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Edited by: Marta Bílková, Malvin Gattinger, Iris van der Giessen, Marianna
Girlando and Yanjing Wang

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Preface

Advances in Modal Logic (AiML) was founded in 1995 as an initiative devoted to presenting an up-to-date picture of research in modal logic and its many applications. It combines a conference series with volumes arising from the conferences, and has become the flagship international forum for work on all aspects of modal logic. Over the past three decades, AiML has both recorded and helped shape developments across the field, bringing together semantic, proof-theoretic, algebraic, topological, computational, philosophical, and applied perspectives on modal and related logics.

The first AiML conference was held in Berlin, Germany, in 1996. Since then, the series has been organized biennially, with meetings in Uppsala, Sweden, in 1998; Leipzig, Germany, in 2000, jointly with ICTL-2000; Toulouse, France, in 2002; Manchester, UK, in 2004; Noosa, Australia, in 2006; Nancy, France, in 2008; Moscow, Russia, in 2010; Copenhagen, Denmark, in 2012; Groningen, the Netherlands, in 2014; Budapest, Hungary, in 2016; Bern, Switzerland, in 2018, jointly with LATD 2018; Helsinki, Finland, in 2020, held online because of the COVID-19 pandemic; Rennes, France, in 2022, co-located with LAMAS&SR 2022; and Prague, Czech Republic, in 2024, co-located with RAMiCS 2024. Information about AiML and related events, including conference proceedings, is available at <http://www.aiml.net>.

Exactly thirty years after the first AiML conference, AiML 2026, the sixteenth conference in the series, is organized by the Institute of Logic, Language and Computation (ILLC) of the University of Amsterdam. The conference takes place in Amsterdam, the Netherlands, from 29 June to 3 July 2026. The website of AiML 2026 can be found at <https://events.illc.uva.nl/aiml2026/>.

This volume contains abstracts of invited talks and full papers accepted for the conference. Beginning with AiML 2026, the proceedings are published open access via Electronic Proceedings in Theoretical Computer Science (EPTCS).

The invited speakers at AiML 2026 are:

- Kit Fine (New York University)
- David Gabelaia (Razmadze Mathematical Institute)
- Mojtaba Mojtabehi (Ghent University)
- Aybüke Özgün (University of Amsterdam)
- Sara Uckelman (Durham University)

Out of 87 regular submissions, the Programme Committee selected 43 full papers for publication in the proceedings and regular presentation at the conference. The volume includes papers on neighborhood and relational semantics, products and fusions of modal logics, constructive and intuitionistic modal logics, tense and temporal logics, (dynamic) epistemic logic, belief revision and belief change, team and inquisitive logics, relevant and substructural modal logics, modal algebras, Gödel modal logic, probability logic, justification logic, causal modalities, topological semantics, modal measurable logics, modal logics over simplicial complexes, modal logics of metric connectedness, bundled modalities, and generalized Medvedev logics. The topics of the papers at AiML 2026 cover axiomatizability, proof theory, model theory, computational aspects, and many other aspects of modal and related logics.

In addition, 15 short presentations out of 25 submissions were accepted for presentation at the conference covering further topics in modal and related logics.

The members of the AiML 2026 Programme Committee were:

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- Yanjing Wang (Peking University, co-chair)

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- Fan Yang (Utrecht University)

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The following 103 subreviewers also assisted with the reviewing process:

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Peter Hawke, Mark Jago, Tobias Kappé, Stanislav Kikot, Johannes Kloibhofer, Miriam Kurtzhals, Serafina Lapenta, David Lehnher, Dazhu Li, Bin Liu, Abuzer Lookmanjee, Paolo Maffezoli, Krishna Balajirao Manoorkar, João Marcos, Sonia Marin, Miguel Martins, Guillaume Massas, Andrea Mazzullo, Mojtaba Mojtahedi, Valentin Müller, Yoshiki Nakamura, Sara Negri, Karl Nygren, Martin Otto, Leonardo Pacheco, Anantha Padmanabha, Fedor Pakhomov, Mina Young Pedersen, Ivo Pezlar, Dominik Pichler, Nicholas Pischke, Wolfgang Poiger, Michele Pra Baldi, Ian Pratt-Hartmann, M. Praveen, Adam Přenosil, Tikhon Pshenitsyn, Joni Puljujärvi, Davide Emilio Quadrellaro, Rojo Randrianomentsoa, Ranjitha Raviprakash, Sayantan Roy, Dmitry Rozplokhas, Saptarshi Sahoo, Sofía Santiago-Fernández, Takahiro Sawasaki, Lutz Schröder, Ian Shillito, Dmitry Shkatov, Sunil Simon, Shawn Standefer, Lutz Straßburger, Youan Su, S. P. Suresh, Mirko Tagliaferri, Tenyo Takahashi, Andrew Tedder, Masanobu Toyooka, Shogo Tsuboi, Sam van Gool, Niels Vooijs, Ruoding Wang, Wei Wang, Zach Weber, Yuanzhe Yang, Michael Zakharyashev, Lukas Zenger, Shengyang Zhong.

The organisers of the conference are grateful to the Institute of Logic, Language and Computation, the University of Amsterdam, the Association for Logic, Language and Information, Humane AI, the E. W. Beth Foundation, and the Association for Symbolic Logic for their support of AiML 2026 and its student travel awards. The organisers are especially grateful to Johan van Benthem and Yde Venema for their anniversary talks marking thirty years of the AiML conference series.

The editors of this volume would like to thank all the members of the Programme Committee and the additional reviewers for their time and effort, all authors of contributed papers and short presentations, and all invited speakers for their cooperation in the production of these proceedings.

20 June 2026

Marta Bílková
Malvin Gattinger
Iris van der Giessen
Marianna Girlando
Yanjing Wang

Invited Talks

1 Aybüke Özgün: The Method of Thought Experiments: Probability and Counterfactuals (Monday)

In this talk I will present a new account of simple counterfactual conditionals, giving acceptability and logical closure conditions for them. We find a simple counterfactual acceptable, it is argued, to the extent that (i) our probability of the consequent under the thought experiment of counterfactually supposing the antecedent is high, (ii) provided the consequent is on-topic with respect to the antecedent. We borrow the expression ‘method of thought experiments’ from Lewis, to use his label for candidate procedures for the assessment of counterfactuals. In the proposed account, counterfactual supposition is modelled via Lewisian imaging and topicality via an algebra of subject matters. A topic-sensitive probabilistic logic is then provided to reason about the acceptability of simple counterfactuals.

This is joint work with Franz Berto.

2 Kit Fine: Modal Logic in the 1960’s, 70’s and 80’s: Some Personal Reminiscences (Tuesday)

(No abstract.)

3 Mojtaba Mojtahedi: Intuitionistic Provability Logic (Wednesday)

Provability logic studies formal notions of provability through the language of modal logic, interpreting the modal operator as “provability in a fixed mathematical theory.” The celebrated theorems of Löb and Solovay show that the propositional provability logic of every sufficiently strong and sufficiently sound classical theory is exactly the Gödel–Löb logic GL. The intuitionistic setting is markedly more subtle, largely because of the existence of admissible but non-derivable rules of inference. In this talk, we survey the research program initiated by Albert Visser and developed by the Dutch school, whose central goal is a complete axiomatization and a decidability theorem for the provability logic of Heyting Arithmetic (HA). We discuss several of the key ideas and techniques involved, including NNIL formulas, unification, projectivity, and admissible rules of intuitionistic logic. We explain how these notions interact in the study of HA-provability, and how the research around them has also provided a positive impetus to the study of admissible rules in intuitionistic logic. The talk concludes with recent progress toward the complete axiomatization and decidability of HA-provability logic.

4 Sara Uckelman: What the history of modal logic can teach us about the future of modal logic (Thursday)

Modal logicians have become increasingly interested in the history of modal logic in recent decades, as there is a wealth of material in the historical development of logic that is directly relevant to our research and practice in the current day. Historical texts in modal logic grapple with the same kinds of questions that modern modal logicians are interested in, but often provide novel ways of approaching familiar problems. I've made a career out of finding these problems and solutions and presenting them to modern logicians; but today I want to argue that there are more reasons for looking to the past than merely looking for inspiration (or to settle priority claims). Understanding the history of the development of logic, generally, and modal logic, more specifically, is crucial for us to understand how our field has come to be in the state that it is in — a state which while increasingly pluralistic, expansive, and open, is nevertheless still a field which can be exclusionary to various groups of people. Where exclusionary structures and practice exist, they do not exist in a vacuum, but are rooted in historical events and developments, which means that if we wish to understand where these exclusionary structures come from and how we can get rid of them, we must understand the history of our field. It is in this way the history of modal logic can teach us about the future of modal logic. Only by looking at historical answers to “who gets to have access to logic and why” can we fully understand where we are at today, and identify ways to change where we are going tomorrow.

5 David Gabelaia: Spaces Wild and Tame: General Topological Semantics for Modal Logic (Friday)

Kripke semantics gives a compelling relational interpretation of modal languages, but ordinary Kripke frames are not fully adequate for capturing all normal modal logics. General frames overcome this limitation by enriching relational structures with a Stone topology, i.e. an algebra of admissible sets. A similar phenomenon arises in topological semantics. Topological spaces alone do not always provide a sufficiently flexible or sufficiently discriminating semantics; what is often needed is a General Space: a topological space equipped with a modal algebra of admissible sets. This talk explores the versatility of such semantic framework. Sometimes the underlying space is familiar and well behaved, while the algebra of admissible sets carries the complexity needed for broad completeness results. Sometimes, as in provability logic, the challenge is to replace highly non-constructive topologies by more natural spaces together with carefully chosen admissible sets. At the other end of the spectrum, admissible sets may be required to be geometrically or model-theoretically tame, leading e.g. to polyhedral or semialgebraic semantics. The resulting picture suggests that general spaces provide not only a robust completeness framework, but also a promising way of extracting logical invariants from topological and geometric structures.

On Modal Logics of Full Products of Neighborhood Frames

Rajab Aghamov¹, Andrey Kudinov^{2,3}, Maik Thanh Nguyen¹, Jakob Piribauer¹

¹Technische Universität Dresden, Dresden, Germany

²Higher School of Modern Mathematics, MIPT, Moscow, Russia

³HSE University, Moscow, Russia

Abstract. On the product of two neighborhood frames, three natural neighborhood functions can be defined: the horizontal one assigning to a point (x, y) the set of all supersets of sets $U \times \{y\}$, where U is a neighborhood of x ; the vertical analog; and the product neighborhood function assigning as neighborhoods all supersets of sets $U \times V$, for neighborhoods U of x and V of y .

We define the tri-modal logics $T \times_n^+ T$ and $D \times_n^+ D$ of classes of full products equipped with all three neighborhood functions of neighborhood frames validating the logic T or D ; thereby extending known product results for $S4$ and $D4$ to weaker systems. Two interaction principles arise: $(\text{sub}) = \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ and $(\text{mix}) = \Box p \rightarrow \Box_1 \Box_2 p \wedge \Box_2 \Box_1 p$, where $\Box$ for the product neighborhood function and $\Box_1, \Box_2$ the horizontal and vertical ones. Namely, we show that $T \times_n^+ T = T \otimes T \otimes T + (\text{mix})$ and $D \times_n^+ D = D \otimes D \otimes D + (\text{mix})$, where $\otimes$ denotes fusion. Notably, (sub) and (mix) are equivalent over $S4 \otimes S4 \otimes S4$ and thus $S4 \otimes S4 \otimes S4 + (\text{mix})$ axiomatizes the logic of full products of topological spaces.

Acknowledgments. The authors were supported by DFG grant 389792660 as part of TRR 248 (see <https://perspicuous-computing.science>) and by the BMBF (Federal Ministry of Education and Research) in DAAD project 57616814 (SECAI, School of Embedded and Composite AI) as part of the program Konrad Zuse Schools of Excellence in Artificial Intelligence.

1 Introduction

Neighborhood semantics generalizes relational Kripke semantics and was introduced in the 1970s by Scott [11] and Montague [8] to study weak, in particular non-normal, modal logics. A neighborhood model $M = (X, \tau, V)$ consists of a domain X , a function $\tau: X \rightarrow 2^{2^X}$ assigning to each $x \in X$ a set of neighborhoods $\tau(x) \subseteq 2^X$, and a valuation $V: AP \rightarrow 2^X$. In this work, each $\tau(x)$ is required to be a filter on X , i.e., a non-empty family of subsets of X closed under supersets and finite intersections, which ensures normality of the resulting logics [9]. The modal clause is given by $M, x \models \Box \varphi$ iff $\llbracket \varphi \rrbracket^M \in \tau(x)$, where $\llbracket \varphi \rrbracket^M = \{y \in X \mid M, y \models \varphi\}$.

Neighborhood semantics is conceptually similar to topological semantics. There are two standard interpretations: the diamond-as-closure (equivalently, box-as-interior) semantics and the diamond-as-derivation semantics; both can be viewed as special cases of neighborhood semantics. Given a topological space (Y, σ) , the closure interpretation assigns to each $y \in Y$ the family of all sets containing some open neighborhood of y . This approach was initiated by McKinsey and Tarski [7] in 1944. In the derivational interpretation, the neighborhood function induced by a topological space assigns to each $y \in Y$ the set of all supersets of $U \setminus \{y\}$, where U is an open neighborhood of y ; this semantics was later studied by Esakia [2] and Shehtman [14]. Unless stated otherwise, “topological semantics” refers to the closure interpretation in this paper.

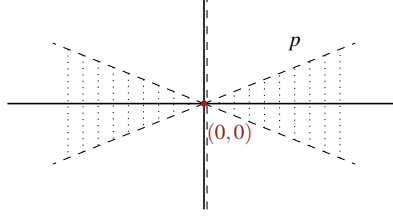


Figure 1: Regions where proposition p is true.

In topological semantics, the axioms $(T) = \Box p \rightarrow p$ and $(4) = \Box p \rightarrow \Box \Box p$ are valid. Hence, the logic of any class of topological spaces subsumes $S4$, the smallest normal modal logic containing (T) and (4) , also denoted $K + (T) + (4)$. In contrast, in neighborhood semantics both axioms can be falsified, making this framework suitable for the study of logics below $S4$. In this paper, in particular the following logics are of interest, where $(D) = \Box p \rightarrow \Diamond p$ is the seriality axiom:

$$D = K + (D), \quad D4 = K + (D) + (4), \quad T = K + (T).$$

Since (D) is derivable in T , all these logics contain (D) ; we therefore refer to them as *serial* logics.

Products of frames. When studying multi-modal systems whose modalities refer to different dimensions, such as space and time, or to different agents, it is often necessary to combine frames for uni-modal logics into a single multi-modal structure. A natural way to do so is via a product construction. Also from the perspective of epistemic logic, for instance, combining models for several agents may be achieved by taking the product of their underlying frames.

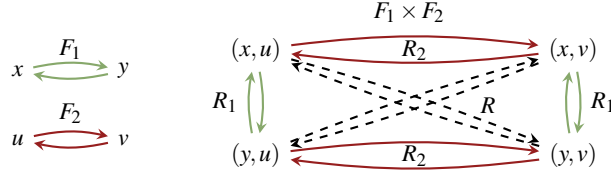
For Kripke frames, the resulting product is given by the Cartesian product of the underlying frames equipped with two accessibility relations. Such relational products were first studied by Segerberg [12] and Shehtman [13]. They validate interaction principles such as commutation (*com*) and convergence known as the Church-Rosser principle (*chr*)

$$(\text{com}) = \Box_1 \Box_2 p \leftrightarrow \Box_2 \Box_1 p \quad \text{and} \quad (\text{chr}) = \Diamond_1 \Box_2 p \rightarrow \Box_2 \Diamond_1 p.$$

In application domains such as epistemic logic, however, these interaction principles are often not desirable. *Topological semantics* provides a way to weaken this built-in interaction: Van Benthem et al. [15] introduce on $X \times Y$ the *horizontal* and *vertical* topologies obtained from the component topologies by fixing one coordinate. These topologies provide a direct analogue of the Kripke product. In neighborhood semantics, analogous products of neighborhood frames resulting in a frame with a *vertical* and *horizontal* neighborhood function were introduced by Sano in [10]. The interaction axioms (*com*) and (*chr*) may fail in products of topological spaces and consequently in the more general products of neighborhood frames. So, topological products and neighborhood products allow one to combine two information dimensions without enforcing interaction principles between the corresponding modalities a priori.

Example 1.1 ([15]). We illustrate how axioms like (*com*) can be falsified in topological semantics. As topological spaces can be viewed as neighborhood frames and the product construction coincides in both views, this also shows how (*com*) can be falsified in neighborhood semantics.

Consider the product of $\mathbb{R}$ with the standard topology σ with itself, resulting in the product $X = (\mathbb{R} \times \mathbb{R}, \sigma_1, \sigma_2)$ where the vertical topology σ_1 contains all sets of the form $U \times \{y\}$ for an open set $U \in \sigma$ and $y \in \mathbb{R}$ and the horizontal topology σ_2 similarly contains all sets of the form $\{x\} \times U$. Now,

Figure 2: Frames F_1 and F_2 and their Kripke product.

consider the valuation $V(p) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |y| < |x|\} \cup \{0\} \times \mathbb{R}$. The region where p holds is depicted in Figure 1. Then, $X, (0, 0) \models \Box_1 \Box_2 p$: For any point $(x, 0)$ in the set $(-1, 1) \times \{0\} \in \sigma_1$, we have $X, (x, 0) \models \Box_2 p$ because, for $x \neq 0$, the set $\{x\} \times (-|x|, |x|)$ is an open neighborhood of $(x, 0)$ in the horizontal topology σ_2 contained in $V(p)$ and for $x = 0$, the set $\{0\} \times (-1, 1)$ has these properties. On the other hand $X, (0, 0) \not\models \Box_2 \Box_1 p$: Any open neighborhood in σ_2 of $(0, 0)$ contains a point $(0, y)$ with $y \neq 0$. The point $(0, y)$ does not have a vertical neighborhood in σ_1 contained in $V(p)$.

Full products. In the same work [15], the authors also study the standard product topology together with the horizontal and vertical topologies, yielding a *full topological product* with three topologies. The modal logic of these products then uses $\Box_1$ and $\Box_2$ for the vertical and horizontal topologies and $\Box$ for the standard product topology. In these full products, the axiom (sub) expressing that standard neighborhoods of a state contain some vertical and horizontal neighborhoods as subsets is valid:

$$(\text{sub}) = \Box p \rightarrow \Box_1 p \wedge \Box_2 p.$$

In Kripke semantics, the analogous full product has three accessibility relations: Namely, the horizontal and vertical accessibility relations R_1 and R_2 acting only in one of the two dimensions and the *synchronous* relation R that corresponds to taking a step in both dimensions, i.e., $R = R_1 \circ R_2 = R_2 \circ R_1$. This strong connection to R_1 and R_2 leads to the fact that $\Box p \leftrightarrow \Box_1 \Box_2 p$ is valid (and $\Box p \leftrightarrow \Box_2 \Box_1 p$ then follows by the validity of (com)) and hence $\Box$ can be seen as pure syntactic sugar in this setting. Note also that, as shown in Example 1.2 below, (sub) is not valid in general on full products of Kripke frames. For reflexive Kripke frames, however, (sub) is valid on their full products.

Example 1.2. Let F_1 and F_2 be the Kripke frames depicted in Figure 2 and consider their full product $F_1 \times^+ F_2$ with horizontal, vertical and synchronous transition relations depicted in Figure 2 as well. Define $V(p) = \{(y, v)\}$. Then $F_1 \times^+ F_2, (x, u) \models \Box p$, but $F_1 \times^+ F_2, (x, u) \not\models \Box_1 p$, so (sub) fails.

In this paper, we consider the analogous full products of neighborhood frames equipped with vertical, horizontal, and the product neighborhood functions. In contrast to the topological setting, (sub) can be falsified in this framework. Indeed, Example 1.2 transfers to neighborhood semantics: passing to the associated neighborhood frames $N(F_1)$ and $N(F_2)$ (see Section 2) preserves validity of modal formulas and yields a counterexample to (sub) for full products of neighborhood frames, too.

Logics of products. We have already stated several axioms that are valid on different kinds of products of frames above. A naturally arising question is what the precise logics of products of frames are in the different settings. As a first ingredient, we need the *fusion* of logics. Given uni-modal logics Λ_1 and Λ_2 , their fusion $\Lambda_1 \otimes \Lambda_2$ is the smallest normal bi-modal logic containing Λ_1 where $\Box$ has been renamed to $\Box_1$ and Λ_2 where $\Box$ has been renamed to $\Box_2$. For three logics, Λ_1 , Λ_2 , and Λ_3 , we keep the use of $\Box$ as it is in Λ_3 , which will refer to the standard topology or product neighborhood function in the analogously defined fusion $\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3$.

Given modal logics Λ_1 and Λ_2 , we denote by $\Lambda_1 \times \Lambda_2$ the logic of the class of Kripke frames $\{F_1 \times F_2 \mid F_1 \models \Lambda_1 \text{ and } F_2 \models \Lambda_2\}$, and by $\Lambda_1 \times^+ \Lambda_2$ the logic of the corresponding full products of such frames. Likewise, we write $\Lambda_1 \times_t \Lambda_2$, $\Lambda_1 \times_t^+ \Lambda_2$, $\Lambda_1 \times_n \Lambda_2$, and $\Lambda_1 \times_n^+ \Lambda_2$ for the logic of (full) products of topological spaces (t) or neighborhood frames (n) satisfying Λ_1 and Λ_2 , respectively.

The first fundamental result on products in Kripke semantics, due to [3], states that

$$K \times K = K \otimes K + (\text{com}) + (\text{chr}).$$

Moreover, for full products the additional modality is characterized by $\Box p \leftrightarrow \Box_1 \Box_2 p$, so the logic of a full product of Kripke frames simply extends $K \otimes K + (\text{com}) + (\text{chr})$ with $\Box p \leftrightarrow \Box_1 \Box_2 p$, which describes $\Box$ in terms of $\Box_1$ and $\Box_2$.

As mentioned before, topological spaces always validate S4. The key results shown in [15] are as follows: $S4 \times_t S4 = S4 \otimes S4$ and $S4 \times_t^+ S4 = S4 \otimes S4 \otimes S4 + (\text{sub})$. Since topological spaces form a special class of neighborhood frames, we have $S4 \times_n S4 = S4 \times_t S4$ and $S4 \times_n^+ S4 = S4 \times_t^+ S4$. Thus,

$$S4 \times_n S4 = S4 \otimes S4, \quad S4 \times_n^+ S4 = S4 \otimes S4 \otimes S4 + (\text{sub}).$$

The logic of full products in derivational semantics was investigated for frames satisfying D4. The main results of [5] are $D4 \times_d D4 = D4 \otimes D4$ and $D4 \times_d^+ D4 = D4 \otimes D4 \otimes D4 + (\text{sub})$. These equalities extend analogously to neighborhood products:

$$D4 \times_n D4 = D4 \otimes D4, \quad D4 \times_n^+ D4 = D4 \otimes D4 \otimes D4 + (\text{sub}).$$

Further, for $L_1, L_2 \in \{S4, D4, D, T\}$, it was shown in [4] that $L_1 \times_n L_2 = L_1 \otimes L_2$. In this paper, we aim to extend this picture on neighborhood products of modal logics by determining the full neighborhood products $T \times_n^+ T$ and $D \times_n^+ D$.

Finally, the following result generalizes some of the results stated before: A normal logic L is called an HTC-logic (Horn preTransitive Closed logic) if it is axiomatizable by closed formulas (i.e., formulas without propositional variables) together with formulas of the form $\Box p \rightarrow \Box^n p$, $n \geq 0$. For HTC-logics L_1 and L_2 one has $L_1 \times_n L_2 = L_1 \otimes L_2 + \Delta$, where $\Delta = \{\varphi \rightarrow \Box_2 \varphi \mid \varphi \text{ closed and } \Box_2\text{-free}\} \cup \{\varphi \rightarrow \Box_1 \varphi \mid \varphi \text{ closed and } \Box_1\text{-free}\}$ [6]. If, moreover, L_1 and L_2 are serial, then $L_1 \times_n L_2 = L_1 \otimes L_2$ [6]. The previously stated results for S4, D4, D, and T are special cases of this result.

Interaction axioms (sub) and (mix). While the axiom (sub) introduced above arises naturally in the context of full topological products, its validity crucially relies on the reflexivity of the underlying frames. If one aims to move below the S4-setting by moving to neighborhood frames, a further structural interaction principle arises:

$$(\text{mix}) = \Box p \rightarrow \Box_1 \Box_2 p \wedge \Box_2 \Box_1 p.$$

Over $S4 \otimes S4 \otimes S4$, the axioms (sub) = $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$ and (mix) are equivalent as shown by the following syntactic derivation (detailed definitions used here are postponed until Section 2):

Proposition 1.3. *Over $S4 \otimes S4 \otimes S4$, the axioms (sub) = $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$ and (mix) = $\Box p \rightarrow \Box_1 \Box_2 p \wedge \Box_2 \Box_1 p$ are equivalent.*

Proof. (sub) $\Rightarrow$ (mix). Assume (sub), i.e. $\Box p \rightarrow \Box_1 p$ and $\Box p \rightarrow \Box_2 p$. From $\Box p \rightarrow \Box_1 p$, by $\Box_2$ -Necessitation and $(K_{\Box_2})$, we obtain $\Box_2 \Box p \rightarrow \Box_2 \Box_1 p$. From $\Box p \rightarrow \Box_2 p$, by substitution $\Box p$ for p , we get $\Box \Box p \rightarrow \Box_2 \Box p$. Using (4), we derive $\Box p \rightarrow \Box_2 \Box p$, and hence $\Box p \rightarrow \Box_2 \Box_1 p$. The derivation of $\Box p \rightarrow \Box_1 \Box_2 p$ is symmetric.

(mix) $\Rightarrow$ (sub). From (mix) we have $\Box p \rightarrow \Box_1 \Box_2 p$. By reflexivity $\Box_1 q \rightarrow q$, substituting $\Box_2 p$ for q , we obtain $\Box_1 \Box_2 p \rightarrow \Box_2 p$, and hence $\Box p \rightarrow \Box_2 p$. The implication $\Box p \rightarrow \Box_1 p$ is obtained symmetrically. $\square$

So, in the context of topological semantics, both axioms may be used interchangeably. Moreover, the derivation of (sub) from (mix) only use (T) and so in fact $(\text{sub}) \in T \otimes T \otimes T + (\text{mix})$, while the derivation of (mix) from (sub) only relies on (4) and so $(\text{mix}) \in K4 \otimes K4 \otimes K4 + (\text{sub})$.

Contributions. Using neighborhood semantics to leave the realm of S4-spaces, we investigate full neighborhood products for the modal logics $T = K + \Box p \rightarrow p$ and $D = K + \Box p \rightarrow \Diamond p$, extending the topological S4-case to genuinely weaker systems. Our main result is:

Theorem 1.4 (Main result). $T \times_n^+ T = T \otimes T \otimes T + (\text{mix})$ and $D \times_n^+ D = D \otimes D \otimes D + (\text{mix})$.

Hidden in this presentation is the fact that the reflexivity axiom $\Box p \rightarrow p$ on neighborhood frames leads to full products satisfying (sub) as $(\text{sub}) \in T \otimes T \otimes T + (\text{mix})$. Since the frame from Example 1.2 is serial, it immediately yields a counterexample to (sub) over non-reflexive serial neighborhood frame products showing $(\text{sub}) \notin D \times_n^+ D$.

2 Preliminaries

We briefly introduce our notation, in particular for Kripke and neighborhood semantics of modal logic.

Sequences. For a non-empty set A , we denote by A^* the set of all finite sequences over A , including the empty sequence ε . For $\vec{a} = a_1 \cdots a_k \in A^*$, we write $|\vec{a}| = k$, with $|\varepsilon| = 0$. Concatenation of sequences is denoted by $\cdot$. We identify sequences of length one with elements of A .

Syntax of modal logic. Multimodal logic with n primitive unary modal operators $(\Box_i)_{i \in \{1, \dots, n\}}$ is called n -modal logic. If $n = 1$, we call the logic uni-modal; if $n = 2$, we call it bi-modal; if $n = 3$, we call it tri-modal. Given a set of atomic propositions AP, the well-formed formulas of n -modal logic are given by the following grammar: $\varphi ::= \perp \mid p \mid \neg \varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi$ where $\perp$ is the Boolean constant false, $p \in \text{AP}$, and $i \in \{1, \dots, n\}$. We use the usual Boolean abbreviations as well as the abbreviation $\Diamond_i \varphi$ for $\neg \Box_i \neg \varphi$.

A normal n -modal logic is a set of formulas Λ that contains all propositional tautologies and the K-axiom $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$ for all $1 \leq i \leq n$ and that is closed under

- modus ponens, i.e., if $\varphi \in \Lambda$ and $\varphi \rightarrow \psi \in \Lambda$, then $\psi \in \Lambda$,
- generalization, i.e., if $\varphi \in \Lambda$, then $\Box_i \varphi \in \Lambda$ for $1 \leq i \leq n$,
- uniform substitution, i.e., if $\varphi \in \Lambda$, then also $\varphi[p \setminus \psi] \in \Lambda$ for all $p \in \text{AP}$ and all formulas ψ where $\varphi[p \setminus \psi]$ is obtained from φ by replacing all occurrences of p with ψ .

We write K_n for the smallest normal n -modal logic over $\{\Box_1, \dots, \Box_n\}$. Given a normal n -modal logic Λ and formulas $\varphi_1, \dots, \varphi_k$, we write $\Lambda + \varphi_1 + \dots + \varphi_k$ to denote the smallest normal n -modal logic containing Λ and $\varphi_1, \dots, \varphi_k$.

Fusion of logics. Let L_1 and L_2 be normal modal logics in the unimodal language with operator $\Box$. Write L'_i for the result of uniformly replacing every occurrence of $\Box$ in formulas of L_i by $\Box_i$ (for $i \in \{1, 2\}$). The fusion $L_1 \otimes L_2$ of L_1 and L_2 is the smallest normal bimodal logic over $\{\Box_1, \Box_2\}$ containing L'_1 and L'_2 . This definition generalizes straightforwardly to arbitrary finite n .

Kripke semantics. Kripke semantics of n -modal logic are given in terms of Kripke frames and models. A Kripke model is a tuple $M = (W, R_1, \dots, R_n, V)$ where W is a non-empty set of *worlds* or *states* called the *domain* or *universe*, $R_i \subseteq W \times W$ is a binary *accessibility relation* for each $i \leq n$, and $V: \text{AP} \rightarrow 2^W$ is a *valuation*. The tuple $F = (W, R_1, \dots, R_n)$ is called a *Kripke frame* and M is said to be based on F . The satisfaction relation $M, x \models \varphi$ is defined recursively by $M, x \models p \Leftrightarrow x \in V(p)$, $M, x \models \Box_i \varphi \Leftrightarrow \forall y (x R_i y \Rightarrow M, y \models \varphi)$, and the standard semantics of $\perp, \neg, \wedge$. We write $\Diamond_i \varphi := \neg \Box_i \neg \varphi$.

Products of Kripke frames. Given two uni-modal Kripke frames $F = (W, S)$ and $F' = (W', S')$ we define the relations R_1 and R_2 on $W \times W'$ as follows: for all $w, w' \in W$ and $v, v' \in W'$, we define $(w, v)R_1(w', v')$ iff wSw' and $v = v'$ as well as $(w, v)R_2(w', v')$ iff $vS'v'$ and $w = w'$. Further, we define $R = R_1 \circ R_2$, which then satisfies $(w, v)R_1(w', v')$ iff wSw' and $vS'v'$ for all $w, w' \in W$ and $v, v' \in W'$. The product of the frames is the bi-modal Kripke frame $F \times F' = (W \times W', R_1, R_2)$ and the full product is the frame $F \times^+ F' = (W \times W', R_1, R_2, R)$.

Neighborhood semantics. Let X be a non-empty set and $k \geq 1$. A k -neighborhood frame (k -n-frame) is a pair $(X, \vec{\tau})$, where $\vec{\tau} = (\tau_1, \dots, \tau_k)$ and each $\tau_i : X \rightarrow 2^{2^X}$ assigns to every $x \in X$ a filter $\tau_i(x)$ on X , i.e., a non-empty collection of subsets of X closed under supersets and finite intersections. A k -neighborhood model (k -n-model) is a triple $(X, \vec{\tau}, \nu)$, where $\nu : AP \rightarrow 2^X$ is a valuation. If $k = 1$ or it is clear from context, we simply speak of neighborhood frames and models. For $M = (X, \tau_1, \dots, \tau_k, \nu)$ and $x \in X$, satisfaction is defined by $M, x \models p$ iff $x \in \nu(p)$ and $M, x \models \Box_i \phi$ iff $\llbracket \phi \rrbracket^M \in \tau_i(x)$, where $\llbracket \phi \rrbracket^M = \{y \in X \mid M, y \models \phi\}$; the clauses for $\perp, \neg, \wedge$ are standard.

