction method for RNmatrices as a local restriction of valuations and the level valuation technique as a global restriction of valuation.

Finally, we note that it is possible to expand the language with additional modal operators. Semantically this can be done by adding more dimensions to the truth-values. Rather than pairs, truth-values would then be n-tuples, depending on the number of additional modal operators.

For example, we can consider a bimodal version of the minimal modal logic **M**, namely the minimal bimodal logic **M2**. This logic is defined over a signature Σ_2 obtained from Σ by replacing $\ominus$ with two modal operators, which will be denoted by $\ominus_1$ and $\ominus_2$. As expected, the snapshots are now triples $z = (z_1, z_2, z_3)$ over **2** in which each coordinate represents a possible truth-value for the formulas A, $\ominus_1 A$ and $\ominus_2 A$, respectively. Hence, eight truth-values are $V_8 = \{(z_1, z_2, z_3) \ : \ z_1, z_2, z_3 \in \{1,0\}\}$, with $D_8 = \{z \in V_8 \ : \ z_1 = 1\}$ being the set of designated values.

The definition of the multioperators over V_8 interpreting the connectives of Σ_2 is a natural generalization of the 4-valued case:

$$\begin{array}{rcl} \tilde{\neg} z & := & (\sim z_1, *, *); \\ \tilde{\ominus}_1 z & := & (z_2, *, *); \\ \tilde{\ominus}_2 z & := & (z_3, *, *); \\ z \tilde{\to} w & := & (z_1 \Rightarrow w_1, *, *). \end{array}$$

Let $\mathscr{M}_2 = \langle V_8, D_8, \mathscr{O}_2 \rangle$ be the obtained 8-valued Nmatrix, where $\mathscr{O}_2(\#) = \tilde{\#}$ for every connective # in Σ_2. Thus, the truth-tables for $\mathscr{M}_2$ are the following (for reasons of limited space we only present the truth-tables for $\ominus_1$ and $\ominus_2$):

A	$(1,1,1)$	$(1,1,0)$	$(1,0,1)$	$(1,0,0)$	$(0,1,1)$	$(0,1,0)$	$(0,0,1)$	$(0,0,0)$
$\ominus_1 A$	$(1,*,*)$	$(1,*,*)$	$(0,*,*)$	$(0,*,*)$	$(1,*,*)$	$(1,*,*)$	$(0,*,*)$	$(0,*,*)$
$\ominus_2 A$	$(0,*,*)$	$(0,*,*)$	$(1,*,*)$	$(1,*,*)$	$(0,*,*)$	$(0,*,*)$	$(1,*,*)$	$(1,*,*)$

While extensions of **M2** wrt Nmatrices have been discussed at large for example in [10, 13, 22, 23, 24, 26, 27, 28], we present restricions for some axioms wrt RNmatrices:

Axiom	Restrictions
$\ominus_2 A \leftrightarrow \neg\ominus_1\neg A$	$v_2(\neg A) = \sim v_3(A)$
$\ominus_1 A \leftrightarrow \neg\ominus_2\neg A$	$v_3(\neg A) = \sim v_2(A)$
$\ominus_1 A \to \ominus_2 A$	$v_2(A) \leq v_3(A)$
$A \to \ominus_1\ominus_2 A$	$v_1(A) \leq v_2(\ominus_2 A)$
$\ominus_2 A \to \ominus_1\ominus_2 A$	$v_3(A) \leq v_2(\ominus_2 A)$
$\ominus_2\ominus_1 A \to \ominus_1\ominus_2 A$	$v_3(\ominus_1 A) \leq v_2(\ominus_2 A)$

We just note in passing that it is not known, whether the last axiom, $\ominus_2\ominus_1 A \to \ominus_1\ominus_2 A$, can be represented in terms of Nmatrices at all. But both approaches to modal logics are certainly capable of producing semantics for a wide range of normal modal logics.

5.2 Scope and limits of the methods

Semantics for modal logics via Nmatrices share the heuristics of systematically eliminating semantical values or non-determinacy from non-deterministic truth-tables to validate desired axioms. This approach is successful in providing a uniform semantics for a broad class of normal and non-normal modal systems, even for some systems that lack possible worlds semantics, cf. [23]. Hence, the proposed framework seems not only more general but also conceptually conservative since the meaning of modal operators was kept uniformly.

However, the technique of eliminating values or non-determinacy has its limitations. It became apparent that not all modal axioms could be straightforwardly represented in a non-deterministic truth-table format, such as the Gödel-Löb axiom $\ominus(\ominus A \to A) \to \ominus A$ (GL) or other non-Sahlquist formulas. The possibility of providing Nmatrices for such formulas remains uncertain.

RNmatrices on the other hand are much more flexible in this regard. In fact, as shown in [12], RNmatrices are stronger than Nmatrices. For example, by analysis similar to the one given for (K), (T) or $(\mathsf{4})$, it is immediate to see that the restriction on the valuation imposed by this axiom is the following: $v_2(\ominus A \to A) \leq v_2(A)$.

However, we are aware of the fact that even RNmatrices are not yet as flexible as Kripke semantics regarding some properties. For example, we have not discussed axiom systems with an infinite number of axioms. While the construction method of RNmatrices for extensions of **M** might give us some arguments that, at least, recursively defined infinite sets of axioms might be expressed in terms of RNmatrices, we might discuss infinite axiom systems, but leave this as a project for future work.

As binary modal operators, like strict implication, the situation is slightly more complicated. For example, in case of strict implication, it seems, the corresponding Kripke semantics implicitly uses a global rule in the definition of the operators, which is something that cannot be expressed in terms of RNmatrices alone, at least not in straight forward manner. We could think of defining strict implication $\strictif$ in terms of $\to$, as follows: $v(A \strictif B) = v(\Box(A \to B))$ and depending on our semantics for $\Box$, we could define different notions of strictness. However, without globally restricting the set of all valuations, none of the sentences $A \strictif B$ would be a tautology. That is because no sentence $A \to B$ would will be assigned the value $(1,1)$ for all valuations. Again, for the moment, we will leave the question of how to define n-ary modal operators open.

Another limitation of our approach at this moment is a property that is called analyticity. In short, if analyticity holds any partial valuation which seems to refute a given formula can be extended to a full valuation (which necessarily refutes that formula too). For example in [22] it was shown that modal logics defined in terms of Nmatrices with global modal rules do not enjoy this property. It is obvious that the failure of analyticity carries over to RNmatrices with global modal rules. Since the failure of analyticity is related to decidability, it seems our presented semantics for modal logics with global modal rules are not decidable. It should of course be mentioned, that in the absence of such global modal rules it can be shown that our semantics are indeed decidable. Needless to say there is gleam of hope. In more recent publications, cf. [13] and [19], it was shown that by a slight adjustment of the level-valuations technique it is possible to regain decidability. The results were proven for the normal modal logics **K**, **KT** and **S4** expressed in terms of Nmatrices. Recently, in [20] the method for **S4** obtained in [13] was adapted to obtain a decidable RNmatrix with level valuations for intuitionistic propositional logic. It is therefore only a matter of time to prove similar results for other modal (or non-classical) logics with global inference rules.

5.3 Philosophical Remarks

The two approaches we presented lead to semantics with sound and complete axiom systems with global modal rules. In that sense, at least with the addition of modal rules, we are justified to claim that we are actually doing modal logics. However, in the absence of such global modal rules, it seems, at the very least, questionable what the status of our operator $\ominus$ might be. Surely, we can define restrictions on the set of valuations that validate well-known modal formulas. But this is not yet an argument in favor of the modal nature of $\ominus$. We could furthermore think of concrete well-studied modal systems such as systems of epistemic of deontic logic, where the rule of necessitation is the source of some paradoxes and therefore not unrestrictedly valid. But even in such systems other global modal rules are present, such as congruentiality.

There are logics, called hyperintensional logics, for which even congruentiality fails to hold, cf. [5], and our approach is certainly able to capture such logics, as well, but we should be very clear, that we are not discussing any particular modal operator. Instead, what can be said in favor of our approach, we are able to capture a multitude of different modal concepts under one and the same umbrella – RNmatrices (or to a lesser extent Nmatrices) with or without global modal rules. Whether this will lead to a new understanding of the concept of modality remains to be open, and needs to be part of a larger investigation and discussion in the future.

Finally, we just remark in passing that the obvious elephant in the room, namely the correlation between Kripke semantics and Nmatrices/ RNmatrices has so far not yet been thoroughly investigated, even though it seems to be a captivating topic. We will leave this subject for future research.

References

[1] Arnon Avron (2007): *Non-deterministic semantics for logics with a consistency operator*. International Journal of Approximate Reasoning 45(2), pp. 271–287, doi:10.1016/j.ijar.2006.06.011.

[2] Arnon Avron & Iddo Lev (2001): *Canonical propositional Gentzen-type systems*. In R. Goré, A. Leitsch & T. Nipkow, editors: Proceedings of the First International Joint Conference on Automated Reasoning (IJCAR '01), Lecture Notes in Artificial Intelligence 2083, pp. 529–544, doi:10.1007/3-540-45744-5_45publisher=Springer-Verlag.

[3] Arnon Avron & Iddo Lev (2005): *Non-Deterministic Multiple-valued Structures*. Journal of Logic and Computation 15(3), pp. 241–261, doi:10.1093/logcom/exi001.

[4] Arnon Avron & Anna Zamansky (2011): *Non-Deterministic Semantics for Logical Systems*. In: Handbook of Philosophical Logic, 16, Springer, pp. 227–304, doi:10.1007/978-94-007-0479-4_4.

[5] Francesco Berto & Daniel Nolan (2023): *Hyperintensionality*. In Edward N. Zalta & Uri Nodelman, editors: The Stanford Encyclopedia of Philosophy, Winter 2023 edition, Metaphysics Research Lab, Stanford University.

[6] Patrick Blackburn, Maarten De Rijke & Yde Venema (2001): *Modal logic*. 53, Cambridge University Press, doi:10.1017/CBO9781107050884.

[7] Walter Alexandre Carnielli & Marcelo Esteban Coniglio (2016): *Paraconsistent logic: Consistency, contradiction and negation*. Logic, Epistemology, and the Unity of Science 40, Springer, doi:10.1007/978-3-319-33205-5_5.

[8] Marcelo E. Coniglio, Luis Fariñas Del Cerro & Newton M. Peron (2015): *Finite Non-Deterministic Semantics for Some Modal Systems*. Journal of Applied Non-Classical Logics 25(1), pp. 20–45, doi:10.1080/11663081.2015.1011543.

[9] Marcelo E. Coniglio, Luis Fariñas Del Cerro & Newton M. Peron (2019): *Modal logic with non-deterministic semantics: Part I—Propositional case*. Logic Journal of the IGPL, pp. 281–315, doi:10.1093/jigpal/jzz027.

[10] Marcelo E. Coniglio, Luis Fariñas Del Cerro & Newton M. Peron (2021): *Modal Logic With Non-Deterministic Semantics: Part II—Quantified Case*. Logic Journal of the IGPL, pp. 695–727, doi:10.1093/jigpal/jzab020.

[11] Marcelo E. Coniglio & Ana Claudia Golzio (2019): *Swap structures semantics for Ivlev-like modal logics*. Soft Computing 23(7), pp. 2243–2254, doi:10.1007/s00500-018-03707-4.

[12] Marcelo E. Coniglio & Guilherme V. Toledo (2021): *Two Decision Procedures for da Costa's C_n Logics Based on Restricted Nmatrix Semantics*. Studia Logica 110(3), pp. 601–642, doi:10.1007/s11225-021-09972-z.

[13] Lukas Grätz (2022): *Truth tables for modal logics T and S4, by using three-valued non-deterministic level semantics*. Journal of Logic and Computation 32(1), pp. 129–157, doi:10.1093/logcom/exab068.

[14] Lloyd Humberstone (2016): *Philosophical Applications of Modal Logic*. College Publications.

[15] Yuri. V. Ivlev (1988): *A semantics for modal calculi*. Bulletin of the Section of Logic 17(3/4), pp. 77–86.

[16] Yuri. V. Ivlev (1991): *Modal logic. (in Russian)*. Moskva: Moskovskij Gosudarstvennyj Universitet.

[17] John Kearns (1981): *Modal Semantics without Possible Worlds*. Journal of Symbolic Logic 46(1), pp. 77–86, doi:10.2307/2273259.

[18] John Kearns (1989): *Leśniewski's strategy and modal logic*. Notre Dame Journal of Formal Logic 30(2), pp. 77–86, doi:10.1305/ndjfl/1093635086.

[19] Ori Lahav & Yoni Zohar (2022): *Effective Semantics for the Modal Logics K and KT via Non-deterministic Matrices*. In J. Blanchette, L. Kovács & D. Pattinson, editors: *Automated Reasoning. Proceedings of IJCAR 2022*, *Lecture Notes in Computer Science* 13385, Springer, Cham, pp. 468–485, doi:10.1007/978-3-031-10769-6_28.

[20] Renato Leme, Marcelo Coniglio & Bruno Lopes (2023): *Intuitionism with Truth Tables: A Decision Procedure for IPL Based on RNmatrices*. arXiv:2308.13664.

[21] David Makinson (1971): *Some embedding theorems for modal logic*. *Notre Dame Journal of Formal Logic* 12(2), pp. 252–254, doi:10.1305/ndjfl/1093894226.

[22] Hitoshi Omori & Daniel Skurt (2016): *More modal semantics without possible worlds*. *IfCoLog Journal of Logics and their Applications* 3(5), pp. 815–846.

[23] Hitoshi Omori & Daniel Skurt (2020): *A Semantics for a Failed Axiomatization of K*. In: *Advances in Modal Logic*, pp. 481–501.

[24] Hitoshi Omori & Daniel Skurt (2021): *Untruth, falsity and non-deterministic semantics*. In: *2021 IEEE 51st International Symposium on Multiple-Valued Logic (ISMVL)*, IEEE, pp. 74–80, doi:10.1109/ISMVL51352.2021.00022.

[25] Hitoshi Omori & Daniel Skurt (2024): *On Ivlev's Semantics for Modality*, pp. 243–275. Springer International Publishing, Cham, doi:10.1007/978-3-031-56595-3_9.

[26] Pawel Pawlowski & Elio La Rosa (2022): *Modular non-deterministic semantics for T, TB, S4, S5 and more*. *Journal of Logic and Computation* 32(1), pp. 158–171, doi:10.1093/logcom/exab079.

[27] Pawel Pawlowski & Daniel Skurt (2024): *8 Valued Non-Deterministic Semantics for Modal Logics*. *Journal of Philosophical Logic*, doi:10.1007/s10992-023-09733-4.

[28] Pawel Pawlowski & Daniel Skurt (2024): *$\Box$ and $\Diamond$ in eight-valued non-deterministic semantics for modal logics*. *Journal of Logic and Computation*, p. exae010, doi:10.1007/s10992-023-09733-4.

[29] Graham Priest (2008): *Introduction to non-classical logics: from ifs to is*, second edition. Cambridge University Press, doi:10.1017/CBO9780511801174.

[30] Karl Krister Segerberg (1971): *An essay in classical modal logic*. Stanford University.

[31] Heinrich Wansing (1989): *Bemerkungen Zur Semantik Nicht-Normaler Möglicher Welten*. *Mathematical Logic Quarterly* 35(6), pp. 551–557, doi:10.1002/malq.19890350611.

Complexity of Nonassociative Lambek Calculus with classical logic

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The Nonassociative Lambek Calculus (NL) represents a logic devoid of the structural rules of exchange, weakening, and contraction, and it does not presume the associativity of its connectives. Its finitary consequence relation is decidable in polynomial time. However, the addition of classical connectives conjunction and disjunction (FNL) makes the consequence relation undecidable. Interestingly, if these connectives are distributive, the consequence relation is decidable in exponential time. This paper provides the proof, that we can merge classical logic and NL (i.e. BFNL), and still the consequence relation is decidable in exponential time.

1 Introduction and preliminaries

Lambek Calculus L was introduced by Lambek [6] under the name *Syntactic Calculus*. L is a propositional logic with three connectives $\otimes$ (product), $\backslash$ and $/$ (residuations of product). Lambek [7] introduced the nonassociative version of this logic, nowadays called Nonassociative Lambek Calculus (NL). From a logical perspective, NL can be seen as the pure logic of residuation, and L as its stronger version for associative product. For both L and NL, J. Lambek provided a sequent system and proved cut elimination [6, 7].

The product for both L and NL derives from conjunction after dropping the structural rules of exchange, weakening, and contraction in terms of sequent systems. NL additionally does not require being an associative operator in terms of algebra. In effect, we obtain a pure operation joining two formulas. This operation may be seen as a binary modality.

Definition 1.1. Let $\mathbf{G} = (G, \otimes, \backslash, /, \leq)$ be a structure such that $(G, \otimes)$ is a groupoid, $(G, \leq)$ is a poset, and the following holds:

$$\text{(RES)} \quad a \otimes b \leq c \text{ iff } b \leq a \backslash c \text{ iff } a \leq c/b$$

for all $a, b, c \in G$. Then $\mathbf{G}$ is called a *residuated groupoid*.

By *groupoid* we mean a set closed under a binary operation without any specific properties required. The residuated groupoids are models of NL. The residuated groupoids where the product is associative are called *residuated semigroups* and are models of L.

The most popular extensions of L and NL are: adding a constant 1 or adding conjunction and disjunction. The constant 1 in algebras is a unit for the product. The conjunction and disjunction replace the partial order with tshe lattice structure and lattice order. We can also add the boundaries, i.e., $\top$ and $\bot$, as respectively, the greatest and lowest elements. In this paper we use the same symbol for both syntactic and semantic purposes and the exact meaning is clear from the context.

Definition 1.2. Let $(G, \otimes, \backslash, /, \leq)$ be a residuated groupoid and let $1 \in G$ be an element such that:

$$1 \otimes a = a = a \otimes 1$$

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for all $a \in G$. Then $(G, \otimes, \backslash, /, 1, \leq)$ is a unital residuated groupoid.

The unital residuated groupoids are models for NL with constant 1 and unital residuated semigroups are models for L with constant 1.

Lambek Calculus with additive connectives (conjunction and disjunction) is called Full Lambek Calculus and denoted FL. Some authors also require the presence of 1 (multiplicative constant) and $\top, \bot$ (additive constants). In this paper, we follow this convention, so FL admits all these constants. Analogously, FNL is an extension of NL with additive connectives and all constants.

Definition 1.3. Let $(G, \otimes, \backslash, /, 1, \leq)$ be a unital residuated groupoid and $(G, \vee, \wedge, \top, \bot, \leq)$ be a bounded lattice. Then, $(G, \otimes, \backslash, /, \vee, \wedge, 1, \top, \bot, \leq)$ is a *residuated lattice*.

The residuated lattices are models for FNL. Residuated lattices where $\otimes$ is associative are models for FL.

Pentus [8] proves that pure L is NP-complete and Buszkowski [1] proves that its finitary consequence relation is undecidable. A similar situation applies if we add the constant 1. FL is a strongly conservative extension[1] of L, so its finitary consequence relation is also undecidable. The same applies to all strongly conservative extensions of L. In this paper, we focus on extensions of NL because of that.

Buszkowski [1] proves that the finitary consequence relation for NL is in *PTIME*. The same applies if we admit the multiplicative constant. Unfortunately, FNL has an undecidable consequence relation [3].

The lattices in the algebras of FNL are not necessarily distributive. If we consider logic with such an axiom for additive connectives, we talk about Distributive Full Nonassociative Lambek Calculus and denote it DFNL. The models for this logic are residuated distributive lattices.

The finitary consequence relation of DFNL is *EXPTIME*-complete if we do not admit the multiplicative constant 1 and is in *EXPTIME* if we admit the constant, which was proved in [9].[2] The lower bound of complexity of the consequence relation for DFNL with constant 1 remains an open problem.

The other interesting extensions of FNL are BFNL and HFNL, i.e., Boolean FNL and Heyting FNL. These logics may be seen as extensions of NL with Boolean and Heyting algebras or as extensions of classical logic and intuitionistic logic with NL. Such logics have been studied by Galatos and Jipsen [4], Buszkowski [2], and others.

Definition 1.4. Let $(G, \otimes, \backslash, /, 1, \leq)$ be a unital residuated groupoid and $(G, \vee, \wedge, \neg, \bot, \top, \leq)$ be a Boolean algebra. Then, $(G, \otimes, \backslash, /, \vee, \wedge, \neg, 1, \top, \bot, \leq)$ is a *residuated Boolean algebra*.

In this paper, we provide the proof of the upper bound of the complexity of the consequence relation for BFNL, extending the results of [9], using the same methods. We also use the results from [10], where distributive lattices, Heyting algebras, and Boolean algebras are considered. The differences between [9, 10] and this paper lay in the details. An experienced reader can easily deduce the results of this paper by reading cited papers, but some changes are subtle, e.g. in some places we do not use families of upsets, but the whole powerset, because we have negation here. Moreover, the results in [9, 10] are described in only algebraic terms and use first-order formulas. Here, we use syntactic notion more directly, still using algebraic methods in proofs.

We show the full proof only for the version with the constant 1 because the proofs for logics without that constant can be easily obtained by omitting some parts.

The proof for HFNL may be done analogously. It is necessary to adjust some definitions and conditions, but the idea remains the same.

[1]A logic $\mathscr{L}_2$, extending $\mathscr{L}_1$, is a (resp. strongly) conservative extension of $\mathscr{L}_1$, if both logics have the same theorems (resp. the same consequence relation) in language of $\mathscr{L}_1$

[2]Shkatov and Van Alten [9] show that the satisfiability problem of quantifier-free first-order formulas in the language of bounded distributive residuated lattices is *EXPTIME*-complete.

Since HFNL and BFNL without 1 are strongly conservative extensions of DFNL,[3] we know their finitary consequence relations are *EXPTIME*-hard and, in effect, are *EXPTIME*-complete. The lower bound for HFNL and BFNL with 1 is still an open problem.

In the second section, we provide the sequent system for BFNL. This system comes from [4], where the authors prove the cut-elimination theorem. In the third section, we study partial structures connected with models of BFNL. We prove important theorems that allow us to check whether a given partial structure is a partial residuated algebra. In the last section, we use these theorems to prove *EXPTIME* complexity of the consequence relation for BFNL.

2 Sequent system

The language of BFNL is defined as follows. We admit a countable set of variables, which we denote by small Latin letters. The formulas are constructed from this set of variables by five binary connectives $(\otimes, \backslash, /, \vee, \wedge)$, one unary connective $(\neg)$ and three constants $(1, \top, \bot)$.

Usual notion of sequents using sequents of formulas is not applicable in nonassociative framework. The comma in sequences is a concatenation operation which is associative. We need to change the structure to something more flexible. Moreover, we need to have two types of commas: one for $\otimes$ and one for $\wedge$ with different properites.

We define bunches. The bunches are elements of free biunital bigroupoid, i.e. the algebra with two binary operations with a unit for both of them, generated from the set of all formulas. We denote first operator by comma and the second one by semicolon. The unit for comma is denoted ε and unit for semicolon is δ.

One may think of bunches as of binary trees in which leaves are formulas or ε or δ and every node besides leaves is labeled by comma or semicolon.

The bunch ε is called an *empty bunch*. All the other bunches are nonempty. We reserve Latin capital letters for formulas and Greek capital letters for bunches. A *context* is a bunch with an anonymous variable. Contexts are denoted by $\Gamma[\_]$, and when we perform the substitution of Δ in place of $\_$, we represent it as $\Gamma[\Delta]$.

A *sequent* is a pair Γ, A, where Γ is a bunch and A is a formula. We write $\Gamma \Rightarrow A$.

The axioms and the rules for BFNL are as follows:

(id) $\quad A \Rightarrow A$

(cut) $\quad \dfrac{\Gamma \Rightarrow A \quad \Delta[A] \Rightarrow C}{\Delta[\Gamma] \Rightarrow C}$

$(\otimes \Rightarrow) \quad \dfrac{\Gamma[(A,B)] \Rightarrow C}{\Gamma[A \otimes B] \Rightarrow C}$

$(\Rightarrow \otimes) \quad \dfrac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}$

$(\backslash \Rightarrow) \quad \dfrac{\Gamma[B] \Rightarrow C \quad \Theta \Rightarrow A}{\Gamma[(\Theta, A\backslash B)] \Rightarrow C}$

$(\Rightarrow \backslash) \quad \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B}$

$(/ \Rightarrow) \quad \dfrac{\Gamma[A] \Rightarrow C \quad \Theta \Rightarrow B}{\Gamma[(A/B, \Theta)] \Rightarrow C}$

$(\Rightarrow /) \quad \dfrac{\Gamma, B \Rightarrow A}{\Gamma \Rightarrow A/B}$

[3]See Remark 5 in [2].

$$(\wedge\Rightarrow)\ \frac{\Gamma[(A;B)]\Rightarrow C}{\Gamma[A\wedge B]\Rightarrow C}\qquad(\Rightarrow\wedge)\ \frac{\Gamma\Rightarrow A\quad\Gamma\Rightarrow B}{\Gamma\Rightarrow A\wedge B}$$

$$(\vee\Rightarrow)\ \frac{\Gamma[A]\Rightarrow C\quad\Gamma[B]\Rightarrow C}{\Gamma[A\vee B]\Rightarrow C}\qquad(\Rightarrow\vee)\ \frac{\Gamma\Rightarrow A}{\Gamma\Rightarrow A\vee B}\quad\frac{\Gamma\Rightarrow B}{\Gamma\Rightarrow A\vee B}$$

$$(\top\Rightarrow)\ \frac{\Gamma[\Delta]\Rightarrow C}{\Gamma[(\top;\Delta)]\Rightarrow C}\ \frac{\Gamma[\Delta]\Rightarrow C}{\Gamma[(\Delta;\top)]\Rightarrow C}\qquad(\Rightarrow\top)\ \Gamma\Rightarrow\top$$

$$(\bot\Rightarrow)\ \Gamma[\bot]\Rightarrow C$$

$$(\wedge\text{-ass})\ \frac{\Gamma[\Delta_1;(\Delta_2;\Delta_3)]\Rightarrow C}{\overline{\Gamma[(\Delta_1;\Delta_2);\Delta_3]\Rightarrow C}}\qquad(\wedge\text{-ex})\ \frac{\Gamma[\Delta;\Theta]\Rightarrow C}{\Gamma[\Theta;\Delta]\Rightarrow C}$$

$$(\wedge\text{-weak})\ \frac{\Gamma[\Delta]\Rightarrow C}{\Gamma[\Delta;\Theta]\Rightarrow C}\qquad(\wedge\text{-cont})\ \frac{\Gamma[\Delta;\Delta]\Rightarrow C}{\Gamma[\Delta]\Rightarrow C}$$

$$(\neg\Rightarrow)\ A\wedge\neg A\Rightarrow\bot\qquad(\Rightarrow\neg)\ \top\Rightarrow A\vee\neg A$$

$$(1\Rightarrow)\ \frac{\Gamma[\Delta]\Rightarrow C}{\Gamma[(1,\Delta)]\Rightarrow C}\quad\frac{\Gamma[\Delta]\Rightarrow C}{\Gamma[(\Delta,1)]\Rightarrow C}\qquad(\Rightarrow 1)\ \varepsilon\Rightarrow 1$$

We shortly describe the semantics of BFNL. The models for BNFL are residuated Boolean algebras. The valuation is a homomorphism μ from the free algebra of formulas to a residuated Boolean algebra **B** extended to bunches inductively as follows:

$$\mu(\varepsilon)=1$$
$$\mu(\delta)=\top$$
$$\mu((\Gamma,\Delta))=\mu(\Gamma)\otimes\mu(\Delta)$$
$$\mu((\Gamma;\Delta))=\mu(\Gamma)\wedge\mu(\Delta)$$

The sequent $\Gamma\Rightarrow A$ is said to be true in **B** under the valuation μ if $\mu(\Gamma)\leq\mu(A)$.

3 Partial residuated Boolean algebras

In this section we provide the notion of partial structures and we prove some properties. The most important result here is Theorem 3.19 which helps in identifying partial residuated Boolean algebras in exponential time in the next section.

3.1 Partial structures

Definition 3.1. A function $f:U\mapsto Y$, where $U\subseteq X$, is called a *partial function* from X to Y (we write $f:X\rightarrow Y$). If $U=X$, then the function is said to be *total*.

We write $f(x)=\infty$, if the function f on the argument x is undefined.

Definition 3.2. Let I,J,K be finite indexing sets. We say $(U,\{f_i^{n_i}\}_{i\in I},\{a_j\}_{j\in J},\{R_k^{m_k}\}_{k\in K})$ is a *partial structure*, if $\{a_j\}_{j\in J}\subseteq U$ and $f_i^{n_i}:U^{n_i}\rightarrow U$ is a partial function for all $i\in I$ and $R_k^{m_k}\subseteq U^{m_k}$ for all $k\in K$. If all operations are total, then we say the structure is *total*.

Definition 3.3. Let I,J,K be finite indexing sets. Let $(U,\{f_i^{n_i}\}_{i\in I},\{a_j\}_{j\in J},\{R_k^{m_k}\}_{k\in K})$ be a partial structure and $(U',\{f_i'^{n_i}\}_{i\in I},\{a_j'\}_{j\in J},\{R_k'^{m_k}\}_{k\in K})$ be a total structure. Let $\iota:U\rightarrow U'$ be an injection. We say ι is an *embedding*, if:

(i) for all $j \in J$ we have $\iota(a_j) = a'_j$,

(ii) for all $i \in I$ and all $x_1, x_2, \ldots, x_{n_i} \in U$, if $f_i^{n_i}(x_1, x_2, \ldots, x_{n_i}) \neq \infty$,
then $\iota(f_i^{n_i}(x_1, x_2, \ldots, x_{n_i}) = f_i'^{n_i}(\iota(x_1), \iota(x_2), \ldots, \iota(x_{n_i}))$,

(iii) for all $k \in K$ we have $(\iota(x_1), \iota(x_2), \ldots, \iota(x_{m_k})) \in (R_k'^{m_k}) \iff (x_1, x_2, \ldots, x_{m_k}) \in R_k^{m_k}$
for all $x_1, x_2, \ldots, x_{m_k} \in U$.

If **A** is a partial structure, **B** is a total structure and there exists an embedding from **A** to **B**, then we say **A** is *embeddable* into **B**. If **A** is embeddable into **B** and $A \subseteq B$, then we say **A** is a *partial substructure* of **B**. Let $\mathscr{K}$ be a class of structures. By $\mathscr{K}^P$ we denote the class of all partial substructures of structures of $\mathscr{K}$.

Definition 3.4. Let $\mathbf{L} = (L, \vee, \wedge, \top, \bot, \leq)$ be a partial structure. We say **L** is a *partial lattice*, if there exists a total lattice $\mathbf{L}'$ such that **L** is embeddable into it. If $\mathbf{L}'$ is distributive, then **L** is a *partial distributive lattice*.

One shows that a partial structure $(L, \vee, \wedge, \top, \bot, \leq)$ is a partial bounded lattice, if $(L, \leq)$ is a poset, $\top$ and $\bot$ are bounds of $\leq$ and $\vee, \wedge$ are compatible with $\leq$, i.e. if $a \vee b \neq \infty$, then $a \vee b$ is the supremum of $\{a, b\}$ with respect to $\leq$ and if $a \wedge b \neq \infty$, then $a \wedge b$ is the infimum of $\{a, b\}$ with respect to $\leq$. See [9].

Definition 3.5. Let $\mathbf{B} = (B, \otimes, \backslash, /, \vee, \wedge, \neg, 1, \top, \bot, \leq)$ be a partial structure. We say **B** is a *partial residuated Boolean algebra*, if there exists a total residuated Boolean algebra such that **B** is embeddable into it and for all $a \in B$ we have $\neg a \neq \infty$, $\neg a \in B$, $a \vee \neg a = \top$ and $a \wedge \neg a = \bot$. One notices that $(B, \otimes, \backslash, /, \vee, \wedge, \top, \bot, \leq)$ is a partial bounded distributive residuated lattice.

3.2 Filters

Let $(P, \leq)$ be a poset and let $A \subseteq P$. We say A is an *upset*, if for all $a \in A$ and all $b \in P$ such that $a \leq b$ we have $b \in A$. Analogously, A is a *downset*, if for all $a \in A$ and $b \in P$ such that $b \leq a$ we have $b \in A$.

For every poset $(P, \leq)$ and every element $a \in P$ we define:

$$[a) = \{b \in P : a \leq b\} \qquad (a] = \{b \in P : b \leq a\}$$

One notices $[a)$ is an upset and $(a]$ is a downset.

Definition 3.6. Let $(L, \vee, \wedge)$ be a lattice and let $F \subseteq L$. We say F is a *filter*, if the following conditions hold:

(F1) if $a \leq b$ and $a \in F$, then $b \in F$

(F2) if $a \in F$ and $b \in F$, then $a \wedge b \in F$

We say F is *proper*, if $F \neq L$. The filter F is *prime*, if it is proper and:

(F3) if $a \vee b \in F$, then $a \in F$ or $b \in F$

Let $(L, \vee, \wedge)$ be a lattice and F be a filter. We use the following notion:

$$F_a = \left\{ y \in L : \underset{x \in F}{\exists} x \wedge a \leq y \right\}$$

One proves F_a is a filter.

If we consider filters on residuated Boolean algebras, then (F3) is replaced with the following condition:

$$\text{(FB)} \quad \neg a \in F \text{ iff } a \notin F$$

Considering filters on partial residuated Boolean algebras, we must change definition. We replace (F2) with the following condition:

$$\text{(F2')} \quad \text{if } a \in F \text{ and } b \in F, \text{ then } a \wedge b \in F \text{ or } a \wedge b = \infty$$

for all $a, b \in B$.

The following properties of filters are useful and may be easily proved.

Lemma 3.7. *Let $(B, \vee, \wedge, \neg, \top, \bot)$ be a Boolean algebra and let $F \subseteq B$ be a proper filter. The filter F is prime if, and only if, $a \in F$ or $\neg a \in F$ for all $a \in B$.*

This lemma remains true for residuated Boolean algebras.

Proof. Let F be a prime filter. Then $a \vee \neg a = \top \in F$ for all $a \in B$, so the condition of lemma holds. Now let $a \in F$ or $\neg a \in F$ for all $a \in B$. Let $a \vee b \in F$ and suppose $a \notin F$ and $b \notin F$. Then $\neg a \in F$ and $\neg b \in F$, by assumption. By (F2), $\neg a \wedge \neg b \in F$. So, $\neg(a \vee b) \in F$. Hence, $(a \vee b) \wedge \neg(a \vee b) = \bot \in F$, by (F2). This is impossible. □

Lemma 3.8. *Let $(L, \vee, \wedge)$ be a distributive lattice and let $F \subseteq L$ be a filter and $b \in L$ be such that $b \notin F$. There exists a prime filter $P \subseteq L$ such that $F \subseteq P$ and $b \notin P$.*

Proof. Let F be a filter, $b \in L$ and $b \notin F$. We construct a prime filter as an extension of F, but we need to avoid adding b.

Let $\mathscr{E}$ be a family of filters of L containing F and not containing b. The family is nonempty, since $F \in \mathscr{E}$. Let $C \subseteq \mathscr{E}$ be any nonempty chain in $\mathscr{E}$. Then $F \subseteq \bigcup C$ and $b \notin \bigcup C$. We show $\bigcup C$ is a filter. Let $c, d \in \bigcup C$, then $c \in G$ and $d \in G'$ for some $G, G' \in C$. Since C is a chain, then $G \subseteq G'$ or $G' \subseteq G$, so both c and d are elements of G or G'. Then, by (F2), $c \wedge d \in G$ or $c \wedge d \in G'$, so $c \wedge d \in \bigcup C$. So $\bigcup C$ satisfies (F2). (F1) is obvious. Hence, $\bigcup C$ is a filter.

By Kuratowski–Zorn's lemma, there exists $P \in \mathscr{E}$, which is a maximal element of $\mathscr{E}$. We need to show P is prime. Let $c, d \notin P$ and $c \vee d \in P$. Since $c \notin P$, then $P \subseteq P_c$, and, since P is a maximal element of $\mathscr{E}$, $P_c \notin \mathscr{E}$. Clearly, $F \subseteq P_c$, so $b \in P_c$. Analogously, since $d \notin P$, then $b \in P_d$.

By definition of P_c, P_d, for some $x, y \in P$ we have $x \wedge c \leq b$ and $y \wedge d \leq b$. Hence, $x \wedge y \wedge c \leq b$ and $x \wedge y \wedge d \leq b$ and so $(x \wedge y \wedge c) \vee (x \wedge y \wedge d) \leq b$. By distributivity, $x \wedge y \wedge (c \vee d) \leq b$. Since $x, y, c \vee d \in P$, then $b \in P$. Thus, if $c, d \notin P$, when $c \vee d \in P$, then $b \in P$, which is impossible by definition of P. □

Corollary 3.9. *Let $(L, \vee, \wedge)$ be a distributive lattice and let $a, b \in L$ be such that $a \not\leq b$. There exists a prime filter $F \subseteq L$ such that $a \in F$ and $b \notin F$.*

Proof. The set $[a)$ is a filter such that $b \notin [a)$. Then, by Lemma 3.8, there exists a prime filter P such that $a \in P$ and $b \notin P$. □

Lemma 3.10. *Let* **LB** *be a total residuated Boolean algebra and let F, G be proper filters of* **B** *and* **H** *be a prime filter of* **H** *such that $\{x \otimes y : x \in F \text{ and } y \in G\} \subseteq H$. Then, there exist prime filters F' and G' such that $F \subseteq F'$ and $G \subseteq G'$ and $\{x \otimes y : x \in F' \text{ and } y \in G\} \subseteq H$ and $\{x \otimes y : x \in F \text{ and } y \in G'\} \subseteq H$.*

Proof. Let F, G be proper filters and H be a prime filter such that $\{x \otimes y : x \in F \text{ and } y \in G\} \subseteq H$. We show there exists a prime filter F' such that $F \subseteq F'$ and $\{x \otimes y : x \in F' \text{ and } y \in G\} \subseteq H$.

Let $\mathscr{E}$ be the family of filters Q of $\mathbf{B}$ such that $\{x \otimes y : x \in Q \text{ and } y \in G\} \subseteq H$. This family is nonempty, since $F \in \mathscr{E}$. Clearly, all filters in $\mathscr{E}$ are proper; otherwise $\bot = \bot \otimes 1 \in H$, which is impossible. We show that $\bigcup C \in \mathscr{E}$ for every nonempty chain $C \subseteq \mathscr{E}$. Now, let $a \in \bigcup C$. Then, for some $Q \in C$ we have $a \in Q$ and $\{x \otimes y : x \in Q \text{ and } y \in G\} \subseteq H$. Hence, for some $y \in G$, we have $a \otimes y \in H$. So, $\bigcup C \in \mathscr{E}$.

By Kuratowski–Zorn's lemma, there exists $P \in \mathscr{E}$, which is a maximal element of $\mathscr{E}$. We show P is a prime filter. Let $a \vee b \in P$ and suppose $a, b \notin P$. We consider P_a, P_b. Clearly, $P \subset P_a$ and $P \subset P_b$. So, since P is a maximal element, $P_a, P_b \notin \mathscr{E}$. So $\{x \otimes y : x \in P_a \text{ and } y \in G\} \nsubseteq H$ and $\{x \otimes y : x \in P_b \text{ and } y \in G\} \nsubseteq H$.

So, for some $x, y \in P$ and some $z_1, z_2 \in G$ we have $(x \wedge a) \otimes z_1 \notin H$ and $(y \wedge b) \otimes z_2 \notin H$. Since $x, y, a \vee b \in P$, then $x \wedge y \wedge (a \vee b) \in P$. So we have $(x \wedge y \wedge (a \vee b)) \otimes (z_1 \wedge z_2) \in H$. But:

$$(x \wedge y \wedge (a \vee b)) \otimes (z_1 \wedge z_2) = ((x \wedge y \wedge a) \vee (x \wedge y \wedge b)) \otimes (z_1 \wedge z_2) = \\ = (x \wedge y \wedge a) \otimes (z_1 \wedge z_2) \vee (x \wedge y \wedge b) \otimes (z_1 \wedge z_2)$$

So, since H is a prime filter, $(x \wedge y \wedge a) \otimes (z_1 \wedge z_2) \in H$ or $(x \wedge y \wedge b) \otimes (z_1 \wedge z_2) \in H$. Because H is a filter, then $(x \wedge a) \otimes z_1 \in H$ or $(y \wedge b) \otimes z_2 \in H$. This contradicts the assumptions. Hence, $a \in P$ or $b \in P$.

We put $F' = P$. We show that there exists G' such that $G \subseteq G'$ and $\{x \otimes y : x \in F \text{ and } y \in G'\} \subseteq H$ analogously. □

Corollary 3.11. *Let* $\mathbf{B}$ *be a total residuated Boolean algebra and let* F, G *be proper filters of* $\mathbf{L}$ *and* $\mathbf{H}$ *be a prime filter of* $\mathbf{H}$ *such that* $\{x \otimes y : x \in F \text{ and } y \in G\} \subseteq H$. *Then, there exist prime filters* F' *and* G' *such that* $F \subseteq F'$ *and* $G \subseteq G'$ *and* $\mathcal{R}_{\mathbf{L}}(F', G', H)$.

Proof. First, we construct F' such that $\{x \otimes y : x \in F' \text{ and } y \in G\} \subseteq H$, by Lemma 3.10. Then, we construct G' such that $\{x \otimes y : x \in F' \text{ and } y \in G'\} \subseteq H$, by Lemma 3.10. Then, by Lemma 3.15, $\mathcal{R}_{\mathbf{L}}(F', G', H)$. □

3.3 Residuated frames

Definition 3.12. Let $\mathfrak{F} = (P, I, R)$. We say $\mathfrak{F}$ is a *residuated frame*, when $I \subset P$ and R is a ternary relation on P and the following conditions hold:

(U1) $\forall_{x,x',y,z \in P} \big(\text{if } R(x,y,z) \text{ and } x' = x, \text{ then } R(x',y,z)\big)$

(U2) $\forall_{x,y,y',z \in P} \big(\text{if } R(x,y,z) \text{ and } y' = y, \text{ then } R(x,y',z)\big)$

(U3) $\forall_{x,y,z,z' \in P} \big(\text{if } R(x,y,z) \text{ and } z = z', \text{ then } R(x,y,z')\big)$

(U4) $\forall_{x \in P} \exists_{y,z \in I} \big(R(x,y,x) \text{ and } R(z,x,x)\big)$

(U5) $\forall_{x,z \in P} \forall_{y \in I} \big(\text{if } R(x,y,z) \text{ or } R(y,x,z), \text{ then } x = z\big)$

Residuated frames are the relational structures similar to groupoids. Instead of a binary operation we use a ternary relation.

Definition 3.13. Let $\mathbf{B} = (B, \otimes, \backslash, /, \vee, \wedge, \neg, 1, \top, \bot, \leq)$ be a partial residuated Boolean algebra. We define the *associated residuated frame* $\mathfrak{F}_{\mathbf{B}} = (\mathcal{F}(B), \mathcal{I}_{\mathbf{B}}, \mathcal{R}_{\mathbf{B}})$, where $\mathcal{F}(B)$ is the set of prime filters of $\mathbf{B}$, $\mathcal{I}_{\mathbf{B}}$ is the set of all prime filters containing 1 and:

$$\begin{aligned}\mathcal{R}_{\mathbf{B}}(F,G,H) \iff & \left(\forall_{a,b\in B} \text{ if } a \in F \text{ and } b \in G, \text{ then } a \otimes b \in H \vee a \otimes b = \infty\right) \\ & \text{and } \left(\forall_{a,b\in B} \text{ if } a \in F \text{ and } a\backslash b \in G \text{ and } a\backslash b \neq \infty, \text{ then } b \in H\right) \\ & \text{and } \left(\forall_{a,b\in B} \text{ if } b/a \in F \text{ and } a \in G \text{ and } a/b \neq \infty, \text{ then } b \in H\right).\end{aligned}$$

Proposition 3.14. *Let* $\mathbf{B}$ *be a residuated Boolean algebra and let* $F \in \mathcal{F}(B)$*. Then, there exist prime filters* $P, Q \in \mathcal{F}(B)$ *such that* $\mathcal{R}_{\mathbf{B}}(F,P,F)$ *and* $\mathcal{R}_{\mathbf{B}}(Q,F,F)$ *and* $1 \in P, 1 \in Q$.

Proof. Let $F \in \mathcal{F}(L)$, we show there exists a prime filter P such that $1 \in P$ and $\mathcal{R}_{\mathbf{L}}(F,P,F)$. The proof for $\mathcal{R}_{\mathbf{L}}(Q,F,F)$ is similar.

Let $\mathcal{E}$ be the family of filters of $\mathbf{L}$ such that for every filter $G \in \mathcal{E}$ we have $1 \in G$ and $f \otimes g \in F$ for all $f \in F$ and $g \in G$. Clearly, all filters in $\mathcal{E}$ are proper. This family is nonempty, since $[1) \in \mathcal{E}$. One shows that $\bigcup C$ is a filter for every nonempty chain $C \subseteq \mathcal{E}$ analogously like in the proof of Lemma 3.8. We show $\bigcup C \in \mathcal{E}$. Clearly, $1 \in \bigcup C$. Let $f \in F$ and $g \in \bigcup C$. Then, $g \in G$ for some $G \in C$. So, $f \otimes g \in F$.

By Kuratowski–Zorn's lemma, there exists $P \in \mathcal{E}$, which is a maximal element of $\mathcal{E}$. We show that P is a prime filter. Assume $a \vee b \in P$. Suppose $a, b \notin P$.

We consider P_a and P_b. Clearly, $P \subset P_a$ and $P \subset P_b$. Since P is a maximal element of $\mathcal{E}$, then $P_a, P_b \notin \mathcal{E}$.

We have $1 \in P_a, P_b$. Then, for some $f_a \in F$ and some $x \in P$, we have $f_a \otimes (x \wedge a) \notin F$ and for some $f_b \in F$ and some $y \in P$ we have $f_b \otimes (y \wedge b) \notin F$. Since $f_a, f_b \in F$, then $f_a \wedge f_b \in F$, by (F2). Since $a \vee b \in P$, then $(x \wedge y) \wedge (a \vee b) = (x \wedge y \wedge a) \vee (x \wedge y \wedge b) \in P$.

So, $(f_a \wedge f_b) \otimes [(x \wedge a) \vee (y \wedge b)] \in F$. As a consequence:

$$(f_a \wedge f_b) \otimes [(x \wedge a) \vee (y \wedge b)] = ((f_a \wedge f_b) \otimes (x \wedge a)) \vee ((f_a \wedge f_b) \otimes (y \wedge b))$$

Because F is a prime filter, then $(f_a \wedge f_b) \otimes (x \wedge a) \in F$ or $(f_a \wedge f_b) \otimes (y \wedge b) \in F$. Assume $(f_a \wedge f_b) \otimes (x \wedge a) \in F$. Then $f_a \otimes (x \wedge a) \in F$, by (F1) and monotonicity of $\otimes$. Assume $(f_a \wedge f_b) \otimes (y \wedge b) \in F$. Then $f_b \otimes (y \wedge b) \in F$. Both possibilites lead to the contradiction with assumptions. Hence, $a \in P$ or $b \in P$.

Therefore, $\mathcal{R}_{\mathbf{L}}(F,P,F)$. □

Lemma 3.15. *Let* $\mathbf{B}$ *be a total residuated Boolean algebra and* $\mathfrak{F}_{\mathbf{B}} = (\mathcal{F}(B), \subseteq, \mathcal{R}_{\mathbf{B}})$ *its associated residuated frame. Then, for* $F, G, H \in \mathcal{F}(B)$*, the following are equivalent:*

(i) if $a \in F$ *and* $b \in G$*, then* $a \otimes b \in H$ *for all* $a, b \in B$

(ii) if $a \in F$ *and* $a\backslash b \in G$*, then* $b \in H$ *for all* $a, b \in B$

(iii) if $b/a \in F$ *and* $a \in G$*, then* $b \in H$ *for all* $a, b \in B$

Proof. We assume (i). Let $a \in F$ and $a\backslash b \in G$. Since $\mathcal{R}_{\mathbf{B}}(F,G,H)$, $a \otimes (a\backslash b) \in H$ and then $b \in H$, because $a \otimes (a\backslash b) \leq b$. Hence (ii) holds. Now we assume (ii). Let $a \in F$ and $b \in G$. Since $b \leq a\backslash(a \otimes b)$, then $a\backslash(a \otimes b) \in G$, so, by (ii), $a \otimes b \in H$ and (i) holds. The proof of equivalence of (i) and (iii) is similar. □

We construct a residuated Boolean algebras from the arbitrary residuated frame $\mathfrak{F} = (P, I, R)$. Let $X, Y \subseteq P$, we define:

$$X \otimes' Y = \left\{ z \in P : \underset{x,y \in P}{\exists} x \in X \text{ and } y \in Y \text{ and } R(x,y,z) \right\}$$
$$X \backslash' Y = \left\{ y \in P : \underset{x,z \in P}{\forall} \text{if } R(x,y,z) \text{ and } x \in X, \text{ then } z \in Y \right\}$$
$$Y /' X = \left\{ x \in P : \underset{y,z \in P}{\forall} \text{if } R(x,y,z) \text{ and } y \in X, \text{ then } z \in Y \right\}$$

Then, $\mathbf{B}_{\mathfrak{F}} = (\mathcal{P}(P), \otimes', \backslash', /', \cup, \cap, ^c, I, P, \emptyset, \subseteq)$ is a residuated Boolean algebra, where $X^c = \mathcal{P}(P) \setminus X$ for all $X \in \mathcal{P}(P)$. We call it the *complex Boolean algebra of the residuated frame* $\mathfrak{F}$.

Lemma 3.16. *Let* $\mathbf{B}$ *be a total residuated Boolean algebra and* $\mathfrak{F}_{\mathbf{B}} = (\mathcal{F}(B), \subseteq, \mathcal{R}_{\mathbf{B}})$ *its associated residuated frame. Let* $a, b \in B$*.*

(1) If $H \in \mathcal{F}(B)$ *and* $a \otimes b \in H$*, then there exist* $F, G \in \mathcal{F}(B)$ *such that* $a \in F$*,* $b \in G$ *and* $\mathcal{R}_{\mathbf{B}}(F, G, H)$*.*

(2) If $G \in \mathcal{F}(B)$ *and* $a \backslash b \notin G$*, then there exist* $F, H \in \mathcal{F}(B)$ *such that* $a \in F$*,* $b \notin H$ *and* $\mathcal{R}_{\mathbf{B}}(F, G, H)$*.*

(3) If $F \in \mathcal{F}(B)$ *and* $b / a \notin F$*, then there exist* $G, H \in \mathcal{F}(B)$ *such that* $a \in G$*,* $b \notin H$ *and* $\mathcal{R}_{\mathbf{B}}(F, G, H)$*.*

Proof. We show (i). Since $a \otimes b \in H$, then $x \otimes y \in H$ for all $a \leq x$ and $b \leq y$. So, $\{x \otimes y : x \in [a) \text{ and } y \in [b)\} \subseteq H$ and, by Corollary 3.11, there exist prime filters F, G such that $\mathcal{R}_{\mathbf{B}}(F, G, H)$.

We show (ii). Let G be a prime filter such that $a \backslash b \notin G$. We consider $aG = \{a \otimes x : x \in G\}$. We extend aG to be filter. Let $Q = \{x \in L : \underset{y \in aG}{\exists} y \leq x\}$. Clearly, (F1) holds. Let $x, y \in Q$. Then, for some $x', y' \in G$ we have $a \otimes x' \leq x$ and $a \otimes y' \leq y$. Since $x', y' \in G$, then $x' \wedge y' \in G$ and $a \otimes (x' \wedge y') \in aG$. So:

$$a \otimes (x' \wedge y') \leq (a \otimes x') \wedge (a \otimes y') \leq x \wedge y$$

Hence, $x \wedge y \in Q$. We show $b \notin Q$. Suppose $b \in Q$, then, for some $x \in G$, $a \otimes x \leq b$. By (RES), $x \leq a \backslash b$. Hence, $a \backslash b \in G$ – contradiction. So, Q is a filter and $b \notin Q$. By Lemma 3.8, there exists a prime filter H such that $Q \subseteq H$ and $b \notin H$. So, we have $\{x \otimes y : x \in [a) \text{ and } y \in G\} \subseteq H$. By Lemma 3.10, there exists a prime filter F such that $\mathcal{R}_{\mathbf{L}}(F, G, H)$.

One shows (iii) analogously. □

Lemma 3.17. *Let* $\mathbf{B}$ *be a partial residuated Boolean algebra and let* $a, b \in L$ *be such that* $a \not\leq b$*. There exists a prime filter* $F \subseteq B$ *such that* $a \in F$ *and* $b \notin F$*.*

Proof. By definition of a partial residuated Boolean algebra, there exists a total residuated Boolean algebra $\mathbf{B}'$ such that ι is an embedding of $\mathbf{B}$ into $\mathbf{B}'$. Then, by Corollary 3.9, there exists a prime filter $F \subseteq B'$ such that $a \in F$ and $b \notin F$. Clearly, $\iota^{-1}(F)$ is a prime filter of $\mathbf{B}$ and $a \in \iota^{-1}(F)$ and $b \notin \iota^{-1}(F)$. □

Proposition 3.18. *Let* $\mathbf{B} = (B, \otimes, \backslash, /, \vee, \wedge, \neg, 1, \top, \bot, \leq)$ *be a partial residuated Boolean algebra. Let* $\mathbf{B}_{\mathfrak{F}_{\mathbf{B}}}$ *be the complex Boolean algebra of the associated residuated frame. We define* $\iota(a) = \{F \in \mathscr{F}_B : a \in F\}$ *for all* $a \in B$*. Then,* ι *is an embedding.*

Proof. Let $a \leq b$. Then, for all $H \in \iota(a)$, we have $b \in H$, so $H \in \iota(b)$. Hence, $\iota(a) \subseteq \iota(b)$. Let $a \nleq b$. By Lemma 3.17, there exists a prime filter H such that $a \in H$ and $b \notin H$. Hence, $\iota(a) \nsubseteq \iota(b)$. Therefore, $a \leq b$ iff $\iota(a) \subseteq \iota(b)$. As a consequence, ι is injective.

Since prime filters are proper filters, $\iota(\bot) = \emptyset$. $\top$ is an element of every filter, so $\iota(\top) = \mathcal{F}(B)$.

Let $a, b \in B$ and $a \otimes b \neq \infty$. By definition:

$$\iota(a) \otimes' \iota(b) = \left\{ H \in \mathcal{F}(B) : \underset{F,G \in \mathcal{F}(B)}{\exists} F \in \iota(a) \text{ and } G \in \iota(b) \text{ and } \mathcal{R}_{\mathbf{B}}(F,G,H) \right\}.$$

We show $\iota(a \otimes b) \subseteq \iota(a) \otimes' \iota(b)$. Let $H \in \iota(a \otimes b)$. Then, $a \otimes b \in H$ and by Lemma 3.16(i), there exist $F, G \in \mathcal{F}(L)$ such that $a \in F$, i.e. $F \in \iota(a)$ and $b \in G$, i.e. $G \in \iota(b)$ and $\mathcal{R}_{\mathbf{B}}(F,G,H)$.

We show $\iota(a) \otimes' \iota(b) \subseteq \iota(a \otimes b)$. Let $H \in \iota(a) \otimes' \iota(b)$. Then, for some $F \in \iota(a)$ and $G \in \iota(b)$ we have $\mathcal{R}_{\mathbf{B}}(F,G,H)$. In particular, $a \in F$, $b \in G$, so $a \otimes b \in H$, by definition of $\mathcal{R}_{\mathbf{B}}$. Hence, $H \in \iota(a \otimes b)$.

For $a \backslash b$ and a / b we prove analogously, using (ii) and (iii) of Lemma 3.16 and Lemma 3.15.

Let $a \vee b \neq \infty$. We show $\iota(a \vee b) \subseteq \iota(a) \cup \iota(b)$. Let $H \in \iota(a \vee b)$, then $a \vee b \in H$. Since H is a prime filter, $a \in H$ or $b \in H$. Hence, $H \in \iota(a)$ or $H \in \iota(b)$. Conversely, let $a \in H$ or $b \in H$. Then, $a \vee b \in H$, by (F1). So, $\iota(a) \cup \iota(b) \subseteq \iota(a \vee b)$.

Let $a \wedge b \neq \infty$. Let $H \in \iota(a \wedge b)$. Then, $a \in H$ and $b \in H$, by (F1). Hence, $H \in \iota(a)$ and $H \in \iota(b)$, i.e. $H \in \iota(a)$. Conversely, let $H \in \iota(a)$. Then, by (F2'), $a \wedge b \in H$, so $H \in \iota(a \wedge b)$. □

The following theorem allows us to identify the partial residuated Boolean algebras. Its proof is a merge of the proofs from [9] and [10]. We skip identical parts and we focus on nontrivial differences.

Theorem 3.19. *Let $\mathbf{B} = (B, \otimes, \backslash, /, \vee, \wedge, \neg, 1, \top, \bot, \leq)$ be a partial structure such that $\neg a \neq \infty$, $\neg a \in B$, $a \vee \neg a = \top$, $a \wedge \neg a = \bot$ and $1 \otimes a = a = a \otimes 1$ for all $a \in B$. Then, $\mathbf{B}$ is a partial unital residuated Boolean algebra if, and only if, it is a partial bounded lattice and there exists a set $\mathscr{F}$ of prime filters of $\mathbf{B}$ and a set $\mathscr{I} \subseteq \mathscr{F}$ such that $1 \in F$ for all $F \in \mathscr{I}$ such that the following conditions hold:*

(S) $$\underset{a,b \in L}{\forall} \left(\text{if } a \nleq b, \text{ then } \underset{F \in \mathscr{F}}{\exists} a \in F \text{ and } b \notin F \right)$$

(M⊗) $$\underset{H \in \mathscr{F}}{\forall} \underset{a,b \in L}{\forall} \left(\text{if } a \otimes b \in H, \text{ then } \underset{F,G \in \mathscr{F}}{\exists} a \in F \text{ and } b \in G \text{ and } \mathcal{R}_{\mathbf{L}}(F,G,H) \right)$$

(M\\) $$\underset{G \in \mathscr{F}}{\forall} \underset{a,b \in L}{\forall} \Big(\text{if } a \backslash b \neq \infty \text{ and } a \backslash b \notin G, \text{ then } \underset{F,H \in \mathscr{F}}{\exists} a \in F \text{ and } b \notin H \text{ and } \mathcal{R}_{\mathbf{L}}(F,G,H) \Big)$$

(M/) $$\underset{F \in \mathscr{F}}{\forall} \underset{a,b \in L}{\forall} \Big(\text{if } a / b \neq \infty \text{ and } a / b \notin F, \text{ then } \underset{G,H \in \mathscr{F}}{\exists} a \in G \text{ and } b \notin H \text{ and } \mathcal{R}_{\mathbf{L}}(F,G,H) \Big)$$

(M1) $$\underset{F \in \mathscr{F}}{\forall} \underset{G_1, G_2 \in \mathscr{I}}{\exists} \left(\mathcal{R}_{\mathbf{L}}(F, G_1, F) \text{ and } \mathcal{R}_{\mathbf{L}}(G_2, F, F) \right)$$

Proof. Let $\mathbf{B} = (B, \otimes, \backslash, /, \vee, \wedge, \neg, 1, \top, \bot, \leq)$ be a partial unital residuated Boolean algebra and let $\mathbf{A} = (A, \otimes', \backslash', /', \vee', \wedge', \neg', 1', \top', \bot', \leq')$ be a total unital residuated Boolean algebra and let ι be an embedding of $\mathbf{B}$ into $\mathbf{A}$. We show that there exists a set $\mathscr{F}$ of prime filters of $\mathbf{B}$ that satisfies (S), (M⊗), (M\\), (M/) and (M1). We define:

$$\mathscr{F} = \{ \iota^{-1}(F) : F \text{ is a prime filter of } \mathbf{A} \}$$

For better readability we use the following notion: let F be a prime filter of $\mathbf{A}$, then $F_\iota = \iota^{-1}(F)$. We prove (S), (M$\otimes$), (M$\backslash$) and (M$/$) like in [9].

We show there exists $\mathscr{I} \subseteq \mathscr{F}$ such that (M1) holds. We define:

$$\mathscr{I} = \{F \in \mathscr{F} : 1 \in F\}$$

Let $F_\iota \in \mathscr{F}$, then, by Proposition 3.14 there exists a prime filter G of $\mathbf{A}$ such that $1 \in G$ and $\mathcal{R}_{\mathbf{A}}(F,G,F)$. Then, $G_\iota \in \mathscr{I}$ and $\mathcal{R}_{\mathbf{B}}(F_\iota, G_\iota, F_\iota)$. Similarly, there exists H such that $H_\iota \in \mathscr{I}$ and $\mathcal{R}_{\mathbf{B}}(H_\iota, F_\iota, F_\iota)$.

Now we assume $\mathbf{B}$ is a partial structure satisfying the assumptions of the theorem. We construct the residuated Boolean algebra $\mathbf{A}$ and the embedding of $\mathbf{B}$ into $\mathbf{A}$. We see $\mathfrak{F} = (\mathscr{F}, \mathscr{I}, \mathcal{R}_{\mathbf{B}})$ satisfies (U1)–(U4). We show (U5). Let $F, H \in \mathscr{F}$ and $G \in \mathscr{I}$ be such that $\mathcal{R}_{\mathbf{B}}(F,G,H)$. Then, for all $a \in F$, since $1 \in G$, we have $a \otimes 1 \in H$, so $F \subseteq H$. Suppose there exists $a \in H$ such that $a \notin F$. Then, by (FB), $\neg a \in F$, which is impossible.

Let $\mathbf{A} = (\mathcal{P}(\mathscr{F}), \otimes, \backslash, /, \cup, \cap, \mathscr{I}, \mathscr{F}, \emptyset, \subseteq)$ be the complex algebra of $\mathfrak{F}$. We define the mapping ι for every $a \in L$ by $\iota(a) = \{F \in \mathscr{F} : a \in F\}$. We show ι is an embedding.