Validity and logic of frames. A formula ϕ is valid on a Kripke or neighborhood frame F , written $F \models \phi$, if $M, x \models \phi$ for all models M based on F and all worlds x of F . For a set of formulas Γ , we write $F \models \Gamma$ if $F \models \phi$ for every $\phi \in \Gamma$. The logic of a class of neighborhood or Kripke frames $\mathcal{C}$ is defined as $Log(\mathcal{C}) = \{\phi \mid F \models \phi \text{ for all } F \in \mathcal{C}\}$. If $\mathcal{C} = \{F\}$ for some F , we write $Log(F)$. We call a frame F an L -frame if $Log(F) \supseteq L$. For a set of formulas Γ , we define $\mathfrak{N}\mathfrak{F}(\Gamma) = \{F \mid F \text{ is a neighborhood frame and } F \models \Gamma\}$ and $\mathfrak{K}\mathfrak{F}(\Gamma) = \{F \mid F \text{ is a Kripke frame and } F \models \Gamma\}$.

Given a Kripke frame $F = (W, R_1, \dots, R_k)$, we let $R_i(w) = \{v \in W \mid wR_i v\}$ for $w \in W$ and $1 \leq i \leq k$ and define an k -n-frame $N(F) = (W, \tau_1, \dots, \tau_k)$ by setting $\tau_i(w) = \{U \mid R_i(w) \subseteq U \subseteq W\}$. Then, it is well-known that $Log(F) = Log(N(F))$ [9].

Neighborhood conditions for (T) and (D). We recall standard semantic characterizations of the axioms (T) and (D) on neighborhood frames (see, e.g., [9]). Let (X, τ) be a neighborhood frame. Then:

- (T): $(X, \tau) \models \Box p \rightarrow p$ iff for all $v \in X$ and all $U \in \tau(v)$, we have $v \in U$.
- (D): $(X, \tau) \models \Box p \rightarrow \Diamond p$ iff for all $v \in X$, $\emptyset \notin \tau(v)$.

Bounded morphism. Let $F = (W, R_1, \dots, R_n)$ and $F' = (W', R'_1, \dots, R'_n)$ be Kripke frames. A function $f : W \rightarrow W'$ is called a *bounded morphism* if

- for each $w, v \in W$ and $i \leq n$ with $wR_i v$, we have $f(w)R'_i f(v)$,
- for each $w \in W$, $v' \in W'$ and $i \leq n$ with $f(w)R'_i v'$, there is a $v \in W$ with $wR_i v$ and $f(v) = v'$.

Let $F = (W, \tau_1, \dots, \tau_k)$ and $F' = (W', \sigma_1, \dots, \sigma_k)$ be neighborhood frames. A function $f : W \rightarrow W'$ is called a bounded morphism if

- for all $1 \leq i \leq k$, for all $w \in W$ and all $U \in \tau_i(w)$, there is a $V \in \sigma_i(f(w))$ with $V = f(U)$,
- for all $1 \leq i \leq k$, for all $w \in W$ and all $V \in \sigma_i(f(w))$, there exists $U \in \tau_i(w)$ such that $f(U) \subseteq V$.

If there is a surjective bounded morphism between neighborhood or Kripke frames F and F' , we call F' a bounded morphic image of F and write $F \twoheadrightarrow F'$. It is well-known that $Log(\{F\}) \subseteq Log(\{F'\})$ in this case.

3 Lower bounds on logics of products of neighborhood frames

Let $\mathcal{X} = (X, \tau_1)$ and $\mathcal{Y} = (Y, \tau_2)$ be two unimodal neighborhood frames. Then the product of these two frames is a bimodal neighborhood frame $\mathcal{X} \times_n \mathcal{Y} = (X \times Y, \tau'_1, \tau'_2)$ defined as follows :

$$\begin{aligned}\tau'_1(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}, \\ \tau'_2(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}.\end{aligned}$$

Here, τ'_1 is called the *horizontal neighborhood function* and τ'_2 is called the *vertical neighborhood function*.

The full product of $\mathcal{X}$ and $\mathcal{Y}$ is the 3-n-frame $\mathcal{X} \times_n^+ \mathcal{Y} = (X \times Y, \tau'_1, \tau'_2, \tau)$, where τ'_1 and τ'_2 are the horizontal and vertical neighborhood functions, respectively, and τ is the *product neighborhood function* given by

$$\tau(x, y) = \{U \subseteq X \times Y \mid \exists W \in \tau_1(x) \exists V \in \tau_2(y) (W \times V \subseteq U)\}.$$

Definition 3.1. For unimodal logics L_1, L_2 , we define their neighborhood product and full neighborhood product: The neighborhood product is the bimodal logic

$$L_1 \times_n L_2 := \text{Log}\{\mathcal{X} \times_n \mathcal{Y} \mid \mathcal{X} \in \mathfrak{NF}(L_1), \mathcal{Y} \in \mathfrak{NF}(L_2)\}.$$

The full neighborhood product is the trimodal logic

$$L_1 \times_n^+ L_2 = \text{Log}\{\mathcal{X} \times_n^+ \mathcal{Y} \mid \mathcal{X} \in \mathfrak{NF}(L_1), \mathcal{Y} \in \mathfrak{NF}(L_2)\}.$$

We use $\Box_1$ and $\Box_2$ as modal operators for the horizontal and vertical neighborhood functions and $\Box$ as the modal operator of the product neighborhood function.

Lower bounds on the full neighborhood products $\mathsf{T} \times_n^+ \mathsf{T}$ and $\mathsf{D} \times_n^+ \mathsf{D}$. As before, we use modal operators $\Box_1$, $\Box_2$, and $\Box$ for the horizontal, vertical, and standard neighborhood functions of full products of neighborhood frames. Consequently, we also use $\Box_1$, $\Box_2$, and $\Box$ as the three modalities of a fusion $L_1 \otimes L_2 \otimes L_3$.

In this paper, we study the full neighborhood products of neighborhood frames satisfying the T-axiom $\Box p \rightarrow p$ and the D-axiom $\Box p \rightarrow \Diamond p$. Over the modal language with modal operators $\Box_1$, $\Box_2$, and $\Box$, we need the axiom (mix) = $\Box p \rightarrow \Box_1 \Box_2 p \wedge \Box_2 \Box_1 p$ as discussed in the introduction. Using $\Box_1$, $\Box_2$, and $\Box$ in the fusions, we define the logics

$$\mathsf{T}_3\mathsf{m} = \mathsf{T} \otimes \mathsf{T} \otimes \mathsf{T} + (\text{mix}) \quad \text{and} \quad \mathsf{D}_3\mathsf{m} = \mathsf{D} \otimes \mathsf{D} \otimes \mathsf{D} + (\text{mix}).$$

First, let us spell out what the logics express about Kripke frames:

Proposition 3.2. For Kripke frames $F = (W, R_1, R_2, R)$, $F \models (\text{mix})$ if and only if $R_1 \circ R_2, R_2 \circ R_1 \subseteq R$.

Proof. First assume F does not satisfy $R_1 \circ R_2, R_2 \circ R_1 \subseteq R$. Then, w.l.o.g., there are x, y such that $x(R_1 \circ R_2)y$ and $\neg xRy$. Choosing a valuation with $v(p) := R(x)$, we get $F, v, x \models \Box p$ while $F, v, x \not\models \Box_1 \Box_2 p$, showing that (mix) is not valid on F . Conversely, assume $F \not\models (\text{mix})$. So, w.l.o.g., there is a valuation v and a world x such that $F, v, x \models \Box p$, but $F, v, x \not\models \Box_1 \Box_2 p$. So, there is a world y with $x(R_1 \circ R_2)y$, but not xRy , showing that F does not satisfy $R_1 \circ R_2, R_2 \circ R_1 \subseteq R$. $\square$

As for any accessibility relation R' on a Kripke frame, the formula (T) is valid iff R' is reflexive, and (D) is valid iff R' is serial, this means the following:

Corollary 3.3. *For Kripke frames $F = (W, R_1, R_2, R)$, we have that*

1. $F \models T_3m$ iff R_1, R_2 , and R are reflexive and $R_1 \circ R_2, R_2 \circ R_1 \subseteq R$ (which implies $R_1, R_2 \subseteq R$).
2. $F \models D_3m$ iff R_1, R_2 , and R are serial and $R_1 \circ R_2, R_2 \circ R_1 \subseteq R$.

Now, we provide lower bounds for the logics of full neighborhood products (proof in the appendix).

Proposition 3.4. *We have $T \times_n^+ T \supseteq T_3m$ and $D \times_n^+ D \supseteq D_3m$.*

Proof. To show $T \times_n^+ T \supseteq T_3m$, let F_1 and F_2 be T -frames and consider their full product $F_1 \times_n^+ F_2$. Fix $(x, y) \in F_1 \times_n^+ F_2$. We first verify the axiom $\Box p \rightarrow p$. Recall the standard fact 2 that

$$F, x \models \Box p \rightarrow p \quad \text{iff} \quad x \in U \text{ for all } U \in \tau(x).$$

Now, assume $U \in \tau(x, y)$. By definition of the standard neighborhood function, there exist $V \in \tau_1(x)$ and $W \in \tau_2(y)$ with $V \times W \subseteq U$. Since $x \in V$ and $y \in W$, we obtain $(x, y) \in V \times W \subseteq U$. The axioms $\Box_1 p \rightarrow p$ and $\Box_2 p \rightarrow p$ are validated by analogous arguments.

We next verify the axiom $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$. Assume that $F_1 \times_n^+ F_2, v, (x, y) \models \Box p$. By definition, there exists a set $U \subseteq v(p)$ such that $U \supseteq V \times W$ for some $V \in \tau_1(x)$ and $W \in \tau_2(y)$. Since $y \in W$, we have $V \times \{y\} \subseteq U$, and hence $U \in \tau'_1(x, y)$. The same argument applies to $\tau'_2(x, y)$. It follows that $F_1 \times_n^+ F_2, (x, y) \models \Box_1 p \wedge \Box_2 p$.

For $\Box p \rightarrow \Box_1 \Box_2 p$, assume $F_1 \times_n^+ F_2, v, (x, y) \models \Box p$. Then there exists $U \in \tau(x, y)$ such that $U \subseteq v(p)$ and $U_1 \times U_2 \subseteq U$ for some $U_1 \in \tau_1(x)$ and $U_2 \in \tau_2(y)$. We show $F_1 \times_n^+ F_2, v, (x, y) \models \Box_1 \Box_2 p$. Since $U_1 \times \{y\} \in \tau'_1(x, y)$, it suffices to prove $U_1 \times \{y\} \subseteq v(\Box_2 p)$. Let $(v', y) \in U_1 \times \{y\}$. As $U_2 \in \tau_2(y)$, we have $\{v'\} \times U_2 \in \tau'_2(v', y)$. Moreover, $\{v'\} \times U_2 \subseteq U_1 \times U_2 \subseteq U \subseteq v(p)$. Hence $(v', y) \models \Box_2 p$. The argument for $\Box p \rightarrow \Box_2 \Box_1 p$ is symmetric.

For $D \times_n^+ D \supseteq D_3^m$, let F_1, F_2 be D -frames and fix (x, y) . To verify $\Box p \rightarrow \Diamond p$, it suffices to show $\emptyset \notin \tau(x, y)$. Let $U \in \tau(x, y)$. Then $V \times W \subseteq U$ for some $V \in \tau_1(x)$ and $W \in \tau_2(y)$. Since $F_1, F_2 \models D$, we have $V \neq \emptyset$ and $W \neq \emptyset$. Hence $U \neq \emptyset$. Thus $\emptyset \notin \tau(x, y)$. This completes the proof, as (mix) holds on all full products. $\square$

4 Upper bounds on the full neighborhood products $T \times_n^+ T$ and $D \times_n^+ D$

In this section, we now prove our main result by showing that $T \times_n^+ T \subseteq T_3m$ and that $D \times_n^+ D \subseteq D_3m$. To this end, we first define two infinite trees $T_{\omega, \omega, \omega}^{r, m}$ and $T_{\omega, \omega, \omega}^{i, m}$ with three relations and show soundness and completeness of T_3m and D_3m with respect to the respective trees (Section 4.1). Based on infinite trees with one relation for the logics T and D , we afterwards define neighborhood frames N_{ω}^r and N_{ω}^i validating T and D , respectively, based on pseudo-infinite extensions of paths in the trees (Section 4.2). The proof is then completed by providing a bounded morphism from $N_{\omega}^r \times_n^+ N_{\omega}^r$ to $N(T_{\omega, \omega, \omega}^{r, m})$ and from $N_{\omega}^i \times_n^+ N_{\omega}^i$ to $N(T_{\omega, \omega, \omega}^{i, m})$ (Section 4.3).

4.1 Completeness of T_3m and D_3m with respect to single trees

We prove that T_3m is sound and complete with respect to a single reflexive infinite tree, and that D_3m is sound and complete with respect to its irreflexive counterpart. To this end, we introduce the infinitely branching trees T_{ω}^r and T_{ω}^i of infinite depth, each equipped with a single relation, where T_{ω}^r is reflexive and T_{ω}^i is irreflexive. We show that T is complete for T_{ω}^r and D for T_{ω}^i , results that will be used in the subsequent section.

Afterwards, we define an infinitely branching, infinite-depth reflexive tree $T_{\omega,\omega,\omega}^{r,m}$ with relations R, R_1, R_2 satisfying $R_1 \circ R_2, R_2 \circ R_1 \subseteq R$, thereby validating T_3m . On the irreflexive counterpart $T_{\omega,\omega,\omega}^{i,m}$, the logic D_3m is valid. We then prove soundness and completeness of T_3m with respect to $T_{\omega,\omega,\omega}^{r,m}$ and of D_3m with respect to $T_{\omega,\omega,\omega}^{i,m}$. In [15], a similar approach based on infinite binary trees with three relations is used to establish a completeness result for $S4 \otimes S4 \otimes S4 + (\text{sub})$. Before establishing completeness, we recall the finite model property.

Definition 4.1 (Finite model property). A logic L has the *finite model property* (FMP) if every formula $\varphi \notin L$ is falsified in some finite Kripke model of L ; that is, there exist a finite model M based on a frame F with $F \models L$ and a world w such that $M, w \not\models \varphi$.

Proposition 4.2. *The logics T_3m and D_3m have the FMP.*

Proof. We first prove the result for T_3m . The same approach works for D_3m as we briefly explain at the end of the proof.

We fix a class of Kripke frames $\mathcal{C}$ w.r.t. which T_3m is sound and complete, and then obtain the FMP by filtration. Let $\mathcal{C}$ be the class of all frames $F = (W, R, R_1, R_2)$, with $F \models T_3m$. Soundness of T_3m w.r.t. $\mathcal{C}$ is immediate, and since all axioms of T_3m are Sahlqvist ([1]), the Sahlqvist theorem yields completeness of T_3m w.r.t. $\mathcal{C}$.

Let $\varphi \notin T_3m$. By completeness, there exist a T_3m -model $M = (W, R, R_1, R_2, V)$ and a world $w \in W$ such that $M, w \not\models \varphi$. Let $\Gamma := \text{Sub}(\varphi)$ be the set of subformulas of φ . A filtration through Γ itself would not guarantee that the condition $R_1 \circ R_2 \cup R_2 \circ R_1 \subseteq R$ is preserved.

To circumvent this problem, we first enlarge Γ to Γ^+ in the following way: for every $\Diamond\psi \in \Gamma$, we add $\Diamond_1\psi$ and $\Diamond_2\psi$. Let now $M^s = (W_{\Gamma^+}, R^s, R_1^s, R_2^s, V_{\Gamma^+})$ be the minimal filtration of M through Γ^+ where $W_{\Gamma^+} = W / \sim_{\Gamma^+}$ and $\sim_{\Gamma^+}$ is the usual Γ^+ -equivalence. To restore the required interaction, we redefine the $\Box$ -relation by

$$R^{s+} := R^s \cup (R_1^s \circ R_2^s) \cup (R_2^s \circ R_1^s),$$

and write $M^{s+} = (F^{s+}, V_{\Gamma^+}) = (W_{\Gamma^+}, R^{s+}, R_1^s, R_2^s, V_{\Gamma^+})$. We show that M^{s+} is still a filtration. Thus it remains to prove that for all $\Diamond\varphi \in \Gamma^+$ and all $[w], [v] \in W_{\Gamma^+}$,

$$[w] R^{s+} [v] \text{ and } M, v \models \varphi \Rightarrow M, w \models \Diamond\varphi.$$

If $[w] R^s [v]$, the claim holds as the minimal filtration M^s is a filtration. Suppose $([w], [v]) \in R_1^s \circ R_2^s$. Then there is $[u] \in W_{\Gamma^+}$ with $[w] R_1^s [u]$ and $[u] R_2^s [v]$. Choose representatives $w' \in [w]$, $u', u'' \in [u]$, $v' \in [v]$ such that $w' R_1 u'$ and $u'' R_2 v'$. Since $M, v \models \varphi$, we have $M, v' \models \varphi$, hence $M, u'' \models \Diamond_2\varphi$ and therefore also $M, u' \models \Diamond_2\varphi$ (as $u' \sim_{\Gamma^+} u''$). It follows that $M, w' \models \Diamond_1\Diamond_2\varphi$. As M validates T_3m , the axiom $\Diamond_1\Diamond_2\psi \rightarrow \Diamond\psi$ holds globally, so $M, w' \models \Diamond\varphi$. Finally, $w' \sim_{\Gamma^+} w$ yields $M, w \models \Diamond\varphi$. The case $([w], [v]) \in R_2^s \circ R_1^s$ is symmetric.

To complete the proof for T_3m , it remains to show that $F^{s+} \models T_3m$. Reflexivity of all relations is preserved by any filtration. Further, $R_1^s \circ R_2^s \subseteq R^{s+}$ and $R_2^s \circ R_1^s \subseteq R^{s+}$ hold by the definition of R^{s+} .

For D_3m , the same filtration works. As seriality is also preserved by any filtration, the constructed filtration M^{s+} when starting with a D_3m -model M is serial and satisfies $R_1^s \circ R_2^s \subseteq R^{s+}$ and $R_2^s \circ R_1^s \subseteq R^{s+}$. $\square$

Trees $T_{\omega}^r, T_{\omega}^i, T_{\omega,\omega,\omega}^{r,m}$ and $T_{\omega,\omega,\omega}^{i,m}$. Formally, $T_{\omega}^r = (W, R)$ is the Kripke frame with $W = \mathbb{N}^*$, the set of all finite sequences over $\mathbb{N}$, and sRt iff there exists $u \in \mathbb{N} \cup \{\varepsilon\}$ such that $t = s \cdot u$. The irreflexive variant is $T_{\omega}^i = (W, R')$, where $sR't$ iff there exists $u \in \mathbb{N}$ such that $t = s \cdot u$.

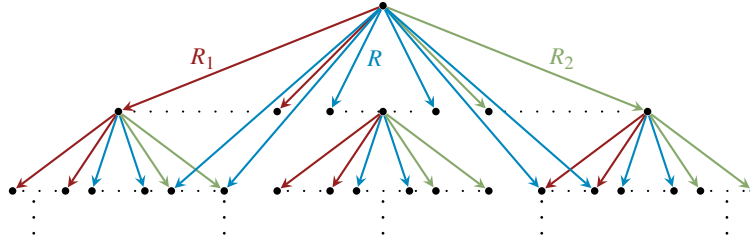


Figure 3: The tree $T_{\omega, \omega, \omega}^{r, m}$ with countably infinite R_1 -, R_2 -, and R -successors, where $R_1, R_2 \subseteq R$ and $R_1 \circ R_2, R_2 \circ R_1 \subseteq R$.

Soundness and completeness of T and D with respect to T_{ω}^r and T_{ω}^i , respectively, follow from the FMP of T and D together with a standard tree unfolding argument:

Proposition 4.3. T is sound and complete w.r.t T_{ω}^r and D is sound and complete w.r.t T_{ω}^i .

Proof. Soundness is clear in both cases. For completeness of T , we use the finite model property of T , which yields $\mathsf{T} = \text{Log}(\{F \mid F \text{ is finite and reflexive}\})$. Hence, to show $\text{Log}(T_{\omega}^r) \subseteq \mathsf{T}$, it suffices to construct a surjective bounded morphism from T_{ω}^r onto an arbitrary finite reflexive frame. Now, let $F = (W', R')$ be such a finite rooted frame with root w . We recursively define an mapping f from the states of T_{ω}^r to states of F . Initially, we assign w to the root of T_{ω}^r . Now suppose that a node $x \in T_{\omega}^r$ is mapped to a node $u = f(x) \in F$. Let $s_0, \dots, s_{k-1}$ be the successors of u , where $k \geq 1$ is ensured by reflexivity. We map the successors $x \cdot 0, x \cdot 1, x \cdot 2, \dots$ of x onto the successors of u in a cyclic manner: for $i = k \cdot m + \ell$ with $m \in \mathbb{N}$ and $\ell < k$, we set $f(x \cdot i) = s_{\ell}$. By construction, f is a bounded morphism.

Completeness of D with respect to T_{ω}^i is shown analogously. $\square$

We now turn to the tree $T_{\omega, \omega, \omega}^{r, m}$, which is also illustrated in Figure 3. At each state, R_1 and R_2 give rise to disjoint countably infinite sets of successors. The relation R strictly extends $R_1 \cup R_2$ by adding a further countably infinite set of immediate successors and by satisfying $R_1 \circ R_2 \subseteq R$ and $R_2 \circ R_1 \subseteq R$. Formally, the domain of $T_{\omega, \omega, \omega}^{r, m}$ is $W = (\mathbb{N} \times \{0, 1, 2\})^*$, where the second component distinguishes the three types of successors described above. We define $T_{\omega, \omega, \omega}^{r, m} = (W, R, R_1, R_2)$ by stipulating, for all $s, t \in W$, that

$$\begin{aligned} sR_1t & \text{ iff } t = s \cdot u \text{ for some } u \in \mathbb{N} \times \{1\} \cup \{\varepsilon\}, \\ sR_2t & \text{ iff } t = s \cdot u \text{ for some } u \in \mathbb{N} \times \{2\} \cup \{\varepsilon\}, \\ sRt & \text{ iff } t = s \cdot u \text{ for some } u \in \mathbb{N} \times \{0\} \cup \{\varepsilon\} \\ & \text{ or there exists } q \in W \text{ such that } sR_1qR_2t \text{ or } sR_2qR_1t. \end{aligned}$$

Note that by the reflexivity of R_1 and R_2 , the last condition also ensures $R_1, R_2 \subseteq R$.

The tree $T_{\omega, \omega, \omega}^{i, m}$ is defined analogously without making the relations reflexive. So, $T_{\omega, \omega, \omega}^{i, m} = (W, R', R'_1, R'_2)$ where for all $s, t \in W$, we let sR'_1t iff $t = s \cdot u$ for some $u \in \mathbb{N} \times \{1\}$; sR'_2t iff $t = s \cdot u$ for some $u \in \mathbb{N} \times \{2\}$; $sR't$ iff $t = s \cdot u$ for some $u \in \mathbb{N} \times \{0\}$ or there is a $q \in W$ with $sR'_1qR'_2t$ or $sR'_2qR'_1t$. Note that here $R'_1, R'_2 \not\subseteq R'$.

Theorem 4.4. T_{3m} is sound and complete w.r.t $T_{\omega, \omega, \omega}^{r, m}$ and D_{3m} is sound and complete w.r.t. $T_{\omega, \omega, \omega}^{i, m}$.

Proof. We start with the proof for T_{3m} . Soundness is immediate: $T_{\omega, \omega, \omega}^{r, m}$ is reflexive with respect to all three relations, and the axiom (mix) is valid since $R_1 \circ R_2 \subseteq R$ and $R_2 \circ R_1 \subseteq R$.

For completeness, we use the finite model property of T_3m . Let $F = (W', R'', R_1'', R_2'')$ be a finite rooted frame with root w such that $F \models T_3m$. We define a bounded morphism $f: T_{\omega, \omega, \omega}^{r, m} \rightarrow F$ recursively, starting with $f(\varepsilon) = w$. Suppose $f(x) = u$. Then the R_1 -successors $x \cdot (m, 1)$ and the R_2 -successors $x \cdot (m, 2)$ ($m \in \mathbb{N}$) are mapped cyclically onto the sets of R_1'' - and R_2'' -successors of u , respectively. As our logics are serial, the sets of R_1'' -, R_2'' -, and R'' -successors of u are nonempty, so the construction can proceed. For the R -successors of x , we first assign only those of the form $x \cdot (m, 0)$, mapping them cyclically onto the R'' -successors of u . The remaining R -successors, namely nodes of the form $x \cdot (m, 2) \cdot (m', 1)$ and $x \cdot (m, 1) \cdot (m', 2)$, are treated in the subsequent step of the construction.

The bounded morphism conditions for R_1 and R_2 are satisfied by construction, since the R_i -successors of any state x are mapped onto the R_i'' -successors of $f(x)$. It remains to verify the conditions for R . Let $x, y \in T_{\omega, \omega, \omega}^{r, m}$ with xRy . If $y = x \cdot (m, i)$ for some $m \in \mathbb{N}$ and $i \in \{0, 1, 2\}$, then $f(y)$ is an R'' -successor of $f(x)$: for $i = 0$ this is immediate, and for $i = 1, 2$ it follows from $R_i'' \subseteq R''$, using $F \models T_3m$. Otherwise, y is of the form $x \cdot (m, 2) \cdot (m', 1)$ or $x \cdot (m, 1) \cdot (m', 2)$ for some $m, m' \in \mathbb{N}$. Without loss of generality, assume $y = x \cdot (m, 2) \cdot (m', 1)$. Then $f(x)R_2''f(x \cdot (m, 2))$ and $f(x \cdot (m, 2))R_1''f(y)$, and since $R_2'' \circ R_1'' \subseteq R''$, we obtain $f(x)R''f(y)$.

For the back condition, let x be a state of $T_{\omega, \omega, \omega}^{r, m}$ and $t \in W'$ with $f(x)R''t$. Then, the step in the procedure mapping the R -successors of x of the form $x \cdot (m, 0)$ alternatingly to the R'' -successors of $f(x)$ ensures that $f(x \cdot (m', 0)) = t$ for some (in fact, infinitely many) $m' \in \mathbb{N}$. Surjectivity of f is immediate from the construction.

As f is a surjective bounded morphism, it follows that $\text{Log}(T_{\omega, \omega, \omega}^{r, m}) \subseteq \text{Log}(F)$. Since $F \models T_3m$ was arbitrary, we obtain $\text{Log}(T_{\omega, \omega, \omega}^{r, m}) \subseteq T_3m$, which completes the proof of completeness.

The proof for D_3m works analogously. $\square$

4.2 Neighborhood frame based on pseudo-infinite extensions of finite sequences

To relate the Kripke completeness results from the previous section to neighborhood products, we make use of some auxiliary machinery described in the sequel.

Remark 4.5. To obtain neighborhood frames F_1 and F_2 validating T with $\text{Log}(F_1 \times_n^+ F_2) = T_3^m$, we cannot use the neighborhood versions of the Kripke frame T_ω^r . Indeed, on full products of Kripke frames the interaction axioms (com) and (chr) are valid, and this validity is preserved when passing to the induced neighborhood frames $N(F)$. To falsify (com) and (chr), we therefore need neighborhood frames that genuinely do not arise from Kripke frames; a necessary condition for this is that some states do not have a smallest neighborhood. We thus introduce a neighborhood frame N_ω^r in which the neighborhoods of each state form an infinite strictly decreasing chain, so that states do not possess a smallest neighborhood. The same construction applies to the irreflexive variant N_ω^i , used in the D-case.

Pseudo-infinite sequences. Based on the tree T_ω^r whose states are finite sequences of natural numbers, we define a neighborhood frame N_ω^r whose worlds are so-called *pseudo-infinite sequences*. Constructions based on pseudo-infinite sequences are also used in [4, 5, 6]. Formally, we define the set of pseudo-infinite sequences of natural numbers as

$$X = \{a_0 a_1 a_2 \dots \in (\mathbb{N} \cup \{\star\})^\omega \mid \text{there is an } N \in \mathbb{N} \text{ with } a_k = \star \text{ for all } k \geq N\}.$$

We define the forget function $\text{forget}: X \rightarrow \mathbb{N}^*$, which removes all occurrences of $\star$ from a sequence in X . We equip X with a neighborhood function to obtain the neighborhood frame N_ω^r . Let S be the successor

relation of the tree T_ω^r , and for $\alpha = a_0a_1a_2 \dots \in X$ define:

$$\begin{aligned} st(\alpha) &= \min\{N \mid \forall k \geq N : a_k = \star\} \\ \alpha|_k &= a_0a_1 \dots a_k \text{ for } k \in \mathbb{N} \\ U_k(\alpha) &= \{\beta \in X \mid \text{forget}(\alpha) S \text{forget}(\beta) \text{ in } T_\omega^r \text{ and } \alpha|_m = \beta|_m, \text{ where } m = \max(k, st(\alpha))\} \end{aligned}$$

These sets $U_k(\alpha)$ define the neighborhoods of the point $\alpha \in X$. So, $N_\omega^r = (X, \tau)$, where the neighborhood function τ is defined by $\tau: X \rightarrow 2^{2^X}$ and

$$\tau(\alpha) = \{U \subseteq X \mid \exists k \in \mathbb{N} (U_k(\alpha) \subseteq U)\}.$$

Note that for any $\alpha \in X$ with $st(\alpha) = n$, we have $U_n(\alpha) = U_j(\alpha)$ for all $j \leq n$. Moreover, $U_k(\alpha) \subseteq U_m(\alpha)$ whenever $k \geq m$. Indeed, let $\beta \in U_k(\alpha)$. Since $\alpha|_k = \beta|_k$ and $k \geq m$, we have $\alpha|_m = \beta|_m$, hence $\beta \in U_m(\alpha)$. Thus, for each $\alpha \in X$, each $U \in \tau(\alpha)$ contains some $U_k(\alpha)$, and $U_0(\alpha) \supseteq U_1(\alpha) \supseteq \dots$. Starting at index $st(\alpha)$, the sequence becomes strictly decreasing. In particular, T is valid on N_ω^r : for every $\alpha \in X$ and every $U \in \tau(\alpha)$, we have $\alpha \in U$, since $\alpha \in U_k(\alpha) \subseteq U$ for some k .

Analogously, T_ω^i induces a neighborhood frame N_ω^i consisting of pseudo-infinite extensions of its nodes. As no state has the empty set as a neighborhood, D is valid on N_ω^i . Analogously to [4, Lemma 5.3], we obtain the following result.

Lemma 4.6. *The forget function defines a surjective bounded morphism from N_ω^r to $N(T_\omega^r)$ and from N_ω^i to $N(T_\omega^i)$. Hence $\text{Log}(N_\omega^r) = \text{T}$ and $\text{Log}(N_\omega^i) = \text{D}$.*

4.3 Surjective bounded morphism from $N_\omega^r \times_n^+ N_\omega^r$ to $N(T_{\omega,\omega,\omega}^{r,m})$ and from $N_\omega^i \times_n^+ N_\omega^i$ to $N(T_{\omega,\omega,\omega}^{i,m})$

Now, we aim to show that $\text{T} \times_n^+ \text{T} \subseteq \text{T}_3\text{m}$. We already know that $\text{T}_3\text{m} = \text{Log}(T_{\omega,\omega,\omega}^{r,m})$. Further, we can view $T_{\omega,\omega,\omega}^{r,m}$ as a neighborhood frame $N(T_{\omega,\omega,\omega}^{r,m})$ (as described in Section 2) without changing the logic. We also know that $N_\omega^r \models \text{T}$ and so $\text{T} \times_n^+ \text{T} \subseteq \text{Log}(N_\omega^r \times_n^+ N_\omega^r)$. Finally, we show $\text{Log}(N_\omega^r \times_n^+ N_\omega^r) \subseteq \text{Log}(N(T_{\omega,\omega,\omega}^{r,m}))$ by providing a surjective bounded morphism from $N_\omega^r \times_n^+ N_\omega^r$ to $N(T_{\omega,\omega,\omega}^{r,m})$.

Using the domain X and the neighborhood function of N_ω^r as in the previous section, we write

$$N_\omega^r \times_n^+ N_\omega^r = (X \times X, \tau_1, \tau_2, \tau) \quad \text{and} \quad N(T_{\omega,\omega,\omega}^{r,m}) = (W, \sigma_1, \sigma_2, \sigma)$$

where $W = (\mathbb{N} \times \{0, 1, 2\})^*$ and, for $x \in W$, the sets $\sigma_1(x)$, $\sigma_2(x)$, and $\sigma(x)$ are the collections of all supersets of the sets of R_1 -, R_2 -, and R -successors of x in $T_{\omega,\omega,\omega}^{r,m}$, respectively.

To define the bounded morphism, we first fix a bijection $h: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. Next, we define function $f: (\mathbb{N} \cup \{\star\}) \times (\mathbb{N} \cup \{\star\}) \rightarrow \mathbb{N} \times \{0, 1, 2\} \cup \{\star\}$ as follows:

$$f(a, b) = \begin{cases} \star, & \text{if } a = b = \star, \\ (a, 1), & \text{if } b = \star, a \in \mathbb{N}, \\ (b, 2), & \text{if } a = \star, b \in \mathbb{N}, \\ (h(a, b), 0), & \text{otherwise.} \end{cases}$$

Recall that X is the domain of N_ω^r . We define a function $g': X \times X \rightarrow (\mathbb{N} \times \{0, 1, 2\} \cup \{\star\})^* \cdot \star^\omega$. For $(\alpha, \beta) \in X \times X$ with $\alpha = a_0a_1 \dots$ and $\beta = b_0b_1 \dots$, we define $g'(\alpha, \beta) = f(a_0, b_0)f(a_1, b_1) \dots$.

Finally, we define $g(\alpha, \beta)$ as the sequence obtained from $g'(\alpha, \beta)$ by deleting all occurrences of $\star$. For $\alpha = a_0a_1 \dots$ and $\beta = b_0b_1 \dots$ in X , there exists $N \in \mathbb{N}$ such that $a_k = b_k = \star$ for all $k \geq N$. Hence $g'(\alpha, \beta)$ is eventually constant $\star$, and therefore $g(\alpha, \beta) \in W = (\mathbb{N} \times \{0, 1, 2\})^*$.

Lemma 4.7. *The function $g : N_\omega^r \times_n^+ N_\omega^r \rightarrow N(T_{\omega,\omega,\omega}^{r,m})$ is a surjective bounded morphism.*

Proof. First, we show that g is surjective. Let $\vec{z} = z_0 z_1 \dots z_n \in W = (\mathbb{N} \times \{0, 1, 2\})^*$. We define two sequences $\alpha = a_0 a_1 \dots a_n \star^\omega$ and $\beta = b_0 b_1 \dots b_n \star^\omega$ as follows. For $i \leq n$, we distinguish the following cases:

- If $z_i = (m, 0)$ for some $m \in \mathbb{N}$, we pick $a_i, b_i \in \mathbb{N}$ such that $h(a_i, b_i) = m$.
- If $z_i = (m, 1)$ for some $m \in \mathbb{N}$, we let $a_i = m$ and $b_i = \star$.
- If $z_i = (m, 2)$ for some $m \in \mathbb{N}$, we let $b_i = m$ and $a_i = \star$.

This choice ensures that $g(\alpha, \beta) = \vec{z}$. Hence, g is surjective.

We now verify the bounded morphism conditions for the neighborhood functions τ and σ . The cases of τ_i and σ_i for $i \in \{1, 2\}$ are analogous and simpler, as fewer case distinctions are required. To verify both conditions, we use the following claim:

Claim: For all $(\alpha, \beta) \in X \times X$ and $m > \max\{st(\alpha), st(\beta)\}$, $R(g(\alpha, \beta)) = g(U_m(\alpha) \times U_m(\beta))$.

$\supseteq$: Let $m > \max\{st(\alpha), st(\beta)\}$ and let $\zeta \in U_m(\alpha), v \in U_m(\beta)$. We show that $g(\zeta, v) \in R(g(\alpha, \beta))$. By definition, we need to find a $\vec{c} \in \mathbb{N} \times \{0, 1, 2\} \cup \{\varepsilon\}$ or $\vec{c} \in (\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\})$ or $\vec{c} \in (\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\})$ s.t. $g(\alpha, \beta) \cdot \vec{c} = g(\zeta, v)$. By the choice of $m > \max\{st(\alpha), st(\beta)\}$, we have $\alpha|_m = \zeta|_m$ and $\beta|_m = v|_m$, whence $g(\alpha, \beta) = g(\zeta|_m \star^\omega, v|_m \star^\omega)$. Furthermore, recalling that S denotes the successor relation on T_ω^r , we have $forget(\alpha) S forget(\zeta)$ and $forget(\beta) S forget(v)$. That means there exist $\vec{d}$ and $\vec{e}$, each of which is either empty (ε) or a single letter from $\mathbb{N}$, such that $forget(\alpha) \cdot \vec{d} = forget(\zeta)$ and $forget(\beta) \cdot \vec{e} = forget(v)$. Thus, the extensions $\vec{d}$ and $\vec{e}$ occur strictly beyond the prefixes $\zeta|_m$ and $v|_m$, respectively. Let ζ' and v' denote the words after $\zeta|_m$ and $v|_m$, i.e., they are of the form: $\zeta' = \star^* \vec{d} \star^\omega$ and $v' = \star^* \vec{e} \star^\omega$. Now there are six cases :

$$\begin{cases} g(\zeta', v') \in \mathbb{N} \times \{1\}, & \text{if } \vec{d} \neq \varepsilon \text{ and } \vec{e} = \varepsilon, \\ g(\zeta', v') \in \mathbb{N} \times \{2\}, & \text{if } \vec{d} = \varepsilon \text{ and } \vec{e} \neq \varepsilon, \\ g(\zeta', v') = \varepsilon, & \text{if } \vec{d} = \vec{e} = \varepsilon, \\ g(\zeta', v') \in \mathbb{N} \times \{0\}, & \text{if } \vec{d}, \vec{e} \neq \varepsilon \text{ occur at the same index in } \zeta', v', \\ g(\zeta', v') \in (\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\}), & \text{if } \vec{d}, \vec{e} \neq \varepsilon \text{ and } \vec{d} \text{ occurs in } \zeta' \text{ before } \vec{e} \text{ in } v', \\ g(\zeta', v') \in (\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\}), & \text{if } \vec{d}, \vec{e} \neq \varepsilon \text{ and } \vec{d} \text{ occurs in } \zeta' \text{ after } \vec{e} \text{ in } v'. \end{cases}$$

It follows that $g(\zeta, v) = g(\zeta|_m \star^\omega, v|_m \star^\omega) \cdot g(\zeta', v') = g(\alpha, \beta) \cdot g(\zeta', v')$. Hence, $g(\zeta, v) \in R(g(\alpha, \beta))$.

$\subseteq$: Let $\vec{w} \in R(g(\alpha, \beta))$ and $m > \max\{st(\alpha), st(\beta)\}$. We will construct $\zeta \in U_m(\alpha)$ and $v \in U_m(\beta)$ s.t. $g(\zeta, v) = \vec{w}$. By definition there exists $\vec{c} \in \mathbb{N} \times \{0, 1, 2\} \cup \{\varepsilon\}$ or $\vec{c} \in (\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\})$ or $\vec{c} \in (\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\})$ such that $g(\alpha, \beta) \cdot \vec{c} = \vec{w}$. We build $\zeta = \alpha|_m \cdot \vec{d} \star^\omega$ and $v = \beta|_m \cdot \vec{e} \star^\omega$ where $\vec{d}$ and $\vec{e}$ depend on the following cases:

$$\begin{cases} \vec{d} = \vec{c}, \vec{e} = \star, & \text{if } \vec{c} \in \mathbb{N} \times \{1\} \\ \vec{d} = \star, \vec{e} = \vec{c}, & \text{if } \vec{c} \in \mathbb{N} \times \{2\} \\ \vec{d} = \vec{e} = \star, & \text{if } \vec{c} = \varepsilon \\ \vec{d} = a, \vec{e} = b, & \text{if } \vec{c} \in \mathbb{N} \times \{0\} \text{ and } h(a, b) = \vec{c} \\ \vec{d} = a, \vec{e} = \star b, & \text{if } \vec{c} \in (\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\}) \text{ with } \vec{c} = a \cdot b \\ \vec{d} = \star a, \vec{e} = b, & \text{if } \vec{c} \in (\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\}) \text{ with } \vec{c} = b \cdot a \end{cases}$$

We have $\zeta \in U_m(\alpha)$ and $v \in U_m(\beta)$. Further, $g(\zeta, v) = g(\zeta|_m, v|_m) \cdot g(\vec{d}\star^\omega, \vec{e}\star^\omega) = g(\alpha, \beta) \cdot \vec{c} = \vec{w}$.

To verify the bounded morphism condition, let $\alpha, \beta \in X$ and $U \in \tau(\alpha, \beta)$. We need to find a $V \in \sigma(g(\alpha, \beta))$ with $V = g(U)$. Choose $m > \max\{st(\alpha), st(\beta)\}$ with $U_m(\alpha) \times U_m(\beta) \subseteq U$. Then, $R(g(\alpha, \beta)) = g(U_m(\alpha) \times U_m(\beta)) \subseteq g(U)$.

For the second condition, let $\alpha, \beta \in X$ and $V \in \sigma(g(\alpha, \beta))$, that is, $R(g(\alpha, \beta)) \subseteq V$. We need to find $U \in \tau(\alpha, \beta)$ with $g(U) \subseteq V$. Choose $m > \max\{st(\alpha), st(\beta)\}$ and set $U := U_m(\alpha) \times U_m(\beta)$. Then, $g(U) = g(U_m(\alpha) \times U_m(\beta)) = R(g(\alpha, \beta)) \subseteq V$.

The arguments for τ_1, σ_1 and τ_2, σ_2 are analogous and simpler, and are therefore omitted. $\square$

Now, we are in the position to complete the proof of our first main result.

Theorem 4.8. *We have $T \times_n^+ T = T_3m$.*

Proof. The inclusion $T \times_n^+ T \supseteq T_3m$ was established in Proposition 3.4. For the converse inclusion, since $T \subseteq \text{Log}(N_\omega^r)$, we obtain $T \times_n^+ T \subseteq \text{Log}(N_\omega^r \times_n^+ N_\omega^r)$. By Lemma 4.7, $\text{Log}(N_\omega^r \times_n^+ N_\omega^r) \subseteq \text{Log}(N(T_{\omega, \omega, \omega}^{r, m}))$, and as observed earlier, passing from $T_{\omega, \omega, \omega}^{r, m}$ to $N(T_{\omega, \omega, \omega}^{r, m})$ does not change the logic. Hence $T \times_n^+ T \subseteq \text{Log}(T_{\omega, \omega, \omega}^{r, m}) = T_3m$ by Theorem 4.4. $\square$

To establish the analogous result for $D \times_n^+ D$, we reason analogously:

Lemma 4.9. *$g : N_\omega^i \times_n^+ N_\omega^i \rightarrow N(T_{\omega, \omega, \omega}^{i, m})$ is a bounded morphism where g is as defined before.*

Proof. Let $\alpha = a_0 a_1 \dots a_n \star^\omega$ and $\beta = b_0 b_1 \dots b_l \star^\omega$. Surjectivity follows as in the previous lemma.

We claim that $R(g(\alpha, \beta)) = g(U_m(\alpha) \times U_m(\beta))$ for some $m > \max\{st(\alpha), st(\beta)\}$. This identity is used to verify the forth and back conditions for τ and σ . Fix such an $m > \max\{st(\alpha), st(\beta)\}$.

$\supseteq$: Assume $\zeta \in U_m(\alpha)$ and $v \in U_m(\beta)$. To show that $g(\zeta, v) \in R(g(\alpha, \beta))$, it suffices to find $\vec{c} \in (\mathbb{N} \times \{0\}) \cup ((\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\})) \cup ((\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\}))$, s.t. $g(\alpha, \beta) \cdot \vec{c} = g(\zeta, v)$. Since $m > \max\{st(\alpha), st(\beta)\}$, we have $g(\alpha, \beta) = g(\zeta|_m, v|_m)$. Moreover, $\text{forget}(\alpha) S \text{forget}(\zeta)$ and $\text{forget}(\beta) S \text{forget}(v)$, where S is the successor relation on T_ω^i . Thus there exist $d, e \in \mathbb{N}$ such that $\text{forget}(\alpha) \cdot d = \text{forget}(\zeta)$ and $\text{forget}(\beta) \cdot e = \text{forget}(v)$. Hence d and e occur strictly after $\zeta|_m$ and $v|_m$. We denote ζ' and v' to be the words after $\zeta|_m$ and $v|_m$ i.e. $\zeta' = \star^* d \star^\omega$ and $v' = \star^* e \star^\omega$. Then we have three cases :

$$\begin{cases} g(\zeta', v') \in \mathbb{N} \times \{0\}, & d \text{ and } e \text{ are aligned in } \zeta' \text{ and } v' \\ g(\zeta', v') \in (\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\}), & d \text{ has a lower word position in } \zeta' \text{ than } e \text{ in } v' \\ g(\zeta', v') \in (\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\}), & d \text{ has a higher word position in } \zeta' \text{ than } e \text{ in } v' \end{cases}$$

That means $g(\zeta, v) \in R(g(\alpha, \beta))$.

$\subseteq$: Let $\vec{w} \in R(g(\alpha, \beta))$. We construct $\zeta \in U_m(\alpha)$ and $v \in U_m(\beta)$ such that $g(\zeta, v) = \vec{w}$. By definition of R , there exists $\vec{c} \in (\mathbb{N} \times \{0\}) \cup ((\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\})) \cup ((\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\}))$ such that $g(\alpha, \beta) \cdot \vec{c} = \vec{w}$. Let $\zeta = \alpha|_m \cdot \vec{d}\star^\omega$ and $v = \beta|_m \cdot \vec{e}\star^\omega$, where $\vec{d}$ and $\vec{e}$ are chosen according to the following cases:

$$\begin{cases} \vec{d} = a, \vec{e} = b & \text{if } \vec{c} \in \mathbb{N} \times \{0\} \text{ and } h(a, b) = \vec{c} \\ \vec{d} = a, \vec{e} = \star b & \text{if } \vec{c} \in (\mathbb{N} \times \{1\}) \cdot (\mathbb{N} \times \{2\}) \text{ with } \vec{c} = ab \\ \vec{d} = \star a, \vec{e} = b & \text{if } \vec{c} \in (\mathbb{N} \times \{2\}) \cdot (\mathbb{N} \times \{1\}) \text{ with } \vec{c} = b \cdot a \end{cases}$$

Thus $\zeta \in U_m(\alpha)$ and $v \in U_m(\beta)$, and $g(\zeta, v) = g(\zeta|_m \star^\omega, v|_m \star^\omega) \cdot g(\vec{d}\star^\omega, \vec{e}\star^\omega) = g(\alpha, \beta) \cdot \vec{c} = \vec{w}$. $\square$

With the reasoning as above, this allows us to conclude the second main result:

Theorem 4.10. *We have $D \times_n^+ D = D_3m$.*

5 Conclusion

We identified the tri-modal logics for full products of neighborhood frames: $T \times_n^+ T = T_3m$ and $D \times_n^+ D = D_3m$. We also showed that these logics possess the finite model property and are therefore decidable. A natural direction for future work is to study the products $L_1 \times_n^+ L_2$ for $L_1, L_2 \in \{K, K4\}$. Since these logics are not serial, the techniques used in this paper do not directly apply.

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Conditionals and Modalities in Constructive Quantum Logics

Juan P. Aguilera

TU Wien, Austria
aguilera@logic.at

Guillaume Massas

Chapman University California, USA
massas@chapman.edu

We investigate logics that generalize both intuitionistic logic and quantum logic. In earlier work, we introduced Ex-logic, an extension of Holliday’s fundamental logic that coincides with the intersection of orthologic and the implication-free fragment of intuitionistic logic. In this paper, we add an implication connective to Ex-logic and axiomatize iEx-logic, the intersection of full intuitionistic logic and orthomodular logic with the implication connective interpreted as the Sasaki hook $a \rightarrow b := \neg a \vee (a \wedge b)$. As a consequence, we obtain a characterization of the lattice of logics extending iEx-logic as the product of the lattice of intermediate logics and the lattice of orthomodular logics. We also explore the robustness of our algebraic approach by briefly discussing extensions of iEx-logic with modal operators.

1 Introduction

Recently introduced by Holliday [9], fundamental logic is a propositional logic that generalizes both constructive reasoning and quantum reasoning. However, the intersection of the consequence relations on the set of formulas in the signature $\{\wedge, \vee, \neg\}$ determined by intuitionistic logic and orthologic respectively is a proper superset of the consequence relation determined by fundamental logic. Answering a question of Holliday, it was shown in [1] that the intersection of intuitionistic logic and orthologic in the signature $\{\wedge, \vee, \neg\}$ is axiomatized over fundamental logic by the following axiom (Ex):

$$\neg \left[a \wedge \left((b \wedge c) \vee (b \wedge d) \right) \right] \wedge a \wedge (c \vee e) \wedge \neg \neg f \vdash \neg \neg (a \wedge f) \\ \wedge \left[(a \wedge c) \vee (a \wedge e) \vee f \right] \wedge \left[\left(b \wedge (c \vee d) \right) \vee \neg \left(b \wedge (c \vee d) \right) \right].$$

This result motivates the study of constructive quantum logics, i.e., propositional logics extending fundamental logic + (Ex). Here, *orthologic* or *minimal quantum logic* is the logic of ortholattices and results from classical logic by removing the axiom of distributivity. The main reason for considering the signature $\{\wedge, \vee, \neg\}$ is that orthologic lacks a reasonable notion of implication (e.g., one satisfying *modus ponens*).

However, some quantum logics extending orthologic do admit reasonable forms of implication. As shown by Dalla Chiara and Giuntini [3], the so-called *Sasaki implication* defined in an ortholattice O by

$$a \rightarrow b := \neg a \vee (a \wedge b)$$

satisfies *modus ponens* if and only if O is orthomodular. The orthomodular law

$$\text{if } a \leq b, \text{ then } a \vee (\neg a \wedge b) = b$$

was already considered by Birkhoff and Von Neumann [2] as a principle for quantum logic. As implication is arguably a crucial component of intuitionistic logic, this motivates the question: *What are the*

jointly valid principles of intuitionistic logic and orthomodular logic with the Sasaki implication? Our main goal here is to answer this question. Note that, when speaking of implication in the context of orthomodular logic below, we shall always mean the Sasaki implication.

In [10], Holliday considers binary operations defined on bounded lattices that he calls preconditionals. Preconditionals are meant to provide a minimal theory of conditionals. Holliday shows that, whenever L is a bounded lattice equipped with a preconditional $\rightarrow$ that satisfies *modus ponens*, L is an orthomodular lattice if and only if $\rightarrow$ satisfies double negation elimination (i.e., the inequality $(a \rightarrow 0) \rightarrow 0 \leq a$), while L is a Heyting algebra if and only if $\rightarrow$ satisfies the *verum ex quodlibet* principle (i.e., the inequality $a \leq b \rightarrow a$). However, it is easy to see that the logic of preconditionals that satisfy *modus ponens* is weaker than the intersection of orthomodular logic and intuitionistic logic. In order to axiomatize the latter, we work in the signature $\{\wedge, \vee, \rightarrow\}$, recasting the negation axioms of fundamental logic in terms of implication via the definition $\neg a := a \rightarrow 0$, and we generalize axiom (Ex) to a new axiom (iEx) (for “*implicative* (Ex)”). Our main theorem is the following:

Theorem 1.1. *The joint validities of intuitionistic logic and orthomodular logic in the full signature $\{\vee, \wedge, \rightarrow\}$ are axiomatized by the following axioms over fundamental logic:*

1. (iEx);
2. $\neg(a \rightarrow b) = \neg(\neg a \vee (a \wedge b))$;
3. $a \wedge (a \rightarrow b) = a \wedge b$;
4. $(a \wedge b) \rightarrow b = 1$;
5. $a \rightarrow (b \wedge c) \leq a \rightarrow b$.

One might argue that the last three of these axioms merely capture the intended meaning of an implication satisfying *modus ponens*, and thus that the “content” of the logic is captured by the first two axioms. We shall also give alternate axiomatizations by decomposing (iEx) into three axioms (Nu), (Vi), and (iCl), the first two of which only involve the connectives $\{\wedge, \vee, \neg\}$ and are already valid in Ex-logic.

We proceed as follows. Section 2 starts by introducing a large class of algebras, called *fundamental algebras*, that essentially correspond to fundamental lattices in a more general signature and generalize Heyting algebras and orthomodular lattices equipped with the Sasaki implication. We focus on a particular class of fundamental algebras, which we call *Ex-algebras*, and introduce the axiom (iEx). In Section 3, we prove our main technical result, the iEx-Embedding Theorem, which establishes that any Ex-algebra subdirectly embeds into the product of an orthomodular algebra and a Heyting algebra. This result is used in Section 4 to show that iEx-logic, the logic of Ex-algebras, coincides with the intersection of intuitionistic logic and orthomodular logic and to characterize the lattice of extensions of iEx-logic. Finally, Section 5 briefly discusses adding modal operators to fundamental algebras.

2 Ex-algebras

In this section, we introduce the algebraic structures that we will be working with. The most general notion is that of a fundamental algebra, which subsumes Heyting algebras and orthomodular lattices. Below, we shall always write $\neg a$ as shorthand for $a \rightarrow 0$.

2.1 Fundamental algebras

Definition 2.1. A structure $(L, \wedge, \vee, 0, 1, \rightarrow)$ is said to be a *fundamental algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and, letting $\neg a := a \rightarrow 0$, $\neg$ is a *weak pseudo-complement* i.e., a unary operation for which the following hold:

1. *antitonicity*, i.e., $a \leq b$ implies $\neg b \leq \neg a$ for all $a, b \in L$;
2. *semi-complementation*, i.e., $a \wedge \neg a = 0$ for all $a \in L$;
3. *double-negation introduction*, i.e., $a \leq \neg\neg a$ for all $a \in L$.

The $\{\wedge, \vee, \neg\}$ -reduct of any fundamental algebra is a fundamental lattice in the sense of [1], also called a weakly pseudo-complemented lattice by Holliday in [9]. We recall that, in any fundamental lattice L , $\neg$ is dually self-adjoint, which means in particular that $\neg a = \neg\neg\neg a$ for any $a \in L$. Since we only impose requirements on the implication $\rightarrow$ when the second component is 0, fundamental algebras generalize Holliday's class of preconditionals satisfying *modus ponens* [10].

Given a weakly pseudo-complemented, bounded lattice L , a *valuation* v into L assigns elements of L to propositional variables. Valuations can be extended to arbitrary propositional formulas in the natural way. Often we abuse notation by omitting mention of v if clear from the context, and identifying formulas with elements of L . We say that $\varphi \vdash \psi$ is *valid* in L , if $\varphi \leq \psi$ holds for all valuations into L .

Fundamental logic can be defined as the logic of the $\{\wedge, \vee, \neg\}$ -reducts of fundamental algebras. Alternatively, it can be defined as the weakest *introduction-elimination* logic, i.e., the weakest logic such that for all formulas φ, ψ, χ , the following are valid:

1. $\varphi \vdash \varphi$;	8. $\varphi \vdash \neg\neg\varphi$;
2. $\perp \vdash \varphi$;	9. $\varphi \wedge \neg\varphi \vdash \perp$;
3. $\varphi \vdash \top$;	10. if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
4. $\varphi \wedge \psi \vdash \varphi$;	11. if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
5. $\varphi \wedge \psi \vdash \psi$;	12. if $\varphi \vdash \chi$ and $\psi \vdash \chi$, then $\varphi \vee \psi \vdash \chi$;
6. $\varphi \vdash \varphi \vee \psi$;	13. if $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.
7. $\psi \vdash \varphi \vee \psi$;	

Definition 2.2. An *orthomodular algebra* is a fundamental algebra $(L, \wedge, \vee, 0, 1, \neg, \rightarrow)$ in which $\neg$ satisfies double-negation elimination $\neg\neg a = a$ for all $a \in L$ and $\rightarrow$ satisfies *modus ponens* $a \wedge (a \rightarrow b) \leq b$ and the Sasaki axiom $a \rightarrow b = \neg a \vee (a \wedge b)$.

It is well known that orthomodular logic with the Sasaki implication is the logic of orthomodular algebras; see Dalla Chiara and Giuntini [3]. For intuitionistic logic, we must impose a strong interaction condition between $\wedge$ and $\rightarrow$:

Definition 2.3. A fundamental algebra $(L, \wedge, \vee, 0, 1, \rightarrow)$ is a *Heyting algebra* if it satisfies the following Residuation Law: for any $a, b, c \in L$, we have

$$(c \wedge a) \leq b \text{ if and only if } c \leq a \rightarrow b.$$

Recall that a lattice is *distributive* if meets distribute over joins, and joins distribute over meets. A standard argument shows that distributivity follows from the Residuation Law, so that a Heyting algebra is always a distributive lattice. As is also well known, intuitionistic logic is the logic of Heyting algebras. We also recall the following result, originally due to Glivenko [8]:

Theorem 2.4 (Glivenko's Theorem). *For any propositional formulas φ and ψ , if $\varphi \leq \psi$ is classically valid, then $\varphi \leq \neg\neg\psi$ is intuitionistically valid.*

We now introduce a particular class of fundamental algebras that we call *Ex-algebras*. As we shall see, Ex-algebras generalize both orthomodular algebras and Heyting algebras. In fact, we shall prove in the next section that the variety of Ex-algebras is the join variety of the variety of orthomodular algebras and the variety of Heyting algebras. We start by introducing a weaker notion.

2.2 The axiom (iEx)

Definition 2.5. A *weak Ex-algebra* is a fundamental algebra in which the following inequality is valid:

$$\left[\left(a \wedge \left((b \wedge c) \vee (b \wedge d) \right) \right) \rightarrow g \right] \wedge a \wedge (c \vee e) \wedge \neg\neg f \leq \neg\neg(a \wedge f) \quad (\text{iEx})$$

$$\wedge \left[(a \wedge c) \vee (a \wedge e) \vee f \right] \wedge \left[(b \wedge (c \vee d)) \vee \left((b \wedge (c \vee d)) \rightarrow g \right) \right].$$

Axiom (iEx) is related to the axiom (Ex) introduced in [1] and, as we shall see below, implies it. Let us first show that weak Ex-algebras generalize both orthomodular algebras and Heyting algebras.

Lemma 2.6. (iEx) is valid in orthomodular logic. Thus, every orthomodular algebra is a weak Ex-algebra.

Proof. We prove that the following strengthening of (iEx) is valid in all orthomodular algebras:

$$a \wedge \neg\neg f \leq \neg\neg(a \wedge f) \wedge f \wedge \left[(b \wedge (c \vee d)) \vee \left((b \wedge (c \vee d)) \rightarrow g \right) \right].$$

We need to show that $a \wedge \neg\neg f$ entails each of the three conjuncts in the consequent of the displayed axiom. First, we have

$$a \wedge \neg\neg f \leq a \wedge f \leq \neg\neg(a \wedge f)$$

by double-negation elimination. Similarly, we have $\neg\neg f \leq f$. Finally, the third conjunct is valid already, as, letting $p := (b \wedge (c \vee d))$, we have

$$1 = p \vee \neg p \leq p \vee (\neg p \vee (p \wedge g)) = p \vee (p \rightarrow g).$$

□

Lemma 2.7. (iEx) is valid in intuitionistic logic. Thus, every Heyting algebra is a weak Ex-algebra.

Proof. We prove that the following strengthening of (iEx) is valid in all Heyting algebras:

$$\left[\left(a \wedge \left((b \wedge c) \vee (b \wedge d) \right) \right) \rightarrow (e \wedge g) \right] \wedge a \wedge (c \vee e) \wedge \neg\neg f$$

$$\leq \neg\neg(a \wedge f) \wedge \left[(a \wedge c) \vee (a \wedge e) \right] \wedge \left[(b \wedge (c \vee d)) \rightarrow g \right].$$

Intuitionistically, we have

$$a \wedge \neg\neg f \leq \neg\neg a \wedge \neg\neg f \leq \neg\neg(a \wedge f). \quad (1)$$

By distributivity, we have

$$a \wedge (c \vee e) \leq (a \wedge c) \vee (a \wedge e). \quad (2)$$

Lastly, we derive the following:

$$\left[\left(a \wedge \left((b \wedge c) \vee (b \wedge d) \right) \right) \rightarrow (e \wedge g) \right] \wedge a \leq \left(b \wedge (c \vee d) \right) \rightarrow g. \quad (3)$$

By residuation, (3) follows from:

$$\left[\left(a \wedge \left((b \wedge c) \vee (b \wedge d) \right) \right) \rightarrow (e \wedge g) \right] \wedge a \wedge \left(b \wedge (c \vee d) \right) \leq g. \quad (4)$$

Let $p = (b \wedge c) \vee (b \wedge d)$, by distributivity and *modus ponens*, we have:

$$\begin{aligned} \left[(a \wedge p) \rightarrow (e \wedge g) \right] \wedge a \wedge \left(b \wedge (c \vee d) \right) &\leq \left[(a \wedge p) \rightarrow (e \wedge g) \right] \wedge a \wedge p \\ &\leq e \wedge g \\ &\leq g, \end{aligned}$$

as desired. Putting together (1) (2), and (3), we derive the strengthening of (iEx) stated at the beginning of the proof. $\square$

We now focus on a particular class of weak Ex-algebras.

Definition 2.8. An *Ex-algebra* is a fundamental algebra in which (iEx) and the following four axioms are valid:

$$\begin{aligned} \neg(a \rightarrow b) &= \neg(\neg a \vee (a \wedge b)) & (\text{nS}) \\ a \wedge (a \rightarrow b) &= a \wedge b & (\text{MP}) \\ (a \wedge b) \rightarrow b &= 1 & (\text{M}) \\ a \rightarrow (b \wedge c) &\leq a \rightarrow b & (\text{W4}) \end{aligned}$$

Just like weak Ex-algebras, Ex-algebras generalize both orthomodular algebras and Heyting algebras.

Lemma 2.9. Axioms (nS), (MP), (M) and (W4) are valid in all orthomodular algebras. Thus, every orthomodular algebra is an Ex-algebra.

Proof. In orthomodular lattices (nS) is equivalent to $a \rightarrow b = \neg a \vee (a \wedge b)$, which holds by definition.

We verify (MP). The inequality $a \wedge (a \rightarrow b) \leq b$ holds by the orthomodular law. Clearly we also have $a \wedge (a \rightarrow b) \leq a$, and thus $a \wedge (a \rightarrow b) \leq a \wedge b$. Conversely, we have $a \wedge b \leq a$ and $a \wedge b \leq \neg a \vee (a \wedge b) = a \rightarrow b$, and thus $a \wedge (a \rightarrow b) = a \wedge b$. For (M), we have

$$\begin{aligned} (a \wedge b) \rightarrow b &= \neg(a \wedge b) \vee ((a \wedge b) \wedge b) \\ &= \neg(a \wedge b) \vee (a \wedge b) = 1, \end{aligned}$$

as desired. Finally, we clearly have that

$$\neg a \vee (a \wedge (b \wedge c)) \leq \neg a \vee (a \wedge b),$$

from which (W4) follows. $\square$

Lemma 2.10. *Axioms (nS), (MP), (M) and (W4) are valid in all Heyting algebras. Thus, every Heyting algebra is an Ex-algebra.*

Proof. We first prove (MP). Notice that $a \wedge (a \rightarrow b) \leq a$ and $a \wedge (a \rightarrow b) \leq b$. Conversely, $a \wedge b \leq a$ and $a \wedge b \leq a \rightarrow b$, where the latter inequality follows from the fact that $a \wedge (a \wedge b) \leq b$ and residuation.

For (nS), we first observe that the equalities

$$\begin{aligned}\neg a \vee (a \wedge b) &= (\neg a \vee a) \wedge (\neg a \vee b) \\ &= \neg a \vee b \\ &= a \rightarrow b\end{aligned}$$

are classically valid, which means that $\neg(a \rightarrow b) = \neg(\neg a \vee (a \wedge b))$ is also classically valid. By Glivenko's theorem, this means that the inequalities

$$\neg(a \rightarrow b) \leq \neg\neg\neg(\neg a \vee (a \wedge b)) = \neg(\neg a \vee (a \wedge b))$$

and

$$\neg(\neg a \vee (a \wedge b)) \leq \neg\neg\neg(a \rightarrow b) = \neg(a \rightarrow b)$$

are intuitionistically valid. This shows that (nS) is valid. For (M), we have that $1 \wedge a \wedge b \leq b$, which by residuation implies $1 \leq (a \wedge b) \rightarrow b$, as desired. Finally, for (W4), by residuation, it is enough to show that $a \wedge (a \rightarrow (b \wedge c)) \leq b$. But *modus ponens* gives us $a \wedge (a \rightarrow (b \wedge c)) \leq b \wedge c \leq b$, as desired. $\square$

2.3 Properties of Ex-algebras and weak Ex-algebras

Let us now derive a number of properties of (weak) Ex-algebras, which will play an important role in the proof of our main theorem in the next section. We start by clarifying the relationship between the axioms (Ex) and (iEx).

Lemma 2.11. *The $\{\wedge, \vee, \neg\}$ -reduct of every Ex-algebra is an Ex-lattice.*

Proof. By setting $g = 0$ in (iEx), we recover (Ex), from which the claim follows. $\square$

Next, we single out three additional axioms that follow directly from (iEx).