Let $a, b \in L$ and $a \leq b$. Then, $\iota(a) \subseteq \iota(b)$, by (F1). Let $a \not\leq b$, then by (S) there exists $F \in \mathscr{F}$ such that $a \in F$ and $b \notin F$, so $\iota(a) \not\subseteq \iota(b)$. Hence $a \leq b$ iff $\iota(a) \subseteq \iota(b)$ and ι is injective.

One shows ι preserves $\otimes, \backslash, /, \vee, \wedge, \top, \bot$, analogously like in [9].

We show $\iota(1) = \mathscr{I}$. The inclusion $\mathscr{I} \subseteq \iota(1)$ is trivial, since 1 belongs to every element of $\mathscr{I}$. Let $F \in \iota(1)$. By (M1), there exists $G \in \mathscr{I}$ such that $\mathcal{R}_{\mathbf{B}}(F,G,F)$. Since $1 \in F$, then $G \subseteq F$. Suppose $a \in F$ and $a \notin G$. Then, by (FB), $\neg a \in G$ and then $\neg a \in F$, which is impossible. So, $G = F$ and $F \in \mathscr{I}$.

Let $a \in B$, then $\iota(\neg a) = \{F \in \mathscr{F} : \neg a \in F\} = \{F \in \mathscr{F} : a \notin F\}$, by (FB). Thus, $\{F \in \mathscr{F} : a \notin F\} = \{F \in \mathscr{F} : a \in F\}^c$. □

4 The upper bound of complexity

In this section we show that the finitary consequence relation for BFNL is decidable in exponential time.

Lemma 4.1. *Let* $\mathbf{B} = (B, \otimes, \backslash, /, \vee, \wedge, \neg, 1, \top, \bot, \leq)$ *be a partial structure. We can verify whether* $\mathbf{B}$ *is a partial residuated Boolean algebra in exponential time (depending on* $|B|$*).*

By definition, $\mathbf{B}$ is a partial residuated Boolean algebra if it is embeddable in a total residuated Boolean algebra. Such a total algebra may have the same set of elements, but may also have additional elements to satisfy all the properties. Hence, to check if $\mathbf{B}$ is a partial residuated Boolean algebra by definition, we need to embed $\mathbf{B}$ in every possible total structure until we find one where all the properties of residuated Boolean algebra hold. Even with the limit on the maximal size of such a structure, it would be 2EXPTIME problem.

Hence, we use Theorem 3.19 to idenify partial residuated Boolean algebras.

Proof. We provide an algorithm to verify whether $\mathbf{B}$ is a partial residuated Boolean algebra. We follow the analogous lemma and its proof from [9].

Step 1. We check whether $\leq$ is a partial order, $\top, \bot$ are bounds and the lattice operators are compatible with $\leq$. If it fails, the algorithm stops with negative answer. It can be done in the polynomial time.

Step 2. We check whether $1 \otimes a = a$ and $a \otimes 1 = a$ for all $a \in L$. If it fails, the algorithm stops with negative answer. It can be done in the polynomial time.

Step 3. We check whether $\neg a \neq \infty$, $\neg a \in B$, $a \vee \neg a = \top$ and $a \wedge \neg a = \bot$ for all $a \in B$. If it fails, the algorithm stops with negative answer. It can be done in the polynomial time.

Step 4. We construct a descreasing sequence of families of filters $\mathscr{F}_n$. We construct the set $\mathscr{F}_0$ of all prime filters of $\mathbf{B}$. For every subset $S \subseteq B$ we check the definition of prime filter. It can be done in $\mathscr{O}(2^{2|B|})$.

We set $i = 0$.

Step 4.1 We define $\mathscr{I}_i = \{F \in \mathscr{F}_i : 1 \in F\}$. For every prime filter $F \in \mathscr{F}_i$ we check (M$\otimes$), (M$\backslash$), (M$/$) and (M1). If every of these condition holds for F, then we add F to set $\mathscr{F}_{i+1}$.

Step 4.2 If $\mathscr{F}_{i+1} = \emptyset$, then the algorithm stops with negative answer. If $\mathscr{F}_i = \mathscr{F}_{i+1}$, then the algorithm proceeds to the next step. Else, the algorithm goes back to Step 4.1 with $i+1$.

Checking conditions for arbitrary F can be done in $\mathscr{O}(2^{3|B|})$. Number of filters in $\mathscr{F}_i$ is $\mathscr{O}(2^{|B|})$. Maximal i does not exceed $2^{|B|}$. So this step can be done in $\mathscr{O}(2^{5|B|})$.

Step 5. We check (S). If (S) does not hold, then the algorithm stops with negative answer. If (S) does not hold for a family of filters, then it does not hold for any smaller family. It can be done in $\mathscr{O}(|B|^2 2^{|B|})$ time.

□

We notice that every sequent $\Gamma \Rightarrow C$ can be represented as $G \Rightarrow C$, where G is a formula arising from Γ by replacing every comma by $\otimes$, every semicolon by $\wedge$, ε by 1 and δ by $\top$. So, we consider only sequents of this form.

Let $G \Rightarrow A$ be a sequent. We define the size of $G \Rightarrow A$ as follows:

$$s(p) = 1 \qquad s(1) = 1$$

$$s(\top) = 1 \qquad s(\bot) = 1$$

$$s(A \otimes B) = s(A) + s(B) + 1$$

$$s(A \backslash B) = s(A) + s(B) + 1 \qquad s(A/B) = s(A) + s(B) + 1$$

$$s(A \wedge B) = s(A) + s(B) + 1 \qquad s(A \vee B) = s(A) + s(B) + 1$$

$$s(\neg A) = s(A) + 1 \qquad s(A \to B) = s(A) + s(B) + 1$$

$$s(G \Rightarrow A) = s(G) + s(A)$$

Definition 4.2. Let $\mathbf{A}$ be a partial residuated Boolean algebra. Let μ be a partial function from the free algebra of $\mathscr{L}$–formulas into $\mathbf{A}$. We say μ is a *valuation*, if the following conditions hold:

- $\mu(\top) = \top$, $\mu(\bot) = \bot$;
- $\mu(1) = 1$;
- if $\mu(D \otimes E) \neq \infty$, then $\mu(D) \neq \infty, \mu(E) \neq \infty$ and $\mu(D \otimes E) = \mu(D) \otimes \mu(E)$;

- if $\mu(D\backslash E) \neq \infty$, then $\mu(D) \neq \infty, \mu(E) \neq \infty$ and $\mu(D\backslash E) = \mu(D)\backslash\mu(E)$;
- if $\mu(D/E) \neq \infty$, then $\mu(D) \neq \infty, \mu(E) \neq \infty$ and $\mu(D/E) = \mu(D)/\mu(E)$;
- if $\mu(D \wedge E) \neq \infty$, then $\mu(D) \neq \infty, \mu(E) \neq \infty$ and $\mu(D \wedge E) = \mu(D) \wedge \mu(E)$;
- if $\mu(D \vee E) \neq \infty$, then $\mu(D) \neq \infty, \mu(E) \neq \infty$ and $\mu(D \vee E) = \mu(D) \vee \mu(E)$;
- if $\mu(\neg D) \neq \infty$, then $\mu(D) \neq \infty$ and $\mu(\neg D) = \neg\mu(D)$;

Let $G \Rightarrow C$ be a sequent and μ be a valuation. We say $G \Rightarrow C$ is satisfied under the valuation μ, if $\mu(G) \neq \infty$, $\mu(C) \neq \infty$ and $\mu(G) \leq \mu(C)$.

Now we are ready to prove the *EXPTIME* complexity of of the consequence relations. The following theorem was formulated in [9] in algebraic terms of satisfiability of quantifier–free first–order formulas of the language of residuated distributive lattices.

Theorem 4.3. *The finitary consequence relation of BFNL is EXPTIME.*

Proof. (1) Let $\mathscr{K}$ be the class of residuated Boolean algebras, $\Phi = \{G_1 \Rightarrow C_1, G_2 \Rightarrow C_2, \ldots, G_k \Rightarrow C_k\}$ be a set of sequents and $G \Rightarrow C$ a sequent. Let:

$$n := 2(s(G_1 \Rightarrow C_1) + s(G_2 \Rightarrow C_2) + \cdots + s(G_k \Rightarrow C_k) + s(G \Rightarrow C)) + 4.$$

We show that Φ entails $G \Rightarrow C$, if, and only if, for all $\mathbf{A} \in \mathscr{K}^P$ such that $|A| \leq n$ and all valuations μ, if all sequents from Φ are satisfied in $\mathbf{A}$ under the valuation μ and both $\mu(G)$ and $\mu(C)$ are defined, then $G \Rightarrow C$ is satisfied in $\mathbf{A}$ under the valuation μ.

(1.1) Let $\mathbf{A} \in \mathscr{K}^P$, $|A| \leq n$ and μ be a valuation. Assume all sequents from Φ are satisfied in $\mathbf{A}$ under the valuation μ and both $\mu(G)$ and $\mu(C)$ are defined, but $G \Rightarrow C$ is not satisfied, i.e. $\mu(G) \not\leq \mu(C)$. Then, for some $\mathbf{A}' \in \mathscr{K}$, we have an embedding ι of $\mathbf{A}$ into $\mathbf{A}'$. Then, $\iota(\mu(G_i)) \leq' \iota(\mu(C_i))$ for all $i = 1, \ldots, k$ and $\iota(\mu(G)) \not\leq' \iota(\mu(C))$ in $\mathbf{A}'$. Hence, for the valuation $\mu' = \iota \circ \mu$ all sequents from Φ are satisfied, but $G \Rightarrow C$ is not satisfied in $\mathbf{A}'$. Thus, Φ does not entail $G \Rightarrow C$.

(1.2) Now let $G \Rightarrow C$ not be satisfied in $\mathbf{A}' \in \mathscr{K}$ under the valuation μ', but all sequents from Φ be satisfied under μ'. We construct $\mathbf{A} \in \mathscr{K}^P$.

First, we define T as the set consisting of $1, \top, \bot$ and all subformulas of $G_1, C_1, \ldots, G_k, C_k, G, C$. We put $A = \{\mu'(D) : D \in T\} \cup \{\neg'\mu'(D) : D \in T\}$. In effect, negation is a total operation, but doing this does not change final complexity. We define partial operations as follows:

- if $D \in T$ and $D = E \otimes F$, then $\mu'(E) \otimes \mu'(F) := \mu'(E \otimes F)$;
- if $D \in T$ and $D = E\backslash F$, then $\mu'(E)\backslash\mu'(F) := \mu'(E\backslash F)$;
- if $D \in T$ and $D = E/F$, then $\mu'(E)/\mu'(F) := \mu'(E/F)$;
- if $D \in T$ and $D = E \vee F$, then $\mu'(E) \vee \mu'(F) := \mu'(E \vee F)$;
- if $D \in T$ and $D = E \wedge F$, then $\mu'(E) \wedge \mu'(F) := \mu'(E \wedge F)$;

We define $1 \otimes a := a$ and $a \otimes 1 := a$ and $\neg a := \neg' a$ and $a \vee \neg a := \top$ and $a \wedge \neg a := \bot$ for all $a \in A$.

We also define $\leq = \leq' \cap A^2$. By the construction, $|A| \leq n$ and $\mathbf{A} \in \mathscr{K}^P$. We define $\mu = \mu'_{|T}$. Clearly, μ satisfies the conditions of Definition 4.2 and $\mu(G_i) \leq \mu(C_i)$ for $i = 1, \ldots, k$ and $\mu(G) \not\leq \mu(C)$ and both $\mu(G)$ and $\mu(C)$ are defined.

(2) Thus, to verify whether $\Phi \vdash G \Rightarrow C$ we check whether $G \Rightarrow C$ is satisfied in all $\mathbf{A} \in \mathscr{K}^P$ under every valuation μ such that $|A| \leq n$ and all sequents from Φ are satisfied in $\mathbf{A}$ under μ and both $\mu(G)$ and $\mu(C)$ are defined.

We construct all partial residuated Boolean algebras with cardinality not exceeding n. Each such a structure can be encoded by matrices. Every binary operation and order is encoded by a matrix of size $\mathscr{O}(n^2)$ and negation is encoded by matrix of size $\mathscr{O}(n)$. Each entry in the matrix can take $\mathscr{O}(n)$ values (including ∞). Hence, we have $\mathscr{O}(2^{Ln^3})$ possibilities, where L is a positive integer. We check whether such a structure is a partial residuated Boolean algebra, using Lemma 4.1. This step can be done in $\mathscr{O}(2^{Ln^3}2^{5n})$.

For a given residuated Boolean algebra $\mathbf{A}$ the number of all possible valuations is $\mathscr{O}(|A|^n)$. Checking if all sequents from Φ and $G \Rightarrow C$ are satisfied under the arbitrary valuation is $\mathscr{O}(n)$. Hence, checking whether Φ entails $G \Rightarrow C$ in $\mathbf{A}$ is $\mathscr{O}(2^{n^3})$.

The time of the whole algorithm is $\mathscr{O}(2^{Ln^3}2^{5n}2^{n^3}) = \mathscr{O}(2^{(L+1)n^3+5n})$.

□

The analogous result for BFL (associative version of BFNL) does not hold. BFL is a strongly conservative extension of L and the consequence relation of L is undecidable [1].

If we exclude the constant 1 from BFNL, the result remains true. Moreover, for 1-free BFNL the lower bound of complexity of the consequence relation is also EXPTIME, since 1-free BFNL is a strongly conservative extension of 1-free DFNL which is EXPTIME-complete [9]. The lower bound of complexity for BFNL or DFNL with 1 remains an open problem.

References

[1] Wojciech Buszkowski (2005): *Lambek calculus with nonlogical axioms*. In Claudia Casadio, Philip J. Scott & Robert A. G. Seely, editors: *Language and Grammar. Studies in Mathematical Linguistics and Natural Language*, pp. 77–93.

[2] Wojciech Buszkowski (2021): *Lambek Calculus with Classical Logic*. In Roussanka Loukanova, editor: *Natural Language Processing in Artificial Intelligence—NLPinAI 2020*, Springer International Publishing, pp. 1–36, doi:10.1007/978-3-030-63787-3_1.

[3] Karel Chvalovskỳ (2015): *Undecidability of consequence relation in full non-associative Lambek calculus*. *The Journal of Symbolic Logic* 80(2), p. 567–586, doi:10.1017/jsl.2014.39.

[4] Nikolaos Galatos & Peter Jipsen (2017): *Distributive residuated frames and generalized bunched implication algebras*. *Algebra universalis* 78(3), pp. 303–336, doi:10.1007/s00012-017-0456-x.

[5] Michał Kozak (2009): *Distributive Full Lambek Calculus Has the Finite Model Property*. *Studia Logica* 91(2), pp. 201–216, doi:10.1007/s11225-009-9172-7.

[6] Joachim Lambek (1958): *The mathematics of sentence structure*. *The American Mathematical Monthly* 65(3), pp. 154–170.

[7] Joachim Lambek (1961): *On the calculus of syntactic types*. In Roman Jakobson, editor: *Structure of Language and Its Mathematical Aspects*, 12, Providence, RI: American Mathematical Society, pp. 166–178.

[8] Mati Pentus (2006): *Lambek calculus is NP-complete*. *Theoretical Computer Science* 357(1), pp. 186–201, doi:https://doi.org/10.1016/j.tcs.2006.03.018.

[9] Dmitry Shkatov & Clint J. Van Alten (2019): *Complexity of the universal theory of bounded residuated distributive lattice-ordered groupoids*. *Algebra Universalis* 80(3), doi:10.1007/s00012-019-0609-1.

[10] Clint J. van Alten (2013): *Partial algebras and complexity of satisfiability and universal theory for distributive lattices, boolean algebras and Heyting algebras*. Theoretical Computer Science 501, pp. 82–92, doi:10.1016/j.tcs.2013.05.012.

A Unified Gentzen-style Framework for Until-free LTL

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A unified Gentzen-style framework for until-free propositional linear-time temporal logic is introduced. The proposed framework, based on infinitary rules and rules for primitive negation, can handle uniformly both a single-succedent sequent calculus and a natural deduction system. Furthermore, an equivalence between these systems, alongside with proofs of cut-elimination and normalization theorems, is established.

1 Introduction

Linear-time temporal logic (LTL) and its fragments and variants have been studied extensively [29, 20, 10, 3, 4, 5, 15, 11, 14, 7, 8, 19, 16, 9]. In particular, many of Gentzen-style sequent calculi for LTL and its until-free fragment have been introduced and investigated [20, 24, 28, 32, 4, 15, 11, 14, 16]. Some natural deduction systems for LTL and its until-free fragment have also been introduced and investigated [3, 5]. This study considers the until-free propositional fragment of LTL as a target logic. A reason for considering this fragment is that it is highly compatible with Gentzen's sequent calculus and natural deduction systems, LJ and NJ, [12, 30] for intuitionistic logic. Namely, the proposed Gentzen-style sequent calculus and Gentzen-style natural deduction system for the fragment can be obtained as modified extensions of LJ and NJ, respectively.

Gentzen-style sequent calculi for LTL have been considered previously in the literature. A sequent calculus LT_ω was introduced by Kawai for first-order until-free LTL, and cut elimination and completeness were proved [20]. A 2-sequent calculus $2\mathrm{S}\omega$ for first-order until-free LTL, with a cut elimination and a completeness proved were given by Baratella and Masini [4]. An equivalence theorem between the propositional fragments of LT_ω and $2\mathrm{S}\omega$ was proved by Kamide [15], with alternative proofs of cut elimination as consequence of the equivalence theorem. Embedding-based proofs of the cut-elimination and completeness theorems for LT_ω and its propositional fragment were presented by Kamide [16]. The present study newly introduces a single-succedent version SLT_ω of LT_ω.

Gentzen-style natural deduction systems PNK and PNJ for classical and intuitionistic until-free LTLs, respectively, were introduced by Baratella and Masini [3]. PNK and PNJ were regarded as extensions of Gentzen's NK and NJ, respectively, and were called by the authors the logics of positions. A natural deduction system PLTL_{ND} was introduced by Bolotov et al. [5] for a full classical propositional LTL with the until operator U. PLTL_{ND} uses labelled formulas of the form $i : \alpha$ and a temporal induction rule concerning the next-time operator X and the "globally in the future" operator G. PNK, PNJ, and PLTL_{ND} use an induction rule and do not use infinite premise rules for temporal operators. In contrast, the proposed natural deduction system uses infinite premise rules and do not use an induction rule. By using this setting, we obtain a unified framework.

In this study, we introduce a unified Gentzen-style framework for the until-free propositional logic LTL that can handle Gentzen-style single-succedent sequent calculus and natural deduction uniformly. We obtain the equivalence among these systems and the fact that cut elimination for the single-succedent sequent calculus implies normalization for the natural deduction system.

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A unified treatment of the systems of sequent calculus and natural deduction is the main aim and the original contribution of this study because a treatment of this type for LTL has not been studied to date, instead, sequent calculus and natural deduction for LTL and its fragments have been studied separately. A uniform handling of these systems eases the import of meta-results from one formalism to another and is a clear theoretical bonus for their applications.

To address the problem of the correspondence between cut elimination and normalization, we need a Gentzen-style single-succedent sequent calculus because the cut-elimination theorem for usual Gentzen-style multiple-succedent sequent calculi for the standard classical LTL does not imply the normalization theorem for the corresponding natural deduction system. The same situation occurs when considering Gentzen's LK and NK for classical logic. On the contrary, it is known that cut elimination for the single-succedent calculus LJ implies normalization for NJ. Thus, we try to obtain an LJ-like single-succedent sequent calculus for the target logic.

To obtain a calculus of this type, we use the following temporal (single-succedent) excluded middle rule:

$$\frac{\mathrm{X}^i\neg\alpha,\Gamma\Rightarrow\gamma \quad \mathrm{X}^i\alpha,\Gamma\Rightarrow\gamma}{\Gamma\Rightarrow\gamma}\ \text{(ex-middle)}$$

where X^i is an i-times nested next-time operator. By using this rule, we can prove the law of excluded middle $\alpha\vee\neg\alpha$. The non-temporal version of this rule, which has no occurrence of X^i, was originally introduced by von Plato [25, 21]. Pursuing the idea of correspondence between cut elimination and normalization, he introduced a single-succedent sequent calculus for classical logic, proved cut elimination, and established normalization for the corresponding natural deduction system. We thus try to extend this idea to the target temporal logic. Actually, the single-succedent sequent calculus SLT_ω proposed in this study can be regarded as a temporal extension of von Plato's calculus and the cut-elimination result for SLT_ω an extension of his cut-elimination result on classical logic.

Moreover, to obtain the corresponding natural deduction system for the target logic, we use the following rules:

$$\frac{\begin{matrix}[\mathrm{X}^i\neg\alpha] & [\mathrm{X}^i\alpha]\\ \vdots & \vdots\\ \gamma & \gamma\end{matrix}}{\gamma}\ \text{(EXM)} \qquad \frac{\mathrm{X}^i\neg\alpha \quad \mathrm{X}^i\alpha}{\gamma}\ \text{(EXP)} \qquad \frac{\begin{matrix}[\mathrm{X}^i\alpha] & [\mathrm{X}^i\alpha]\\ \vdots & \vdots\\ \mathrm{X}^j\neg\gamma & \mathrm{X}^j\gamma\end{matrix}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

where (EXM) corresponds to (ex-middle). As mentioned above, the non-temporal version of (EXM), which has no occurrence of X^i, was originally introduced by von Plato [25, 21] and the non-temporal version of (EXP) and (¬I) were originally introduced by Gentzen. For more information on these rules, see [26, 27]. (EXP) has also been used by Bolotov and Shangin [6] for constructing the paracomplete logic PCont, by Kürbis and Petrukhin [23] for developing some natural deduction systems for a family of many-valued logics including N3, and by Kamide and Negri [17, 18] for formalizing Gurevich logic [13] and Nelson logic [22, 2]. Some similar rules to (EXP) were proposed by Priest [31] for constructing natural deduction systems for logics in the FDE (First Degree Entailment) family. (EXP) is regarded as a counterpart rule of (EXM) and is useful for appropriately handling natural deduction systems with negation as a primitive connective (instead of negation defined through implication and the falsity constant). The proposed natural deduction system NLT_ω in this study can thus be regarded as a modified temporal extension of von Plato's classical system with the addition of the use of (EXP) and (¬I), and the normalization result for NLT_ω an extension of the normalization result by von Plato for classical logic.

2 Sequent calculus and cut elimination

Formulas of the logic discussed in this study are constructed using countably many propositional variables, the logical connectives $\rightarrow$ (implication), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), G (globally in the future), F (eventually in the future), and X (next-time). We use small letters $p, q, \ldots$ to denote propositional variables and Greek small letters $\alpha, \beta, \ldots$ to denote formulas. We use Greek capital letters $\Gamma, \Delta, \ldots$ to denote finite (possibly empty) sets of formulas. For any $\sharp \in \{\mathrm{G}, \mathrm{F}, \mathrm{X}\}$, we use an expression $\sharp\Gamma$ to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$. The symbol $\equiv$ is used to denote definitional equality. The symbol ω is used to represent the set of natural numbers. An expression $\mathrm{X}^i\alpha$ for any $i \in \omega$ is defined inductively by $\mathrm{X}^0\alpha \equiv \alpha$ and $\mathrm{X}^{n+1}\alpha \equiv \mathrm{X}^n\mathrm{X}\alpha$. We use lower-case letters i, j and k to denote any natural numbers.

We will define Kawai's sequent calculus LT_ω [20] and a new alternative single-succedent sequent calculus SLT_ω. Prior to defining these sequent calculi, we need to define some notions and notations.

Definition 2.1 *A* sequent *for* LT_ω *is an expression of the form* $\Gamma \Rightarrow \Delta$*, and a sequent for* SLT_ω *is an expression of the form* $\Gamma \Rightarrow \gamma$ *where* γ *is a formula or the empty set. We use the expression* $L \vdash S$ *to express the fact that a sequent* S *is derivable in a sequent calculus* L*. We say that a rule* R *is* admissible *in a sequent calculus* L *if the following condition is satisfied: For any instance* $\frac{S_1 \cdots S_n}{S}$ *of* R*, if* $L \vdash S_i$ *for all* i*, then* $L \vdash S$*. The* height *of a derivation in* L *is the number of nodes in a maximal branch of a derivation minus one. A rule* R *is* height-preserving admissible *if whenever the premises* $S_1 \cdots S_n$ *are derivable with height at most* n *then also the conclusion* S *is derivable with the same bound on the derivation height. Furthermore, we say that* R *is* derivable *in* L *if there is a derivation in* L *of* S *from* $S_1, \cdots, S_n$*.*

Definition 2.2 (LT_ω) *In the following definitions, i and k represent any natural numbers.*
The initial sequents of LT_ω *are of the form* $\mathrm{X}^i p \Rightarrow \mathrm{X}^i p$ *for any propositional variable* p*.*
The structural rules of LT_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}) \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The logical rules of LT_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta}\ (\rightarrow\text{left}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \rightarrow \beta)}\ (\rightarrow\text{right}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \wedge \beta)}\ (\wedge\text{right}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}) \quad \frac{\{\, \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha \,\}_{j \in \omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright}) \quad \frac{\{\, \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta \,\}_{j \in \omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

Remark 2.3 *The calculus* LT_ω *introduced here is a slightly modified propositional version of Kawai's sequent calculus* [20] *for until-free first-order linear-time temporal logic. The following cut-elimination theorem holds for* LT_ω*. The rule* (cut) *is admissible in cut-free* LT_ω*. We will use this theorem in the following discussion. The cut-elimination theorem for (the original first-order)* LT_ω *was proved by Kawai in* [20].

Next, we introduce SLT_ω. We use the same names for the rules of SLT_ω as those of LT_ω, although the forms of the rules are different.

Definition 2.4 (SLT_ω) *In the following definitions, i and k represent any natural numbers and* γ *represents a formula or the empty set.*
The initial sequents of SLT_ω *are of the form* $\mathrm{X}^i p, \Gamma \Rightarrow \mathrm{X}^i p$ *for any propositional variable* p*.*

The structural rules of SLT_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha,\Sigma \Rightarrow \gamma}{\Gamma,\Sigma \Rightarrow \gamma}\ (\text{cut}) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\text{we-right}).$$

The logical rules of SLT_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta,\Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha{\rightarrow}\beta),\Gamma \Rightarrow \gamma}\ ({\rightarrow}\text{left}) \quad \frac{\mathrm{X}^i\alpha,\Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\rightarrow}\beta)}\ ({\rightarrow}\text{right}) \quad \frac{\Gamma \Rightarrow \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha,\Gamma \Rightarrow}\ (\neg\text{left}) \quad \frac{\mathrm{X}^i\alpha,\Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^i\neg\alpha,\Gamma \Rightarrow \gamma \quad \mathrm{X}^i\alpha,\Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle}) \quad \frac{\mathrm{X}^i\alpha,\mathrm{X}^i\beta,\Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha\wedge\beta),\Gamma \Rightarrow \gamma}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha\wedge\beta)}\ (\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha,\Gamma \Rightarrow \gamma \quad \mathrm{X}^i\beta,\Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha\vee\beta),\Gamma \Rightarrow \gamma}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \mathrm{X}^i\alpha}{\Gamma \Rightarrow \mathrm{X}^i(\alpha\vee\beta)}\ (\vee\text{right1}) \quad \frac{\Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha\vee\beta)}\ (\vee\text{right2})$$

$$\frac{\mathrm{X}^{i+k}\alpha,\Gamma \Rightarrow \gamma}{\mathrm{X}^i\mathrm{G}\alpha,\Gamma \Rightarrow \gamma}\ (\text{Gleft}) \quad \frac{\{\,\Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\,\}_{j\in\omega}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright}) \quad \frac{\{\,\mathrm{X}^{i+j}\alpha,\Gamma \Rightarrow \gamma\,\}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha,\Gamma \Rightarrow \gamma}\ (\text{Fleft}) \quad \frac{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

Proposition 2.5 *Let L be* LT_ω *or* SLT_ω*. The sequents of the form* $\mathrm{X}^i\alpha,\Gamma \Rightarrow \mathrm{X}^i\alpha$ *for any formula* α *and any natural number i are derivable in L.*

Proof. By induction on α. ■

Proposition 2.6 *The following rule is height-preserving admissible in cut-free* SLT_ω*:*

$$\frac{\Gamma \Rightarrow \gamma}{\alpha,\Gamma \Rightarrow \gamma}\ (\text{we-left}).$$

Proof. By straightforward induction on the height of the derivation since weakening is in-built in initial sequents and all the rules have an arbitrary context on the left. ■

Next, we show the cut-elimination theorem for SLT_ω using the method by Africk [1]. We also prove a theorem that establishes an equivalence between SLT_ω and LT_ω. Prior to proving these theorems, we show the following proposition and lemmas.

Proposition 2.7 *The following rule is derivable in cut-free* SLT_ω*:*

$$\frac{\mathrm{X}^i\neg\alpha,\Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{X}^i\alpha}\ (\neg\text{left}^{-1}).$$

Proof. By using (ex-middle), (we-right), and Proposition 2.5. ■

Lemma 2.8 *For any sequent* $\Gamma \Rightarrow \Delta$*, if* $\mathrm{LT}_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \Delta$*, then* $\mathrm{SLT}_\omega - (\text{cut}) \vdash \neg\Delta,\Gamma \Rightarrow$.

Proof. By induction on the derivations $\mathscr{D}$ of $\Gamma \Rightarrow \Delta$ in cut-free LT_ω. We distinguish the cases according to the last inference of $\mathscr{D}$. We show only the case of ($\vee$right) as follows. The last inference of $\mathscr{D}$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta,\mathrm{X}^i\alpha,\mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta,\mathrm{X}^i(\alpha\vee\beta)}\ (\vee\text{right}).$$

By induction hypothesis, we have $\mathrm{SLT}_\omega - (\text{cut}) \vdash \neg\mathrm{X}^i\alpha,\neg\mathrm{X}^i\beta,\neg\Delta,\Gamma \Rightarrow$. Then, we obtain the required derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ Ind.hyp.}{\neg\mathrm{X}^i\alpha,\neg\mathrm{X}^i\beta,\neg\Delta,\Gamma \Rightarrow}}{\neg\mathrm{X}^i\beta,\neg\Delta,\Gamma \Rightarrow \mathrm{X}^i\alpha}\ (\neg\text{left}^{-1})}{\neg\mathrm{X}^i\beta,\neg\Delta,\Gamma \Rightarrow \mathrm{X}^i(\alpha\vee\beta)}\ (\vee\text{right1})}{\neg\mathrm{X}^i(\alpha\vee\beta),\neg\mathrm{X}^i\beta,\neg\Delta,\Gamma \Rightarrow}\ (\neg\text{left})}{\neg\mathrm{X}^i(\alpha\vee\beta),\neg\Delta,\Gamma \Rightarrow \mathrm{X}^i\beta}\ (\neg\text{left}^{-1})$$

$$\dfrac{\dfrac{\neg\mathrm{X}^i(\alpha\vee\beta),\neg\Delta,\Gamma \Rightarrow \mathrm{X}^i\beta}{\neg\mathrm{X}^i(\alpha\vee\beta),\neg\Delta,\Gamma \Rightarrow \mathrm{X}^i(\alpha\vee\beta)}\ (\vee\text{right2})}{\neg\mathrm{X}^i(\alpha\vee\beta),\neg\mathrm{X}^i(\alpha\vee\beta),\neg\Delta,\Gamma \Rightarrow}\ (\neg\text{left})$$

where $\neg X^i(\alpha \vee \beta), \neg X^i(\alpha \vee \beta), \neg\Delta, \Gamma \Rightarrow$ is equivalent to $\neg X^i(\alpha \vee \beta), \neg\Delta, \Gamma \Rightarrow$ (because the antecedent of the sequent is a set of formulas) and ($\neg$left^{-1}) is derivable in cut-free SLT_ω by Proposition 2.7. ■

Lemma 2.9 *For any sequent* $\Gamma \Rightarrow \gamma$*, if* $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \gamma$*, then* $\mathrm{LT}_\omega \vdash \Gamma \Rightarrow \gamma$*.*

Proof. By induction on the derivations $\mathscr{D}$ of $\Gamma \Rightarrow \gamma$ in SLT_ω. We distinguish the cases according to the last inference of $\mathscr{D}$. An initial sequent of SLT_ω, i.e. of the form $X^i p, \Gamma \Rightarrow X^i p$, is derived from an initial sequent of LT_ω using weakening steps. Next, we show only the critical case of (ex-middle) as follows. The last inference of $\mathscr{D}$ is fo the form:

$$\dfrac{\begin{matrix}\vdots \\ X^i\neg\alpha, \Gamma \Rightarrow \gamma\end{matrix} \quad \begin{matrix}\vdots \\ X^i\alpha, \Gamma \Rightarrow \gamma\end{matrix}}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle}).$$

By induction hypotheses, we have $\mathrm{LT}_\omega \vdash X^i\neg\alpha, \Gamma \Rightarrow \gamma$ and $\mathrm{LT}_\omega \vdash X^i\alpha, \Gamma \Rightarrow \gamma$. We then obtain the required derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ Prop.\ 2.5 \\ X^i\alpha \Rightarrow X^i\alpha\end{matrix}}{\Rightarrow X^i\alpha, X^i\neg\alpha}\ (\neg\text{right}) \quad \begin{matrix}\vdots\ Ind.hyp. \\ X^i\neg\alpha, \Gamma \Rightarrow \gamma\end{matrix}}{\Gamma \Rightarrow \gamma, X^i\alpha}\ (\text{cut}) \quad \begin{matrix}\vdots\ Ind.hyp. \\ X^i\alpha, \Gamma \Rightarrow \gamma\end{matrix}}{\Gamma \Rightarrow \gamma}\ (\text{cut})$$

■

Theorem 2.10 (Cut elimination for $\mathbf{SLT}_\omega$) *The rule* (cut) *is admissible in cut-free* SLT_ω*.*

Proof. Suppose $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \gamma$. Then, we obtain $\mathrm{LT}_\omega \vdash \Gamma \Rightarrow \gamma$ by Lemma 2.9. Thus, we have $\mathrm{LT}_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ by the cut-elimination theorem for LT_ω [20, 15]. Thus, we obtain $\mathrm{SLT}_\omega - (\text{cut}) \vdash \neg\gamma, \Gamma \Rightarrow$ by Lemma 2.8. We thus obtain the required fact $\mathrm{SLT}_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ by applying ($\neg$left^{-1}) to $\neg\gamma, \Gamma \Rightarrow$, where ($\neg$left^{-1}) is derivable in cut-free SLT_ω by Proposition 2.7. ■

Theorem 2.11 (Equivalence between $\mathbf{SLT}_\omega$ and $\mathbf{LT}_\omega$) *For any formula* α*,* $\mathrm{SLT}_\omega \vdash \Rightarrow \alpha$ *iff* $\mathrm{LT}_\omega \vdash \Rightarrow \alpha$*.*

Proof. ($\Longrightarrow$): By Lemma 2.9. ($\Longleftarrow$): Suppose $\mathrm{LT}_\omega \vdash \Rightarrow \alpha$. Then, we obtain $\mathrm{LT}_\omega - (\text{cut}) \vdash \Rightarrow \alpha$ by the cut-elimination theorem for LT_ω [20, 15]. We then obtain $\mathrm{SLT}_\omega - (\text{cut}) \vdash \neg\alpha \Rightarrow$ by Lemma 2.8. Thus, we obtain the required fact $\mathrm{SLT}_\omega \vdash \Rightarrow \alpha$ by applying ($\neg$left^{-1}) to $\neg\alpha \Rightarrow$, where ($\neg$left^{-1}) is derivable in cut-free SLT_ω by Proposition 2.7. ■

3 Natural deduction

As usual in the definition of a natural deduction system, the notation $[\alpha]$ denotes that the formula α is a discharged assumption by the underlying logical inference rule.

We define a Gentzen-style natural deduction system NLT_ω for until-free propositional LTL.

Definition 3.1 ($\mathbf{NLT}_\omega$) *Let i and k be any natural numbers. The logical rules of* NLT_ω *are of the following form, where in* ($\to$I) *the discharge can be vacuous:*

$$\dfrac{\begin{matrix}[X^i\alpha] \\ \vdots \\ X^i\beta\end{matrix}}{X^i(\alpha\to\beta)}\ (\to\text{I}) \qquad \dfrac{X^i(\alpha\to\beta) \quad X^i\alpha}{X^i\beta}\ (\to\text{E}) \qquad \dfrac{X^i\neg\alpha \quad X^i\alpha}{\gamma}\ (\text{EXP}) \qquad \dfrac{\begin{matrix}[X^i\neg\alpha] \\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[X^i\alpha] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\text{EXM})$$

$$\dfrac{\begin{matrix}[\mathrm{X}^i\alpha] & [\mathrm{X}^i\alpha]\\ \vdots & \vdots \\ \mathrm{X}^j\neg\gamma & \mathrm{X}^j\gamma\end{matrix}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I}) \quad \dfrac{\mathrm{X}^i\alpha \quad \mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\wedge\beta)}\ (\wedge\mathrm{I}) \quad \dfrac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\alpha}\ (\wedge\mathrm{E1}) \quad \dfrac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\beta}\ (\wedge\mathrm{E2})$$

$$\dfrac{\mathrm{X}^i\alpha}{\mathrm{X}^i(\alpha\vee\beta)}\ (\vee\mathrm{I1}) \quad \dfrac{\mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\vee\beta)}\ (\vee\mathrm{I2}) \quad \dfrac{\begin{matrix} & [\mathrm{X}^i\alpha] & [\mathrm{X}^i\beta]\\ & \vdots & \vdots \\ \mathrm{X}^i(\alpha\vee\beta) & \gamma & \gamma\end{matrix}}{\gamma}\ (\vee\mathrm{E})$$

$$\dfrac{\{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI}) \quad \dfrac{\mathrm{X}^i\mathrm{G}\alpha}{\mathrm{X}^{i+k}\alpha}\ (\mathrm{GE}) \quad \dfrac{\mathrm{X}^{i+k}\alpha}{\mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{FI}) \quad \dfrac{\begin{matrix} & [\mathrm{X}^{i+j}\alpha] \\ & \vdots \\ \mathrm{X}^i\mathrm{F}\alpha & \{\ \gamma\ \}_{j\in\omega}\end{matrix}}{\gamma}\ (\mathrm{FE}).$$

Remark 3.2 (EXP), (EXM)*, and* (¬I) *are characteristic rules in* NLT_ω*. The rule* (EXP) *and* (¬I) *are temporal generalizations of the original rules introduced by Gentzen. The rule* (EXM) *is a temporal generalization of the original rule introduced by von Plato* [25, 21]*. The non-temporal versions of* (EXP), (EXM)*, and* (¬I) *were also used by Kamide and Negri in* [18] *for constructing natural deduction systems for logics with strong negation. Using* (EXP) *and* (EXM)*, we can prove the formulas of the form* $(\neg\alpha\wedge\alpha)\rightarrow\gamma$ *and* $\neg\alpha\vee\alpha$*, respectively. Using* (¬I) *and* (EXP)*, we can prove the formulas of the form* $\alpha\rightarrow\neg\neg\alpha$ *and* $\neg\neg(\alpha\rightarrow\alpha)$ *by:*

$$\dfrac{\dfrac{\dfrac{[\neg\alpha]^2 \quad [\alpha]^1}{\neg\alpha}\ (\mathrm{EXP}) \quad \dfrac{[\neg\alpha]^2 \quad [\alpha]^1}{\alpha}\ (\mathrm{EXP})}{\neg\neg\alpha}\ (\neg\mathrm{I})^2}{\alpha\rightarrow\neg\neg\alpha}\ (\rightarrow\mathrm{I})^1 \qquad \dfrac{\dfrac{\dfrac{[\alpha]^3}{\alpha\rightarrow\alpha}\ (\rightarrow\mathrm{I})^3 \quad [\neg(\alpha\rightarrow\alpha)]^1}{\alpha\rightarrow\alpha}\ (\mathrm{EXP}) \quad \dfrac{\dfrac{[\alpha]^2}{\alpha\rightarrow\alpha}\ (\rightarrow\mathrm{I})^2 \quad [\neg(\alpha\rightarrow\alpha)]^1}{\neg(\alpha\rightarrow\alpha)}\ (\mathrm{EXP})}{\neg\neg(\alpha\rightarrow\alpha)}\ (\neg\mathrm{I})^1.$$

Next, we define some notions for NLT_ω.

Definition 3.3 *The rules* (→I), (∧I), (∨I1), (∨I2), (¬I), (GI), (FI)*, and* (EXM) *are called* introduction rules*, and the rules* (→E), (∧E1), (∧E2), (∨E), (GE), (FE)*, and* (EXP) *are called* elimination rules*. The notions of* major *and* minor premises *of the rules without* (EXM) *and* (EXP) *are defined as usual. If* $\mathrm{X}^i\neg\alpha$ *and* $\mathrm{X}^i\alpha$ *are both premises of* (EXP)*, then* $\mathrm{X}^i\neg\alpha$ *and* $\mathrm{X}^i\alpha$ *are called the major and minor premises of* (EXP)*, respectively. The notions of* derivation*,* (open and discharged) assumptions *of a derivation, and* end-formula *of a derivation are also defined as usual. For a derivation* $\mathscr{D}$*, we use the expression* $\mathrm{oa}(\mathscr{D})$ *to denote the set of open assumptions of* $\mathscr{D}$ *and the expression* $\mathrm{end}(\mathscr{D})$ *to denote the end-formula of* $\mathscr{D}$*. A formula* α *is said to be* provable *in a natural deduction system L if there exists a derivation of L with no open assumption whose end-formula is* α*.*

Remark 3.4 *There are no notions of major and minor premises of* (EXM) *and* (¬I)*. Namely, the premises of* (EXM) *and* (¬I) *are neither major nor minor premises. In this study,* (EXP) *is treated as an elimination rule, and* (EXM) *is treated as an introduction rule.*

Next, we define a reduction relation $\gg$ on the set of derivations in NLT_ω. Prior to defining $\gg$, we define some notions concerning $\gg$.

Definition 3.5 *Let* α *be a formula occurring in a derivation* $\mathscr{D}$ *in* NLT_ω*. Then,* α *is called a* maximum formula *in* $\mathscr{D}$ *if* α *satisfies the following conditions: (1)* α *is the conclusion of an introduction rule,* (∨E)*, or* (EXP) *and (2)* α *is the major premise of an elimination rule. A derivation is said to be* normal *if it contains no maximum formula. The notion of substitution of derivations for assumptions is defined as usual. We assume that the set of derivations is closed under substitution.*

Definition 3.6 (Reduction relation) *Let γ be a maximum formula in a derivation that is the conclusion of a rule R. The definition of the reduction relation $\gg$ at γ in NLT_ω is obtained by the following conditions.*

1. *R is* $(\to\mathrm{I})$ *and γ is* $\mathrm{X}^i(\alpha\to\beta)$*:*

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathscr{D}\\ \mathrm{X}^i\beta\end{matrix}}{\mathrm{X}^i(\alpha\to\beta)}\,(\to\mathrm{I}) \qquad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\alpha\end{matrix}}{\mathrm{X}^i\beta}\,(\to\mathrm{E}) \qquad\gg\qquad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\alpha\\ \vdots\ \mathscr{D}\\ \mathrm{X}^i\beta.\end{matrix}$$

2. *R is* (EXP)*:*

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathscr{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{D}_2\\ \mathrm{X}^i\delta\end{matrix}}{\gamma}\,(\mathrm{EXP}) \quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \pi_1\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \pi_2\end{matrix}}{\pi}\,R' \qquad\gg\qquad \dfrac{\begin{matrix}\vdots\ \mathscr{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{D}_2\\ \mathrm{X}^i\delta\end{matrix}}{\pi}\,(\mathrm{EXP})$$

where R' is an arbitrary rule, and both $\mathscr{E}_1$ and $\mathscr{E}_2$ are derivations of the minor premises of R' if they exist.

3. *R is* $(\neg\mathrm{I})$*, γ is* $\mathrm{X}^i\neg\alpha$*, and β is the conclusion of* (EXP)*:*

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathscr{D}_1\\ \mathrm{X}^j\neg\delta\end{matrix} \quad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathscr{D}_2\\ \mathrm{X}^j\delta\end{matrix}}{\mathrm{X}^i\neg\alpha}\,(\neg\mathrm{I}) \quad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\alpha\end{matrix}}{\beta}\,(\mathrm{EXP}) \qquad\gg\qquad \dfrac{\begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\alpha\\ \vdots\ \mathscr{D}_1\\ \mathrm{X}^j\neg\delta\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\alpha\\ \vdots\ \mathscr{D}_2\\ \mathrm{X}^j\delta\end{matrix}}{\beta}\,(\mathrm{EXP}).$$

4. *R is* $(\neg\mathrm{I})$*, γ is* $\mathrm{X}^i\neg\delta$*, and* $\mathrm{X}^i\delta$ *is the conclusion of* (EXP)*:*

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\delta]\\ \vdots\ \mathscr{D}_1\\ \mathrm{X}^j\neg\beta\end{matrix} \quad \begin{matrix}[\mathrm{X}^i\delta]\\ \vdots\ \mathscr{D}_2\\ \mathrm{X}^j\beta\end{matrix}}{\mathrm{X}^i\neg\delta}\,(\neg\mathrm{I}) \quad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\delta\end{matrix}}{\mathrm{X}^i\delta}\,(\mathrm{EXP}) \qquad\gg\qquad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\delta\end{matrix}$$

5. *R is* (EXM) *and γ is* $\mathrm{X}^i(\gamma_1\to\gamma_2)$*,* $\mathrm{X}^i(\gamma_1\wedge\gamma_2)$*, or* $\mathrm{X}^i(\gamma_1\vee\gamma_2)$*:*

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\neg\alpha]\\ \vdots\ \mathscr{D}_1\\ \gamma\end{matrix} \quad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\mathrm{EXM}) \quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \qquad\gg\qquad \dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\neg\alpha]\\ \vdots\ \mathscr{D}_1\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad \dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\mathrm{EXM})$$

where R' is $(\to\mathrm{E})$*,* $(\wedge\mathrm{E1})$*,* $(\wedge\mathrm{E2})$*, or* $(\vee\mathrm{E})$*, and both $\mathscr{E}_1$ and $\mathscr{E}_2$ are derivations of the minor premises of R' if they exist.*

6. *R is* (EXM)*, γ is* $\mathrm{X}^i\neg\delta$*, and* $\mathrm{X}^i\delta$ *is the conclusion of* (EXP)*:*

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\neg\alpha]\\ \vdots\ \mathscr{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix} \quad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathscr{D}_2\\ \mathrm{X}^i\neg\delta\end{matrix}}{\mathrm{X}^i\neg\delta}\,(\mathrm{EXM}) \quad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\delta\end{matrix}}{\mathrm{X}^i\delta}\,(\mathrm{EXP}) \qquad\gg\qquad \begin{matrix}\vdots\ \mathscr{E}\\ \mathrm{X}^i\delta.\end{matrix}$$

7. *R is* $(\wedge\mathrm{I})$ *and* γ *is* $\mathrm{X}^i(\alpha_1\wedge\alpha_2)$*:*

$$\dfrac{\dfrac{\overset{\vdots\,\mathscr{D}_1}{\mathrm{X}^i\alpha_1}\qquad \overset{\vdots\,\mathscr{D}_2}{\mathrm{X}^i\alpha_2}}{\mathrm{X}^i(\alpha_1\wedge\alpha_2)}\,(\wedge\mathrm{I})}{\mathrm{X}^i\alpha_i}\,(\wedge\mathrm{E}i) \quad\gg\quad \overset{\vdots\,\mathscr{D}_i}{\mathrm{X}^i\alpha_i}\qquad \textit{where } i \textit{ is } 1 \textit{ or } 2.$$

8. *R is* $(\vee\mathrm{I}1)$ *or* $(\vee\mathrm{I}2)$ *and* γ *is* $\mathrm{X}^i(\alpha_1\vee\alpha_2)$*:*

$$\dfrac{\dfrac{\overset{\vdots\,\mathscr{D}}{\mathrm{X}^i\alpha_i}}{\mathrm{X}^i(\alpha_1\vee\alpha_2)}\,(\vee\mathrm{I}i)\qquad \begin{matrix}[\mathrm{X}^i\alpha_1]\\ \vdots\,\mathscr{E}_1\\ \delta\end{matrix}\qquad \begin{matrix}[\mathrm{X}^i\alpha_2]\\ \vdots\,\mathscr{E}_2\\ \delta\end{matrix}}{\delta}\,(\vee\mathrm{E}) \quad\gg\quad \begin{matrix}\vdots\,\mathscr{D}\\ \mathrm{X}^i\alpha_i\\ \vdots\,\mathscr{E}_i\\ \delta\end{matrix}\qquad \textit{where } i \textit{ is } 1 \textit{ or } 2.$$

9. *R is* $(\vee\mathrm{E})$*:*

$$\dfrac{\dfrac{\overset{\vdots\,\mathscr{D}_1}{\mathrm{X}^i(\alpha\vee\beta)}\qquad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathscr{D}_2\\ \pi\end{matrix}\qquad \begin{matrix}[\mathrm{X}^i\beta]\\ \vdots\,\mathscr{D}_3\\ \pi\end{matrix}}{\pi}\,(\vee\mathrm{E})\qquad \overset{\vdots\,\mathscr{E}_n}{\{\,\delta_n\,\}}}{\delta}\,R' \quad\gg\quad \dfrac{\overset{\vdots\,\mathscr{D}_1}{\mathrm{X}^i(\alpha\vee\beta)}\qquad \dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathscr{D}_2\\ \pi\end{matrix}\qquad \overset{\vdots\,\mathscr{E}_n}{\{\,\delta_n\,\}}}{\delta}\,R'\qquad \dfrac{\begin{matrix}[\mathrm{X}^i\beta]\\ \vdots\,\mathscr{D}_3\\ \pi\end{matrix}\qquad \overset{\vdots\,\mathscr{E}_n}{\{\,\delta_n\,\}}}{\delta}\,R'}{\delta}\,(\vee\mathrm{E})$$

where R' *is an arbitrary rule, and* $\mathscr{E}_1$, $\mathscr{E}_2$, ... , $\mathscr{E}_n$, ... *are derivations of the minor premises of* R' *if they exist.*

10. *R is* (GI) *and* γ *is* $\mathrm{X}^i\mathrm{G}\alpha$*:*

$$\dfrac{\dfrac{\overset{\vdots\,\mathscr{D}_j}{\{\,\mathrm{X}^{i+j}\alpha\,\}_{j\in\omega}}}{\mathrm{X}^i\mathrm{G}\alpha}\,(\mathrm{GI})}{\mathrm{X}^{i+k}\alpha}\,(\mathrm{GE}) \quad\gg\quad \overset{\vdots\,\mathscr{D}_k}{\mathrm{X}^{i+k}\alpha}\qquad \textit{where } k\in\omega.$$

11. *R is* (FI) *and* γ *is* $\mathrm{X}^i\mathrm{F}\alpha$*:*

$$\dfrac{\dfrac{\overset{\vdots\,\mathscr{D}_k}{\mathrm{X}^{i+k}\alpha}}{\mathrm{X}^i\mathrm{F}\alpha}\,(\mathrm{FI})\qquad \begin{matrix}[\mathrm{X}^{i+j}\alpha]\\ \vdots\,\mathscr{E}_j\\ \{\,\delta\,\}_{j\in\omega}\end{matrix}}{\delta}\,(\mathrm{FE}) \quad\gg\quad \begin{matrix}\vdots\,\mathscr{D}_k\\ \mathrm{X}^{i+k}\alpha\\ \vdots\,\mathscr{E}_k\\ \delta\end{matrix}\qquad \textit{where } k\in\omega.$$

12. *R is* (FE)*:*

$$\dfrac{\dfrac{\overset{\vdots\,\mathscr{D}}{\mathrm{X}^i\mathrm{F}\alpha}\qquad \begin{matrix}[\mathrm{X}^{i+j}\alpha]\\ \vdots\,\mathscr{D}_j\\ \{\,\pi\,\}_{j\in\omega}\end{matrix}}{\pi}\,(\mathrm{FE})\qquad \overset{\vdots\,\mathscr{E}_n}{\{\,\delta_n\,\}}}{\delta}\,R' \quad\gg\quad \dfrac{\overset{\vdots\,\mathscr{D}}{\mathrm{X}^i\mathrm{F}\alpha}\qquad \dfrac{\begin{matrix}[\mathrm{X}^{i+j}\alpha]\\ \vdots\,\mathscr{D}_j\\ \pi\end{matrix}\qquad \overset{\vdots\,\mathscr{E}_n}{\{\,\delta_n\,\}}}{\{\,\delta\,\}_{j\in\omega}}\,R'}{\delta}\,(\mathrm{FE})$$

where R' *is an arbitrary rule, and* $\mathscr{E}_1$, $\mathscr{E}_2$, ... , $\mathscr{E}_n$, ... *are derivations of the minor premises of* R' *if they exist.*

13. *The set of derivations are closed under* $\gg$*.*

Definition 3.7 *If* $\mathscr{D}'$ *is obtained from* $\mathscr{D}$ *by the reduction relation of Definition 3.6, we write* $\mathscr{D}\gg\mathscr{D}'$*. A sequence* $\mathscr{D}_0,\mathscr{D}_1,\ldots$ *of derivations is called a* reduction sequence *if it satisfies the following conditions: (1)* $\mathscr{D}_i\gg\mathscr{D}_{i+1}$ *for all* $i\geq 0$*, and (2) the last derivation in the sequence is normal if the sequence is finite. A derivation* $\mathscr{D}$ *is called* normalizable *if there is a finite reduction sequence starting from* $\mathscr{D}$*.*

4 Equivalence and normalization

In the following discussion, a derivation of $\Gamma \Rightarrow$ in SLT_ω is interpreted as a derivation $\mathscr{D}$ in NLT_ω such that $\mathrm{oa}(\mathscr{D}) = \Gamma$ and $\mathrm{end}(\mathscr{D}) = \neg p \wedge p$.

Lemma 4.1 *We have the following statements.*

1. *If $\mathscr{D}$ is a derivation in NLT_ω such that $\mathrm{oa}(\mathscr{D}) = \Gamma$ and $\mathrm{end}(\mathscr{D}) = \beta$, then $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \beta$,*
2. *If $\mathrm{SLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \beta$, then we obtain a derivation $\mathscr{D}'$ in NLT_ω such that (a) $\mathrm{oa}(\mathscr{D}') = \Gamma$, (b) $\mathrm{end}(\mathscr{D}') = \beta$, and (c) $\mathscr{D}'$ is normal.*

Proof.

1. We prove 1 by induction on the derivations $\mathscr{D}$ of NLT_ω such that $\mathrm{oa}(\mathscr{D}) = \Gamma$ and $\mathrm{end}(\mathscr{D}) = \beta$. We distinguish the cases according to the last inference of $\mathscr{D}$. We show some cases. Observe that we shall use (we-left), which is admissible by Proposition 2.6.

   (a) Case ($\rightarrow$I): We show only the following subcase, which has no discharged assumption $[\mathrm{X}^i\alpha]$. $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}\\ \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i(\alpha\rightarrow\gamma)}\ (\rightarrow\mathrm{I})$$

   where $\mathrm{oa}(\mathscr{D}) = \Gamma$ and $\mathrm{end}(\mathscr{D}) = \gamma$. By induction hypothesis, we have $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\gamma$. Then, we obtain that $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i(\alpha\rightarrow\gamma)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \Gamma \Rightarrow \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma}\ (\text{we-left})}{\Gamma \Rightarrow \mathrm{X}^i(\alpha\rightarrow\gamma)}\ (\rightarrow\text{right}).$$

   (b) Case ($\neg$I): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_1\\ \vdots\ \mathscr{D}_1\\ \mathrm{X}^j\neg\gamma\end{array}\qquad \begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_2\\ \vdots\ \mathscr{D}_2\\ \mathrm{X}^j\gamma\end{array}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

   where $\mathrm{oa}(\mathscr{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathscr{D}) = \mathrm{X}^i\neg\alpha$. By induction hypotheses, we have $\mathrm{SLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_1 \Rightarrow \mathrm{X}^j\neg\gamma$ and $\mathrm{SLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \mathrm{X}^j\gamma$. Then, we obtain that $\mathrm{SLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \mathrm{X}^i\neg\alpha$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \mathrm{X}^i\alpha,\Gamma_1 \Rightarrow \mathrm{X}^j\neg\gamma\end{array}\qquad \dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \mathrm{X}^i\alpha,\Gamma_2 \Rightarrow \mathrm{X}^j\gamma\end{array}}{\mathrm{X}^j\neg\gamma, \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow}\ (\neg\text{left})}{\mathrm{X}^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow}\ (\text{cut})}{\Gamma_1, \Gamma_2 \Rightarrow \mathrm{X}^i\neg\alpha}\ (\neg\text{right}).$$

   (c) Case (EXP): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathscr{E}_1\\ \mathrm{X}^i\neg\alpha\end{array}\qquad \begin{array}{c}\Gamma_2\\ \vdots\ \mathscr{E}_2\\ \mathrm{X}^i\alpha\end{array}}{\beta}\ (\mathrm{EXP})$$

where oa($\mathscr{D}$) = $\Gamma_1 \cup \Gamma_2$ and end($\mathscr{D}$) = β. By induction hypotheses, we have $\mathrm{SLT}_\omega \vdash \Gamma_1 \Rightarrow \mathrm{X}^i\neg\alpha$ and $\mathrm{SLT}_\omega \vdash \Gamma_2 \Rightarrow \mathrm{X}^i\alpha$. Then, we obtain that $\mathrm{SLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \beta$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \Gamma_2 \Rightarrow \mathrm{X}^i\alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \Gamma_1 \Rightarrow \mathrm{X}^i\neg\alpha\end{array} \quad \dfrac{\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \mathrm{X}^i\alpha \Rightarrow}\,(\neg\text{left})}{\mathrm{X}^i\alpha, \Gamma_1 \Rightarrow}\,(\text{cut})}{\Gamma_1, \Gamma_2 \Rightarrow}\,(\text{cut})}{\Gamma_1, \Gamma_2 \Rightarrow \beta}\,(\text{we-right}).$$

(d) Case (EXM): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\Gamma_1\\ \vdots\ \mathscr{E}_1\\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_2\\ \vdots\ \mathscr{E}_2\\ \gamma\end{array}}{\gamma}\,(\text{EXM})$$

where oa($\mathscr{D}$) = $\Gamma_1 \cup \Gamma_2$ and end($\mathscr{D}$) = γ. By induction hypotheses, we have $\mathrm{SLT}_\omega \vdash \mathrm{X}^i\neg\alpha, \Gamma_1 \Rightarrow \gamma$ and $\mathrm{SLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \gamma$. Then, we obtain that $\mathrm{SLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \gamma$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \mathrm{X}^i\neg\alpha, \Gamma_1 \Rightarrow \gamma\\ \vdots\ (\text{we-left})\\ \mathrm{X}^i\neg\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \gamma\\ \vdots\ (\text{we-left})\\ \mathrm{X}^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\,(\text{ex-middle}).$$

(e) Case (GI): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_j\\ \vdots\ P_j\\ \{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}\end{array}}{\mathrm{X}^i\mathrm{G}\alpha}\,(\text{GI})$$

where oa($\mathscr{D}$) = $\Gamma = \bigcup_{j\in\omega} \Gamma_j$ and end($\mathscr{D}$) = $\mathrm{X}^i\mathrm{G}\alpha$. By induction hypotheses, we have $\mathrm{SLT}_\omega \vdash \Gamma_j \Rightarrow \mathrm{X}^{i+j}\alpha$ for all $j \in \omega$. Then, we obtain that $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \Gamma_j \Rightarrow \mathrm{X}^{i+j}\alpha\\ \vdots\ (\text{we-left})\\ \{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}\end{array}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha}\,(\text{Gright}).$$

Note that the induction hypothesis is applied for each of the denumerable set of premises.