Lemma 2.12. *The following axioms are valid in all weak Ex-algebras:*

$$(iCl) \quad \left[(a \wedge ((b \wedge c) \vee (b \wedge d))) \rightarrow g \right] \wedge a \leq (b \wedge (c \vee d)) \vee \left[(b \wedge (c \vee d)) \rightarrow g \right];$$

$$(iCl_1) \quad ((a \wedge b) \rightarrow g) \wedge a \leq b \vee (b \rightarrow g);$$

$$(iCl_2) \quad ((b \wedge c) \vee (b \wedge d)) \rightarrow g \leq (b \wedge (c \vee d)) \vee \left[(b \wedge (c \vee d)) \rightarrow g \right];$$

Proof. We prove each item in order. Substituting $e = f = 1$ in (iEx), we have

$$\left[(a \wedge ((b \wedge c) \vee (b \wedge d))) \rightarrow g \right] \wedge a \leq \neg\neg a \wedge \left[((b \wedge (c \vee d)) \vee ((b \wedge (c \vee d)) \rightarrow g)) \right],$$

which clearly implies (iCl).

Moreover, (iCl₁) follows from substituting $c = d = 1$ in (iCl), and (iCl₂) follows from substituting $a = 1$ in (iCl). $\square$

Finally, we prove two useful properties of Ex-algebras.

Lemma 2.13. *Suppose L is an Ex-algebra. Then, for all $a, b \in L$, we have*

$$a \leq b \text{ if and only if } a \rightarrow b = 1.$$

Proof. Suppose that $a \leq b$. Then by (M) we have $1 = (a \wedge b) \rightarrow b = a \rightarrow b$. Conversely, suppose that $a \rightarrow b = 1$. By (MP), we have $a \wedge b = a \wedge (a \rightarrow b) = a$, which implies $a \leq b$. $\square$

Lemma 2.14. *Suppose that L is an Ex-algebra. Then, for all $a, b \in L$, we have*

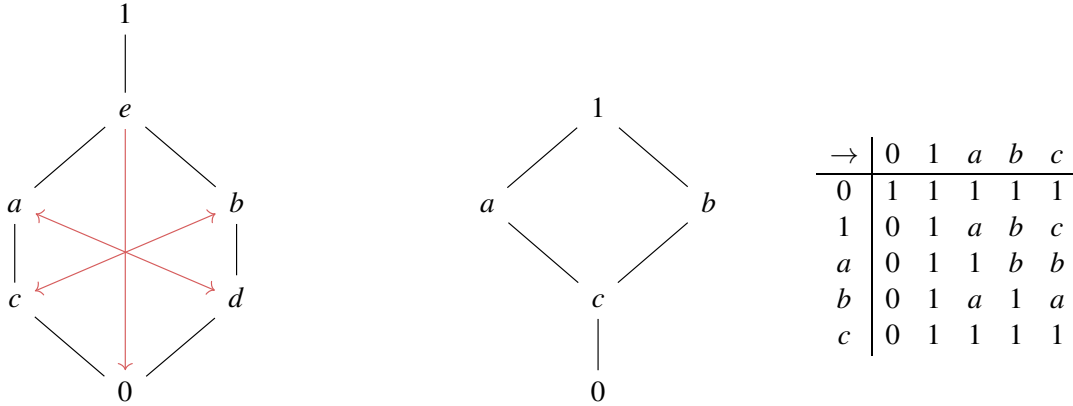
$$a \wedge b \leq a \rightarrow b.$$

Proof. By Lemma 2.13, we have that $a \wedge b \leq a \rightarrow b$ implies $1 \leq (a \wedge b) \rightarrow (a \rightarrow b)$. Now we reason as follows:

$$\begin{aligned} 1 &\leq (a \wedge b) \rightarrow (a \rightarrow b) \\ &\leq (a \wedge b) \rightarrow ((a \rightarrow b) \wedge a), && \text{by (MP)} \\ &\leq (a \wedge b) \rightarrow (a \rightarrow b), && \text{by (W4),} \end{aligned}$$

from which the inequality follows by Lemma 2.13 again. $\square$

We conclude this section with two examples. The 7-element lattice depicted on the left below is an Ex-lattice that cannot be turned into an Ex-algebra. Red arrows represent the negation operation, with the convention that $\neg 0 = 1$ and $\neg 1 = 0$. The 5-element lattice on the right, with the implication given by the adjacent table, is an Ex-algebra that is neither a Heyting algebra nor an orthomodular algebra. In particular, it has a “trivial” negation, where $\neg 0 = 1$ and $\neg x = 0$ for any $x \neq 0$.



3 The iEx-Embedding Theorem

Throughout this section, a map between fundamental algebras that preserves all the operations will be called a *fundamental algebra homomorphism*, and a map between the $\{\wedge, \vee, \neg\}$ -reducts of two fundamental algebras that preserve all three operations will be called a *fundamental homomorphism*. We are now ready to prove our main theorem. Recall that a homomorphism $f : L \rightarrow M_1 \times M_2$ is a subdirect embedding if f is injective and the composition of f with the projections $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ yields surjective homomorphisms.

Theorem 3.1 (The iEx-Embedding Theorem). *Let $(L, \wedge, \vee, 0, 1, \rightarrow)$ be an Ex-algebra. Then there are an orthomodular algebra O_L , a Heyting algebra I_L , and a subdirect fundamental algebra embedding $e : L \rightarrow O_L \times I_L$.*

The theorem will be proved using the (lattice) Ex-Embedding Theorem from [1]. There, we proved that if $(L, \wedge, \vee, 0, 1, \neg)$ is an Ex-lattice (i.e., a fundamental lattice satisfying (Ex)), then there are an ortholattice O_L , a Heyting lattice I_L , and a subdirect fundamental lattice embedding $e : L \rightarrow O_L \times I_L$.

We shall first recall this construction from [1]. Let L be an Ex-lattice. We define an ortholattice O_L and a Heyting lattice (i.e., a weakly pseudo-complemented distributive lattice) I_L which are both homomorphic images of L . An element $a \in L$ is then mapped to a pair $e(a) = (a_O, a_I) \in O_L \times I_L$ which we describe below. The ortholattice O_L is defined as a quotient of L by the equivalence relation

$$a \sim b \leftrightarrow \neg a = \neg b.$$

We shall write $[a]_O$ for the equivalence class of a (i.e., for a_O) if this helps with clarity. It was shown in [1] that O_L is indeed an ortholattice and that this equivalence relation is congruent with the operations $\wedge, \vee$, and $\neg$, i.e., that $[a \wedge b]_O = a_O \wedge b_O$, $[a \vee b]_O = a_O \vee b_O$, and $\neg a_O = [\neg a]_O$.

The Heyting lattice I_L is defined as follows. Letting $(P(L), \supseteq)$ be the set of (proper) prime filters of L ordered by reverse inclusion, we consider the $\rightarrow$ -free reduct of the Heyting algebra $Dn(P(L))$ of open subsets of $(P(L), \supseteq)$ endowed with the downset topology. We then define I_L to be the Heyting sublattice of $Dn(P(L))$ generated by sets of the form $a_I = \{P \in P(L) \mid a \in P\}$ for any $a \in L$, and show that the map $a \mapsto a_I$ is a surjective fundamental homomorphism.

We now extend these results to the setting of Ex-algebras. For the rest of this section, fix an Ex-algebra $(L, \wedge, \vee, 0, 1, \rightarrow)$. We first define O_L as the quotient of L determined by the equivalence relation $a \sim b \leftrightarrow \neg a = \neg b$ and write a_O for the equivalence class of an element a modulo $\sim$. We first claim that $\sim$ is a congruence relation on L . Because $(L, \wedge, \vee, 0, 1, \neg)$, with $\neg a$ defined as $a \rightarrow 0$, is an Ex-lattice, we know already that $\sim$ preserves the operation $\wedge, \vee$ and $\neg$. We now claim the following:

Lemma 3.2. *For all $a, b, c, d \in L$, if $a_O = c_O$ and $b_O = d_O$, then $[a \rightarrow b]_O = [c \rightarrow d]_O$.*

Proof. Suppose that $a_O = c_O$ and $b_O = d_O$. We must show that $\neg(a \rightarrow b) = \neg(c \rightarrow d)$. Because L is an Ex-algebra, it satisfies (nS). Hence we have $\neg(a \rightarrow b) = \neg(\neg a \vee (a \wedge b))$ and $\neg(c \rightarrow d) = \neg(\neg c \vee (c \wedge d))$. Since $\sim$ preserves $\wedge, \vee$ and $\neg$, we have that $a \sim c$ and $b \sim d$ entails

$$a \rightarrow b \sim \neg a \vee (a \wedge b) \sim \neg c \vee (c \wedge d) \sim c \rightarrow d,$$

as desired. This shows that $[a \rightarrow b]_O = [c \rightarrow d]_O$. □

Lemma 3.3. *$(O_L, \wedge, \vee, 0, 1, \rightarrow)$ is a homomorphic image of $(L, \wedge, \vee, 0, 1, \rightarrow)$. Moreover, it is an orthomodular algebra.*

Proof. For the first part, we know already that the map $a \mapsto a_O$ is a surjective lattice homomorphism that preserves $\neg$. Hence we only need to show that $[a \rightarrow b]_O = a_O \rightarrow b_O$ for any $a, b \in L$. Since

$$a_O \rightarrow b_O = \neg a_O \vee (a_O \wedge b_O) = [\neg a \vee (a \wedge b)]_O,$$

this amounts to showing that $a \rightarrow b \sim \neg a \vee (a \wedge b)$, which holds by (nS). For the second part, we know already that O_L is an ortholattice. By the result of Dalla Chiara and Giuntini [3], it then suffices to show

that the principle $a \wedge (\neg a \vee (a \wedge b)) \leq b$ is valid in O_L . But this is immediate from the fact that (MP) holds in L together with the first part. Hence O_L is an orthomodular algebra. $\square$

We have shown that the mapping $a \mapsto a_O$ is a (surjective) fundamental algebra homomorphism. We now turn to I_L . Following [1], we consider first the Heyting algebra $Dn(P(L))$ of open subsets of the topological space $(P(L), \supseteq)$, where $P(L)$ is the set of all (proper) prime filters on L and open sets are subsets of $P(L)$ which are downward-closed (here, a prime filter Q is below P if $P \subseteq Q$). We let $a \mapsto a_I$ maps any $a \in L$ to the set $a_I = \{P \in P(L) \mid a \in P\}$, and define I_L as the Heyting subalgebra of $Dn(P(L))$ generated by sets of the form a_I for some $a \in A$. We now claim that the map $a \mapsto a_I$ is a surjective fundamental algebra homomorphism. It is routine to check that this map preserves $\wedge$ and $\vee$, but we must show that $[a \rightarrow b]_I = a_I \rightarrow b_I$, where $a_I \rightarrow b_I$ is the Heyting implication in $Dn(P(L))$, i.e.:

$$a_I \rightarrow b_I = \{P \in P(L) \mid \forall Q((Q \supseteq P \wedge a \in Q) \rightarrow b \in Q)\}.$$

We start by proving the following restricted version of the Prime Filter Theorem.

Lemma 3.4. *Let L be an Ex-algebra, P a prime filter on L and F and I a filter and an ideal on L respectively. If $P \subseteq F$ and $F \cap I = \emptyset$, then there is a prime filter Q on L such that $F \subseteq Q$ and $Q \cap I = \emptyset$.*

Proof. Let P , F and I as in the statement of the lemma, and let Q be a maximal filter extending F and disjoint from I , which exists by Zorn's Lemma. We claim that Q is prime. To see this, suppose that $c \notin Q$ and $d \notin Q$. We want to show that $c \vee d \notin Q$. Since Q is maximal among filters disjoint from I , there must be $a \in Q$ and $b \in I$ such that $a \wedge c \leq b$ and $a \wedge d \leq b$. This means that $(a \wedge c) \vee (a \wedge d) \leq b$ and thus, by, Lemma 2.13 together with iCl2, we have

$$1 \leq ((a \wedge c) \vee (a \wedge d)) \rightarrow b \leq (a \wedge (c \vee d)) \vee ((a \wedge (c \vee d)) \rightarrow b).$$

Since $1 \in P$ and P is prime, this means that either $a \wedge (c \vee d) \in P$ or $(a \wedge (c \vee d)) \rightarrow b \in P$. In the first case, we have that $c \vee d \in P$, and hence either $c \in P \subseteq Q$ or $d \in P \subseteq Q$, contradicting our assumption. In the second case, we have that $(a \wedge (c \vee d)) \rightarrow b \in P \subseteq Q$. But then, since $a \in Q$, (MP) entails that $c \vee d \in Q$ implies $b \in Q$. Hence $c \vee d \notin Q$, and Q is prime, as desired. $\square$

The following is a straightforward application of the previous lemma.

Lemma 3.5. *Let L be an Ex-algebra. Suppose that P is a prime filter on L such that $a \rightarrow b \notin P$. Then, there is a prime filter Q extending P such that $a \in Q$ but $b \notin Q$.*

Proof. Suppose P is a prime filter not containing $a \rightarrow b$. If $a \vee (a \rightarrow b) \in P$, then $a \in P$, as P is prime. We must already have $b \notin P$, as otherwise we would have $a \rightarrow b \in P$, since $a \wedge b \leq a \rightarrow b$ by Lemma 2.14. Thus, we may assume that $a \vee (a \rightarrow b) \notin P$. We claim that $P \cup \{a\}$ generates a filter which does not contain b . If not, then there is $c \in P$ such that $a \wedge c \leq b$, so $(a \wedge c) \rightarrow b = 1$, by Lemma 2.13. Thus, $((a \wedge c) \rightarrow b) \wedge c \in P$. By (iCl₁), we have $((a \wedge c) \rightarrow b) \wedge c \leq a \vee (a \rightarrow b)$, so $a \vee (a \rightarrow b) \in P$, contrary to the assumption above.

Thus, $P \cup \{a\}$ generates a filter disjoint from the ideal generated by $\{b\}$. By Lemma 3.4, there is a prime filter Q extending $P \cup \{a\}$ that does not contain b , as desired. $\square$

Lemma 3.6. *The mapping $a \mapsto a_I$ is a surjective fundamental algebra homomorphism. Moreover, if $\neg a = \neg b$ and $a \not\leq b$, then $a_I \not\subseteq b_I$.*

Proof. For the first part, we only need to show that if $P \in P(L)$, $a \rightarrow b \in P$ if and only if for all Q extending P with $a \in Q$, we have $b \in Q$. The right-to-left direction is Lemma 3.5. The left-to-right direction follows because any such Q is a filter on L containing $a \rightarrow b$ and a , thus also $a \wedge (a \rightarrow b)$. Since $a \wedge (a \rightarrow b) \leq b$, Q also contains b . For the second part, Lemma 4.7 in [1] establishes that whenever $\neg a = \neg b$ and $a \not\leq b$ in an Ex-lattice L , there is a prime filter P over L containing a and not containing b . Since Ex-algebras are also Ex-lattices, the result applies here as well. $\square$

Our main theorem now follows: since each mapping $a \mapsto a_O$ and $a \mapsto a_I$ is a fundamental algebra homomorphism, the map $e : L \rightarrow O_L \times I_L$ given by $e(a) = (a_O, a_I)$ is also a fundamental algebra homomorphism. To see that it is injective, suppose that $a \not\leq b$. We distinguish two cases. If $\neg a \neq \neg b$ then $a_O \not\leq b_O$. Otherwise, by Lemma 3.6, we have $a_I \not\leq b_I$. Either way, it follows that $e(a) \not\leq e(b)$, as desired. Lastly, since the composition of e with the projections $\pi_1 : O_L \times I_L \rightarrow O_L$ and $\pi_2 : O_L \times I_L \rightarrow I_L$ coincides with the surjective maps $a \mapsto a_O$ and $a \mapsto a_I$ respectively, it follows that e is a subdirect embedding. This completes the proof of the iEx-Embedding Theorem.

4 Implicative Constructive Quantum Logics

In this section, we derive some consequences of the iEx-Embedding Theorem established above for constructive quantum logics equipped with an implication. We shall call such logics *implicative constructive quantum logics* (ICQ logics for short).

4.1 The Minimal ICQ Logic

We start by axiomatizing the intersection of intuitionistic logic and orthomodular logic, which can arguably be considered the minimal ICQ logic.

Theorem 4.1. *Let φ and ψ be formulas. The following are equivalent:*

1. $\varphi \vdash \psi$ is valid in all Heyting algebras and in all orthomodular algebras.
2. $\varphi \vdash \psi$ is valid in all Ex-algebras.

Proof. It was verified in Section 2.2 that every Heyting algebra and every orthomodular algebra is an Ex-algebra. Conversely, suppose that $\varphi \vdash \psi$ is not valid in all Ex-algebras, so that $v(\varphi) \not\leq v(\psi)$ holds for some Ex-algebra L and some valuation v . By the iEx-Embedding Theorem, there is an orthomodular algebra O_L a Heyting Algebra I_L and a subdirect embedding $e : L \rightarrow O_L \times I_L$. By a standard argument in algebraic logic, this implies that the entailment $\varphi \vdash \psi$ is falsifiable in either O_L or I_L , as desired. $\square$

As a consequence of Theorem 4.1, we immediately obtain:

Theorem 4.2. *The joint validities of intuitionistic logic and orthomodular logic are axiomatized over Fundamental Logic with implication by (iEx) together with the following principles:*

- (nS) $\neg(a \rightarrow b) = \neg(\neg a \vee (a \wedge b))$;
- (MP) $a \wedge (a \rightarrow b) = a \wedge b$;
- (M) $(a \wedge b) \rightarrow b = 1$;
- (W4) $a \rightarrow (b \wedge c) \leq a \rightarrow b$.

Proof. This follows from Theorem 4.1 together with the definition of Ex-algebras, the completeness of intuitionistic logic with respect to Heyting algebras, and the completeness of orthomodular logic with respect to orthomodular algebras (see Dalla Chiara and Giuntini [3]). $\square$

Analyzing the proof of the iEx-Embedding Theorem, we also obtain the following:

Theorem 4.3. *The joint validities of intuitionistic logic and orthomodular logic with implication are axiomatized over Fundamental Logic with implication by the following principles:*

- (nS) $\neg(a \rightarrow b) = \neg(\neg a \vee (a \wedge b))$.
- (MP) $a \wedge (a \rightarrow b) = a \wedge b$.
- (M) $(a \wedge b) \rightarrow b = 1$.
- (W4) $a \rightarrow (b \wedge c) \leq a \rightarrow b$.
- (iCl₁) $a \wedge ((a \wedge b) \rightarrow c) \leq b \vee (b \rightarrow c)$.
- (iCl₂) $((b \wedge c) \vee (b \wedge d)) \rightarrow a \leq (b \wedge (c \vee d)) \vee ((b \wedge (c \vee d)) \rightarrow a)$.
- (Nu) $\neg\neg a \wedge \neg\neg b \leq \neg\neg(a \wedge b)$.
- (Vi) $a \wedge (b \vee c) \wedge \neg\neg d \leq (a \wedge b) \vee (a \wedge c) \vee d$.

Proof. It was verified throughout Section 2.3 that the principles (iCl₁), (iCl₂), (Nu), and (Vi) can all be obtained as instances of (iEx).

Conversely, examining the proof of the iEx-Embedding Theorem, we track down all uses of (iEx): We see that the proof in Section 3 only makes use of the principles (MP), (nS), (iCl₁), and (iCl₂). The principles (Nu) and (Vi) are used implicitly when appealing to the lattice Ex-Embedding Theorem of [1]. Finally, the proof makes use of Lemma 2.13, which only requires (M) and (MP), and Lemma 2.14, which again only makes use of (M) and (MP), as well as (W4). $\square$

4.2 The Lattice of ICQ Logics

Next, we generalize our previous result by providing an axiomatization for the intersection $L_O \cap L_I$ of any pair of a superintuitionistic logic L_I and a superorthomodular logic L_O . As a corollary, we also obtain a characterization of the lattice of all ICQ logics, i.e., the lattice of all extensions of iEx-logic. A similar result was already established for extensions of Ex-logic, the lattice of which is isomorphic to the product of the lattice of quantum logics and the lattice of extensions of the implication-free fragment of intuitionistic logic [1, Thm. 6.7]. The proof of the result presented here follows the same strategy. We start with the following two lemmas.

Lemma 4.4. *Let L_I be a superintuitionistic logic axiomatized by $\{\varphi_i \vdash \psi_i\}_{i \in I}$ and L_O be a superorthomodular logic axiomatized by $\{\varphi_j \vdash \psi_j\}_{j \in J}$. Then $L_I \cap L_O$ is axiomatized by extending iEx-logic with the set of axioms*

$$\{\varphi_i \vdash \psi_i \vee \neg\psi_i\}_{i \in I} \cup \{\varphi_j \vdash \neg\neg\psi_j\}_{j \in J}.$$

Proof. Fix a superintuitionistic logic L_I axiomatized by $\{\varphi_i \vdash \psi_i\}_{i \in I}$ and a superorthomodular logic L_O axiomatized by $\{\varphi_j \vdash \psi_j\}_{j \in J}$. We first check that the set $\{\varphi_i \vdash \psi_i \vee \neg\psi_i\}_{i \in I} \cup \{\varphi_j \vdash \neg\neg\psi_j\}_{j \in J}$ is a set of valid axioms in $L_I \cap L_O$. Fix some $i \in I$. Clearly, L_I contains the axiom $\varphi_i \vdash \psi_i \vee \neg\psi_i$, since $\varphi_i \vdash \psi_i \in L_I$. Moreover, $\varphi_i \vdash \psi_i \vee \neg\psi_i$ follows from the excluded middle, so it also belongs to L_O . Now consider an axiom of the form $\varphi_j \vdash \neg\neg\psi_j$ for some $j \in J$. Over orthomodular logic, this axiom is equivalent to

$\varphi_j \vdash \psi_j$, hence it also belongs to L_O . Moreover, since L_O is a subclassical logic, $\varphi_j \vdash \psi_j$ is classically valid. By Glivenko's theorem, this means that $\varphi_j \vdash \neg\neg\psi_j$ is valid intuitionistically, and therefore belongs to L_I , as desired. This shows that $\{\varphi_i \vdash \psi_i \vee \neg\psi_i\}_{i \in I} \cup \{\varphi_j \vdash \neg\neg\psi_j\}_{j \in J}$ is a set of axioms in $L_I \cap L_O$.

Let now L be the extension of iEx axiomatized by this set of axioms. Clearly, we have that $L \subseteq L_I \cap L_O$. We now claim that $L_I \cap L_O \subseteq L$. To show this, suppose that $\varphi \vdash \psi$ is not valid in L . Then there is an L -lattice L such that $\varphi \vdash \psi$ is not valid in L . By the iEx-Embedding Theorem, L subdirectly embeds into $O_L \times I_L$. Since O_L is a homomorphic image of L , the axiom $\varphi_j \vdash \neg\neg\psi_j$ is valid on O_L for any $j \in J$. But since O_L is an ortholattice, this means that $\varphi_j \vdash \psi_j$ is also valid on O_L for any $j \in J$, and therefore O_L is an L_O -lattice. Similarly, we have that I_L is a homomorphic image of L , and therefore $\varphi_i \vdash \psi_i \vee \neg\psi_i$ is valid in I_L for any $i \in I$. Fix some $i \in I$. Since $\varphi_i \vdash \psi_i$ is classically valid, we have that $\vdash \neg(\varphi_i \wedge \neg\psi_i)$ is also classically valid, and thus also intuitionistically valid by Glivenko's theorem. This means that, for any valuation V on I_L , we have $V(\varphi_i) \leq V(\psi_i) \vee V(\neg\psi_i)$ and $(V(\varphi_i) \wedge \neg V(\psi_i)) \leq 0$. By distributivity, this gives us:

$$V(\varphi_i) \leq V(\varphi_i) \wedge V(\psi_i \vee \neg\psi_i) \leq (V(\varphi_i) \wedge V(\psi_i)) \vee (V(\varphi_i) \wedge \neg V(\psi_i)) \leq V(\varphi_i) \wedge V(\psi_i),$$

from which it follows that $\varphi_i \vdash \psi_i$ is valid on I_L for any $i \in I$. Hence I_L is an L_I -lattice. Since $\varphi \vdash \psi$ is not valid on L and embeds into $O_L \times I_L$, $\varphi \vdash \psi$ is not valid on $O_L \times I_L$, which means that it is not valid on either O_L or I_L . Either way, it follows that $\varphi \vdash \psi \notin L_O \cap L_I$. This shows that $L_O \cap L_I \subseteq L$, as desired. $\square$

The previous lemma shows how to axiomatize the intersection of any pair of a superorthomodular logic and a superintuitionistic logic as an extension of iEx-logic. In fact, every extension of iEx-logic arises as such an intersection of logics, as the following establishes.

Lemma 4.5. *Let L be an extension of iEx, and let $\mathbb{V}(L)$ be the variety of fundamental algebras on which L is valid. Let O_L be the logic of the class of algebras $\{O_L \mid L \in \mathbb{V}(L)\}$ and I_L be the logic of the class of algebras $\{I_L \mid L \in \mathbb{V}(L)\}$, where, for any Ex-algebra L , O_L and I_L are as in the iEx-Embedding Theorem. Then $L = O_L \cap I_L$.*

Proof. Clearly, since $O_L \models L$ and $I_L \models L$ whenever $L \models L$, we have that $L \subseteq O_L \cap I_L$. For the converse, note that, if $\varphi \vdash \psi \in O_L \cap I_L$, then $\varphi \vdash \psi$ is valid on $O_L \times I_L$ for any $L \in \mathbb{V}(L)$. By the iEx-Embedding theorem, L subdirectly embeds into $O_L \times I_L$, so $\varphi \vdash \psi$ is valid on L . This shows that $O_L \cap I_L \subseteq L$. $\square$

As a straightforward consequence of the previous two lemmas, we can provide a characterization of the lattice of extensions of iEx-logic.

Theorem 4.6. *Let $i\mathbf{SI}$ be the lattice of superintuitionistic logics, $i\mathbf{SO}$ the lattice of superorthomodular logics, and $i\mathbf{SE}$ the lattice of ICQ logics. Then $i\mathbf{SE} \simeq i\mathbf{SO} \times i\mathbf{SI}$.*

Proof. Consider the map $(L_O, L_I) \mapsto L_O \cap L_I$ from the lattice $i\mathbf{SO} \times i\mathbf{SI}$ to the lattice $i\mathbf{SE}$. Clearly, this map is monotone, and it is surjective by Lemma 4.5. Hence, it is an isomorphism if we can also show injectivity. Suppose that $L_I \subseteq L'_I \in i\mathbf{SI}$ and $L_O \subseteq L'_O \in i\mathbf{SO}$ are such that $(L_O, L_I) \not\leq_{i\mathbf{SO} \times i\mathbf{SI}} (L'_O, L'_I)$. Then either $L'_O \not\subseteq L_O$ or $L'_I \not\subseteq L_I$. In the first case, this means that there is an L_O -orthomodular algebra O and an axiom $\varphi \vdash \psi \in L'_O$ that is not valid on O . By Lemma 4.4, we have that $\varphi \vdash \neg\neg\psi$ is not valid on O , yet $\varphi \vdash \neg\neg\psi \in L'_O \cap L'_I$. In the second case, there is a L_I -Heyting algebra A and an axiom $\varphi \vdash \psi$ that is not valid on A . By 4.4 again, we have that $\varphi \vdash \psi \vee \neg\psi$ is not valid on A , yet $\varphi \vdash \psi \vee \neg\psi \in L'_O \cap L'_I$. Either way, this shows that $L'_O \cap L'_I \not\subseteq L_O \cap L_I$, as desired. $\square$

5 Adding Modalities

In this final section, we consider extensions of fundamental algebras with modal operators. In particular, we are interested in using the iEx-Embedding Theorem to axiomatize the intersection of a reasonable intuitionistic modal logic and a reasonable quantum modal logic. Intuitionistic modal logic is a sprawling and fast growing field [14, 15]. Prominent candidates to the title of “minimal intuitionistic modal logic” include Wijesekera’s logic CK [17] and Fischer Servi’s logic IK [7]. The field of quantum modal logics is comparatively less explored [6, 12, 16]. Here, we do not intend to make any claim regarding minimal intuitionistic modal logic or minimal quantum modal logic, but we are rather interested in logics whose intersection can be axiomatized using the techniques presented here. We leave a more thorough investigation of constructive quantum modal logics, e.g. along the lines of Holliday’s fundamental modal logic [11], for future work.

From the point of view of the iEx-Embedding Theorem, the addition of modalities to the signature of fundamental algebras introduces some significant difficulties. In particular, the representation of a fundamental modal algebra as a subalgebra of the product of a quantum modal algebra and an intuitionistic modal algebra obtained as quotients of the original algebra is only possible if the quotient maps preserve the modal operators. Here, we run into some significant issues. In algebras corresponding to Fischer Servi’s logic IK (and, *a fortiori*, in algebras corresponding to the weaker logic CK), it is not always the case that the equivalence relation $\sim$ defined above preserves $\Box$. In other words, $\neg a = \neg b$ does not always imply that $\neg\Box a = \neg\Box b$. Similarly, the proof that the map $a \mapsto a_I$ also preserves modal operators requires a stronger version of the restricted Prime Filter Theorem 3.4 that may not hold in modal orthomodular lattices. Here, we overcome these issues by restricting ourselves to classes of algebras in which they do not arise, leaving the development of a more elegant solution for future work.

We start with the following definitions.

Definition 5.1. A $\Box$ -involutive Fischer Servi algebra (*IFS algebra* for short) is a tuple $(A, \wedge, \vee, 0, 1, \rightarrow, \Box, \Diamond)$ such that $(A, \wedge, \vee, 0, 1, \rightarrow)$ is a Heyting algebra and $\Box, \Diamond$ are unary operators satisfying the following:

$$(M1) \quad \Box 1 = 1, \Box(a \wedge b) = \Box a \wedge \Box b;$$

$$(M2) \quad \Diamond 0 = 0, \Diamond(a \vee b) = \Diamond a \vee \Diamond b;$$

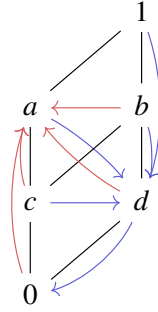
$$(FS1) \quad \Diamond a \rightarrow \Box b \leq \Box(a \rightarrow b);$$

$$(FS2) \quad \Diamond(a \rightarrow b) \leq \Box a \rightarrow \Diamond b;$$

$$(I1) \quad 1 \leq \Box(\neg\neg a \rightarrow a).$$

The class of modal Heyting algebras satisfying the first four axioms corresponds to Fischer Servi’s logic IK. As mentioned above, axiom (I1) is need to guarantee that $\neg a = \neg b$ implies $\neg\Box a = \neg\Box b$, as evidenced by the Fischer Servi algebra depicted below, where $\neg 1 = \neg b$, but $\Box\neg 1 \neq \Box\neg b$. Blue and red

arrows respectively represent the operators $\Diamond$ and $\Box$:



Definition 5.2. A $\Diamond$ -distributive orthomodular algebra (DOM algebra for short) is a tuple $(A, \wedge, \vee, 0, 1, \rightarrow, \Box, \Diamond)$ such that $(A, \wedge, \vee, 0, 1, \rightarrow)$ is an orthomodular algebra and $\Box, \Diamond$ are unary operators satisfying axioms (M1) and (M2) as well as the following:

$$(M3) \quad \Diamond \neg a = \neg \Box a;$$

$$(D1) \quad \Diamond a \wedge \Box b \leq \Diamond(a \wedge b);$$

$$(D2) \quad \Diamond(a \wedge (b \vee c)) \leq \Diamond(a \wedge b) \vee \Diamond(a \wedge c).$$

Here again, the definition is more restrictive than in standard modal orthologic (see, e.g., [12]). Axiom (D1) is a principle of positive modal logic, and (D2) amounts to assuming a form of “modal distributivity” which we will need later on. We now define the class of fundamental modal algebras that we will focus on from now on.

Definition 5.3. A modal iEx algebra (miEx algebra for short) is a tuple $(A, \wedge, \vee, 0, 1, \rightarrow, \Box, \Diamond)$ such that $(A, \wedge, \vee, 0, 1, \rightarrow)$ is an iEx lattice and $\Box, \Diamond$ are unary operators satisfying axioms (M1), (M2), (I1), (D1), (D2) and the following:

$$(MC1) \quad \Box(((a \wedge b) \vee (a \wedge c)) \rightarrow d) \leq \Box((a \wedge (b \vee c)) \rightarrow d) \vee \Diamond(a \wedge b) \vee \Diamond(a \wedge c);$$

$$(M4) \quad \Diamond \neg a \leq \neg \Box a;$$

$$(M5) \quad \neg \Diamond a = \Box \neg a;$$

$$(M6) \quad \neg \Diamond \neg a \leq \Box a;$$

$$(FS3) \quad \Diamond(a \rightarrow b) \leq (\Box a \rightarrow \Diamond b) \vee \Box a.$$

We first verify that *miEx* algebras generalize both DOM algebras and IFS algebras.

Lemma 5.4. Any DOM algebra is a *miEx* algebra.

Proof. Note first that (I1) follows from (M1) together with $1 \leq a \rightarrow a = \neg \neg a \rightarrow a$, which holds because $\neg$ is an involution. Moreover, (M4), (M5) and (M6) all clearly follow from (M3) in any ortholattice. For (FS3), we have that $\neg \Box a \leq \Box a \rightarrow \Diamond b$, so the right-hand side also follows from the excluded middle. Hence it only remains to be shown that (MC1) holds. We claim that

$$\neg \Diamond(a \wedge (b \vee c)) \leq \Box((a \wedge (b \vee c)) \rightarrow d).$$

By (D2), it is enough, since then we have

$$1 \leq \neg \Diamond(a \wedge (b \vee c)) \vee \Diamond(a \wedge b \vee c) \leq \Box((a \wedge (b \vee c)) \rightarrow d) \vee \Diamond(a \wedge b) \vee \Diamond(a \wedge c).$$

For the proof of the claim, letting $p = (a \wedge (b \vee c))$, note that

$$\neg \Diamond p \leq \Box \neg p \leq \Box (\neg p \vee (p \wedge d)) \leq \Box (p \rightarrow d),$$

as desired. $\square$

Lemma 5.5. *Any IFS algebra is a miEx algebra.*

Proof. For (D1), note first that (M2) and $a \leq b \rightarrow (a \wedge b)$ yield $\Diamond a \rightarrow \Diamond(b \rightarrow (a \wedge b))$. By (FS2), this means that

$$\Diamond a \leq \Diamond(b \rightarrow (a \wedge b)) \leq \Box b \rightarrow \Diamond(a \wedge b),$$

and (D1) now follows from residuation. (D2) clearly follows from distributivity and (D2). For (MCI), this follows from

$$\Box(((a \wedge b) \vee (a \wedge c)) \rightarrow d) \leq \Box((a \wedge (b \vee c)) \rightarrow d),$$

which itself holds in any IFS algebra by distributivity. For (M4), by (FS2), we have

$$\Diamond(a \rightarrow 0) \wedge \Box a \leq \Box a \wedge (\Box a \rightarrow \Diamond 0) \leq \Diamond 0 \leq 0,$$

whence $\Diamond \neg a \leq \neg \Box a$. For (M5), notice first that (M4) entails that $\Box \neg a \leq \neg \Diamond \neg \neg a \leq \neg \Diamond a$, where the second inequality holds because $\Diamond$ is monotone. For the converse, we have that

$$\neg \Diamond a \leq \Diamond a \rightarrow \Box 0 \leq \Box(a \rightarrow 0) = \Box \neg a.$$

This shows that (M5) holds. For (M6), given (M5), we have that $\neg \Diamond \neg a = \Box \neg \neg a$. By (I1), (M1) and *modus ponens*, we get:

$$\neg \Diamond \neg a \leq \Box \neg \neg a \wedge \Box(\neg \neg a \rightarrow a) \leq \Box(\neg \neg a \wedge (\neg \neg a \rightarrow a)) \leq \Box a,$$

as desired. Finally, (FS3) follows from (FS2). $\square$

We will now prove a version of the iEx-Embedding Theorem for miEx algebras.

Lemma 5.6. *Let L be a miEx algebra and O_L its quotient orthomodular algebra. Letting $\Box_O a_O = [\Box a]_O$ and $\Diamond_O a_O = [\Diamond a]_O$ for any $a \in L$, we have that the map $a \mapsto a_O$ is a surjective fundamental modal algebra homomorphism from $(L, \Box, \Diamond)$ to $(O_L, \Box_O, \Diamond_O)$. Moreover, $(O_L, \Box_O, \Diamond_O)$ is a DOM algebra.*

Proof. By Lemma 3.3, we know that O_L is an orthomodular algebra and that the map $a \mapsto a_O$ is a fundamental algebra homomorphism. We first show that it also preserves modalities. Let $a, b \in L$ be such that $a \sim b$. Then

$$\neg \Diamond a = \Box \neg a = \Box \neg b = \neg \Diamond b,$$

where the first and last equalities hold by (M5). This show that $\Diamond a \sim \Diamond b$. Moreover, we claim that $\neg \Box a = \neg \neg \Diamond \neg a$. The left-to-right direction follows from (M6) and the fact that $\neg$ is antitone, and, for the converse, by (M4), we have that $\neg \neg \Diamond \neg a \leq \neg \neg \neg \Box a \leq \neg \Box a$. By the claim, we now have that

$$\neg \Box a = \neg \neg \Diamond \neg a = \neg \neg \Diamond \neg b = \neg \Box b.$$

Hence $\sim$ is a congruence relation with respect to $\Box$ and $\Diamond$ as well, which implies that $a \mapsto a_O$ is a fundamental modal algebra homomorphism from $(L, \Box, \Diamond)$ to $(O_L, \Box_O, \Diamond_O)$.

Finally, we show that $(O_L, \Box_O, \Diamond_O)$ is a DOM algebra. Since it is a homomorphic image of L , we already know that D_1 and D_2 hold, so we only need to check (M3). This amounts to showing that $\neg \Diamond \neg a = \neg \neg \Box a$, which already follows from the claim above. $\square$

This gives us the construction of O_L . For I_L , we first need to introduce the following restriction of Fischer Servi frames [4, 7]:

Definition 5.7. An IFS frame is a triple $(X, \leq, R)$ such that $(X, \leq)$ is a poset and R a relation on X satisfying the following conditions for any $x, y, z \in X$:

(FC2) $z \leq x$ and xRy together imply that there is $y' \leq y$ such that zRy' ;

(IFC) xRy and $y \geq z$ together imply $y = z$.

Note that (IFC) is a strengthening of the usual (FC1) condition on Fischer Servi frames, which states that $xRy \geq z$ implies that there is $x' \leq x$ such that $x'Rz$. As is to be expected, the downward closed subsets on an IFS frame form an IFS algebra.

Lemma 5.8. Let $(X, \leq, R)$ be an IFS frame. For any downset $U \subseteq X$, let $\Box_R U = \{x \in X \mid \forall y, z (x \geq yRz \rightarrow z \in U)\}$ and $\Diamond_R U = \{x \in X \mid \exists y : xRy \wedge y \in U\}$. Then $(Dn(X), \Box_R, \Diamond_R)$ is an IFS algebra.

Proof. It is well known that (M1), (M2), (FS1) and (FS2) hold on $(Dn(X), \Box_R, \Diamond_R)$ whenever $(X, \leq, R)$ satisfies conditions (FC1) and (FC2) (see, e.g., [4]). Hence we only need to check that (I1) holds. Fix $x \in X$ and a downset U . We must show that, whenever $x \geq yRz$, $z \in \neg\neg U \rightarrow U$. Since yRz , we know that z is minimal, so we only need to verify that either $z \notin \neg\neg U$ or $z \in U$. Again, since z is minimal, we have that, if $z \notin U$, then $z \in \neg U$, so that $z \notin \neg\neg U$, as desired. $\square$

We can now lift the map $a \mapsto a_I$ to a fundamental modal homomorphism.

Lemma 5.9. Let L be a miEx algebra. Let $(P(L), \supseteq, R)$ be the set of all proper prime filters on L , ordered by reverse inclusion and endowed with the following relation R :

$$PRQ \Leftrightarrow ((\Box a \in P \Rightarrow a \in Q) \wedge (a \in Q \Rightarrow \Diamond a \in P)).$$

Then $(Dn(P(L)), \Box_R, \Diamond_R)$ is an IFS algebra. Moreover, the map $a \mapsto a_I$ is a fundamental modal algebra homomorphism.

Proof. For the first part, by Lemma 5.8, it is enough to show that $(P(L), \supseteq, R)$ satisfies (FC2) and (IFC). Let P, P' and Q be prime filters on L such that $P' \supseteq P$ and PRQ . Because P' is a prime filter and (M1), (M2) hold on L , note first that $F = \{a \mid \Box a \in P'\}$ is a filter and $I = \{b \mid \Diamond b \notin P'\}$ is an ideal. Let F' be the filter generated by $Q \cup F$. We claim that $F' \cap I = \emptyset$. If this is true, then, since L is an iEx-algebra, we can apply 3.4 and obtain a prime filter Q' extending F' and disjoint from I . By construction, for such a filter Q' we have that $Q' \supseteq Q$ and $P'RQ'$, as desired. For the proof of the claim, suppose, towards a contradiction, that we have $c \in Q$, $a \in F$ and $b \in I$ such that $c \wedge a \leq b$. Then $1 \leq (c \wedge a) \rightarrow b$, which means that $(c \wedge a) \rightarrow b \in Q$. We must then have $c \wedge ((c \wedge a) \rightarrow b) \in Q$, so $a \vee (a \rightarrow b) \in Q$ by (iC1₁). Since Q is prime, either $a \in Q$ or $a \rightarrow b \in Q$. If $a \in Q$, we have that $(c \wedge a) \wedge ((c \wedge a) \rightarrow b)$, and hence $b \in Q$. Since PRQ , this implies that $\Diamond b \in P \subseteq P'$, contradicting our assumption. Hence we may assume that $a \notin Q$. We must then have $(a \rightarrow b) \in Q$, and hence $\Diamond(a \rightarrow b) \in P$. By (FS3), we get that $\Box a \in P$ or $\Box a \rightarrow \Diamond b \in P$. In the first case, it follows that $a \in Q$, which we have ruled out already. So $\Box a \rightarrow \Diamond b \in P \subseteq P'$. But since $\Box a \in P'$, this means that $\Diamond b \in P'$, a contradiction. This completes the proof of the claim, and thus that (FC2) holds. For (IFC), we argue as follows. Suppose that P, Q, Q' are prime filters such that PRQ and $Q \subseteq Q'$. Letting $a \in L$, recall first that $1 \leq \neg\neg(a \vee \neg a)$ in any fundamental lattice. By (I1), we have that $\Box(\neg\neg(a \vee \neg a) \rightarrow (a \vee \neg a)) \in P$, so $\neg\neg(a \vee \neg a) \rightarrow (a \vee \neg a) \in Q$. Since we also have $\neg\neg(a \vee \neg a) \in Q$, it follows that $a \vee \neg a \in Q$. Because Q is prime, it follows that $a \in Q$ or $\neg a \in Q$ for any $a \in L$. But this clearly means that $Q \subseteq Q'$ implies $Q' = Q$. Hence (IFC) holds as well.

This shows that $(Dn(P(L)), \Box_R, \Diamond_R)$ is an IFS algebra. For the second part, we know already that the map $a \mapsto a_I$ is a fundamental algebra homomorphism, but we must show that this map also preserves the modal operators. This amounts to showing the following for any $a \in L$ and any $P \in P(L)$:

1. $\Diamond a \in P$ if and only if there is $Q \in P(L)$ such that $a \in Q$ and PRQ ;
2. $\Box a \in P$ if and only if, for any $P', Q \in P(L)$ such that $P \subseteq P'$ and $P'RQ$, $a \in Q$.

We start from the first item. The right-to-left direction is clear by the definition of the relation R . For the converse, we assume that $\Diamond a \in P$, and we first claim that the filter F generated by the set $\{a\} \cup \{b \mid \Box b \in P\}$ is disjoint from the ideal $I = \{d \mid \Diamond d \notin P\}$. Assume towards a contradiction that we have $a \wedge b \leq d$ for some $d \in I$ and b such that $\Box b \in P$. By (M2), we get that $\Diamond(a \wedge b) \leq \Diamond d$. Moreover, by (D1), we have that $\Diamond a \wedge \Box b \leq \Diamond(a \wedge b) \leq \Diamond d$. Since $\Diamond a$ and $\Box b$ belong to P by assumption, this means that $\Diamond d \in P$, a contradiction. Now let Q be a filter extending F that is maximal among those disjoint from I . We claim that Q is prime. If true, this clearly completes the proof of the first item above. For the proof of the claim, suppose that $c_1, c_2 \notin Q$. By a standard argument, there must be $d \in I$ and b such that $\Box b \in P$ such that: $(a \wedge b) \wedge c_1 \leq d$ and $(a \wedge b) \wedge c_2 \leq d$. By standard reasoning using (M1), (M2) and Lemma 2.13, this implies that:

- $1 \leq \Box(((a \wedge b) \wedge c_1) \vee ((a \wedge b) \wedge c_2)) \rightarrow d$;
- $\Diamond((a \wedge b) \wedge c_1) \leq \Diamond d$;
- $\Diamond((a \wedge b) \wedge c_2) \leq \Diamond d$.

Together with (MCI), the first inequality implies that

$$\Box(((a \wedge b) \wedge (c_1 \vee c_2)) \rightarrow d) \vee \Diamond((a \wedge b) \wedge c_1) \vee \Diamond((a \wedge b) \wedge c_2)$$

is in P . Since P is prime, one of $\Box(((a \wedge b) \wedge (c_1 \vee c_2)) \rightarrow d)$, $\Diamond((a \wedge b) \wedge c_1)$ or $\Diamond((a \wedge b) \wedge c_2)$ must belong to P . In the first case, this implies that $((a \wedge b) \wedge (c_1 \vee c_2)) \rightarrow d \in Q$. Since $a \wedge b \in Q$ and $d \notin Q$, this means that $c_1 \vee c_2 \notin Q$. In the second and third case, the two inequalities above imply that $\Diamond d \in P$, a contradiction. Therefore $c_1 \vee c_2 \notin Q$, and Q is prime, as desired.

Finally, we show that item 2 above also holds. The left-to-right direction clearly follows from the definition of R . For the converse, suppose that $\Box a \notin P$. By (M6), this means that $\neg \Diamond \neg a \notin P$. Since L is an Ex-algebra, we can apply Lemma 3.5, and obtain a prime filter P' extending P and containing $\Diamond \neg a$. By the proof of item 1 above, $\Diamond \neg a \in P'$ implies that there is a prime filter Q such that $\neg a \in Q$ and $P'RQ$. Clearly, $a \notin Q$, which is what we needed. This completes the proof. $\square$

Putting Lemmas 5.6 and 5.9 together, we can now prove a modal version of the iEx-Embedding Theorem.

Theorem 5.10. *Let L be a miEx algebra. Then there are a DOM algebra O_L , an IFS algebra I_L and a subdirect fundamental modal algebra embedding $e : L \rightarrow O_L \times I_L$.*

Proof. Fix an miEx algebra L . We let O_L be the quotient ortholattice of L , and I_L be the fundamental modal subalgebra of $(Dn(P(L)), \Box_R, \Diamond_R)$ generated by sets of the form a_I for $a \in L$. By Lemmas 5.6 and 5.9, O_L is a DOM algebra and I_L is an IFS algebra, and both are fundamental modal homomorphic images of L . By the same reasoning as the proof of the iEx-Embedding Theorem, the map $e : L \rightarrow O_L \times I_L$ given by $e(a) = (a_O, a_I)$ is therefore a fundamental modal algebra subdirect embedding. $\square$

Finally, by the same reasoning as in the proof of Theorem 4.1, we now get the following:

Corollary 5.11. *Let DOM be the modal logic of DOM algebras and IFS the modal logic of IFS algebras. Then $\text{DOM} \cap \text{IFS}$ is the logic of miEx algebras.*

6 Conclusion

We conclude with some open problems about constructive quantum logics motivated by the results presented here:

Question 1 What is the complexity of iEx-logic?

Question 2 Are all the axioms listed in Theorem 4.3 independent over Holliday's preconditional logic?

Question 3 Is there a finite axiomatization of iEx-logic with axioms that contain at most three variables?

Question 4 What are the joint validities of basic modal orthologic and the implication-free fragment of IK or CK?

Question 5 What are the joint validities of basic modal orthomodular logic and IK or CK?

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Uniform Local Tabularity in Intuitionistic Logic

Rodrigo Nicolau Almeida

ILLC - UvA - Amsterdam - Netherlands

r.dacruzsilvapinadealmeida@uva.nl

By contrast with S4, the analysis of local tabularity above IPC has provided a difficult challenge. This paper studies a strengthening of local tabularity — *uniform local tabularity* — where one demands that all formulas be equivalent to formulas of a given implication depth. Algebraically, this amounts to considering Heyting algebras generated by finitely many iterations of the implication operation. It is shown that in contrast with locally finite Heyting algebras, n -uniformly locally finite Heyting algebras always form a variety, and an explicit axiomatization of the variety of n -uniform locally finite Heyting algebras for $n \leq 3$ is given. In connection with this analysis, it is shown that there exist locally tabular logics which are not uniformly locally tabular, answering a question of Shehtman — an example of a pre-uniformly locally tabular logic is presented, which is shown to be the unique pre-uniformly locally tabular extension of the system KG.

1 Introduction

In intuitionistic and modal logics, one often looks for properties which ensure the system is “sufficiently simple” to analyze in depth. Local tabularity ensures this by saying that over finitely many propositional variables there can only be finitely many such formulas, and in turn, that the semantics of the system cannot be too complicated. This appears as a strengthening of ‘tabularity’ — saying that a single finite structure captures the full semantics of the logic — which is quite well-understood in intuitionistic and modal logics (see [10, Chapter 12]). Moreover, it is a classical result due to Segerberg [22] and Maksimova [18] that an extension of S4 is locally tabular if and only if it is of finite depth. This provides an easy criterion and algorithm to verify local tabularity, with there being a single “pre-locally tabular” logic [10, Sec. 12.4].

By contrast, in the system IPC, the situation is quite different. It was shown by Mardaev [19, 20] that there is a continuum of pre-locally tabular logics, all above KC. It is open whether every locally tabular logic is contained in a pre-locally tabular one and whether local tabularity is decidable. In terms of algebraic models — which correspond in algebraic logical terms to the logics — it is thus open whether the category of locally finite Heyting algebra admits any easy and logically insightful description (see [17] for recent discussion).

Given these difficulties it is reasonable to seek for strengthenings of local tabularity. In [24], Shehtman observed that one can use bisimulation games to characterize when certain logics are locally tabular, introducing a property called “finite formula depth”, which corresponds to the existence of a bound on the nestings of modal operators or implications. In [1], we dubbed this property “uniform local tabularity”, with the idea being that the bound on modal depth or implication depth provides a uniform bound on the size of formulas. It was shown by Shehtman that above S4 local tabularity implies uniform local tabularity, via finite depth, and hence the ‘niceness’ of local tabularity may be attributed to the uniform version. The relationship between local tabularity and its uniform version was likewise noted in [23], where generalizations of the finite height criterion were discussed. The concept of uniform local tabularity also has deep connections to the theory of free Heyting algebras and free constructions of Heyting

algebras: as discussed in [2, Chapter 5], uniformity of local tabularity appears to be connected to the existence of simple representations of duals of free algebras.

This article provides a first systematic investigation of this property in the context of superintuitionistic logics. Our main contributions are the following:

1. We provide an explicit axiomatization of the varieties of n -uniform Heyting algebras for $n \leq 3$; for $n = 1$, this corresponds to the logic CPC, and for $n = 2$, the extensions of logic LC, the Gödel-Dummett calculus [11, 16]; for $n = 3$, such a logic is introduced here, and called 3Uni, being the least 3-uniform logic, and sitting between well-known logics:

$$\text{wPL} \subsetneq 3\text{Uni} \subsetneq \text{LC},$$

where wPL is the logic of the Weak Peirce's Law (see Section 3.1).

2. We provide an example of a locally tabular logic which is not uniformly locally tabular, answering a problem left open in [25]. Indeed, we show that the logic Box (see Section 4), is pre-uniformly locally tabular — all of its extensions are uniformly locally tabular, though Box itself is not so. We also show that it is the unique logic with this property above the system KG [3].

2 Preliminaries and basic theory

2.1 Superintuitionistic logics and Kripke models

Throughout we will assume that we are working in the signature $\mathcal{L} = (\wedge, \vee, \rightarrow, \perp, \top)$ of Heyting algebras, the language of intuitionistic logic. We use the letters $\varphi, \psi, \dots$ for $\mathcal{L}$ -formulas and letters $p, q, \dots$ or $x, y, \dots$ for variables; we compactly represent a tuple of distinct variables as $\bar{p}$. We write $\varphi(\bar{p})$ to mean that all the variables of φ occur amongst $\bar{p}$.

Definition 2.1. The *implication-depth* of φ , denoted by $d(\varphi)$, is defined by induction:

- $d(p) = d(\perp) = d(\top) = 0$;
- $d(\varphi \wedge \psi) = d(\varphi \vee \psi) = \max(d(\varphi), d(\psi))$;
- $d(\varphi \rightarrow \psi) = \max(d(\varphi), d(\psi)) + 1$.

Given a poset P , we write $P \Vdash \varphi$ to mean that P validates φ in the usual Kripke semantics (see e.g., [10, Chapter 3]). We denote by $\text{Ext}(\text{IPC})$ the lattice of its extensions; elements are called *superintuitionistic logics*. Given a poset P , we denote by:

$$\text{Log}(P) = \{\varphi : P \Vdash \varphi\},$$

the *logic of P* , which is an element of $\text{Ext}(\text{IPC})$. If $P \Vdash L$ for all $\varphi \in L$ we say that P is a *frame for L* .

Given a poset P , a subset $U \subseteq P$ is called an *upset* if whenever $x \in U$ and $x \leq y$ then $y \in U$. We denote the set of upsets of P by $\text{Up}(P)$. Given a subset C , we denote by $\uparrow C$ the least upset containing it, and for $x \in P$, $\uparrow x := \uparrow \{x\}$. We say that Q is a *rooted upset* of P if there is $x \in P$ such that $Q = \uparrow x$. Given $x \in P$, we also denote $\uparrow x$ by P_x .

Definition 2.2. A *p-morphism* between posets $f : P \rightarrow Q$ is an order-preserving function such that for each $x \in P$ and $y \in Q$ if $f(x) \leq y$ then there is some $x' \geq x$ such that $f(x') = y$; we say that Q is a *p-morphic image* of P if there is a surjective p-morphism from P onto Q .

We now recall some relevant notions concerning finite Kripke models; the reader can find all necessary background in [15, Chapter 4.1].

Definition 2.3. We say that a triple $(\mathfrak{M}, x) = (P, \leq, c, x)$ where $(P, \leq)$ is a rooted poset, with root x , and $c : P \rightarrow \mathcal{P}(\bar{p})$ is an order-preserving function, is a *model over the propositional letters $\bar{p}$* . We say that $(\mathfrak{M}, x)$ is a model of a logic L (over the propositional letters $\bar{p}$) if $(P, \leq)$ is a frame for L . Moreover, given $\mathbf{K}$ a class of posets, we say that $(\mathfrak{M}, x)$ a model over $\bar{p}$ is a $\bar{p}$ -*model of $\mathbf{K}$* if there is some poset $Q \in \mathbf{K}$ such that P is a p-morphic image of a rooted upset of Q . We denote by $\mathcal{M}_{\bar{p}}(\mathbf{K})$ the class of $\bar{p}$ -generated models over $\mathbf{K}$. We also denote by $M_{\mathbf{K}}(\varphi) = \{(\mathfrak{M}, x) \in \mathcal{M}_{\bar{p}}(\mathbf{K}) : (\mathfrak{M}, x) \Vdash \varphi\}$.

Definition 2.4. Given two models $(\mathfrak{M}, x) = (P, \leq, c, x)$ and $(\mathfrak{N}, y) = (Q, \leq, c', y)$ over $\bar{p}$, we say that they are *bisimilar* if there exists a relation $S \subseteq P \times Q$ such that (1) $(x, y) \in S$; (2) Whenever wSw' then $c(w) = c'(w')$; (3) whenever wSw' and $w \leq v$ there is some $w' \leq v'$ such that vSv' ; (4) whenever wSw' and $w' \leq v'$ there is some $w \leq v$ such that vSv' . We say that two points $w \in P$ and $w' \in Q$ are bisimilar if (P_w, w) is bisimilar with $(Q_{w'}, w')$.

Remark 2.1.

1. (**Notation**) Whenever we do not wish to emphasize the specific propositional letters used in the language, we will identify the set $\{p_1, \dots, p_n\}$ with the natural number n , and consider models over $\bar{p}$ to be maps $c : P \rightarrow 2^n$; the image for a point $x \in P$ will be called its *color*, denoted by a tuple $c(x) = \langle i_1, \dots, i_n \rangle$. We will further shorten tuples of the form $\langle 1, \dots, 1, 0, \dots, 0 \rangle$, consisting of a sequence of 1 followed by a sequence of 0, by the notation $1^{n-k}0^k$. When n is given, we will often omit the order and the valuation, and write simply (P, x) for a model. We occasionally write $\mathcal{M}_n(\mathbf{K})$ for the class of models over n from $\mathbf{K}$.
2. All statements given below for posets (P, x) or models thereof assume that P is *rooted at x* . This is implicit in our notion of model, but we repeat it for the sake of the reader familiar with frame theoretic arguments.
3. Given an n , say that a model (P, x) over n is *n-generated* if P does not contain distinct bisimilar points. A well-known characterization of n -generation which we will use throughout is through α and β -reductions (see [7, Chapter 3]). Namely, given a model P , we say that we can apply an *alpha-reduction* (α) whenever $x \in P$ has an immediate successor x' such that $\uparrow x = (\uparrow x' \cup \{x\})$ and $c(x) = c(x')$, obtained by identifying x with x' ; we say we can perform a β -reduction (β) if x, x' are two points, such that $\uparrow x - \{x\} = \uparrow x' - \{x'\}$, and $c(x) = c(x')$, obtained by identifying x and x' . A model is n -generated precisely when there are no α or β -reductions which can be performed on the model. A convenient fact which we will exploit throughout is that for n -generated models, *bisimulation implies isomorphism*.

Definition 2.5. Let $(P, \leq, c, x)$ and $(Q, \leq, c', y)$ be two models over n . We say that a relation $S_k \subseteq P \times Q$ is a *k-bisimulation* based on (x, y) if there are relations $S_k \subseteq \dots \subseteq S_j \subseteq \dots \subseteq S_0 \subseteq P \times Q$ for each $0 \leq j \leq k$, such that $xS_k y$ and for each $w \in P, w' \in Q$:

1. Whenever $wS_0 w'$ then $c(w) = c'(w')$;
2. For each $j < k$, whenever $wS_{j+1} w'$ and $w \leq v$ there is some $w' \leq v'$ such that $vS_j v'$.
3. For each $j < k$, whenever $wS_{j+1} w'$ and $w' \leq v'$ there is some $w \leq v$ such that $vS_j v'$.

We say that (P, x) is *k-bisimilar* with (Q, y) if there exists a k -bisimulation between them. We write $(P, x) \leq_0 (Q, y)$ if $c(x) \leq c(y)$ and $(P, x) \leq_{k+1} (Q, y)$ if for each $y \leq w'$ there is some $x \leq w$ such that $(P_w, w) \sim_k (Q_{w'}, w')$. We also write $(P, x) \leq_\infty (Q, y)$ if whenever $y \leq w'$ there is some $x \leq w$ such that (P, w) is bisimilar with (Q, w') .

As described in [15, Chapter 4.1], n -bisimulations can be characterized in game-theoretic terms: given a pair (P, x) and (Q, y) , Player I wins a 0-bisimulation game if $c(x) \neq c(y)$ and otherwise Player II wins; and Player I wins an $n+1$ -bisimulation game if there is $z \in P$, such that Player II cannot choose any $w \in Q$ and win an n -bisimulation game with (P_z, z) and (Q_w, w) .

The following is a basic property of n -bisimulation:

Proposition 2.6. For each pair of models (P, x) and (Q, y) over $\bar{p}$ the following are equivalent:

1. $(P, x) \leq_n (Q, y)$;
2. Whenever $d(\varphi(\bar{p})) \leq n$ and $(P, x) \Vdash \varphi$ then $(Q, y) \Vdash \varphi$.

It follows from this that for finite models, $(P, x) \leq_\infty (Q, y)$ if and only if for every formula φ , $(P, x) \Vdash \varphi$ then $(Q, y) \Vdash \varphi$. The following is completely analogous to [15, Chapter 4, Exercise 11], where we relativise to a specific subclass of posets, to capture different equivalence in different logics:

Proposition 2.7. Let $\mathbf{K}$ be a class of finite posets, $L = \text{Log}(\mathbf{K})$ and let $\bar{p}$ be a set of propositional variables. Then we have:

1. For each $\varphi(\bar{p}), \psi(\bar{p})$, we have:

$$L \vdash \varphi \leftrightarrow \psi \iff \mathcal{M}_{\mathbf{K}}(\varphi) = \mathcal{M}_{\mathbf{K}}(\psi).$$

2. The following are equivalent:

- (a) $S \subseteq \mathcal{M}_{\bar{p}}(\mathbf{K})$ is the class of models from $\mathcal{M}_{\bar{p}}(\mathbf{K})$ satisfying a formula ψ with $d(\psi) \leq n$;
- (b) S is closed under $\leq_n$ -bisimulation within $\mathcal{M}_{\bar{p}}(\mathbf{K})$: whenever $(P, x) \in S$ and $(Q, y) \in \mathcal{M}_{\bar{p}}(\mathbf{K})$ and $(P, x) \leq_n (Q, y)$ then $(Q, y) \in S$

Proof. (1) is an immediate consequence of L being the logic of a class of finite posets, and therefore, if $\varphi \leftrightarrow \psi \notin L$, there being a model of φ which is not a model of ψ .

(2) If S is the class of models satisfying a formula ψ with $d(\psi) \leq n$, then by Proposition 2.6, we have that S is closed under $\leq_n$ -bisimulation. Conversely, assume that S is closed under $\leq_n$ -bisimulation, and consider:

$$\text{Th}_n(S) = \{\psi(\bar{p}) : d(\psi) \leq n, \forall (P, x) \in S, (P, x) \Vdash \psi\}.$$

Since there are only finitely many formulas of implication depth at most n , $\text{Th}(S)$ is finite, and so we can consider $\chi(S) = \bigwedge \text{Th}(S)$. We claim that S is precisely the class of models from $\mathcal{M}_{\bar{p}}(\mathbf{K})$ satisfying $\chi(S)$. It is clear that if $(P, x) \in S$, $(P, x) \Vdash \chi(S)$. On the other hand, assume that $(Q, y) \notin S$. Then for each $(P, x) \in S$, $(P, x) \not\leq_n (Q, y)$, which by Proposition 2.6 means that there is a formula $\varphi(\bar{p})$ of implication depth at most n such that $(P, x) \Vdash \varphi$ and $(Q, y) \not\Vdash \varphi$. Let $\mu_P = \bigvee_{P \in S} \varphi_P$; note that up to equivalence, μ_P is finite, and $\mu_P \in \text{Th}_n(S)$. Since $(Q, y) \not\Vdash \mu_P$, then $(Q, y) \not\Vdash \chi(S)$, which concludes the proof. ■

2.2 Local tabularity, local finiteness and their uniform versions

To investigate these notions it will be convenient to work with Heyting algebras. For the connection between superintuitionistic logics and varieties of Heyting algebras, and the algebraic semantics therein, we refer to [10, Chapter 8]. Moreover, though the reader will not explicitly need any facts about the duality of finite Heyting algebras and finite posets, this will be implicit in the following results.

Given a formula φ , we write $H \models \varphi$ to mean that the Heyting algebra H validates φ ; notations are extended to classes in the usual way. Given a class of Heyting algebras $\mathbf{K}$, we write:

$$\text{Log}(\mathbf{K}) = \{\varphi : \mathbf{K} \models \varphi\};$$

conversely, given a logic L , we write:

$$\text{Alg}(L) = \{H \in \mathbf{HA} : H \models L\}.$$

We now turn to the basic concepts in our analysis.

Definition 2.8. Let $L \supseteq \text{IPC}$ be a superintuitionistic logic. We say that L is:

- *locally tabular* if given $\bar{p} = \{p_1, \dots, p_n\}$ propositional variables, there are only finitely many formulas over $\bar{p}$ up to L -provability. We denote by LTab the class of locally tabular logics;
- *n -uniform* if each formula φ is equivalent to a formula of implication depth n over L . We say that L is *uniformly locally tabular* if L is n -uniform for some n . We denote by ULTab_n the class of n -uniform logics, and by ULTab their union, the class of all uniformly locally tabular logics;
- of *finite depth* if $L \supseteq \text{BD}_n$ for some n , where:

$$\text{bd}_0 = p_0 \text{ and } \text{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \rightarrow \text{bd}_n),$$

and

$$\text{BD}_n = \text{IPC} \oplus \text{bd}_n.$$

We denote by FinDep the class of logics of finite depth.

- *tabular* if there is a finite poset P such that $L = \text{Log}(P)$. We write Tab for the class of tabular logics.