(f) Case (FE): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma'\\ \vdots\ \mathscr{D}'\\ \mathrm{X}^i\mathrm{F}\alpha\end{array} \quad \begin{array}{c}[\mathrm{X}^{i+j}\alpha]\Gamma_j\\ \vdots\ \mathscr{D}_j\\ \{\ \gamma\ \}_{j\in\omega}\end{array}}{\gamma}\,(\text{FE})$$

where oa($\mathscr{D}$) = $\Gamma' \cup \Gamma$ with $\Gamma = \bigcup_{j\in\omega} \Gamma_j$ and end($\mathscr{D}$) = γ. By induction hypotheses, we have $\mathrm{SLT}_\omega \vdash \Gamma' \Rightarrow \mathrm{X}^i\mathrm{F}\alpha$ and $\mathrm{SLT}_\omega \vdash \mathrm{X}^{i+j}\alpha, \Gamma_j \Rightarrow \gamma$ for all $j \in \omega$. Then we obtain that $\mathrm{SLT}_\omega \vdash \Gamma', \Gamma \Rightarrow \gamma$ by the following derivation where the induction hypothesis is applied for each of the denumerable set of premises:

$$\dfrac{\begin{array}{c}\vdots\ Ind.hyp.\\ \Gamma' \Rightarrow \mathrm{X}^i\mathrm{F}\alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\ Ind.hyp.\\ \mathrm{X}^{i+j}\alpha,\Gamma_j\Rightarrow\gamma\\ \vdots\ (\text{we-left})\\ \{\,\mathrm{X}^{i+j}\alpha,\Gamma\Rightarrow\gamma\,\}_{j\in\omega}\end{array}}{\mathrm{X}^i\mathrm{F}\alpha,\Gamma\Rightarrow\gamma}\,(\text{Fleft})}{\Gamma',\Gamma\Rightarrow\gamma}\,(\text{cut}).$$

2. We prove 2 by induction on the derivations $\mathscr{D}$ of $\Gamma\Rightarrow\beta$ in SLT_ω − (cut). We distinguish the cases according to the last inference of $\mathscr{D}$. We show some cases.
   (a) Case (we-right): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathscr{D}'\\ \Gamma\Rightarrow\end{array}}{\Gamma\Rightarrow\alpha}\,(\text{we-right})$$

   By induction hypothesis, we have a normal derivation $\mathscr{E}'$ in NLT_ω of the form:

$$\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}'\\ \neg p\wedge p\end{array}$$

   where $\mathrm{oa}(\mathscr{E}')=\Gamma$ and $\mathrm{end}(\mathscr{E}')=\neg p\wedge p$. Then, we obtain a required normal derivation $\mathscr{E}$ by:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}'\\ \neg p\wedge p\end{array}}{\neg p}\,(\wedge\mathrm{E1}) \quad \dfrac{\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}'\\ \neg p\wedge p\end{array}}{p}\,(\wedge\mathrm{E2})}{\alpha}\,(\mathrm{Exp})$$

   where $\mathrm{oa}(\mathscr{E})=\Gamma$ and $\mathrm{end}(\mathscr{E})=\alpha$.
   (b) Case ($\neg$left): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathscr{D}'\\ \Gamma\Rightarrow\mathrm{X}^i\alpha\end{array}}{\mathrm{X}^i\neg\alpha,\Gamma\Rightarrow}\,(\neg\text{left}).$$

   By induction hypothesis, we have a normal derivation $\mathscr{E}'$ in NLT_ω of the form:

$$\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}'\\ \mathrm{X}^i\alpha\end{array}$$

   where $\mathrm{oa}(\mathscr{E}')=\Gamma$ and $\mathrm{end}(\mathscr{E}')=\mathrm{X}^i\alpha$. Then, we obtain a required normal derivation $\mathscr{E}$ by:

$$\dfrac{\mathrm{X}^i\neg\alpha \quad \begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}'\\ \mathrm{X}^i\alpha\end{array}}{\neg p\wedge p}\,(\mathrm{EXP})$$

   where $\mathrm{oa}(\mathscr{E})=\{\mathrm{X}^i\neg\alpha\}\cup\Gamma$ and $\mathrm{end}(\mathscr{E})=\neg p\wedge p$ (i.e., $\bot$). We remark that the last inference (EXP) in $\mathscr{E}$ cannot be replaced with ($\to$E), because using ($\to$E) entails a possibility of developing a non-normal derivation. Namely, there is a possibility of the case that the last inference of $\mathscr{E}'$ is ($\to\mathrm{I}^*$).
   (c) Case (ex-middle): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1\\ \mathrm{X}^i\neg\alpha,\Gamma\Rightarrow\gamma\end{array} \quad \begin{array}{c}\vdots\ \mathscr{D}_2\\ \mathrm{X}^i\alpha,\Gamma\Rightarrow\gamma\end{array}}{\Gamma\Rightarrow\gamma}\,(\text{ex-middle}).$$

By induction hypotheses, we have normal derivations $\mathscr{E}_1$ and $\mathscr{E}_2$ in NLT_ω of the form:

$$\begin{array}{cc} \mathrm{X}^i\neg\alpha, \Gamma & \mathrm{X}^i\alpha, \Gamma \\ \vdots\ \mathscr{E}_1 & \vdots\ \mathscr{E}_2 \\ \gamma & \gamma \end{array}$$

where $\mathrm{oa}(\mathscr{E}_1) = \{\mathrm{X}^i\neg\alpha\} \cup \Gamma$, $\mathrm{oa}(\mathscr{E}_2) = \{\mathrm{X}^i\alpha\} \cup \Gamma$, $\mathrm{end}(\mathscr{E}_1) = \gamma$, and $\mathrm{end}(\mathscr{E}_2) = \gamma$. Then, we obtain a required normal derivation $\mathscr{E}$ by:

$$\dfrac{\begin{array}{cc} [\mathrm{X}^i\neg\alpha]\Gamma & [\mathrm{X}^i\alpha]\Gamma \\ \vdots\ \mathscr{E}_1 & \vdots\ \mathscr{E}_2 \\ \gamma & \gamma \end{array}}{\gamma}\ (\mathrm{EXM})$$

where $\mathrm{oa}(\mathscr{E}) = \Gamma$ and $\mathrm{end}(\mathscr{E}) = \gamma$.

(d) Case (Fleft): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathscr{D}' \\ \{\ \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma\ \}_{j\in\omega} \end{array}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \gamma}\ (\mathrm{Fleft}).$$

By induction hypotheses, we have normal derivations $\mathscr{E}_j$ for all $j \in \omega$ in NLT_ω of the form:

$$\begin{array}{c} \mathrm{X}^{i+j}\alpha \quad \Gamma_j \\ \vdots\ \mathscr{E}_j \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathscr{E}_j) = \{\mathrm{X}^{i+j}\alpha\} \cup \Gamma_j$ with $\Gamma = \bigcup_{j\in\omega} \Gamma_j$ and $\mathrm{end}(\mathscr{E}_j) = \gamma$. Then, we obtain a required normal derivation $\mathscr{E}$ by:

$$\dfrac{\mathrm{X}^i\mathrm{F}\alpha \qquad \begin{array}{c} [\mathrm{X}^{i+j}\alpha] \quad \Gamma_j \\ \vdots\ \mathscr{E}_j \\ \{\ \gamma\ \}_{j\in\omega} \end{array}}{\gamma}\ (\mathrm{FE})$$

where $\mathrm{oa}(\mathscr{E}) = \{\mathrm{X}^i\mathrm{F}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathscr{E}) = \gamma$.

■

Theorem 4.2 (Equivalence between NLT_ω and SLT_ω) *For any formula α, $\mathrm{SLT}_\omega \vdash\ \Rightarrow \alpha$ iff α is derivable in NLT_ω.*

Proof. Taking $\emptyset$ as Γ in Lemma 4.1, we obtain the required fact. ■

Theorem 4.3 (Normalization for NLT_ω) *All derivations in NLT_ω are normalizable. More precisely, if a derivation $\mathscr{D}$ in NLT_ω is given, then we obtain a normal derivation $\mathscr{E}$ in NLT_ω such that $\mathrm{oa}(\mathscr{E}) = \mathrm{oa}(\mathscr{D})$ and $\mathrm{end}(\mathscr{E}) = \mathrm{end}(\mathscr{D})$.*

Proof. Suppose that a derivation $\mathscr{D}$ in NLT_ω is given, and suppose that $\mathrm{oa}(\mathscr{D}) = \Gamma$ and $\mathrm{end}(\mathscr{D}) = \beta$. Then, by Lemma 4.1 (1), we obtain $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \beta$. By the cut-elimination theorem for SLT_ω, we obtain $\mathrm{SLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \beta$. Then, by Lemma 4.1 (2), we obtain a normal derivation Q in NLT_ω such that $\mathrm{oa}(\mathscr{E}) = \mathrm{oa}(\mathscr{D})$ and $\mathrm{end}(\mathscr{E}) = \mathrm{end}(\mathscr{D})$. ■

5 Concluding remarks and acknowledgments

In this paper we introduced a unified Gentzen-style framework for the until-free propositional logic LTL. In this framework, based on infinitary rules and rules for primitive negation, sequent calculus and natural deduction can be treated in a uniform way, that eases a proof of their deductive equivalence and a proof of normalization for the natural deduction system. More specifically, natural deduction derivations are translated to sequent calculus derivations with cuts, and cut-free derivations are translated to normal derivations in natural deduction. In this way, cut elimination provides the bridge to an indirect proof normalization. In future work, we plan to improve the correspondence between cut elimination and normalization to a bi-directional one with the use of general elimination rules (as in [21, Chapter 8]). This should also address a question posed by one of the referees (who are gratefully acknowledged for their valuable comments) on the correspondence between steps of cut elimination and reduction steps in a normalization sequence. Other desiderata for further work include a direct proof of normalization, and an inquiry on strong normalization and the Church-Rosser theorem.

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References

[1] Henry Africk (1992): *Classical logic, intuitionistic logic, and the Peirce rule*. *Notre Dame Journal of Formal Logic* 33 (2), pp. 229-235, doi:10.1305/ndjfl/1093636101.

[2] Ahmad Almukdad & David Nelson (1984): *Constructible falsity and inexact predicates*. *Journal of Symbolic Logic* 49 (1), pp. 231-233, doi:10.2307/2274105.

[3] Stefano Baratella & Andrea Masini (2003): *A proof-theoretic investigation of a logic of positions*. *Annals of Pure and Applied Logic* 123, pp. 135-162, doi:10.1016/S0168-0072(03)00021-6.

[4] Stefano Baratella & Andrea Masini (2004): *An approach to infinitary temporal proof theory*. *Archive for Mathematical Logic* 43 (8), pp. 965-990, doi:10.1007/S00153-004-0237-Z.

[5] Alexander Bolotov, Artie Basukoski, Oleg M. Grigoriev & Vasilyi Shangin (2006): *Natural deduction calculus for linear-time temporal logic*. *Proceedings of the 10th European Conference on Logics in Artificial Intelligence (JELIA 2006), Lecture Notes in Computer Science 4160*, pp. 56-68, doi:10.1007/11853886_7.

[6] Alexander Bolotov & Vasilyi Shangin (2012): *Natural deduction system in paraconsistent setting: Proof search for PCont*. *Journal of Intelligent Systems* 21 (1), pp. 1-24, doi:10.1515/JISYS-2011-0021.

[7] Bianca Boretti & Sara Negri (2009): *Decidability for Priorean linear time using a fixed-point labelled calculus*. *Proceedings of the 18th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX), Lecture Notes in Computer Science 5607*, pp. 108-122, doi:10.1007/978-3-642-02716-1_9.

[8] Bianca Boretti & Sara Negri (2010): *On the finitization of Priorean linear time*. In D'Agostino et al., editor: *New Essays in Logic and Philosophy of Science*, College Publications, London.

[9] Serenella Cerrito, Valentin Goranko & Sophie Paillocher (2023): *Partial model checking and partial model synthesis in LTL using a Tableau-based approach*. Proceedings of the 8th International Conference on Formal Structures for Computation and Deduction (FSCD), pp. 23:1-23:21, doi:10.4230/LIPICS.FSCD.2023.23.

[10] E. Allen Emerson (1990): *Temporal and modal logic*. In: Handbook of Theoretical Computer Science, Formal Models and Semantics (B), Jan van Leeuwen (Ed.), pp. 995-1072, Elsevier and MIT Press, doi:10.1016/B978-0-444-88074-1.50021-4.

[11] Joxe Gaintzarain, Montserrat Hermo, Paqui Lucio, Marisa Navarro & Fernando Orejas (2007): *A cut-free and invariant-free sequent calculus for PLTL*. Proceedings of the 21st International Workshop on Computer Science Logic, Lecture Notes in Computer Science 4646, pp. 481-495, doi:10.1007/978-3-540-74915-8_36.

[12] Gerhard Gentzen (1969): *Collected papers of Gerhard Gentzen*. M.E. Szabo (ed.), Studies in logic and the foundations of mathematics, North-Holland (English translation, doi:10.2307/2272429.

[13] Yuri Gurevich (1977): *Intuitionistic logic with strong negation*. Studia Logica 36, pp. 49–59, doi:10.1007/BF02121114.

[14] Kai Brünnler & Martin Lange (2008): *Cut-free sequent systems for temporal logic*. Journal of Logic and Algebraic Methods in Programming 76 (2), pp. 216-225, doi:10.1016/J.JLAP.2008.02.004.

[15] Norihiro Kamide (2006): *An equivalence between sequent calculi for linear-time temporal logic*. Bulletin of the Section of the Logic 35(4), pp. 187–194.

[16] Norihiro Kamide (2015): *Embedding theorems for LTL and its variants*. Mathematical Structures in Computer Science 25(1), pp. 83–134, doi:10.1017/S0960129514000048.

[17] Norihiro Kamide (2023): *Natural deduction with explosion and excluded middle*. Proceedings of the 53rd IEEE International Symposium on Multiple-valued Logic (ISMVL 2023), pp. 24–29, doi:10.1109/ISMVL57333.2023.00016.

[18] Norihiro Kamide & Sara Negri (2024): *Unified natural deduction for logics of strong negation*. Draft.

[19] Norihiro Kamide & Heinrich wansing (2011): *A paraconsistent linear-time temporal logic*. Fundamenta Informaticae 106 (1), pp. 1-23, doi:10.3233/FI-2011-374.

[20] Hiroya Kawai (1987): *Sequential calculus for a first order infinitary temporal logic*. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 33, pp. 423-432, doi:10.1002/MALQ.19870330506.

[21] Sara Negri & Jan von Plato (2001): *Structural Proof Theory*. Cambridge University Press, doi:10.1017/CBO9780511527340.

[22] David Nelson (1949): *Constructible falsity*. Journal of Symbolic Logic 14, pp. 16-26, doi:10.2307/2268973.

[23] Nils Kürbis & Yaroslav Petrukhin (2021): *Normalisation for some quite interesting many-valued logics*. Logic and Logical Philosophy 30(3), pp. 493–534, doi:10.12775/LLP.2021.009.

[24] Barbara Paech (1988): *Gentzen-systems for propositional temporal logics*. Lecture Notes in Computer Science 385, pp. 240-253, doi:10.1007/BFB0026305.

[25] Jan von Plato (1999): *Proof theory of full classical propositional logic*. Manuscript, 16 pages.

[26] Jan von Plato (2014): *Elements of Logical Reasoning*. Cambridge University Press, doi:10.1017/CBO9781139567862.

[27] Jan von Plato (2017): *Saved from the Cellar: Gerhard Gentzen's Shorthand Notes on Logic & Foundations of Mathematics*. Springr.

[28] Regimantas Pliuškevičius (1991): *Investigation of finitary calculus for a discrete linear time logic by means of infinitary calculus*. Lecture Notes in Computer Science 502, pp. 504-528, doi:10.1007/BFB0019366.

[29] Amir Pnueli (1977): *The temporal logic of programs*. Proceedings of the 18th IEEE Symposium on Foundations of Computer Science, pp. 46-57, doi:10.1109/SFCS.1977.32.

[30] Dag Prawitz (1965): *Natural deduction: a proof-theoretical study*. *Almqvist and Wiksell, Stockholm*, doi:10.2307/2271676.

[31] G. Priest (2019): *Natural deduction systems for logics in the FDE family*. *New Essays on Belnap–Dunn Logic (Synthese Library 418), pp. 279-292*, doi:10.1007/978-3-030-31136-0_16.

[32] Manfred E. Szabo (1980): *A sequent calculus for Kröger logic*. *Lecture Notes in Computer Science 148, pp. 295-303*, doi:10.1007/3-540-11981-7_21.

Kamide is in America, Moisil and Leitgeb are in Australia*

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It is not uncommon for a logic to be invented multiple times, hinting at its robustness. This trend is followed also by the expansion **BD+** of Belnap-Dunn logic by Boolean negation. Ending up in the same logic, however, does not mean that the semantic interpretations are always the same as well. In particular, different interpretations can bring us to different logics, once the basic setting is moved from a classical one to an intuitionistic one. For **BD+**, two such paths seem to have been taken; one (**BDi**) by N. Kamide along the so-called American plan, and another (**HYPE**) by G. Moisil and H. Leitgeb along the so-called Australian plan. The aim of this paper is to better understand this divergence. This task is approached mainly by (i) formulating a semantics for first-order **BD+** that provides an Australian view of the system; (ii) showing connections of the less explored (first-order) **BDi** with neighbouring systems, including an intermediate logic and variants of Nelson's logics.

1 Introduction

Since the birth of modern logic, with an enormous help from mathematical tools, we have seen many important and interesting formal theories being developed. Among the vast number of formal theories in the literature, those that are based on classical logic and intuitionistic logic have been particularly successful and explored in great depth.

Soon after the initial developments of intuitionistic logic and theories based on it, there were a number of attempts in comparing the theories based on classical logic and theories based on intuitionistic logic. These comparisons, in many cases, are highly non-trivial, and sometimes even surprising. For example, take one of the most famous modal logic **S5**. Then, it turns out that there are *uncountably* many systems of intuitionistic version of **S5** that will all collapse into classical **S5** once one of the familiar formulas (e.g. the law of excluded middle, elimination of double negation, or Peirce's law, and others) are added to the intuitionistic versions (cf. [29, Corollary 2.4]). Corresponding intuitionistic versions, therefore, of various formal theories may come along with a lot of surprising results, and also seem to bring us some new insights towards a deeper understanding of theories based on classical logic.

In the present article, we will focus on the system **BDi** developed by Norihiro Kamide in [16]. In brief, **BDi** is an intuitionistic version of the system **BD+** which can be seen in at least two different ways: (i) as an expansion of classical logic by de Morgan negation, or (ii) as an expansion of **FDE** (or Belnap-Dunn logic), expanded by Boolean negation. As we shall point out later in some more details, various systems that are definitionally equivalent to the system **BD+** have been developed independently

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by various authors, and that seems to partly confirm the naturalness and importance of the system **BD+**. Therefore, Kamide's attempt of investigating the intuitionistic version of **BD+** seems to be of importance.

Furthermore, as the title may already make some of the readers guess, there are interesting ways to connect Kamide's **BDi** to yet another expansion of intuitionistic logic that has been known and studied by a few authors. Very roughly put, what is nowadays best known as **HYPE**, (re)introduced by Hannes Leitgeb in [22], though already introduced by Grigore Constantin Moisil in 1942, can be seen as another system that can be seen as an intuitionistic counterpart of **BD+** (see [10] for a detailed view of Moisil's work). Somewhat more precisely, Kamide's **BDi** can be viewed as an intuitionistic counterpart of **BD+** in light of the American plan for negation in **FDE**, while the system explored by Moisil and Leitgeb can be viewed as an intuitionistic counterpart of **BD+** in light of the Australian plan for negation in **FDE**.

Against these backgrounds, the aim of this article is twofold. First, we will clarify the relations of systems **BD+**, **HYPE**, and **BDi**. To this end, we will present another semantics for **BD+** that offers a systematic view on the systems related to **BD+**. Second, we will explore a few extensions and variations of **BDi**, and in particular, establish some basic results for the extension of **BDi** obtained by adding the *ex contradictione quodlibet*. Most of our results are obtained for the language with first-order quantifiers.

2 Semantics and proof system for BD+

The predicate language $\mathscr{L}_Q$ consists of connectives $\{\bot, \sim, \wedge, \vee, \to\}$, quantifiers $\{\forall, \exists\}$, countable sets of constants $\mathsf{Con} = \{c_1, c_2, \dots\}$, variables $\mathsf{Var} = \{v_1, v_2, \dots\}$ and n-ary predicates $\mathsf{Pred} = \{P^n_1, P^n_2, \dots : n \in \mathbb{N}\}$. A *term* is either a constant or a variable. The set of formulas in $\mathscr{L}_Q$ will be denoted by Form_Q.

2.1 Preliminaries

Let us recall the semantics in [18, Definition 18], for which we take $\bot$ and not $\neg$ as primitive here.

Definition 1. A **QBD+**-Dunn-model for the language $\mathscr{L}_Q$ is a pair $\langle D, V\rangle$ where $D \supseteq \mathsf{Con}$ is a non-empty set and we assign both the *extension* $V^+(P^n) \subseteq D^n$ and the *anti-extension* $V^-(P^n) \subseteq D^n$ to each n-ary predicate symbol P^n. Valuations V are then extended to interpretations I for all the sentences of $\mathscr{L}_Q$ (Sent_Q) expanded by D inductively as follows: as for the atomic *sentences*,

- $1 {\in} I(P^n(t_1, ..., t_n))$ iff $\langle t_1, \dots, t_n\rangle {\in} V^+(P^n)$,
- $0 {\in} I(P^n(t_1, ..., t_n))$ iff $\langle t_1, \dots, t_n\rangle {\in} V^-(P^n)$.

The rest of the clauses are as follows:

$$
\begin{array}{lll@{\quad}lll}
1 \notin I(\bot), & & & 0 \in I(\bot), & & \\
1 \in I(\sim A) & \text{iff} & 0 \in I(A), & 0 \in I(\sim A) & \text{iff} & 1 \in I(A), \\
1 \in I(A \wedge B) & \text{iff} & 1 \in I(A) \text{ and } 1 \in I(B), & 0 \in I(A \wedge B) & \text{iff} & 0 \in I(A) \text{ or } 0 \in I(B), \\
1 \in I(A \vee B) & \text{iff} & 1 \in I(A) \text{ or } 1 \in I(B), & 0 \in I(A \vee B) & \text{iff} & 0 \in I(A) \text{ and } 0 \in I(B), \\
1 \in I(A \to B) & \text{iff} & 1 \notin I(A) \text{ or } 1 \in I(B), & 0 \in I(A \to B) & \text{iff} & 0 \notin I(A) \text{ and } 0 \in I(B), \\
1 \in I(\forall x A) & \text{iff} & 1 \in I(A(d)), \text{for all } d \in D, & 0 \in I(\forall x A) & \text{iff} & 0 \in I(A(d)), \text{for some } d \in D, \\
1 \in I(\exists x A) & \text{iff} & 1 \in I(A(d)), \text{for some } d \in D, & 0 \in I(\exists x A) & \text{iff} & 0 \in I(A(d)), \text{for all } d \in D.
\end{array}
$$

Finally, let $\Gamma \cup \{A\}$ be any set of sentences. Then, A is a **BD+**-*semantic consequence* from Γ ($\Gamma \models A$) iff for all **QBD+**-Dunn-models $\langle D, V\rangle$, $1 \in I(A)$ if $1 \in I(B)$ for all $B \in \Gamma$.

Remark 2. Note that the unary operation $\neg A$ defined as $A \to \bot$ is Boolean Negation in the sense that:

- $1 \in I(\neg A)$ iff $1 \notin I(A)$, and $0 \in I(\neg A)$ iff $0 \notin I(A)$.

For a discussion on the notion of classical negation in **FDE** and their extensions, see [8].[1]

Moreover, note that we have the following equivalences.

- $1 \in I(\sim(\sim B \to \sim A))$ iff $1 \in I(A)$ and $1 \notin I(B)$, and $0 \in I(\sim(\sim B \to \sim A))$ iff $0 \in I(A)$ or $0 \notin I(B)$.

Therefore, the connective $\leftarrow$ of the system **SPL** introduced by Kamide and Wansing in [19] is definable in **BD+**. This implies that **SPL** and **BD+** are definitionally equivalent.

Remark 3. As already observed in [8, §3.5], there are a few systems in the literature that are definitionally equivalent to **BD+**. Those include, the system **PM4N** formulated in the language $\{\neg, \wedge, \vee, \Box\}$ by Jean-Yves Béziau in [6], and the system **FDEP** formulated in the language $\{\sim, \to\}$ by Dmitry Zaitsev in [43]. We already added another system **SPL** in the previous remark, and we may add another more recent rediscovery by Arnon Avron. More specifically, Avron, in [4], introduces the system **SE4** in the context of exploring expansions of **FDE** by a conditional that are self-extensional.

We now turn to the proof system, again recalling the definition and completeness theorem from [18].

Definition 4. Consider the following axioms and rules where $\neg A$ and $A \leftrightarrow B$ abbreviate $A \to \bot$ and $(A \to B) \wedge (B \to A)$ respectively:

$A \to (B \to A)$	(Ax1)
$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	(Ax2)
$((A \to B) \to A) \to A$	(Ax3)
$(A \wedge B) \to A$	(Ax4)
$(A \wedge B) \to B$	(Ax5)
$(C \to A) \to ((C \to B) \to (C \to (A \wedge B)))$	(Ax6)
$A \to (A \vee B)$	(Ax7)
$B \to (A \vee B)$	(Ax8)
$(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$	(Ax9)
$\bot \to A$	(Ax10)
$\dfrac{A \quad A \to B}{B}$	(MP)

$A(t) \to \exists x A$	(Ax11)
$\forall x(A \to B) \to (\exists y A(y) \to B)$	(Ax12)
$\forall x(B \to A) \to (B \to \forall x A)$	(Ax13)
$\forall x A \to A(t)$	(Ax14)
$A \to \sim\bot$	(Ax15)
$\sim\sim A \leftrightarrow A$	(Ax16)
$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$	(Ax17)
$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$	(Ax18)
$\sim(A \to B) \leftrightarrow (\neg\sim A \wedge \sim B)$	(Ax19)
$\sim\forall x A \leftrightarrow \exists x \sim A$	(Ax20)
$\sim\exists x A \leftrightarrow \forall x \sim A$	(Ax21)
$\dfrac{A}{\forall x A}$	(Gen)

We write $\Gamma \vdash A$ if there is a finite list $B_1, \ldots, B_n \equiv A$ such that each B_i is either an element of Γ, an instance of one of the axioms, or obtained from previous items in the list by (MP) or (Gen).

Theorem 1. For all $\Gamma \cup \{A\} \subseteq \mathsf{Sent}_Q$, $\Gamma \vdash A$ iff $\Gamma \models A$.

2.2 Another semantics

Before moving ahead, let us introduce another semantics for **BD+**.[2]

Definition 5. A **QBD+**-star-model for the language $\mathscr{L}_Q$ is a quadruple $\langle W, *, D, V \rangle$ where W is a non-empty set (of states); $*$ is a function on W with $w^{**} = w$ for all $w \in W$; $D \supseteq \mathsf{Con}$ is a non-empty set and we assign the *extension* $V(w, P^n) \subseteq D^n$ to each n-ary predicate symbol P^n and $w \in W$. Valuations V are then extended to interpretations I for all the state-sentence pairs of $\mathscr{L}$ expanded by D inductively as follows: as for the atomic *sentences*,

[1] For those who are ready to accept *non-deterministic* classical negation, see also [39].
[2] The propositional fragment is already introduced briefly in [27].

- $I(w, P^n(t_1, ..., t_n)) = 1$ iff $\langle t_1, \ldots, t_n \rangle \in V(w, P^n)$.

The rest of the clauses are as follows:

- $I(w, \bot) \neq 1$,
- $I(w, {\sim}A) = 1$ iff $I(w^*, A) \neq 1$,
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$,
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$,
- $I(w, A \to B) = 1$ iff $I(w, A) \neq 1$ or $I(w, B) = 1$,
- $I(w, \forall x A) = 1$ iff $I(w, A(d)) = 1$, for all $d \in D$,
- $I(w, \exists x A) = 1$ iff $I(w, A(d)) = 1$, for some $d \in D$.

Finally, let $\Gamma \cup \{A\}$ be any set of sentences. Then, A is a **BD+**-*star-semantic consequence* from Γ ($\Gamma \models_* A$) iff for all **QBD+**-star-models $\langle W, *, D, V \rangle$, and for all $w \in W$, $I(w, A) = 1$ if $I(w, B) = 1$ for all $B \in \Gamma$.

Then, we obtain the following result.

Proposition 6. For all $\Gamma \cup \{A\} \subseteq \mathsf{Sent}_Q$, $\Gamma \vdash A$ iff $\Gamma \models_* A$.

Proof. For the soundness direction, we will only check the case for (Ax19). For all $A, B \in \mathsf{Sent}_Q$ and for all $w \in W$: $I(w, {\sim}(A \to B)) = 1$ iff $I(w^*, A \to B) \neq 1$ iff $I(w^*, A) = 1$ and $I(w^*, B) \neq 1$ iff $I(w, \neg{\sim}A) = 1$ and $I(w, {\sim}B) = 1$ iff $I(w, \neg{\sim}A \wedge {\sim}B) = 1$. Therefore, we obtain the desired result.

For the completeness direction, it suffices to show that $\Gamma \models_* A$ only if $\Gamma \models A$ by Theorem 1. Suppose $\Gamma \not\models A$. Then, there is a **QBD+**-Dunn-model $\langle D_0, V_0 \rangle$ such that $1 \notin I_0(A)$ and $1 \in I_0(B)$ for all $B \in \Gamma$. Define a **QBD+**-star-model $\langle W_1, *_1, D_1, V_1 \rangle$ as follows: $W_1 := \{a, b\}$; $a^* = b, b^* = a$; $D_1 := D_0$; $V_1(a, P^n) := V_0^+(P^n)$, $V_1(b, P^n) := D^n \setminus V_0^-(P^n)$. Then, we can show that the following holds for all sentences:

- $I_1(a, A) = 1$ iff $1 \in I_0(A)$ and $I_1(b, A) = 1$ iff $0 \notin I_0(A)$

We can prove this by induction, but the details are straightforward and safely left to the readers. We are then ready to conclude that $\Gamma \not\models_* A$ since we have $I_1(a, A) \neq 1$ and $I_1(a, B) = 1$ for all $B \in \Gamma$ in the **QBD+**-star-model $\langle W_1, *_1, D_1, V_1 \rangle$. This completes the proof. □

Remark 7. Both for **SPL** and **SE4**, the status of the contraposition rule is highlighted, and this becomes even clearer once we have the star semantics. We may also add that our proof can be seen as an alternative proof to the result on the admissibility of contraposition rule in **BD+** established by Kamide in [17, Theorem 16] in which two sequent calculi are made use of.

Moreover, the star semantics makes the relation between **HYPE** and **BD+** (and its definitionally equivalent systems) explicit. Indeed, by building on the semantics for **HYPE** presented by Sergei Odintsov and Heinrich Wansing in [25], it is easy to see that **BD+** is obtained by trivialising the partial order which is necessary to capture the constructive conditional.

3 N3-style extension of BDi

In [16], Norihiro Kamide presented an intuitionistic version of the system **BD+**. This variant **BDi** can also be seen as a variant of the system **N4** of Almukdad and Nelson [2], obtained by changing the falsity condition for implication. It then is a natural question to study an extension of **BDi** with the characteristic axiom for **N3** [23], the explosive variant of **N4**. We shall see that this extension, henceforth called **BDi3**, validates the principle of *potential omniscience* investigated by Ichiro Hasuo and Ryo Kashima [15], in contrast to the case for **N3**. This motivates us to consider **BDi3** as a predicate logic **QBDi3**, since potential omniscience implies the *double negation shift* (a.k.a. *Kuroda's conjecture*) $\forall x \neg\neg A \to \neg\neg\forall x A$.

3.1 Semantics

Definition 8. A **QBDi3**-model for the language $\mathscr{L}_Q$ is a quadruple $\langle W, \leq, D, V\rangle$, where W is a non-empty set (of states); $\leq$ is a partial ordering on W; D is a mapping that assigns to each $w \in W$ a set $D(w) \supseteq \mathsf{Con}$, with a proviso that $x \geq w$ implies $D(x) \supseteq D(w)$. As an additional condition, $(W, \leq)$ has to satisfy $\forall w \in W \exists x \geq w(\forall y(y \geq x \Rightarrow y = x))$, i.e. any state has a maximal successor.

V assigns both the *extension* $V^+(w,P^n) \subseteq (D(w))^n$ and the *anti-extension* $V^-(w,P^n) \subseteq (D(w))^n$ to each n-ary predicate symbol P^n and a state w, such that $V^+(w,P^n) \cap V^-(w,P^n) = \emptyset$. Moreover, V^+ and V^- must be monotone: $\langle d_1, \dots d_n\rangle \in V^*(w,P^n)$ and $x \geq w$ implies $\langle d_1, \dots d_n\rangle \in V^*(x,P^n)$ for $* \in \{+,-\}$. Additionally, we assume V to be *potentially omniscient*, i.e. for all $w \in W$ and $\langle d_1, \dots, d_n\rangle \in (D(w))^n$: for all $x \geq w$ there exists $y \geq x$: $\langle d_1, \dots, d_n\rangle \in V^+(y,P^n) \cup V^-(y,P^n)$. V is extended to the interpretation I to state-sentence pairs (of $\mathsf{Sent}_{\mathbf{D}}$, i.e. $\mathscr{L}_Q$ extended with $\mathbf{D} := \cup_{w \in W} D(w)$) by the following conditions:

- $1 \in I(w, P(d_1, \dots, d_n))$ iff $\langle d_1, \dots, d_n\rangle \in V^+(x,P^n)$,
- $0 \in I(w, P(d_1, \dots, d_n))$ iff $\langle d_1, \dots, d_n\rangle \in V^-(x,P^n)$,
- $1 \notin I(w, \bot)$ and $0 \in I(w, \bot)$,
- $1 \in I(w, {\sim}A)$ iff $0 \in I(w,A)$,
- $0 \in I(w, {\sim}A)$ iff $1 \in I(w,A)$,
- $1 \in I(w, A \wedge B)$ iff $1 \in I(w,A)$ and $1 \in I(w,B)$,
- $0 \in I(w, A \wedge B)$ iff $0 \in I(w,A)$ or $0 \in I(w,B)$,
- $1 \in I(w, A \vee B)$ iff $1 \in I(w,A)$ or $1 \in I(w,B)$,
- $0 \in I(w, A \vee B)$ iff $0 \in I(w,A)$ and $0 \in I(w,B)$,
- $1 \in I(w, A \to B)$ iff for all $x \in W : (w \leq x$ only if $(1 \notin I(x,A)$ or $1 \in I(x,B)))$,
- $0 \in I(w, A \to B)$ iff for all $x \in W : ((w \leq x$ only if $0 \notin I(x,A))$ and $0 \in I(w,B))$,
- $1 \in I(w, \forall xA)$ iff for all $x \in W : (w \leq x$ only if $1 \in I(x, A(d))$ for all $d \in D(x))$,
- $0 \in I(w, \forall xA)$ iff $0 \in I(w, A(d))$ for some $d \in D(w)$,
- $1 \in I(w, \exists xA)$ iff $1 \in I(w, A(d))$ for some $d \in D(w)$,
- $0 \in I(w, \exists xA)$ iff for all $x \in W : (w \leq x$ only if $0 \in I(x, A(d))$ for all $d \in D(x))$.

Finally, the semantic consequence is defined as follows: $\Gamma \models_{i3} A$ iff for all **QBDi3**-models $\langle W, \leq, D, V\rangle$, and for all $w \in W$: $1 \in I(w,A)$ if $1 \in I(w,B)$ for all $B \in \Gamma$.

Remark 9. Let $\mathscr{L}_{int}$ be a language consisting of $\{\bot, {\sim}\bot, \wedge, \vee, \to, \forall, \exists\}$ and containing additional predicates P', Q', etc. corresponding to P, Q, etc. We include ${\sim}\bot$ for the sake of convenience in the proof of completeness. Then a model of intuitionistic logic *plus* double negation shift, known as **MH**, can be defined by restricting the language to $\mathscr{L}_{int}$, removing references to V^-, ${\sim}$-related clauses and 0 in the interpretation and adding the clause that $1 \in I(w, {\sim}\bot)$. We shall use $\models_{mh}$ to denote the consequence.

The following proposition can be established by induction on the complexity of A.

Proposition 10. In a **QBDi3**-model, for all $A \in \mathsf{Sent}_{\mathbf{D}}$, if $1 \in I(w,A)$ and $w \leq x$ then $1 \in I(x,A)$.

Proposition 11. In a **QBDi3**-model, for all $w \in W$ the following statements hold.
(i) For no $A(\vec{d}) \in \mathsf{Sent}_{\mathbf{D}}$ s.t. $\vec{d} \in D(w)$, $1 \in I(w, A(\vec{d}))$ and $0 \in I(w, A(\vec{d}))$,
(ii) For all $A(\vec{d}) \in \mathsf{Sent}_{\mathbf{D}}$ s.t. $\vec{d} \in D(w)$, for all $x \geq w$ there exists $y \geq x : (1 \in I(y, A(\vec{d}))$ or $0 \in I(y, A(\vec{d})))$.

Proof. By simultaneous induction on the complexity of A. Here we shall look at the case for $\to$ and $\forall$.
For implication: (i) Suppose $1 \in I(w, B \to C)$ and $0 \in I(w, B \to C)$. By IH, for all $x \geq w$ there exists $y \geq x$ such that $1 \in I(y,B)$ or $0 \in I(y,B)$. But since $0 \notin I(x,B)$ for any $x \geq w$, it has to be that for all $x \geq w$

there exists $y \geq x$ such that $1 \in I(y,B)$. Thus by supposition, for all $x \geq w$ there exists $y \geq x$ such that $1 \in I(y,C)$. But this contradicts with $0 \in I(w,C)$; so our supposition cannot hold. (ii) We want to show

$$\forall x \geq w \exists y \geq x (1 \in I(y, B \to C) \text{ or } 0 \in I(y, B \to C)).$$

Let $x \geq w$. Then by IH there is $y \geq x$ s.t. $1 \in I(y,B)$ or $0 \in I(y,B)$. Now again by IH there is $z \geq y$ s.t. $1 \in I(z,C)$ or $0 \in I(z,C)$ as well as $1 \in I(z,B)$ or $0 \in I(z,B)$ by monotonicity. Then if $1 \in I(z,C)$ or $0 \in I(z,B)$, we infer $1 \in I(z, B \to C)$: the latter case follows from the IH of (i) for B. On the other hand, if $1 \in I(z,B)$ and $0 \in I(z,C)$, then from the former $0 \notin I(u,B)$ for all $u \geq z$. Hence $0 \in I(z, B \to C)$.
For universal quantifier: (i) If $1 \in I(w, \forall xA)$, then $1 \in I(w, A(d))$ for all $d \in D(w)$. So by IH $0 \notin I(w, A(d))$ for all $d \in D(w)$. Hence $0 \notin I(w, \forall xA)$. (ii) Given $w \in W$, by frame condition there is a $x \geq w$ that is maximal. By IH and maximality, for all $d \in D(x)$, either $1 \in I(x, A(d))$ or $0 \in I(x, A(d))$. Thus $1 \in I(x, A(d))$ for all $d \in D(x)$ or $0 \in I(x, A(d))$ for some $d \in D(x)$. So $1 \in I(x, \forall xA)$ or $0 \in I(x, \forall xA)$. □

3.2 Proof system

Definition 12. The logic **QBDi3** is a system in $\mathscr{L}_Q$ defined by (Ax1)–(Ax21) (except for (Ax3)), (MP),(Gen) as well as the following axioms. (We shall use $\Gamma \vdash_{i3} A$ for the derivability relation.)

$$\forall x \neg\neg A \to \neg\neg \forall xA \quad \text{(i1)}$$

$$\sim A \to \neg A \quad \text{(i2)}$$

$$\neg\neg(A \vee \sim A) \quad \text{(i3)}$$

Remark 13. If we change the language to $\mathscr{L}_{int}$ and axioms to non-$\sim$-related ones (except (Ax15)), then we obtain the intermediate logic **MH** [12]. We shall use $\vdash_{mh}$ to denote the derivability in **MH**.

(i3) is an axiom schema known as *potential omniscience*, which was investigated in [15] as one of the additional axiom to **N3**. In comparison, we have the following remark on the status of (i3) in **QBDi3**.
Remark 14. We note that (i3) is in fact redundant in **QBDi3**: consider a subsystem of **QBDi3** without (i3), and take an instance $\sim\neg A \to \neg\neg A$ of (i2). This is equivalent to $\neg(\neg\sim A \wedge \neg A)$, and so to the schema for (i3). Alternatively, we may drop (i2) instead of (i3) in obtaining an equivalent system: an instance $\neg\neg(\neg A \vee \sim\neg A)$ of (i3) is equivalent to $\neg\neg\sim A \to \neg A$, so (i2) is derivable. In spite of these observations, We posit both of the axioms because it is more convenient for the proof of the completeness theorem.
Remark 15. It is immediate from the above remark that the addition of $A \vee \sim A$ to **BDi** results in the collapse of $\neg A$ and $\sim A$, as well as the classicalisation of the positive fragment. This can be contrasted with **N4**, for which the same addition makes the positive fragment of the logic classical, but not $\sim$ [5].
Remark 16. It is shown in [15] that the combination of (i2) and (i3) proves (i1). To see this, note $\forall x \neg\neg A$ derives $\neg\exists x \neg A$ and so $\neg\exists x \sim A$ by (i2). This is equivalent to $\neg\sim\forall xA$ and thus by (i3) $\neg\neg\forall xA$. Therefore (i1) is also redundant. We retain it again for convenience in the completeness proof.

3.3 Completeness

In order to establish the completeness of **QBDi3**, we first introduce the notion of *reduction* [14].
Definition 17. We define a *reduction* $f : \mathsf{Form}_Q \to \mathsf{Form}_Q$ by the following clauses:

$$\begin{array}{lll} f(P) = P, & f(\sim P) = \sim P, & f(\sim\exists xA) = \forall x f(\sim A), \\ f(\bot) = \bot, & f(\sim\bot) = \sim\bot, & f(\sim(A \wedge B)) = f(\sim A) \vee f(\sim B), \\ f(A \circ B) = f(A) \circ f(B), & f(\sim\sim A) = f(A), & f(\sim(A \vee B)) = f(\sim A) \wedge f(\sim B), \\ f(QxA) = Qx f(A), & f(\sim\forall xA) = \exists x f(\sim A), & f(\sim(A \to B)) = \neg f(\sim A) \wedge f(\sim B). \end{array}$$

where $\circ \in \{\wedge, \vee, \to\}$ and $Q \in \{\forall, \exists\}$. We then let $f(\Gamma) = \{f(B) : B \in \Gamma\}$ for a set Γ of formulas.

Recall that a *prime* formula is either atomic or $\bot$. The next proposition is then readily checkable.

Proposition 18. For all $A \in \mathsf{Form}_Q$, any B in a subformula $\sim B$ of $f(A)$ is a prime formula.

We shall call a formula *reduced* if it is of the form $f(A)$. We shall often write $A[\sim P_1, \ldots, \sim P_n]$ to denote the occurrences of subformulas of the form $\sim B$. If all formulas in a proof are reduced, then we shall call it a *reduced proof*, and use the notation $\vdash_r$. Then the proposition below is shown easily.

Proposition 19. For all $A \in \mathsf{Form}_Q$, $\vdash_{i3} A \leftrightarrow f(A)$.

Proposition 20. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}_Q$, if $\Gamma \vdash_{i3} A$ then $f(\Gamma) \vdash_r f(A)$.

Proof. By induction on the length of a proof. For cases concerning (i2) and (i3), we show

$$\vdash_r f(\sim A \to \neg A) \text{ and } \vdash_r f(\neg\neg(A \vee \sim A))$$

by simultaneous induction on the complexity of A. When A is prime, $\sim A \to \neg A$ and $\neg\neg(A \vee \sim A)$ are already reduced. When $A \equiv \sim B$, $f(\sim A \to \neg A) = f(B) \to \neg f(\sim B)$, which is equivalent to $f(\sim B \to \neg B)$. Hence by IH there is a reduced proof. Similarly for $f(\neg\neg(A \vee \sim A))$.

For conjunction: When $A \equiv B \wedge C$, we have to show:

1. $\vdash_r f(\sim B) \vee f(\sim C) \to \neg(f(B) \wedge f(C))$,
2. $\vdash_r \neg\neg((f(B) \wedge f(C)) \vee f(\sim B) \vee f(\sim C))$.

By IH, there are reduced derivations for:

1. $f(\sim B) \to \neg f(B)$ and $f(\sim C) \to \neg f(C)$,
2. $\neg\neg(f(B) \vee f(\sim B))$ and $\neg\neg(f(C) \vee f(\sim C))$.

For (1), the formula follows from $\vdash_r (\neg f(B) \vee \neg f(C)) \to \neg(f(B) \wedge f(C))$. For (2), the formula follows from $\vdash_r ((f(B) \vee f(\sim B)) \wedge (f(C) \vee f(\sim C))) \to ((f(B) \wedge f(C)) \vee f(\sim B) \vee f(\sim C))$. The case for $\vee$ is similar.

For implication: When $A \equiv B \to C$, we have to show:

1. $\vdash_r (\neg f(\sim B) \wedge f(\sim C)) \to \neg(f(B) \to f(C))$.
2. $\vdash_r \neg\neg((f(B) \to f(C)) \vee (\neg f(\sim B) \wedge f(\sim C)))$.

For (1), we shall show $\vdash_r (f(\sim C) \wedge (f(B) \to f(C))) \to \neg\neg f(\sim B)$. First, by IH $\vdash_r (f(B) \to f(C)) \to (f(\sim C) \to \neg f(B))$. Then note $\vdash_r \neg\neg(f(B) \vee f(\sim B)) \to (\neg f(B) \to \neg\neg f(\sim B))$. Hence by IH the desired formula follows. For (2), we first note that $\neg((f(B) \to f(C)) \vee (\neg f(\sim B) \wedge f(\sim C)))$ is equivalent to $\neg\neg f(B) \wedge \neg f(C) \wedge (\neg f(\sim B) \to \neg f(\sim C))$. (Recall $\neg(A \to B) \leftrightarrow (\neg\neg A \wedge \neg B)$ is an intuitionistic theorem.) Now by IH, $\vdash_r \neg\neg f(B) \to \neg f(\sim B)$; so $\neg f(C) \wedge \neg f(\sim C)$ follows from the above formula. But by IH we also have $\vdash_r \neg f(C) \to \neg\neg f(\sim C)$. Thus: $\vdash_r \neg((f(B) \to f(C)) \vee (\neg f(\sim B) \wedge f(\sim C))) \to (\neg f(\sim C) \wedge \neg\neg f(\sim C))$ and so the desired formula follows by an intuitionistic inference.

For universal quantifier: When $A \equiv \forall x B$, we have to show:

1. $\vdash_r \exists x f(\sim B) \to \neg \forall x f(B)$.
2. $\vdash_r \neg\neg(\forall x f(B) \vee \exists x f(\sim B))$.

For (1), from IH we can derive $\vdash_r \exists x f(\sim B) \to \exists x \neg f(B)$. Then use the fact that $\exists x \neg C \to \neg \forall x C$ is intuitionistically derivable. For (2), by IH, (Gen) and (i1), $\vdash_r \neg\neg\forall x(\neg f(B) \to \exists x f(\sim B))$. Hence using (Ax12) and contraposing the inside, we obtain $\vdash_r \neg\neg(\neg\exists x f(\sim B) \to \neg\exists x \neg f(B))$. Using the equivalence between $\neg\exists x C$ and $\forall x \neg C$ as well as (i1), this implies $\vdash_r \neg\neg(\neg\exists x f(\sim B) \to \neg\neg\forall x f(B))$. Since $C \to \neg\neg D$ is equivalent to $\neg\neg(C \to D)$, $\vdash_r \neg\neg(\neg\exists x f(\sim B) \to \forall x f(B))$. Therefore $\vdash_r \neg\neg(\exists x f(\sim B) \vee \forall x f(B))$, using $(\neg C \to D) \to \neg\neg(C \vee D)$. So the desired formula follows. The case for $\exists$ is similar. $\square$

Given a set of reduced formulas Γ, we define a set of formula E_Γ in $\mathscr{L}_{int}$ by:

$$E_\Gamma := \{\forall \vec{x}(P' \to \neg P) : \sim P \text{ occurs in some } B \in \Gamma\} \cup \{\forall \vec{x} \neg\neg(P' \vee P) : \sim P \text{ occurs in some } B \in \Gamma\}$$

Given a reduced formula $A[\sim P_1, \ldots \sim P_n]$, we define A' to be the formula obtained by replacing the occurrences of $\sim P_i$ with P_i'. We then define $\Gamma' = \{B' : B \in \Gamma\}$ for a set Γ of reduced formulas.

Proposition 21. Let $\Gamma \cup \{A\} \subseteq \mathsf{Form}_Q$ be reduced. Then $\Gamma \vdash_{i3} A$ if and only if $\Gamma', E_{\Gamma \cup \{A\}} \vdash_{mh} A'$.

Proof. For arguing left-to-right, by proposition 20 we can assume that the derivation of A from Γ to be reduced.[3] then by induction the length of a proof, we can show that **MH** can replicate the derivation of **BDi3**. In particular, for (i2) and (i3), the formulas negated by $\sim$ must be prime, and we have:

$$\forall\vec{x}(P' \to \neg P) \vdash_{mh} (\sim P \to \neg P)[\sim P/P'] \text{ and } \forall\vec{x}\neg\neg(P \vee P') \vdash_{mh} \neg\neg(P \vee \sim P)[\sim P/P'].$$

Similarly for the case of $\perp$. For arguing right-to-left, by replacing atomic formulas of the form P' by $\sim P$ in the proof of $\Gamma', E_{\Gamma\cup\{A\}} \vdash_{mh} A'$, we obtain a proof for $\Gamma \vdash_{i3} A$. □

We move on to the completeness theorem after stating one more lemma that is easily checkable.

Lemma 22. In a **BDi3**-model and $A(\vec{d}) \in \mathsf{Sent}_{\mathbf{D}}$ s.t. $\vec{d} \in D(w)$, the next equivalences hold.

(i) $1 \in I(w, A(\vec{d}))$ iff $1 \in I(w, f(A(\vec{d})))$. (ii) $0 \in I(w, A(\vec{d}))$ iff $1 \in I(w, f(\sim A(\vec{d})))$.

Theorem 2 (Soundness and completeness of **QBDi3**)**.** For all $\Gamma \cup \{A\} \in \mathsf{Sent}_Q$, $\Gamma \vdash_{i3} A$ iff $\Gamma \models_{i3} A$.

Proof. The soundness follows by induction on the length of derivation (by substituting free variables with elements in the relevant domain). In particular, the cases for (i2), (i3) follow from Proposition 10.

For completeness, we show by contraposition. Assume $\Gamma \nvdash_{i3} A$. Then by Proposition 19, $f(\Gamma) \nvdash_{i3} f(A)$, and so $f(\Gamma)', E_{f(\Gamma\cup\{A\})} \nvdash_{mh} f(A)'$ by Proposition 21. Hence by the strong completeness for **MH** [3, 12], $f(\Gamma)', E_{f(\Gamma\cup\{A\})} \not\models_{mh} f(A)'$. Consequently, there is a model $\langle W, \leq, D, V\rangle$ of **MH** such that for some $w \in W$, $1 \in I(w, B)$ for all $B \in f(\Gamma)' \cup E_{f(\Gamma\cup\{A\})}$ but $1 \notin I(w, f(A)')$ for some $x \in W$.

Define a **QBDi3**-model $\langle W, \leq, D, V_2\rangle$ such that for $\vec{d} \in D(w)$:

$$\vec{d} \in V_2^+(w, P) \text{ iff } \vec{d} \in V^+(w, P), \text{ and } \vec{d} \in V_2^-(w, P) \text{ iff } \vec{d} \in V^+(w, P').$$

We have to check that $\langle W, \leq, D, V_2\rangle$ is indeed a **QBDi3**-model. If $\vec{d} \in V_2^+(w, P)$ and $\vec{d} \in V_2^-(w, P)$, then $\vec{d} \in V^+(w, P)$ and $\vec{d} \in V^+(w, P')$. But then $1 \in I(w, B)$ for all B in $\langle W, \leq, D, V\rangle$, a contradiction. Next, since $1 \in I(w, \neg\neg(P(\vec{d}) \vee P'(\vec{d})))$ for $\vec{d} \in D(w)$, for any $w \in W$: $\forall x \geq w \exists y \geq x(\vec{d} \in V^+(y, P) \cup V^+(y, P')))$. Hence for any $\vec{d} \in D(w)$, we have $\forall x \geq w \exists y \geq x(\vec{d} \in V_2^+(y, P) \cup V_2^-(y, P))$ in $\langle W, \leq, D, V_2\rangle$.

We shall now observe that $1 \in I(w, B')$ iff $1 \in I_2(w, B)$ for any closed subformulas of $f(\Gamma \cup \{A\})$ with constants in $D(w)$. In particular, when $B \equiv \sim C$, $C \equiv P(\vec{d})$ for some P which occurs in $E_{f(\Gamma\cup\{A\})}$. Then $1 \in I(w, (\sim P(\vec{d}))')$ iff $0 \in I_2(w, P(\vec{d}))$ iff $1 \in I_2(w, \sim P(\vec{d}))$.

It now follows that $1 \in I_2(x, f(B))$ for all $f(B) \in f(\Gamma)$ but $1 \notin I_2(x, f(A))$. Therefore from Lemma 22, we infer that $1 \in I_2(x, B)$ for all $B \in \Gamma$ but $1 \notin I_2(x, A)$. Hence $\Gamma \not\models_{i3} A$. □

3.4 Constructive properties

Constructivity for **BDi** has been observed in [16] by establishing the disjunction and constructible falsity properties. These properties constitute an important difference from **HYPE**, for which they fail, as Odintsov and Wansing [25] observed through Drobyshevich's formula [9]. On the other hand, for **MH**, the disjunction and existence properties have been established by Komori [21]. It is therefore of interest to check these properties for **QBDi3**. Here, we adopt an approach via *Aczel slash* [1].

Definition 23. For $A \in \mathsf{Sent}_Q$. we define its *slashes* $|^+A$ and $|^-A$ by the following clause.

[3] We may assume the subformulas of the form $\sim P$ in $\Gamma \cup \{A\}$ exhaust all formulas of the form in the derivation, for otherwise we can take $A \wedge (\sim P \to \sim P)$ instead. A similar remark applies to the right-to-left case.

- $|^+ P(t_1,\dots t_n)$ iff $\vdash_{i3} P(t_1,\dots t_n)$.
- $|^- P(t_1,\dots t_n)$ iff $\vdash_{i3} {\sim} P(t_1,\dots t_n)$.
- $\not|^+ \bot$.
- $|^- \bot$.
- $|^+ {\sim} A$ iff $|^- A$.
- $|^- {\sim} A$ iff $|^+ A$.
- $|^+ A \wedge B$ iff $|^+ A$ and $|^+ B$.
- $|^- A \wedge B$ iff $|^- A$ or $|^- B$.
- $|^+ A \vee B$ iff $|^+ A$ or $|^+ B$.
- $|^- A \vee B$ iff $|^- A$ and $|^- B$.
- $|^+ A \to B$ iff $\vdash_{i3} A \to B$ and ($|^+ A$ implies $|^+ B$).
- $|^- A \to B$ iff $\vdash_{i3} \neg{\sim} A$ and $|^- B$.
- $|^+ \forall x A$ iff $\vdash_{i3} \forall x A$ and ($|^+ A(c)$ for all $c \in \mathsf{Con}$).
- $|^- \forall x A$ iff $|^- A(c)$ for some $c \in \mathsf{Con}$.
- $|^+ \exists x A$ iff $|^+ A(c)$ for some $c \in \mathsf{Con}$.
- $|^- \exists x A$ iff $\vdash_{i3} {\sim}\exists x A$ and ($|^- A(c)$ for all $c \in \mathsf{Con}$).

We proceed to show a couple of lemmas. The first one has a handy consequence that $|^+ \neg A$ iff $\vdash_{i3} \neg A$.

Lemma 24. Let $A \in \mathsf{Sent}_Q$. Then $|^+ A$ implies $\vdash_{i3} A$.

Proof. By induction on the complexity of A. When A is strongly negated, we further divide into cases depending on the complexity of the negand. As an example, consider the case $A \equiv {\sim}(B \to C)$. Assume $|^+ {\sim}(B \to C)$: then $|^- (B \to C)$ and so $\vdash_{i3} \neg{\sim} B$ and $|^- C$. The latter implies $|^+ {\sim} C$, which by IH implies $\vdash_{i3} {\sim} C$. Thus $\vdash_{i3} {\sim}(B \to C)$ follows from (Ax19). □

Before stating the next lemma, we expand the (+ve) slash to Form_Q, by stipulating $|^+ A$ if $|^+ A'$ for any A' obtained from A by substituting its free variables by constants.

Lemma 25. Let $A \in \mathsf{Sent}_Q$. Then $\vdash_{i3} A$ implies $|^+ A$.

Proof. By induction on the length of proof, using the expanded notion of slash. Here we treat a couple of cases as examples. For cases of intuitionistic axioms and rules, see e.g. [40, Theorem 3.5.9]. Moreover, in view of Remark 14, 16, it suffices to consider a simpler axiomatisation of **QBDi3** without (i1), (i3).

For (Ax19), we need to show $|^+ {\sim}(A \to B) \to (\neg{\sim} A \wedge {\sim} B)$ and $|^+ (\neg{\sim} A \wedge {\sim} B) \to {\sim}(A \to B)$ for $A, B \in \mathsf{Sent}_Q$. Consider the former. By definition, it is equivalent to:

$$\vdash_{i3} {\sim}(A \to B) \to (\neg{\sim} A \wedge {\sim} B) \text{ and } (|^+ {\sim}(A \to B) \text{ implies } |^+ \neg{\sim} A \wedge {\sim} B).$$

The former conjunct is one direction of (Ax19); the latter conjunct follows immediately from the handy consequence we noted above. The other direction similarly follows.

For (i2), we must show $|^+ {\sim} A \to \neg A$ for $A \in \mathsf{Sent}_Q$. This follows since $|^+ {\sim} A$ implies $\vdash_{i3} {\sim} A$ and thus $\vdash_{i3} \neg A$ by the previous lemma and (i2): now use again the handy consequence to conclude $|^+ \neg A$. □

We obtain disjunction, existence and constructible falsity property for **QBDi3** as consequences.

Theorem 3. Let $A, B \in \mathsf{Sent}_Q$. Then:

(i) $\vdash_{i3} A \vee B$ implies $\vdash_{i3} A$ or $\vdash_{i3} B$.
(ii) $\vdash_{i3} \exists x A$ then $\vdash_{i3} A(c)$ for some $c \in \mathsf{Con}$.
(iii) $\vdash_{i3} {\sim}(A \wedge B)$ implies $\vdash_{i3} {\sim} A$ or $\vdash_{i3} {\sim} B$.
(iv) $\vdash_{i3} {\sim}\forall x A$ then $\vdash_{i3} {\sim} A(c)$ for some $c \in \mathsf{Con}$.

Proof. (i) If $\vdash_{i3} A \vee B$, then by Lemma 25 $|^+ A \vee B$, and so either $|^+ A$ or $|^+ B$. Thus either $\vdash_{i3} A$ or $\vdash_{i3} B$ by Lemma 25. (ii) is shown analogously. (iii) and (iv) then follow form (i) and (ii), respectively. □

Remark 26. Despite Theorem 3, **QBDi3** may be unacceptable to some constructivists, as the double negation shift contradicts principles of some schools of constructivism[4] [40, Corollary 6.3.4.2, 6.6.4].

[4] For an analysis of the double negation shift and its variants in the mathematical setting, see e.g. [11].

4 Comparisons with systems related to BDi3

4.1 Two-state case as a four-valued logic

Let $\mathscr{L}$ be $\mathscr{L}_Q$ without quantifiers. Consider the extension of propositional **BDi3** with an axiom schema:

$$A \vee (A \to B) \vee \neg B. \tag{AxG}$$

For intuitionistic logic, the addition of (AxG) results in a system called **G3**, which is sound and strongly complete with respect to the class of linear Kripke frames with ≤ 2 elements: cf. [7, 30, 33]. The semantics can be represented by the three-valued truth tables below.

$A \wedge B$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**i**	**i**	**0**
0	**0**	**0**	**0**

$A \vee B$	**1**	**i**	**0**
1	**1**	**1**	**1**
i	**1**	**i**	**i**
0	**1**	**i**	**0**

$A \to B$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**1**	**1**	**0**
0	**1**	**1**	**1**

$\neg A$	
1	**0**
i	**0**
0	**1**

We shall use $\vdash_{i3g3}$ for the consequence in **BDi3**+(AxG), and $\models_{i3g3}$ for the semantical consequence of the class of linear propositional **BDi3**-frames with ≤ 2 elements. Then using the strong completeness of **G3**, we can show the completeness theorem by arguing analogously to the previous subsection.

Theorem 4. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{i3g3} A$ iff $\Gamma \models_{i3g3} A$.

Given this correspondence, it is of interest to ask what kind of truth tables can characterize this extension. We claim that the following 4-valued truth tables are adequate ($\perp$ has the constant value **0**).

$A \wedge B$	**1**	**i**	**j**	**0**
1	**1**	**i**	**j**	**0**
i	**i**	**i**	**j**	**0**
j	**j**	**j**	**j**	**0**
0	**0**	**0**	**0**	**0**

$A \vee B$	**1**	**i**	**j**	**0**
1	**1**	**1**	**1**	**1**
i	**1**	**i**	**i**	**i**
j	**1**	**i**	**j**	**j**
0	**1**	**i**	**j**	**0**

$A \to B$	**1**	**i**	**j**	**0**
1	**1**	**i**	**j**	**0**
i	**1**	**1**	**j**	**0**
j	**1**	**1**	**1**	**1**
0	**1**	**1**	**1**	**1**

$\neg A$	
1	**0**
i	**0**
j	**1**
0	**1**

$\sim A$	
1	**0**
i	**j**
j	**i**
0	**1**

Let $V_4 : \mathsf{Prop} \longrightarrow \{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{0}\}$ be a four-valued assignment and I_4 be the interpretation extending it according to the tables. We write $\Gamma \models_4 A$ if $I_4(B) = \mathbf{1}$ for all $B \in \Gamma$ implies $I_4(A) = \mathbf{1}$ for all interpretations.

Theorem 5. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models_{i3g3} A$ then $\Gamma \models_4 A$.

Proof. For the left-to-right direction, let V_4 be an assignment s.t. $I_4(B) = \mathbf{1}$ for all $B \in \Gamma$. We define a linear **BDi3**-model with 2 elements $\langle \{x,y\}, \{(x,x),(x,y),(y,y)\}, V \rangle$ by:

$$V(x,p) := \begin{cases} \{1\} & \text{if } V_4(p) = \mathbf{1}. \\ \{0\} & \text{if } V_4(p) = \mathbf{0}. \\ \emptyset & \text{otherwise.} \end{cases} \quad V(y,p) := \begin{cases} \{1\} & \text{if } V_4(p) = \mathbf{1} \text{ or } \mathbf{i}. \\ \{0\} & \text{otherwise.} \end{cases}$$

We can then show that V is monotone and potentially omniscient, and for all $A \in \mathsf{Form}$:

- $I(x,A) = \{1\} \Longleftrightarrow I_4(A) = \mathbf{1}$.
- $I(x,A) = \{0\} \Longleftrightarrow I_4(A) = \mathbf{0}$.
- $I(x,A) = \emptyset \Longleftrightarrow I_4(A) = \mathbf{i}$ or $\mathbf{j}$.
- $I(y,A) = \{1\} \Longleftrightarrow I_4(A) = \mathbf{1}$ or $\mathbf{i}$.
- $I(y,A) = \{0\} \Longleftrightarrow I_4(A) = \mathbf{j}$ or $\mathbf{0}$.

Now by assumption, $1 \in I(x,B)$ for all $B \in \Gamma$ and so $1 \in I(x,A)$; hence $I_4(A) = \mathbf{1}$. Thus $\Gamma \models_4 A$. □

Theorem 6. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models_4 A$ then $\Gamma \models_{i3g3} A$.