All of these concepts correspond algebraically to specific properties:

Definition 2.9. Let H be an algebra. We say that H is *locally finite* if whenever $A \leq H$ is a finitely generated algebra, then A is finite. We say that a class of algebras $\mathbf{K}$ is *locally finite* if every algebra in $\mathbf{K}$ is locally finite.

Recall that given H a Heyting algebra, $A \subseteq H$ a subset, the least Heyting subalgebra of H containing A can be constructed as:

$$\langle A \rangle = \{t(b_1, \dots, b_k) : b_i = a_i \text{ or } b_i = \neg a_i, a_i \in A, t(x_1, \dots, x_n) \text{ is a formula}\}.$$

Local finiteness means precisely that for each H , and A a finite subset, there is an n such that $\langle A \rangle$ is generated by formulas of implication depth at most n . This is where we can impose uniformity:

Definition 2.10. Let H be an algebra. We say that H is *n -uniformly finite*, or simply *n -uniform*, if whenever $A \subseteq H$ is finite, then:

$$\langle A \rangle = \{t(a_1, \dots, a_k) : a_i \in A, t(x_1, \dots, x_k) \text{ has implication depth } \leq n\}.$$

In this case we say that $\langle A \rangle$ is *n -uniformly generated*. We say that a class of algebras $\mathbf{K}$ is *n -uniformly finite* if every algebra in $\mathbf{K}$ is n -uniformly finite.

The motivation for the former definition lies in a certain modal understanding of Heyting algebras: one sees the implications as modalities and their nesting as a way to parametrize the complexity of a given logical system or algebra. A different way of measuring such complexity goes through the structure of frames:

Definition 2.11. Given $x \in P$ a poset, the *depth of x* , denoted $d_p(x)$, is the largest n such that there exists a chain $x = x_1 \leq \dots \leq x_n$, and there is no such chain of size $n + 1$, or ∞ if such an n does not exist. If each point in P has a depth, we denote by $d_p(P) = \sup\{d_p(x) : x \in P\}$.

The following is all well-known and provides the algebraic translation of the above properties:

Proposition 2.12. The following holds for L a superintuitionistic logic:

1. L is locally tabular if and only if $\text{Alg}(L)$ is locally finite.
2. L is n -uniform if and only if $\text{Alg}(L)$ is n -uniformly finite.
3. L is of finite depth k if and only if whenever $P \Vdash L$ is a poset, then P is of depth at most k , if and only if L contains the formula bd_k , defined recursively by:

$$\text{bd}_1 = p_1 \vee \neg p_1 \text{ and } \text{bd}_{j+1} = p_{j+1} \vee (p_{j+1} \rightarrow \text{bd}_j)$$

4. L is tabular if and only if $\text{Alg}(L)$ is finitely generated as a variety (i.e., it is the least variety containing a given finite algebra).

Moreover, the following inclusions hold between classes of Heyting algebras:

$$\text{Tab} \subseteq \text{FinDep} \subseteq \text{ULTab} \subseteq \text{LTab}.$$

Proof. The equivalences of (1-3) are easy to prove. We concentrate on the inclusions. $\text{Tab} \subseteq \text{FDep}$, since each finite poset P satisfies bd_k where k is its height; the fact that $\text{FDep} \subseteq \text{ULTab}$ was shown by Shehtman [25], and amounts to the fact that for posets of finite depth, there is a uniform bound for how complex n -bisimulation can be; the fact that $\text{ULTab} \subseteq \text{LTab}$ is obvious. ■

Most converses of the inclusions of Proposition 2.12 are known not to hold. Namely, the logic BD_2 is known not to be tabular, even though it is of finite depth. The logic LC is axiomatized as:

$$\text{LC} := (p \rightarrow q) \vee (q \rightarrow p),$$

is known to be uniformly locally tabular (see [25], see also [1] for a proof) but is not of finite depth. We will take a look at the last converse — $\text{LTab} \subseteq \text{ULTab}$ in Section 3.

2.3 Basic properties and criteria for uniform local tabularity

In order to establish n -uniform local tabularity we will make use of the following characterization in terms of n -bisimulations¹:

Proposition 2.13. Let $\mathbf{K}$ be a class of finite posets. Then $L = \text{Log}(\mathbf{K})$ is $n + 1$ -uniformly locally tabular if and only if for each $\bar{p}$, whenever (P, x) and (Q, y) are two models in $\mathcal{M}_{\bar{p}}(\mathbf{K})$ and $(P, x) \leq_{n+1} (Q, y)$, then $(P, x) \leq_{\infty} (Q, y)$. Moreover, if n -bisimilarity implies full bisimilarity, the latter holds.

Proof. If L is $n + 1$ -uniformly locally tabular, then all formulas are equivalent to formulas of implication depth $\leq n + 1$. Let $(P, x), (Q, y) \in \mathcal{M}_{\bar{p}}(\mathbf{K})$ be two models, and assume that $(P, x) \leq_{n+1} (Q, y)$. Then whenever $(P, x) \Vdash \varphi$ for any φ , $(Q, y) \Vdash \varphi$. Thus $(P, x) \leq_{\infty} (Q, y)$ by our remark after Proposition 2.6.

¹We note for the reader that the criterion below makes use of the relation $\leq_n$ as opposed to n -bisimilarity. In [25], the corresponding result does not mention this relation, however, we have not been able to reproduce such a result when omitting this relation.

Conversely, let $\varphi(\bar{p})$ be an arbitrary formula. Consider $S = \{(P, x) \in \mathcal{M}_{\bar{p}}(\mathbf{K}) : (P, x) \Vdash \varphi\}$. Now assume that $(P, x) \leq_{n+1} (Q, y)$. By assumption, $(P, x) \leq_{\infty} (Q, y)$, and consequently, $(Q, y) \Vdash \varphi$ as well. This means that S is closed under $\leq_{n+1}$ -bisimulation within $\mathcal{M}_{\bar{p}}(\mathbf{K})$, which by Proposition 2.7.(2) that S is the class of models from $\mathcal{M}_{\bar{p}}(\mathbf{K})$ satisfying a formula ψ with $d(\psi) \leq n+1$. By Proposition 2.7.(1), this means that $L \vdash \varphi \leftrightarrow \psi$, i.e., φ is equivalent to a formula of implication depth $\leq n+1$. Thus L is $n+1$ -uniformly locally tabular, as desired.

If we assume that n -bisimilarity implies full bisimilarity instead, then if we assume that $(P, x) \leq_{n+1} (Q, y)$, by considering $y \leq y$, we have that there is some $x \leq w$ such that $(P_w, w) \sim_n (Q, y)$. Since $(P, x) \Vdash \varphi$, then $(P_w, w) \Vdash \varphi$ as well; since n -bisimilarity implies full bisimilarity, by invariance of truth under bisimilarity, $(Q, y) \Vdash \varphi$. We thus obtain the same conclusion. ■

We close this section by noting an interesting property of n -uniform Heyting algebras. Let $\mathbf{HA}_{lf}$ be the category of locally finite Heyting algebras with Heyting algebra homomorphisms, and let $\mathbf{HA}_{lf}^n$ be the category of n -uniformly finite Heyting algebras with the same maps.

Proposition 2.14. The category $\mathbf{HA}_{lf}$ is closed under subalgebras, homomorphic images and finite products. For each $n \in \omega$, $\mathbf{HA}_{lf}^n$ is a variety.

Proof. The fact about subalgebras is trivial. To see the fact about homomorphic images, say H is locally finite (n -uniformly finite), and A is a homomorphic image. Let $\{b_1, \dots, b_n\} \subseteq A$ be finitely many elements. Then we can find finitely many elements $\{h_1, \dots, h_n\} \subseteq H$, mapping surjectively onto b_1 ; by restricting the homomorphism from H to A , we find that $\langle \{h_1, \dots, h_n\} \rangle$ maps surjectively to $\langle \{b_1, \dots, b_n\} \rangle$. Thus the latter is finite (n -uniformly generated) by hypothesis.

To see the fact about products, note that if H_0, H_1 are locally finite, and $(a_0, b_0), \dots, (a_n, b_n)$ where $a_i \in H_0$ and $b_i \in H_1$, then

$$\langle (a_0, b_0), \dots, (a_n, b_n) \rangle = \{t((a_0, b_0), \dots, (a_n, b_n)) : d(t) \text{ is the maximum of that of } \langle A \rangle \text{ and } \langle B \rangle\}.$$

The same argument works for infinite products if we assume the algebras are n -uniformly finite, since then, rather than taking the maximum of $\langle A \rangle$ and $\langle B \rangle$ one can take the bound n . ■

The above Proposition entails that there is the least n -uniform logic for each n . On the other hand, $\mathbf{HA}_{lf}$ is not a variety — it contains every finite Heyting algebra, and hence the smallest variety containing it is $\mathbf{HA}$ itself.

3 The hierarchy of locally tabular logics

In this section we will study the hierarchy of locally tabular logics — the *top* of the Kuznetsov hierarchy [6]. In 3.1 by characterizing the 0, 1, 2, 3-uniformly locally tabular logics. We then give in Section 3.2 an example of a locally tabular but not uniformly locally tabular logic.

3.1 0-uniform, 1-uniform, 2-uniform, 3-uniform logics

The reader will notice that due to our definition, the only superintuitionistic logic which is 0-uniform is the trivial logic. 1-uniformity turns out to be quite a restrictive property: only CPC has it. 2-uniformity will be characteristic of extensions of LC, whilst 3-uniformity and beyond will turn out to have a rather complex structure.

We assume the reader is familiar with the Yankov theorem (see [7, Chapter 3] for an extensive discussion) associating a formula $\mathcal{J}(Q)$ to a rooted poset Q ; the key property we will need is the following:

Theorem 3.1 (Yankov's theorem). *For a finite poset P and a rooted poset Q , $P \not\models \mathcal{J}(Q)$ if and only if Q is a rooted subframe of a p -morphic image of P .*

We briefly recall the structure of the lattice of extensions of superintuitionistic logics; all proofs of these facts can be found in [10, Chapter 4, Chapter 8]:

1. Given logics $L' \subseteq L$ we say that L' is a *cover* of L if L is an immediate successor of L' , i.e., $L' \subseteq K \subseteq L$ implies that either $L' = K$ or $K = L$.
2. CPC is the largest consistent extension of IPC;
3. CPC has a unique cover $\text{Sme} = \text{IPC} \oplus (\neg p \vee q \vee (q \rightarrow p)) = \text{Log}(2)$, the logic of the 2-element poset.
4. Sme is an extension of LC, in fact the greatest extension different from CPC.
5. The logic LC is axiomatized by $\mathcal{J}(2F)$ and $\mathcal{J}(\text{Dia})$ where 2F is the 2-fork (the frame underlying the model M_2 in Figure 1), and Dia is the diamond frame (the poset Q_6 in Figure 2).

Then we have:

Theorem 3.2. *For L a superintuitionistic logic, we have:*

1. L is 0-uniform if and only if L is inconsistent.
2. L is 1-uniform if and only if $L = \text{CPC}$;
3. L is 2-uniform if and only if $L \supseteq \text{LC}$.

Proof. As noted, the only 0-uniform logic is the inconsistent one. Note that CPC is 1-uniform: by Definition 2.1, every formula in CPC is equivalent to a conjunction of disjunctions of formulas $p, \neg p$. On the other hand, the logic Sme is not 1-uniform: we see in Figure 1 models M_0 and M_1 , based on frames validating Sme, and $M_1 \leq_1 M_0$, certainly, but $M_1 \not\leq_\infty M_0$, since no point in M_1 labelled with 0 is 1-bisimilar with the singleton containing 0. By Proposition 2.13, this means that the logic is not 1-uniform, and neither is any logic extended by it.

As noted before, it was shown by Shehtman [25] that LC is 2-uniform; it is also not difficult to see: if (P, x) and (Q, y) are two linearly ordered posets rooted at x, y , they are n -generated, and they are 1-bisimilar, then notice that by n -generation, there cannot be successive nodes with the same color (otherwise we could identify them). But then by an easy induction we see that the two chains $\uparrow x$ and $\uparrow y$ must be isomorphic. It follows that if $(P, x) \sim_1 (Q, y)$ for n -generated models, then (P, x) and (Q, y) are isomorphic, and consequently, fully bisimilar. We can then appeal to the same proposition as above.

Now assume that L is some logic and $\text{LC} \not\subseteq L$; then $L \not\models \mathcal{J}(2F)$ or $L \not\models \mathcal{J}(\text{Dia})$, so either the 2-fork or the diamond are frames for L . If 2F is a frame for L , note that in Figure 1, we have the models M_1 and M_2 , and $M_2 \leq_2 M_1$: if we pick in M_1 the top node labelled with 1, there is a node in M_2 with the same label, obtaining an isomorphism; and if we pick in M_1 the root, node that $M_1 \sim_1 M_2$. However, $M_2 \not\leq_3 M_1$, since by picking the root of M_1 , there is no 2-bisimilar node in M_2 . Similarly, the models M_3 and M_4 (the frame of which is a p -morphic image of M_3) have the property that $M_3 \leq_2 M_4$ (there is an isomorphic node in M_3 for each node but the root, and the roots are 1-bisimilar), but $M_2 \not\leq_3 M_4$. This shows that L cannot be 2-uniformly locally tabular again by Proposition 2.13. ■

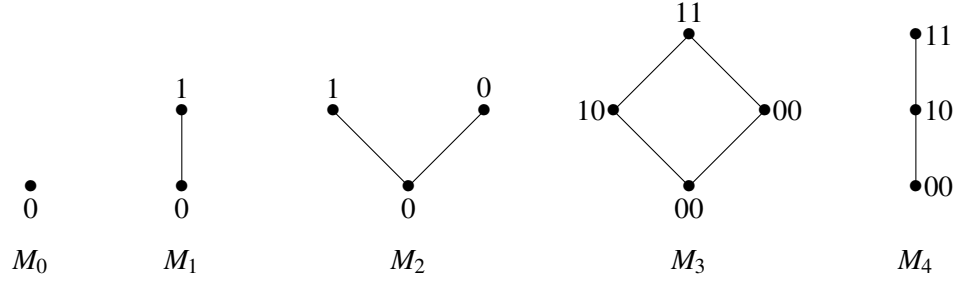


Figure 1: Models witnessing degrees of uniformity

The case of 3-uniform logics is a fair bit more complicated and deserves further consideration. For that purpose, recall the logic wPL:

$$\text{wPL} := \text{IPC} \oplus (q \rightarrow p) \vee ((p \rightarrow q) \rightarrow p) \rightarrow p.$$

It is known (see [12, A.9]) that a Heyting algebra H validates wPL if and only if it is a *cascade Heyting algebra*. Their variety was named by Esakia CHL, and was studied in [5] in the context of analyzing profiniteness and representability of Heyting algebras. The finite members of such a variety are the “sums” of Boolean algebras; we will make use of the dual posets of such algebras:

Definition 3.3. A finite (rooted) poset $(P, \leq)$ is called a *Boolean sum* if $d_p(P) = n$, and:

1. Whenever $x \prec y$, i.e., y is an immediate successor of x , then $d_p(x) = d_p(y) + 1$;
2. Each point x of depth $k + 1$ sees all points of depth k for each k .

When working with Boolean sums, note that the notion of depth of a point captures precisely its *level*: for P a Boolean sum, $\text{Lev}_k(P) = \{x \in P : d_p(x) = k\}$ consists of incomparable points which are all seen by the points in $\text{Lev}_{k-1}(P)$, and all see the points in $\text{Lev}_{k+1}(P)$. The following is a consequence of [12, A.9.2]

Proposition 3.4. A finite rooted poset P is a frame for wPL if and only if it is a Boolean sum.

The following was shown by Mardaev [21, Theorem 5]:

Theorem 3.5. The logic wPL can be axiomatized as:

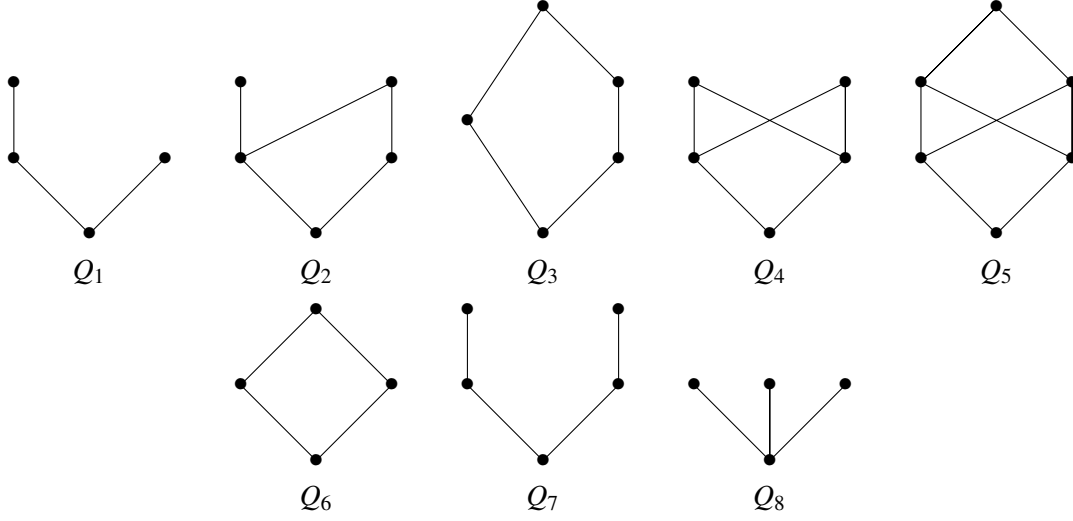
$$\text{wPL} := \text{IPC} \oplus \mathcal{J}(Q_1) \oplus \mathcal{J}(Q_2) \oplus \mathcal{J}(Q_3),$$

where $\mathcal{J}(Q_i)$ is the Yankov formula for the posets in Figure 2. Moreover, this logic is locally tabular.

We will be concerned with special extensions of this logic. For that we will need some definitions:

Definition 3.6. Let P be a Boolean sum of depth n . For each n , let c_n be the cardinality of the set of elements of depth n . Given $\langle c_1, \dots, c_n \rangle$ we say that a consecutive subsequence $c_i, \dots, c_{i+k}$ where $c_{i+j} > 1$ for $0 \leq j \leq k$ is a *k-stack*. We say that P has *stack-depth* k if there exists a k -stack but not a $k + 1$ -stack in P .

The stack depth of a Boolean sum thus measures how many consecutive levels there can be which contain more than one point.

Figure 2: Q_i posets

Remark 3.1. In [5] a subvariety of CHL was considered in connection to the problem of profinite completions, called the variety of *Diamond Heyting algebras* DHL. Such algebras are of width 2, have a maximal element, and they satisfy the Yankov formula $\mathcal{J}(Q_5)$. Thus, their finite rooted posets are precisely Boolean sums of stack depth 1, width 2 having a greatest element (see also Section 4, where the associated logic will be called Box).

Lemma 3.7. Let (P, x) be a finite poset which is a Boolean sum. Then $P \Vdash \mathcal{J}(Q_i)$, for $i \in \{4, 5\}$ in Figure 2, if and only if P has stack-depth 1.

Proof. First note that both of the posets Q_4, Q_5 have stack depth 2. If Q_4, Q_5 is a rooted subframe of Q , which is a p-morphic image of P , by pulling along the p-morphism we will see that P has stack depth at least 2.

Conversely, assume that P has stack depth at least 2. Let $x \in P$ be a maximal point such that $\uparrow x$ has stack-depth 2. We may thus assume that x has at least two successors y_0, y_1 , and that each of them has two successors (otherwise we could have picked y_i as our maximal point) z_0, z_1 . Then there are two cases: either z_0 and z_1 have no common successor, or they do. In the first case, we define a map to Q_4 as follows: we send z_0 to the left top point in Q_4 , z_1 and all remaining points of that depth to the right top point; y_0 to the left middle point, y_1 and all remaining points of that depth to the middle right point; and x to itself. This is a surjective p-morphism onto Q_4 . The case of Q_5 is similar, except we send all points above z_0, z_1 to the maximal point. ■

Given a rooted frame (P, x) based on a Boolean sum with stack depth 1, let us say that a point $y \in P$ is *solitary* if y is the only point of depth k for some k . Otherwise we say that y is *accompanied* by a point y' . The importance of stack depth for us comes from the following result:

Lemma 3.8. Let (P, x) be an n -generated model based on a finite Boolean sum of stack depth 1. If $(P, x) \sim_1 (P_y, y)$ for some $y \in \uparrow x$, then $x = y$.

Proof. Assume first that x is a solitary point. Then for each y , either $x \leq y$ or $y \leq x$ must hold. If $y \leq x$, note that by n -generation, any predecessor of x must have a color strictly smaller than x , and thus we

cannot have $(P, y) \sim_1 (P, x)$. On the other hand, if $x \leq y$, let $S(x)$ be the set of immediate successors of x . If $S(x)$ is a singleton, we likewise have that the color of x must be strictly smaller, and so $(P, y) \sim_1 (P, x)$ cannot hold. If $S(x) = \{x_0, x'_0\}$, then since the stack depth is 1, these points must have distinct color, and either $c(x_0) \neq y$ or $c(x'_0) \neq y$; say $c(x_0) \neq y$ without loss of generality. If $(P, y) \sim_1 (P, x)$, then y would have to have a successor $y_0 \leq y$ such that $c(y_0) = c(x_0)$. However, note that if z is the unique immediate successor of x_0, x'_0 , $c(x_0), c(x'_0) < c(z)$, so $y_0 = x_0$ or $y_0 = x'_0$. Thus if $x \neq y$, we would have a contradiction, since $x \leq y \leq y_0$ would force $y = y_0$ (since y_0 must be an immediate successor of x), whilst we chose $c(y) \neq c(y_0)$. It follows that $(P, x) \sim_1 (P, y)$ can only hold if $x = y$.

Next assume that x is accompanied by x' . If $x < y$, then by n -generation, $c(x) \neq c(y)$. Moreover, $c(x) \neq c(x')$, so $(P, x) \sim_1 (P, x')$ cannot hold. Finally if $y < x$, then by the same arguments as in the previous case, if $(P, x) \sim_1 (P, y)$, we would obtain a contradiction. We thus obtain the result. ■

Proposition 3.9. Let P, Q be two finite Boolean sum frames of stack depth 1. Assume that $(P, x) \sim_2 (Q, y)$ are n -generated models which are 2-bisimilar. Then they are isomorphic.

Proof. Let $S_2 \subseteq S_1 \subseteq S_0 \subseteq P \times Q$ be a 2-bisimulation. We claim that S_1 defines an isomorphism, mapping x to y . Indeed, assume that $y, y' \in P$ and $(y, z) \in S_1$ and $(y', z) \in S_1$. Then we have that $(P, y) \sim_1 (Q, z) \sim_1 (P, y')$, so $(P, y) \sim_1 (P, y')$. By Lemma 3.8, we have that $y = y'$. Thus S_1 is injective; a similar argument applied to Q shows that S_1 is functional, and the bisimulation clauses guarantee that it is a surjective relation. Thus S_1 is an isomorphism as desired. ■

Let 2Uni be the following logic:

$$2\text{Uni} := \text{IPC} \oplus \{ \mathcal{J}(Q_i) : i \in \{1, \dots, 5\} \} = \text{wPL} \oplus \mathcal{J}(Q_4) \oplus \mathcal{J}(Q_5).$$

Then we have:

Theorem 3.10. The logic 2Uni is the least 3-uniform logic.

Proof. First, note that in Figure 3, we see that all the posets Q_i for $1 \leq i \leq 5$ admit 2-bisimilar models (obtained by p-morphic images of Q_i , denoted Q_i^l and Q_i^r whether they occur on the right or left) which violate our criterion. Namely, we have that $Q_i^l \leq_3 Q_i^r$, since for each node except the root in Q_i^r there is a point isomorphic to it in Q_i^l , and the root of Q_i^r is 2-bisimilar to Q_i^l ; but $Q_i^l \not\leq_4 Q_i^r$, because the root in Q_i^r has no 3-bisimilar point on the left models.

Hence, no logic which contains them can be 2-uniform. By our axiomatization of the logic 2Uni, any logic which is not an extension of it will therefore contain one of these posets by Yankov's theorem.

On the other hand, by Lemma 3.7, Proposition 3.9 and Corollary 2.13, we have that 2Uni is 3-uniform. ■

3.2 A Non-Uniform Locally Tabular Logic

The analysis of 3-uniformity sheds some light on the possibilities for logics to be n -uniform but not $(n+1)$ -uniform. Namely, note that $\text{Log}(Q_4)$ will be an extension of wPL, and it will be 4-uniform but not 3-uniform by what was just shown. A more general fact holds which will allow us to provide an example of a locally tabular logic which is not uniformly locally tabular.

Definition 3.11. Let $n > 2$ and $k+1 < n$. Let M_n^k, N_n^k for $k \leq n$ be the models depicted in Figure 5. Formally, we define:

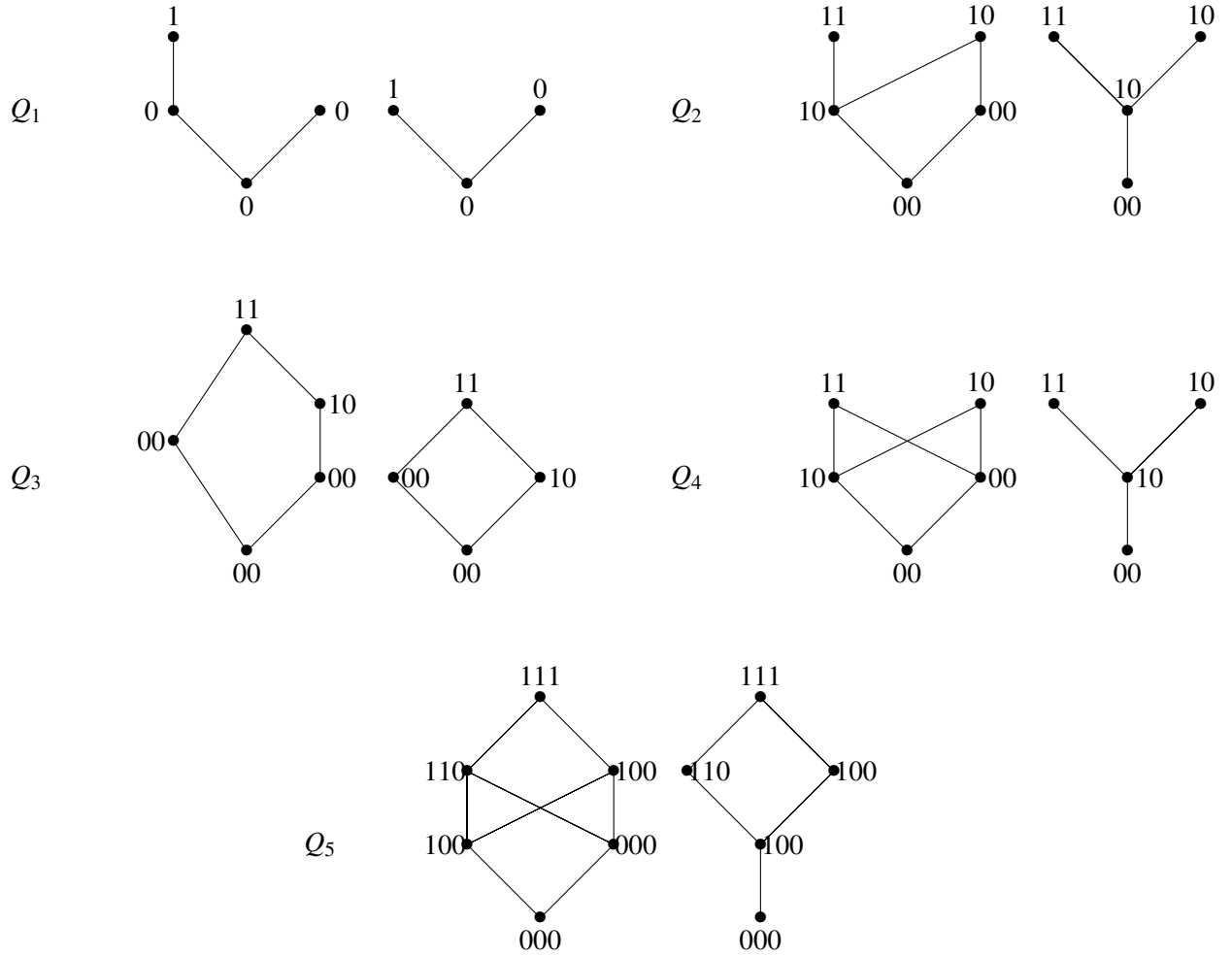


Figure 3: Models witnessing failure of 3-uniformity

1. M_n^k has $k + 1$ many layers of two elements; if $L_j(M_n^k)$ is the j -th layer for $0 \leq j \leq k$, then the left element in $L_j(M_n^k)$ is labelled with $1^{n-j}0^j$, and the right element with $1^{n-(j+1)}0^{j+1}$. There is also a root element labelled with $1^{n-k-1}0^{k+1}$.
2. N_n^k has likewise $k + 1$ many layers of two elements; if $L_j(N_n^k)$ is the j -th layer for $0 \leq j \leq k$, then the left element in $L_j(N_n^k)$ is labelled with $1^{n-j}0^j$, and the right element with $1^{n-(j+1)}0^{j+1}$. There is also a root element labelled with $1^{n-k-2}0^{k+2}$.

Thus the only difference between M_n^k and N_n^k is in the label of the root. The order in both cases is given by stacking the layers on top of each other.

Note that these models are always n -generated. Indeed, the key difference between these models is that the root of N_n^k goes down in color, relative to its right immediate successor, whilst in M_n^k the color is the same. Figure 4 shows the cases for $k = 1$, for any $n \geq 2$

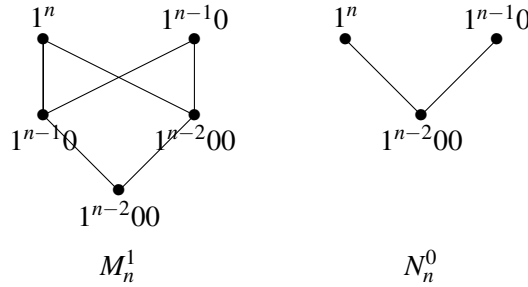


Figure 4: Models for wPL

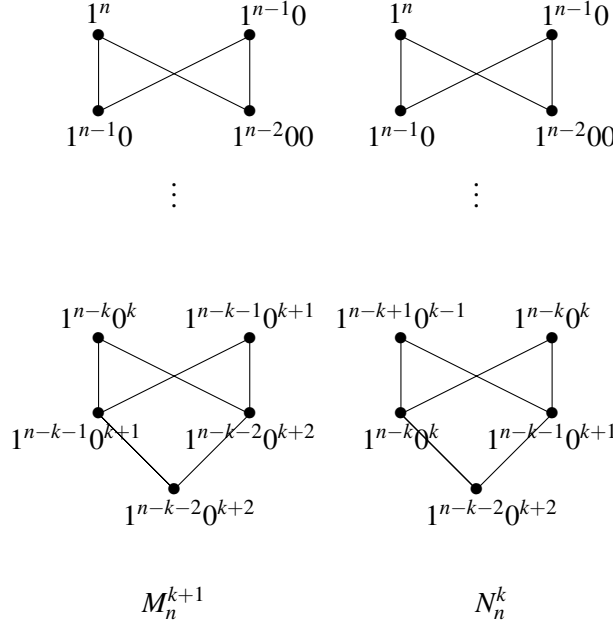
The following lemma provides the relationship between such models:

Lemma 3.12. Let $n > 2$ be arbitrary, and $k + 1 < n$. Then M_n^k is k -bisimilar, but not $(k + 1)$ -bisimilar with N_n^{k-1} .

Proof. This is shown by induction. Note that for $k = 1$, this is easy to verify, since N_n^1 will be of depth 2: looking at Figure 4, we have that the two roots have the same color, and they see points with the same colors, so they will be 1-bisimilar. However, assume that they were 2-bisimilar; let x be the point in M_n^1 with the label $1^{n-1}0$ which is the successor of the root. Then by 2-bisimilarity, there is some point in N_n^0 which is 1-bisimilar with it; but the only point which could be is the maximal point, which is a contradiction.

Now, assume that this holds for k . Consider M_n^{k+1} and N_n^k . Note that they have roots of the same color, labelled with $1^{n-k-2}0^{k+2}$, and indeed, if z is the immediate successor of M_n^{k+1} with the same color, $\uparrow z$ is isomorphic to N_n^k . Thus if Player I picks the root of N_n^k or any of its successors, Player II has an obvious response by choosing the element inside of $\uparrow z$. Thus the only way that Player I can hope to win the game is to pick either the root of M_n^k , or the left successor of the root. If Player I picks the left successor of the root, say y_0 , note that $\uparrow y_0 \cong M_n^k$; on the other hand, the right successor of the root of N_n^k is isomorphic to N_n^{k-1} , so Player II wins by induction hypothesis. Finally, if Player I picks the root of M_n^{k+1} , Player II is forced to show that this is k -bisimilar to the root of N_n^k ; by the arguments just shown Player I is in turn forced to play the root of M_n^{k+1} for all k many steps of the game, and loses when Player II finally picks the root, given the two roots have the same color. This shows that M_n^{k+1} is k -bisimilar to N_n^k as desired.