Proof. Let $\langle W, \leq, V \rangle$ be a linear **BDi3**-model with ≤ 2 elements such that $1 \in I(w,B)$ for all $B \in \Gamma$. As the case when $|W| = 1$ is immediate, we turn our attention to the case when $|W| = 2$. Let $W = \{x,y\}$, $\leq = \{(x,x),(x,y),(y,y)\}$ and $w = x$. We define an assignment V_4 by the following clauses.

$$V_4(p) = \begin{cases} \mathbf{1} & \text{if } V(x,p) = \{1\}. \\ \mathbf{i} & \text{if } V(x,p) = \emptyset \text{ and } V(y,p) = \{1\}. \\ \mathbf{j} & \text{if } V(x,p) = \emptyset \text{ and } V(y,p) = \{0\}. \\ \mathbf{0} & \text{if } V(x,p) = \{0\}. \end{cases}$$

This can be checked to generalise to all $A \in \mathsf{Form}$. Now by assumption, $I_4(B) = \mathbf{1}$ for all $B \in \Gamma$ and thus $I_4(A) = \mathbf{1}$. Hence $I(x,A) = \{1\}$. Therefore $\models_{g3i3} A$. □

Therefore we conclude that **BDi3**+(AxG) is sound and complete with respect to the above tables:

Corollary 27. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{i3g3} A$ iff $\Gamma \models_4 A$.

4.2 Some subsystems of BDi3

Here we make some observations regarding the predicate expansions of other systems related to **QBDi3**.

Firstly, we consider the predicate version **QBDi** of the system **BDi**. A major difference of **QBDi** from **QBDi3** is that there is no need to posit the double negation shift axiom.

Definition 28. A **QBDi**-model is a quadruple $\langle W, \leq, D, V \rangle$ defined like that of **QBDi3**, except that:

- The condition about the existence of maximal elements is dropped.
- The condition $V^+(w,P^n) \cap V^-(w,P^n) = \emptyset$ and the assumption of potential omniscience are dropped.

We shall use $\models_i$ in denoting the semantic consequence.

Definition 29. The logic **QBDi** is a system in $\mathscr{L}_Q$ defined by removing (i1),(i2),(i3) from the axiomatisation of **QBDi3**. We shall use $\vdash_i$ to denote the derivability in **QBDi**.

Theorem 7. For all $\Gamma \cup \{A\} \subseteq \mathsf{Sent}_Q$, $\Gamma \vdash_i A$ iff $\Gamma \models_i A$.

Proof. The argument is analogous to Theorem 2. We do not need an analogue of Proposition 10, and the proof of the analogue of Proposition 20 is much simplified. For the analogue of Proposition 21 and elsewhere, we do not need to appeal to $E_{\Gamma \cup \{A\}}$. In the proof of the theorem itself, we appeal to the strong completeness of intuitionistic logic, rather than of **MH**. □

Constructive properties of **QBDi** can be observed as well, by arguing analogously to Theorem 3. Next, we consider the predicate expansions **QDN3** and **QDN4** of the systems **DN3** and **DN4** [24]. **QDN4** is defined from **QBDi** by replacing (Ax19) with $\sim(A \to B) \leftrightarrow (\neg\neg A \wedge \sim B)$. A Kripke model for **QDN4** is obtained from that of **QBDi** by changing the clauses for $0 \in I(w, A \to B)$ to:

- $0 \in I(w, A \to B)$ iff for all $x \geq w$ there is $y \geq x(1 \in I(y,A)))$ and $0 \in I(w,B)$.

QDN3 and its models are defined by imposing (i2) and the condition $V^+(w,P^n) \cap V^-(w,P^n) = \emptyset$.

Let us use subscripts $_{d3}$ and $_{d4}$ for the syntactic and semantic consequences in these systems. Then we obtain the following completeness theorems (cf. also [24] for the propositional case.)

Theorem 8. Let $k \in \{3,4\}$. For all $\Gamma \cup \{A\} \subseteq \mathsf{Sent}_Q$, $\Gamma \vdash_{dk} A$ iff $\Gamma \models_{dk} A$.

Proof. For **QDN4**, the argument is the same as the case for **QBDi**. The only major difference is that we have to use the clause $f(\sim(A \to B)) = \neg\neg f(A) \wedge f(\sim B)$ for reduction. For **QDN3**, the outline is almost identical to the case of **QBDi3**. Aside from the difference in reduction, and using the completeness of intuitionistic logic rather than of **MH**, we take E_Γ to be $\{\forall \vec{x}(P' \to \neg P) : \exists B \in \Gamma(\sim P \text{ occurs in } B)\}$. □

Remark 30. A motivation for **DN3** and **DN4** is to brings strong and intuitionistic negation closer: $\sim(A \to B) \to A$ holds in **N4**, but its analogue does not hold w.r.t. $\neg$. This may appear too demanding for a refutation of implication, and is thus avoided in the systems of [24]. This approach is also more thoroughly pursued in *quasi-nelson algebras* [32]: notice a similarity with the clause for $\to$ in *nucleus-based quasi-Nelson twist-algebra* [31], where $\Box$ is a *nucleus* (a generalisation of double negation):

- $\langle a_1, a_2 \rangle \to \langle b_1, b_2 \rangle = \langle a_1 \to b_1, \Box a_1 \wedge b_2 \rangle$.

Constructive properties of **QDN3** and **QDN4** can be checked again analogously to Theorem 3, by changing the clause for $|^{-} A \to B$ by $\vdash \neg\neg A$ and $|^{-} B$. Next, we observe that **QBDi3** and **DN3** are related in an essential way; indeed, the difference is exactly the potential omniscience axiom.

Proposition 31. $\mathbf{QBDi3} = \mathbf{QDN3} + \text{(i3)}$.

Proof. It suffices to show that $\neg{\sim}A \leftrightarrow \neg\neg A$ in each system, for then the two conditions for negated implications become inter-derivable. For **QBDi3**, it follows from (i2) using $\neg{\sim}A \leftrightarrow {\sim}\neg A$. For **QDN3** + (i3), one direction follows from (i2) and the other direction is equivalent to (i3). $\Box$

Remark 32. This also means that another advantage of **DN3** over **N3** claimed in [24], namely that contraposition is available in a limited form $(\neg A \to B) \to ({\sim}B \to {\sim}\neg A)$, also holds for **QBDi3**.

On the other hand, **QDN4** is not a subsystem of **QBDi**; that would imply $\vdash_i {\sim}\neg A \leftrightarrow \neg\neg A$ and thus $\vdash_i \neg{\sim}A \to \neg\neg A$, i.e. (i3) that separates **QBDi** from **QBDi3**.

Remark 33. In [24], we observed another extension of **DN4** by the axiom schema $A \vee {\sim}A$. At the propositional level, this already derives the weak law of excluded middle $\neg\neg A \vee \neg A$. If we consider a predicate expansion of this logic, then for the semantics to validate $\forall x A \vee {\sim}\forall x A$ we seem to require that a model has a constant domain.[5] This suggests the adoption of the constant domain axiom $\forall x(A(x) \vee C) \to (\forall x A(x) \vee C)$ in the expansion. On the other hand, the combination of the weak excluded middle and the constant domain axiom is known to cause Kripke incompleteness in intermediate logics [13, 35]. So an adequate treatment of the predicate system for this extension is expected to need more sophistications.

4.3 A connexive variant?

One of the most well-known variant of **N4** is the logic **C** introduced by Wansing [41]. This is obtained by replacing the conjunction in the **N4** condition ${\sim}(A \to B) \leftrightarrow (A \wedge {\sim}B)$ by implication. As a result of this change, **C** validates *Aristotle's theses* ${\sim}(A \to {\sim}A)$, ${\sim}({\sim}A \to A)$ and *Boethius' theses* $(A \to B) \to {\sim}(A \to {\sim}B)$ and $(A \to {\sim}B) \to {\sim}(A \to B)$ characteristic to connexive logic [42].

We can also test what happens if a similar change is made to **BDi**. In this case, (Ax19) becomes ${\sim}(A{\to}B){\to}(\neg{\sim}A{\to}{\sim}B)$ and otherwise the axiomatisation is kept intact. Then the theses become equivalent to $\neg{\sim}A{\to}A$, $\neg A{\to}{\sim}A$ (for Aristotle's theses) and $(A{\to}B){\to}(\neg{\sim}A{\to}B)$, $(A{\to}{\sim}B){\to}(\neg{\sim}A{\to}{\sim}B)$ (for Boethius' theses). So the resulting system is not connexive, but only humbly connexive (cf. [20]).

Another characteristic of **C** is that it is non-trivial but *negation inconsistent*, i.e. it validates a formula and its (strong) negation. That this would also be negation inconsistent in our variant of **BDi** is evident as ${\sim}\bot$ is one of the axioms. We also find a witness for negation inconsistency even in the absence of this axiom: e.g. both $(p \wedge {\sim}\neg{\sim}p) \to {\sim}\neg{\sim}p$ and ${\sim}((p \wedge {\sim}\neg{\sim}p) \to {\sim}\neg{\sim}p)$ turn out to be derivable. This system (and its extension with the variants of the connexive theses) remains *non-trivial*; this is checkable with the classical truth tables which in addition assigns every formula of the form ${\sim}A$ the value **1**.

[5] This situation is similar to the case for the predicate extension **QC3** of a connexive logic **C3**. [26, 28]

5 Concluding remarks

Our main motivation was to connect **BD+** and its intuitionistic counterpart **BDi** (in the first-order setting) with neighbouring systems. We firstly focused on establishing the picture of **BDi** and **HYPE** as sibling systems, through the formulation of star semantics for **QBD+**. Our suggestion there was to understand the two systems as results of constuctivising **BD+** along different (American/Australian) semantical contours. One question that remains, connecting back to the example of **S5** in the introduction, is whether there are other siblings for the two systems: i.e. a logic with the intuitionistic positive part, whose extension by Peirce's law coincides with **BD+**. Another venue would be to compare **BDi** and **HYPE** in more details, by e.g. introducing star semantics for **BDi** following ones for **N4** by Routley [34].

The second focus in this article was to compare **QBD+** from a more Nelsonian viewpoint. For this purpose an explosive system **QBDi3** was introduced. We observed a remarkable feature of this system that the falsity condition for implication now settles the status of potential omniscience and double negation shift. Since the motivations for these principles are by themselves not too clear, the falsity condition can provide another route to analyse their desirability. A further understanding of the falsity condition may be facilitated by comparison with the *strong implication* $A \Rightarrow B := (A \to B) \land ({\sim}B \to {\sim}A)$ in **BDi** and **BDi3** (also for **DN4** and **DN3**), following the approach for **N3/N4** in [36, 37, 38].

References

[1] Peter H. G. Aczel (1968): *Saturated intuitionistic theories*. In: *Studies in Logic and the Foundations of Mathematics*, 50, Elsevier, pp. 1–11, doi:10.1016/S0049-237X(08)70515-9.

[2] Ahmad Almukdad & David Nelson (1984): *Constructible falsity and inexact predicates*. *The Journal of Symbolic Logic* 49(1), pp. 231–233, doi:10.2307/2274105.

[3] Mohammad Ardeshir & S. Mojtaba Mojtahedi (2014): *Completeness of intermediate logics with doubly negated axioms*. *Mathematical Logic Quarterly* 60(1-2), pp. 6–11, doi:10.1002/malq.201200083.

[4] Arnon Avron (2020): *The normal and self-extensional extension of Dunn–Belnap logic*. *Logica Universalis* 14(3), pp. 281–296, doi:10.1007/s11787-020-00254-1.

[5] Diderik Batens & Kristof De Clercq (2004): *A rich paraconsistent extension of full positive logic*. *Logique et Analyse* 185–188, pp. 227–257.

[6] Jean-Yves Béziau (2011): *A New Four-Valued Approach to Modal Logic*. *Logique et Analyse* 54(213), pp. 109–121.

[7] Alexander Chagrov & Michael Zakharyaschev (1997): *Modal Logic*. Oxford, doi:10.1093/oso/9780198537793.001.0001.

[8] Michael De & Hitoshi Omori (2015): *Classical negation and expansions of Belnap-Dunn logic*. *Studia Logica* 103(4), pp. 825–851, doi:10.1007/s11225-014-9595-7.

[9] Sergey Drobyshevich (2015): *Double negation operator in logic N^**. *Journal of Mathematical sciences* 205(3), pp. 389–403, doi:10.1007/s10958-015-2254-3.

[10] Sergey Drobyshevich, Sergei Odintsov & Heinrich Wansing (2022): *Moisil's modal logic and related systems*. In Katalin Bimbó, editor: *Relevance Logics and Other Tools for Reasoning. Essays in Honour of Michael Dunn*, College Publications, pp. 150–177.

[11] Makoto Fujiwara & Ulrich Kohlenbach (2018): *Interrelation between weak fragments of double negation shift and related principles*. *The Journal of Symbolic Logic* 83(3), pp. 991–1012, doi:10.1017/jsl.2017.63.

[12] Dov M Gabbay (1972): *Applications of trees to intermediate logics*. *The Journal of Symbolic Logic* 37(1), pp. 135–138, doi:10.2307/2272556.

[13] Silvio Ghilardi (1989): *Presheaf semantics and independence results for some non-classical first-order logics*. Archive for Mathematical Logic 29(2), pp. 125–136, doi:10.1007/BF01620621.

[14] Yuri Gurevich (1977): *Intuitionistic logic with strong negation*. Studia Logica: An International Journal for Symbolic Logic 36(1/2), pp. 49–59, doi:10.1007/bf02121114.

[15] Ichiro Hasuo & Ryo Kashima (2003): *Kripke completeness of first-order constructive logics with strong negation*. Logic Journal of IGPL 11(6), pp. 615–646, doi:10.1093/jigpal/11.6.615.

[16] Norihiro Kamide (2021): *Modal and Intuitionistic Variants of Extended Belnap–Dunn Logic with Classical Negation*. Journal of Logic, Language and Information 30, p. 491–531, doi:10.1007/s10849-021-09330-1.

[17] Norihiro Kamide (2022): *Herbrand and contraposition-elimination theorems for extended first-order Belnap-Dunn logic*. In Katalin Bimbó, editor: Relevance Logics and other Tools for Reasoning: Essays in Honor of J. Michael Dunn, College Publications, pp. 237–260.

[18] Norihiro Kamide & Hitoshi Omori (2017): *An extended first-order Belnap-Dunn Logic with classical negation*. In: International Workshop on Logic, Rationality and Interaction, Springer, pp. 79–93, doi:10.1007/978-3-662-55665-8_6.

[19] Norihiro Kamide & Heinrich Wansing (2010): *Symmetric and dual paraconsistent logics*. Logic and Logical Philosophy 19(1-2), pp. 7–30, doi:10.12775/LLP.2010.002.

[20] Andreas Kapsner (2019): *Humble connexivity*. Logic and Logical Philosophy 28(3), pp. 513–536, doi:10.12775/LLP.2019.001.

[21] Yuichi Komori (1983): *Some results on the super-intuitionistic predicate logics*. Reports on Mathematical Logic 15, pp. 13–31.

[22] Hannes Leitgeb (2019): *HYPE: A system of hyperintensional logic (with an application to semantic paradoxes)*. Journal of Philosophical Logic 48(2), pp. 305–405, doi:10.1007/s10992-018-9467-0.

[23] David Nelson (1949): *Constructible falsity*. The Journal of Symbolic Logic 14(1), pp. 16–26, doi:10.2307/2268973.

[24] Satoru Niki (2023): *Improving Strong Negation*. The Review of Symbolic Logic 16(3), p. 951–977, doi:10.1017/S1755020321000290.

[25] Sergei Odintsov & Heinrich Wansing (2021): *Routley star and hyperintensionality*. Journal of Philosophical Logic 50(1), pp. 33–56, doi:10.1007/s10992-020-09558-5.

[26] Grigory K Olkhovikov (2023): *On the completeness of some first-order extensions of C*. Journal of Applied Logics-IfCoLog Journal 10(1), pp. 57–114, doi:10.13154/294-9815.

[27] Hitoshi Omori & Daniel Skurt (2019): *SIXTEEN$_3$ in Light of Routley Stars*. In: Proceedings of WoLLIC 2019, Springer, pp. 516–532, doi:10.1007/978-3-662-59533-6_31.

[28] Hitoshi Omori & Heinrich Wansing (2020): *An Extension of Connexive Logic C*. In Nicola Olivetti, Rineke Verbrugge, Sara Negri & Gabriel Sandu, editors: Advances in Modal Logic, 13, College Publications, pp. 503–522.

[29] Hiroakira Ono (1977): *On some intuitionistic modal logics*. Publications of the Research Institute for Mathematical Sciences 13(3), pp. 687–722, doi:10.2977/prims/1195189604.

[30] Hiroakira Ono (2019): *Proof Theory and Algebra in Logic*. Springer, doi:10.1007/978-981-13-7997-0.

[31] Umberto Rivieccio (2022): *Fragments of quasi-Nelson: The algebraizable core*. Logic Journal of the IGPL 30(5), pp. 807–839, doi:10.1093/jigpal/jzab023.

[32] Umberto Rivieccio & Matthew Spinks (2019): *Quasi-Nelson algebras*. Electronic Notes in Theoretical Computer Science 344, pp. 169–188, doi:10.1016/j.entcs.2019.07.011.

[33] Gemma Robles (2014): *A simple Henkin-style completeness proof for Gödel 3-valued logic G3*. Logic and Logical Philosophy 23(4), pp. 371–390, doi:10.12775/LLP.2014.001.

[34] Richard Routley (1974): *Semantical analyses of propositional systems of Fitch and Nelson*. *Studia Logica* 33(3), pp. 283–298, doi:10.1007/BF02123283.

[35] Valentin Shehtman & Dmitrij Skvortsov (1990): *Semantics of non-classical first order predicate logics*. *Mathematical logic*, pp. 105–116, doi:10.1007/978-1-4613-0609-2_9.

[36] Matthew Spinks & Robert Veroff (2008): *Constructive logic with strong negation is a substructural logic. I*. *Studia Logica* 88, pp. 325–348, doi:10.1007/s11225-008-9113-x.

[37] Matthew Spinks & Robert Veroff (2008): *Constructive logic with strong negation is a substructural logic. II*. *Studia Logica* 89, pp. 401–425, doi:10.1007/s11225-008-9138-1.

[38] Matthew Spinks & Robert Veroff (2018): *Paraconsistent constructive logic with strong negation as a contraction-free relevant logic*. *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science*, pp. 323–379, doi:10.1007/978-3-319-74772-9_13.

[39] Damian Szmuc & Hitoshi Omori (2022): *Liberating classical negation from falsity conditions*. In: *2022 IEEE 52th International Symposium on Multiple-Valued Logic*, 52, IEEE, pp. 131–136, doi:10.1109/ISMVL52857.2022.00027.

[40] Anne Sjerp Troelstra & Dirk van Dalen (1988): *Constructivism in Mathematics: An Introduction*. I, Elsevier.

[41] Heinrich Wansing (2005): *Connexive Modal Logic*. In Renate Schmidt, Ian Pratt-Hartmann, Mark Reynolds & Heinrich Wansing, editors: *Advances in Modal Logic. Volume 5*, King's College Publications, pp. 367–383.

[42] Heinrich Wansing (2023): *Connexive Logic*. In Edward N. Zalta, editor: *The Stanford Encyclopedia of Philosophy*, Summer 2023 edition, Metaphysics Research Lab, Stanford University.

[43] Dmitry Zaitsev (2012): *Generalized relevant logic and models of reasoning*. *Moscow State Lomonosov University doctoral (Doctor of Science) dissertation.*

Semi-Substructural Logics à la Lambek

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This work studies the proof theory of left (right) skew monoidal closed categories and skew monoidal bi-closed categories from the perspective of non-associative Lambek calculus. Skew monoidal closed categories represent a relaxed version of monoidal closed categories, where the structural laws are not invertible; instead, they are natural transformations with a specific orientation. Uustalu et al. used sequents with stoup (the leftmost position of an antecedent that can be either empty or a single formula) to deductively model left skew monoidal closed categories, yielding results regarding proof identities and categorical coherence. However, their syntax does not work well when modeling right skew monoidal closed and skew monoidal bi-closed categories.

We solve the problem by constructing cut-free sequent calculi for left skew monoidal closed and skew monoidal bi-closed categories, reminiscent of non-associative Lambek calculus, with trees as antecedents. Each calculus is respectively equivalent to the sequent calculus with stoup (for left skew monoidal categories) and the axiomatic calculus (for skew monoidal bi-closed categories). Moreover, we prove that the latter calculus is sound and complete with respect to its relational models. We also prove a correspondence between frame conditions and structural laws, providing an algebraic way to understand the relationship between the left and right skew monoidal (closed) categories.

1 Introduction

Substructural logics are logic systems that lack at least one of the structural rules, weakening, contraction, and exchange. Joachim Lambek's syntactic calculus [16] is a well-known example that disallows weakening, contraction, and exchange. Another example, linear logic, proposed by Jean-Yves Girard [12], is a substructural logic in which weakening and contraction are in general disallowed but can be recovered for some formulae via modalities. Substructural logics have been found in numerous applications from computational analysis of natural languages to the development of resource-sensitive programming languages.

Left skew monoidal categories [23] are a weaker variant of MacLane's monoidal categories where the structural morphisms of associativity and unitality are not required to be bidirectional, they are natural transformations with a particular orientation. Therefore, they can be seen as *semi-associative* and *semi-unital* variants of monoidal categories. Left skew monoidal categories arise naturally in the semantics of programming languages [2], while the concept of semi-associativity is connected with combinatorial structures like the Tamari lattice and Stasheff associahedra [32, 19].

In recent years, Tarmo Uustalu, Niccolò Veltri, and Noam Zeilberger started a research project on *semi-substructural* logics, which is inspired by a series of developments on left skew monoidal categories and related variants by Szlachányi, Street, Bourke, Lack and others [23, 14, 22, 15, 8, 5, 6, 7].

We call the languages of left skew monoidal categories and their variants *semi-substructural* logics, because they are intermediate logics between (certain fragments of) non-associative and associative intuitionistic linear logic (or Lambek calculus). Semi-associativity and semi-unitality are encoded as follows.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics. Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 195–213, doi:10.4204/EPTCS.415.18

Sequents are in the form $S \mid \Gamma \vdash A$, where the antecedent consists of an optional formula S, called stoup, adapted from Girard [13], and an ordered list of formulae Γ. The succedent is a single formula A. We restrict the application of introduction rules in an appropriate way to allow only one of the directions of associativity and unitality.

This approach has successfully captured languages for a variety of categories, including (*i*) left skew semigroup [32], (*ii*) left skew monoidal [28], (*iii*) left skew (prounital) closed [26], (*iv*) left skew monoidal closed categories [24, 30], and (*v*) left distributive skew monoidal categories with finite products and coproducts [31] through skew variants of the fragments of non-commutative intuitionistic linear logic consisting of combinations of connectives $(\mathsf{I}, \otimes, \multimap, \wedge, \vee)$. Additionally, discussions have covered partial normality conditions, in which one or more structural morphisms are allowed to have an inverse [27], as well as extensions with skew exchange à la Bourke and Lack [29, 31].

In all of the aforementioned works, internal languages of left skew monoidal categories and their variants are characterized in a similar way which we call sequent calculus à la Girard. These calculi with sequents of the form $S \mid \Gamma \vdash A$ are cut-free and by their rule design, they are decidable. Moreover, they all admit sound and complete subcalculi inspired by Andreoli's focusing [3] in which rules are restricted to be applied in a specific order. A focused calculus provides an algorithm to solve both the proof identity problems for its non-focused calculus and coherence problems for its corresponding variant of left skew monoidal category.

By reversing all structural morphisms and modifying coherence conditions in left skew monoidal closed categories, right skew monoidal closed categories emerge [25]. Moreover, skew monoidal bi-closed categories are defined by appropriately integrating left and right skew monoidal closed structures. It is natural for us to consider sound sequent calculi for these categories. However, the implication rules are not well-behaved when just modeling right skew monoidal closed categories with sequent calculus à la Girard.

The problem stems from the skew structure concealed within the flat antecedent of $S \mid \Gamma \vdash A$. While the antecedent $S \mid \Gamma$ is defined similarly to an ordered list, it is actually a tree associating to the left. We start in Section 2, by introducing the sequent calculus à la Girard (`LSkG`) for left skew monoidal closed categories from [24] and its equivalent sequent calculus à la Lambek (`LSkT`), which is inspired by sequent calculus for non-associative Lambek calculus [9, 20] with trees as antecedents.

In Section 3, we introduce definitions of left (right) skew monoidal closed categories and skew monoidal bi-closed categories, and normality conditions for skew categories. In Section 4, we describe two calculi that characterize skew monoidal bi-closed categories: one is an axiomatic calculus (`SkMBiCA`), while the other is a sequent calculus (`SkMBiCT`) similar to the multimodal non-associative Lambek calculus [18]. In Section 5, we introduce the relational semantics for `SkMBiCA` via preordered sets of possible worlds with ternary relations. Furthermore, we show a correspondence theorem (Theorem 5.7) between conditions on ternary relations and structural laws on any frame. The theorem allows us to prove a thin version of main theorems in [25].

2 Sequent Calculus

We recall the sequent calculus à la Girard for left skew monoidal closed categories from [24], which is a skew variant of non-commutative multiplicative intuitionistic linear logic.

Formulae (Fma) in `LSkG` are inductively generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \otimes B \mid A \multimap B$, where X comes from a set At of atoms, I is a multiplicative unit, $\otimes$ is multiplicative conjunction and $\multimap$ is a linear implication.

A sequent is a triple of the form $S \mid \Gamma \vdash_{\mathsf{G}} A$, where the antecedent splits into: an optional formula S, called *stoup* [13], and an ordered list of formulae Γ and succedent A is a single formula. The symbol S consistently denotes a stoup, meaning S can either be a single formula or empty, indicated as $S = -$; furthermore, X, Y, and Z always represent atomic formulae.

Definition 2.1. Derivations in LSkG are generated recursively by the following rules:

$$\frac{}{A \mid \ \vdash_{\mathsf{G}} A}\ \mathsf{ax} \qquad \frac{- \mid \Gamma \vdash_{\mathsf{G}} A \quad B \mid \Delta \vdash_{\mathsf{G}} C}{A \multimap B \mid \Gamma, \Delta \vdash_{\mathsf{G}} C}\ {\multimap}\mathsf{L} \qquad \frac{- \mid \Gamma \vdash_{\mathsf{G}} C}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{G}} C}\ \mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash_{\mathsf{G}} C}{A \otimes B \mid \Gamma \vdash_{\mathsf{G}} C}\ \otimes\mathsf{L}$$

$$\frac{A \mid \Gamma \vdash_{\mathsf{G}} C}{- \mid A, \Gamma \vdash_{\mathsf{G}} C}\ \mathsf{pass} \qquad \frac{S \mid \Gamma, A \vdash_{\mathsf{G}} B}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\ {\multimap}\mathsf{R} \qquad \frac{}{- \mid \ \vdash_{\mathsf{G}} \mathsf{I}}\ \mathsf{IR} \qquad \frac{S \mid \Gamma \vdash_{\mathsf{G}} A \quad - \mid \Delta \vdash_{\mathsf{G}} B}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A \otimes B}\ \otimes\mathsf{R}$$

The inference rules of LSkG are similar to the ones in the sequent calculus for non-commutative multiplicative intuitionistic linear logic (NMILL) [1], but with some crucial differences:

1. The left logical rules IL, $\otimes$L and $\multimap$L, read bottom-up, are only allowed to be applied on the formula in the stoup position.
2. The right tensor rule $\otimes$R, read bottom-up, splits the antecedent of a sequent $S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A \otimes B$ and in the case where S is a formula, S is always moved to the stoup of the left premise, even if Γ is empty.
3. The presence of the stoup distinguishes two types of antecedents, $A \mid \Gamma$ and $- \mid A, \Gamma$. The structural rule pass (for 'passivation'), read bottom-up, allows the moving of the leftmost formula in the context to the stoup position whenever the stoup is empty.
4. The logical connectives of NMILL (and associative Lambek calculus) typically include two ordered implications $\diagdown$ and $\diagup$, which are two variants of linear implication arising from the removal of the exchange rule from intuitionistic linear logic. In LSkG, only the right residuation ($B \diagup A = A \multimap B$) of Lambek calculus is present.

For a more detailed explanation and a linear logical interpretation of LSkG, see [24, Section 2].

Theorem 2.2. LSkG *is cut-free, i.e. the rules*

$$\frac{\overset{f}{S \mid \Gamma \vdash_{\mathsf{G}} A} \quad \overset{g}{A \mid \Delta \vdash_{\mathsf{G}} C}}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} C}\ \mathsf{scut} \qquad \frac{\overset{f}{- \mid \Gamma \vdash_{\mathsf{G}} A} \quad \overset{g}{S \mid \Delta_0, A, \Delta_1 \vdash_{\mathsf{G}} C}}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash_{\mathsf{G}} C}\ \mathsf{ccut}$$

are admissible.

Proof. The proof proceeds by induction on the height of derivations and the complexity of cut formulae. Specifically, for scut, we first perform induction on the left premise f, and if necessary, we perform subinduction on g or the complexity of the cut formula A. For ccut, we start by performing induction on the right premise g instead. The cases other than $\multimap$L and $\multimap$R have been discussed in [28, Lemma 5], so we will only elaborate on the cases of $\multimap$.

We first deal with scut. If $f = {\multimap}\mathsf{L}(f', f'')$, then we permute scut up, i.e.

$$\frac{\dfrac{\overset{f'}{- \mid \Gamma \vdash_{\mathsf{G}} A'} \quad \overset{f''}{B' \mid \Delta \vdash_{\mathsf{G}} A}}{A' \multimap B' \mid \Gamma, \Delta \vdash_{\mathsf{G}} A}\ {\multimap}\mathsf{L} \quad \overset{g}{A \mid \Lambda \vdash_{\mathsf{G}} C}}{A' \multimap B' \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\ \mathsf{scut} \quad \mapsto \quad \frac{\overset{f'}{- \mid \Gamma \vdash_{\mathsf{G}} A'} \quad \dfrac{\overset{f''}{B' \mid \Delta \vdash_{\mathsf{G}} A} \quad \overset{g}{A \mid \Lambda \vdash_{\mathsf{G}} C}}{B' \mid \Delta, \Lambda \vdash_{\mathsf{G}} C}\ \mathsf{scut}}{A' \multimap B' \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\ {\multimap}\mathsf{L}$$

If $f = {\multimap}\mathsf{R}\ f'$, then we perform a subinduction on g:

– If $g = {\multimap}\mathsf{L}(g', g'')$, then

$$\dfrac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B}}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\,{\multimap}\mathsf{R} \quad \dfrac{\overset{g'}{- \mid \Delta \vdash_{\mathsf{G}} A} \quad \overset{g''}{B \mid \Lambda \vdash_{\mathsf{G}} C}}{A \multimap B \mid \Delta, \Lambda \vdash_{\mathsf{G}} C}\,{\multimap}\mathsf{L}}{S \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{scut} \quad \mapsto \quad \dfrac{\overset{g'}{- \mid \Delta \vdash_{\mathsf{G}} A} \quad \dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B} \quad \overset{g''}{B \mid \Lambda \vdash_{\mathsf{G}} C}}{S \mid \Gamma, A, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{scut}}{S \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{ccut}$$

where the complexity of the cut formulae is reduced.

– For other rules, we permute scut up. For example, if $g = {\multimap}\mathsf{R}\ g'$, then

$$\dfrac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B}}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\,{\multimap}\mathsf{R} \quad \dfrac{\overset{g'}{A \multimap B \mid \Delta, A' \vdash_{\mathsf{G}} B'}}{A \multimap B \mid \Delta \vdash_{\mathsf{G}} A' \multimap B'}\,{\multimap}\mathsf{R}}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A' \multimap B'}\,\mathsf{scut} \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B}}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\,{\multimap}\mathsf{R} \quad \overset{g'}{A \multimap B \mid \Delta, A' \vdash_{\mathsf{G}} B'}}{S \mid \Gamma, \Delta, A' \vdash_{\mathsf{G}} B'}\,\mathsf{scut}}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A' \multimap B'}\,{\multimap}\mathsf{R}$$

For ccut, if $g = {\multimap}\mathsf{R}\ g'$, then we permute ccut up. If $g = {\multimap}\mathsf{L}(g', g'')$, we permute ccut up as well, but depending on where the cut formula is placed, we either apply ccut on f and g' or f and g''. □

Moreover, LSkG is sound and complete wrt. left skew monoidal closed categories [24, Theorem 3.2].

By soundness and completeness, similar to the result in [28] for skew monoidal categories, we mean that LSkG is deductively equivalent to the axiomatic characterization of the free left skew monoidal closed category.

$$\dfrac{}{A \vdash_{\mathsf{L}} A}\,\mathsf{id} \qquad \dfrac{A \vdash_{\mathsf{L}} B \quad B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} C}\,\mathsf{comp} \qquad \dfrac{A \vdash_{\mathsf{L}} C \quad B \vdash_{\mathsf{L}} D}{A \otimes B \vdash_{\mathsf{L}} C \otimes D}\,\otimes \qquad \dfrac{C \vdash_{\mathsf{L}} A \quad B \vdash_{\mathsf{L}} D}{A \multimap B \vdash_{\mathsf{L}} C \multimap D}\,\multimap$$

$$\dfrac{}{\mathsf{I} \otimes A \vdash_{\mathsf{L}} A}\,\lambda \qquad \dfrac{}{A \vdash_{\mathsf{L}} A \otimes \mathsf{I}}\,\rho \qquad \dfrac{}{(A \otimes B) \otimes C \vdash_{\mathsf{L}} A \otimes (B \otimes C)}\,\alpha \qquad \dfrac{A \otimes B \vdash_{\mathsf{L}} C}{\overline{\overline{A \vdash_{\mathsf{L}} B \multimap C}}}\,\pi$$

In particular, this is a semi-unital and semi-associative variation of Moortgat and Oehrle's calculus [20, Chapter 4] of non-associative Lambek calculus (NL), where only right residuation is present. We only care about sequent derivability in this section, therefore we omit the congruence relations on sets of derivations $A \vdash_{\mathsf{L}} B$ and $S \mid \Gamma \vdash_{\mathsf{G}} A$ that identify certain pairs of derivations. However, the congruence relations are essential for these calculi being correct characterizations of the free left skew monoidal closed category.

The calculus LSkG, being an equivalent presentation of a skew version of NL, provides an effective procedure to determine formulae derivability in LSkNL. In other words, for any formula A, $\vdash_{\mathsf{L}} A$ if and only if $- \mid \ \vdash_{\mathsf{G}} A$. Exhaustive proof search in LSkG always terminates, so for any A, either it finds a proof or it fails and there is no proof

Adapted from [20], we define trees inductively by the grammar $T ::= \mathsf{Fma} \mid - \mid (T, T)$, where $-$ is an empty tree. A context is a tree with a hole defined recursively as $\mathscr{C} ::= [\cdot] \mid (\mathscr{C}, T) \mid (T, \mathscr{C})$. The substitution of a tree into a hole is defined recursively:

$$\begin{array}{rcl} subst([\cdot], U) &=& U \\ subst((T', \mathscr{C}), U) &=& (T', subst(\mathscr{C}, U)) \\ subst((\mathscr{C}, T'), U) &=& (subst(\mathscr{C}, U), T') \end{array}$$

We use $T[\cdot]$ to denote a context and $T[U]$ to abbreviate $subst(T[\cdot], U)$. Sometimes we omit parentheses for trees when it does not cause ambiguity. Sequents in LSkT are in the form $T \vdash_{\mathsf{T}} A$ where T is a tree

and A is a single formula.
Derivations in LSkT are generated recursively by following rules:

$$\frac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}$$

$$\text{(logical rules)} \quad \frac{T[-] \vdash_{\mathsf{T}} C}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL} \quad \frac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \frac{T[A,B] \vdash_{\mathsf{T}} C}{T[A \otimes B] \vdash_{\mathsf{T}} C}\ \otimes\mathsf{L} \quad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T,U \vdash_{\mathsf{T}} A \otimes B}\ \otimes\mathsf{R}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap B, U] \vdash_{\mathsf{T}} C}\ \multimap\mathsf{L} \quad \frac{T, A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap B}\ \multimap\mathsf{R}$$

$$\text{(structural rules)} \quad \frac{T[U_0,(U_1,U_2)] \vdash_{\mathsf{T}} C}{T[(U_0,U_1),U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc} \quad \frac{T[U] \vdash_{\mathsf{T}} C}{T[-,U] \vdash_{\mathsf{T}} C}\ \mathsf{unitL} \quad \frac{T[U,-] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR}$$

This calculus is similar to the ones for NL [20] and NL with unit [9] but with semi-associative (assoc) and semi-unital (unitL and unitR) rules. The structural rule unitL, read bottom-up, removes an empty tree from the left. It helps us to correctly characterize the axiom λ in LSkT, i.e. $\mathsf{I} \otimes A \vdash_{\mathsf{T}} A$ is derivable while $A \vdash_{\mathsf{T}} \mathsf{I} \otimes A$ is not. Analogously for the rule unitR, from a bottom-up perspective, adds an empty tree from the right, and we cannot capture ρ in LSkT without unitR (a double question mark ?? means that there is no rule can be applied):

$$\frac{\frac{\frac{\frac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}}{-,A \vdash_{\mathsf{T}} A}\ \mathsf{unitL}}{\mathsf{I},A \vdash_{\mathsf{T}} A}\ \mathsf{IL}}{\mathsf{I} \otimes A \vdash_{\mathsf{T}} A}\ \otimes\mathsf{L} \qquad \frac{\frac{\overset{??}{X \vdash_{\mathsf{T}} \mathsf{I}} \quad \overset{??}{- \vdash_{\mathsf{T}} X}}{X,- \vdash_{\mathsf{T}} \mathsf{I} \otimes X}\ \otimes\mathsf{R}}{X \vdash_{\mathsf{T}} \mathsf{I} \otimes X}\ \mathsf{unitR} \qquad \frac{\frac{\frac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \frac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR}}{A,- \vdash_{\mathsf{T}} A \otimes \mathsf{I}}\ \otimes\mathsf{R}}{A \vdash_{\mathsf{T}} A \otimes I}\ \mathsf{unitR} \qquad \frac{\frac{\overset{??}{X,- \vdash_{\mathsf{T}} X}}{X,\mathsf{I} \vdash_{\mathsf{T}} X}\ \mathsf{IL}}{X \otimes \mathsf{I} \vdash_{\mathsf{T}} X}\ \otimes\mathsf{L}$$

Theorem 2.3. LSkT *is cut-free, i.e. the rule*

$$\frac{\overset{f}{U \vdash_{\mathsf{T}} A} \quad \overset{g}{T[A] \vdash_{\mathsf{T}} C}}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

is admissible.

Proof. We perform induction on the structure of derivation f of the left premise, and if necessary, we perform subinduction on the derivation g or the complexity of the cut formula A. Cases of logical rules $\mathsf{ax}, \otimes\mathsf{L}, \otimes\mathsf{R}, \multimap\mathsf{L}$, and $\multimap\mathsf{R}$ have been discussed in [20], so we only elaborate on the new cases arising in LSkT.

- The first new case is that $f = \mathsf{IR}$, then we inspect the structure of g.
  - If $g = \mathsf{ax} : \mathsf{I} \vdash_{\mathsf{T}} \mathsf{I}$, then we define $\mathsf{cut}(\mathsf{IR}, \mathsf{ax}) = \mathsf{IR}$.
  - If $g = \mathsf{IL}\ g'$, then there are two subcases:
    * if the I introduced by IL is the cut formula, then we define

$$\frac{\frac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \frac{\overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}}{T[-] \vdash_{\mathsf{T}} C}\ \mathsf{cut} \quad \mapsto \quad \overset{g'}{T[-] \vdash_{\mathsf{T}} C}$$

* if the I introduced by IL is not the cut formula, then we define

$$\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \dfrac{\overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}}{T^{\{\mathsf{I}:=-\}}[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{cut} \quad \mapsto \quad \dfrac{\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{ax} \quad \overset{g'}{T - |\vdash_{\mathsf{T}} C}}{T^{\{\mathsf{I}:=-\}}[-] \vdash_{\mathsf{T}} C}\ \mathsf{cut}}{T^{\{\mathsf{I}:=-\}}[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}$$

where $T^{\{\mathsf{I}:=-\}}[\cdot]$ means that a formula occurrence I at some fixed position in the context has been replaced by $-$.

– If $g = \mathscr{R}\ g'$, where $\mathscr{R}$ is a one-premise rule other than IL, then $\mathsf{cut}(\mathsf{IR}, \mathscr{R}\ g') = \mathscr{R}(\mathsf{cut}(\mathsf{IR}, g'))$.

– The cases of an arbitrary two-premises rule are similar.

- Other new cases (IL and structural rules) are in the type of one-premise left rules, where we can permute cut up. For example, if $f = \mathsf{unitL}\ f'$, then we define

$$\dfrac{\dfrac{\overset{f'}{T'[U] \vdash_{\mathsf{T}} A}}{T'[-, U] \vdash_{\mathsf{T}} A}\ \mathsf{unitL} \quad \overset{g}{T[A] \vdash_{\mathsf{T}} C}}{T[T'[-, U]] \vdash_{\mathsf{T}} C}\ \mathsf{cut} \quad \mapsto \quad \dfrac{\dfrac{\overset{f'}{T'[U] \vdash_{\mathsf{T}} A} \quad \overset{g}{T[A] \vdash_{\mathsf{T}}}}{T[T'[U]] \vdash_{\mathsf{T}} C}\ \mathsf{cut}}{T[T'[-, U]] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}$$

The other cases are similar.

□

The proof of equivalence relies on the following admissible rule, lemma and definition.

$$\dfrac{T[\mathsf{I}] \vdash_{\mathsf{T}} C}{T[-] \vdash_{\mathsf{T}} C}\ \mathsf{IL}^{-1}$$

Lemma 2.4. *Given a context $T[\cdot]$ and a derivation $f : A \mid\ \vdash_{\mathsf{G}} B$, there exists a derivation $f^* : T[A]^* \mid\ \vdash_{\mathsf{G}} T[B]^*$, where T^* transforms a tree into a formula by replacing commas with $\otimes$ and $-$ with I, respectively.*

Proof. Proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then we have $T[A]^* = A$ and $T[B]^* = B$, and $f : A \mid\ \vdash_{\mathsf{G}} B$ by assumption.
If $T[\cdot] = T'[\cdot], T''$, then by inductive hypothesis, we have $f^* : T'[A]^* \mid\ \vdash_{\mathsf{G}} T'[B]^*$ and following derivation:

$$\dfrac{\dfrac{\overset{f^*}{T'[A]^* \mid\ \vdash_{\mathsf{G}} T'[B]^*} \quad \dfrac{\dfrac{}{T''^* \mid\ \vdash_{\mathsf{G}} T''^*}\ \mathsf{ax}}{- \mid T''^* \vdash_{\mathsf{G}} T''^*}\ \mathsf{pass}}{T'[A]^* \mid T''^* \vdash_{\mathsf{G}} T'[B]^* \otimes T''^*}\ \otimes\mathsf{R}}{T'[A]^* \otimes T''^* \mid\ \vdash_{\mathsf{G}} T'[B]^* \otimes T''^*}\ \otimes\mathsf{L}$$

The other case ($T[\cdot] = T'', T'[\cdot]$) is symmetric. □

Definition 2.5. We define an encoding function $[\![- \mid -]\!]$ that transforms a tree and an ordered list of formulae into a tree associating to the left:

$$[\![T \mid [\,]]\!] = T$$
$$[\![T \mid B, \Gamma]\!] = [\![(T, B) \mid \Gamma]\!]$$

With the above lemmata, definition, and functions $s(S)$ that maps a stoup to a formula (i.e. $s(S) = I$ if $S = -$ or $s(S) = B$ if $S = B$) and T^* that transforms trees into formulae, we can state and prove the equivalence between `LSkG` and `LSkT`.

Theorem 2.6. *The calculi* `LSkG` *and* `LSkT` *are equivalent, meaning that the two statements below are true:*

- *For any derivation* $f : S \mid \Gamma \vdash_{\mathsf{G}} C$, *there exists a derivation* $\mathsf{G2T} f : [\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} C$.
- *For any derivation* $f : T \vdash_{\mathsf{T}} C$, *there exists a derivation* $\mathsf{T2G} f : T^* \mid \ \vdash_{\mathsf{G}} C$.

Proof. Both $\mathsf{G2T}$ and $\mathsf{T2G}$ are proved by induction on height of f.

For $\mathsf{G2T}$, the interesting cases are $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$. For example, if $f = \otimes\mathsf{R}(f', f'')$, then by inductive hypothesis, we have two derivations $\mathsf{G2T}\ f' : [\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} A$ and $\mathsf{G2T}\ f'' : [\![\mathsf{I} \mid \Delta]\!] \vdash_{\mathsf{T}} B$. Our goal sequent is $[\![[\![s(S) \mid \Gamma]\!] \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B$, which is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{cc}\mathsf{G2T}\ f' & \mathsf{G2T}\ f'' \\ [\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} A & [\![\mathsf{I} \mid \Delta]\!] \vdash_{\mathsf{T}} B\end{array}}{[\![s(S) \mid \Gamma]\!], [\![\mathsf{I} \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \otimes\mathsf{R}}{[\![[\![s(S) \mid \Gamma]\!], \mathsf{I} \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \mathsf{assoc}^*}{[\![[\![s(S) \mid \Gamma]\!], - \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \mathsf{IL}^{-1}}{[\![[\![s(S) \mid \Gamma]\!] \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \mathsf{unitR}$$

where assoc^* means multiple applications of assoc. The case of $\multimap\mathsf{L}$ is similar.

For $\mathsf{T2G}$, the proof relies on Lemma 2.4 heavily. For example, when $f = \mathsf{unitR}\ g$, where we have $g : T[U, -] \vdash_{\mathsf{T}} C$. By inductive hypothesis, we have $\mathsf{T2G}\ g : T[U^* \otimes \mathsf{I}]^* \mid \ \vdash_{\mathsf{G}} C$. With Lemma 2.4, we construct the desired derivation as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{U^* \mid \ \vdash_{\mathsf{G}} U^*}\ \mathsf{ax} \qquad \dfrac{}{- \mid \ \vdash_{\mathsf{G}} \mathsf{I}}\ \mathsf{IR}}{U^* \mid \ \vdash_{\mathsf{G}} U^* \otimes \mathsf{I}}\ \otimes\mathsf{R}}{T[U^*]^* \mid \ \vdash_{\mathsf{G}} T[U^* \otimes \mathsf{I}]^*}\ \text{Lemma 2.4} \qquad \begin{array}{c}\mathsf{T2G}\ g \\ T[U^* \otimes \mathsf{I}]^* \mid \ \vdash_{\mathsf{G}} C\end{array}}{T[U^*]^* \mid \ \vdash_{\mathsf{G}} C}\ \mathsf{scut}$$

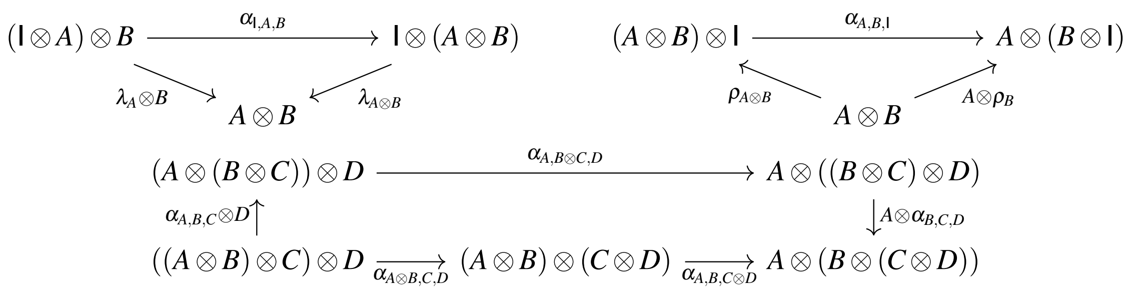
The other cases are similar. □

3 Skew Categories

In this section, we present the definitions of left (right) skew monoidal closed categories, skew monoidal bi-closed categories, and various terms that will be used in the following section for discussion.

Definition 3.1. A *left skew monoidal closed category* $\mathbb{C}$ is a category with a unit object I and two functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ forming an adjunction $- \otimes B \dashv B \multimap -$ for all B, and three natural transformations λ, ρ, α typed $\lambda_A : \mathsf{I} \otimes A \to A$, $\rho_A : A \to A \otimes \mathsf{I}$ and $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, satisfying coherence conditions on morphisms due to Mac Lane [17]:

$$\begin{array}{ccccc} & & \mathsf{I} \otimes \mathsf{I} & & \\ & \overset{\rho_{\mathsf{I}}}{\nearrow} & & \overset{\lambda_{\mathsf{I}}}{\searrow} & \\ \mathsf{I} & & = \!=\!=\!= & & \mathsf{I} \end{array} \qquad \begin{array}{ccc} (A \otimes \mathsf{I}) \otimes B & \xrightarrow{\alpha_{A,\mathsf{I},B}} & A \otimes (\mathsf{I} \otimes B) \\ {\scriptstyle \rho_A \otimes B} \uparrow & & \downarrow {\scriptstyle A \otimes \lambda_B} \\ A \otimes B & = \!=\!=\!= & A \otimes B \end{array}$$



Left skew monoidal closed category has other equivalent characterizations [22, 25], because natural transformations (λ, ρ, α) are in bijective correspondence with tuples of (extra)natural transformations (j, i, L) typed $j_A : \mathsf{I} \to A \multimap A$, $i_A : \mathsf{I} \multimap A \to A$, and $L_{A,B,C} : B \multimap C \to (A \multimap B) \multimap (A \multimap C)$. In particular, in a left skew *non-monoidal* closed category, (λ, ρ, α) are not available and one has to work with (j, i, L) and corresponding equations.

Definition 3.2. A *right skew monoidal closed category* $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ is defined with the same objects and adjoint functors as a in left skew monoidal closed category but three natural transformations λ^{R}, ρ^{R}, α^{R} are typed $\lambda^{\mathsf{R}}_A : A \to \mathsf{I} \otimes A$, $\rho^{\mathsf{R}}_A : A \otimes \mathsf{I} \to A$ and $\alpha^{\mathsf{R}}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$. The equations on morphisms are analogous but modified to fit the definition.

Similar to left skew monoidal closed categories, natural transformations $(\lambda^{\mathsf{R}}, \rho^{\mathsf{R}}, \alpha^{\mathsf{R}})$ are in bijective correspondence with tuples $(j^{\mathsf{R}}, i^{\mathsf{R}}, L^{\mathsf{R}})$ typed $j^{\mathsf{R}}_{A,B} : \mathbb{C}(\mathsf{I}, A \multimap B) \to \mathbb{C}(A, B)$, $i^{\mathsf{R}}_A : A \to \mathsf{I} \multimap A$, and $L^{\mathsf{R}}_{A,B,C,D} : \mathbb{C}(A, B \multimap (C \multimap D)) \to \int^X X.\mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X)$, where $\int^X$ is a coend, cf. [25, Section 4], and $\mathbb{C}(A,B)$ means the set of morphisms from A to B. In parts of the next sections, where we only work with thin categories (for any two objects A and B, $\mathbb{C}(A,B)$ is either empty or a singleton set), it is safe to replace $\int^X$ with an existential quantifier.
In the rest of the paper, we usually omit subscripts of natural transformations.

Definition 3.3. A left skew monoidal closed category is

– *associative normal* if α is a natural isomorphism;

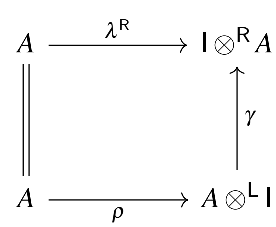
– *left unital normal* if λ is a natural isomorphism;

– *right unital normal* if ρ is a natural isomorphism.

The (j, i, L) version is similar. The case of right skew monoidal closed categories is analogous.

Definition 3.4. A category $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{L}}, \otimes^{\mathsf{R}}, \multimap^{\mathsf{R}})$ is skew monoidal bi-closed (SkMBiC) if there exists a natural isomorphism $\gamma : A \otimes^{\mathsf{L}} B \to B \otimes^{\mathsf{R}} A$, $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{L}})$ is left skew monoidal closed such that right skew structural rules are dictated by the left skew ones via γ.
This definition combines concepts from skew bi-monoidal and bi-closed categories as introduced in [25].

Example 3.5. λ^{R} is defined as $\gamma \circ \rho$, diagrammatically:



In contrast to the categorical model of associative Lambek calculus, the monoidal bi-closed category, we do not have both left ($\diagdown$) and right residuation ($\diagup$), but instead have two right residuations corresponding to different tensor products. However, with the natural isomorphism γ, and selecting a specific tensor, we can simulate both left and right residuations.

In the remainder of the paper, we will develop axiomatic and sequent calculi for SkMBiC and explore its relational semantics.

4 Calculi for SkMBiC

By defining new formulae and adding rules in LSkNL, we can have an axiomatic calculus SkMBiCA, where formulae (Fma) are inductively generated by the grammar $A,B ::= X \mid \mathsf{I} \mid A \otimes^{\mathsf{L}} B \mid A \multimap^{\mathsf{L}} B \mid A \otimes^{\mathsf{R}} B \mid A \multimap^{\mathsf{R}} B$. X and I adhere to the definitions provided in Section 2, and $\otimes^{\mathsf{L}}$ and $\multimap^{\mathsf{L}}$ ($\otimes^{\mathsf{R}}$ and $\multimap^{\mathsf{R}}$) represent left (right) skew multiplicative conjunction and implication, respectively.
Derivations in SkMBiCA are inductively generated by following rules:

$$\frac{}{A \vdash_{\mathsf{L}} A}\ \mathsf{id} \qquad \frac{A \vdash_{\mathsf{L}} B \quad B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} C}\ \mathsf{comp}$$

$$\frac{A \vdash_{\mathsf{L}} C \quad B \vdash_{\mathsf{L}} D}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}} \qquad \frac{C \vdash_{\mathsf{L}} A \quad B \vdash_{\mathsf{L}} D}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{L}} C \multimap^{\mathsf{L}} D}\ \multimap^{\mathsf{L}} \qquad \frac{C \vdash_{\mathsf{L}} A \quad B \vdash_{\mathsf{L}} D}{A \multimap^{\mathsf{R}} B \vdash_{\mathsf{L}} C \multimap^{\mathsf{R}} D}\ \multimap^{\mathsf{R}}$$

$$\frac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} A}\ \lambda \qquad \frac{}{A \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} \mathsf{I}}\ \rho \qquad \frac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \alpha$$

$$\frac{}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{R}} A}\ \gamma \qquad \frac{}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{L}} A}\ \gamma^{-1} \qquad \frac{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} B \multimap^{\mathsf{L}} C}\ \pi \qquad \frac{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} B \multimap^{\mathsf{R}} C}\ \pi^{\mathsf{R}}$$

For any $f : A \vdash_{\mathsf{L}} B$ and $g : C \vdash_{\mathsf{L}} D$, we define $f \otimes^{\mathsf{R}} g$ as $\gamma \circ (g \otimes^{\mathsf{L}} f) \circ \gamma^{-1}$. λ^{R}, ρ^{R}, and α^{R} are also derivable.
Similar to the constructions in [28, 27, 26, 29, 24], SkMBiCA generates the free SkMBiC (FSkMBiC(At)) over a set At in the following way:

– Objects of FSkMBiC(At) are formulae (Fma).
– Morphisms between formulae A and B are derivations of sequents $A \vdash_{\mathsf{L}} B$ and identified up to the congruence relation $\doteq$:

(category laws)	$\mathsf{id} \circ f \doteq f \qquad f \doteq f \circ \mathsf{id} \qquad (f \circ g) \circ h \doteq f \circ (g \circ h)$
($\otimes^{\mathsf{L}}$ functorial)	$\mathsf{id} \otimes^{\mathsf{L}} \mathsf{id} \doteq \mathsf{id} \qquad (h \circ f) \otimes^{\mathsf{L}} (k \circ g) \doteq h \otimes^{\mathsf{L}} k \circ f \otimes^{\mathsf{L}} g$
($\multimap^{\mathsf{L}}$ functorial)	$\mathsf{id} \multimap^{\mathsf{L}} \mathsf{id} \doteq \mathsf{id} \qquad (f \circ h) \multimap^{\mathsf{L}} (k \circ g) \doteq h \multimap^{\mathsf{L}} k \circ f \multimap^{\mathsf{L}} g$
($\multimap^{\mathsf{R}}$ functorial)	$\mathsf{id} \multimap^{\mathsf{R}} \mathsf{id} \doteq \mathsf{id} \qquad (f \circ h) \multimap^{\mathsf{R}} (k \circ g) \doteq h \multimap^{\mathsf{R}} k \circ f \multimap^{\mathsf{R}} g$
(λ, ρ, α nat. trans.)	$\lambda \circ \mathsf{id} \otimes^{\mathsf{L}} f \doteq f \circ \lambda$ $\rho \circ f \doteq f \otimes^{\mathsf{L}} \mathsf{id} \circ \rho$ $\alpha \circ (f \otimes^{\mathsf{L}} g) \otimes^{\mathsf{L}} h \doteq f \otimes^{\mathsf{L}} (g \otimes^{\mathsf{L}} h) \circ \alpha$
(Mac Lane axioms)	$\lambda \circ \rho \doteq \mathsf{id} \qquad \mathsf{id} \doteq \mathsf{id} \otimes^{\mathsf{L}} \lambda \circ \alpha \circ \rho \otimes^{\mathsf{L}} \mathsf{id}$ $\lambda \circ \alpha \doteq \lambda \otimes^{\mathsf{L}} \mathsf{id} \qquad \alpha \circ \rho \doteq \mathsf{id} \otimes^{\mathsf{L}} \rho$ $\alpha \circ \alpha \doteq \mathsf{id} \otimes^{\mathsf{L}} \alpha \circ \alpha \circ \alpha \otimes^{\mathsf{L}} \mathsf{id}$
(γ isomorphism)	$\gamma \circ \gamma^{-1} \doteq \mathsf{id} \qquad \gamma^{-1} \circ \gamma \doteq \mathsf{id}$
($\pi^{(\mathsf{R})}$ nat. trans.)	$\pi f \circ g \doteq \pi(f \circ (g \otimes^{\mathsf{L}} \mathsf{id})) \qquad \pi(f \circ g) \doteq (\mathsf{id} \multimap^{\mathsf{L}} f) \circ \pi g$ $\pi(\mathsf{id} \otimes^{\mathsf{L}} f) \doteq (g \multimap^{\mathsf{L}} \mathsf{id}) \circ \pi \mathsf{id} \qquad \pi^{\mathsf{R}}(\mathsf{id} \otimes^{\mathsf{R}} f) \doteq (g \multimap^{\mathsf{R}} \mathsf{id}) \circ \pi^{\mathsf{R}} \mathsf{id}$ $\pi^{\mathsf{R}} f \circ g \doteq \pi^{\mathsf{R}}(f \circ (g \otimes^{\mathsf{R}} \mathsf{id})) \qquad \pi^{\mathsf{R}}(f \circ g) \doteq (\mathsf{id} \multimap^{\mathsf{R}} f) \circ \pi^{\mathsf{R}} g$
($\pi^{(\mathsf{R})}$ isomorphism)	$\pi(\pi^{-1} f) \doteq f \qquad \pi^{-1}(\pi f) \doteq f \qquad \pi^{\mathsf{R}}(\pi^{\mathsf{R}-1} f) \doteq f \qquad \pi^{\mathsf{R}-1}(\pi^{\mathsf{R}} f) \doteq f$

Notice that by the definition of $f \otimes^{\mathsf{R}} g$ and γ being an isomorphism, γ and γ^{-1} are natural transformations. For example, $\gamma \circ f \otimes^{\mathsf{L}} g \doteq \gamma \circ f \otimes^{\mathsf{L}} g \circ \mathsf{id} \doteq \gamma \circ f \otimes^{\mathsf{L}} g \circ \gamma^{-1} \circ \gamma = g \otimes^{\mathsf{R}} f \circ \gamma$. Similarly, naturality of $(\lambda^{\mathsf{R}}, \rho^{\mathsf{R}}, \alpha^{R})$ and corresponding Mac Lane axioms hold as well.

Given a skew monoidal bi-closed category $\mathbb{D}$ with function $G : \mathsf{At} \to \mathbb{D}$, we can define functions $\overline{G}_0 : \mathsf{Fma} \to \mathbb{D}_0$ ($\mathbb{D}_0$ is the collection of objects in $\mathbb{D}$) and $\overline{G}_1 : \mathsf{FSkMBiC(At)}(A,B) \to \mathbb{D}(\overline{G}_0(A), \overline{G}_0(B))$ by induction on complexity of formulae and height of derivations respectively. This construction uniquely specifies a strict skew monoidal bi-closed functor $\overline{G} : \mathsf{FSkMBiC} \to \mathbb{D}$ satisfying $\overline{G}(X) = G(X)$.

However, it remains unclear how to construct a sequent calculus à la Girard for SkMBiC. A simpler scenario to consider is the sequent calculus for right skew monoidal closed categories. In this context, recalling Definition 3.2, where natural transformations are in an opposite direction compared to left skew monoidal closed categories. One approach is to propose a dual sequent calculus to `LSkG`. Here, sequents would be of the form $\Gamma \mid S \vdash_{\mathsf{G}} A$, indicating a reversal of stoup and context, with all left rules applicable solely to the stoup. We should think of the antecedents as trees associating to the right, structured as $(A_n, (\ldots, (A_1, A_0)) \ldots)$. Nevertheless, $\multimap^{\mathsf{R}}$, by definition, is again a right residuation, implying that $\multimap^{\mathsf{R}}\mathsf{L}$ and $\multimap^{\mathsf{R}}\mathsf{R}$ should resemble those in `LSkG`. This requirement then necessitates contexts to appear on the right-hand side of the stoup.

Fortunately, we can develop a sequent calculus, denoted as `SkMBiCT`, which is inspired by `LSkT` to characterize SkMBiC categories. Specifically, `SkMBiCT` is an instantiation of Moortgat's multimodal Lambek calculus [18] with unit, semi-unital, and semi-associative structural rules.

Trees in `SkMBiCT` are inductively defined by the grammar $T ::= \mathsf{Fma} \mid - \mid (T,T) \mid (T;T)$. What we have defined are trees with two different ways of linking nodes: through the use of commas and semicolons, corresponding to $\otimes^{\mathsf{L}}$ and $\otimes^{\mathsf{R}}$, respectively. Contexts and substitution are defined analogously to those of `LSkT`. Sequents are in the form $T \vdash_{\mathsf{T}} A$ analogous to those in Section 2.

Derivations in `SkMBiCT` are generated recursively by following rules:

$$\frac{}{A \vdash_{\mathsf{T}} A}\,\mathsf{ax} \qquad \frac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\,\mathsf{IR} \qquad \frac{T[-] \vdash_{\mathsf{T}} C}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\,\mathsf{IL}$$

(logical rules)

$$\frac{T[A,B] \vdash_{\mathsf{T}} C}{T[A \otimes^{\mathsf{L}} B] \vdash_{\mathsf{T}} C}\,\otimes^{\mathsf{L}}\mathsf{L} \qquad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T,U \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} B}\,\otimes^{\mathsf{L}}\mathsf{R} \qquad \frac{T[A;B] \vdash_{\mathsf{T}} C}{T[A \otimes^{\mathsf{R}} B] \vdash_{\mathsf{T}} C}\,\otimes^{\mathsf{R}}\mathsf{L} \qquad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T;U \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\,\otimes^{\mathsf{R}}\mathsf{R}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{L}} B, U] \vdash_{\mathsf{T}} C}\,\multimap^{\mathsf{L}}\mathsf{L} \qquad \frac{T, A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{L}} B}\,\multimap^{\mathsf{L}}\mathsf{R} \qquad \frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{R}} B; U] \vdash_{\mathsf{T}} C}\,\multimap^{\mathsf{R}}\mathsf{L} \qquad \frac{T; A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{R}} B}\,\multimap^{\mathsf{R}}\mathsf{R}$$

(structural rules)

$$\frac{T[U_0,(U_1,U_2)] \vdash_{\mathsf{T}} C}{T[(U_0,U_1),U_2] \vdash_{\mathsf{T}} C}\,\mathsf{assoc}^{\mathsf{L}} \qquad \frac{T[U_0,U_1] \vdash_{\mathsf{T}} C}{T[U_1;U_0] \vdash_{\mathsf{T}} C}\,\otimes\mathsf{comm} \qquad \frac{T[(U_0;U_1);U_2] \vdash_{\mathsf{T}} C}{T[U_0;(U_1;U_2)] \vdash_{\mathsf{T}} C}\,\mathsf{assoc}^{\mathsf{R}}$$

$$\frac{T[U] \vdash_{\mathsf{T}} C}{T[-,U] \vdash_{\mathsf{T}} C}\,\mathsf{unitL}^{\mathsf{L}} \qquad \frac{T[U,-] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\,\mathsf{unitR}^{\mathsf{L}} \qquad \frac{T[U] \vdash_{\mathsf{T}} C}{T[U;-] \vdash_{\mathsf{T}} C}\,\mathsf{unitL}^{\mathsf{R}} \qquad \frac{T[-;U] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\,\mathsf{unitR}^{\mathsf{R}}$$

We can think of these rules as originating from two separate calculi: `LSkT` (the red part with ax, IR, and IL) and another for right skew monoidal closed categories (`RSkT`, the blue part with ax, IR, and IL), linked by $\otimes\mathsf{comm}$, in other words, we can mimic all the blue rules in the style of `LSkT` (only commas appear in antecedents) and vice versa. For example, we can express $\otimes^{\mathsf{R}}\mathsf{L}$, $\otimes^{\mathsf{R}}\mathsf{R}$ and $\multimap^{\mathsf{R}}\mathsf{L}$ in the style of `LSkT`:

$$\frac{T[A,B] \vdash_{\mathsf{T}} C}{T[B \otimes^{\mathsf{R}} A] \vdash_{\mathsf{T}} C}\,\otimes^{\mathsf{R}}\mathsf{L}' \quad = \quad \frac{\dfrac{T[A,B] \vdash_{\mathsf{T}} C}{T[B;A] \vdash_{\mathsf{T}} C}\,\otimes\mathsf{comm}}{T[B \otimes^{\mathsf{R}} A] \vdash_{\mathsf{T}} C}\,\otimes^{\mathsf{R}}\mathsf{L} \qquad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{U,T \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\,\otimes^{\mathsf{R}}\mathsf{R}' \quad = \quad \frac{\dfrac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T;U \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\,\otimes^{\mathsf{R}}\mathsf{L}}{U,T \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\,\otimes\mathsf{comm}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[U, A \multimap^{\mathsf{R}} B] \vdash_{\mathsf{T}} C}\,\multimap^{\mathsf{R}}\mathsf{L}' \quad = \quad \frac{\dfrac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{R}} B; U] \vdash_{\mathsf{T}} C}\,\multimap^{\mathsf{R}}\mathsf{L}}{T[U, A \multimap^{\mathsf{R}} B] \vdash_{\mathsf{T}} C}\,\otimes\mathsf{comm}$$

Theorem 4.1. *Similar to* LSkT*,* cut *is admissible in* SkMBiCT*.*

$$\frac{U \vdash_{\mathsf{T}} A \quad T[A] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

Proof. The proof proceeds similarly to that of Theorem 2.3. In particular, the new rules (⊗comm and the structural rules in blue) are all one-premise left rules, allowing us to permute cut upwards. □

The equivalence between SkMBiCA and SkMBiCT can be proved by induction on height of derivations with a lemma similar to Lemma 2.4 and the following admissible rules:

$$\frac{T[A \otimes^{\mathsf{L}} B] \vdash_{\mathsf{L}} C}{T[A,B] \vdash_{\mathsf{L}} C}\ \otimes^{\mathsf{L}}\mathsf{L}^{-1} \quad \frac{T \vdash_{\mathsf{L}} A \multimap^{\mathsf{L}} B}{T,A \vdash_{\mathsf{L}} B}\ \multimap^{\mathsf{L}}\mathsf{R}^{-1} \quad \frac{T[A \otimes^{\mathsf{R}} B] \vdash_{\mathsf{L}} C}{T[A;B] \vdash_{\mathsf{L}} C}\ \otimes^{\mathsf{R}}\mathsf{L}^{-1} \quad \frac{T \vdash_{\mathsf{L}} A \multimap^{\mathsf{R}} B}{T;A \vdash_{\mathsf{L}} B}\ \multimap^{\mathsf{R}}\mathsf{R}^{-1}$$

Theorem 4.2. SkMBiCT *is equivalent to* SkMBiCA*, meaning that the following two statements are true:*

- *For any derivation $f : A \vdash_{\mathsf{L}} C$, there exists a derivation* $\mathsf{A2G}f : A \vdash_{\mathsf{T}} C$*.*
- *For any derivation $f : T \vdash_{\mathsf{T}} C$, there exists a derivation* $\mathsf{G2A}f : T^{\#} \vdash_{\mathsf{L}} C$*, where $T^{\#}$ transforms a tree into a formula by replacing commas with $\otimes^{\mathsf{L}}$ and semicolons with $\otimes^{\mathsf{R}}$, and $-$ with* I*, respectively.*

5 Relational Semantics of SkMBiCA and Application

In this section, we present the relational semantics of SkMBiCA. Furthermore, the relational semantics for SkMBiCA is characterized modularly, allowing us to construct models for semi-substructural logics step by step by incorporating additional structural conditions into the frame. The modularity allows us to provide an algebraic proof for the main theorems concerning the interdefinability of a series of skew categories as discussed in [25].

A preordered ternary frame with a special subset is $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$, where W is a set, $\leq$ is a preorder relation on W, $\mathbb{I}$ is a downwards closed subset of W, and $\mathbb{L}$ is an arbitrary ternary relation on W, where $\mathbb{L}$ is upwards closed on the first two arguments and downwards closed on the last argument with respect to $\leq$.

Definition 5.1. We list properties of ternary relations which we will focus on.

Left Skew Associativity (LSA)	$\forall a,b,c,d,x \in W, \mathbb{L}abx \,\&\, \mathbb{L}xcd \longrightarrow \exists y \in W$ such that $\mathbb{L}bcy \,\&\, \mathbb{L}ayd$.
Left Skew Left Unitality (LSLU)	$\forall a,b \in W, e \in \mathbb{I}, \mathbb{L}eab \longrightarrow b \leq a$.
Left Skew Right Unitality (LSRU)	$\forall a \in W, \exists e \in \mathbb{I}$ such that $\mathbb{L}aea$.
Right Skew Associativity (RSA)	$\forall a,b,c,d,x \in W, \mathbb{L}bcx \,\&\, \mathbb{L}axd \longrightarrow \exists y \in W$ such that $\mathbb{L}aby \,\&\, \mathbb{L}ycd$.
Right Skew Left Unitality (RSLU)	$\forall a \in W, \exists e \in \mathbb{I}$ such that $\mathbb{L}eaa$.
Right Skew Right Unitality (RSRU)	$\forall a,b \in W, e \in \mathbb{I}, \mathbb{L}aeb \longrightarrow b \leq a$.

Given another ternary relation $\mathbb{R}$, we define

$$\mathbb{L}\mathbb{R}\text{-reverse} \quad \forall a,b,c \in W, \mathbb{L}abc \longleftrightarrow \mathbb{R}bac.$$

The associativity and unitality conditions are adapted from the theory of relational monoids [21] and relational semantics for Lambek calculus [11].

An SkMBiCA frame is a quintuple $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$, where $\mathbb{L}\mathbb{R}$-reverse is satisfied, $\mathbb{L}$ satisfies LSA, LSLU, LSRU, and $\mathbb{R}$ automatically satisfies RSA, RSLU, RSRU because of $\mathbb{L}\mathbb{R}$-reverse.

Unlike studies in NL e.g. [11, 18, 20], where two associativity conditions simultaneously hold for a relation or not, we explore two relations where one satisfies LSA and the other satisfies RSA. Another distinction from the existing studies on semantics for NL with unit [9] (or non-commutative linear logic [1]) is that while W is commonly assumed to be an unital groupoid (or monoid in the case of linear logic), here, we should consider that the unit behaves differently for different relations.

We denote the set of downwards closed subsets of W as $\mathscr{P}_{\downarrow}(W)$.

Definition 5.2. A function $v : \mathsf{Fma} \to \mathscr{P}_{\downarrow}(W)$ on a SkMBiCA frame is a valuation if it satisfies:

$$\begin{array}{ll} v(\mathsf{I}) & = \mathbb{I} \\ v(A \otimes^{\mathsf{L}} B) & = \{c : \exists a \in v(A), b \in v(B),\ \mathbb{L}abc\} \\ v(A \multimap^{\mathsf{L}} B) & = \{c : \forall a \in v(A), b \in W,\ \mathbb{L}cab \Rightarrow b \in v(B)\} \\ v(A \otimes^{\mathsf{R}} B) & = \{c : \exists a \in v(A), b \in v(B),\ \mathbb{R}abc\} \\ v(A \multimap^{\mathsf{R}} B) & = \{c : \forall a \in v(A), b \in W,\ \mathbb{R}cab \Rightarrow b \in v(B)\} \end{array}$$

We define a SkMBiCA model to be a SkMBiCA frame with a valuation function, i.e. $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$. A sequent $A \vdash_{\mathsf{L}} B$ is valid in a model $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ if $v(A) \subseteq v(B)$ and is valid in a frame if for any v for that frame, $v(A) \subseteq v(B)$.

Theorem 5.3 (Soundness)**.** *If a sequent $A \vdash_{\mathsf{L}} B$ is provable in* SkMBiCA *then it is valid in any* SkMBiCA *model.*

Proof. The proof is adapted from [11, 20], where the cases of α and α^{R} have been discussed. Therefore, we only elaborate on new cases arising in SkMBiCA.

- If the derivation is the axiom $\lambda : \mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} A$, then for any SkMBiCA model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $a \in v(\mathsf{I} \otimes^{\mathsf{L}} A)$, there exist $e \in \mathbb{I}$, $a' \in v(A)$, and $\mathbb{L}ea'a$. By LSLU, we know that $a \leq a'$, and then $a \in v(A)$.
- If the derivation is the axiom $\rho : A \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} \mathsf{I}$, then for any SkMBiCA model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $a \in v(A)$, by LSRU, there exists $e \in \mathbb{I}$ such that $\mathbb{L}aea$, which means that $a \in v(A \otimes^{\mathsf{L}} \mathsf{I})$.
- If the derivation is the axiom $\gamma : A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{R}} A$, then for any SkMBiCA model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $c \in v(A \otimes^{\mathsf{L}} B)$, there exist $a \in v(A)$ and $b \in v(B)$ such that $\mathbb{L}abc$. By $\mathbb{L}\mathbb{R}$-reverse, we have $\mathbb{R}bac$, therefore $c \in v(B \otimes^{\mathsf{R}} A)$.
- The case of γ^{-1} is similar.

□

Definition 5.4. The canonical model of SkMBiCA is $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ where

- $W = \mathsf{Fma}$ and $A \leq B$ if and only if $A \vdash_{\mathsf{L}} B$,
- $\mathbb{I} = v(\mathsf{I})$,
- $\mathbb{L}ABC$ if and only if $C \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} B$,
- $\mathbb{R}ABC$ if and only if $C \vdash_{\mathsf{L}} A \otimes^{\mathsf{R}} B$, and
- $v(A) = \{B \mid B \vdash_{\mathsf{L}} A \text{ is provable in SkMBiCA}\}$.

Lemma 5.5. *The canonical model is a* SkMBiCA *model.*

Proof.

- The set $(\mathsf{Fma}, \vdash_{\mathsf{L}})$ is a preorder because of the rules id and comp, and the set $\mathbb{I}$ is downwards closed because of comp. The relations $\mathbb{L}$ and $\mathbb{R}$ are downwards closed on their last argument because of the rule comp. They are upwards closed on their first two arguments due to the rules $\otimes^{\mathsf{L}}$ and $\otimes^{\mathsf{R}}$, respectively. These facts ensure that $\langle \mathsf{Fma}, \vdash_{\mathsf{L}}, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$ is a ternary frame.

- We show two cases (LSRU and LSRU) of the proof that $\mathbb{L}, \mathbb{R}$ satisfy their corresponding conditions, while other cases are similar.

(LSLU) Given any two formulae A and B, and $J \in \mathbb{I}$ with $\mathbb{L}JAB$, we have $J \vdash_{\mathsf{L}} \mathsf{I}$, and $B \vdash_{\mathsf{L}} J \otimes^{\mathsf{L}} A$, then we can construct $B \vdash_{\mathsf{L}} A$ as follows:

$$\dfrac{\dfrac{B \vdash_{\mathsf{L}} J \otimes^{\mathsf{L}} A \quad \dfrac{J \vdash_{\mathsf{L}} \mathsf{I} \quad \dfrac{}{A \vdash_{\mathsf{L}} A}\,\mathsf{id}}{J \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} \mathsf{I} \otimes^{\mathsf{L}} A}\,\otimes^{\mathsf{L}}}{B \vdash_{\mathsf{L}} \mathsf{I} \otimes^{\mathsf{L}} A}\,\mathsf{comp} \quad \dfrac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} A}\,\lambda}{B \vdash_{\mathsf{L}} A}\,\mathsf{comp}$$

(LSRU) By the axiom ρ, for any formula A, we have $A \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} \mathsf{I}$, i.e. $\mathbb{L}AIA$.

- The valuation v is downwards closed because of the rule comp. The other conditions on connectives are satisfied by definition.

Therefore, $\langle \mathsf{Fma}, \vdash_{\mathsf{L}}, \mathbb{I}, \mathbb{L}, \mathbb{R}, v \rangle$ is a `SkMBiCA` model. □

Theorem 5.6 (Completeness)**.** *If $A \vdash_{\mathsf{L}} B$ is valid in any* `SkMBiCA` *model, then it is provable in* `SkMBiCA`.

Proof. If $A \vdash_{\mathsf{L}} B$ is valid in any `SkMBiCA` model, then it is valid in the canonical model, i.e. $v(A) \subseteq v(B)$ in the canonical model. From $A \vdash_{\mathsf{L}} A$, by definition of v, we have $A \in v(A)$, and because $v(A) \subseteq v(B)$, we know that $A \in v(B)$, therefore $A \vdash_{\mathsf{L}} B$. □

We show a correspondence between frame conditions and the validity of structural laws in frames.

Theorem 5.7. *For any ternary frame $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R} \rangle$,*

	$\mathbb{LR}$-reverse holds	$\longleftrightarrow$	*γ and γ^{-1}valid*	
$\alpha^{(\mathsf{R})}$ valid	$\longleftrightarrow$	*LSA (RSA) holds*	$\longleftrightarrow$	*$L^{(\mathsf{R})}$ valid*
$\lambda^{(\mathsf{R})}$ valid	$\longleftrightarrow$	*LSLU (RSLU) holds*	$\longleftrightarrow$	*$j^{(\mathsf{R})}$ valid*
$\rho^{(\mathsf{R})}$ valid	$\longleftrightarrow$	*LSRU (RSRU) holds*	$\longleftrightarrow$	*$i^{(\mathsf{R})}$ valid*

Proof. The first case is that $\mathbb{LR}$-reverse holds if and only if γ and γ^{-1} are valid, i.e. $v(A \otimes^{\mathsf{L}} B) = v(B \otimes^{\mathsf{R}} A)$.

($\longrightarrow$) For any $x \in v(A \otimes^{\mathsf{L}} B) \subseteq W$, there exists $a \in v(A), b \in v(B)$ and $\mathbb{L}abx$. By $\mathbb{LR}$-reverse, we have $\mathbb{R}bax$ meaning that $x \in v(B \otimes^{\mathsf{R}} A)$. The other way around is similar.

($\longleftarrow$) Suppose that for any v, A, B, we have $v(A \otimes^{\mathsf{L}} B) = v(B \otimes^{\mathsf{R}} A)$. Consider any $a, b, x \in W$ such that $\mathbb{L}abx$. We take $v(A) = a\!\downarrow$ and $v(B) = b\!\downarrow$ for some $A, B \in \mathsf{At}$. By the definition of v and assumption, x belongs to $v(A \otimes^{\mathsf{L}} B)$ and $v(B \otimes^{\mathsf{R}} A)$, therefore $\mathbb{R}bax$. The other direction is similar.

λ : LSLU holds if and only if λ is valid.

($\longrightarrow$) This is similar to case of λ in the proof of Theorem 5.3.

($\longleftarrow$) Suppose that λ is valid, i.e. for any A and v, we have $v(\mathsf{I} \otimes^{\mathsf{L}} A) \subseteq v(A)$. Consider any $a, b \in W$, $e \in \mathbb{I}$ such that $\mathbb{L}eab$. We take $v(A) = a\!\downarrow$ for some $A \in \mathsf{At}$. By $\mathbb{L}eab$ and the assumption, we know that $b \in v(A)$, which means that $b \leq a$.

ρ : LSRU holds if and only if ρ is valid.

($\longrightarrow$) This is similar to case of ρ in the proof of Theorem 5.3.

- $(\longleftarrow)$ Suppose ρ is valid, i.e. for any A and v, $v(A) \subseteq v(A \otimes^{\mathsf{L}} \mathsf{I})$. Consider any $a \in W$. We take $v(A) = a{\downarrow}$ for some $A \in \mathsf{At}$. By the assumption, there exist $a' \in v(A)$ and $e \in \mathbb{I}$ such that $\mathbb{L}a'ea$. Because $\mathbb{L}$ is upwards closed, we know that $\mathbb{L}aea$.

α : LSA holds if and only if α is valid.

- $(\longrightarrow)$ For any $s \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$, there exists $a \in v(A), b \in v(B), x \in v(A \otimes^{\mathsf{L}} B), c \in v(C), \mathbb{L}abx$, and $\mathbb{L}xcs$. By LSA, there exists $y \in W$ such that $\mathbb{L}bcy$ and $\mathbb{L}ays$, then by definition of v, $y \in v(B \otimes^{\mathsf{L}} C)$ and $s \in v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$.
- $(\longleftarrow)$ Suppose that α is valid, i.e. for any A,B,C,v, we have $v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C) \subseteq v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$. Consider any $a,b,x,c,d \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. We take $v(A) = a{\downarrow}, v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$ for some $A,B,C \in \mathsf{At}$, then we know that $x \in v(A \otimes^{\mathsf{L}} B)$ and $d \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} B)$. By the assumption, d belongs to $v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$ as well, which means that there exist $a',b',y,c' \in W$ such that $\mathbb{L}b'c'y$ and $\mathbb{L}a'yd$. Because $\mathbb{L}$ is upwards closed, we have $\mathbb{L}bcy$ and $\mathbb{L}ayd$ as desired.

L : LSA holds if and only if for any A,B,C and v, $v(B \multimap^{\mathsf{L}} C) \subseteq v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$.

- $(\longrightarrow)$ For any $s \in v(B \multimap^{\mathsf{L}} C)$, we show $s \in v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. By definition, from assumptions $x \in v(A \multimap^{\mathsf{L}} B)$, $\mathbb{L}sxy$, $y \in v(A \multimap^{\mathsf{L}} C)$, $a \in A$, $c \in W$, and $\mathbb{L}yac$, we have to prove that $c \in C$. By LSA, there exists $x' \in W$ such that $\mathbb{L}xax'$ and $\mathbb{L}sx'c$. We get $x' \in B$ due to $x \in v(A \multimap^{\mathsf{L}} B)$. Thus, we have $c \in C$ because $s \in v(B \multimap^{\mathsf{L}} C)$.
- $(\longleftarrow)$ Suppose that for any A,B,C and v, we have $v(B \multimap^{\mathsf{L}} C) \subseteq v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. Consider $a,b,x,c,d \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. Take $v(A) = c{\downarrow}$, $v(B) = \{y \mid \mathbb{L}bcy\}$, and $v(C) = \{d' \mid \exists y \in v(B), \mathbb{L}ayd'\}$ for some $A,B,C \in \mathsf{At}$. Given any $y \in v(B)$ and any $d' \in W$, if $\mathbb{L}ayd'$, then by definition of $v(C)$, $d' \in v(C)$, therefore $a \in v(B \multimap^{\mathsf{L}} C)$. By assumption, $a \in v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$ as well, which means that, for any $b' \in v(A \multimap^{\mathsf{L}} B)$, $x' \in W$ $c' \in v(A)$ and $d' \in W$, if $\mathbb{L}ab'x'$, then $x' \in v(A \multimap^{\mathsf{L}} C)$, and if $\mathbb{L}x'c'd'$, then $d' \in C$. By the definition of $v(B)$ and assumptions $\mathbb{L}abx$ and $\mathbb{L}xcd$, we have $b \in v(A \multimap^{\mathsf{L}} B)$, $x \in v(A \multimap^{\mathsf{L}} C)$, therefore $d \in v(C)$, which means that there exists $y \in W$ such that $\mathbb{L}bcy$ and $\mathbb{L}ayd$.

j^{R} : RSLU holds if and only if for any A,B and v, if $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$, then $v(A) \subseteq v(B)$.

- $(\longrightarrow)$ By RSLU, for all $a \in v(A)$, there exists $e \in \mathbb{I}$ such that $\mathbb{R}eaa$, then we have $a \in v(B)$ because $e \in v(A \multimap^{\mathsf{R}} B)$.
- $(\longleftarrow)$ Suppose that for any A,B and v, if $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$, then $v(A) \subseteq v(B)$. Consider any $a \in W$. We take $v(A) = a{\downarrow}$ and $v(B) = \{b \mid \exists e \in \mathbb{I}, \mathbb{R}eab\}$ for some $A,B \in \mathsf{At}$. For any $e' \in \mathbb{I}$, $a' \in v(A)$, and $b' \in W$, if $\mathbb{R}e'a'b'$, then because $\mathbb{R}$ is upwards closed, we have $b' \in v(B)$, which means $e' \in v(A \multimap^{\mathsf{R}} B)$. Therefore $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$. From the assumption, we can now conclude that $v(A) \subseteq v(B)$. In particular, $a \in v(B)$, which means that there exists $e \in \mathbb{I}$ such that $\mathbb{R}eaa$.

L^{R} : RSA holds if and only if for any A,B,C,D and v, if $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$ then there exists X such that $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$.

- $(\longrightarrow)$ We expand the assumption first.
  For any A,B,C,D, $a \in v(A)$, and $b,z \in W$, if $b \in v(B)$ and $\mathbb{R}abz$ then $z \in v(C \multimap^{\mathsf{R}} D)$ and for all $z \in v(C \multimap^{\mathsf{R}} D)$, for all $c,d \in W$ if $c \in v(C)$ and $\mathbb{R}zcd$, then $d \in v(D)$. In other words, for any $z,d \in W$, if there are $a \in v(A)$, $b \in v(B)$, $c \in v(C)$, $\mathbb{R}abz$, and $\mathbb{R}zcd$, then $d \in v(D)$.
  We show that $B \otimes^{\mathsf{R}} C$ satisfies following two statements:
  - For any $a \in v(A)$, we show that $a \in v((B \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$. For any $x \in v(B \otimes^{\mathsf{R}} C)$ and $d \in W$, if $\mathbb{R}axd$, then by definition of $\otimes^{\mathsf{R}}$, we have $\mathbb{R}bcx$, where $b \in v(B)$ and $c \in v(C)$.

By RSA, there exists $z \in W$ such that $\mathbb{R}abz$, and $\mathbb{R}zcd$. By the expanded assumption, $d \in v(D)$. Therefore $a \in v((B \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$.

– For any $b \in v(B)$, $c \in v(C)$, and $x \in W$, suppose $\mathbb{R}bcx$, then $x \in v(B \otimes^{\mathsf{R}} C)$ by definition of $\otimes^{\mathsf{R}}$. Therefore $b \in v(C \multimap^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$.

($\longleftarrow$) Assume that for any A,B,C,D and v, if $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$, then there exists X such that $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$. Suppose that we have $a,b,c,d,x \in W$ such that $\mathbb{R}axd$ and $\mathbb{R}bcx$, then we take $v(A) = a\downarrow$, $v(B) = b\downarrow$, $v(C) = c\downarrow$, and $v(D) = \{d' \mid \exists y, \mathbb{R}aby \& \mathbb{R}ycd'\}$ for some $A,B,C,D \in \mathsf{At}$. For any $a' \in v(A)$, given any $b' \in v(B)$, $x' \in W$, $c' \in v(C)$, $d' \in W$ such that $\mathbb{R}a'b'x'$ and $\mathbb{R}x'c'd'$. Because $\mathbb{R}$ is upwards closed, by the definition of $v(D)$, we have $d' \in v(D)$, which means $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$. By the assumption, there exists X such that

(1) $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$, which means that for any $a' \in v(A)$, given any $x' \in X$, $d' \in W$, if $\mathbb{R}a'x'd'$, then $d' \in v(D)$, and

(2) $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$, which means that for any $b' \in v(B)$, given any $c' \in v(C)$ and $x' \in W$, if $\mathbb{R}b'c'x'$, then $x' \in v(X)$.

By $\mathbb{R}bcx$, and (2), we know that $x \in v(X)$. By $\mathbb{R}axd$, and (1), we know that $d \in v(D)$, which means that there exists $y \in W$ such that $\mathbb{R}aby$ and $\mathbb{R}ycd$.

The other cases are similar to the arguments above. □

A frame $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is left (right) skew associative if $\mathbb{L}$ satisfies LSA (RSA). For other conditions, the naming is similar. If $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ satisfies LSA, LSLU, and LSRU (respectively RSA, RSLU, RSRU), then it is a left (respectively right) skew.

We can think of a `SkMBiCA` frame $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$ as a combination of two ternary frames $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ (left skew frame) and $\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ (right skew frame) sharing the same set of possible worlds, where the ternary relations are interdefinable by $\mathbb{LR}$-reverse. Whenever $\mathbb{LR}$-reverse holds, then $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is left skew if and only if $\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ is right skew. In fact, we have:

$$\begin{array}{lcl}
\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle \text{ left skew associative} & \longleftrightarrow & \langle W, \leq, \mathbb{I}, \mathbb{R}\rangle \text{ right skew associative} \\
\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle \text{ left skew left unital} & \longleftrightarrow & \langle W, \leq, \mathbb{I}, \mathbb{R}\rangle \text{ right skew right unital} \\
\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle \text{ left skew right unital} & \longleftrightarrow & \langle W, \leq, \mathbb{I}, \mathbb{R}\rangle \text{ right skew left unital}
\end{array}$$

If we state the structural laws semantically rather than sequents, we can reformulate Theorem 5.7 without referring to sequents and valuations. For example, we can define $\otimes^{\mathsf{L}}$ on downwards closed sets of worlds as $A \otimes^{\mathsf{L}} B = \{c : \exists a \in A \ \&\ b \in B \ \&\ \mathbb{L}abc\}$ and express α as $(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \subseteq A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)$. It is the case that α holds in a frame if and only if it satisfies LSA.

We construct a thin SkMBiC from the frame $\langle W, \leq, \mathsf{I}, \mathbb{L}, \mathbb{R}\rangle$ and provide algebraic proofs for main theorems in [25]. The objects in the category are downwards closed subsets of W and for A, B, we have a map $A \to B$ if and only if $A \subseteq B$.

Corollary 5.8. *The category $(\mathscr{P}_{\downarrow}(W), \subseteq)$ generated from any* `SkMBiCA` *frame is a thin* SkMBiC.

A frame $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is associative normal if it satisfies LSA and RSA simultaneously, and left (right) unital normal if LSLU and RSLU (LSRU and RSRU) are satisfied. Therefore, by Theorem 5.7, we have a thin version of the main results in [25].

Corollary 5.9. *Given any frame, for the category $(\mathscr{P}_{\downarrow}(W), \subseteq)$ generated from the frame we have:*

$$\begin{array}{lcl}
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ left skew monoidal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ left skew closed} \\
(\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ right skew monoidal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ right skew closed}
\end{array}$$

Moreover, if the frame satisfies $\mathbb{LR}$*-reverse then:*

$$\begin{array}{lcl}
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ left skew monoidal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ right skew monoidal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ left skew closed} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ right skew closed} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ associative normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ associative normal} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ left unital normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ right unital normal} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ right unital normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ left unital normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ associative normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ associative normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ left unital normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ right unital normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ right unital normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ left unital normal}
\end{array}$$

6 Concluding remarks

This paper discusses sequent calculi for left (right) skew monoidal categories and skew monoidal bi-closed categories in the style of non-associative Lambek calculus. Compared to the sequent calculi with stoup, although the calculi à la Lambek are not immediately decidable but are more flexible in the sense that the sequent calculi for right skew monoidal closed categories (`RSkT`) and skew monoidal bi-closed categories (`SkMBiCT`) are presentable. Moreover, we show that they are cut-free and equivalent to the calculus with stoup (Theorem 2.6) and the axiomatic calculus (Theorem 4.2).

In the last section, we focus on the relational semantics of `SkMBiCA` via the ternary frame $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$ where $\mathbb{L}$ and $\mathbb{R}$ are connected by $\mathbb{LR}$-reverse and therefore if $\mathbb{L}$ satisfies left skew structural conditions then $\mathbb{R}$ satisfies right skew structural conditions automatically. By Theorem 5.7, for any `SkMBiCA` model, we can construct a thin skew monoidal bi-closed category $(\mathscr{P}_{\downarrow}(W), \subseteq)$. In addition, we can obtain algebraic proofs of main theorems in [25].

A future project is to explore Craig interpolation [10] for semi-substructural logics. In `LSkT`, the situation is more complicated than either associative or fully non-associative Lambek calculi because we only allow semi-associativity. Consider the statement:

> Given a derivation, $f : T[U] \vdash_{\mathsf{T}} C$, then there exist a formula D and two derivations $f_0 : U \vdash_{\mathsf{T}} D$ and $f_1 : T[D] \vdash_{\mathsf{T}} C$, and $\mathsf{var}(D) \subseteq \mathsf{var}(\delta(U)) \cap \mathsf{var}(\delta(T[-]), C)$, where δ is a function that transforms a tree into a list of formulae.

If we try to prove by induction on f, then there is a critical case

$$\frac{\begin{array}{c} f \\ T[U_0, (U_1, U_2)] \vdash_{\mathsf{T}} C \end{array}}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc}$$

where $U = U_0, U_1$, therefore, the goal is to find a formula D and two derivations $g : U_0, U_1 \vdash_{\mathsf{T}} D$ and $T[D, U_2] \vdash_{\mathsf{T}} C$. However, we cannot directly apply the inductive hypothesis twice on f, because the procedure of finding an interpolant formula and corresponding derivations is not height preserving. Therefore, proving the interpolation property for semi-substructural logics is more subtle than expected.

Another possible direction is to incorporate modalities (exponentials in linear logical terminology) with semi-substructural logic as in [18] (modalities) and [4] (subexponentials) with non-associative Lambek calculus and non-commutative and non-associative linear logic.

Similar to the equational theories for `SkMBiCA` discussed in Section 4, we also plan to investigate the equational theories on the derivations of `LSkT` and `SkMBiCT` in the future.

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References

[1] Vito Michele Abrusci (1990): *Non-Commutative Intuitionistic Linear Logic*. Mathematical Logic Quarterly 36(4), pp. 297–318, doi:10.1002/malq.19900360405.

[2] Thorsten Altenkirch, James Chapman & Tarmo Uustalu (2015): *Monads Need Not Be Endofunctors*. Logical Methods in Computer Science 11(1):3, doi:10.2168/lmcs-11(1:3)2015.

[3] Jean-Marc Andreoli (1992): *Logic Programming with Focusing Proofs in Linear Logic*. Journal of Logic and Computation 2(3), pp. 297–347, doi:10.1093/logcom/2.3.297.

[4] Eben Blaisdell, Max Kanovich, Stepan L. Kuznetsov, Elaine Pimentel & Andre Scedrov (2022): *Non-Associative, Non-Commutative Multi-Modal Linear Logic*, pp. 449–467. Springer International Publishing, doi:10.1007/978-3-031-10769-6_27.

[5] John Bourke (2017): *Skew Structures in 2-Category Theory and Homotopy Theory*. Journal of Homotopy and Related Structures 12(1), pp. 31–81, doi:10.1007/s40062-015-0121-z.

[6] John Bourke & Stephen Lack (2018): *Skew Monoidal Categories and Skew Multicategories*. Journal of Algebra 506, pp. 237–266, doi:10.1016/j.jalgebra.2018.02.039.

[7] John Bourke & Stephen Lack (2020): *Braided Skew Monoidal Categories*. Theory and Applications of Categories 35(2), pp. 19–63. Available at http://www.tac.mta.ca/tac/volumes/35/2/35-02abs.html.

[8] Michael Buckley, Richard Garner, Stephen Lack & Ross Street (2015): *The Catalan Simplicial Set*. Mathematical Proceedings of Cambridge Philosophical Society 158(2), pp. 211–222, doi:10.1017/s0305004114000498.

[9] Maria Bulińska (2009): *On the Complexity of Nonassociative Lambek Calculus with Unit*. Studia Logica 93(1), pp. 1–14, doi:10.1007/s11225-009-9205-2.

[10] William Craig (1957): *Three Uses of the Herbrand-Gentzen Theorem in Relating Model Theory and Proof Theory*. Journal of Symbolic Logic 22(3), pp. 269–285, doi:10.2307/2963594.

[11] Kosta Došen (1992): *A Brief Survey of Fames for the Lambek Calculus*. Mathematical Logic Quarterly 38(1), pp. 179–187, doi:10.1002/malq.19920380113.

[12] Jean-Yves Girard (1987): *Linear Logic*. Theoretical Computer Science 50, pp. 1–102, doi:10.1016/0304-3975(87)90045-4.

[13] Jean-Yves Girard (1991): *A New Constructive Logic: Classical Logic*. Mathematical Structures in Computer Science 1(3), pp. 255–296, doi:10.1017/s0960129500001328.

[14] Stephen Lack & Ross Street (2012): *Skew Monoidales, Skew Warpings and Quantum Categories*. Theory and Applications of Categories 26, pp. 385–402. Available at http://www.tac.mta.ca/tac/volumes/26/15/26-15abs.html.

[15] Stephen Lack & Ross Street (2014): *Triangulations, Orientals, and Skew Monoidal Categories*. Advances in Mathematics 258, pp. 351–396, doi:10.1016/j.aim.2014.03.003.

[16] Joachim Lambek (1958): *The Mathematics of Sentence Structure*. American Mathematical Monthly 65(3), pp. 154–170, doi:10.2307/2310058.

[17] Saunders Mac Lane (1963): *Natural Associativity and Commutativity*. Rice University Studies 49(4), pp. 28–46. Available at http://hdl.handle.net/1911/62865.

[18] Michael Moortgat (1996): *Multimodal Linguistic Inference*. Journal of Logic, Language and Information 5(3–4), pp. 349–385, doi:10.1007/bf00159344.

[19] Michael Moortgat (2020): *The Tamari Order for D^3 and Derivability in Semi-Associative Lambek-Grishin Calculus*. Talk at 16th Workshop on Computational Logic and Applications, CLA 2020. Slides available at: http://cla.tcs.uj.edu.pl/history/2020/pdfs/CLA_slides_Moortgat.pdf.

[20] Richard Moot & Christian Retoré (2012): *The Logic of Categorial Grammars: A Deductive Account of Natural Language Syntax and Semantics*. Springer Berlin Heidelberg, doi:10.1007/978-3-642-31555-8.

[21] Kimmo Rosenthal (1997): *Relational Monoids, Multirelations, and Quantalic Recognizers*. Cahiers de Topologie et Géométrie Différentielle Catégoriques 38(2), pp. 161–171.

[22] Ross Street (2013): *Skew-Closed Categories*. Journal of Pure and Applied Algebra 217(6), pp. 973–988, doi:10.1016/j.jpaa.2012.09.020.

[23] Kornél Szlachányi (2012): *Skew-Monoidal Categories and Bialgebroids*. Advances in Mathematics 231(3–4), pp. 1694–1730, doi:10.1016/j.aim.2012.06.027.

[24] Tarmo Uustalu, Niccolò Veltri & Cheng-Syuan Wan (2022): *Proof Theory of Skew Non-Commutative MILL*. In Andrzej Indrzejczak & Michal Zawidzki, editors: Proceedings of 10th International Conference on Non-classical Logics: Theory and Applications, NCL 2022, Electronic Proceedings in Theoretical Computer Science 358, Open Publishing Association, pp. 118–135, doi:10.4204/eptcs.358.9.

[25] Tarmo Uustalu, Niccolò Veltri & Noam Zeilberger (2020): *Eilenberg-Kelly Reloaded*. In Patricia Johann, editor: Proceedings of the 36th Conference on the Mathematical Foundations of Programming Semantics, MFPS 2020, Electronic Notes in Theoretical Computer Science 352, pp. 233–256, doi:10.1016/j.entcs.2020.09.012.

[26] Tarmo Uustalu, Niccolò Veltri & Noam Zeilberger (2021): *Deductive Systems and Coherence for Skew Prounital Closed Categories*. In Claudio Sacerdoti Coen & Alwen Tiu, editors: Proceedings of 15th Workshop on Logical Frameworks and Meta-Languages: Theory and Practice, LFMTP 2020, Electronic Proceedings in Theoretical Computer Science 332, Open Publishing Association, pp. 35–53, doi:10.4204/eptcs.332.3.

[27] Tarmo Uustalu, Niccolò Veltri & Noam Zeilberger (2021): *Proof Theory of Partially Normal Skew Monoidal Categories*. In David I. Spivak & Jamie Vicary, editors: Proceedings of 3rd Annual International Applied Category Theory Conference 2020, ACT 2020, Electronic Proceedings in Theoretical Computer Science 333, Open Publishing Association, pp. 230–246, doi:10.4204/eptcs.333.16.

[28] Tarmo Uustalu, Niccolò Veltri & Noam Zeilberger (2021): *The Sequent Calculus of Skew Monoidal Categories*. In Claudio Casadio & Philip J. Scott, editors: Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics, Outstanding Contributions to Logic 20, Springer, pp. 377–406, doi:10.1007/978-3-030-66545-6_11.

[29] Niccolò Veltri (2021): *Coherence Via Focusing for Symmetric Skew Monoidal Categories*. In Alexandra Silva, Renata Wassermann & Ruy de Queiroz, editors: Proceedings of 27th International Workshop on Logic, Language, Information, and Computation, WoLLIC 2021, Lecture Notes in Computer Science 13028, Springer, pp. 184–200, doi:10.1007/978-3-030-88853-4_12.

[30] Niccolò Veltri (2023): *Maximally Multi-Focused Proofs for Skew Non-Commutative MILL*. In Helle Hvid Hansen, Andre Scedrov & Ruy J. G. B. de Queiroz, editors: Proceedings of 29th International Workshop on Logic, Language, Information, and Computation, WoLLIC 2023, Lecture Notes in Computer Science 13923, Springer, pp. 377–393, doi:10.1007/978-3-031-39784-4_24.

[31] Niccolò Veltri & Cheng-Syuan Wan (2023): *Semi-Substructural Logics with Additives*. In David Monniaux Temur Kutsia, Daniel Ventura & José F. Morales, editors: Proceedings of 18th International Workshop on Logical and Semantic Frameworks, with Applications and 10th Workshop on Horn Clauses for Verification and Synthesis, LSFA/HCVS 2023, Electronic Proceedings in Theoretical Computer Science, Open Publishing Association, pp. 63–80, doi:10.4204/eptcs.402.8.

[32] Noam Zeilberger (2019): *A Sequent Calculus for a Semi-Associative Law*. *Logical Methods in Computer Science* 15(1):9, doi:10.23638/lmcs-15(1:9)2019.

Unified Gentzen Approach to Connexive Logics over Wansing's C

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Gentzen-style sequent calculi and Gentzen-style natural deduction systems are introduced for a family (C-family) of connexive logics over Wansing's basic connexive logic C. The C-family is derived from C by incorporating the Peirce law, the law of excluded middle, and the generalized law of excluded middle. Theorems establishing equivalence between the proposed sequent calculi and natural deduction systems are demonstrated. Cut-elimination and normalization theorems are established for the proposed sequent calculi and natural deduction systems, respectively.

1 Introduction

Connexive logics are recognized as philosophically plausible paraconsistent logics [3, 21, 36, 37]. A distinguishing feature of connexive logics is their validation of the so-called Boethius' theses: $(\alpha\to\beta)\to\sim(\alpha\to\sim\beta)$ and $(\alpha\to\sim\beta)\to\sim(\alpha\to\beta)$. On one hand, the roots of connexive logics can be traced back to Aristotle and Boethius. On the other hand, modern perspectives on connexive logics were established by Angell [3] and McCall [21].

A basic constructive connexive logic referred to as C, considered a variant of Nelson's paraconsistent logic N4 [2, 23, 18], was introduced by Wansing in [36]. Additionally, C was extended by Wansing in [36] to introduce a constructive connexive modal logic, serving as a constructive connexive analogue of the smallest normal modal logic K. For further details on connexive logics, refer to, for example, [3, 21, 36, 4, 16, 19, 37, 29, 25] and the references therein.

In this study, a unified Gentzen-style framework is employed to investigate several connexive logics over Wansing's C. The term "unified Gentzen-style framework" means that we can handle Gentzen-style sequent calculus and Gentzen-style natural deduction system uniformly, with an equivalence between them. The logics under consideration include Omori and Wansing's connexive logic C3 [29], material connexive logic MC [37], and Cantwell's connexive logic CN [4]. C3 is obtained from C by adding the law of excluded middle $\neg\alpha\vee\alpha$, MC is obtained from C by adding the Peirce law $((\alpha\to\beta)\to\alpha)\to\alpha$, and CN is obtained from C3 by adding the Peirce law.

On one hand, Gentzen-style or G3-style sequent calculi for C, C3, CN and some intermediate logics between C and C3 have been introduced and investigated [36, 29, 6, 24], along with a Gentzen-style natural deduction system for the implicational fragment of C [13]. On the other hand, a unified Gentzen-style framework for C, C3, MC, and CN has not been established. Therefore, we construct such a framework in this study. This framework enables an integrated proof-theoretical treatment of these logics and establishes a natural correspondence between sequent calculi and natural deduction systems for them.

We now discuss some related works on sequent calculi for connexive logics. The cut-elimination theorem for a Gentzen-style sequent calculus, referred to as sC, was proved by Wansing in [36], although the name sC was not used by him. The cut-elimination theorems for G3-style sequent calculi, namely G3C and G3C3at for C and C3, respectively, were established by Omori and Wansing in [29]. In this

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics. Theory and Applications (NCL'24).
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context, G3C3at is a sequent calculus that incorporates the rule of atomic excluded middle (at-ex-middle) in place of the rule of excluded middle (ex-middle). The admissibility of (ex-middle) in G3C3at was also demonstrated by them. Consequently, the cut-elimination theorem for a G3-style sequent calculus, referred to as G3C3, which is obtained from G3C3at by replacing (at-ex-middle) with (ex-middle), was also demonstrated by them in [25]. Additionally, the first-order extensions of G3C, G3C3at, and G3C3 were also introduced and investigated by them. The systems G3C, G3C3at, and G3C3 were also used by Niki and Wansing in [25] to explore the provable contradictions of C and C3.

Several sequent calculi for some intermediate logics between C and C3 have recently been studied by Niki in [24]. A three-sided sequent calculus for CN, under the name CC/TTm, has recently been introduced and investigated by Égré et al. in [6]. A natural deduction system, NC2, and a two-sorted typed λ-calculus, 2λ, were introduced and investigated by Wansing in [39] for the bi-connexive propositional logic 2C. Natural deduction systems for two variants of connexive logics concerning non-classical interpretations of a certain kind between negation and implication were studied by Francez in [8]. In addition, some extensions of C were studied by Olkhovikov in [26, 27] and by Omori in [28], although these studies are not concerned with sequent calculus or natural deduction system.

The structure of this paper is as follows. In Section 2, we introduce Gentzen-style sequent calculi sC, sC3, sMC, and sCN for C, C3, MC, and CN, respectively. Additionally, we prove the cut-elimination theorems for these calculi. The calculi sC3, sMC, and sCN are obtained from sC by adding the excluded middle rule (ex-middle), the Peirce rule (Peirce), and both (ex-middle) and (Peirce), respectively. Moreover, we introduce alternative Gentzen-style sequent calculi sMC* and sCN* for MC and CN, respectively. These calculi are obtained from sC by adding the generalized excluded middle rule (g-ex-middle) and both (ex-middle) and (g-ex-middle), respectively. We then obtain a theorem establishing cut-free equivalence between sMC* (sCN*) and sMC (sCN, resp.), along with presenting the cut-elimination theorem for sMC* and sCN*. In Section 3, we introduce Gentzen-style natural deduction systems nC, nC3, nMC, and nCN for C, C3, MC, and CN, respectively. Additionally, we prove a theorem establishing equivalence between nC, (nC3, nMC, and nCN), and sC, (sC3, sMC*, and nCN*, resp.). Furthermore, we prove the normalization theorems for nC, nC3, nMC, and nCN.

2 Gentzen-style sequent calculi

Formulas of connexive logics [3, 21, 36, 37] are constructed using countably many propositional variables, the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\sim$ (connexive negation). We use small letters $p, q, \ldots$ to denote propositional variables, Greek small letters $\alpha, \beta, \ldots$ to denote formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ to denote finite (possibly empty) sets of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \gamma$. We use the expression $L \vdash S$ to represent the fact that a sequent S is provable in a sequent calculus L. We say that "a rule R of inference is *admissible* in a sequent calculus L" if the following condition is satisfied: For any instance $\frac{S_1 \cdots S_n}{S}$ of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Furthermore, we say that "R is *derivable* in L" if there is a derivation from $S_1, \cdots, S_n$ to S in L.

We introduce Gentzen-style sequent calculi LJ$^+$ [9], sC [36], sC3, sMC, and sCN for positive intuitionistic logic, C [36], C3 [29], MC [37], and CN [4], respectively.

Definition 2.1 (LJ$^+$, sC, sC3, sMC, and sCN)

1. LJ$^+$ *is defined by the initial sequents and structural and logical inference rules of the following form, for any propositional variable p:*

$$p, \Gamma \Rightarrow p \ (\text{init1}) \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma} \ (\text{cut})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta,\Delta \Rightarrow \gamma}{\alpha \to \beta,\Gamma,\Delta \Rightarrow \gamma}\ (\to\text{left}) \quad \frac{\alpha,\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\to\text{right}) \quad \frac{\alpha,\beta,\Gamma \Rightarrow \gamma}{\alpha \wedge \beta,\Gamma \Rightarrow \gamma}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha,\Gamma \Rightarrow \gamma \quad \beta,\Gamma \Rightarrow \gamma}{\alpha \vee \beta,\Gamma \Rightarrow \gamma}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right1}) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right2}).$$

2. sC *is obtained from* LJ^+ *by adding the initial sequents and logical inference rules of the form:*

$$\sim p,\Gamma \Rightarrow \sim p\ (\text{init2}) \quad \frac{\alpha,\Gamma \Rightarrow \gamma}{\sim\sim\alpha,\Gamma \Rightarrow \gamma}\ (\sim\text{left}) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim\sim\alpha}\ (\sim\text{right})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \sim\beta,\Delta \Rightarrow \gamma}{\sim(\alpha \to \beta),\Gamma,\Delta \Rightarrow \gamma}\ (\sim\to\text{left}) \quad \frac{\alpha,\Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha \to \beta)}\ (\sim\to\text{right})$$

$$\frac{\sim\alpha,\Gamma \Rightarrow \gamma \quad \sim\beta,\Gamma \Rightarrow \gamma}{\sim(\alpha \wedge \beta),\Gamma \Rightarrow \gamma}\ (\sim\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \sim\alpha}{\Gamma \Rightarrow \sim(\alpha \wedge \beta)}\ (\sim\wedge\text{right1}) \quad \frac{\Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha \wedge \beta)}\ (\sim\wedge\text{right2})$$

$$\frac{\sim\alpha,\sim\beta,\Gamma \Rightarrow \gamma}{\sim(\alpha \vee \beta),\Gamma \Rightarrow \gamma}\ (\sim\vee\text{left}) \quad \frac{\Gamma \Rightarrow \sim\alpha \quad \Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha \vee \beta)}\ (\sim\vee\text{right}).$$

3. sC3 *and* sMC *are obtained from* sC *by adding the following excluded middle rule and Peirce rule, respectively:*

$$\frac{\sim\alpha,\Gamma \Rightarrow \gamma \quad \alpha,\Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle}) \quad \frac{\alpha \to \beta,\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha}\ (\text{Peirce}).$$

4. sCN *is obtained from* sC3 *by adding* (Peirce).

Remark 2.2

1. *It is known that single-succedent Gentzen-style sequent calculi for classical logic are obtained from Gentzen's sequent calculus* LJ *(or other variants such as the G3-style sequent calculus* G3ip*) for intuitionistic logic by adding one of* (ex-middle), (Peirce)*, and their variants. These single-succeddent calculi have been studied by several researchers* [5, 7, 10, 1, 30, 22, 12, 15]. *For a survey on these calculi, see, for example,* [12, 15].
2. (ex-middle)*, which corresponds to the law of excluded middle* $\sim\alpha \vee \alpha$*, was introduced and investigated by von Plato* [30, 22]*, although the name* (ex-middle) *was not used by him. He showed that* (ex-middle) *can be restricted to the inference rule of the form:*

$$\frac{\sim p,\Gamma \Rightarrow \gamma \quad p,\Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{at-ex-middle})$$

   where p is a propositional variable. Namely, (at-ex-middle) *and* (ex-middle) *are equivalent over intuitionistic logic. He proved the cut-elimination theorems for some sequent calculi with* (at-ex-middle) *or* (ex-middle).

3. (Peirce)*, which corresponds to the Peirce law* $((\alpha \to \beta) \to \alpha) \to \alpha$*, was introduced and investigated by Curry* [5]*, Felscher* [7]*, Gordeev* [10]*, and Africk* [1]. *The cut-elimination theorem for* LJ + (Peirce) *was proved by them. Specifically, Africk* [1] *obtained a simple embedding-based proof of the cut-elimination theorem for* LJ + (Peirce)*. The subformula property for a version of* LJ + (Peirce) *without the falsity constant* $\perp$ *was shown by Gordeev. Specifically, he proved in* [10] *that* β *in* (Peirce) *can be restricted to a subformula of some formulas in* (Γ,α).
4. *Gentzen's* LK *for classical logic,* LJ + (ex-middle)*, and* LJ + (Peirce) *are theorem-equivalent within the language* $\{\wedge,\vee,\to,\neg,\perp\}$*. However,* sC3, sMC*, and* sCN *(and their corresponding logics* C3, MC*, and* CN*) are not logically-equivalent. This fact will be shown in Theorem 2.7.*

Proposition 2.3 *Let L be* LJ^+, sC, sC3, sMC*, or* sCN*. For any formula* α *and any set* Γ *of formulas, we have:* $L \vdash \alpha,\Gamma \Rightarrow \alpha$.

Proof. By induction on α. ■

Proposition 2.4 *Let L be* LJ^+*,* sC*,* sC3*,* sMC*, or* sCN*. The following rule is admissible in cut-free L:*

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\mathrm{we}).$$

Proof. By induction on the proofs P of $\Gamma \Rightarrow \gamma$ of (we) in cut-free L. ■

The following cut-elimination theorems for LJ^+ and sC are well-known.

Theorem 2.5 (Cut-elimination for LJ^+ and sC [9, 36]) *Let L be* LJ^+ *or* sC*. The rule* (cut) *is admissible in cut-free L.*

We now show the cut-elimination theorems for sC3, sMC, and sCN.

Theorem 2.6 (Cut-elimination for sC3, sMC, and sCN) *Let L be* sC3*,* sMC*, or* sCN*. The rule* (cut) *is admissible in cut-free L.*

Proof. (Sketch). We give a sketch of the proof.

• First, we show the cut-elimination theorem for sC3. It is known that the cut-elimination theorem for the G3-style sequent calculus G3C3 for C3, which has (ex-middle), holds [29]. Then, we can show the cut-free equivalence between G3C3 and sC3. Thus, from this equivalence and the cut-elimination theorem for G3C3, we obtain the cut-elimination theorem for sC3.

• Second, we show the cut-elimination theorem for sMC. It is known that the cut-elimination theorem for LJ + (Peirce) holds. This theorem was proved directly and indirectly by using the methods by Gordeev [10] and Africk [1]. Thus, the cut-elimination theorem for the negation-less fragment (i.e., LJ^+ + (Peirce)) of LJ + (Peirce) holds because LJ + (Peirce) is a conservative extension of LJ^+ + (Peirce) by the cut-elimination theorem for LJ + (Peirce). Then, we can show a theorem for embedding (cut-free) sMC into (cut-free) LJ^+ + (Peirce), and by using this theorem, we can show the cut-elimination theorem for sMC. We will show this in the following.

Prior to showing the embedding theorem, we introduce a translation of sMC to LJ^+ + (Peirce). Let Φ be a set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. Then, the language $\mathscr{L}_{\mathrm{MC}}$ of sMC is defined using Φ, $\wedge$, $\vee$, $\rightarrow$, and $\sim$. The language $\mathscr{L}_{\mathrm{Int+}}$ of LJ^+ is obtained from $\mathscr{L}_{\mathrm{MC}}$ by replacing $\sim$ with Φ'. A mapping f from $\mathscr{L}_{\mathrm{MC}}$ to $\mathscr{L}_{\mathrm{Int+}}$ is defined inductively by: (1) for any $p \in \Phi$, $f(p) := p$ and $f({\sim}p) := p' \in \Phi'$; (2) $f(\alpha \sharp \beta) := f(\alpha) \sharp f(\beta)$ with $\sharp \in \{\wedge, \vee, \rightarrow\}$; (3) $f({\sim}{\sim}\alpha) := f(\alpha)$; (4) $f({\sim}(\alpha \wedge \beta)) := f({\sim}\alpha) \vee f({\sim}\beta)$; (5) $f({\sim}(\alpha \vee \beta)) := f({\sim}\alpha) \wedge f({\sim}\beta)$; and (6) $f({\sim}(\alpha \rightarrow \beta)) := f(\alpha) \rightarrow f({\sim}\beta)$. An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. We remark that a similar translation defined as above has been used by Gurevich [11], Rautenberg [32] and Vorob'ev [34] to embed Nelson's constructive logic [2, 23] into positive intuitionistic logic.

We then obtain the following theorem for embedding sMC into $\mathrm{LJ}^+ +$ (Peirce):

1. $\mathrm{sMC} \vdash \Gamma \Rightarrow \gamma$ iff $\mathrm{LJ}^+ + (\mathrm{Peirce}) \vdash f(\Gamma) \Rightarrow f(\gamma)$,
2. $\mathrm{sMC} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \gamma$ iff $\mathrm{LJ}^+ + (\mathrm{Peirce}) - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\gamma)$.

The proof of this theorem is almost the same as that for the theorem for embedding sC or a Gentzen-style sequent calculus for Nelson's paraconsistent four-valued logic N4 into LJ^+. For more information on these embedding theorems, see, for example, [17, 18, 16, 19, 14].

We are ready to prove of the cut-elimination theorem for sMC. Suppose that sMC $\vdash \Gamma \Rightarrow \gamma$. Then, we have LJ$^+$ + (Peirce) $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by the statement (1) of the theorem, and hence LJ$^+$ + (Peirce) $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by the cut-elimination theorem for LJ$^+$ + (Peirce). Then, by the statement (2) of the theorem, we obtain sMC $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$.

• Finally, the cut-elimination theorem for sCN can be proved in a similar way as for sMC. ■

Theorem 2.7 (Separation of C, C3, MC, and CN) *The logics* C, C3, MC, *and* CN *are not logically-equivalent.*

Proof. By Theorem 2.6. ■

Next, we introduce alternative Gentzen-style sequent calculi sMC* and sCN* for MC and CN, respectively. These calculi will be used to prove the normalization theorems for the natural deduction systems nMC and nCN for MC and CN, respectively.

Definition 2.8 (sMC* and sCN*)

1. sMC* *is obtained from* sC *by adding the generalized excluded middle rule of the form:*

$$\frac{\alpha{\rightarrow}\beta, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma} \text{ (g-ex-middle)}.$$

2. sCN* *is obtained from* sC3 *by adding* (g-ex-middle).

Remark 2.9

1. (g-ex-middle)*, which corresponds to the generalized law of excluded middle* $(\alpha{\rightarrow}\beta)\vee\alpha$*, was introduced and investigated by Kamide in* [12]*, although the name* (g-ex-middle) *was not used by him. He proved the cut-elimination theorem for* LJ + (g-ex-middle) *using the method by Africk* [1].
2. LJ + (g-ex-middle) *is regarded as a sequent calculus for classical logic. Actually,* (g-ex-middle) *and* (ex-middle) *are equivalent over positive intuitionistic logic.* (g-ex-middle) *is regarded as a generalization of* (ex-middle) *if we assume the falsity constant* $\bot$ *and the definition* $\sim\alpha := \alpha{\rightarrow}\bot$. (g-ex-middle) *is also regarded as a generalization of* (Peirce) *and it was referred to as generalized Peirce rule (named* (g-Peirce)*) in* [12].

Proposition 2.10 *Let L be* sMC* *or* sCN*. *For any formula* α *and set* Γ *of formulas, we have:* $L \vdash \alpha, \Gamma \Rightarrow \alpha$.

Proof. By induction on α. ■

Proposition 2.11 *Let L be* sMC* *or* sCN*. *The rule* (we) *is admissible in cut-free L.*

Proof. Similar to the proof of Proposition 2.4. ■

Theorem 2.12 (Equivalence between sMC (sCN) and sMC* (sCN*)) *Let* L_1 *and* L_2 *be the sequent calculi* sMC *and* sCN*, respectively. Let* L_1^* *and* L_2^* *be the sequent calculi* sMC* *and* sCN**, respectively. For any* $i \in \{1,2\}$*, we have:*

1. $L_i \vdash \Gamma \Rightarrow \gamma$ *iff* $L_i^* \vdash \Gamma \Rightarrow \gamma$,
2. L_i $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$ *iff* L_i^* $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$.

Proof. Straightforward. ■

Theorem 2.13 (Cut-elimination for sMC* and sCN*) *Let L be* sMC* *or* sCN*. *The rule* (cut) *is admissible in cut-free L.*

Proof. By Theorems 2.6 and 2.12. ■

3 Gentzen-style natural deduction systems

We now define Gentzen-style natural deduction systems NJ^{+}, nC, nC3, nMC, and nCN for positive intuitionistic logic, C, C3, MC, and CN, respectively. We use the notation $[\alpha]$ in the definitions of natural deduction systems to denote the discharged assumption (i.e., the formula α is a discharged assumption by the underlying logical inference rule).

Definition 3.1 (NJ^{+}, nC, nC3, nMC, and nCN)

1. NJ^{+} *is defined as the logical inference rules of the form, where in* ($\to$I) *the discharge can be vacuous:*

$$\dfrac{\begin{matrix}[\alpha]\\ \vdots \\ \beta\end{matrix}}{\alpha\to\beta}\ (\to\mathrm{I}) \quad \dfrac{\alpha\to\beta \quad \alpha}{\beta}\ (\to\mathrm{E}) \quad \dfrac{\alpha \quad \beta}{\alpha\wedge\beta}\ (\wedge\mathrm{I}) \quad \dfrac{\alpha\wedge\beta}{\alpha}\ (\wedge\mathrm{E1}) \quad \dfrac{\alpha\wedge\beta}{\beta}\ (\wedge\mathrm{E2})$$

$$\dfrac{\alpha}{\alpha\vee\beta}\ (\vee\mathrm{I1}) \quad \dfrac{\beta}{\alpha\vee\beta}\ (\vee\mathrm{I2}) \quad \dfrac{\alpha\vee\beta \quad \begin{matrix}[\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\vee\mathrm{E}).$$

2. nC *is obtained from* NJ^{+} *by adding the negated logical inference rules of the form:*

$$\dfrac{\alpha}{\sim\sim\alpha}\ (\sim\sim\mathrm{I}) \quad \dfrac{\sim\sim\alpha}{\alpha}\ (\sim\sim\mathrm{E}) \quad \dfrac{\begin{matrix}[\alpha]\\ \vdots \\ \sim\beta\end{matrix}}{\sim(\alpha\to\beta)}\ (\sim\to\mathrm{I}) \quad \dfrac{\sim(\alpha\to\beta) \quad \alpha}{\sim\beta}\ (\sim\to\mathrm{E})$$

$$\dfrac{\sim\alpha}{\sim(\alpha\wedge\beta)}\ (\sim\wedge\mathrm{I1}) \quad \dfrac{\sim\beta}{\sim(\alpha\wedge\beta)}\ (\sim\wedge\mathrm{I2}) \quad \dfrac{\sim(\alpha\wedge\beta) \quad \begin{matrix}[\sim\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\sim\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\sim\wedge\mathrm{E})$$

$$\dfrac{\sim\alpha \quad \sim\beta}{\sim(\alpha\vee\beta)}\ (\sim\vee\mathrm{I}) \quad \dfrac{\sim(\alpha\vee\beta)}{\sim\alpha}\ (\sim\vee\mathrm{E1}) \quad \dfrac{\sim(\alpha\vee\beta)}{\sim\beta}\ (\sim\vee\mathrm{E2}).$$

3. nC3 *and* nMC *are obtained from* nC *by adding the following excluded middle rule and generalized excluded middle rule, respectively:*

$$\dfrac{\begin{matrix}[\sim\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\alpha]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\mathrm{EM}) \qquad \dfrac{\begin{matrix}[\alpha\to\beta]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\alpha]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\mathrm{GEM}).$$

4. nCN *is obtained from* nC3 *by adding* (GEM).