Now assume that M_n^{k+1} is $(k + 1)$ -bisimilar to N_n^k . Then Player I can pick the left successor of the root of M_n^{k+1} , y_0 , labelled with $1^{n-k-1}0^{k+1}$, and Player II is forced to respond with the right successor of N_n^k ,

Figure 5: General models M_n^{k+1} and N_n^k

y_1 ; as noted $\uparrow y_0 \cong M_n^k$ and $\uparrow y_1 \cong N_n^{k-1}$. Since by induction hypothesis these models are not k -bisimilar, Player II then loses, meaning that M_n^{k+1} and N_n^k are not $(k+1)$ -bisimilar. ■

Hence we have the following:

Theorem 3.13. *The logic wPL is locally tabular but not uniformly locally tabular.*

Proof. All the models M_n^k for each n, k are based on frames which validate wPL; hence by Corollary 2.13, wPL is not uniformly locally tabular, since we notice that $M_n^{k+1} \leq_{k+1} N_n^k$ but not $M_n^{k+1} \leq_{k+2} N_n^k$. On the other hand, it is locally tabular by Theorem 3.5. ■

4 Pre-uniform local tabularity

As noted before the problem of working with locally tabular intermediate logics presents a challenge since, as shown by Mardaev [19], there are uncountably many such pre-locally tabular logics. By contrast, there are exactly three pre-tabular logics [13], and every non-tabular logic is contained in one of them, and all of them are decidable. Together, this makes the problem of deciding, given an axiom φ , whether $\text{IPC} \oplus \varphi$ is tabular or not decidable. We thus have the following interesting problem:

Problem 1. *Is uniform local tabularity decidable over IPC?*

Intrinsic in this is the idea of “pre-local tabularity”. The idea of “pre- P ”-logics, for P a property, has been extensively used by several authors (e.g., [13, 22, 18]) to characterize certain logical properties. We can make such a definition also in our setting:

Definition 4.1. Let L be a superintuitionistic logic. We say that L is *pre-uniformly locally tabular* if L is not uniformly locally tabular but all of its extensions are so.

We will now characterize pre-uniform local tabularity amongst extensions of KG. For that, first recall the following well-known formula,

$$\text{bw}_2 := (p_0 \rightarrow (p_1 \vee p_2)) \vee (p_1 \rightarrow (p_0 \vee p_2)) \vee (p_2 \rightarrow (p_0 \vee p_1)),$$

the *bounded width 2* formula [10, Proposition 2.39]; a finite frame validates this axiom if and only if they have width at most 2, i.e., for each point x in such a frame, $\uparrow x$ contains antichains of size at most 2. Recall also the formula:

$$\neg p \vee \neg \neg p,$$

the weak law of excluded middle; it is also well-known [10, Proposition 2.37] that frames for such a formula are exactly those containing a greatest point above each point.

Definition 4.2. The logic Box is defined to be the extension of wPL by the $\text{Box} = \text{wPL} \oplus \text{bw}_2 \oplus (\neg p \vee \neg \neg p)$.

We will see that rooted frames for Box look very much like those from Figure 5, with a point in the maximum — if one thinks of four adjacent points there, these look like stacks of boxes.

Remark 4.1. Note that Box, as an extension of wPL is locally tabular. For that reason, we will assume throughout that all logics L extending it are locally tabular, and consequently have the finite model property.

We will now show that Box is pre-uniformly locally tabular. For that purpose, let S_n be the result of adding a maximal point to the underlying frame of N_n^{n-2} (see Figure 5). We denote by $\text{Lev}_j(S_k) = \{x \in S_k : d_p(x) = j\}$, the j -th level. Then first we show:

Proposition 4.3. We have that $\text{Box} = \text{wPL} \oplus \text{bw}_2 \oplus (\neg p \vee \neg \neg p) = \text{Log}(\{S_n : n \in \omega\})$.

Proof. It is clear that each S_n validates all the axioms of Box, since they have a maximal point, have width at most 2, and are frames for wPL as noted in Theorem 3.13. Conversely, by the semantic characterizations given of these formulas, any finite rooted poset P validating Box will be a Boolean sum (as it validates wPL), it will have width at most 2 by bw_2 , and a maximal element by the axiom $\neg p \vee \neg \neg p$. Let k be the depth of the root of P . Then for any $j \leq k$, either there is a unique point x such that $d_p(x) = j$, or there are two. This defines a p-morphism from S_k onto P . Thus, every finite rooted poset of Box can be obtained by p-morphisms from one of the form S_n , showing the result. ■

Recall the notion of stack depth (Definition 3.6); for frames with bounded width 2, having stack depth k amounts to having a bound of k on the number of consecutive levels with two points. Thus we obtain:

Proposition 4.4. Let $\text{Box} \subsetneq L'$ be a logic of infinite depth. Then there is some n such that posets in L' have stack depth n .

Proof. If we assume that there is no bound on the stack depth of posets for L' , then for each n , let P be a frame for L' containing a stack of at least $n + 2$ many elements. Let p be one of the minimal elements of the stack, and m be an element situated at n many levels above p ; then $\uparrow p$ is a frame for L' . Define a p-morphism onto S_n by sending m , the other element in its level, and all their successors to the top element in S_n , and all elements in the stack mapped through an isomorphism to S_n . Thus $S_n \Vdash L'$. Since this holds for any n , by Proposition 4.3 we have that $L' = \text{Box}$. ■

We thus obtain that the extensions of Box are controlled by the depth, which can be infinite or finite, and the stack-depth, which can be finite or infinite. We show the following:

Proposition 4.5. Assume that $L \supseteq \text{Box}$ is a logic of infinite depth such that all its elements have at most stack-depth $k \geq 1$. Then L is $(k + 2)$ -uniformly tabular. Consequently, Box is pre-uniformly locally tabular.

Proof. The fact that Box is not uniformly locally tabular follows by a simple modification of Theorem 3.13, adding a point on top to the models M_n^k with a distinguished color.

Assume that $L \supseteq \text{Box}$ is a logic such that its models have stack depth at most k . Then we show the following modification of Lemma 3.8: if (P, x) is a Boolean sum of stack depth at most k and $(P, x) \sim_k (P_y, y)$ then $x = y$. This proceeds by a double induction on k and m , where m is the number of stacks above x of size at least k . For $k = 1$ and arbitrary m , this was provided by Lemma 3.8. Assume that it holds for k , and we first show it for $k + 1$ and $m = 1$; In this case, note that since $(P, x) \sim_{k+1} (P_y, y)$, we can pick an immediate successor of x , x' , and by k -bisimilarity, $(P_{x'}, x') \sim_k (P_{y'}, y')$ for some successor of y ; then the stack above $P_{x'}$ must be of size at most k , so by induction hypothesis, $x' = y'$ must hold. Since this holds for any immediate successor of x , we conclude that $x = y$. Assuming this for arbitrary m , we then show it for $m + 1$, by noting that when we move to an immediate successor, the immediate successor will see one fewer $k + 1$ -stack, and so the induction hypothesis can be carried through.

From this fact, we proceed as in Proposition 3.9: assuming we have $(P, x), (Q, y)$, n -generated models based on frames for L , where L has a bound on the stack-depth of k , such that $(P, x) \sim_{k+1} (Q, y)$. Then we have an S_{k+1} -bisimulation, defined by a chain $S_{k+1} \subseteq S_k \subseteq \dots \subseteq S_0$; then the relation S_k defines an isomorphism, and so (P, x) and (Q, y) are bisimilar. From this it follows by the same arguments as in Proposition 3.9 that L is $k + 2$ -uniformly locally tabular. ■

4.1 Pre-uniform local tabularity in KG

We will provide a solution to Problem 1 in the case of extensions of the system KG of Kuznetsov-Gerciu [3] (for an overview of the system see [7, Chapter 4]). Such a system has a very broad range of extensions, and a rich semantics, which has made it useful for several purposes in the study of superintuitionistic logic (e.g. [4]). One of its key extensions is the logic $\text{RN} = \text{Log}(\text{RN})$ — the logic of the frame of the Rieger-Nishimura ladder (see Figure 6), which is dual to the *Rieger-Nishimura lattice*, the free 1-generated Heyting algebra.

Recall from [14, 9] (see also [8] and [1]) that RN , the Rieger-Nishimura ladder, can be constructed as the inverse limit of a sequence:

$$P_1 \xleftarrow{r_1} P_2 \xleftarrow{r_2} P_3 \xleftarrow{r_3} \dots$$

where r_i is a map which is a p-morphism with respect to “sufficiently many” parts of P_3 . In fact these admit an explicit description:

1. The posets P_i are precisely the upsets as labelled in Figure 6.
2. The map $r_{2n+1} : P_{2n+2} \rightarrow P_{2n+1}$ sends the P_{2n+2} -root point, and the P_{2n} -root point, to P_{2n+1} , and leaves the rest untouched;
3. The map r_{2n+2} sends the P_{2n+3} and the P_{2n+1} -root points to P_{2n+2} , and leaves everything else untouched.

For our purposes we will need the following lemma, shown essentially in [7, Chapter 4]:

Lemma 4.6. For each n , the poset S_n is an upset of a p-morphic image of $P_{2n+1} \oplus 1$.

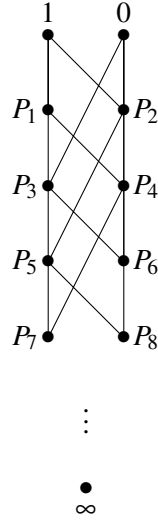


Figure 6: Rieger-Nishimura Ladder RN

Proof. One proceeds inductively as follows: first identify the point labelled 1 in Figure 6 with the top element. This produces one top level containing the maximum, followed by a level containing P_1 and 0, seen by P_3 . Then to obtain the next layer, identify P_2 with its now unique successor, and we obtain a new layer, consisting of P_3 and P_4 , seen by P_6 . We thus proceed in this way recursively, obtaining all boxes as desired. ■

To characterize uniform local tabularity, we will rely on the fact that already local tabularity above KG is characterized. In principle this is not necessary, as uniform local tabularity is potentially simpler; however in this simple case, it will be convenient to rely on the following, shown in [7, Theorem 4.6.5]:

Theorem 4.7. *A logic $\text{KG} \subseteq L$ is not locally tabular if and only if $L \subseteq \text{RN.KC} = \text{RN} \oplus (\neg p \vee \neg \neg p)$.*

We will also need one more technical notion:

Definition 4.8. We say that a poset P has *degree of uniformity* k if whenever (P, x) is an n -generated model, and (Q, y) is another n -generated model such that $S_k \subseteq P \times Q$, is a k -bisimulation between x and y , then S_k is an isomorphism.

Remark 4.2. The reader should notice that for a fixed poset P , a degree of uniformity need not be minimal — for the purposes where we will use it, the notion of a minimal degree of uniformity will not be needed.

It is well-known that any finite poset has a degree of uniformity (see e.g. [15, Proposition 4.5]). Recall that given rooted posets P, Q , we write $P \oplus Q$ for the *vertical sum* (see [7, Def. 4.1.13]) the poset based on the disjoint union of P, Q , where for each $x \in P$ and $y \in Q$, $x \leq y$. We denote the repeated vertical sum as $P_1 \oplus \dots \oplus P_n$.

Lemma 4.9. If $P_1, \dots, P_n$ is a sequence of posets with degree of uniformity k_i , then $P_1 \oplus \dots \oplus P_n$ has degree of uniformity $k' = \max(\{k_i : i \leq n\}) + 1$.

Proof. Write $P = P_1 \oplus \dots \oplus P_n$. Let (Q, y) be an arbitrary n -generated model, and assume that $(P, x) \sim_{k'} (Q, y)$. Let x_n be the root of P_n ; by k' -bisimilarity, we have that there is some $y_n \in Q$, such that $(P_{x_n}, x_n) \sim_{k'-1} (Q_{y_n}, y_n)$; since $k_n \leq k' - 1$, this relation defines an isomorphism. Note that any predecessor of x_n or y_n must have a properly smaller color. Let $Q_n = \uparrow y_n$. By restricting the relation $S_{k'}$ to

$P \setminus (P_n \setminus \{x_n\})$ and $Q \setminus (Q_n \setminus \{y_n\})$, we claim that we still have a k' -bisimulation: for each $x' \geq x$, such that $x' \in P \setminus (P_n \setminus \{x_n\})$, there must be $y' \geq y$ which is $k' - 1$ -bisimilar, and this must lie also in $Q \setminus (Q_n \setminus \{y_n\})$. Thus we can proceed by induction, showing that each such restriction of S_k is an isomorphism. ■

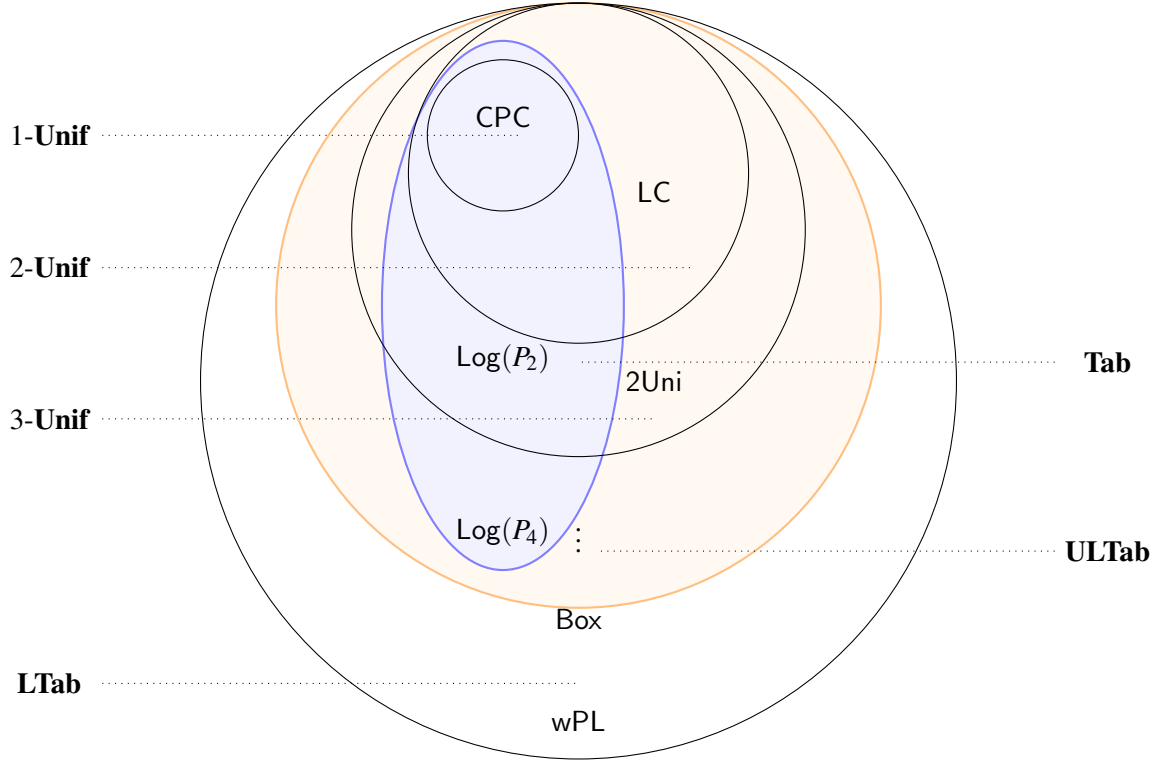


Figure 7: Hierarchy of Locally Tabular Superintuitionistic Logics; blue oval covers the tabular logics.

We can thus conclude:

Theorem 4.10. *If $L \supseteq \text{KG}$ is any logic, then L is not pre-uniformly locally tabular if and only if $L \subseteq \text{Box}$.*

Proof. If $\text{KG} \subseteq L$, and L is not locally tabular, we thus have by Theorem 4.7 that $L \subseteq \text{RN} \subseteq \text{Box}$. Hence assume that L is locally tabular but not uniformly locally tabular. We can assume by Theorem 2.12 that L is not of finite depth. Since L is locally tabular, there must be an n which is maximal such that $\uparrow P'_n$ is a frame for the logic. Hence by [7, Corollary 4.3.9], the finitely generated frames for L must be finite vertical sums of p-morphic images of upsets of P_n for $k \leq n$ (not necessarily rooted).

Now as noted in Lemma 4.6, for large enough P_n we can obtain S_n , and for large enough vertical sums, we can obtain all the S_n . Suppose that there is some n such that we cannot obtain S_n as a frame for L . Then there must be a bound on the size of the posets Q with a root and a maximal element which appear in vertical sums in L (otherwise using the technique of Lemma 4.6 we could obtain arbitrarily large S_n). Using Lemma 4.9 we can conclude that posets for L will have a maximal degree of uniformity k for some k — which implies (and is stronger than the fact that) whenever $(P, x) \leq_k (Q, y)$ for (P, x) and (Q, y) two models based on frames for L , then $(P, x) \leq_\infty (Q, y)$. By Proposition 2.13 we obtain that L is uniformly locally tabular — a contradiction.

So we must assume that L contains arbitrarily large vertical sums of P_n as frames. Thus as noted, we can obtain S_n for each n . By Proposition 4.3 we have that $L \subseteq \text{Box}$ as desired. ■

Corollary 4.11. *Above KG, uniform local tabularity is decidable: if $L = KG \oplus \phi$, it is decidable whether L is uniformly locally tabular.*

Proof. Since the logic Box is finitely axiomatized and locally finite, it is decidable. Given L a decidable logic, one simply checks whether $L \subseteq \text{Box}$, by checking whether Box contains the axiom ϕ , which is enough by Theorem 4.10. ■

5 Summary

As a conclusion to the paper, we note a visual summary of our results in Figure 7.

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Coequivalence Relations and Descent in Modal Logic

Rodrigo Nicolau Almeida

ILLC-UvA
Amsterdam, the Netherlands
r.dacruzsilvapinadealmeida@uva.nl

Matteo De Berardinis

ILLC-UvA
Amsterdam, the Netherlands
Università degli Studi di Salerno
Salerno, Italy
mdeberardinis@unisa.it

A coequivalence relation over a modal logic L is a formula $\rho(\bar{p}_1, \bar{p}_2)$ such that the logic L proves that ρ is an equivalence relation. They were introduced by Ghilardi and Zawadowski in the context of the categorical study of non-classical logics. A coequivalence relation is said to *separate variables* or to be *separating* if it corresponds to a conjunction of formulas of the form $\psi(\bar{p}_1) \leftrightarrow \psi(\bar{p}_2)$, which serve as explicit definitions of quotients. A logic L where all coequivalence relations are separating is said to have the *coequivalence separation property* (CoSP). Ghilardi and Zawadowski showed that CoSP fails for IPC. In previous work, the second author showed that such a phenomenon happens already in presumably simpler systems like S5. Ghilardi and Zawadowski therefore raised the question whether a weaker property, formulated in categorical terms related to descent theory, was still true. In this paper, we identify the logical meaning of such a property in relation to CoSP. We introduce the notion of *local coequivalence relations*, which have the additional structure of a *local transition term*, $\xi(\bar{p}, \bar{q}_1, \bar{q}_2, \bar{r})$, intuitively capturing the structure of elements lying in the same fiber. We introduce the *local coequivalence separation property* (LCoSP), and prove it to be equivalent, in good cases, to the almost Barr-exactness of $\text{Alg}(L)_{fp}^{op}$. We conclude by showing that S5 has the LCoSP.

1 Introduction

In algebraic logic it has long been a subject of interest to correspond categorical properties of classes of models, on one hand, to logical properties of calculi, especially definability properties: Beth definability theorems and regularity properties [15] as well as uniform interpolation and the existence of adjoints between free algebras [9, Chapter 3], are two notable examples. In [9, Chapter 4.9], Ghilardi and Zawadowski introduced the concept of a *coequivalence relation*, to serve as a logical correspondent to the notion of an *equivalence relation*¹ on the dual of the class of finitely presented algebras. In the simplest case, these are formulas $\rho(\bar{p}_1, \bar{p}_2)$ over disjoint tuples of variables $\bar{p}_1, \bar{p}_2$ of the same length, such that the logic L proves the axioms of an equivalence relation for ρ . The most basic kind of coequivalence relation is one determined by a conjunction of formulas of the form $\psi(\bar{p}_1) \leftrightarrow \psi(\bar{p}_2)$ – such coequivalence relations are said to *separate variables*. A logic is said to have the *coequivalence separation property* (CoSP) if all of its coequivalence relations separate variables. This stands in analogy with Beth definability, in the following sense:

1. **Beth definability** says that whenever $\Gamma(\bar{p}, q)$ proves q to be constant for any $\bar{p}$ (in the sense that $\Gamma(\bar{p}, q_1), \Gamma(\bar{p}, q_2) \vdash_L q_1 \leftrightarrow q_2$), then there exists an explicit term $\psi(\bar{p})$ which Γ proves gives the values of q . In a slogan: *implicit functions can be made explicit*.

¹The precise connection is with *r*-equivalence relations – which are based on *regular* subobjects, instead of arbitrary subobjects. See Remark 2.1 for more context on this.

2. **Coequivalence separation** says that whenever a logic L can prove a term $\rho(\bar{p}_1, \bar{p}_2)$ to be an equivalence, there are explicit function terms $\psi_i(\bar{p})$ which compute equivalence classes. In a slogan: *implicit quotients can be made explicit*.

In model theoretic terms, this property is an instance of *uniform elimination of imaginaries* [17]. In categorical terms, it amounts to the property that equivalence relations in the category $\text{Alg}(L)_{fp}^{op}$ are *effective*.

Ghilardi and Zawadowski, using a sheaf-theoretic representation of the category $\mathbf{HA}_{fp}^{op}$, showed that IPC fails to have the CoSP. Moreover, in [6], the second author studied coequivalence relations in the context of a natural infinitary extension of modal logic. For locally tabular logics, the results from that framework can be transferred, yielding that normal modal logic S5, much like IPC, has non-separating coequivalence relations.

Ghilardi and Zawadowski raised the question of whether a weakening of Barr-exactness – namely the property that all regular epimorphisms are of “effective descent type” – holds for $\mathbf{HA}_{fp}^{op}$, and left open what the logical or algebraic interpretation of such a property should be. The property in question is *almost Barr-exactness*² [13] of the category $\mathbf{HA}_{fp}^{op}$, or more generally, of $\text{Alg}(L)_{fp}^{op}$ for L a normal modal logic or superintuitionistic logic. In the categorical literature, the property of all regular epimorphisms being of effective descent type has been extensively studied in connection to descent theory [13].

In this paper we introduce the concepts of a *local transition term* $\xi(\bar{p}, \bar{q}_1, \bar{q}_2, \bar{r})$, and of *local coequivalences* induced by such terms, as well as the *local coequivalence separation property* (LCoSP). Local transition terms take their name from the fact that they encode local compatibility between elements, with local coequivalence relations essentially corresponding to the identification of elements lying inside the orbit of local transitions. Local coequivalence relations should therefore be thought of as inducing *implicit covering maps*, by analogy with the geometric picture where local homeomorphisms and covering maps come equipped with additional local patching data. Formally, the LCoSP corresponds to all local coequivalences separating variables, and consequently, being given by explicit terms.

In Section 2, we give all necessary preliminaries concerning the logical, algebraic and category-theoretic concepts used throughout the paper. In Section 3 we define coequivalence relations and their relationship to categorical equivalence relations. In Section 4 we move to the question of effective descent, and show that it corresponds to LCoSP. In Section 5 we use a concrete representation of equivalence relations in S5 frames to prove the LCoSP for this system, proving a first of this kind of results.

2 Logic and Algebra

Throughout we will be working in the language $\mathcal{L}$ of basic modal logic, namely containing the symbols $(\wedge, \neg, \Box, \perp, \top)$. We work throughout with a set of $\mathcal{L}$ -terms (also *propositional formulas* or just *formulas*) over a fixed but arbitrary set Prop . We use letter $p, q, r, \dots$ for variables, and write $\bar{p}$ for tuples of distinct variables. Whenever we are given a tuple $\bar{p}$, we will write $\bar{p}_1, \bar{p}_2, \dots$ to mean disjoint copies of $\bar{p}$, and for $p \in \bar{p}$, write p_i to mean the propositional letter in such a tuple. We use the letters $\varphi, \psi, \chi, \dots$ for formulas. The notation $\varphi(\bar{p})$ means that the formula φ has free variables included in the tuple; note that when we write $\varphi(\bar{p}, \bar{q})$ we always assume all variables to be distinct as they occur in φ .

We assume familiarity with the algebraic semantics of normal modal logics (see e.g. [5, Chapter 8]). By default a *logic* will mean a normal modal logic, unless otherwise specified. Given a logic L , we denote by $\text{Alg}(L)$ the class modal algebras validating the axioms in L , also called L -algebras. We

²This terminology was introduced by Janelidze and Sobral in [12].

write $\alpha \vdash_L \beta$ to mean that β is derivable from α using the axioms in L and the rules of Modus Ponens, Uniform substitution (the latter only applied to axioms), and possibly Necessitation. Given a (finite) set of generators X , we write $\mathcal{F}_L(X)$ to mean the free X -generated L -algebra. We write $\text{Alg}(L)_{fp}$ for the full subcategory of $\text{Alg}(L)$ given by *finitely presented* L -algebras (see [9, Chapter 2] for facts about finitely presented L -algebras and their duals). These can be identified with quotients of the form $\mathcal{F}_L(\bar{p})/\alpha(\bar{p})$, where $\alpha(\bar{p})$ is an arbitrary formula; the quotient is constructed by taking the quotient under $\sim_\alpha$, given by, for $\varphi(\bar{p}), \psi(\bar{p})$ two formulas:

$$\varphi \sim_\alpha \psi \iff \alpha(\bar{p}) \vdash_L \varphi(\bar{p}) \leftrightarrow \psi(\bar{p}).$$

A morphism of finitely presented L -algebras $\mathcal{F}_L(\bar{p})/\alpha(\bar{p}) \rightarrow \mathcal{F}_L(\bar{q})/\beta(\bar{q})$ is specified by a tuple of formulas $\bar{\theta} = (\theta_p(\bar{q}) : p \in \bar{p})$ in the variables $\bar{q}$ such that $\beta(\bar{q}) \vdash_L \alpha(\bar{\theta}/\bar{p})$ (we denote by $\alpha(\bar{\theta}/\bar{p})$, or simply by $\alpha(\bar{\theta})$, the substitution of $\theta_p(\bar{q})$ in every instance of p in α). If $\bar{\theta}'$ is another such tuple of formulas, then $\bar{\theta}$ and $\bar{\theta}'$ specify the same morphism if and only if $\beta(\bar{q}) \vdash_L \theta_p(\bar{q}) \leftrightarrow \theta'_p(\bar{q})$ for all $p \in \bar{p}$. If we fix the domain $\mathcal{F}_L(\bar{p})/\alpha(\bar{p})$, any morphism of finitely presented L -algebras out of $\mathcal{F}_L(\bar{p})/\alpha(\bar{p})$ can be seen as an extension of the theory $\alpha(\bar{p})$, both in the propositional letters and in the relations imposed; in fact, each one of them can be presented as

$$\mathcal{F}_L(\bar{p})/\alpha(\bar{p}) \rightarrow \mathcal{F}_L(\bar{p}, \bar{q})/\beta(\bar{p}, \bar{q})$$

with $\beta(\bar{p}, \bar{q}) \vdash_L \alpha(\bar{p})$ (the substitution being $\theta_p(\bar{p}, \bar{q}) = p$). Among these, the quotients (regular epimorphisms, in the language of category theory) are those extensions for which no new propositional variable is added; in other words, they are given by $\beta(\bar{p})$ such that $\beta(\bar{p}) \vdash_L \alpha(\bar{p})$.

We write $\text{Alg}(L)_{fp}^{op}$ for the dual category of $\text{Alg}(L)_{fp}$. This category will be identified via duality, when convenient, with a category of relational spaces and frames.

Notation 2.1. When proving the correspondence between syntactic and categorical properties, we will use the following convenient notation. We identify a formula $\alpha(\bar{p})$ with the object in $\text{Alg}(L)_{fp}^{op}$ dual to the finitely presented algebra $\mathcal{F}_L(\bar{p})/\alpha(\bar{p})$. Moreover, given a substitution $\bar{\theta} = (\theta_p(\bar{q}) : p \in \bar{p})$ such that $\beta(\bar{q}) \vdash_L \alpha(\bar{\theta}/\bar{p})$, we let the derivation $\beta(\bar{q}) \vdash_L \alpha(\bar{\theta}/\bar{p})$ denote the morphism in $\text{Alg}(L)_{fp}^{op}$ dual to the morphism of finitely presented algebras $\mathcal{F}_L(\bar{p})/\alpha(\bar{p}) \rightarrow \mathcal{F}_L(\bar{q})/\beta(\bar{q})$ specified by $\bar{\theta}$. Observe that compositions are obtained by composing substitutions: the composite of $\gamma(\bar{r}) \vdash_L \beta(\bar{\mu}/\bar{q})$ and $\beta(\bar{q}) \vdash_L \alpha(\bar{\theta}/\bar{p})$ is $\gamma(\bar{r}) \vdash_L \alpha(\bar{\theta}(\bar{\mu}/\bar{q})/\bar{p})$.

In many cases, we will assume that $\text{Alg}(L)_{fp}^{op}$ is a *regular category*. This means a category with finite limits, pullback-stable regular epimorphisms, admitting coequalizers of kernel pairs (see e.g. [4, Volume 2, Chapter 4]). In logical terms, the condition of regularity can be reformulated as follows.

Definition 2.1 (Beth definability). We say that a logic L has the *Beth definability property* if for any finite sets $\bar{p}, \bar{q}$ of propositional variables, and formula $\varphi(\bar{p}, \bar{q})$, whenever $\varphi(\bar{p}, q_1) \wedge \varphi(\bar{p}, q_2) \vdash_L q_1 \leftrightarrow q_2$, then there exists a formula $\psi(\bar{p})$ such that $\varphi(\bar{p}, q) \vdash_L q \leftrightarrow \psi(\bar{p})$.

Definition 2.2 ($\exists_{\text{MIP}}$). We say that a logic L has the *right-uniform Maehara interpolation property* ($\exists_{\text{MIP}}$) if for any finite sets $\bar{p}, \bar{q}$ of propositional variables, and formula $\varphi(\bar{p}, \bar{q})$, there is a formula $\mu(\bar{q})$ such that for any formulas $\psi(\bar{q}, \bar{r})$ and $\chi(\bar{q}, \bar{r})$:

$$\varphi(\bar{p}, \bar{q}) \wedge \psi(\bar{q}, \bar{r}) \vdash_L \chi(\bar{q}, \bar{r}) \iff \mu(\bar{q}) \wedge \psi(\bar{q}, \bar{r}) \vdash_L \chi(\bar{q}, \bar{r})$$

Remark 2.1. The Beth property ensures that the r -version of regularity coincides with the standard one (as explained in [9, Chapter 2.3]). Throughout the paper we will assume our logic L to have the Beth definability property; for this reason we will not make any distinction between r -regularity and regularity, or between r -equivalence relations and equivalence relations.

The property $\exists_{\text{MIP}}$ entails regularity of $\text{Alg}(L)_{fp}^{op}$: see e.g. [1, Appendix B]. We then have the following:

Proposition 2.3. If L is a logic with the Beth property and with $\exists_{\text{MIP}}$, then $\text{Alg}(L)_{fp}^{op}$ is a regular category.

3 Coequivalence relations and equivalence relations

We move on to the key definition of a coequivalence relation. The following facts are noted in [9, Chapter 4.9] and [7, Section 5.4]. We fix throughout a given normal modal logic L with the *Beth definability property*.

Definition 3.1 (Coequivalence relation). Let $\rho(\bar{p}_1, \bar{p}_2)$ be a formula over tuples of the same length $\bar{p}_1, \bar{p}_2$. Given a formula $\tau(\bar{p})$ we say that ρ is a τ -*coequivalence relation* if $\rho(\bar{p}_1, \bar{p}_2) \vdash_L \tau(\bar{p}_1) \wedge \tau(\bar{p}_2)$, and it satisfies:

1. (Reflexivity) $\tau(\bar{p}) \vdash_L \rho(\bar{p}, \bar{p})$;
2. (Symmetry) $\rho(\bar{p}_1, \bar{p}_2) \vdash_L \rho(\bar{p}_2, \bar{p}_1)$;
3. (Transitivity) $\rho(\bar{p}_1, \bar{p}_2) \wedge \rho(\bar{p}_2, \bar{p}_3) \vdash_L \rho(\bar{p}_1, \bar{p}_3)$.