Remark 3.2 (EM) *and its restricted version* (EM-at) *with the propositional variable discharged assumptions were originally introduced by von Plato* [30]*, and called there* Gem *and* Gem-at*, respectively. He proved normalization theorems for systems with* (EM) *or* (EM-at)*.*

Next, we define some notions for the natural deduction systems.

Definition 3.3 ($\to$I)*,* ($\wedge$I)*,* ($\vee$I1)*,* ($\vee$I2)*,* ($\sim\sim$I)*,* ($\sim\to$I)*,* ($\sim\wedge$I1)*,* ($\sim\wedge$I2)*,* ($\sim\vee$I)*,* (EM)*, and* (GEM) *are called* introduction rules*, and* ($\to$E)*,* ($\wedge$E1)*,* ($\wedge$E2)*,* ($\vee$E)*,* ($\sim\sim$E)*,* ($\sim\to$E)*,* ($\sim\wedge$E)*,* ($\sim\vee$E1)*, and* ($\sim\vee$E2) *are called* elimination rules*. The notions of* major *and* minor premises *of the inference rules without* (EM) *and* (GEM) *are defined as usual. The notions of* derivation*,* (open and discharged) assumptions *of derivation, and* end-formula *of derivation are also defined as usual. Any derivation starts with an* *assumption α can be considered a derivation of α from itself. For a derivation $\mathscr{D}$, we use the expression* oa($\mathscr{D}$) *to denote the set of open assumptions of $\mathscr{D}$ and the expression* end($\mathscr{D}$) *to denote the end-formula of $\mathscr{D}$. A formula α is said to be* provable *in a natural deduction system N if there exists a derivation in N with no open assumptions whose end-formula is α.*

Remark 3.4 *There are no notions of major and minor premises of* (EM) *and* (GEM). *Namely, both the premises of* (EM) *and* (GEM) *are neither major nor minor premise. In this study,* (EM) *and* (GEM) *are treated as introduction rules.*

Next, we define a reduction relation $\gg$ on the set of derivations in the natural deduction systems. Prior to defining $\gg$, we define some notions concerning $\gg$.

Definition 3.5 *Let L be* nC, nC3, nMC, *or* nCN. *Let α be a formula occurring in a derivation $\mathscr{D}$ in L. Then, α is called a* maximum formula *in $\mathscr{D}$ if α satisfies the following conditions:*

1. *α is the conclusion of an introduction rule,* ($\vee$E), *or* ($\sim\wedge$E),
2. *α is the major premise of an elimination rule.*

A derivation is said to be normal *if it contains no maximum formula. The notion of substitution of derivations to assumptions is defined as usual. We assume that the set of derivations is closed under substitution.*

Definition 3.6 (Reduction relation) *Let γ be a maximum formula in a derivation that is the conclusion of an inference rule R.*

1. *The definition of the reduction relation $\gg$ at γ in* nC *is obtained by the following conditions.*
   (a) *R is* ($\to$I) *and γ is $\alpha\to\beta$:*

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathscr{D}\\ \beta\end{matrix}}{\alpha\to\beta}\,(\to\mathrm{I}) \qquad \begin{matrix}\vdots\ \mathscr{E}\\ \alpha\end{matrix}}{\beta}\,(\to\mathrm{E}) \qquad\gg\qquad \begin{matrix}\vdots\ \mathscr{E}\\ \alpha\\ \vdots\ \mathscr{D}\\ \beta.\end{matrix}$$

   (b) *R is* ($\wedge$I) *and γ is $\alpha_1\wedge\alpha_2$:*

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathscr{D}_1\\ \alpha_1\end{matrix}\qquad\begin{matrix}\vdots\ \mathscr{D}_2\\ \alpha_2\end{matrix}}{\alpha_1\wedge\alpha_2}\,(\wedge\mathrm{I})}{\alpha_i}\,(\wedge\mathrm{E}i) \qquad\gg\qquad \begin{matrix}\vdots\ \mathscr{D}_i\\ \alpha_i\end{matrix} \qquad \textit{where } i \textit{ is } 1 \textit{ or } 2.$$

   (c) *R is* ($\vee$I1) *or* ($\vee$I2) *and γ is $\alpha_1\vee\alpha_2$:*

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathscr{D}\\ \alpha_i\end{matrix}}{\alpha_1\vee\alpha_2}\,(\vee\mathrm{I}i) \qquad \begin{matrix}[\alpha_1]\\ \vdots\ \mathscr{E}_1\\ \delta\end{matrix} \qquad \begin{matrix}[\alpha_2]\\ \vdots\ \mathscr{E}_2\\ \delta\end{matrix}}{\delta}\,(\vee\mathrm{E}) \qquad\gg\qquad \begin{matrix}\vdots\ \mathscr{D}\\ \alpha_i\\ \vdots\ \mathscr{E}_i\\ \delta\end{matrix} \qquad \textit{where } i \textit{ is } 1 \textit{ or } 2.$$

   (d) *R is* ($\vee$E)*:*

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathscr{D}_1\\ \alpha\vee\beta\end{matrix} \quad \begin{matrix}[\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}[\beta]\\ \vdots\ \mathscr{D}_3\\ \gamma\end{matrix}}{\gamma}\,(\vee\mathrm{E}) \quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad\gg\quad \dfrac{\begin{matrix}\vdots\ \mathscr{D}_1\\ \alpha\vee\beta\end{matrix} \quad \dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad \dfrac{\begin{matrix}[\beta]\\ \vdots\ \mathscr{D}_3\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\vee\mathrm{E})$$

   where R' is an arbitrary inference rule, and both $\mathscr{E}_1$ and $\mathscr{E}_2$ are derivations of the minor premises of R' if they exist.

(e) *R is* $(\sim\sim\mathrm{I})$*, and* γ *is* $\sim\sim\alpha$*:*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathscr{D}\\ \alpha\end{array}}{\sim\sim\alpha}\,(\sim\sim\mathrm{I})}{\alpha}\,(\sim\sim\mathrm{E}) \qquad \gg \qquad \begin{array}{c}\vdots\ \mathscr{D}\\ \alpha.\end{array}$$

(f) *R is* $(\sim\!\to\!\mathrm{I})$ *and* γ *is* $\sim(\alpha\to\beta)$*:*

$$\dfrac{\dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathscr{D}\\ \sim\beta\end{array}}{\sim(\alpha\to\beta)}\,(\sim\!\to\!\mathrm{I}) \qquad \begin{array}{c}\vdots\ \mathscr{E}\\ \alpha\end{array}}{\sim\beta}\,(\sim\!\to\!\mathrm{E}) \qquad \gg \qquad \begin{array}{c}\vdots\ \mathscr{E}\\ \alpha\\ \vdots\ \mathscr{D}\\ \sim\beta.\end{array}$$

(g) *R is* $(\sim\wedge\mathrm{I1})$ *or* $(\sim\wedge\mathrm{I2})$ *and* γ *is* $\sim(\alpha_1\wedge\alpha_2)$*:*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathscr{D}\\ \sim\alpha_i\end{array}}{\sim(\alpha_1\wedge\alpha_2)}\,(\sim\wedge\mathrm{I1}) \qquad \begin{array}{c}[\sim\alpha_1]\\ \vdots\ \mathscr{E}_1\\ \delta\end{array} \qquad \begin{array}{c}[\sim\alpha_2]\\ \vdots\ \mathscr{E}_2\\ \delta\end{array}}{\delta}\,(\sim\wedge\mathrm{E}) \qquad \gg \qquad \begin{array}{c}\vdots\ \mathscr{D}\\ \sim\alpha_1\\ \vdots\ \mathscr{E}_i\\ \delta\end{array} \quad \textit{where } i \textit{ is } 1 \textit{ or } 2.$$

(h) *R is* $(\sim\wedge\mathrm{E})$*:*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1\\ \sim(\alpha\wedge\beta)\end{array} \quad \begin{array}{c}[\sim\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{array} \quad \begin{array}{c}[\sim\beta]\\ \vdots\ \mathscr{D}_3\\ \gamma\end{array}}{\gamma}\,(\sim\wedge\mathrm{E}) \quad \begin{array}{c}\vdots\ \mathscr{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_2\\ \delta_2\end{array}}{\delta}\,R' \qquad \gg \qquad \dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1\\ \sim(\alpha\wedge\beta)\end{array} \quad \dfrac{\begin{array}{c}[\sim\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_2\\ \delta_2\end{array}}{\delta}\,R' \quad \dfrac{\begin{array}{c}[\sim\beta]\\ \vdots\ \mathscr{D}_3\\ \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\sim\wedge\mathrm{E})$$

where R' *is an arbitrary inference rule, and both* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *are derivations of the minor premises of* R' *if they exist.*

(i) *R is* $(\sim\vee\mathrm{I})$ *and* γ *is* $\sim(\alpha_1\vee\alpha_2)$*:*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1\\ \sim\alpha_1\end{array} \qquad \begin{array}{c}\vdots\ \mathscr{D}_2\\ \sim\alpha_2\end{array}}{\sim(\alpha_1\vee\alpha_2)}\,(\sim\vee\mathrm{I})}{\sim\alpha_i}\,(\sim\vee\mathrm{E}i) \qquad \gg \qquad \begin{array}{c}\vdots\ \mathscr{D}_i\\ \sim\alpha_i\end{array} \quad \textit{where } i \textit{ is } 1 \textit{ or } 2.$$

(j) *The set of derivations is closed under* $\gg$*.*

2. *The definition of the reduction relation* $\gg$ *at* γ *in* nC3 *is obtained from the conditions for the reduction relation* $\gg$ *at* γ *in* nC *by adding the following condition.*

   (a) *R is* (EM) *and* γ *is* $\gamma_1\to\gamma_2$*,* $\gamma_1\wedge\gamma_2$*,* $\gamma_1\vee\gamma_2$*,* $\sim\sim\gamma'$*,* $\sim(\gamma_1\to\gamma_2)$*,* $\sim(\gamma_1\wedge\gamma_2)$*, or* $\sim(\gamma_1\vee\gamma_2)$*:*

$$\dfrac{\dfrac{\begin{array}{c}[\sim\alpha]\\ \vdots\ \mathscr{D}_1\\ \gamma\end{array} \quad \begin{array}{c}[\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{array}}{\gamma}\,(\mathrm{EM}) \quad \begin{array}{c}\vdots\ \mathscr{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_2\\ \delta_2\end{array}}{\delta}\,R' \qquad \gg \qquad \dfrac{\dfrac{\begin{array}{c}[\sim\alpha]\\ \vdots\ \mathscr{D}_1\\ \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_2\\ \delta_2\end{array}}{\delta}\,R' \quad \dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathscr{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\mathrm{EM})$$

   where R' *is* $(\to\mathrm{E})$*,* $(\wedge\mathrm{E1})$*,* $(\wedge\mathrm{E2})$*,* $(\vee\mathrm{E})$*,* $(\sim\sim\mathrm{E})$ $(\sim\!\to\!\mathrm{E})$*,* $(\sim\wedge\mathrm{E})$*,* $(\sim\vee\mathrm{E1})$*, or* $(\sim\vee\mathrm{E2})$*, and both* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *are derivations of the minor premises of* R' *if they exist.*

3. *The definition of the reduction relation* $\gg$ *at* γ *in* nMC *is obtained from the conditions for the reduction relation* $\gg$ *at* γ *in* nC *by adding the following condition.*

(a) *R is* (GEM) *and* γ *is* $\gamma_1{\to}\gamma_2$, $\gamma_1{\wedge}\gamma_2$, $\gamma_1{\vee}\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\to}\gamma_2)$, ${\sim}(\gamma_1{\wedge}\gamma_2)$, *or* ${\sim}(\gamma_1{\vee}\gamma_2)$*:*

$$\dfrac{\dfrac{\begin{matrix}[\alpha\to\beta]\\ \vdots\ \mathscr{D}_1\\ \gamma\end{matrix}\quad \begin{matrix}[\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\text{GEM})\quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix}\quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad \gg \quad \dfrac{\dfrac{\begin{matrix}[\alpha\to\beta]\\ \vdots\ \mathscr{D}_1\\ \gamma\end{matrix}\quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix}\quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad \dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathscr{D}_2\\ \gamma\end{matrix}\quad \begin{matrix}\vdots\ \mathscr{E}_1\\ \delta_1\end{matrix}\quad \begin{matrix}\vdots\ \mathscr{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\text{GEM})$$

where R' *is* $(\to\text{E})$, $(\wedge\text{E1})$, $(\wedge\text{E2})$, $(\vee\text{E})$, $({\sim}{\sim}\text{E})$ $({\sim}{\to}\text{E})$, $({\sim}\wedge\text{E})$, $({\sim}\vee\text{E1})$, *or* $({\sim}\vee\text{E2})$*, and both* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *are derivations of the minor premises of* R' *if they exist.*

4. *The definition of the reduction relation* $\gg$ *at* γ *in* nCN *is obtained from the conditions for the reduction relation* $\gg$ *at* γ *in* nC3 *by adding the other conditions of* nMC*. Namely, it is defined as all the conditions for both* nC3 *and* nMC*.*

Prior to proving the normalization theorems for nC, nC3, nMC, and nCN, we need the following lemma.

Lemma 3.7 *Let* N_1*,* N_2*,* N_3*, and* N_4 *be* nC*,* nC3*,* nMC*, and* nCN*, respectively. Let* S_1*,* S_2*,* S_3*, and* S_4 *be* sC*,* sC3*,* sMC**, and* sCN**, respectively. For any* $i \in \{1,2,3,4\}$*, the following hold.*

1. *If* $\mathscr{D}$ *is a derivation in* N_i *such that* $\text{oa}(\mathscr{D}) = \Gamma$ *and* $\text{end}(\mathscr{D}) = \beta$*, then* $S_i \vdash \Gamma \Rightarrow \beta$*,*
2. *If* $S_i - (\text{cut}) \vdash \Gamma \Rightarrow \beta$*, then we can obtain a derivation* $\mathscr{D}'$ *in* N_i *such that (a)* $\text{oa}(\mathscr{D}') = \Gamma$*, (b)* $\text{end}(\mathscr{D}') = \beta$*, and (c)* $\mathscr{D}'$ *is normal.*

Proof.

1. We prove 1 by induction on the derivations $\mathscr{D}$ of N_i such that $\text{oa}(\mathscr{D}) = \Gamma$ and $\text{end}(\mathscr{D}) = \beta$. We distinguish the cases according to the last inference of $\mathscr{D}$. We show some cases.
   (a) Case (EM): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{matrix}[{\sim}\alpha]\Gamma_1\\ \vdots\ \mathscr{D}_1\\ \gamma\end{matrix}\qquad \begin{matrix}[\alpha]\Gamma_2\\ \vdots\ \mathscr{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\text{EM})$$

where $\text{oa}(\mathscr{D}) = \Gamma_1 \cup \Gamma_2$ and $\text{end}(\mathscr{D}) = \gamma$. By induction hypothesis, we have $S_i \vdash {\sim}\alpha, \Gamma_1 \Rightarrow \gamma$ and $S_i \vdash \alpha, \Gamma_2 \Rightarrow \gamma$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow \gamma$:

$$\dfrac{\begin{matrix}\vdots\ \mathit{Ind.hyp.}\\ {\sim}\alpha, \Gamma_1 \Rightarrow \gamma\\ \vdots\ (\text{we})\\ {\sim}\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{matrix}\qquad \begin{matrix}\vdots\ \mathit{Ind.hyp.}\\ \alpha, \Gamma_2 \Rightarrow \gamma\\ \vdots\ (\text{we})\\ \alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{matrix}}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\,(\text{ex-middle})$$

where (we) is admissible in $S_i - (\text{cut})$ by Propositions 2.4 and 2.11.

   (b) Case $({\sim}{\to}\text{I})$: We divide this case into two subcases.
      i. Subcase 1: $\mathscr{D}$ is of the form:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\ \mathscr{D}'\\ {\sim}\beta\end{matrix}}{{\sim}(\alpha\to\beta)}\,({\sim}{\to}\text{I})$$

where $\text{oa}(\mathscr{D}) = \Gamma$ and $\text{end}(\mathscr{D}) = {\sim}(\alpha\to\beta)$. By induction hypothesis, we have $S_i \vdash \Gamma \Rightarrow {\sim}\beta$. Then, we obtain that $S_i \vdash \Gamma \Rightarrow {\sim}(\alpha\to\beta)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathit{Ind.hyp.}\\ \Gamma \Rightarrow {\sim}\beta\end{matrix}}{\alpha, \Gamma \Rightarrow {\sim}\beta}\,(\text{we})}{\Gamma \Rightarrow {\sim}(\alpha\to\beta)}\,({\sim}{\to}\text{right})$$

where (we) is admissible in $S_i - (\text{cut})$ by Propositions 2.4 and 2.11.

ii. Subcase 2: $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\alpha]\ \Gamma \\ \vdots\ \mathscr{D}' \\ {\sim}\beta\end{array}}{{\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{I})$$

where $\mathrm{oa}(\mathscr{D}) = \Gamma$ and $\mathrm{end}(\mathscr{D}) = {\sim}(\alpha{\to}\beta)$. By induction hypothesis, we have $S_i \vdash \alpha, \Gamma \Rightarrow {\sim}\beta$. Then, we obtain the required fact $S_i \vdash \Gamma \Rightarrow {\sim}(\alpha{\to}\beta)$:

$$\dfrac{\begin{array}{c}\vdots\ Ind.hyp. \\ \alpha, \Gamma \Rightarrow {\sim}\beta\end{array}}{\Gamma \Rightarrow {\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{right}).$$

(c) Case $({\sim}{\to}\mathrm{E})$: $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1 \\ \vdots\ \mathscr{D}_1 \\ {\sim}(\alpha{\to}\beta)\end{array} \quad \begin{array}{c}\Gamma_2 \\ \vdots\ \mathscr{D}_2 \\ \alpha\end{array}}{{\sim}\beta}\ ({\sim}{\to}\mathrm{E})$$

where $\mathrm{oa}(\mathscr{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathscr{D}) = {\sim}\beta$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow {\sim}(\alpha{\to}\beta)$ and $S_i \vdash \Gamma_2 \Rightarrow \alpha$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow {\sim}\beta$:

$$\dfrac{\begin{array}{c}\vdots\ Ind.hyp. \\ \Gamma_2 \Rightarrow \alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\ Ind.hyp. \\ \Gamma_1 \Rightarrow {\sim}(\alpha{\to}\beta)\end{array} \quad \dfrac{\begin{array}{c}\vdots\ Prop.2.3 \\ \alpha \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\ Prop.2.3 \\ {\sim}\beta \Rightarrow {\sim}\beta\end{array}}{{\sim}(\alpha{\to}\beta), \alpha \Rightarrow {\sim}\beta}\ ({\sim}{\to}\mathrm{left})}{\alpha, \Gamma_1 \Rightarrow {\sim}\beta}\ (\mathrm{cut})}{\Gamma_1, \Gamma_2 \Rightarrow {\sim}\beta}\ (\mathrm{cut}).$$

(d) Case $({\sim}{\wedge}\mathrm{E})$: $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1 \\ \vdots\ \mathscr{D}_1 \\ {\sim}(\alpha{\wedge}\beta)\end{array} \quad \begin{array}{c}[{\sim}\alpha]\Gamma_2 \\ \vdots\ \mathscr{D}_2 \\ \gamma\end{array} \quad \begin{array}{c}[{\sim}\beta]\Gamma_3 \\ \vdots\ \mathscr{D}_3 \\ \gamma\end{array}}{\gamma}\ ({\sim}{\wedge}\mathrm{E})$$

where $\mathrm{oa}(\mathscr{D}) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\mathrm{end}(\mathscr{D}) = \gamma$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow {\sim}(\alpha{\wedge}\beta)$, $S_i \vdash {\sim}\alpha, \Gamma_2 \Rightarrow \gamma$, and $S_i \vdash {\sim}\beta, \Gamma_3 \Rightarrow \gamma$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma$:

$$\dfrac{\begin{array}{c}\vdots\ Ind.hyp. \\ \Gamma_1 \Rightarrow {\sim}(\alpha{\wedge}\beta)\end{array} \quad \dfrac{\begin{array}{c}\vdots\ Ind.hyp. \\ {\sim}\alpha, \Gamma_2 \Rightarrow \gamma \\ \vdots\ (\mathrm{we}) \\ {\sim}\alpha, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ Ind.hyp. \\ {\sim}\beta, \Gamma_3 \Rightarrow \gamma \\ \vdots\ (\mathrm{we}) \\ {\sim}\beta, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array}}{{\sim}(\alpha{\wedge}\beta), \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ ({\sim}{\wedge}\mathrm{left})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\mathrm{cut})$$

where (we) is admissible in $S_i - (\text{cut})$ by Propositions 2.4 and 2.11.

2. We prove 2 by induction on the derivations $\mathscr{D}$ of $\Gamma \Rightarrow \beta$ in $S_i - (\text{cut})$. We distinguish the cases according to the last inference of $\mathscr{D}$. We show some cases.

(a) Case (ex-middle): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1 \\ {\sim}\alpha, \Gamma \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathscr{D}_2 \\ \alpha, \Gamma \Rightarrow \gamma\end{array}}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle})$$

By induction hypotheses, we have normal derivations $\mathscr{E}_1$ and $\mathscr{E}_2$ in N_i of the form:

$$\begin{array}{ccc} \sim\alpha,\Gamma & \quad & \alpha,\Gamma \\ \vdots\ \mathscr{E}_1 & & \vdots\ \mathscr{E}_2 \\ \gamma & & \gamma \end{array}$$

where oa($\mathscr{E}_1$) = $\{\sim\alpha\}\cup\Gamma$, oa($\mathscr{E}_2$) = $\{\alpha\}\cup\Gamma$, end($\mathscr{E}_1$) = γ, and end($\mathscr{E}_2$) = γ. Then, we obtain a required normal derivation $\mathscr{D}'$ by:

$$\dfrac{\begin{array}{c}[\sim\alpha]\Gamma \\ \vdots\ \mathscr{E}_1 \\ \gamma\end{array} \quad \begin{array}{c}[\alpha]\Gamma \\ \vdots\ \mathscr{E}_2 \\ \gamma\end{array}}{\gamma}\ (\text{EM})$$

where oa($\mathscr{D}'$) = Γ and end($\mathscr{D}'$) = γ.

(b) Case ($\sim\sim$left): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ P' \\ \alpha,\Gamma\Rightarrow\gamma\end{array}}{\sim\sim\alpha,\Gamma\Rightarrow\gamma}\ (\sim\sim\text{left})$$

By induction hypothesis, we have a normal derivation Q' in N_i of the form:

$$\begin{array}{c}\alpha,\Gamma \\ \vdots\ Q' \\ \gamma\end{array}$$

where oa(Q') = $\{\alpha\}\cup\Gamma$ and end(Q') = γ. Then, we obtain a required normal derivation $\mathscr{D}'$ by:

$$\begin{array}{c}\dfrac{\sim\sim\alpha}{\alpha}\ (\sim\sim\text{E}) \quad \Gamma \\ \vdots\ Q' \\ \gamma\end{array}$$

where oa($\mathscr{D}'$) = $\{\sim\sim\alpha\}\cup\Gamma$ and end($\mathscr{D}'$) = γ.

(c) Case ($\sim\!\to$left): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1 \\ \Gamma\Rightarrow\alpha\end{array} \quad \begin{array}{c}\vdots\ \mathscr{D}_2 \\ \sim\beta,\Delta\Rightarrow\gamma\end{array}}{\sim(\alpha\to\beta),\Gamma,\Delta\Rightarrow\gamma}\ (\sim\!\to\text{left})$$

By induction hypotheses, we have normal derivations $\mathscr{E}_1$ and $\mathscr{E}_2$ in N_i of the form:

$$\begin{array}{ccc} \Gamma & \quad & \sim\beta,\Delta \\ \vdots\ \mathscr{E}_1 & & \vdots\ \mathscr{E}_2 \\ \alpha & & \gamma \end{array}$$

where oa($\mathscr{E}_1$) = Γ, oa($\mathscr{E}_2$) = $\{\beta\}\cup\Delta$, end($\mathscr{E}_1$) = α, and end($\mathscr{E}_2$) = γ. Then, we obtain a required normal derivation $\mathscr{D}'$ by:

$$\begin{array}{c}\dfrac{\sim(\alpha\to\beta) \quad \begin{array}{c}\Gamma \\ \vdots\ \mathscr{E}_1 \\ \alpha\end{array}}{\sim\beta}\ (\sim\!\to\text{E}) \quad \Delta \\ \vdots\ \mathscr{E}_2 \\ \gamma\end{array}$$

where oa($\mathscr{D}'$) = $\{\sim(\alpha\to\beta)\}\cup\Gamma\cup\Delta$ and end($\mathscr{D}'$) = γ.

(d) Case ($\sim\wedge$left): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1\\ \sim\alpha,\Gamma\Rightarrow\gamma\end{array}\quad\begin{array}{c}\vdots\ \mathscr{D}_2\\ \sim\beta,\Gamma\Rightarrow\gamma\end{array}}{\sim(\alpha\wedge\beta),\Gamma\Rightarrow\gamma}\ (\sim\wedge\text{left})$$

By induction hypotheses, we have normal derivations $\mathscr{E}_1$ and $\mathscr{E}_2$ in N_i of the form:

$$\begin{array}{c}\sim\alpha,\Gamma\\ \vdots\ \mathscr{E}_1\\ \gamma\end{array}\qquad\begin{array}{c}\sim\beta,\Gamma\\ \vdots\ \mathscr{E}_2\\ \gamma\end{array}$$

where oa($\mathscr{E}_1$) = $\{\sim\alpha\}\cup\Gamma$, oa($\mathscr{E}_2$) = $\{\sim\beta\}\cup\Gamma$, and end($\mathscr{E}_1$) = end($\mathscr{E}_2$) = γ. Then, we obtain a required normal derivation $\mathscr{D}'$ by:

$$\dfrac{\sim(\alpha\wedge\beta)\quad\begin{array}{c}[\sim\alpha]\Gamma\\ \vdots\ \mathscr{E}_1\\ \gamma\end{array}\quad\begin{array}{c}[\sim\beta]\Gamma\\ \vdots\ \mathscr{E}_2\\ \gamma\end{array}}{\gamma}\ (\sim\wedge\text{E})$$

where oa($\mathscr{D}'$) = $\{\sim(\alpha\wedge\beta)\}\cup\Gamma$ and end($\mathscr{D}'$) = γ.

(e) Case ($\sim\vee$right): $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathscr{D}_1\\ \Gamma\Rightarrow\sim\alpha\end{array}\quad\begin{array}{c}\vdots\ \mathscr{D}_2\\ \Gamma\Rightarrow\sim\beta\end{array}}{\Gamma\Rightarrow\sim(\alpha\vee\beta)}\ (\sim\vee\text{right})$$

By induction hypotheses, we have normal derivations $\mathscr{E}_1$ and $\mathscr{E}_2$ in N_i of the form:

$$\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}_1\\ \sim\alpha\end{array}\qquad\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}_2\\ \sim\beta\end{array}$$

where oa($\mathscr{E}_1$) = oa($\mathscr{E}_2$) = Γ, end($\mathscr{E}_1$) = $\sim\alpha$, and end($\mathscr{E}_2$) = $\sim\beta$. Then, we obtain a required normal derivation $\mathscr{D}'$ by:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}_1\\ \sim\alpha\end{array}\quad\begin{array}{c}\Gamma\\ \vdots\ \mathscr{E}_2\\ \sim\beta\end{array}}{\sim(\alpha\vee\beta)}\ (\sim\vee\text{I})$$

where oa($\mathscr{D}'$) = Γ and end($\mathscr{D}'$) = $\sim(\alpha\vee\beta)$.

■

Theorem 3.8 (Equivalence between nC-family and sC-family) *Let N_1, N_2, N_3, and N_4 be* nC, nC3, nMC, *and* nCN, *respectively. Let S_1, S_2, S_3, and S_4 be* sC, sC3, sMC*, *and* sCN*, *respectively. For any formula α and any $i\in\{1,2,3,4\}$, $S_i\vdash\ \Rightarrow\alpha$ iff α is provable in N_i.*

Proof. Taking $\emptyset$ as Γ in Lemma 3.7, we obtain the claim. ■

Theorem 3.9 (Normalization for nC, nC3, nMC, and nCN) *Let N be* nC, nC3, nMC, *or* nCN. *All derivations in N are normalizable. More precisely, if a derivation $\mathscr{D}$ in N is given, then we can obtain a normal derivation $\mathscr{D}'$ in N such that* oa($\mathscr{D}'$) = oa($\mathscr{D}$) *and* end($\mathscr{D}'$) = end($\mathscr{D}$).

Proof. Let N_1, N_2, N_3, and N_4 be nC, nC3, nMC, and nCN, respectively. Let S_1, S_2, S_3, and S_4 be sC, sC3, sMC*, and sCN*, respectively. Let i be 1, 2, 3, or 4. Suppose that a derivation $\mathscr{D}$ in N_i is given, and suppose that oa($\mathscr{D}$) = Γ and end($\mathscr{D}$) = β. Then, by Lemma 3.7 (1), we obtain $L_i\vdash\Gamma\Rightarrow\beta$. By the cut-elimination theorem for S_i (i.e., Theorems 2.5, 2.6, and 2.13), we obtain $S_i - (\text{cut}) \vdash \Gamma \Rightarrow \beta$. Then, by Lemma 3.7 (2), we can obtain a normal derivation $\mathscr{D}'$ in N_i such that oa($\mathscr{D}'$) = oa($\mathscr{D}$) and end($\mathscr{D}'$) = end($\mathscr{D}$). ■

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References

[1] Henry Africk (1992): *Classical logic, intuitionistic logic, and the Peirce rule*. Notre Dame Journal of Formal Logic 33 (2), pp. 229-235, doi:10.1305/ndjfl/1093636101.

[2] Ahmad Almukdad & David Nelson (1984): *Constructible falsity and inexact predicates*. Journal of Symbolic Logic 49 (1), pp. 231-233, doi:10.2307/2274105.

[3] Richard Angell (1962): *A propositional logics with subjunctive conditionals*. Journal of Symbolic Logic 27, pp. 327-343, doi:10.2307/2964651.

[4] John Cantwell (2008): *The logic of conditional negation*. Notre Dame Journal of Formal Logic 49, pp. 245-260, doi:10.1215/00294527-2008-010.

[5] Haskell B. Curry (1963): *Foundations of mathematical logic*. McGraw-Hill, New York.

[6] Paul Égré, Lorenzo Rossi & Jan Sprenger (2021): *De finettian logics of indicative conditionals Part II: Proof theory and algebraic semantics*. Journal of Philosophical Logic 50, pp. 215-247, doi:10.1007/S10992-020-09572-7.

[7] Walter Felscher (1975): *Kombinatorische Konstruktionen mit beweisen und schnittelimination*. Lecture Notes in Mathematics 500, pp. 119-151, doi:10.1007/BFb0079549.

[8] Nissim Francez (2016): *Natural deduction for two connexive logics*. IfCoLog Journal of Logics and their Applications 3(3), pp. 479-504.

[9] Gerhard Gentzen (1969): *Collected papers of Gerhard Gentzen*. M.E. Szabo (ed.), Studies in logic and the foundations of mathematics, North-Holland (English translation), doi:10.2307/2272429.

[10] Lew Gordeev (1987): *On cut elimination in the presence of Peirce rule*. Archiv für Mathematische Logik und Grundlagenforsch 26, pp. 147-164, doi:10.1007/BF02017499.

[11] Yuri Gurevich (1977): *Intuitionistic logic with strong negation*. Studia Logica 36, pp. 49-59, doi:10.1007/BF02121114.

[12] Norihiro Kamide (2005): *Cut-free single-succedent systems revisited*. Bulletin of the Section of Logic 34 (3), pp. 165-175.

[13] Norihiro Kamide (2005): *Natural deduction systems for Nelson's paraconsistent logic and its neighbors*. Journal of Applied Non-Classical Logics 15 (4), pp. 405-435, doi:10.3166/JANCL.15.405-435.

[14] Norihiro Kamide (2022): *Embedding friendly first-order paradefinite and connexive logics*. Journal of Philosophical logic 51 (5), pp. 1055-1102, doi:10.1007/S10992-022-09659-3.

[15] Norihiro Kamide (2024): *Rules of explosion and excluded middle: Constructing a unified single-succedent Gentzen-style framework for classical, paradefinite, paraconsistent, and paracomplete logics*. Journal of Logic, Language and Information, Published as an online first article, doi:10.1007/s10849-024-09416-6.

[16] Norihiro Kamide & Heinrich Wansing (2011): *Connexive modal logic based on positive S4*. In: Logic without Frontiers: Festschrift for Walter Alexandre Carnielli on the occasion of his 60th Birthday, (Jean-Yves

Beziau and Marcelo Esteban Coniglio, eds.), Volume 17 of Tribute Series, pp. 389-410, College Publications. London.

[17] Norihiro Kamide & Heinrich Wansing (2012): *Proof theory of Nelson's paraconsistent logic: A uniform perspective*. Theoretical Computer Science 415, pp. 1-38, doi:10.1016/J.TCS.2011.11.001.

[18] Norihiro Kamide & Heinrich Wansing (2015): *Proof theory of N4-related paraconsistent logics*. Studies in Logic, Volume 54, College Publications, pp. 1-401, doi:10.1007/s11225-017-9729-9.

[19] Norihiro Kamide & Heinrich Wansing (2016): *Completeness of connexive Heyting-Brouwer logic*. IFCoLog Journal of Logics and their Applications 3 (3), pp. 441-466.

[20] Nils Kürbis & Yaroslav Petrukhin (2021): *Normalisation for some quite interesting many-valued logics*. Logic and Logical Philosophy 30 (3), pp. 493-534, doi:10.12775/LLP.2021.009.

[21] Storrs McCall (1966): *Connexive implication*. Journal of Symbolic Logic 31, pp. 415-433, doi:10.2307/2270458.

[22] Sara Negri & Jan von Plato (2001): *Structural Proof Theory*. doi:10.1017/CBO9780511527340.

[23] David Nelson (1949): *Constructible falsity*. Journal of Symbolic Logic 14, pp. 16-26, doi:10.2307/2268973.

[24] Satoru Niki (2024): *Intuitionistic views on connexive constructible falsity*. Journal of Applied Logics 11 (2), pp. 125-157.

[25] Satoru Niki & Heinrich Wansing (2023): *On the provable contradictions of the connexive logics C and C3*. Journal of Philosophical Logic 52 (5), pp. 1355-1383, doi:10.1007/S10992-023-09709-4.

[26] Grigory K. Olkhovikov (2002): *On a new three-valued paraconsistent logic*. In: Logic of Law and Tolerance, Yekaterinburg: Ural State University Press, 96-113, 2002. It was translated by T.M. Ferguson in IfCoLog Journal of Logics and their Applications 3(3), pp. 317-334, 2016.

[27] Grigory K. Olkhovikov (2016): *A complete, correct, and independent axiomatization of the first-order fragment of a three-valued paraconsistent logic*. IfCoLog Journal of Logics and their Applications 3 (3), pp. 335-340.

[28] Hitoshi Omori (2016): *From paraconsistent logic to dialetheic logic*. In: H. Andreas and P. Verdee (eds.), Logical Studies of Paraconsistent Reasoning in Science and Mathematics, Berlin: Springer, pp. 111-134, doi:10.1007/978-3-319-40220-8_8.

[29] Hitoshi Omori & Heinrich Wansing (2020): *An extension of connexive logic C*. Advances in Modal Logic, pp. 503-522.

[30] Jan von Plato (1999): *Proof theory of full classical propositional logic*. Manuscript, 16 pages.

[31] Dag Prawitz (1965): *Natural deduction: a proof-theoretical study*. Almqvist and Wiksell, Stockholm, doi:10.2307/2271676.

[32] Wolfgang Rautenberg (1979): *Klassische und nicht-klassische Aussagenlogik*. Vieweg, Braunschweig, doi:10.1007/978-3-322-85796-5.

[33] Allard M. Tamminga & Koji Tanaka (1999): *A natural deduction system for first degree entailment*. Notre Dame Journal of Formal Logic 40 (2), pp. 258-272, doi:10.1305/NDJFL/1038949541.

[34] N.N. Vorob'ev (1952): *A constructive propositional calculus with strong negation (in Russian)*. Doklady Akademii Nauk SSSR 85, pp. 465-468.

[35] Heinrich Wansing (1993): *The logic of information structures*. Lecture Notes in Artificial Intelligence 681, 163 pages, doi:10.1007/3-540-56734-8.

[36] Heinrich Wansing (2005): *Connexive modal logic*. Advances in Modal Logic 5, pp. 367-385.

[37] Heinrich Wansing (2016): *Connexive logic*. Stanford Encyclopedia of Philosophy, 2020: http://plato.stanford.edu/entries/logic-connexive/.

[38] Heinrich Wansing (2016): *Falsification, natural deduction and bi-intuitionistic logic*. Journal of Logic and Computation 26 (1), pp. 425-450, doi:10.1093/LOGCOM/EXT035.

[39] Heinrich Wansing (2016): *Natural deduction for bi-connexive logic and a two-sorted typed λ-calculus*. *If-CoLog Journal of Logics and their Applications 3(3), pp. 413-440.*

A note on Grigoriev and Zaitsev's system $\mathbf{CNL}_4^2$*

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The present article examines a system of four-valued logic recently introduced by Oleg Grigoriev and Dmitry Zaitsev. In particular, besides other interesting results, we will clarify the connection of this system to related systems developed by Paul Ruet and Norihiro Kamide. By doing so, we discuss two philosophical problems that arise from making such connections quite explicit: first, there is an issue with how to make intelligible the meaning of the connectives and the nature of the truth values involved in the many-valued setting employed — what we have called 'the Haackian theme'. We argue that this can be done in a satisfactory way, when seen according to the classicist's light. Second, and related to the first problem, there is a complication arising from the fact that the proof system advanced may be made sense of by advancing at least four such different and incompatible readings — a sharpening of the so-called 'Carnap problem'. We make explicit how the problems connect with each other precisely and argue that what results is a kind of underdetermination by the deductive apparatus for the system.

1 Introduction

By its very nature and purpose, a non-classical system of logic is a system that deviates from classical standards on some regards. Most of us believe we can make some sense of what classical connectives mean, and of what classical logical consequence means. Given that for a long time now classical logic has set the standards for the understanding of connectives and logical consequence, whenever some non-classical system of logic is advanced, questions concerning the meaning of the connectives, and what the logical consequence relation is telling us, come to the front. One interesting way to address these questions was suggested some time ago by Susan Haack [6, chap.11]:[1] most of the mysteries of at least some non-classical systems disappears if one can advance a reasonable 'classical-like' reading of the connectives. A similar classically-oriented story may be told for logical consequence and for the understanding of the truth values assumed. Having such readings accounts for the classicists' intelligibility of such systems, although, it must be recognize, it deprives them of much of their revolutionary character.

Given that background, in this paper, we shall be concerned with a system of four-valued logic recently introduced by Oleg Grigoriev and Dmitry Zaitsev in [5], called $\mathbf{CNL}_4^2$. Our plan is to apply the strategy suggested by Susan Haack which we have just described, henceforth the 'Haackian strategy', in order to make it comfortable for classicists; we shall discuss also whether such application helps us in shedding some light on the system. As part of the implementation, we shall highlight the fact that the strategy may be implemented in different, incompatible ways, generating a scenario where meaning

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[1] For recent developments, see [13, 11].

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is not properly fixed by the system of logic under scrutiny. Although this is not new, the flexibility on meaning allowed by this system raises interesting questions we shall also discuss.

That leads us to the next major philosophical theme we shall concentrate on. The problem we have just touched on regarding meaning flexibility via our attempts to increase the intelligibility here connect to the so-called *Carnap problem*, the fact that a given proof system for a system of logic does not single out *one unique intended interpretation*. As we have indicated, the problem has some very deep ramifications related to the system to be considered here because, as we shall see, given the results to be presented, the same proof system may have at least four different readings. But that is not all yet: those readings are indeed readings of the same ingredients comprising 'the' formal semantics, but they are also different enough to suggest that they are instantiations of different approaches to the very understanding of the workings of truth, falsity, the meaning of the connectives and the consequence relation of the underlying system. As a result, they actually seem to count as radically different semantic understandings for the same deductive system. The outcome of this scenario is that two persons using the same system may be having radically different understandings of the references of the logical apparatus, without disagreeing on what follows from what in the system.

As we have already mentioned, on our way to address the problem of the meaning of the connectives and the intelligibility of the distinct readings proposed for the proof theory, we shall provide for two possible ways to endow $\mathbf{CNL}_4^2$ with a more or less classical reading. The first one is obtained by providing for direct re-readings of the truth values of the original four-valued system. The second one, which will actually instantiate the Haackian strategy, is directly related to a reformulation of the semantics in terms of relational semantics (or Dunn semantics). That will make completely explicit the use of the two classical truth values, and will also illustrate more clearly the different possible readings available in classical terms. As an additional resource for the classicist, given that negation is one of the most controversial connectives, we appeal to functional completeness and to the definability of classical negation inside the system. So, in a sense, the classical logician can gain intelligibility of the working of the system by appeal to a classical behavior that is also available in $\mathbf{CNL}_4^2$.

The rest of the paper is structured as follows. After a brief preliminaries in §2 recalling the four-valued semantics for $\mathbf{CNL}_4^2$ explored by Grigoriev and Zaitsev, we add some basic results in §3. Building on these results, we turn to the theme from Carnap in §4. This will be followed by §5 in which we discuss matters in light of the theme from Haack. We shall add some further reflections in §6, and the paper will be concluded by §7 with some brief final remarks.

2 Preliminaries

The language $\mathscr{L}$ consists of a set $\{\sim, \wedge, \vee\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. We denote by Form the set of formulas defined as usual in $\mathscr{L}$. We denote a formula of $\mathscr{L}$ by A, B, C, etc. and a set of formulas of $\mathscr{L}$ by Γ, Δ, Σ, etc.

Let us now recall the semantics introduced in [5]. For the purpose of this article, we will slightly change the notation to keep the values free of intuitive readings.

Definition 1 (Grigoriev & Zaitsev)**.** *A* $\mathbf{CNL}_4^2$*-interpretation of* $\mathscr{L}$ *is a function I from* Prop *to* $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{0}\}$. *Given a* $\mathbf{CNL}_4^2$*-interpretation I, this is extended to a valuation V that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:*

A	$\sim A$
1	**i**
i	**0**
j	**1**
0	**j**

$A \wedge B$	**1**	**i**	**j**	**0**
1	**1**	**i**	**j**	**0**
i	**i**	**i**	**0**	**0**
j	**j**	**0**	**j**	**0**
0	**0**	**0**	**0**	**0**

$A \vee B$	**1**	**i**	**j**	**0**
1	**1**	**1**	**1**	**1**
i	**1**	**i**	**1**	**i**
j	**1**	**1**	**j**	**j**
0	**1**	**i**	**j**	**0**

Definition 2. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{CNL}^2_4} A$ *iff for all* $\mathbf{CNL}^2_4$*-interpretations* I, $V(A) \in \mathscr{D}$ *if* $V(B) \in \mathscr{D}$ *for all* $B \in \Gamma$ *where* $\mathscr{D} = \{\mathbf{1}, \mathbf{i}\}$.

Remark 3. *Note that Grigoriev and Zaitsev also consider another four-valued logic called* $\mathbf{CNLL}^2_4$ *in which four values are* linearly ordered. *We shall not, however, consider the other system since there are already plenty of topics to discuss for* $\mathbf{CNL}^2_4$.

3 Basic observations

3.1 An alternative proof system

In [5], a binary proof system is defined by Grigoriev and Zaitsev, but here we will present a natural deduction system.

Definition 4. *The natural deduction rules* $\mathscr{R}_{\mathbf{CNL}^2_4}$ *for* $\mathbf{CNL}^2_4$ *are all the following rules:*

$$\frac{A \quad B}{A\wedge B} \quad \frac{A\wedge B}{A} \quad \frac{A\wedge B}{B} \quad \frac{A}{A\vee B} \quad \frac{B}{A\vee B} \quad \frac{A\vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C} \quad \frac{A \quad \sim\sim A}{B}\ (\sim\sim 1) \quad \frac{}{A\vee\sim\sim A}\ (\sim\sim 2)$$

$$\frac{\sim A \quad \sim B}{\sim(A\wedge B)} \quad \frac{\sim(A\wedge B)}{\sim A} \quad \frac{\sim(A\wedge B)}{\sim B} \quad \frac{\sim A}{\sim(A\vee B)} \quad \frac{\sim B}{\sim(A\vee B)} \quad \frac{\sim(A\vee B) \quad \begin{matrix}[\sim A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[\sim B]\\ \vdots \\ C\end{matrix}}{C}$$

Based on these, given any set $\Sigma \cup \{A\}$ *of formulas,* $\Sigma \vdash A$ *iff for some finite* $\Sigma' \subseteq \Sigma$, *there is a derivation of* A *from* Σ' *in the calculus whose rule set is* $\mathscr{R}_{\mathbf{CNL}^2_4}$.

Then, the soundness direction is tedious, but standard, so we only state it without a proof.

Theorem 1 (Soundness). *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash A$ *only if* $\Gamma \models_{\mathbf{CNL}^2_4} A$.

For the completeness direction, we prepare some well known notions and lemmas.

Definition 5. *Let* Σ *be a set of formulas. Then,* Σ *is a* theory *iff* $\Sigma \vdash A$ *implies* $A \in \Sigma$, *and* Σ *is* prime *iff* $A\vee B \in \Sigma$ *implies* $A \in \Sigma$ *or* $B \in \Sigma$.

Lemma 1 (Lindenbaum). *If* $\Sigma \nvdash A$, *then there is* $\Sigma' \supseteq \Sigma$ *such that* $\Sigma' \nvdash A$ *and* Σ' *is a prime theory.*

We now define the canonical valuation in the following manner.

Definition 6. *For any* $\Sigma \subseteq \mathsf{Form}$, *let* v_Σ *from* Prop *to* $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{0}\}$ *be defined as follows:*

$$v_\Sigma(p) := \begin{cases} \mathbf{1} \text{ iff } \Sigma \vdash p \text{ and } \Sigma \vdash \sim p; \\ \mathbf{i} \text{ iff } \Sigma \vdash p \text{ and } \Sigma \nvdash \sim p; \\ \mathbf{j} \text{ iff } \Sigma \nvdash p \text{ and } \Sigma \vdash \sim p; \\ \mathbf{0} \text{ iff } \Sigma \nvdash p \text{ and } \Sigma \nvdash \sim p. \end{cases}$$

Remark 7. *Note that the above definition is different from the more familiar definition when the four values are understood as in* **FDE***.*

The following lemma is the key for the completeness result.

Lemma 2. *If Σ is a prime theory, then the following hold for all $B \in \mathsf{Form}$.*

$$v_\Sigma(B) = \begin{cases} \mathbf{1} \text{ iff } \Sigma \vdash B \text{ and } \Sigma \vdash {\sim}B; \\ \mathbf{i} \text{ iff } \Sigma \vdash B \text{ and } \Sigma \nvdash {\sim}B; \\ \mathbf{j} \text{ iff } \Sigma \nvdash B \text{ and } \Sigma \vdash {\sim}B; \\ \mathbf{0} \text{ iff } \Sigma \nvdash B \text{ and } \Sigma \nvdash {\sim}B. \end{cases}$$

Proof. Note first that the well-definedness of v_Σ is obvious. Then the desired result is proved by induction on the construction of B. The base case, for atomic formulas, is obvious by the definition. For the induction step, the cases are split based on the connectives. The details are spelled out in the Appendix. □

We are now ready to prove the completeness result.

Theorem 2 (Completeness)**.** *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{CNL}_4^2} A$ only if $\Gamma \vdash A$.*

Proof. Assume $\Gamma \nvdash A$. Then, by Lemma 1, there is a $\Sigma \supseteq \Gamma$ such that Σ is a prime theory and $A \notin \Sigma$, and by Lemma 2, a four-valued valuation v_Σ can be defined with $I_\Sigma(B) \in \mathscr{D}$ for every $B \in \Gamma$ and $I_\Sigma(A) \notin \mathscr{D}$. Thus it follows that $\Gamma \not\models_{\mathbf{CNL}_4^2} A$, as desired. □

3.2 Functional completeness

We now turn to show that the matrix that characterizes the system is functionally complete. To this end, we will first introduce some related notions.

Definition 8 (Functional completeness)**.** *An algebra $\mathfrak{A} = \langle A, f_1, \ldots, f_n \rangle$, is said to be* functionally complete *provided that every finitary function $f \colon A^m \to A$ is definable by compositions of the functions $f_1, \ldots, f_n$ alone. A matrix $\langle \mathfrak{A}, \mathscr{D} \rangle$ is* functionally complete *if $\mathfrak{A}$ is functionally complete.*

Definition 9 (Definitional completeness)**.** *A logic* **L** *is* definitionally complete *if there exists a functionally complete matrix that is strongly sound and complete for L.*

For the characterization of the functional completeness, the following theorem of Jerzy Słupecki is elegant and useful. In order to state the result, we need the following definition.

Definition 10. *Let $\mathfrak{A} = \langle A, f_1, \ldots, f_n \rangle$ be an algebra, and f be a binary operation defined in $\mathfrak{A}$. Then, f is* unary reducible *iff for some unary operation g definable in $\mathfrak{A}$, $f(x,y) = g(x)$ for all $x, y \in A$ or $f(x,y) = g(y)$ for all $x, y \in A$. And f is* essentially binary *if f is not unary reducible.*

Theorem 3 (Słupecki, [21])**.** *$\mathfrak{A} = \langle \langle \mathscr{V}, f_1, \ldots, f_n \rangle, \mathscr{D} \rangle$ ($|\mathscr{V}| \geq 3$) is functionally complete iff in $\langle \mathscr{V}, f_1, \ldots, f_n \rangle$* (1) *all unary functions on $\mathscr{V}$ are definable, and* (2) *at least one surjective and essentially binary function on $\mathscr{V}$ is definable.*

This elegant characterization by Słupecki can be simplified even further in case of expansions of the algebra related to **FDE** (cf. [15, Theorem 4.8]).

Theorem 4. *Given any expansion $\mathscr{F}$ of the algebra $\langle \{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{0}\}, \wedge, \vee \rangle$, the following are equivalent:*

(1) *$\mathscr{F}$ is functionally complete;*

(2) *all of the* δ_a*s as well as* C_a*s* ($a \in \{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{0}\}$) *are definable, where* $\delta_a(b)=\mathbf{1}$*, if* $a=b$*, otherwise* $\delta_a(b)=\mathbf{0}$*; and* $\mathsf{C}_a(b)=a$*, for all* $a,b \in \mathscr{V}$*.*

Building on this result, we obtain the following.

Theorem 5. $\mathbf{CNL}_4^2$ *is definitionally complete.*

Proof. In view of the above theorem, it suffices to prove that all of the δ_as as well as C_as ($a \in \{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{0}\}$) are definable in $\langle\{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{0}\},\sim,\wedge,\vee\rangle$, and this can be done as follows:

$$\delta_{\mathbf{1}}(x):=\neg(\sim\sim x\vee\sim\sim\sim x),\quad \delta_{\mathbf{j}}(x):=\neg(\sim\sim\sim x\vee\neg\neg x),\quad \mathsf{C}_{\mathbf{1}}(x):=x\vee\sim\sim x,\quad \mathsf{C}_{\mathbf{j}}(x):=\sim(x\wedge\sim\sim x),$$
$$\delta_{\mathbf{i}}(x):=\neg(\sim\sim x\vee\neg\sim\sim\sim x),\quad \delta_{\mathbf{0}}(x):=\neg\neg(\sim\sim x\wedge\sim\sim\sim x),\quad \mathsf{C}_{\mathbf{i}}(x):=\sim(x\vee\sim\sim x),\quad \mathsf{C}_{\mathbf{0}}(x):=x\wedge\sim\sim x,$$

where $\neg x:=\sim\sim\sim(\sim\sim((x\wedge\sim\sim\sim x)\wedge\sim\sim\sim(x\wedge\sim\sim\sim x))\wedge((x\wedge\sim\sim\sim x)\vee\sim\sim\sim(x\wedge\sim\sim\sim x)))$. □

Finally, we add a brief remark on the Post completeness.

Definition 11. *The logic* $\mathbf{L}$ *is* Post complete *iff for every formula* A *such that* $\nvdash A$*, the extension of* $\mathbf{L}$ *by* A *becomes trivial, i.e.,* $\vdash_{\mathbf{L}\cup\{A\}} B$ *for any* B*.*

Theorem 6 (Tokarz, [23])**.** *Definitionally complete logics are Post complete.*

In view of Theorems 5 and 6, we obtain the following result.

Corollary 1. $\mathbf{CNL}_4^2$ *is Post complete.*

3.3 A few more results

Before moving further, we list some valid/derivable inferences, as well as invalid/non-derivable ones.

Proposition 1. *The following hold in* $\mathbf{CNL}_4^2$*.*

$$B \models_{\mathbf{CNL}_4^2} (A\vee\sim\sim A),\quad B \models_{\mathbf{CNL}_4^2} \sim(A\vee\sim\sim A),\quad A\wedge\sim\sim A \models_{\mathbf{CNL}_4^2} B,\quad \sim(A\wedge\sim\sim A) \models_{\mathbf{CNL}_4^2} B.$$

Proof. It suffices to observe that $V(A\vee\sim\sim A)=\mathbf{1}$, and $V(\sim(A\vee\sim\sim A))=\mathbf{i}$ for the first two items, and that $V(A\wedge\sim\sim A)=\mathbf{0}$, and $V(\sim(A\wedge\sim\sim A))=\mathbf{j}$ for the latter two items. □

Proposition 2. *The following also hold in* $\mathbf{CNL}_4^2$*.*

$$q \not\models_{\mathbf{CNL}_4^2} p\vee\sim p,\quad p\wedge\sim p \not\models_{\mathbf{CNL}_4^2} q,\quad \sim\sim p \not\models_{\mathbf{CNL}_4^2} p,\quad p \not\models_{\mathbf{CNL}_4^2} \sim\sim p.$$

Proof. Interpretations such that $I_1(p)=\mathbf{0}, I_2(q)=\mathbf{1}$ for the first item, $I_2(p)=\mathbf{1}, I_2(q)=\mathbf{0}$ for the second item, $I_3(p)=\mathbf{0}$ for the third item, and $I_4(p)=\mathbf{1}$ for the last item will establish the desired results. □

Remark 12. *One might be already tempted to discuss features of* $\mathbf{CNL}_4^2$ *based on the above observations. In particular, one may be tempted to refer to* $\sim$ *as negation. This, however, is a rather delicate matter, and we will return to this point in §6 after some discussions on the interpretations of the four values.*

4 Carnapian theme: Four interpretations of one truth table

We now present the options for the readings of the truth values, according to the two strategies we mentioned before, viz., the re-readings for the original four-valued semantics, and the relational semantics.

4.1 Option 1

The first two options will be to interpret **1** and **0** as **t** and **f** of **FDE**, respectively, and make a choice between options in interpreting the intermediate values. Let us start by following the choice made by Grigoriev and Zaitsev, that is, the values **i** and **j** are interpreted as **b** and **n**, respectively.

A	$\sim A$
t	**b**
b	**f**
n	**t**
f	**n**

$A \wedge B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$A \vee B$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

Then, this makes it very clear that the resulting truth tables are those introduced by Paul Ruet in [20]. Moreover, the resulting logic is obtained by considering the truth preservation by building on the above truth tables, and therefore, it is the same logic introduced by Ruet.[2][3]

In order to observe the differences of interpretations, let us apply the mechanical procedure described in [14], and offer an alternative presentation of the interpretation in terms of truth and falsity conditions, assuming that we rewrite the four values $\mathbf{t}, \mathbf{b}, \mathbf{n}$ and $\mathbf{f}$ as $\{1\}, \{1,0\}, \emptyset$ and $\{0\}$, respectively. For the present case, we obtain the following truth and falsity conditions.

$1 \in V(\sim A)$ iff $0 \notin V(A)$; $\quad 0 \in V(\sim A)$ iff $1 \in V(A)$;

$1 \in V(A \wedge B)$ iff $1 \in V(A)$ and $1 \in V(B)$; $\quad 0 \in V(A \wedge B)$ iff $0 \in V(A)$ or $0 \in V(B)$;

$1 \in V(A \vee B)$ iff $1 \in V(A)$ or $1 \in V(B)$; $\quad 0 \in V(A \vee B)$ iff $0 \in V(A)$ and $0 \in V(B)$.

Therefore, it becomes very clear that the truth and falsity conditions are almost the same with **FDE**. Indeed, $\wedge$ and $\vee$ are interpreted as in **FDE**, and for $\sim$, the truth condition is the only condition that is deviating from **FDE**.[4]

4.2 Option 2

Let us now turn to the other option. That is, the values **i** and **j** are interpreted as **n** and **b**, respectively. Then, as a result of rewriting the values, we obtain the following truth table.

A	$\sim A$
t	**n**
n	**f**
b	**t**
f	**b**

$A \wedge B$	**t**	**n**	**b**	**f**
t	**t**	**n**	**b**	**f**
n	**n**	**n**	**f**	**f**
b	**b**	**f**	**b**	**f**
f	**f**	**f**	**f**	**f**

$A \vee B$	**t**	**n**	**b**	**f**
t	**t**	**t**	**t**	**t**
n	**t**	**n**	**t**	**n**
b	**t**	**t**	**b**	**b**
f	**t**	**n**	**b**	**f**

If we rewrite it slightly, for the purpose of making the comparison easier, we obtain the following tables.

A	$\sim A$
t	**n**
b	**t**
n	**f**
f	**b**

$A \wedge B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$A \vee B$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

[2] To be more precise, Ruet added the unary operator $\sim$ (or ↻ in Ruet's notation) on top of Belnap-Dunn logic. Therefore, for the purpose of establishing the definitional equivalence of the system of Ruet and the system of Grigoriev and Zaitsev, we need to check that de Morgan negation is definable in $\mathbf{CNL}_4^2$. However, this is an immediate corollary of the functional completeness result. Therefore, the desired result is established.

[3] Note that Ruet's system is also discussed in [3] under the name **dCP** by Grigoriev and Zaitsev together with Alex Belikov. Even though Grigoriev and Zaitsev do not state it explicitly in [5], $\mathbf{CNL}_4^2$ is definitionally equivalent to **dCP**.

[4] Note that $\sim\sim$ behaves as the Boolean complement. For discussions on such a kind of connective, see [8, 9, 10, 15, 18, 16].

Then, this makes it clear that the resulting truth tables are those introduced by Norihiro Kamide in [9, 10], and explored in [15, 16, 18].[5] Moreover, we obtain the following truth and falsity conditions.

$1 \in V(\sim A)$ iff $0 \in V(A)$; $0 \in V(\sim A)$ iff $1 \notin V(A)$;
$1 \in V(A \wedge B)$ iff $1 \in V(A)$ and $1 \in V(B)$; $0 \in V(A \wedge B)$ iff $0 \in V(A)$ or $0 \in V(B)$;
$1 \in V(A \vee B)$ iff $1 \in V(A)$ or $1 \in V(B)$; $0 \in V(A \vee B)$ iff $0 \in V(A)$ and $0 \in V(B)$.

Compared to **FDE**, the only difference lies in the falsity conditions for $\sim$.

Note, however, that the resulting logic is *not* the same since the designated values are **t** and **n**, not **t** and **b**. In other words, we are considering the consequence relation in terms of non-falsity preservation, rather than truth preservation.[6]

4.3 Option 3

Seen in the light of the natural deduction system, we may also think of regarding the binary connectives as *information* connectives, rather than *truth* connectives. This will correspond to interpret **1** and **0** as **b** and **n** of **FDE**, respectively. Then, there are again two options in interpreting the intermediate values. Let us begin with the case in which we interpret **i** and **j** as **t** and **f** of **FDE**, respectively. Then, as a result of rewriting the values, we obtain the following truth table.

A	$\sim A$
b	**t**
t	**n**
f	**b**
n	**f**

$A \wedge B$	**b**	**t**	**f**	**n**
b	**b**	**t**	**f**	**n**
t	**t**	**t**	**n**	**n**
f	**f**	**n**	**f**	**n**
n	**n**	**n**	**n**	**n**

$A \vee B$	**b**	**t**	**f**	**n**
b	**b**	**b**	**b**	**b**
t	**b**	**t**	**b**	**t**
f	**b**	**b**	**f**	**f**
n	**b**	**t**	**f**	**n**

If we again rewrite it slightly, then we obtain the following truth tables.

A	$\sim A$
t	**n**
b	**t**
n	**f**
f	**b**

$A \wedge B$	**t**	**b**	**n**	**f**
t	**t**	**t**	**n**	**n**
b	**t**	**b**	**n**	**f**
n	**n**	**n**	**n**	**n**
f	**n**	**f**	**n**	**f**

$A \vee B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**t**	**b**
b	**b**	**b**	**b**	**b**
n	**t**	**b**	**n**	**f**
f	**b**	**b**	**f**	**f**

Then, this makes it clear that the resulting truth tables are obtained by putting Kamide's unary operator together with information meet and join connectives. Moreover, we obtain the following truth and falsity conditions.

$1 \in V(\sim A)$ iff $0 \in V(A)$; $0 \in V(\sim A)$ iff $1 \notin V(A)$;
$1 \in V(A \wedge B)$ iff $1 \in V(A)$ and $1 \in V(B)$; $0 \in V(A \wedge B)$ iff $0 \in V(A)$ and $0 \in V(B)$;
$1 \in V(A \vee B)$ iff $1 \in V(A)$ or $1 \in V(B)$; $0 \in V(A \vee B)$ iff $0 \in V(A)$ or $0 \in V(B)$.

Now, compared to **FDE**, the truth conditions are exactly the same for all the connectives. However, the falsity conditions are different, and in particular, for $\wedge$ and $\vee$, those are taken as in the information meet and join, respectively.

Finally, since the designated values are **t** and **b**, the resulting logic is obtained by considering the truth preservation building on the above truth tables.

[5] To be precise, there are some differences in the language. Indeed, in [9, 10, 15], a classical conditional is added, while informational join and meet are added in [18]. The language in [16] is the same as here.

[6] The case of consequence relation defined in terms of truth preservation within the same language is explored in [16].

4.4 Option 4

Let us now turn to the other option. That is, we interpret **i** and **j** as **f** and **t** of **FDE**, respectively. Then, as a result of rewriting the values, we obtain the following truth table.

A	$\sim A$
b	**f**
f	**n**
t	**b**
n	**t**

$A \wedge B$	**b**	**f**	**t**	**n**
b	**b**	**f**	**t**	**n**
f	**f**	**t**	**n**	**n**
t	**t**	**n**	**t**	**n**
n	**n**	**n**	**n**	**n**

$A \vee B$	**b**	**f**	**t**	**n**
b	**b**	**b**	**b**	**b**
f	**b**	**f**	**b**	**f**
t	**b**	**b**	**t**	**t**
n	**b**	**f**	**t**	**n**

If we again rewrite it slightly, then we obtain the following truth tables.

A	$\sim A$
t	**b**
b	**f**
n	**t**
f	**n**

$A \wedge B$	**t**	**b**	**n**	**f**
t	**t**	**t**	**n**	**n**
b	**t**	**b**	**n**	**f**
n	**n**	**n**	**n**	**n**
f	**n**	**f**	**n**	**f**

$A \vee B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**t**	**b**
b	**b**	**b**	**b**	**b**
n	**t**	**b**	**n**	**f**
f	**b**	**b**	**f**	**f**

Then, this makes it clear that the resulting truth tables are obtained by putting Ruet's unary operator together with information meet and join connectives. Moreover, we obtain the following truth and falsity conditions.

$1 \in V(\sim A)$ iff $0 \notin V(A)$; $\quad 0 \in V(\sim A)$ iff $1 \in V(A)$;
$1 \in V(A \wedge B)$ iff $1 \in V(A)$ and $1 \in V(B)$; $\quad 0 \in V(A \wedge B)$ iff $0 \in V(A)$ and $0 \in V(B)$;
$1 \in V(A \vee B)$ iff $1 \in V(A)$ or $1 \in V(B)$; $\quad 0 \in V(A \vee B)$ iff $0 \in V(A)$ or $0 \in V(B)$.

Now, compared to **FDE**, the falsity condition is the same for $\sim$, and the truth conditions are exactly the same for $\wedge$ and $\vee$. However, the other conditions are different, and in particular, for $\wedge$ and $\vee$, the falsity condition is taken as in the information meet and join, respectively.

Moreover, since the designated values are **b** and **f**, the resulting logic is obtained by considering the falsity preservation building on the above truth tables.

4.5 A summary of the four options

The readings may be summarized in the following correspondence table:

Options	$O1$	$O2$	$O3$	$O4$
1	**t**	**t**	**b**	**b**
0	**f**	**f**	**n**	**n**
i	**b**	**n**	**t**	**f**
j	**n**	**b**	**f**	**t**

There are at least four different ways of interpreting the truth-values for the system. Notice that this is not just a matter of re-interpreting them with different names, but the fact that *different accounts of the truth values* may be exchanged, while, at the same time, the meaning of the connectives changes, without a difference being made at the level of the deductive rules for the system.

As a result of this process of re-interpreting truth values and connectives, we obtain a strengthened version of the Carnap problem for the system under consideration. The matter is that a quite radical form of underdetermination arises that is internal to a fixed formal apparatus selected for the semantics; the same truth-values are able to exchange their roles and that change can go quite unnoticed from the point of view of the deductive behavior. This situation seems to be even more complex, or, at least, to add a layer of complexity to typical situations where categoricity is lacking, given that what is at stake here is not that additional surplus truth values are being added without making a difference for the consequence relation, rather a sort of underdetermination of meaning that can cause serious problems; radical misunderstanding can arise without being noticed.

That is precisely where intelligibility seems to be threatened, and one needs some common background from which to access the many options. To discuss this, the idea that the common more or less classical background offered by the relational semantics may be of help (see also [11] for additional discussion). That is, at least, the suggestion by Susan Haack, as we take it in the Haackian strategy, and as we shall discuss next.

5 Haackian theme: another representation for classicists

Although there is such a deep underdetermination of meaning between the four reading options, and one may be asking what kind of truth values are being dealt with here and what the connectives actually mean, there is also a sense in which, once an option is fixed, a classical logician can make sense of what is being advanced and of the kinds of disagreement that are at stake when it comes to deal with the other remaining readings. The first step for a better understanding of these problems concerns recognizing that not everything is lost once there are many conflicting options. As Susan Haack claimed, the first step is to notice that the addition of new truth values is *not always* accompanied by the rejections of bivalence or two-valuedness:

> Not surprisingly, it has sometimes been supposed that the use of a many-valued logic would inevitably involve a claim to the effect that there are more than two truth-values [...] But in fact, I think it is clear that a many-valued logic needn't require the admission of one or more extra truth-values over and above 'true' and 'false', and indeed, that it needn't even require the rejection of bivalence. [6, p.213]

The claim here is that one should avoid new or 'sui generis' truth values that should be understood on their own terms, such as 'paradoxical', or 'meaningless', because the addition of such truth values is incompatible with the purported aims of logic (the investigation of which inferences are legitimate, in the sense of truth preservation) and they are, in the end, quite obscure if they are to have a proper meaning, not couched in terms of usual truth values. With those exceptions out of the way, one may sometimes provide for readings of the new truth values that are compatible with two-valuedness, preserving the intelligibility of the system (see a discussion in [13, 11]).

That strategy can certainly be applied in scenarios involving four truth values, as the one we have been discussing so far. That happens because truth values such as 'neither true nor false' are actually just the lack of classical truth-values, and values such as 'both true and false' just indicates that both classical values are attributed to a formula. So, in a sense, the truth values required here are not of a *sui generis* kind. Adding 'neither' and 'both' is certainly not very classical, but their readings are clear enough to meet Haack's standards. Discussing the case of **K3**, where the third truth value may be read as 'neither', Haack explains:

> Assignment of the third truth value to a wff [wellformed formula] indicates that it has no truth value, not that it has a non-standard, third truth value. [6, p.213]

The situation for the four-valued system we are considering here, then, gets clearer when we use the Dunn semantics presented for the four options discussed above. There is then a common ground of truth values that allows one to make sense of the different readings of the system and of the meanings of the connectives. Notice, nothing in Haack's strategy requires that one and the same system cannot have different readings in such more intelligible terms, it is only required that at least one such reading exists.

One can make the case for the difference in understanding of the meanings clearer by selecting, for instance, options 1 and 2 (a similar problem arises for any pair of options, of course). For fixing on a more specific problem, let us be concerned with negation according to these two options. This gives rise to problems that may look quite similar to cases available in the literature concerning the similarity of gaps and gluts (see [2]). Suppose that two adherents, one of option 1, and another one of option 2, agree that a given proposition A is true (i.e. $1 \in V(A)$). The supporter of reading 1 would claim that $\sim A$ is both 1 and 0, while the friend of reading 2 would understand it as being neither 1 nor 0. The disagreement persists with $\sim\sim A$; the problem, however, is that they would happily agree that $A \vee \sim\sim A$. So, nothing changes, from a logical point of view if we focus only on the deductive behavior, although there is an abyssal difference in understanding the meaning of the truth values and the meaning of negation. We can, however, clearly make sense of the differences once the terminology of Dunn semantics is employed. A sort of common background is offered for discussion, although there is no *purely logical* grounds for distinguishing those readings.

6 Reflections

6.1 Meaning of the connectives

One of the most important themes in the philosophy of logic concerns the problem of whether changing a system of logic would require a corresponding change of the meaning of the connectives, leading to failure in legitimate rivalry between such systems. This is the famous *meaning variance thesis*, commonly associated with Quine (see [19, p.81], see also [17, 7]), and it will be fruitful to present it here to elaborate on a contrast with the problem that we are highlighting in our paper. According to the meaning variance problem, for instance, if two systems disagree on the validity of the law of excluded middle, they may be understanding the meanings of the connectives involved in different terms:

> ...the best explanation of this meaning change is that one or more of the logical constants occurring in the sentence have changed their meaning. This thought can be spelled out in a number of roughly equivalent ways, but all of them involve the idea that, for example, meaning what the classical logician means by "not" and "or" *suffices* for acceptance of any instance of excluded middle whatsoever, at least potentially. So, if some particular instance of excluded middle isn't accepted, it must be because either "not" or "or" (or both) are being understood in a non-classical fashion. ([24, p.423])

This is a difficult problem, and it is not even clear whether Quine himself would have endorsed the typical conclusion leading to scepticism about substantial disputes between different systems. It could actually be the case that there is such a radical meaning variation, while still it being the case that dispute concerning the appropriateness of choice of one of the systems may happen in the open, with disagreement about the validity specific laws and inferences; the case is that such dispute may be conduced according to a dispute on different reasons for accepting or rejecting a system involving the disputed laws and inferences, i.e. one may have a reason to prefer one system over the other. As Quine himself famously put:

> [W]hoever denies the law of excluded middle changes the subject. This is not to say that he is wrong in so doing. In repudiating 'p or $\sim p$' he is indeed giving up classical negation, or perhaps alternation, or both; and he may have his reasons. [19, p.83]

So, even though there may be disagreement on meaning, there is a dispute that can be conducted according to some kind of exchange of reasons pro and con each system. Quine mentions the simplification of quantum mechanics as one possible reason to revise classical logic (although he himself, of course, did not recommend taking that route).

All of that is well known. However, that scenario provides for a nice platform from which to consider the problem we have been advancing here, which is way more radical. Given that we have at least four reading options for the same system, what results is that we actually have 'change of subject', but without having the option to advance reasons, because we are not changing the logic. That is, there is a sense in which a change in the underlying meaning of the connectives does not carry over to a change in logic, so that one cannot carry the dispute with reasons for or against certain laws, because all the parties involved accept precisely the same logic (in fact, precisely even the same truth tables). So, the situation here is that we have change of meaning, without change of logic. One could not, for instance, argue that one reading is better because it simplifies quantum mechanics, while the other does not, given that from the point of view of logical consequence, any reading will do exactly the same as the others would.

The result is similar to Quinean scenarios of indeterminacy of translation; we may never be sure whether our understanding of the logical vocabulary of $\mathbf{CNL}_4^2$ is the intended one by some other user. Someone whose knowledge of the meaning of the connectives were obtained exclusively from the familiarity with the derivation rules of the system would be able to learn completely incompatible lessons from the same teachings. Also, she would not be able to be sure that, whenever someone else uses the same logic, that someone is actually using the logic in the same meaning. This was illustrated before with the particular case of negation.

That leads us directly to a worry that is also related to the meaning variance problem, which is the problem of determining whether the connectives we are labelling as conjunction, disjunction and negation are actually such logical connectives. Typically, the Quinean conclusion that a change of meaning engenders that we are no longer dealing with 'the' logical connectives is quite well known. In a debate between a classical logician and a paraconsistent logician, remember, Quine famously wrote:

> My view of the dialogue is that neither party knows what he is talking about. They think that they are talking about negation, '$\sim$', 'not'; but surely the notion ceased to be recognisable as negation when they took to regarding some conjunctions of the form '$p\ .{\sim}p$' as true, and stopped regarding such sentences as implying all others. Here, evidently, is the deviant logician's predicament: when he tries to deny the doctrine he only changes the subject. (Quine [19, p.81])

So, if we set aside the part of the comment involving failure of some inference (explosion), what Quine is saying is that negation sign is not a negation if it allows some contradictions to be true. That certainly applies to our case, where some readings of negation do allow for such scenarios, while others allow that negation changes truth values in ways incompatible with the expected truth to falsity behavior. That problem can also be extended to conjunction and disjunction, for sure. Probably, one could remark that a conjunction and a disjunction not satisfying the classical truth and falsity conditions are not the proper logical connectives.

But once one adopts the thesis that meaning is defined in model theoretic terms, related to truth and falsity conditions, the possibility opens up for some kind of rescue of the two binary connectives in our four options. A classicist like Quine would require precisely the classical truth and falsity conditions for these connectives to be identified. However, once one leaves aside the assumption that the classical characterization is the correct one, still we can provide for a kind of minimal meaning conditions by restricting ourselves to the truth conditions for conjunction and disjunction. As one can easily check, the truth conditions for these connectives are held constant —indeed, they are the classical conditions— in all four options, leaving some flexibility for the falsity conditions (for more on the separation of such conditions while preserving the connective, see also the discussion in [12]). That is, if we can identify the connectives solely by their truth conditions, then, there is a sense in which these connectives are actually conjunction and disjunction.

This is even clearer from the Haackian perspective that we are adopting here. Given the two-valued relational semantics, we can not only explain the differences in reading of the truth values in each case, but also make explicit the common truth conditions, and the diverging falsity conditions. There is a sense in which a classical part of the meaning of the connectives is preserved, with such truth conditions. It is not as easy to say the same about negation. Let us discuss it explicitly.

6.2 Is $\sim$ a negation?

The debate gets even more complicated when it comes to deal with negation: there is a question if $\sim$ is negation or not. Remember, if we follow the strategy made use of in [15, 16], by applying the mechanical procedure described in [14], then we obtain the following truth and falsity conditions, assuming that we rewrite the four values $\mathbf{t}, \mathbf{b}, \mathbf{n}$ and $\mathbf{f}$ as $\{1\}, \{1,0\}, \emptyset$ and $\{0\}$, respectively.

Then, for options 1 and 4, we have the following conditions:

$1 \in V(\sim A)$ iff $0 \notin V(A)$; $\quad 0 \in V(\sim A)$ iff $1 \in V(A)$.

In view of these conditions, seen in the light of classicists' background assumption that holds for our Haackian strategy, the connective $\sim$ may be regarded as negation thanks to the falsity condition for negation. Remember, we are assuming that partial satisfaction of the classical truth conditions is enough for meaning attribution.

On the other hand, for options 2 and 3, we have the following conditions:

$1 \in V(\sim A)$ iff $0 \in V(A)$; $\quad 0 \in V(\sim A)$ iff $1 \notin V(A)$.

Now, that means the classical truth condition is satisfied. Again, that would ensure that $\sim$ can be regarded as negation by building on the classicists' understanding.[7]

Once the preferred option is settled, we are ready to interpret the results from Proposition 1. Still, we need to take into account of the differences in the definition of the consequence relation. For example, options 2 and 3 do agree on the truth and falsity conditions for negation, but they disagree on the definition of the consequence relation since option 2 has non-falsity preservation in mind, whereas option 3 has truth preservation mind. Therefore, the first two items from the proposition will imply negation incompleteness of $\mathbf{CNL}_4^2$ when option 2 is taken, whereas the same items imply negation inconsistency of $\mathbf{CNL}_4^2$ for those preferring option 3.

If one requires more to regard a unary connective as negation, for example requiring both truth and falsity condition to be the same with classical logic, or even requiring a different truth and/or falsity condition, then the unary connective $\sim$ will not be regarded as negation.[8] An alternative route to discuss the issue of negation is related to the possibility of defining different connectives that behave as classical negation inside the system.[9] In this case, negation is a connective inside the system, but that seems to conclude that the actual connective $\sim$ is not a negation. In this case, it seems that the question of how to interpret the connective $\sim$ seems to remain.

[7] This is the understanding of negation taken, for example, in [1, 16, 12].

[8] This seems to be the direction pursued in [3].

[9] For some discussions on classical negation in the context of **FDE**, see [4, 22].

7 Concluding remarks

In this paper, we have explored how themes from Carnap and Haack display in the system $\mathbf{CNL}_4^2$ advanced by Oleg Grigoriev and Dmitry Zaitsev. The themes are a direct consequence of some re-workings we provided for the system. We have not only provided for an alternative proof system, but also discussed how four options of readings for the semantics are available, connecting them with other systems available in the literature. The very idea that four readings are available for the original four truth values of $\mathbf{CNL}_4^2$ gives rise to the Carnap problem: distinct approaches to truth and falsity are available and are put on the top of the same formal semantics. That gives rise to incompatible readings that still fit the system, in a sense that, broadly put, makes the system compatible with quite incompatible readings of the truth values it intends to deal with. Our proposed strategy to make sense of this diversity is to use Haack's claim that the use of a two-valued setting is appropriate to confer intelligibility to the system. That was achieved through a relational semantics, which made clearer, from a classical perspective, how the options differ. We have also discussed, on these lights, the status of negation in the system, connecting the truth and falsity conditions to the classicists' demands, but also identifying candidates for a classical negation inside the system, as provided by functional completeness.

Appendix

Here are the details of the proof of Lemma 2. For the case of negation, it goes as follows.

- $v_\Sigma({\sim}B)=\mathbf{1}$ iff $v_\Sigma(B)=\mathbf{j}$ (by the definition of v_Σ) iff $\Sigma \nvdash B$ and $\Sigma \vdash {\sim}B$ (by IH) iff $\Sigma \vdash {\sim}B$ and $\Sigma \vdash {\sim}{\sim}B$ (by $({\sim}{\sim}2)$ for the left-to-right direction and $({\sim}{\sim}1)$ for the other direction).
- $v_\Sigma({\sim}B)=\mathbf{i}$ iff $v_\Sigma(B)=\mathbf{1}$ (by the definition of v_Σ) iff $\Sigma \vdash B$ and $\Sigma \vdash {\sim}B$ (by IH) iff $\Sigma \vdash {\sim}B$ and $\Sigma \nvdash {\sim}{\sim}B$ (by $({\sim}{\sim}1)$ for the left-to-right direction and $({\sim}{\sim}2)$ for the other direction).
- $v_\Sigma({\sim}B)=\mathbf{j}$ iff $v_\Sigma(B)=\mathbf{0}$ (by the definition of v_Σ) iff $\Sigma \nvdash B$ and $\Sigma \nvdash {\sim}B$ (by IH) iff $\Sigma \nvdash {\sim}B$ and $\Sigma \vdash {\sim}{\sim}B$ (by $({\sim}{\sim}2)$ for the left-to-right direction and $({\sim}{\sim}1)$ for the other direction).
- $v_\Sigma({\sim}B)=\mathbf{0}$ iff $v_\Sigma(B)=\mathbf{i}$ (by the definition of v_Σ) iff $\Sigma \vdash B$ and $\Sigma \nvdash {\sim}B$ (by IH) iff $\Sigma \nvdash {\sim}B$ and $\Sigma \nvdash {\sim}{\sim}B$ (by $({\sim}{\sim}1)$ for the left-to-right direction and $({\sim}{\sim}2)$ for the other direction).

For the case of conjunction, it goes as follows.

- $v_\Sigma(B\wedge C)=\mathbf{1}$ iff $v_\Sigma(B)=\mathbf{1}$ and $v_\Sigma(C)=\mathbf{1}$ (by the definition of v_Σ) iff $\Sigma \vdash B$ and $\Sigma \vdash {\sim}B$ and $\Sigma \vdash C$ and $\Sigma \vdash {\sim}C$ (by IH) iff $\Sigma \vdash B\wedge C$ and $\Sigma \vdash {\sim}(B\wedge C)$.
- $v_\Sigma(B\wedge C)=\mathbf{i}$ iff ($v_\Sigma(B)=\mathbf{1}$ and $v_\Sigma(C)=\mathbf{i}$) or ($v_\Sigma(B)=\mathbf{i}$ and $v_\Sigma(C)=\mathbf{i}$) or ($v_\Sigma(B)=\mathbf{i}$ and $v_\Sigma(C)=\mathbf{1}$) (by the definition of v_Σ) iff ($\Sigma \vdash B$ and $\Sigma \vdash {\sim}B$ and $\Sigma \vdash C$ and $\Sigma \nvdash {\sim}C$) or ($\Sigma \vdash B$ and $\Sigma \nvdash {\sim}B$ and $\Sigma \vdash C$ and $\Sigma \nvdash {\sim}C$) or ($\Sigma \vdash B$ and $\Sigma \nvdash {\sim}B$ and $\Sigma \vdash C$ and $\Sigma \vdash {\sim}C$) (by IH) iff $\Sigma \vdash B\wedge C$ and $\Sigma \nvdash {\sim}(B\wedge C)$.
- $v_\Sigma(B\wedge C)=\mathbf{j}$ iff ($v_\Sigma(B)=\mathbf{1}$ and $v_\Sigma(C)=\mathbf{j}$) or ($v_\Sigma(B)=\mathbf{j}$ and $v_\Sigma(C)=\mathbf{j}$) or ($v_\Sigma(B)=\mathbf{j}$ and $v_\Sigma(C)=\mathbf{1}$) (by the definition of v_Σ) iff ($\Sigma \vdash B$ and $\Sigma \vdash {\sim}B$ and $\Sigma \nvdash C$ and $\Sigma \vdash {\sim}C$) or ($\Sigma \nvdash B$ and $\Sigma \vdash {\sim}B$ and $\Sigma \nvdash C$ and $\Sigma \vdash {\sim}C$) or ($\Sigma \nvdash B$ and $\Sigma \vdash {\sim}B$ and $\Sigma \vdash C$ and $\Sigma \vdash {\sim}C$) (by IH) iff $\Sigma \nvdash B\wedge C$ and $\Sigma \vdash {\sim}(B\wedge C)$.
- $v_\Sigma(B\wedge C)=\mathbf{0}$ iff $v_\Sigma(B)=\mathbf{0}$ or $v_\Sigma(C)=\mathbf{0}$ or ($v_\Sigma(B)=\mathbf{i}$ and $v_\Sigma(C)=\mathbf{j}$) or ($v_\Sigma(B)=\mathbf{j}$ and $v_\Sigma(C)=\mathbf{i}$) (by the definition of v_Σ) iff ($\Sigma \nvdash B$ and $\Sigma \nvdash {\sim}B$) or ($\Sigma \nvdash C$ and $\Sigma \nvdash {\sim}C$) or ($\Sigma \vdash B$ and $\Sigma \nvdash {\sim}B$ and $\Sigma \nvdash C$ and $\Sigma \vdash {\sim}C$) or ($\Sigma \nvdash B$ and $\Sigma \vdash {\sim}B$ and $\Sigma \vdash C$ and $\Sigma \nvdash {\sim}C$) (by IH) iff $\Sigma \nvdash B\wedge C$ and $\Sigma \nvdash {\sim}(B\wedge C)$.

The case for disjunction is similar to the case for conjunction.

References

[1] Arnon Avron (2005): *A non-deterministic view on non-classical negations*. *Studia Logica* 80(2-3), pp. 159–194, doi:10.1007/s11225-005-8468-5.

[2] JC Beall & David Ripley (2004): *Analetheism and Dialetheism*. *Analysis* 64(1), pp. 30–35, doi:10.1093/analys/64.1.30.

[3] Alex Belikov, Oleg Grigoriev & Dmitry Zaitsev (2022): *On Connegation*. In Katalin Bimbó, editor: *Relevance Logics and other Tools for Reasoning. Essays in Honour of J. Michael Dunn*, College Publications, pp. 73–88.

[4] Michael De & Hitoshi Omori (2015): *Classical negation and expansions of Belnap-Dunn logic*. *Studia Logica* 103, pp. 825–851, doi:10.1007/s11225-014-9595-7.

[5] Oleg Grigoriev & Dmitry Zaitsev (2022): *Basic Four-Valued Systems of Cyclic Negations*. *Bulletin of the Section of Logic* 51(4), pp. 507–533, doi:10.18778/0138-0680.2022.21.

[6] Susan Haack (1978): *Philosophy of Logics*. Cambridge University Press, doi:10.1017/CBO9780511812866.

[7] Ole Thomassen Hjortland (2022): *Disagreement About Logic*. *Inquiry: An Interdisciplinary Journal of Philosophy* 65(6), pp. 660–682, doi:10.1080/0020174x.2019.1651084.

[8] Lloyd Humberstone (1995): *Negation by iteration*. *Theoria* 61(1), pp. 1–24, doi:10.1111/j.1755-2567.1995.tb00489.x.

[9] Norihiro Kamide (2016): *Paraconsistent double negation that can simulate classical negation*. In: *Proceedings of the 46th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2016)*, pp. 131–136, doi:10.1109/ISMVL.2016.12.

[10] Norihiro Kamide (2017): *Paraconsistent Double Negations as Classical and Intuitionistic Negations*. *Studia Logica* 105(6), pp. 1167–1191, doi:10.1007/s11225-017-9731-2.

[11] Hitoshi Omori & Jonas R. B. Arenhart (2023): *Change of Logic, Without Change of Meaning*. *Theoria* 89(4), pp. 414–431, doi:10.1111/theo.12459.

[12] Hitoshi Omori & Jonas R. B. Arenhart (2024): *Why* **FDE** *Might Be Too Strong for Beall*. *Asian Journal of Philosophy* 3(1), pp. 1–16, doi:10.1007/s44204-023-00133-8.

[13] Hitoshi Omori & Jonas Rafael Becker Arenhart (2022): *Haack meets Herzberger and Priest*. In: *2022 IEEE 52nd International Symposium on Multiple-Valued Logic (ISMVL)*, pp. 137–144, doi:10.1109/ISMVL52857.2022.00028.

[14] Hitoshi Omori & Katsuhiko Sano (2015): *Generalizing Functional Completeness in Belnap-Dunn Logic*. *Studia Logica* 103(5), pp. 883–917, doi:10.1007/s11225-014-9597-5.

[15] Hitoshi Omori & Heinrich Wansing (2018): *On contra-classical variants of Nelson logic N4 and its classical extension*. *The Review of Symbolic Logic* 11(4), pp. 805–820, doi:10.1017/s1755020318000308.

[16] Hitoshi Omori & Heinrich Wansing (2022): *Varieties of negation and contra-classicality in view of Dunn semantics*. In Katalin Bimbó, editor: *Relevance Logics and other Tools for Reasoning. Essays in Honour of J. Michael Dunn*, College Publications, pp. 309–337, doi:10.13154/294-10519.

[17] Francesco Paoli (2003): *Quine and Slater on Paraconsistency and Deviance*. *Journal of Philosophical Logic* 32(5), pp. 531–548, doi:10.1023/a:1025648722188.

[18] Francesco Paoli (2019): *Bilattice Logics and Demi-Negation*. In Hitoshi Omori & Heinrich Wansing, editors: *New Essays on Belnap-Dunn Logic*, Synthese Library, Springer, pp. 233–253, doi:10.1007/978-3-030-31136-0_14.

[19] W. V. Quine (1986): *Philosophy of Logic (2nd Edition)*. Harvard University Press, Cambridge, MA, doi:10.4159/9780674042445.

[20] Paul Ruet (1996): *Complete set of connectives and complete sequent calculus for Belnap's logic*. Technical Report, Ecole Normale Superieure. *Logic Colloquium* 96, Document LIENS-96-28.

[21] Jerzy Słupecki (1972): *A criterion of fullness of many-valued systems of propositional logic*. *Studia Logica* 30, pp. 153–157, doi:10.1007/BF02120845.

[22] Damian Szmuc & Hitoshi Omori (2022): *Liberating classical negation from falsity conditions*. In: *2022 IEEE 52th International Symposium on Multiple-Valued Logic*, 52, IEEE Computer Society, pp. 131–136, doi:10.1109/ISMVL52857.2022.00027.

[23] Marek Tokarz (1973): *Connections between some notions of completeness of structural propositional calculi*. *Studia Logica* 32(1), pp. 77–89, doi:10.1007/BF02123818.

[24] Jared Warren (2018): *Change of Logic, Change of Meaning*. *Philosophy and Phenomenological Research* 96(2), pp. 421–442, doi:10.1111/phpr.12312.

The Power of Generalized Clemens Semantics*

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In this paper, we elaborate on the ordered-pair semantics originally presented by Matthew Clemens for **LP** (Priest's *Logic of Paradox*). For this purpose, we build on a generalization of Clemens semantics to the case of n-tuple semantics, for every n. More concretely, i) we deal with the case of a language with quantifiers, and ii) we consider philosophical implications of the semantics. The latter includes, first, a reading of the semantics in epistemic terms, involving multiple agents. Furthermore, we discuss the proper understanding of many-valued logics, namely **LP** and **K3** (Kleene strong 3-valued logic), from the perspective of classical logic, along the lines suggested by Susan Haack. We will also discuss some applications of the semantics to issues related to informative contradictions, i.e. contradictions involving quantification over different respects a vague predicate may have, as advanced by Paul Égré, and also to the mixed consequence relations, promoted by Pablo Cobreros, Paul Égré, David Ripley and Robert van Rooij.

1 Introduction

It goes without saying that the addition of truth values to the traditional pair consisting of 'truth' and 'falsity' brings several interesting technical consequences to the center of the stage. Understanding what such additional truth values mean, and how they affect the resulting logical system, however, constitutes a deep philosophical challenge. The intelligibility of such systems and, consequently, of their applications, both philosophical and in general, hangs on the prior understanding of such notions.

The idea that there are difficulties related to the appropriate understanding of logical concepts is not new, although it has not always received the appropriate attention in the context of philosophical use of many-valued systems. In the literature about the subject, the topic has been forcefully discussed by Susan Haack in [8, chap.11]. Haack advanced one specific proposal to achieve a clear picture of such systems. The first point of the proposal consists in preserving two-valuedness. As she puts it:

> I think it is clear that a many-valued logic needn't require the admission of one or more extra truth-values over and above 'true' and 'false', and indeed, that it needn't even require the rejection of bivalence. [8, p.213]

The second step in the proposal advanced by Haack consists in offering an explanation of how, in a many-valued scenario, one will be able to retain two-valuedness (and, sometimes, even bivalence) and actually dispense with additional *sui generis* truth values. The plan is actually quite simple: whatever seems to be prima facie an additional truth value should actually be explained away; it should be read in terms of the classical truth values and some additional epistemic or semantic ingredient that accounts for

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the distinct option in such settings. Discussing three-valued logics, for instance, Haack suggests that in some cases the third truth value may be understood as involving *epistemic and/or semantic* restrictions to classical truth values, such as 'true *and known to be true by an agent O*', or 'true *and analytic*'; in other cases, its meaning needs not *go beyond* the two truth values, such as when one is attributing to a proposition the value 'neither true nor false', which is not an extra truth value. By building our understanding of many-valued systems solely on the already available two truth values, we use these previously intelligible notions of truth and falsity to endow such many-valued systems with the much needed clarification in terms of notions understood beforehand.

In this paper, we adopt that general Haackian strategy as a means to increase intelligibility of a family of many-valued logics.[1] Our aim is to provide a clear meaning for some such logics by endowing our target systems with a common underlying ordered-pair semantics and generalizations of it. We shall discuss how close to retaining bivalence and two-valuedness such a strategy is. However, our overall claim is that such a semantics does contribute to advance the Haackian desideratum of granting understanding for many-valued logics. In particular, we discuss not only cases of Tarskian consequence relations, but also the notoriously obscure cases of mixed consequence relations.

The rest of the paper is structured as follows. After briefly recalling the original semantics proposed by Clemens and a generalization within the propositional language in §2, we generalize the semantics for the language with quantifiers in §3. We will also discuss the theme from Haack for the semantics proposed by Clemens. These will be followed by §4 in which we discuss three applications of the semantics. These include the issues related to the mixed consequence relations. Finally, the paper will be concluded with some brief remarks in §5.

2 Revisiting ordered-pair semantics

In this section, we first briefly recall the original Clemens' semantics, as presented in [1]. We then go on to present a generalization as advanced in [10].

The language $\mathscr{L}_0$ consists of a set $\{\neg, \wedge, \vee\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathscr{L}_0$. We denote a formula of $\mathscr{L}_0$ by A, B, C, etc. and a set of formulas of $\mathscr{L}_0$ by Γ, Δ, Σ, etc.

We first revisit the ordered pair semantics as it was set out by Clemens.

Definition 1. *A four-valued interpretation of $\mathscr{L}_0$ is a function v from* Prop *to* $\{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$. *Given a four-valued interpretation v, this is extended to a function I* : Form$\to\{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$ *as follows:*

	$\neg$
$\langle 1,1\rangle$	$\langle 0,0\rangle$
$\langle 1,0\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$
$\langle 0,0\rangle$	$\langle 1,1\rangle$

$\wedge$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$

$\vee$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$
$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 0,1\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$

Remark 2. *Note that truth tables for conjunction and disjunction result from adapting* min *and* max *definitions, respectively, with the following order on the values:* $\langle 0,0\rangle < \langle 0,1\rangle < \langle 1,0\rangle < \langle 1,1\rangle$.

[1] For further discussion of the Haackian strategy in a different context, see [11, 12].

Definition 3. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_3 A$ *iff for all four-valued interpretations* v*,* $I(A) \in \mathscr{D}$ *if* $I(B) \in \mathscr{D}$ *for all* $B \in \Gamma$*, where* $\mathscr{D} = \{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$.[2]

We now recall the standard three-valued semantics for **LP** (see [14, §7.4]).

Definition 4. *A three-valued interpretation of* $\mathscr{L}_0$ *is a function* $v_3 : \mathsf{Prop} \to \{\mathbf{t},\mathbf{i},\mathbf{f}\}$*. Given a three-valued interpretation* v_3*, this is extended to a function* I_3 *from* Form *to* $\{\mathbf{t},\mathbf{i},\mathbf{f}\}$ *by truth functions depicted in the form of truth tables as follows:*

	$\neg$
t	**f**
i	**i**
f	**t**

$\wedge$	**t**	**i**	**f**
t	**t**	**i**	**f**
i	**i**	**i**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**i**	**f**
t	**t**	**t**	**t**
i	**t**	**i**	**i**
f	**t**	**i**	**f**

Definition 5. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{\mathbf{LP}} A$ *iff for all three-valued interpretations* v_3*,* $I_3(A) \in \mathscr{D}$ *if* $I_3(B) \in \mathscr{D}$ *for all* $B \in \Gamma$*, where* $\mathscr{D} = \{\mathbf{t},\mathbf{i}\}$.

Based on these, Clemens established the following result in [1].

Fact 6 (Clemens)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_3 A$ *iff* $\Gamma \models_{\mathbf{LP}} A$.

Let us now present a generalization of the above result, presented in [10]. For this purpose, we refer to the two-element Boolean algebra as **2**.

Definition 7. *For* $n \geq 2$*, we define* $\mathbf{2}^n$ *as the n-ary Cartesian product of* **2** *with the lexicographical order. Given* $\langle x_1,\dots,x_n\rangle \in \mathbf{2}^n$*, we define a unary operation* $- : \mathbf{2}^n \to \mathbf{2}^n$ *as follows:* $-\langle x_1,\dots,x_n\rangle := \langle 1-x_1,\dots,1-x_n\rangle$.

Definition 8. *An n-interpretation of* $\mathscr{L}_0$ *is a function* $v : \mathsf{Prop} \to \mathbf{2}^n$*. Given an n-interpretation* v*, this is extended to a function* $I : \mathsf{Form} \to \mathbf{2}^n$ *as follows:* $I(p) = v(p)$*;* $I(\neg A) = -I(A)$*;* $I(A \wedge B) = \min(I(A), I(B))$*;* $I(A \vee B) = \max(I(A), I(B))$.

Definition 9. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{n,t} A$ *(tolerant consequence based on n-interpretations) iff for all n-interpretations* v*,* $I(A) \in \mathscr{D}$ *if* $I(B) \in \mathscr{D}$ *for all* $B \in \Gamma$*, where* $\mathscr{D} = \mathbf{2}^n \setminus \{\langle 0,0,\dots,0\rangle\}$.

We are now ready to recall a generalization of Clemens' observation.

Theorem 1. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{n,t} A$ *iff* $\Gamma \models_{\mathbf{LP}} A$.

Although the motivation of Clemens in [1] was focused exclusively on **LP**, with a special emphasis on negation, we obtain an interesting insight into the relation between **LP** and other related systems, namely **CL** (classical logic) and **K3**. For the purpose of clarifying our point, however, we shall state the results in the language with quantifiers, and this is the goal of the next section.[3]

3 More on ordered-pair semantics

3.1 Basic observations

The language $\mathscr{L}_1$ consists of the following vocabulary: a set $\{\neg, \wedge, \vee\}$ of propositional connectives, the universal and particular quantifiers $\forall$ and $\exists$, a countable set $\{x_0, x_1, \dots\}$ of variables, a countable set $\{c_0, c_1, \dots\}$ of constant symbols, and a countable set $\{P_0, P_1, \dots\}$ of predicate symbols, where we

[2] We are using the subscript 3 just to indicate that there are three designated values. See §3.2 for details concerning the intended reading of such truth values.

[3] For a discussion of the propositional cases, see [10].

associate each predicate P_k with a fixed finite arity. We regard 0-ary predicate symbols as propositional letters. We define the set of formulas in $\mathscr{L}_1$ as follows:

$$A ::= P(t_1,\ldots,t_n)\,|\,\neg A\,|\,A\wedge B\,|\,A\vee B\,|\,\forall xA\,|\,\exists xA,$$

where t_i is a *term*, namely a variable or a constant symbol. We say that a formula is *propositional* if it is constructed from propositional letters (i.e., 0-ary predicate symbols) by using the propositional connectives. We define the notions of *free* and *bound* variable, and *sentence* as usual. We write $A_x(t)$ to mean the result of substituting all the occurrences of free variable x in A by the term t, renaming the bound variables, if necessary, to avoid variable-clashes. We denote sets of formulas by Γ, Σ, etc.

Let us first recall the three-valued semantics for **K3** and **LP**.

Definition 10. *A* three-valued interpretation $\mathscr{I}$ *for* $\mathscr{L}_1$ *is a pair* $\langle D,v\rangle$ *where* D *is a non-empty set and we assign* $v(c)\in D$ *to each constant* c*, assign an* i*-place function* $v(P): D^i \longrightarrow \{0,1/2,1\}$ *to each* i*-ary predicate symbol* P*. Given any interpretation* $\langle D,v\rangle$*, we can define Clemens-valuation* $\overline{v}$ *for all the sentences of* $\mathscr{L}_1$ *expanded by* $\{k_d : d\in D\}$ *inductively as follows: as for the atomic* sentences*,*

- $\overline{v}(P(t_1,...,t_n)) = v(P)(v(t_1),\ldots,v(t_n))$.

The rest of the clauses are as follows:

- $\overline{v}(\neg A) = 1-\overline{v}(A)$
- $\overline{v}(A\wedge B) = \min(\overline{v}(A),\overline{v}(B))$
- $\overline{v}(A\vee B) = \max(\overline{v}(A),\overline{v}(B))$
- $\overline{v}(\forall xA) = \min(\{\overline{v}(A_x(k_d)) : d\in D\})$
- $\overline{v}(\exists xA) = \max(\{\overline{v}(A_x(k_d)) : d\in D\})$

Definition 11. *For all sets of sentences* $\Gamma\cup\{A\}$*,* $\Gamma\models_i A$ *iff for all three-valued interpretations* $\mathscr{I}$*,* $\overline{v}(A)\in\mathscr{D}_i$ *if* $\overline{v}(B)\in\mathscr{D}_i$ *for all* $B\in\Gamma$*, where* $i\in\{k,l\}$ *and*

- $\mathscr{D}_k = \{1\}$ *(k for* **K3***),*
- $\mathscr{D}_l = \{1,1/2\}$ *(l for* **LP***).*

Moreover, building on the notation above, we introduce an instance of the p-consequence relation (cf. [7]), as follows.

Definition 12. *For all sets of sentences* $\Gamma\cup\{A\}$*,* $\Gamma\models_{st} A$ *iff for all three-valued interpretations* $\mathscr{I}$*,* $\overline{v}(A)\in\mathscr{D}_l$ *if* $\overline{v}(B)\in\mathscr{D}_k$ *for all* $B\in\Gamma$*.*

Finally, we refer to the semantic consequence relation based on the standard two-valued interpretations for classical logic as $\models_2$.

We now turn to introduce the semantics inspired by Clemens.

Definition 13. *A* Clemens interpretation $\mathscr{I}$ *for* $\mathscr{L}_1$ *is a pair* $\langle D,v\rangle$ *where* D *is a non-empty set and we assign* $v(c)\in D$ *to each constant* c*, assign an* i*-place function* $v(P): D^i \longrightarrow \mathbf{2}^n$ *to each* i*-ary predicate symbol* P*. Given any interpretation* $\langle D,v\rangle$*, we can define Clemens-valuation* $\overline{v}$ *for all the sentences of* $\mathscr{L}_1$ *expanded by* $\{k_d : d\in D\}$ *inductively as follows: as for the atomic* sentences*,*

$$\overline{v}(P(t_1,...,t_n)) = v(P)(v(t_1),\ldots,v(t_n)).$$

The rest of the clauses are as follows (recall Definition 7):

- $\overline{v}(\neg A) = -\overline{v}(A)$
- $\overline{v}(A \wedge B) = \min(\overline{v}(A), \overline{v}(B))$
- $\overline{v}(A \vee B) = \max(\overline{v}(A), \overline{v}(B))$
- $\overline{v}(\forall xA) = \min(\{\overline{v}(A_x(k_d)) : d \in D\})$
- $\overline{v}(\exists xA) = \max(\{\overline{v}(A_x(k_d)) : d \in D\})$

Definition 14. *For all set of sentences $\Gamma \cup \{A\}$, $\Gamma \models_{n,i} A$ iff for all Clemens interpretations $\mathscr{I}$, $\overline{v}(A) \in \mathscr{D}_i$ if $\overline{v}(B) \in \mathscr{D}_i$ for all $B \in \Gamma$, where $i \in \{s,b,t\}$ and*

- $\mathscr{D}_s = \{\langle 1,1,\dots,1\rangle\}$ *(s for strict),*
- $\mathscr{D}_b = \{\langle 1,x_2,\dots,x_n\rangle : x_2,\dots,x_n \in \mathbf{2}\}$ *(b for bossy), and*
- $\mathscr{D}_t = \mathbf{2}^n \setminus \{\langle 0,0,\dots,0\rangle\}$ *(t for tolerant).*

Moreover, building on the notation above, we introduce another instance of the p-consequence relation as follows.

Definition 15. *For all set of sentences $\Gamma \cup \{A\}$, $\Gamma \models_{n,s,t} A$ iff for all Clemens interpretations $\mathscr{I}$, $\overline{v}(A) \in \mathscr{D}_t$ if $\overline{v}(B) \in \mathscr{D}_s$ for all $B \in \Gamma$.*

In what follows, we will establish the equivalence of consequence relations based on the two semantics.

Lemma 1. *Given a Clemens-interpretation $\langle D,v\rangle$, define the three-valued interpretation $\langle D',v'\rangle$ as follows:*

- $D' := D$
- *For each constant c, $v'(c)$:=$v(c)$ and for each i-ary predicate symbol P,*
  - $v'(P)(d_1,\dots d_i)$=1 *if* $v(P)(d_1,\dots d_i) = \langle 1,1,\dots,1\rangle$
  - $v'(P)(d_1,\dots d_i)$=1/2 *if* $v(P)(d_1,\dots d_i) \in \mathbf{2}^n \setminus \{\langle 1,1,\dots,1\rangle, \langle 0,0,\dots,0\rangle\}$
  - $v'(P)(d_1,\dots d_i)$=0 *if* $v(P)(d_1,\dots d_i) = \langle 0,0,\dots,0\rangle$

Then, for all sentences A, (a) $\overline{v'}(A)$=1 *iff* $\overline{v}(A)$=$\langle 1,1,\dots,1\rangle$*;* (b) $\overline{v'}(A)$=0 *iff* $\overline{v}(A)$=$\langle 0,0,\dots,0\rangle$.

Proof. By induction on the complexity of A. □

Lemma 2. *Given a three-valued interpretation $\langle D,v\rangle$, define the Clemens interpretation $\langle D',v'\rangle$ as follows:*

- $D' := D$
- *For each constant c, $v'(c)$:=$v(c)$ and for each i-ary predicate symbol P,*
  - $v'(P)(d_1,\dots d_i)$=$\langle 1,1,\dots,1,1\rangle$ *if* $v(P)(d_1,\dots d_i)$=1
  - $v'(P)(d_1,\dots d_i)$=$\langle 1,1,\dots,1,0\rangle$ *if* $v(P)(d_1,\dots d_i)$=1/2
  - $v'(P)(d_1,\dots d_i)$=$\langle 0,0,\dots,0,0\rangle$ *if* $v(P)(d_1,\dots d_i)$=0

Then, for all sentences A, (a) $\overline{v'}(A)$=$\langle 1,1,\dots,1\rangle$ *iff* $\overline{v}(A)$=1*;* (b) $\overline{v'}(A)$=$\langle 0,0,\dots,0\rangle$ *iff* $\overline{v}(A)$=0.

Proof. By induction on the complexity of A. □

Theorem 2. *For all set of sentences* $\Gamma \cup \{A\}$, (i) $\Gamma \models_{n,s} A$ *iff* $\Gamma \models_k A$, (ii) $\Gamma \models_{n,b} A$ *iff* $\Gamma \models_2 A$, *and* (iii) $\Gamma \models_{n,t} A$ *iff* $\Gamma \models_l A$.[4]

Proof. Ad. (i): For the right-to-left direction, suppose $\Gamma \not\models_{n,s} A$. Then, there is a Clemens-interpretation $\mathscr{I}$ such that $\overline{v}(B)=\langle 1,1,\dots,1\rangle$ for all $B\in\Gamma$ and $\overline{v}(A)\neq\langle 1,1,\dots,1\rangle$. By making use of (a) of Lemma 1, there is a three-valued interpretation $\mathscr{I}' = \langle D', v'\rangle$ such that we obtain that $\overline{v'}(B)=1$ for all $B\in\Gamma$ and $\overline{v'}(A)\neq 1$, that is $\Gamma \not\models_k A$. For the other way around, suppose $\Gamma \not\models_k A$. Then, there is a three-valued interpretation $\mathscr{I}$ such that $\overline{v}(B)=1$ for all $B\in\Gamma$ and $\overline{v}(A)\neq 1$. By making use of (a) of Lemma 2, there is a Clemens-valued interpretation $\mathscr{I}' = \langle D', v'\rangle$ such that we obtain that $\overline{v'}(B)=\langle 1,1,\dots,1,1\rangle$ for all $B\in\Gamma$ and $\overline{v'}(A)\neq\langle 1,1,\dots,1,1\rangle$, that is $\Gamma \not\models_{n,s} A$.

Ad (ii): The proof runs in the above manner, but we make use of lemmas that are obtained by making some obvious modifications to Lemmas 1 and 2.

Ad (iii): The proof again runs in the above manner, but we make use of (b), instead of (a), of Lemmas 1 and 2. This completes the proof. □

Theorem 3. *For all set of sentences* $\Gamma \cup \{A\}$, $\Gamma \models_{n,s,t} A$ *iff* $\Gamma \models_{st} A$.

Proof. Suppose $\Gamma \not\models_{n,s,t} A$. Then, there is a Clemens-interpretation $\mathscr{I}$ such that $\overline{v}(B)=\langle 1,1,\dots,1\rangle$ for all $B\in\Gamma$ and $\overline{v}(A)=\langle 0,0,\dots,0\rangle$. By making use of Lemma 1, there is a three-valued interpretation $\mathscr{I}' = \langle D', v'\rangle$ such that we obtain that $\overline{v'}(B)=1$ for all $B\in\Gamma$ and $\overline{v'}(A)=0$, that is $\Gamma \not\models_{st} A$. For the other way around, suppose $\Gamma \not\models_{st} A$. Then, there is a three-valued interpretation $\mathscr{I}$ such that $\overline{v}(B)=1$ for all $B\in\Gamma$ and $\overline{v}(A)=0$. By making use of Lemma 2, there is a Clemens-valued interpretation $\mathscr{I}' = \langle D', v'\rangle$ such that we obtain that $\overline{v'}(B)=\langle 1,1,\dots,1,1\rangle$ for all $B\in\Gamma$ and $\overline{v'}(A)=\langle 0,0,\dots,0,0\rangle$, that is $\Gamma \not\models_{n,s,t} A$. □

3.2 Clemens in view of Haack

Now that the basics of the generalized and first-order Clemens semantics is presented, we may return to the problem of providing for understanding of many-valued logics, as raised by Susan Haack, in view of the Clemens semantics. More explicitly, we need to address how the Clemens semantics contributes to fulfil the explicit demand for intelligibility advanced by Haack. As we have commented in the introduction, Haack's strategy for the understanding of some prima facie candidate for a *sui generis* truth value is as follows: in order to endow a system of many-valued logic with intelligibility, we should attempt to 'read' such truth values in terms of the already known and understood classical truth values, possibly with additional semantic or epistemic contours. If that can be done, the need for additional truth values is actually avoided, and we have explained them away, in a sense.

Given that demand, the next natural question is: can one such 'classical reading' of the truth values be attributed to the generalization of the semantics advanced by Clemens? It is our contention now that this is perfectly possible, and more, that the framework presented is quite classical, in a sense. Let us focus on the simple ordered-pair semantics as originally presented by Clemens, where the set of truth values is $\{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$ (it is a simple matter to extend the readings to more general cases). Clearly, given the order established for the truth values, and the division between designated and non-designated truth values in order to define the classical consequence relation, it is not difficult to see *the first component* of the pairs as playing a more prominent role than all the others. That is, there is a natural reading of the truth values where the first component marks a division between truth and untruth, regardless of what the second component adds to it.

[4]Note that one can also obtain similar results by building on the framework due to Hans Herzberger, presented in [9]. For some discussions related to the results, cf. [12].

One may consider this reading favoring the first component as being elaborated according to a kind of realist approach to truth and falsity that classical logic is said to promote anyway. There is a sense in which propositions in a classical setting are defined as to their truth or falsity independently of whether any one agent knows the relevant facts about such a distribution of truth values (a discussion is to be found in [5]). In this kind of reading, the classical meaning of the connectives may be properly understood as available through the first component of the truth values, while the other components add an epistemic dimensions, that is, they add, and that is one possibility, appreciations of different agents that may disagree on the truth value of some proposition. A first shot on understanding what is going on, suggested by Clemens himself ([1, p.202]), advanced the readings as follows:

$\langle 1,1\rangle$ = true, and true only; $\langle 0,1\rangle$ = false, but also true;
$\langle 1,0\rangle$ = true, but also false; $\langle 0,0\rangle$ = false, and false only.

That makes for an interesting first attempt in the direction of a better understanding of the truth values involved in terms of the already available classical truth values, conferring also classical intelligibility to **K3** and **LP**. It is clear that reading the truth values like that requires that some sentences receive two of such classical truth values some times, but that is not a problem; as Haack comments on what concerns the case of truth-value gaps in [8, p.213]:

> Assignment of the third truth value to a wff [well-formed formula] indicates that it has *no* truth value, not that it has a non-standard, third truth value.

So, in the case of gaps, intelligibility is preserved. By parity of reasoning, of course, attributing two classical truth values to a formula is also not the attribution of a non-standard truth value; it is merely attribution of two of the available truth values to a formula (for further discussions, see [10]).

As a result, in the sense required by Haackian demands of intelligibility, the semantics presented by Clemens does seem to stay very close to a classical semantics, allowing for readings in terms of truth and falsity that stay very close to classical logic. The division between two groups of truth and false sentences then contribute to the idea that no additional *sui generis* truth values has been added.

We have also seen that one may generalize the semantics from ordered pairs to n-tuples of classical truth values. In a sense, that causes no additional complication on what concerns the understanding of such truth values. The order attributed to the n-tuples does the work in getting the appropriate division between truths and falsehoods when it comes to obtain classical logic: truth is whatever has truth as its first component; false is whatever has falsity as its first component.

In the cases of non-classical systems — **K3** and **LP**— the motivations for the choice of designated truth values may come from different fronts, regarding the role of the order of the truth values and additional suppositions that a more nuanced choice may be motivated. One may see **K3** as involving choice of certified or strict truth as designated, while **LP** may be seen as involving a tolerant approach where only certified falsity is excluded.

The role of the order of truth values will play a prominent role in selecting each kind of systems available, and providing for interesting uses of such systems. In the next section, we shall follow Haack and suggest more epistemic-oriented readings of the truth values

4 Reflections

In this section, we discuss various topics concerning the proper understanding of the semantics proposed here, and how the Haackian demand for intelligibility may be achieved by using such a semantics. In particular, we focus on how the framework developed here sheds light on some not so clear issues concerning many-valued logics and their applications in connection with mixed consequence relations.

4.1 Topic I - Agent reading

To begin with, besides the original Clemens reading, we introduce one additional possible reading for the truth values available in the generalization of Clemens semantics. Doing so will offer a more epistemic reading of the truth values, which justifies our claim that we are following the Haackian strategy of reading typical additional truth values in terms of the classical truth values with epistemic restrictions on them. Such reading also motivates a plainly classical understanding of the different consequence relations available, as well as motivates a discussion on the meaning of the connectives (again, the reader may also see the discussion in [10, 12]).[5] We can think of roughly the following kind of intuitive reading that is seen as the result of epistemic qualification to the classical truth values.

A specifically epistemic reading may be conferred to the order of the truth values in the n-tuples available for the Clemens semantics if we approach it in terms of n distinct agents, each of whom is supposed to evaluate the classical truth value of any given atomic proposition. Each element of an n-tuple then corresponds to the evaluation of the n-th agent. In order to make sense of the order of truth values, we can rank agents confidence in their evaluation too, so that we may think of going from specialists on a topic — the first entry from left to right — to someone who is not actually specialist on the topic —the first entry from right to left. Collectively, once the evaluations are performed for the atomic propositions, we may compute the truth values of complex propositions by evaluating the Boolean connectives.

Besides including agents, one can also think of an epistemic reading that is less focused on human beings, and more focused on procedures, reading the positions on the truth values as different tests that may be applied to check the application of a given predicate, with tests varying on their rigour or confidence. So, with n tests, an n-tuple would fill with 1 or 0 the n-th position depending on whether the n test is positive or negative for the application of the predicate (this reading is a generalization and adaptation of a discussion by Newton da Costa in [4, p.131]).[6] The order of the truth values would rank the degree of confidence we accept for a test in granting that the predicate does apply.

By focusing on the agent reading for the sake of simplicity, the distinct consequence relations should be read:

1. **K3** is the logic resulting from preserving only what all the agents agree on being true. In this sense, this logic requires unanimity if a proposition is to follow from a unanimous set of premises;
2. **CL** requires that the first agent should be seen as having privileged epistemic abilities, so that validity is related to whatever that particular agent judges as true. One may also consider that the first component is a kind of God's eye point of view, never failing, and the other ones are fallible human beings;
3. **LP** results when one is more tolerant towards all of the agents opinions; validity is prevented only in cases where there is consensus about falsity.

4.2 Topic II: Respects

Another interesting application of the generalized Clemens semantics may be found in relation to Paul Égré's discussion on acceptable contradictions in [6] (we omit some of the more fine grained details of Égré's exposition). The discussion is related to the dialetheist claim that some contradictions may be actually true (and also false). The major example of one such contradiction, of course, is derived from

[5] Haack did not, in fact, extend her discussion of the understanding of the additional truth values to the consequence relations in scenarios involving such truth values.

[6] The first edition of da Costa's book is from 1980.

discussions on the Liar paradox (see [13] for the *locus classicus*). In one possible presentation, the Liar sentence may be presented by introducing a sentence λ that says of itself that it is false:

λ : The sentence λ is false.

With very simple logical derivations usually available, one may then derive that the Liar sentence is both true and false.

According to Égré, if one is going to accept some contradictions, as a dialetheist is motivated to, one should attempt to make clear sense of such contradictions. In particular, it may happen, as dialetheists argue, that some contradictions are actually informative, not empty of content. Égré then goes on to define an *acceptable contradiction* as an informative sentence of the form 'x is P and x is not P'. The contradictions are understood as involving a kind of vagueness, they hide some additional information regarding the assertion of a predicate and its denial; literally, one is asserting the predicate according to some regards or respects, while at the same time denying it according to other respects:

> the acceptability of contradictions involving adjectives in particular (including "true") might indeed be grounded in the availability of multiple respects of application, but provided those respects of comparison are closely related to each other in a way that is constitutive of the vagueness of the expression in question. [6, p.41]

This looks quite similar to the above criteria of application of a predicate; the same predicate had to have different criteria of application, which could result in different verdicts concerning the appropriateness of application of the predicate. Now, instead of criteria of application, what we have is different respects associated with the same predicate. Contradictory sentences involve quantification over respects available for the application of the terms that are involved in generating the contradiction. This may be the case for adjectives, like 'good', 'intelligent', 'tall'. A contradiction like

- John is rich and John is not rich

is then understood as involving quantification over respects, with the latter indicating that John may be rich in some respects, but not rich in other (different) respects. For example, it may be that, in regard of academic professors, John is actually rich, while, at the same time, according to the standard used to compute latest list of billionaires in the world, John is not even close to being rich.

The idea that acceptable contradictions involve quantification over respects applies not only to adjectives, but also to nouns, for example:

- Mario is a man and is not a man,

where the first occurrence of 'man' designates 'man with respect to gender', and the second one designates 'man with respect to satisfaction of some stereotype of masculinity'. We may quantify over respects also in the case of verbs:

- I like fish and I don't like fish

which indicates that there are some respects according to which I like fish, let us say, as animals, while it also indicates that I don't like fish in every respect, let us say, as an option for a meal. The plan, remember, is that "each time contradictions can be paraphrased by means of an explicit specification of distinct respects of application." ([6, p.44])

As a template of the analysis of informative contradictions in terms of the proposed paraphrase using different respects, we have the following scheme:

- x is P [in some respects], and x is not P [in some respects].

Égré prefers the following way of putting it (this will be relevant for us soon):

- x is P [in some respects], and x is not [in all respects] P.

Given this account of informative contradictions, the informativeness is accounted for by the fact that "the respects relevant to the second conjunct are distinct from the respects relevant to the first" ([6, p.46]). That means that it is different information that is being dealt with in the affirmation and in the negation.

In summary again, the plan is the following:

> The basic idea is that relevant respects determine different extents to which a property can be satisfied, and those extents can be quantified over. [6, p.50]

This availability of different respects for application of a predicate opens the door for application of Clemens semantics. Using the generalized Clemens semantics, we can fix some n and interpret the places in the n-tuples representing truth values as the different respects available for a given noun, adjective or verb. The values 1 and 0 indicate whether a given object qualifies as having the corresponding noun, adjective or verb in the corresponding respect. Let us fix on the discussion of the example "John is a man and John is not a man". For the sake of simplicity, let us suppose we have two respects, the first one is related to being a man in respect to John's gender, the second one is related to being a man as concerned with a given stereotype of masculinity. Then, we have the four options:

- $\langle 1,1\rangle$: John is a man according to gender, and according to the stereotype;
- $\langle 1,0\rangle$: John is a man according to gender, but not according to the stereotype;
- $\langle 0,1\rangle$: John is not a man according to gender, but satisfies the man stereotype;
- $\langle 0,0\rangle$: John is not a man according to gender, and also not according to the stereotype.

This nicely illustrates the idea that we can have different respects that can be quantified over; basically, for any n we can have a semantics with n respects. It also captures the claim, by Égré, that a contradiction is informative when some predicate is not the case for all respects, so that it can be applied in relation to some respects, but not to others. That matches well the idea that if a proposition is the case for all respects (it receives a block of 1s), then its negation will not be the case (it will receive a block of 0s). The conjunction the will be just completely false for each respect. There is a sense in which such contradictions say nothing, they exclude the applicability of the predicate according to any respect. In this specific case, there is disagreement as related to every respect.

The distinct consequence relations that can be defined on the top of the Clemens semantics also acquire an interesting reading with that kind of approach. Let us briefly check:

- **K3** is the logic obtained when consequence must preserve satisfaction of all the respects; not contradictions allowed, even if informative;
- **LP** is the logic obtained when informative contradictions are allowed; uninformative contradictions should be ruled out;
- **CL** is the logic where the first respect has a priority over others, so that it is this one that must be preserved.

One final point before we leave this particular application. It is interesting to remark once again that a fixed order for the different respects is required, if Clemens semantics is to be used in this case. That means that some regards are considered to be more important than others, at least in each context. This is not completely unrealistic, given that depending on a context, one may privilege some respects as more important than others. In a certain sense, given the classical reading of the consequence relation, privileging the first regard could be read in a kind of epistemic approach to vagueness, where vagueness is only in language, and the first regard is a kind of universal standard (God's knowledge of borders). So, the other regards would play a role similar to the one different agents played in the agent reading.

4.3 Topic III - Mixed consequence

Now, let us consider the effect that Clemens semantics may have on topics related to mixed consequence relations. From a technical point of view, such consequence relations are simple to obtain on the top of a three-valued semantics for **K3** and **LP**. However, given that for the cases of these logics such a semantics is typically interpreted in different terms, with the third truth value as meaning either gaps or gluts, respectively, the understanding of the mixed consequence relations seems to face some difficulties: the meaning of the third truth value seems to fluctuate between gap and glut, depending on whether it is taken strictly or tolerantly. Let us be a bit more explicit about it: when the set of truth values is considered from a strict point of view, the third truth value is not designated, and so, is not to be counted as truth, acquiring the features of a gap as per **K3**; when considered from the tolerant point of view, the third truth values is read as per **LP**, looking like a glut. In a sense, then, the third truth values has a kind of chameleon nature.

In order to face some of such difficulties, a distinct reading for the semantics is offered in [2]:

> A second feature of our target semantics is that, while it coincides with the predictions of the many-valued logics LP and K3, it answers to a distinct motivation. Rather than seeing truth as a unified notion to which sentences might answer in three (or more) different ways, our approach posits distinct notions of truth, each of which a sentence may have or fail to have, but none of which is many-valued. [2, p.365]

Although the idea is to provide understanding of the semantic concepts involved, the strategy is requiring that truth be understood as a multiplicity of concepts. That is clearly a very non-classical reading of the notions of truth and falsity, illustrating what one may take as the addition of some new *sui generis* truth values. The result is that those like Haack, who are not sympathetic to such additions of truth values would be intrigued by what ‘distinct notions of truth’ could mean. What Clemens semantics does, in this case, is to provide for chances of uniform readings of the semantic values for mixed consequence cases, just as for standard ones. The intuitive reading advanced by Clemens, or else the agent reading, present before, are nice illustrations. The readings are there before the consequence relation is defined, so they can be used to illuminate the system independently of what kind of consequence relation one plugs in the semantics.

But besides using the already offered readings for the Clemens semantics for the understanding of the three-valued presentations of mixed consequence relation cases, we can also benefit from those readings to make a sharper sense of what ‘strict’ and ‘tolerant’ mean as per [3]. It is suggested that there are two ways of understanding sentences when it comes to classify them as strict or tolerant: there is a pragmatic way, according to which an *assertion* is qualified in terms of strict and tolerant, and there is an approach through *meaning*, where a sentence may have strict or tolerant meaning. Concerning the pragmatic approach:

> [...] we can see a direct connection between model-theoretic value and assertibility. A sentence is either both strictly and tolerantly assertible (value 1), tolerantly but not strictly assertible (value $\frac{1}{2}$), or not assertible at all (value 0). We do not allow for sentences that are strictly but not tolerantly assertible; strict assertion, on this picture, is a (strictly) stronger speech act than tolerant assertion. [3, pp.857-858]

Notice that as an explanation of what ‘strict’ and ‘tolerant’ mean, those are a bit circular: if we wanted to know what ‘strict’ and ‘tolerant’ mean, the explanation comes in terms already using ‘strict’ and ‘tolerant’. Those terms gain interesting meanings when one uses the Clemens semantics, and also, one obtains a more fine-grained distinction, allowing a distinction of two kinds of tolerant assertions.[7] A

[7] More kinds are allowed, of course, when ordered n-tuples are used, for $2 < n$.

sentence is strictly assertible, according to the agent reading, if it is asserted by both agents (considering the case of two agents). It is tolerantly assertible, but not strictly assertible, in two distinct scenarios: when only the first agent asserts it, or else when only the second agent asserts it. It is neither strictly nor tolerantly assertible when no agent asserts it. Here, 'strict' and 'tolerant' qualify the assertions made by each of the agents, which are previously understood in classical terms (thus satisfying Haack's demands).

The 'meaning approach' to strict and tolerant offered by [3] is equally dependent on our having grasped the meaning of 'strict' and 'tolerant' beforehand:

> The other approach works at the level of *meaning*. Rather than supposing that there are two distinct speech acts of assertion, this approach supposes that each sentence has two distinct meanings (or two distinct aspects of its meaning, if you like) that can be asserted: its strict meaning and its tolerant meaning. Understanding meanings as dividing the space of models in two, we can understand a sentence's strict meaning as one drawing a division between those models on which the sentence takes value 1 and those on which it takes some value less than 1, and we can understand a sentence's tolerant meaning as one drawing a division between those models on which the sentence takes some value greater than 0 and those on which it takes value 0. [3, p.858]

Again, according to Clemens semantics, a sentence may have meaning, only tolerant but not strict, or neither. However, tolerant meaning may be qualified in different guises, just as in the case of tolerant truth. These may be cashed in terms of the agent reading, or of the original reading by Clemens, among others. They do confer a nice illustration of how those notions may be understood in terms of the classical concepts, even though this understanding deviates from the original one proposed by [3].

5 Concluding remarks

In this paper, we have expanded on a semantic framework advanced originally by Matthew Clemens. In particular, we have presented a Clemens semantics for first-order logic, and we also considered the use of such a framework to deal with mixed consequence relations. The benefits of such an investigation were explored through the lenses of a demand formerly expressed by Susan Haack, according to which many-valued logics become more intelligible when additional truth values are analysed in terms of bivalent truth and falsity. We have provided for some readings of Clemens semantics that satisfy such a requirement, and indicated how such readings impact on current attempts to use many-valued logics to deal with some interesting philosophical problems.

References

[1] Matthew Clemens (2020): *Ordered pair semantics and negation in LP*. The Australasian Journal of Logic 17(5), pp. 201–205, doi:10.26686/ajl.v17i5.4080.

[2] Pablo Cobreros, Paul Égré, David Ripley & Robert van Rooij (2012): *Tolerant, classical, strict*. Journal of Philosophical Logic 41(2), pp. 347–385, doi:10.1007/s10992-010-9165-z.

[3] Pablo Cobreros, Paul Égré, David Ripley & Robert Van Rooij (2013): *Reaching transparent truth*. Mind 122(488), pp. 841–866, doi:10.1093/mind/fzt110.

[4] Newton Carneiro Affonso da Costa (2008): *Ensaio sobre os fundamentos da lógica*. Hucitec.

[5] Michael Dummett (1984): *The philosophical basis of intuitionistic logic*. In Paul Benacerraf & Hilary Putnam, editors: Philosophy of Mathematics: Selected Readings, 2 edition, Cambridge University Press, p. 97–129, doi:10.1017/CBO9781139171519.007.

[6] Paul Égré (2019): *Respects for Contradictions*. In Can Başkent & Thomas Macaulay Ferguson, editors: *Graham Priest on Dialetheism and Paraconsistency*, Springer International Publishing, Cham, pp. 39–57, doi:10.1007/978-3-030-25365-3_4.

[7] Szymon Frankowski (2004): *Formalization of a plausible inference*. *Bulletin of the Section of Logic* 33(1), pp. 41–52.

[8] Susan Haack (1978): *Philosophy of Logics*. Cambridge University Press, doi:10.1017/CBO9780511812866.

[9] Hans G. Herzberger (1973): *Dimensions of truth*. *Journal of Philosophical Logic* 2(4), pp. 535–556, doi:10.1007/bf00262954.

[10] Hitoshi Omori & Jonas R. B. Arenhart (2021): *A Generalization of Ordered-Pair Semantics*. In: *Logic, Rationality, and Interaction: 8th International Workshop, LORI 2021, Xi'ian, China, October 16-18, 2021, Proceedings 8*, Springer, pp. 149–157, doi:10.1007/978-3-030-88708-7_12.

[11] Hitoshi Omori & Jonas Rafael Becker Arenhart (2022): *Haack meets Herzberger and Priest*. In: *2022 IEEE 52nd International Symposium on Multiple-Valued Logic (ISMVL)*, pp. 137–144, doi:10.1109/ISMVL52857.2022.00028.

[12] Hitoshi Omori & Jonas Rafael Becker Arenhart (2023): *Change of logic, without change of meaning*. *Theoria* 89(4), pp. 414–431, doi:10.1111/theo.12459.

[13] Graham Priest (2006): *In Contradiction: A Study of the Transconsistent*. Oxford University Press, doi:10.1093/acprof:oso/9780199263301.001.0001.

[14] Graham Priest (2008): *Introduction to non-classical logics: from ifs to is*, second edition. Cambridge University Press, doi:10.1017/CBO9780511801174.

The Disjunction-Free Fragment of $\mathbf{D}_2$ is Three-Valued*

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In this article, the disjunction-free fragment of Jaśkowski's discussive logic $\mathbf{D}_2$ in the language of classical logic is shown to be complete with respect to three- and four-valued semantics. As a byproduct, a rather simple axiomatization of the disjunction-free fragment of $\mathbf{D}_2$ is obtained. Some implications of this result are also discussed.

1 Introduction

Stanisław Jaśkowski is known to be one of the modern founders of paraconsistent logic, together with Newton C. A. da Costa. The most important contribution of Jaśkowski is that he clearly distinguished two notions for a theory, namely a theory being *contradictory* (or *inconsistent* in [18]) and a theory being *trivial* (or *overfilled* in [18]). In addition to this distinction, he also presented a system of paraconsistent logic known as $\mathbf{D}_2$ which is often referred to as discursive logic or discussive logic (cf. [18, 19]).

In this article, the disjunction-free fragment of Jaśkowski's discussive logic is shown to be complete with respect to three- and four-valued semantics. Note here that $\mathbf{D}_2$ is known to be not complete with respect to any finitely many-valued semantics, which is proved by Jerzy Kotas in [20]. As a byproduct of the main result, a simple axiomatization of the disjunction-free fragment of Jaśkowski's discussive logic in the language of classical logic is obtained. For the problem of axiomatization of $\mathbf{D}_2$, see [24].

2 Semantics and proof theory

The propositional languages in this article consist of a finite set S of propositional connectives and a countable set Prop of propositional variables. The languages are referred to as $\mathscr{L}$, $\mathscr{L}_r^-$, $\mathscr{L}_r$, $\mathscr{L}_l^-$ and $\mathscr{L}_l$ when S are $\{\sim, \to_d, \wedge, \vee\}$, $\{\sim, \to_d, \wedge_d^r\}$, $\{\sim, \to_d, \wedge_d^r, \vee\}$, $\{\sim, \to_d, \wedge_d^l\}$, and $\{\sim, \to_d, \wedge_d^l, \vee\}$, respectively. Note that the languages $\mathscr{L}$ and $\mathscr{L}_r$ were introduced by Jaśkowski in [18] and [19], respectively.[1]

*The main result was presented with a rather different narrative at *Logic in Bochum 2*, *CCPEA 2016* in Seoul, *Paradoxes, Logic and Philosophy* in Beijing, *V Workshop on Philosophical of Logic* in Buenos Aires, *Prague Seminar on Paraconsistent Logic*, a colloquium in Munich and *ISRALOG17* in Haifa. I owe a special debt of gratitude to Dave Ripley whose comments led me to rethink the overall presentation of the main result. An earlier version of this article was presented at: *Non-classical modalities* in Mexico City, *the Eleventh Smirnov Readings in Logic* in Moscow, *CoPS-FaM-2019* in Gdańsk, *Paris-Bochum-Moscow Workshop in Mathematical Philosophy* in Paris and another colloquium in Munich. I would like to thank the organizers of these events for their kind invitations, warm hospitality and helpful discussions, as well as the audiences at these meeting for useful comments. I would also like to thank Jonas Rafael Becker Arenhart and Fabio De Martin Polo for helpful discussions and comments. Finally, but not the least, I would like to thank the referees for their very kind, detailed, and supportive comments that improved the presentation of the paper. The preparation of an earlier version of this article was supported by a Sofja Kovalevskaja Award of the Alexander von Humboldt-Foundation, funded by the German Ministry for Education and Research.

[1] As correctly pointed out by a referee, Jaśkowski also included the discussive biconditional as a *primitive* connective. However, in view of [24, Proposition 1], I will treat the discussive biconditional as a *defined* connective.

A. Indrzejczak, M. Zawidzki (Eds.): Non-Classical Logics.
Theory and Applications (NCL'24).
EPTCS 415, 2024, pp. 257–270, doi:10.4204/EPTCS.415.22

The language $\mathscr{L}_l$ has been considered in a number of papers including [10, 38]. The language $\mathscr{L}_r^-$ is the main one dealt with in this paper, but I will also refer to the other languages when it is helpful.[2] The set of formulas defined as usual in $\mathscr{L}$, $\mathscr{L}_r^-$ and $\mathscr{L}_l^-$, are denoted by Form, Form_r^- and Form_l^-, respectively. Moreover, a formula is denoted by A, B, C, etc. and a set of formulas by Γ, Δ, Σ, etc.

2.1 Semantics for the disjunction-free fragment of $\mathbf{D}_2$

The original semantics of Jaśkowski can be precisified by making use of translations into modal language, but here I follow Janusz Ciuciura (cf. [7]) who stated the semantics without the help of translation.

Definition 1 ($\mathbf{D}_2^-$**-model).** $\mathbf{D}_2^-$-model for $\mathscr{L}_r^-$ is a pair $\langle W, v\rangle$ where W is a non-empty set and $v : W \times \mathsf{Prop} \longrightarrow \{0,1\}$, an assignment of truth values to state-variable pairs. Valuations v are then extended to interpretations I to state-formula pairs by the following conditions.

- $I(w,p) = v(w,p)$, for all $w \in W$ and for all $p \in \mathsf{Prop}$;
- $I(w,{\sim}A) = 1$ iff $I(w,A) = 0$;
- $I(w,A\wedge_d^r B) = 1$ iff $I(w,A) = 1$ and for some $x \in W(I(x,B) = 1)$;
- $I(w,A{\rightarrow_d}B) = 1$ iff for all $x \in W(I(x,A) = 0)$ or $I(w,B) = 1$.

Furthermore, $\Gamma \models_d A$ iff for every $\mathbf{D}_2^-$-model $\langle W, v\rangle$, if for all $B \in \Gamma$, there is $x \in W$ such that $I(x,B) = 1$, then $I(y,A) = 1$ for some $y \in W$.

Remark 2. Note that the semantic consequence relation is defined in an unusual way, which is not a mistake, but a definition that reflects the original idea of Jaśkowski.

Now, by considering a special case of the Kripke semantics in which the cardinality of W is two, the following four-valued semantics is obtained.

Definition 3. A *four-valued* $\mathbf{D}_2^-$*-interpretation* of $\mathscr{L}_r^-$ is a function $v : \mathsf{Prop} \longrightarrow \{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{0}\}$. Given a four-valued $\mathbf{D}_2^-$-interpretation v, this is extended to a function I that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

A	$\sim A$
1	**0**
i	**j**
j	**i**
0	**1**

$A\wedge_d^r B$	**1**	**i**	**j**	**0**
1	**1**	**1**	**1**	**0**
i	**i**	**i**	**i**	**0**
j	**j**	**j**	**j**	**0**
0	**0**	**0**	**0**	**0**

$A{\rightarrow_d}B$	**1**	**i**	**j**	**0**
1	**1**	**i**	**j**	**0**
i	**1**	**i**	**j**	**0**
j	**1**	**i**	**j**	**0**
0	**1**	**1**	**1**	**1**

Note that the set of designated values, denoted by $\mathscr{D}_4$, is $\{\mathbf{1},\mathbf{i},\mathbf{j}\}$. The semantic consequence relation $\models_4^-$ is defined in terms of preservation of designated values.

Remark 4. Assume that $W = \{w_1, w_2\}$. Then,

- $v(A) = \mathbf{1}$ corresponds to $v(w_1,A) = 1$ and $v(w_2,A) = 1$,
- $v(A) = \mathbf{i}$ corresponds to $v(w_1,A) = 1$ and $v(w_2,A) = 0$,
- $v(A) = \mathbf{j}$ corresponds to $v(w_1,A) = 0$ and $v(w_2,A) = 1$,
- $v(A) = \mathbf{0}$ corresponds to $v(w_1,A) = 0$ and $v(w_2,A) = 0$.

Note also that the unusual definition of the semantic consequence relation is here reflected as having three designated values.

[2] My emphasis on the languages $\mathscr{L}_r^-$ and $\mathscr{L}_r$ is a personal choice paying my respect to Jaśkowski for introducing the first discussive conjunction in [19]. However, the main observation of the paper carries over for other languages, and some of the details are spelled out in §5.1 and §5.2.

In the above semantics, the intermediate values are representing the two possibilities depending on which of the two states or worlds falsifies the sentence. In fact, these two possibilities can be "merged", and the third value can stand for the case in which the two states or worlds disagree. As a result, the following three-valued semantics is obtained.

Definition 5. A *three-valued* $\mathbf{D}_2^-$*-interpretation* of $\mathscr{L}_r^-$ is a function $v : \mathsf{Prop} \longrightarrow \{\mathbf{1},\mathbf{i},\mathbf{0}\}$. Given a three-valued $\mathbf{D}_2^-$-interpretation v, this is extended to a function I that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

A	$\sim A$
1	**0**
i	**i**
0	**1**

$A \wedge_d^r B$	**1**	**i**	**0**
1	**1**	**1**	**0**
i	**i**	**i**	**0**
0	**0**	**0**	**0**

$A \rightarrow_d B$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**1**	**i**	**0**
0	**1**	**1**	**1**

Note that the set of designated values, denoted by $\mathscr{D}_3$, is $\{\mathbf{1},\mathbf{i}\}$. The semantic consequence relation $\models_3$ is defined in terms of preservation of designated values.

Remark 6. From a purely technical viewpoint, the above truth table for negation is exactly the one for the three-valued logic developed by Łukasiewicz, as well as for the Logic of Paradox (cf. [29]). Moreover, the truth table for conditional is identical with the one in $\mathbf{RM}_3^{\supset}$ (cf. [2]), **LFI1** (cf. [4]) and **CLuNs** (cf. [3]), among many other systems.

Remark 7. Note that in view of a general result established by Arnon Avron, Ofer Arieli and Anna Zamansky, it follows that $\models_3$ is maximally paraconsistent in the strong sense, and thus maximal with respect to extended classical logic, by [1, Corollary 3.6].

2.2 Proof system for the disjunction-free fragment of $\mathbf{D}_2$

I now turn to the proof theory which is presented in terms of a Hilbert-style calculus.

Definition 8. The system $\mathbf{D}_2^-$ consists of the following axiom schemata and a rule of inference, where $A \leftrightarrow_d B$ abbreviates $(A \rightarrow_d B) \wedge_d^r (B \rightarrow_d A)$.

$$A \rightarrow_d (B \rightarrow_d A) \quad \text{(Ax1)}$$
$$(A \rightarrow_d (B \rightarrow_d C)) \rightarrow_d ((A \rightarrow_d B) \rightarrow_d (A \rightarrow_d C)) \quad \text{(Ax2)}$$
$$((A \rightarrow_d B) \rightarrow_d A) \rightarrow_d A \quad \text{(Ax3)}$$
$$(A \wedge_d^r B) \rightarrow_d A \quad \text{(Ax4)}$$
$$(A \wedge_d^r B) \rightarrow_d B \quad \text{(Ax5)}$$
$$(C \rightarrow_d A) \rightarrow_d ((C \rightarrow_d B) \rightarrow_d (C \rightarrow_d (A \wedge_d^r B))) \quad \text{(Ax6)}$$
$$(\sim A \rightarrow_d A) \rightarrow_d A \quad \text{(Ax7)}$$
$$\sim\sim A \leftrightarrow_d A \quad \text{(Ax8)}$$
$$\sim(A \wedge_d^r B) \leftrightarrow_d (B \rightarrow_d \sim A) \quad \text{(Ax9)}$$
$$\sim(A \rightarrow_d B) \leftrightarrow_d (A \wedge_d^r \sim B) \quad \text{(Ax10)}$$
$$\frac{A \quad A \rightarrow_d B}{B} \quad \text{(MP)}$$

Finally, $\Gamma \vdash A$ iff there is a sequence of formulas $B_1, \ldots, B_n, A$ ($n \geq 0$), called *a derivation*, such that every formula in the sequence either (i) belongs to Γ; (ii) is an axiom of $\mathbf{D}_2^-$; (iii) is obtained by (MP) from formulas preceding it in the sequence.

Remark 9. Note that the only unusual axiom in the literature of paraconsistent logic is (Ax9).

Before moving further, note that the deduction theorem holds for $\vdash$.

Proposition 10. For all $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_r^-$, $\Gamma, A \vdash B$ iff $\Gamma \vdash A \rightarrow_d B$.

3 Soundness and Completeness for the three-valued semantics

I now turn to prove that the proof system introduced in the previous section is sound and complete with respect to the three-valued semantics.

3.1 Soundness

I begin with the soundness which is easy as usual.

Proposition 11 (Soundness)**.** For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}_r^-$, if $\Gamma \vdash A$ then $\Gamma \models_3 A$.

Proof. By a straightforward verification that each instance of each axiom schema always takes a designated value, and that (MP) preserves designated values. □

3.2 Completeness

For the completeness, some terminologies are needed. To this end, I deploy those from [34] with a slightly different term using *non-trivial* instead of *consistent*.

Definition 12 (Schumm)**.** For $\Sigma \cup \{B\} \subseteq \mathsf{Form}_r^-$, Σ is *maximally non-trivial* iff (i) $\Sigma \nvdash A$ for some $A \in \mathsf{Form}_r^-$ and (ii) for every $A \in \mathsf{Form}_r^-$, if $A \notin \Sigma$ then $\Sigma \cup \{A\} \vdash B$ for all $B \in \mathsf{Form}_r^-$.

Remark 13. Note that if Σ is maximally non-trivial, then Σ is a theory, i.e. closed under $\vdash$.

Then the following well-known lemma is obtained. The proof is given in [34, Theorem 8].

Lemma 14 (Schumm)**.** For all $\Sigma \cup \{A\} \subseteq \mathsf{Form}_r^-$, suppose that $\Sigma \nvdash A$. Then, there is a $\Pi \supseteq \Sigma$ such that Π is maximally non-trivial and $A \notin \Pi$.

Moreover, the following lemma, which will be useful later, is also easy to prove.

Lemma 15. If Σ is maximally non-trivial, then $\Sigma \vdash A \rightarrow_d B$ iff ($\Sigma \nvdash A$ or $\Sigma \vdash B$).

Definition 16. Let Σ be maximally non-trivial. Then, let v_Σ from Prop to $\{\mathbf{1}, \mathbf{i}, \mathbf{0}\}$ be defined as follows:

$$v_\Sigma(p) = \mathbf{1} \text{ iff } \Sigma \nvdash {\sim} p \quad \text{and} \quad v_\Sigma(p) = \mathbf{i} \text{ iff } \Sigma \vdash p \text{ and } \Sigma \vdash {\sim} p \quad \text{and} \quad v_\Sigma(p) = \mathbf{0} \text{ iff } \Sigma \nvdash p$$

I need one more lemma which is the key for the completeness result.

Lemma 17. If Σ is maximally non-trivial, then the following holds for all $B \in \mathsf{Form}_r^-$.

$$v_\Sigma(B) = \mathbf{1} \text{ iff } \Sigma \nvdash {\sim} B \quad \text{and} \quad v_\Sigma(B) = \mathbf{i} \text{ iff } \Sigma \vdash B \text{ and } \Sigma \vdash {\sim} B \quad \text{and} \quad v_\Sigma(B) = \mathbf{0} \text{ iff } \Sigma \nvdash B$$

Proof. Note first that the well-definedness of v_Σ is obvious. Then the desired result is proved by induction on the the construction of B. The base case, for atomic formulas, is obvious by the definition. For the induction step, the cases are split based on the connectives.

Case 1. If $B = {\sim} C$, then there are the following three cases.

$$\begin{aligned} v_\Sigma({\sim} C) = \mathbf{1} &\text{ iff } v_\Sigma(C) = \mathbf{0} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \nvdash C && \text{by IH} \\ &\text{ iff } \Sigma \nvdash {\sim}{\sim} C && \text{by (Ax8)} \end{aligned}$$

$$\begin{aligned} v_\Sigma({\sim}C) = \mathbf{i} &\text{ iff } v_\Sigma(C) = \mathbf{i} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \vdash {\sim}C \text{ and } \Sigma \vdash C && \text{by IH} \\ &\text{ iff } \Sigma \vdash {\sim}C \text{ and } \Sigma \vdash {\sim}{\sim}C && \text{by (Ax8)} \end{aligned}$$

$$\begin{aligned} v_\Sigma({\sim}C) = \mathbf{0} &\text{ iff } v_\Sigma(C) = \mathbf{1} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \nvdash {\sim}C && \text{by IH} \end{aligned}$$

Case 2. If $B = C{\rightarrow_d}D$, then there are the following three cases.

$$\begin{aligned} v_\Sigma(C{\rightarrow_d}D) = \mathbf{1} &\text{ iff } v_\Sigma(C) = \mathbf{0} \text{ or } v_\Sigma(D) = \mathbf{1} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \nvdash C \text{ or } \Sigma \nvdash {\sim}D && \text{by IH} \\ &\text{ iff } \Sigma \nvdash (C \wedge {\sim}D) && \text{by } \Sigma \text{ is a theory} \\ &\text{ iff } \Sigma \nvdash {\sim}(C{\rightarrow_d}D) && \text{by (Ax10)} \end{aligned}$$

$$\begin{aligned} v_\Sigma(C{\rightarrow_d}D) = \mathbf{i} &\text{ iff } v_\Sigma(C) \neq \mathbf{0} \text{ and } v_\Sigma(D) = \mathbf{i} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \vdash C \text{ and } (\Sigma \vdash D \text{ and } \Sigma \vdash {\sim}D) && \text{by IH} \\ &\text{ iff } (\Sigma \nvdash C \text{ or } \Sigma \vdash D) \text{ and } \Sigma \vdash (C \wedge {\sim}D) && \Sigma \text{ is a theory} \\ &\text{ iff } \Sigma \vdash (C{\rightarrow_d}D) \text{ and } \Sigma \vdash {\sim}(C{\rightarrow_d}D) && \text{by Lemma 15 and (Ax10)} \end{aligned}$$

$$\begin{aligned} v_\Sigma(C{\rightarrow_d}D) = \mathbf{0} &\text{ iff } v_\Sigma(C) \neq \mathbf{0} \text{ and } v_\Sigma(D) = \mathbf{0} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \vdash C \text{ and } \Sigma \nvdash D && \text{by IH} \\ &\text{ iff } \Sigma \nvdash (C{\rightarrow_d}D) && \text{by Lemma 15} \end{aligned}$$

Case 3. If $B = C{\wedge}D$, then there are the following three cases.

$$\begin{aligned} v_\Sigma(C \wedge D) = \mathbf{1} &\text{ iff } v_\Sigma(C) = \mathbf{1} \text{ and } v_\Sigma(D) \neq \mathbf{0} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \vdash D \text{ and } \Sigma \nvdash {\sim}C && \text{by IH} \\ &\text{ iff } \Sigma \nvdash D{\rightarrow_d}{\sim}C && \text{by Lemma 15} \\ &\text{ iff } \Sigma \nvdash {\sim}(C \wedge D) && \text{by (Ax9)} \end{aligned}$$

$$\begin{aligned} v_\Sigma(C{\wedge}D) = \mathbf{i} &\text{ iff } v_\Sigma(C) = \mathbf{i} \text{ and } v_\Sigma(D) \neq \mathbf{0} && \text{by the definition of } v_\Sigma \\ &\text{ iff } (\Sigma \vdash C \text{ and } \Sigma \vdash {\sim}C) \text{ and } \Sigma \vdash D && \text{by IH} \\ &\text{ iff } (\Sigma \vdash C \text{ and } \Sigma \vdash D) \text{ and } (\Sigma \nvdash D \text{ or } \Sigma \vdash {\sim}C) && \text{by simple calculation} \\ &\text{ iff } (\Sigma \vdash C \text{ and } \Sigma \vdash D) \text{ and } \Sigma \vdash D{\rightarrow_d}{\sim}C && \text{by Lemma 15} \\ &\text{ iff } \Sigma \vdash (C \wedge D) \text{ and } \Sigma \vdash {\sim}(C \wedge D) && \Sigma \text{ is a theory and by (Ax10)} \end{aligned}$$

$$\begin{aligned} v_\Sigma(C \wedge D) = \mathbf{0} &\text{ iff } v_\Sigma(C) = \mathbf{0} \text{ or } v_\Sigma(D) = \mathbf{0} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \nvdash C \text{ or } \Sigma \nvdash D && \text{by IH} \\ &\text{ iff } \Sigma \nvdash (C \wedge D) && \Sigma \text{ is a theory} \end{aligned}$$

This completes the proof. □

Theorem 1 (Completeness)**.** For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}_r^-$, if $\Gamma \models_3 A$ then $\Gamma \vdash A$.

Proof. Assume $\Gamma \nvdash A$. Then, by Lemma 14, there is a $\Pi \supseteq \Gamma$ such that Π is maximally non-trivial and $A \notin \Pi$, and by Lemma 17, a three-valued $\mathbf{D}_2^-$-valuation v_Π can be defined with $I_\Pi(B) \in \mathscr{D}_3$ for every $B \in \Gamma$ and $I_\Pi(A) \notin \mathscr{D}_3$. Thus it follows that $\Gamma \not\models_3 A$, as desired. □

4 The main result

By making use of the result in the previous section, I prove the main result of this article. To this end, I need one more lemma.

Lemma 18. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}_r^-$, if $\Gamma \models_4 A$ then $\Gamma \models_3 A$.

Proof. Suppose $\Gamma \not\models_3 A$. Then there is a three-valued $\mathbf{D}_2^-$-interpretation v_0 such that $I_0(B) \in \mathscr{D}_3$ for all $B \in \Gamma$ and $I_0(A) \notin \mathscr{D}_3$. Now, let v_1 be a four-valued $\mathbf{D}_2^-$-interpretation such that $v_1(p) = v_0(p)$. Then, it holds that $I_1(A) = \mathbf{1}$ iff $I_0(A) = \mathbf{1}$ and $I_1(A) = \mathbf{0}$ iff $I_0(A) = \mathbf{0}$. This can be proved by a simple induction on the complexity of A.

- The base case when $A \in \mathsf{Prop}$ is obvious by definition.
- For induction step, consider the following two cases.
  - If A is of the form $\sim B$, then by IH,
    - $I_1(B) = \mathbf{1}$ iff $I_0(B) = \mathbf{1}$ and
    - $I_1(B) = \mathbf{0}$ iff $I_0(B) = \mathbf{0}$.

    Then, by the truth table, it follows that $I_1(\sim B)=\mathbf{0}$ iff (by the truth table) $I_1(B)=\mathbf{1}$ iff (by IH) $I_0(B)=\mathbf{1}$ iff (by the truth table) $I_0(\sim B)=\mathbf{0}$. Moreover, $I_1(\sim B)=\mathbf{1}$ iff (by the truth table) $I_1(B)=\mathbf{0}$ iff (by IH) $I_0(B)=\mathbf{0}$ iff (by the truth table) $I_0(\sim B)=\mathbf{1}$.
  - If A is of the form $B \to_d C$, then by IH,
    - $I_1(B) = \mathbf{1}$ iff $I_0(B) = \mathbf{1}$, $I_1(B) = \mathbf{0}$ iff $I_0(B) = \mathbf{0}$, and
    - $I_1(C) = \mathbf{1}$ iff $I_0(C) = \mathbf{1}$, $I_1(C) = \mathbf{0}$ iff $I_0(C) = \mathbf{0}$.

    Then, by the truth table, it follows that $I_1(B \to_d C)=\mathbf{0}$ iff (by the truth table) $I_1(B) \neq \mathbf{0}$ and $I_1(C)=\mathbf{0}$ iff (by IH) $I_0(B) \neq \mathbf{0}$ and $I_0(C)=\mathbf{0}$ iff (by the truth table) $I_0(B \to_d C)=\mathbf{0}$. Moreover, $I_1(B \to_d C)=\mathbf{1}$ iff (by the truth table) $I_1(B)=\mathbf{0}$ or $I_1(C)=\mathbf{1}$ iff (by IH) $I_0(B)=\mathbf{0}$ and $I_0(C)=\mathbf{1}$ iff (by the truth table) $I_0(B \to_d C)=\mathbf{1}$.

  The case for conjunction is similar to the case for $\to_d$. This completes the proof.

Once this is established it is easy to see that the desired result holds since $I_1(A) = \mathbf{0}$ iff $I_0(A) = \mathbf{0}$ is equivalent to $I_1(A) \notin \mathscr{D}_4$ iff $I_0(A) \notin \mathscr{D}_3$. □

I am now ready to prove the main result.

Theorem 2 (Main Theorem)**.** For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}_r^-$, $\Gamma \models_3 A$ iff $\Gamma \models_d A$.

Proof. For the left-to-right direction, if $\Gamma \models_3 A$ then $\Gamma \vdash A$ by Theorem 1. One may then check that if $\Gamma \vdash A$ then $\Gamma \models_d A$. This is tedious but not difficult. For the other direction, if $\Gamma \models_d A$ then it immediately implies that $\Gamma \models_4 A$, by recalling Remark 4. Thus, together with Lemma 18, the desired result is proved. □

As a corollary of Proposition 11 and Theorems 1 and 2, the following result is obtained.

Corollary 19. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}_r^-$, $\Gamma \vdash A$ iff $\Gamma \models_d A$.

5 Reflections

5.1 The language $\mathscr{L}_l$

In the later works related to discussive logics, the language $\mathscr{L}_l$ has been also studied intensively. Here, I note that the above observations carry over to $\mathscr{L}_l^-$.

- First, the truth condition for the left discussive conjunction within the Kripke semantics is as follows.

$$(\wedge_d^l) \quad v(w, A \wedge_d^l B) = 1 \text{ iff for some } x \in W(v(x,A) = 1) \text{ and } v(w,B) = 1.$$

- Second, the three- and four-valued truth tables for the left discussive conjunction are as follows. Of course, the four-valued truth table is obtained by considering the special case of the Kripke semantics (recall Remark 4), and the three-valued truth table is obtained by "merging" the intermediate values in the four-valued truth table.

$A \wedge_d^l B$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**1**	**i**	**0**
0	**0**	**0**	**0**

$A \wedge_d^l B$	**1**	**i**	**j**	**0**
1	**1**	**i**	**j**	**0**
i	**1**	**i**	**j**	**0**
j	**1**	**i**	**j**	**0**
0	**0**	**0**	**0**	**0**

- Third, for the proof system, (Ax9) is replaced by the following.

$$\sim(A \wedge_d^l B) \leftrightarrow_d (A \rightarrow_d \sim B) \qquad \text{(Ax9')}$$

Based on these, the equivalence of the discussive semantics and the three-valued semantics may be established in a similar manner. For those who are interested in the details, note that for the purpose of establishing the result corresponding to Theorem 2, it suffices to check the following three items.

- Lemma 17, for the completeness result, i.e. if $\Gamma \models_3 A$ then $\Gamma \vdash A$.
- $\Gamma \vdash A$ then $\Gamma \models_d A$.
- Lemma 18, i.e. $\Gamma \models_4 A$ then $\Gamma \models_3 A$.

For the first item, it is enough to check the case related to conjunction, in particular the following two cases.

$$\begin{aligned}
v_\Sigma(C \wedge D) = \mathbf{1} &\text{ iff } v_\Sigma(C) \neq \mathbf{0} \text{ and } v_\Sigma(D) = \mathbf{1} && \text{by the definition of } v_\Sigma \\
&\text{ iff } \Sigma \vdash C \text{ and } \Sigma \nvdash \sim D && \text{by IH} \\
&\text{ iff } \Sigma \nvdash C \rightarrow_d \sim D && \text{by Lemma 15} \\
&\text{ iff } \Sigma \nvdash \sim(C \wedge D) && \text{by (Ax9')}
\end{aligned}$$

$$\begin{aligned}
v_\Sigma(C \wedge D) = \mathbf{i} &\text{ iff } v_\Sigma(C) \neq \mathbf{0} \text{ and } v_\Sigma(D) = \mathbf{i} && \text{by the definition of } v_\Sigma \\
&\text{ iff } \Sigma \vdash C \text{ and } (\Sigma \vdash D \text{ and } \Sigma \vdash \sim D) && \text{by IH} \\
&\text{ iff } (\Sigma \vdash C \text{ and } \Sigma \vdash D) \text{ and } (\Sigma \nvdash C \text{ or } \Sigma \vdash \sim D) && \text{by simple calculation} \\
&\text{ iff } (\Sigma \vdash C \text{ and } \Sigma \vdash D) \text{ and } \Sigma \vdash C \rightarrow_d \sim D && \text{by Lemma 15} \\
&\text{ iff } \Sigma \vdash (C \wedge D) \text{ and } \Sigma \vdash \sim(C \wedge D) && \Sigma \text{ is a theory and by (Ax9')}
\end{aligned}$$

For the second item, this is immediate in view of the new truth condition for the left discussive conjunction within the Kripke semantics.

Finally, for the third item, it is again enough to check the case for conjunction, and the proof runs as follows. If A is of the form $B \wedge_d^l C$, then by IH,

- $I_1(B) = \mathbf{1}$ iff $I_0(B) = \mathbf{1}$, $I_1(B) = \mathbf{0}$ iff $I_0(B) = \mathbf{0}$, and
- $I_1(C) = \mathbf{1}$ iff $I_0(C) = \mathbf{1}$, $I_1(C) = \mathbf{0}$ iff $I_0(C) = \mathbf{0}$.

Then, by the truth table, it follows that $I_1(B \wedge^l_d C)=\mathbf{0}$ iff (by the truth table) $I_1(B)=\mathbf{0}$ or $I_1(C)=\mathbf{0}$ iff (by IH) $I_0(B)=\mathbf{0}$ and $I_0(C)=\mathbf{0}$ iff (by the truth table) $I_0(B \wedge^l_d C)=\mathbf{0}$. Moreover, $I_1(B \wedge^l_d C)=\mathbf{1}$ iff (by the truth table) $I_1(B)\neq\mathbf{0}$ or $I_1(C)=\mathbf{1}$ iff (by IH) $I_0(B)\neq\mathbf{0}$ and $I_0(C)=\mathbf{1}$ iff (by the truth table) $I_0(B \wedge^l_d C)=\mathbf{1}$.

Based on these, the proof of Theorem 2 can be repeated to establish the desired result.

5.2 The language $\mathscr{L}$

If one considers the very first discussive language $\mathscr{L}$ in which the only discussive connective is conditional, a similar result is obtained by considering the negation-conditional fragment. More specifically, the concerned fragment is equivalent to the three-valued semantics induced by the following truth tables:

A	$\sim A$
1	**0**
i	**i**
0	**1**

$A\to_d B$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**1**	**i**	**0**
0	**1**	**1**	**1**

This can be confirmed by carefully removing the cases for conjunction in the proof of the main result. For those who are interested in the details, note once again that for the purpose of establishing the result corresponding to Theorem 2, it suffices to check the following three items.

- Lemma 17, for the completeness result, i.e. if $\Gamma \models_3 A$ then $\Gamma \vdash A$.
- $\Gamma \vdash A$ then $\Gamma \models_d A$.
- Lemma 18, i.e. $\Gamma \models_4 A$ then $\Gamma \models_3 A$.

In particular, it is enough to check that the previous proofs are not essentially relying on conjunction. For the first item, note first that (Ax10) needs to be replaced by the following three axioms.

$$\sim(A\to_d B)\to_d A \quad \text{(Ax10.1)}$$

$$\sim(A\to_d B)\to_d \sim B \quad \text{(Ax10.2)}$$

$$A\to_d(\sim B\to_d \sim(A\to_d B)) \quad \text{(Ax10.3)}$$

Then, it suffices to check that if Σ is maximally non-trivial, then $\Sigma \vdash \sim(A\to_d B)$ iff ($\Sigma \vdash A$ and $\Sigma \vdash \sim B$). This of course holds even without the maximal non-triviality. For the second and the third items, there is nothing to be checked since they are both already established.

Based on these, the proof of Theorem 2 can be repeated to establish the desired result.

5.3 Discussive negation

Another variation of the main result is obtained by considering a discussive interpretation of negation, a suggestion made by Jerzy Perzanowski as one of the comments of the translator in [19, p.59], and explored further by Ciuciura in [6].[3] Here, once again, I note that the above observations carry over to this variant.

- First, the truth condition for the discussive negation within the Kripke semantics is as follows.

$$(\sim_d) \quad v(w, \sim_d A) = 1 \text{ iff for some } x \in W, v(x,A) = 0.$$

[3] There is, unfortunately, a problem with one of the main results in [6]. See the appendix of [24] for the details.

- Second, the three- and four-valued truth tables for the discussive negation are as follows.

A	$\sim_d A$
1	**0**
i	**1**
0	**1**

A	$\sim_d A$
1	**0**
i	**1**
j	**1**
0	**1**

- Third, for the proof system, (Ax8) is replaced by the following.

$$\sim_d A \to_d (\sim_d \sim_d A \to_d B)$$

Based on these, the equivalence of the discussive semantics and the three-valued semantics is established in a similar manner. I first note here that given the proof system, the following is obtained.

Lemma 20. If Σ is maximally non-trivial, then $\Sigma \vdash \sim_d \sim_d A$ iff $\Sigma \nvdash \sim_d A$.

Then, for the purpose of establishing the result corresponding to Theorem 2, it suffices to check the following three items.

- Lemma 17, for the completeness result, i.e. if $\Gamma \models_3 A$ then $\Gamma \vdash A$.
- $\Gamma \vdash A$ then $\Gamma \models_d A$.
- Lemma 18, i.e. $\Gamma \models_4 A$ then $\Gamma \models_3 A$.

For the first item, it is enough to check the case related to negation, in particular the following case since negated formula *never* takes the value **i**, and the case when negated formula takes the value **0** is already covered by the original Lemma 17.

$$\begin{aligned} v_\Sigma(\sim C) = \mathbf{1} \text{ iff } & v_\Sigma(C) \neq \mathbf{1} && \text{by the definition of } v_\Sigma \\ \text{iff } & \Sigma \vdash \sim C && \text{by IH} \\ \text{iff } & \Sigma \nvdash \sim\sim C && \text{by Lemma 20} \end{aligned}$$

For the second item, this is immediate in view of the new truth condition for the discussive negation within the Kripke semantics.

Finally, for the third item, it is sufficient to check the case for negation, and the proof runs as follows. If A is of the form $\sim B$, then by IH,

- $I_1(B) = \mathbf{1}$ iff $I_0(B) = \mathbf{1}$ and
- $I_1(B) = \mathbf{0}$ iff $I_0(B) = \mathbf{0}$.

Then, by the truth table, it follows that $I_1(\sim B)=\mathbf{0}$ iff (by the truth table) $I_1(B)=\mathbf{1}$ iff (by IH) $I_0(B)=\mathbf{1}$ iff (by the truth table) $I_0(\sim B)=\mathbf{0}$. Moreover, $I_1(\sim B)=\mathbf{1}$ iff (by the truth table) $I_1(B)\neq\mathbf{1}$ iff (by IH) $I_0(B)\neq\mathbf{1}$ iff (by the truth table) $I_0(\sim B)=\mathbf{1}$.

Based on these, the proof of Theorem 2 can be repeated to establish the desired result.

5.4 Disjunction

One may wonder about the possibility of adding disjunction to the many-valued semantics. In the case of three-valued semantics, one can prove the completeness in a similar manner.

- First, let $\mathbf{D}_2^+$ be the expansion of $\mathbf{D}_2^-$ obtained by adding the following axiom schemata.

$$A \rightarrow_d (A \vee B) \quad \text{(Ax13)} \qquad (A \rightarrow_d C) \rightarrow_d ((B \rightarrow_d C) \rightarrow_d ((A \vee B) \rightarrow_d C)) \quad \text{(Ax15)}$$
$$B \rightarrow_d (A \vee B) \quad \text{(Ax14)} \qquad \sim(A \vee B) \leftrightarrow_d (\sim A \wedge_d^r \sim B) \quad \text{(Ax16)}$$

  The consequence relation $\vdash_{\mathbf{D}_2^+}$ is defined as before.
- Second, the three-valued truth tables for $\mathbf{D}_2^+$-*valuation* are as follows:

A	$\sim A$
1	**0**
i	**i**
0	**1**

$A \vee B$	**1**	**i**	**0**
1	**1**	**1**	**1**
i	**1**	**i**	**i**
0	**1**	**i**	**0**

$A \wedge_d^r B$	**1**	**i**	**0**
1	**1**	**1**	**0**
i	**i**	**i**	**0**
0	**0**	**0**	**0**

$A \rightarrow_d B$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**1**	**i**	**0**
0	**1**	**1**	**1**

  The designated values are **1** and **i**, and the semantic consequence relation $\models_3^+$ is defined in terms of preservation of designated values.

Then, the main result will carry over to this expansion of $\mathbf{D}_2^-$. This time, I first note here that given the proof system, the following is obtained.

Lemma 21. If Σ is maximally non-trivial, then $\Sigma \vdash_{\mathbf{D}_2^+} A \vee B$ iff $\Sigma \vdash_{\mathbf{D}_2^+} A$ or $\Sigma \vdash_{\mathbf{D}_2^+} B$.

Then, for the purpose of establishing the soundness and completeness results, the soundness is straightforward. For the completeness result, it suffices to check the additional case for Lemma 17 related to disjunction since other cases are already covered. If $B = C \vee D$, then there are the following three cases.

$$\begin{aligned} v_\Sigma(C \vee D) = \mathbf{1} &\text{ iff } v_\Sigma(C) = \mathbf{1} \text{ or } v_\Sigma(D) = \mathbf{1} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \nvdash \sim C \text{ or } \Sigma \nvdash \sim D && \text{by IH} \\ &\text{ iff } \Sigma \nvdash (\sim C \wedge \sim D) && \text{by } \Sigma \text{ is a theory} \\ &\text{ iff } \Sigma \nvdash \sim(C \vee D) && \text{by (Ax16)} \end{aligned}$$

$$\begin{aligned} v_\Sigma(C \vee D) = \mathbf{i} &\text{ iff } (v_\Sigma(C) \neq \mathbf{1} \text{ and } v_\Sigma(D) = \mathbf{i}) \text{ or} \\ &\qquad (v_\Sigma(D) \neq \mathbf{1} \text{ and } v_\Sigma(C) = \mathbf{i}) && \text{by the def. of } v_\Sigma \\ &\text{ iff } (\Sigma \vdash \sim C \text{ and } (\Sigma \vdash D \text{ and } \Sigma \vdash \sim D)) \text{ or} \\ &\qquad (\Sigma \vdash \sim D \text{ and } (\Sigma \vdash C \text{ and } \Sigma \vdash \sim C)) && \text{by IH} \\ &\text{ iff } (\Sigma \vdash C \text{ or } \Sigma \vdash D) \text{ and } \Sigma \vdash (\sim C \wedge \sim D) && \Sigma \text{ is a theory} \\ &\text{ iff } \Sigma \vdash C \vee D \text{ and } \Sigma \vdash \sim(C \vee D) && \text{by Lemma 21 and (Ax16)} \end{aligned}$$

$$\begin{aligned} v_\Sigma(C \vee D) = \mathbf{0} &\text{ iff } v_\Sigma(C) = \mathbf{0} \text{ and } v_\Sigma(D) = \mathbf{0} && \text{by the definition of } v_\Sigma \\ &\text{ iff } \Sigma \nvdash C \text{ and } \Sigma \nvdash D && \text{by IH} \\ &\text{ iff } \Sigma \nvdash (C \vee D) && \text{by Lemma 21} \end{aligned}$$

Based on these, the desired result is obtained.

Note finally that neither $\mathbf{D}_2^+$ nor $\mathbf{D}_2$ contains the other. Indeed, the following may be verified.

- $\vdash_{\mathbf{D}_2} \sim(A \vee \sim A) \rightarrow_d B$ but $\nvdash_{\mathbf{D}_2^+} \sim(A \vee \sim A) \rightarrow_d B$,
- $\vdash_{\mathbf{D}_2^+} \sim(A \vee B) \leftrightarrow_d (\sim A \wedge_d^r \sim B)$ but $\nvdash_{\mathbf{D}_2} \sim(A \vee B) \leftrightarrow_d (\sim A \wedge_d^r \sim B)$.

5.5 An application

The main result was obtained rather surprisingly by looking at the semantics for discussive logics without any aim of bridging discussive logics and many-valued logics. However, in view of the relation between discussive semantics and many-valued semantics, one may change the perspective to regard discussive semantics as a tool to make sense of some of the many-valued logics. What I have in mind here are the semantic frameworks such as Michael Dunn's relational semantics (cf. [11]), Richard and Valerie Routley's star semantics (cf. [33]), and Graham Priest's plurivalent semantics (cf. [30, 32]). These can be seen as offering alternative two-valued semantics for many-valued logics, and by doing so these frameworks offer different ways to give intuitive readings to the additional truth values, and understand the semantics for the connectives. Indeed, the first two frameworks offer alternative semantics for the four-valued logic **FDE**, and the last framework offers alternative semantics for **LP** and weak Kleene logic, among many others.[4]

In fact, the idea is already applied successfully to $\mathbf{P}^1$ of Antonio Sette which is one of the oldest three-valued paraconsistent logics introduced in [35]. More specifically, with the help of discussive semantics, one may intuitively read the three values with some discussive flavor, and moreover understand the paraconsistent negation as a negative modality. Further details, including a comparison to the so-called society semantics for $\mathbf{P}^1$ devised by Walter Carnielli and Mamede Lima-Marques in [5], can be found in [23].

What I would like to add here is one more instance that seems to offer an alternative perspective to a variant of **FDE**, called **NFL** in [37], and compare with **FDE** as well as **ETL**, introduced in [28] (see also [22]). The rest of this subsection is devoted to spell out the details. Note that the language of **FDE**, which consists of a finite set $\{\sim, \wedge, \vee\}$ of propositional connectives and a countable set Prop of propositional variables, is referred to as $\mathscr{L}_{\mathbf{FDE}}$. Moreover, as expected, the set of formulas defined as usual in $\mathscr{L}_{\mathbf{FDE}}$ is denoted by $\mathsf{Form}_{\mathbf{FDE}}$.

Definition 22. A *four-valued Belnap-Dunn-valuation* for $\mathscr{L}_{\mathbf{FDE}}$ is a homomorphism from $\mathsf{Form}_{\mathbf{FDE}}$ to $\{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$, induced by the following matrices:

A	$\sim A$
t	**f**
b	**b**
n	**n**
f	**t**

$A \vee B$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

$A \wedge B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

Then, the semantic consequence relation for **FDE**, $\models_{\mathbf{FDE}}$, is defined in terms of preservation of values **t** and **b** for all four-valued Belnap-Dunn-valuations. Moreover, the semantic consequence relations for **NFL**, $\models_{\mathbf{NFL}}$, and **ETL**, $\models_{\mathbf{ETL}}$, are defined by preserving values **t**, **b** and **n** and the value **t**, respectively, for all four-valued Belnap-Dunn-valuations.

For the purpose of presenting an alternative semantics for **NFL**, I make use of Routleys' invention.

Definition 23. A *Routley interpretation* for $\mathscr{L}_{\mathbf{FDE}}$ is a structure $\langle W, g, *, v\rangle$ where W is a set of worlds, $g \in W$, $* : W \longrightarrow W$ is a function with $w^{**} = w$, and $v : W \times \mathsf{Prop} \longrightarrow \{0,1\}$. The function v is extended to $I : W \times \mathsf{Form}_{\mathbf{FDE}} \longrightarrow \{0,1\}$ as follows:

- $I(w, p) = v(w, p)$,
- $I(w, \sim A) = 1$ iff $I(w^*, A) \neq 1$,
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$,

[4] For some of the recent discussions on this theme, see [25, 26, 27] which build heavily on [15, 14].

- $I(w, A \vee B) = 1$ iff $I(w,A) = 1$ or $I(w,B) = 1$.

Based on Routley interpretations, three consequence relations can be defined as follows.

Definition 24. For all $A, B \in \mathsf{Form}_{\mathbf{FDE}}$,

- $A \models_{*,\forall} B$ iff for all Routley interpretations $\langle W, g, *, v\rangle$, if $I(w,A) = 1$ for all $w \in W$, then $I(w,B) = 1$ for all $w \in W$.
- $A \models_{*,g} B$ iff for all Routley interpretations $\langle W, g, *, v\rangle$, if $I(g,A){=}1$, then $I(g,B){=}1$.
- $A \models_{*,\exists} B$ iff for all Routley interpretations $\langle W, g, *, v\rangle$, if $I(w,A) = 1$ for some $w \in W$, then $I(w,B) = 1$ for some $w \in W$.

Then, the following results are obtained (the second item is due to Routleys).

Theorem 3. For all $A, B \in \mathsf{Form}_{\mathbf{FDE}}$, (i) $A \models_{*,\forall} B$ iff $A \models_{\mathbf{ETL}} B$; (ii) $A \models_{*,g} B$ iff $A \models_{\mathbf{FDE}} B$; (iii) $A \models_{*,\exists} B$ iff $A \models_{\mathbf{NFL}} B$.

Proof. The strategy is exactly the same as I did for the main result of the paper. I only note that for the first item, a Hilbert-style proof system introduced in [28, §3] can be used. Therefore, I will only outline the case for the third item.

For the left-to-right direction, one should simply consider the Routley interpretations in which the cardinality of W is two. Then, by unpacking the definition of Routley interpretations, $\models_{\mathbf{NFL}}$ is obtained. For the other way around, one may use of the proof system for **NFL**, for example the one presented in [36]. Then, what remains to be done is to check the soundness, and this is tedious but not difficult. □

Remark 25. In view of the recent revival of p- and q-consequence relations (cf. [21, 12, 13]), through a series of papers by Pablo Cobreros, Paul Egré, Dave Ripley, and Robert van Rooij (e.g. [8, 9]), the above result seems to imply that Jaśkowski's idea can be exported to enrich the p- and q-consequence relations by modal vocabularies that are characterized in terms of Kripke models. Whether this is the case, and if so then how this might be developed remains to be seen, and is left as a topic for further investigations.

6 Concluding remarks

Discussive logics are often characterized as typical paraconsistent logics in which the rule of adjunction fails. The failure of adjunction is of course true for the non-discussive conjunction, but false for discussive conjunction. In fact, the negation-free fragment of $\mathscr{L}_r$ and $\mathscr{L}_l$ are both completely classical.

What I hope to have pointed out, as an application of the main result, is an aspect of discussive logics beyond the failure of adjunction. More specifically, it seems that the discussive semantics can be seen as a tool to make sense of certain many-valued semantics that may look rather difficult to have an intuitive grasp of. The key feature of the discussive semantics is this: just require one of the points in the model to force formulas in order to define the validity. Of course, the rule of adjunction will fail for non-discussive conjunction because of this key feature. But, its effect goes well beyond the failure of adjunction since one may consider discussive semantics for languages without conjunction, such as the negation-conditional fragment of $\mathbf{D}_2$. It therefore seems that there is more to discussive logics than the failure of adjunction.

Finally, building on this view of discussive logics, there seem to be a number of future directions. For instance, thanks to the simplicity of the key feature, discussive variants can be considered for a wide range of logics with Kripke models. A systematic investigation of this question from both technical as well as philosophical perspective remains to be seen. For the former, a first step is marked by Lloyd Humberstone in [16]. For the latter, the discussion by Priest on Jaina logic in [31] seems to be promising, beside the topics related to p- and q-consequence relations mentioned above.

References

[1] Ofer Arieli, Arnon Avron & Anna Zamansky (2011): *Maximal and Premaximal Paraconsistency in the Framework of Three-Valued Semantics*. Studia Logica 97, pp. 31–60, doi:10.1007/s11225-010-9296-9.

[2] Arnon Avron (1986): *On An Implication Connective of RM*. Notre Dame Journal of Formal Logic 27(2), pp. 201–209, doi:10.1305/NDJFL/1093636612.

[3] Diderik Batens & Kristof De Clercq (2004): *A Rich Paraconsistent Extension of Full Positive Logic*. Logique et Analyse 185-188, pp. 227–257.

[4] Walter Carnielli, Joao Marcos & Sandra de Amo (2000): *Formal Inconsistency and Evolutionary Databases*. Logic and Logical Philosophy 8, pp. 115–152, doi:10.12775/LLP.2000.008.

[5] Walter A Carnielli & Mamede Lima-Marques (1999): *Society semantics and multiple-valued logics*. In: Contemporary Mathematics, 235, American Mathematical Society, pp. 33–52, doi:10.1090/conm/235/03464.

[6] Janusz Ciuciura (2006): *A Quasi-Discursive System ND_2^+*. Notre Dame Journal of Formal Logic 47, pp. 371–384, doi:10.1305/ndjfl/1163775444.

[7] Janusz Ciuciura (2008): *Frontiers of the discursive logic*. Bulletin of the Section of Logic 37(2), pp. 81–92.

[8] Pablo Cobreros, Paul Egré, David Ripley & Robert van Rooij (2012): *Tolerant, classical, strict*. Journal of Philosophical Logic 41(2), pp. 347–385, doi:10.1007/s10992-010-9165-z.

[9] Pablo Cobreros, Paul Égré, David Ripley & Robert Van Rooij (2013): *Reaching transparent truth*. Mind 122(488), pp. 841–866, doi:10.1093/mind/fzt110.

[10] Newton C. A. da Costa & Lech Dubikajtis (1977): *On Jaśkowski's Discussive Logic*. In A. I. Arruda, N. C. A. da Costa & R. Chuaqui, editors: Non Classical Logics, Model Theory and Computability, North-Holland, pp. 37–56, doi:10.1016/S0049-237X(08)70644-X.

[11] Michael Dunn (1976): *Intuitive semantics for first-degree entailments and 'coupled trees'*. Philosophical studies 29(3), pp. 149–168, doi:10.1007/BF00373152.

[12] Szymon Frankowski (2004): *Formalization of a plausible inference*. Bulletin of the Section of Logic 33(1), pp. 41–52.

[13] Szymon Frankowski (2004): *p-consequence Versus q-consequence Operations*. Bulletin of the Section of Logic 33(4), pp. 197–207.

[14] Susan Haack (1978): *Philosophy of Logics*. Cambridge University Press, doi:10.1017/CBO9780511812866.

[15] Hans G. Herzberger (1973): *Dimensions of truth*. Journal of Philosophical Logic 2(4), pp. 535–556, doi:10.1007/bf00262954.

[16] Lloyd Humberstone (2008): *Modal formulas true at some point in every model*. The Australasian Journal of Logic 6, pp. 70–82, doi:10.26686/ajl.v6i0.1794.

[17] Stanisław Jaśkowski (1969): *Propositional Calculus for Contradictory Deductive Systems*. Studia Logica 24, pp. 143–157, doi:10.1007/BF02134311.

[18] Stanisław Jaśkowski (1999): *A Propositional Calculus for Inconsistent Deductive Systems*. Logic and Logical Philosophy 7, pp. 35–56, doi:10.12775/LLP.1999.003. A new translation of [17].

[19] Stanisław Jaśkowski (1999): *On the Discussive Conjunction in the Propositional Calculus for Inconsistent Deductive Systems*. Logic and Logical Philosophy 7, pp. 57–59, doi:10.12775/LLP.1999.004.

[20] Jerzy Kotas (1974): *On Quantity of Logical Values in the Discussive D_2 System and in Modular Logic*. Studia Logica 33(3), pp. 273–275, doi:doi.org/10.1007/BF02123281.

[21] Grzegorz Malinowski (1990): *Q-consequence operation*. Reports on Mathematical Logic 24, pp. 49–59.

[22] João Marcos (2011): *The value of the two values*. In J.-Y. Béziau & M.E. Coniglio, editors: *Logic without Frontiers: Festschrift for Walter Alexandre Carnielli on the occasion of his 60th birthday*, College Publication, pp. 277–294.

[23] Hitoshi Omori (2017): *Sette's Logics, Revisited*. In Alexandru Baltag, Jeremy Seligman & Tomoyuki Yamada, editors: *Proceedings of LORI 2017*, pp. 451–465, doi:10.1007/978-3-662-55665-8_31.

[24] Hitoshi Omori & Jesse Alama (2018): *Axiomatizing Jaśkowski's Discussive Logic* $\mathbf{D}_2$. *Studia Logica* 106(6), pp. 1163–1180, doi:10.1007/s11225-017-9780-6.

[25] Hitoshi Omori & Jonas R. B. Arenhart (2022): *Haack meets Herzberger and Priest*. In: *2022 IEEE 52nd International Symposium on Multiple-Valued Logic (ISMVL)*, pp. 137–144, doi:10.1109/ISMVL52857.2022.00028.

[26] Hitoshi Omori & Jonas R. B. Arenhart (2023): *Change of logic, without change of meaning*. *Theoria* 89(4), pp. 414–431, doi:10.1111/theo.12459.

[27] Hitoshi Omori & Jonas R. B. Arenhart (2024): *Is the de Finetti conditional a conditional?* *Argumenta*, doi:10.14275/2465-2334/20230.omo.

[28] Andreas Pietz & Umberto Rivieccio (2013): *Nothing but the truth*. *Journal of Philosophical Logic* 42(1), pp. 125–135, doi:10.1007/s10992-011-9215-1.

[29] Graham Priest (1979): *The Logic of Paradox*. *Journal of Philosophical Logic* 8(1), pp. 219–241, doi:10.1007/BF00258428.

[30] Graham Priest (1984): *Hyper-contradictions*. *Logique et Analyse* 27(107), pp. 237–243.

[31] Graham Priest (2008): *Jaina logic: A contemporary perspective*. *History and Philosophy of Logic* 29(3), pp. 263–278, doi:10.1080/01445340701690233.

[32] Graham Priest (2014): *Plurivalent Logics*. *The Australasian Journal of Logic* 11(1), pp. 1–13, doi:10.26686/ajl.v11i1.1830.

[33] Richard Routley & Valerie Routley (1972): *Semantics for first degree entailment*. *Noûs* 6, pp. 335–359, doi:10.2307/2214309.

[34] George F. Schumm (1975): *A Henkin-style completeness proof for the pure implicational calculus*. *Notre Dame Journal of Formal Logic* 16(3), pp. 402–404, doi:10.1305/ndjfl/1093891803.

[35] Antonio Sette (1973): *On the propositional calculus* P^1. *Mathematica Japonicae* 18(3), pp. 173–180.

[36] Yaroslav Shramko (2019): *First-Degree Entailment and Structural Reasoning*. In Hitoshi Omori & Heinrich Wansing, editors: *New Essays on Belnap-Dunn Logic*, *Synthese Library* 418, Springer, pp. 311–324, doi:10.1007/978-3-030-31136-0_18.

[37] Yaroslav Shramko, Dmitry Zaitsev & Alexander Belikov (2017): *First-Degree Entailment and its Relatives*. *Studia Logica* 105(6), pp. 1291–1317, doi:10.1007/s11225-017-9747-7.

[38] Vladimir L. Vasyukov (2001): *A New Axiomatization of Jaśkowski's Discussive Logic*. *Logic and Logical Philosophy* 9, pp. 35–46, doi:10.12775/LLP.2001.003.