As a simple example, notice that for a tuple $\psi_1(\bar{p}), \dots, \psi_m(\bar{p})$ of formulas, the formula $\bigwedge_{i=1}^m \psi_i(\bar{p}_1) \leftrightarrow \psi_i(\bar{p}_2)$ is a coequivalence relation.

Coequivalence relations coincide precisely with equivalence relations, in a categorical sense, as we proceed to explain.

Notation 3.1. Throughout, given a category $\mathbf{C}$ with finite limits, and objects $X, Y, Z \in \mathbf{C}$:

1. We write $X \otimes Y$ to mean the categorical product;
2. Given morphisms $f: X \rightarrow Z$ and $g: Y \rightarrow Z$, we write $X \otimes_Z Y$ to mean the categorical pullback.

Definition 3.2 (Equivalence relation). Let $\mathbf{C}$ be a category with all finite limits, and X an object there. An *equivalence relation* over X is a subobject $(\pi_1, \pi_2): R \hookrightarrow X \otimes X$, equipped with the following morphisms:

1. *internal reflexivity*, $r: X \rightarrow R$, which is a section of both π_i , i.e., $\pi_1 r = \pi_2 r = 1_X$;
2. *internal symmetry*, $s: R \rightarrow R$, such that $\pi_1 s = \pi_2$ and $\pi_2 s = \pi_1$;
3. *internal transitivity*, $t: R \otimes_X R \rightarrow R$, such that $\pi_1 t = \pi_1 \kappa_1$ and $\pi_2 t = \pi_2 \kappa_2$, where $R \otimes_X R$ is the pullback

$$\begin{array}{ccc} R \otimes_X R & \xrightarrow{\kappa_1} & R \\ \kappa_2 \downarrow & & \downarrow \pi_2 \\ R & \xrightarrow{\pi_1} & X \end{array}$$

Definition 3.3 (Effective equivalence). Given a category $\mathbf{C}$ with finite limits, and an object X , we say that an equivalence relation R over X is *effective* if there exists a morphism $f: X \rightarrow Q$ such that R is (isomorphic to) the pullback $X \otimes_Q X$, called the *kernel pair* of f .

Remark 3.1. The categorical product $X \otimes X$ is always an effective equivalence relation over X ; the maps r , s and t , witnessing the internal properties of this equivalence relation, are induced by the universal property of the product. One can therefore alternatively say that a subobject R of $X \otimes X$ satisfies:

1. internal reflexivity if and only if $r: X \rightarrow X \otimes X$ factorizes through R ;

2. internal symmetry if and only if the restriction of $s: X \otimes X \rightarrow X \otimes X$ to R factorizes through R ;
3. internal transitivity if and only if the restriction of $t: (X \otimes X) \otimes_X (X \otimes X) \rightarrow X \otimes X$ to $R \otimes_X R$ (which is a subobject of $(X \otimes X) \otimes_X (X \otimes X)$) factorizes through R .

The effectiveness of equivalence relations is often considered for regular categories. Such a combination gives rise to the following important notion:

Definition 3.4 (Barr-exact category). We say that a category is *Barr-exact* if it is regular and all equivalence relations are effective.

We briefly sketch the correspondence between coequivalence relations and equivalence relations in $\text{Alg}(L)_{fp}^{op}$, as outlined in [9, Chapter 4.8]. Recall our convention from 2.1. The product in $\text{Alg}(L)_{fp}^{op}$ of $\tau(\overline{p})$ with itself is given by $\tau(\overline{p}_1) \wedge \tau(\overline{p}_2)$, with projections $\tau(\overline{p}_1) \wedge \tau(\overline{p}_2) \vdash_L \tau(\overline{p}_1/\overline{p})$ and $\tau(\overline{p}_1) \wedge \tau(\overline{p}_2) \vdash_L \tau(\overline{p}_2/\overline{p})$.

A subobject of $\tau(\overline{p}_1) \wedge \tau(\overline{p}_2)$ is given by $\rho(\overline{p}_1, \overline{p}_2)$ such that $\rho(\overline{p}_1, \overline{p}_2) \vdash_L \tau(\overline{p}_1) \wedge \tau(\overline{p}_2)$ (as noted in 2.1, quotients in $\text{Alg}(L)_{fp}$ are given by those extensions of theories for which no new propositional variable is added).

Proposition 3.5. A finitely presented algebra $\mathcal{F}_L(\overline{p}_1, \overline{p}_2)/\rho(\overline{p}_1, \overline{p}_2)$ is dual to an equivalence relation over $\tau(\overline{p})$ in $\text{Alg}(L)_{fp}^{op}$ if and only if $\rho(\overline{p}_1, \overline{p}_2)$ is a τ -coequivalence relation. More specifically:

1. an internal reflexivity map exists if and only if $\tau(\overline{p}) \vdash_L \rho(\overline{p}, \overline{p})$;
2. an internal symmetry map exists if and only if $\rho(\overline{p}_1, \overline{p}_2) \vdash_L \rho(\overline{p}_2, \overline{p}_1)$;
3. an internal transitivity map t exists if and only if $\rho(\overline{p}_1, \overline{p}_2) \wedge \rho(\overline{p}_2, \overline{p}_3) \vdash_L \rho(\overline{p}_1, \overline{p}_3)$.

Proof. We prove 1., which illustrates the arguments for the other cases. First recall the restricted projections,

$$\rho(\overline{p}_1, \overline{p}_2) \vdash_L \tau(\overline{p}_1) \wedge \tau(\overline{p}_2) \vdash_L \tau(\overline{p}_1/\overline{p}) \quad \text{and} \quad \rho(\overline{p}_1, \overline{p}_2) \vdash_L \tau(\overline{p}_1) \wedge \tau(\overline{p}_2) \vdash_L \tau(\overline{p}_2/\overline{p}).$$

Internal reflexivity is given by a map $\tau(\overline{p}) \vdash_L \rho(\overline{\mu}_1/\overline{p}_1, \overline{\mu}_2/\overline{p}_2)$, whose compositions with such restricted projections, namely:

$$\tau(\overline{p}) \vdash_L \tau(\overline{\mu}_1/\overline{p}_1) \quad \text{and} \quad \tau(\overline{p}) \vdash_L \tau(\overline{\mu}_2/\overline{p}_2),$$

are the identity. This means that we can equivalently rewrite the map witnessing internal reflexivity simply as $\tau(\overline{p}) \vdash_L \rho(\overline{p}/\overline{p}_1, \overline{p}/\overline{p}_2)$, i.e., $\tau(\overline{p}) \vdash_L \rho(\overline{p}, \overline{p})$. ■

Definition 3.6 (Coeq. separating variables). A τ -coequivalence relation $\rho(\overline{p}_1, \overline{p}_2)$ is said to *separate variables* or simply be *separating* if there exists a sequence of formulas $\psi_1(\overline{p}), \dots, \psi_m(\overline{p})$, such that

$$\rho(\overline{p}_1, \overline{p}_2) \dashv\vdash_L \tau(\overline{p}_1) \wedge \tau(\overline{p}_2) \wedge \bigwedge_{i=1}^m \psi_i(\overline{p}_1) \leftrightarrow \psi_i(\overline{p}_2).$$

The following was first observed by Ghilardi and Zawadowski [9, pp.101]:

Proposition 3.7. A finitely presented algebra $\mathcal{F}_L(\overline{p}_1, \overline{p}_2)/\rho(\overline{p}_1, \overline{p}_2)$ is dual to an effective equivalence relation over $\tau(\overline{p})$ in $\text{Alg}(L)_{fp}^{op}$ if and only if $\rho(\overline{p}_1, \overline{p}_2)$ is a separating τ -coequivalence relation.

Proof. It is sufficient to observe that the formula in the definition above represents the kernel pair of a morphism $\tau(\overline{p}) \vdash_L \sigma(\overline{\psi}/\overline{\tau})$ in $\text{Alg}(L)_{fp}^{op}$. ■

Definition 3.8 (CoSP). A logic L is said to have the *coequivalence separation property* (CoSP) if for each formula $\tau(\bar{p})$, each τ -coequivalence relation is separating.

From Proposition 3.5 and Proposition 3.7, the following is the categorical meaning of the CoSP:

Proposition 3.9. A logic L with the Beth property has the CoSP if and only if in $\text{Alg}(L)_{fp}^{op}$ all equivalence relations are effective.

Remark 3.2. In model theoretic terms, the CoSP can be related to the framework of *imaginary elements* [17]. This was introduced by Bruno Poizat as a generalization of classical Galois theory, with the intention of providing a model-theoretic framework which generalizes the fundamental theorem of Galois theory. Our coequivalence relations are precisely equivalence formulas. The existence of formulas $\theta(\bar{x})$ witnessing the separation corresponds to the existence of a uniform schema for eliminating imaginaries – that is, picking equivalence classes in algebraic models. It would be interesting, in this vein, to determine the precise connection between uniform elimination of imaginaries and the CoSP, in a similar way to the known connections between interpolation and amalgamation (e.g., [16]), or uniform interpolation and model completions (e.g., [10]).

We now give a few examples to illustrate the meaning of the CoSP:

Example 3.10 (CPC has the CoSP). We will show that every coequivalence relation over CPC is separating, through elementary methods³. Assume that $\rho(\bar{p}_1, \bar{p}_2)$ is a coequivalence relation (over $\top$). The conditions of being a coequivalence relation means three things:

- (reflexivity) for any truth-assignment $v: \bar{p}_1, \bar{p}_2 \rightarrow \{0, 1\}$ where for each $p \in \bar{p}$, $v(p_1) = v(p_2)$, then v makes $\rho(\bar{p}_1, \bar{p}_2)$ true;
- (symmetry) if v makes $\rho(\bar{p}_1, \bar{p}_2)$ true, then if $v^\dagger$ is defined by setting, for each $p \in \bar{p}$, $v^\dagger(p_1) = v(p_2)$ and $v^\dagger(p_2) = v(p_1)$, we have that $v^\dagger$ also makes $\rho(\bar{p}_1, \bar{p}_2)$ true;
- (transitivity) if v makes $\rho(\bar{p}_1, \bar{p}_2)$ true and $v': \bar{p}_2, \bar{p}_3 \rightarrow \{0, 1\}$ is another valuation which makes $\rho(\bar{p}_2, \bar{p}_3)$ true, and such that for each $p \in \bar{p}$, $v(p_2) = v'(p_2)$, then the valuation $v'' = v \cup v'$ makes $\rho(\bar{p}_1, \bar{p}_3)$ true.

Given such a coequivalence relation, define an equivalence relation on the set $\text{Val}(\bar{p})$ of valuations v over $\bar{p}$ as follows: $v \sim v'$ if and only if there is a valuation v_0 over $\bar{p}_1, \bar{p}_2$ such that $v_0 \upharpoonright_{\bar{p}_1} = v$ and $v_0 \upharpoonright_{\bar{p}_2} = v'$ and v_0 validates $\rho(\bar{p}_1, \bar{p}_2)$. This is an equivalence relation by the properties just listed. For each equivalence class C_i , let ψ_i be the formula given by the disjunction of the (atoms corresponding to the) valuations in C_i , so that a valuation v over $\bar{p}$ validates ψ_i if and only if $v \in C_i$. Then we claim that:

$$\rho(\bar{p}_1, \bar{p}_2) \dashv\vdash \bigwedge_{i=1}^m \psi_i(\bar{p}_1) \leftrightarrow \psi_i(\bar{p}_2).$$

Notice that a valuation v on $\bar{p}_1$ and $\bar{p}_2$ validating the conjunction means that “ $v \upharpoonright_{\bar{p}_1} \in C_i$ if and only if $v \upharpoonright_{\bar{p}_2} \in C_i$ ” for all i ; this is equivalent to say that $v \upharpoonright_{\bar{p}_1} \sim v \upharpoonright_{\bar{p}_2}$, since $v \upharpoonright_{\bar{p}_1}$ must belong to some C_i . Then, by construction any valuation that validates ρ validates the conjunction. Vice versa, if v is a valuation validating the conjunction, then $v \upharpoonright_{\bar{p}_1} \sim v \upharpoonright_{\bar{p}_2}$. This means by assumption that there is some v_0 satisfying ρ , such that $v_0 \upharpoonright_{\bar{p}_1} = v \upharpoonright_{\bar{p}_1}$ and $v_0 \upharpoonright_{\bar{p}_2} = v \upharpoonright_{\bar{p}_2}$. This forces $v = v_0$, hence v validates ρ .

It will be useful to contrast this picture with the setting of S5. For that purpose we recall the frames for S5.

³Notice that by Proposition 3.9, it would suffice to observe that in $\mathbf{BA}_{fp}^{op}$ all equivalence relations are effective, which is well-known since such a category is equivalent to **FinSet**, and the latter category is a topos, where all equivalence relations are always effective.

Definition 3.11 (S5-frames). A pair (X, R) is called an *S5-frame* if R is an equivalence relation over a set X . We say that the frame is *locally finite* if for each $x \in X$, $R[x] = \{y \in X : xRy\}$ is finite. It is *finite* if X is finite. Obviously, finite frames are locally finite. Given such a frame, a subset $C \subseteq X$ is called a *cluster* if it is an R -equivalence class. We say that X is a *simple* frame if X itself is a cluster.

Definition 3.12 (S5-model). An *S5-model* over $\bar{p}$ is a pair $\mathfrak{M}, x$, where $\mathfrak{M}$ is a pair consisting of an S5-frame (X, R) , together with a function $X \rightarrow \text{Val}(\bar{p})$ (we denote by $v_y: \bar{p} \rightarrow \{0, 1\}$ the valuation associated to $y \in X$), and $x \in X$. If $\bar{q} \subseteq \bar{p}$, then $\mathfrak{M}^{\bar{q}}, x$ denotes the obvious restriction of $\mathfrak{M}, x$.

We assume the reader is familiar with the usual Kripke semantics for modal logic. Recall also that by Kripke completeness of S5 with respect to finite simple frames, we have that for any pair of formulas φ, ψ :

$$\varphi \vdash_{\text{S5}} \psi \iff \forall \mathfrak{M} \text{ based on a finite cluster, } (\forall x \in \mathfrak{M}, \mathfrak{M}, x \Vdash \varphi) \Rightarrow (\forall x \in \mathfrak{M}, \mathfrak{M}, x \Vdash \psi).$$

Recall the following representation of modal formulas:

Definition 3.13 (Nabla form). Let $\chi_1, \dots, \chi_n$ be a collection of formulas. We denote by $\nabla(\chi_1, \dots, \chi_n)$, the *nabla form*, the formula $\bigwedge_{i=1}^n \Diamond \chi_i \wedge \Box (\bigvee_{i=1}^n \chi_i)$.

Example 3.14 (S5 contains non-separating coequivalences). Let p be a propositional variable; we call a valuation v over p_1, p_2 *matching*, or (p_1, p_2) -*matching*, if $v(p_1) = v(p_2)$. Denote by $\text{Val}(p_1, p_2)$ the set of valuations, and by $\text{Mat}(p_1, p_2)$ the set of matching valuations over these variables. Then consider the formula:

$$\rho(p_1, p_2) = \bigvee_{S \subseteq \text{Mat}(p_1, p_2)} \nabla(v: v \in S) \vee \bigvee_{T \subseteq \text{Val}(p_1, p_2) \setminus \text{Mat}(p_1, p_2)} \nabla(v: v \in T),$$

Then ρ is a coequivalence relation over S5. To see this, note that $\rho(p, p)$ is logically equivalent to the disjunction of all possible (irreducible) simple S5-models over the variable p , hence it is a tautology. Moreover, $\rho(p_1, p_2)$ is logically equivalent to $\rho(p_2, p_1)$. Finally assume that $\mathfrak{M}, x$ is a model over p_1, p_2, p_3 , that validates $\rho(p_1, p_2)$ and $\rho(p_2, p_3)$. Let $\mathfrak{M}^{p_1, p_2}, x$ and $\mathfrak{M}^{p_2, p_3}, x$ be the two restrictions of this model to the specified propositional letters. By considering the four cases, we see that $\mathfrak{M}^{p_1, p_3}$ will be matching whenever both pairs match or both do not match, and will be non-matching otherwise. Thus transitivity follows.

Now we show that this coequivalence is not separating. Assume that there is a sequence of formulas $\psi_1(p), \dots, \psi_m(p)$ such that $\rho \vdash \bigwedge_{i=1}^m \psi_i(p_1) \leftrightarrow \psi_i(p_2)$. We prove that $\bigwedge_{i=1}^m \psi_i(p_1) \leftrightarrow \psi_i(p_2) \vdash \rho$ cannot be the case. Let $\mathfrak{M}$ be a model over a (finite) cluster, containing exactly one point $y \in \mathfrak{M}$ such that $v_y = v$, for each $v \notin \text{Mat}(p_1, p_2)$. By assumption, $\mathfrak{M} \Vdash \rho(p_1, p_2)$, so $\mathfrak{M} \Vdash \bigwedge_{i=1}^m \psi_i(p_1) \leftrightarrow \psi_i(p_2)$. Let $\mathfrak{M}^+$ be the model obtained by adding to the cluster $\mathfrak{M}$ a point e corresponding to a single fixed valuation v which is matching. We claim that then it is still the case that $\mathfrak{M}^+ \Vdash \bigwedge_{i=1}^m \psi_i(p_1) \leftrightarrow \psi_i(p_2)$. To see this, observe that, for each $x \in \mathfrak{M}$ (i.e. $x \in \mathfrak{M}^+$ different from e), $(\mathfrak{M}^+)^{p_1}, x \cong \mathfrak{M}^{p_1}, x$: this is because $(\mathfrak{M}^+)^{p_1}, x$ admits a surjective p-morphism to $\mathfrak{M}^{p_1}, x$ ($y \in \mathfrak{M}^+$ different from e is sent to itself, while e is sent to a non-matching point e^* , whose valuation restricted to p_1 coincides with that of e). Similarly $(\mathfrak{M}^+)^{p_2}, x \cong \mathfrak{M}^{p_2}, x$. Thus we have that for each ψ_i , and $x \in \mathfrak{M}$, by composing these equivalences with the assumption that $\mathfrak{M}, x \Vdash \psi_i(p_1) \leftrightarrow \psi_i(p_2)$:

$$\mathfrak{M}^+, x \Vdash \psi_i(p_1) \iff \mathfrak{M}^+, x \Vdash \psi_i(p_2).$$

Moreover, $(\mathfrak{M}^+)^{p=p_1}, e \cong (\mathfrak{M}^+)^{p=p_2}, e$ (via the isomorphism sending e to itself and each $x \in \mathfrak{M}$ to its symmetric $x^\dagger$ exchanging the valuations of p_1 and p_2). Reasoning as before, we have that $\mathfrak{M}^+, e \Vdash \psi_i(p_1)$ if and only if $\mathfrak{M}^+, e \Vdash \psi_i(p_2)$. We conclude that $\mathfrak{M}^+ \Vdash \bigwedge_{i=1}^m \psi_i(p_1) \leftrightarrow \psi_i(p_2)$.

However, $\mathfrak{M}^+, e \not\models \rho(p_1, p_2)$, since the point e neither sees only matching valuations nor does it see only valuations which do not match, by construction. Thus we have that ρ does not separate variables.

Remark 3.3. From Example 3.14, we see that the CoSP – at least in the case of locally tabular logics like S5 – is related to a sheaf-like condition, similar to the uniqueness of gluing. The reason why the CoSP holds in CPC is the obvious fact that a valuation v over $\bar{p}_1$ and $\bar{p}_2$ is uniquely determined by its restrictions $v|_{\bar{p}_1}$ and $v|_{\bar{p}_2}$. This fails in S5: there can be non-bisimilar models $\mathfrak{M}, x$ and $\mathfrak{N}, y$ over $\bar{p}_1$ and $\bar{p}_2$ such that $\mathfrak{M}^{\bar{p}_1}, x \cong \mathfrak{N}^{\bar{p}_1}, y$ and $\mathfrak{M}^{\bar{p}_2}, x \cong \mathfrak{N}^{\bar{p}_2}, y$. More so, as observed in [6, Theorem 6.10], in the extension S5₂ of S5 where clusters are restricted to at most 2-elements, the property again holds for similar reasons.

As remarked in the introduction, Ghilardi and Zawadowski showed that IPC fails to have the CoSP, and subsequently asked whether a natural weakening of it – categorically corresponding to *almost Barr-exactness* – actually holds. In light of Example 3.14, such a question can be asked already for S5. Moreover, they left open what the logical and algebraic content of such a property should be. We turn to this in the next section.

4 Local coequivalence relations and effective descent

4.1 Local transition terms

In the previous section we saw that coequivalence relations encode in a logical formula some implicit identification of elements. This is a general phenomenon in mathematics: in topology, for instance, equivalence relations on a space likewise provide implicit identifications between elements. Nevertheless, there might be equivalences which identify elements in terms of some algebraic data – such as what happens in the setting of topology with local homeomorphisms, and sheaves over a space.

In a logical setting, one can likewise ask whether such local kinds of equivalences appear. For this purpose, we introduce the main technical tool arising in the context of our work. For ease of notation, given $\alpha(\bar{p})$, we write $\text{Ext}(\alpha) = \{\beta(\bar{p}, \bar{q}) : \beta \vdash_L \alpha\}$.

Definition 4.1. Let $\gamma(\bar{p}, \bar{q}, \bar{r}) \in \text{Ext}(\beta(\bar{p}, \bar{q}))$. A sequence $\bar{\xi}$ of formulas over the variables $\bar{p}, \bar{q}_1, \bar{q}_2, \bar{r}$, namely $\xi_r(\bar{p}, \bar{q}_1, \bar{q}_2, \bar{r})$ for each $r \in \bar{r}$, is called a sequence of *local transition terms* for γ if the following hold:

1. (γ -compatibility): $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{p}, \bar{q}_2, \bar{\xi})$.
2. (Identity): For each $r \in \bar{r}$, $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L r \leftrightarrow \xi_r(\bar{p}, \bar{q}, \bar{q}, \bar{r})$.
3. (Cocycle): For each $r \in \bar{r}$, $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \wedge \beta(\bar{p}, \bar{q}_3) \vdash_L \xi_r(\bar{p}, \bar{q}_1, \bar{q}_3, \bar{r}) \leftrightarrow \xi_r(\bar{p}, \bar{q}_2, \bar{q}_3, \bar{\xi})$.

To obtain some intuition for these conditions, assume that $\bar{p} = \emptyset$, $\bar{q}_1 = \{q_1\}$, $\bar{q}_2 = \{q_2\}$ and $\bar{r} = \{r\}$. Then one can think of ξ as defining, for every pair of terms b, b' , a function $\xi_{b,b'}(r)$, which we can think of as a “local transition from b to b' ”. In this context the above conditions have the following meaning:

1. γ -compatibility means that ξ -transitions occur in the context of γ and β ;
2. Identity says that the transition from b to b is the identity;
3. The cocycle condition simply ensures that transitions compose.

Example 4.2 (Local transition term in S5). We provide an example of a local transition term in S5. Let $\beta(q) := \Diamond q \wedge \Diamond \neg q$, and $\gamma(q, r) := \gamma_M(q, r) \vee \gamma_U(q, r)$, where

$$\gamma_M(q, r) := \nabla(q \wedge r, \neg q \wedge \neg r) \quad \text{and} \quad \gamma_U(q, r) := \nabla(q \wedge \neg r, \neg q \wedge r).$$

Given propositional variables q and r , define $\mu(q, r) := (q \wedge r) \vee (\neg q \wedge \neg r)$; observe that $\neg\mu(q, r) = (q \wedge \neg r) \vee (\neg q \wedge r)$. We show that

$$\xi(q_1, q_2, r) := \mu(q_2, \mu(q_1, r)) = (q_2 \wedge \mu(q_1, r)) \vee (\neg q_2 \wedge \neg\mu(q_1, r))$$

is a local transition term for γ .

Note that for a valuation v , we have that $v(\xi) = 1$ whenever “ $v(q_2) = 1$ if and only if $v(q_1) = v(r)$ ”; from this, (Identity) and (Cocycle) follow by unfolding the definitions. We show (γ -compatibility). We prove that $\gamma(q_1, r) \wedge \beta(q_2) \vdash \gamma(q_2, \xi)$. Let $\mathfrak{M}, x$ be a model satisfying $\gamma(q_1, r)$ and $\beta(q_2)$; without loss of generality, assume that $\mathfrak{M}, x$ satisfies $\gamma_M(q_1, r) = \nabla(q_1 \wedge r, \neg q_1 \wedge \neg r)$. To show that $\mathfrak{M}, x$ satisfies $\gamma(q_2, \xi)$, we prove that $\mathfrak{M}, x \Vdash \gamma_M(q_2, \xi)$. For what we said before,

$$\neg\xi(q_1, q_2, r) = \neg\mu(q_2, \mu(q_1, r)) = (q_2 \wedge \neg\mu(q_1, r)) \vee (\neg q_2 \wedge \mu(q_1, r)),$$

hence $\gamma_M(q_2, \xi) = \nabla(q_2 \wedge \xi, \neg q_2 \wedge \neg\xi) = \nabla(q_2 \wedge \mu(q_1, r), \neg q_2 \wedge \mu(q_1, r))$. Given $x' \in \mathfrak{M}$, by $\mathfrak{M}, x' \Vdash \gamma_M(q_1, r)$, we have that $\mathfrak{M}, x'$ satisfies one between: $q_1 \wedge r$ and q_2 ; $q_1 \wedge r$ and $\neg q_2$; $\neg q_1 \wedge \neg r$ and q_2 ; $\neg q_1 \wedge \neg r$ and $\neg q_2$. As a consequence, $\mathfrak{M}, x'$ satisfies one between $q_2 \wedge \mu(q_1, r)$ and $\neg q_2 \wedge \mu(q_1, r)$. Vice versa, using that $\mathfrak{M}, x$ satisfies $\Diamond q_2 \wedge \Diamond \neg q_2$, we can find $x' \in \mathfrak{M}$ such that $\mathfrak{M}, x'$ satisfies q_2 ; using that $\mathfrak{M}, x$ satisfies $\gamma_M(q_1, r) = \nabla(q_1 \wedge r, \neg q_1 \wedge \neg r)$, we also get that $\mathfrak{M}, x'$ satisfies $\mu(q_1, r)$; putting everything together, $\mathfrak{M}, x' \Vdash q_2 \wedge \mu(q_1, r)$. A similar argument proves the existence of $x'' \in \mathfrak{M}$ such that $\mathfrak{M}, x'' \Vdash \neg q_2 \wedge \mu(q_1, r)$.

Definition 4.3. Let $\alpha(\bar{p}), \beta(\bar{p}, \bar{q}) \in \text{Ext}(\alpha)$ and $\gamma(\bar{p}, \bar{q}, \bar{r}) \in \text{Ext}(\beta)$, equipped with a sequence of local transition terms ξ . The *local coequivalence relation* $\rho_{\gamma, \xi}$ associated with γ and ξ , is the coequivalence relation over γ ,

$$\rho_{\gamma, \xi} = \gamma(\bar{p}_1, \bar{q}_1, \bar{r}_1) \wedge \gamma(\bar{p}_2, \bar{q}_2, \bar{r}_2) \wedge \bigwedge_{p \in \bar{p}} (p_1 \leftrightarrow p_2) \wedge \bigwedge_{r \in \bar{r}} (\xi_r(\bar{p}_1, \bar{q}_1, \bar{q}_2, \bar{r}_1) \leftrightarrow r_2).$$

Note that a local coequivalence relation $\rho_{\gamma, \xi}$ is indeed a γ -coequivalence relation: it is in the domain γ by γ -compatibility, satisfies reflexivity by (Identity), and is symmetric and transitive by (Cocycle).

Remark 4.1. As noted in Remark 3.3, the CoSP corresponds seemingly to a sheaf-like condition, ensuring the uniqueness of gluings of models over their restrictions. Local transition terms allow us to weaken this condition: rather than requiring the uniqueness of gluings for models over an even splitting of the propositional variables, we require it only for models which lie in the same path of the local transition, where a model $\mathfrak{M}_{1, x_1}$ over $\bar{p}_1, \bar{q}_1, \bar{r}_1$ has a transition to $\mathfrak{M}_{2, x_2}$ over $\bar{p}_2, \bar{q}_2, \bar{r}_2$ if there exists $\mathfrak{N}, y$ over the variables $\bar{p}, \bar{q}_1, \bar{q}_2, \bar{r}$, such that $\mathfrak{N}^{\bar{p}_1 = \bar{p}, \bar{q}_1 = \bar{r}} y \cong \mathfrak{M}_{1, x_1}$ and $\mathfrak{N}^{\bar{p}_2 = \bar{p}, \bar{q}_2 = \bar{r}} y \cong \mathfrak{M}_{2, x_2}$.

From this we extract the following property:

Definition 4.4. A logic L has the *local coequivalence separation property* (LCoSP) if for each formula $\alpha(\bar{p})$, each $\beta(\bar{p}, \bar{q}) \in \text{Ext}(\alpha)$, and each $\gamma(\bar{p}, \bar{q}, \bar{r}) \in \text{Ext}(\beta)$ together with a sequence of local transition terms ξ for γ , the local coequivalence relation $\rho_{\gamma, \xi}$ separates variables.

4.2 Descent data and Barr-exactness

Local coequivalence relations correspond to a natural weakening of Barr-exactness [13], using the notion of “descent data”, which we now introduce. We refer to [13, 14] for the following notions.

Let $\mathbf{C}$ be a category with pullbacks and $p : E \rightarrow B$ a morphism in $\mathbf{C}$. For an object B in $\mathbf{C}$, let $\mathbf{C}/B$ denote the slice category over B , that is the category whose objects are pairs (Y, g) , with $g : Y \rightarrow B$ in $\mathbf{C}$,

and whose morphisms $(Y, g) \rightarrow (Y', g')$ are morphisms $h: Y \rightarrow Y'$ in $\mathbf{C}$ such that $g'h = g$. There is a pair of adjoint functors

$$p_! \dashv p^*: \mathbf{C}/B \longrightarrow \mathbf{C}/E$$

(with p^* pulling back along p and with $p_!$ composing with p from the left). This induces a *monad*⁴ on $\mathbf{C}/E$. We denote by $\text{Des}(p)$ the *category of descent data for p* , i.e. the category of *algebras*, in the sense of Eilenberg-Moore, for such a monad. An object of $\text{Des}(p)$, called *descent datum for p* , is then a triple $(X, f; \xi)$, with $(X, f) \in \mathbf{C}/E$ and $\xi: X \otimes_B E \rightarrow X$ in $\mathbf{C}$ satisfying the following compatibility conditions:

$$\begin{array}{ccc} X & \xrightarrow{(1_X, f)} & X \otimes_B E \\ 1_X \downarrow & \begin{array}{c} (2) \searrow \xi \\ (1) \swarrow \end{array} & \downarrow \pi_2 \\ X & \xrightarrow{f} & E \end{array} \qquad \begin{array}{ccc} (X \otimes_B E) \otimes_B E & \xrightarrow{\xi \otimes_B 1_E} & X \otimes_B E \\ \pi_1 \otimes_B 1_E \downarrow & (3) & \downarrow \xi \\ X \otimes_B E & \xrightarrow{\xi} & X \end{array} \quad (\text{D})$$

A morphism $(X, f; \xi) \rightarrow (X', f'; \xi')$ in $\text{Des}(p)$ is a morphism $h: (X, f) \rightarrow (X', f')$ in $\mathbf{C}/E$ such that

$$\begin{array}{ccc} X \otimes_B E & \xrightarrow{h \otimes_B 1_E} & X' \otimes_B E \\ \xi \downarrow & & \downarrow \xi' \\ X & \xrightarrow{h} & X' \end{array}$$

commutes. This induces a comparison functor $\Phi^p: \mathbf{C}/B \rightarrow \text{Des}(p)$, sending $(Y, g) \in \mathbf{C}/B$ to the descent data $(Y \otimes_B E, \pi_2; \xi)$, with ξ given by $\pi_1 \otimes_B 1_E: (Y \otimes_B E) \otimes_B E \rightarrow Y \otimes_B E$.

Definition 4.5. We say that p is a *descent morphism* if Φ^p is full and faithful. If Φ^p is an equivalence of categories, then p is said to be an *effective descent morphism*.

Remark 4.2. If p is an (effective) descent morphism, then it is a universal regular epimorphism, meaning that its pullback along any morphism is a regular epimorphism. If $\mathbf{C}$ is Barr-exact, then the converse holds (see [14]). However, even assuming regularity, the fact that all regular epimorphisms are effective descent morphisms is not sufficient to ensure that $\mathbf{C}$ is Barr-exact [13, pp.395,(ii)].

Definition 4.6. A regular category $\mathbf{C}$ is said to be *almost Barr-exact* if all regular epimorphisms are effective descent morphisms.

Almost Barr-exactness of duals of categories of algebras has been studied in the case of loops, magmas and groups [19]. In regular categories, we have a nice characterization of effective descent morphisms:

Theorem 4.7. *Let $\mathbf{C}$ be a regular category, $p: E \rightarrow B$ a morphism. Then:*

1. *Given $(X, f; \xi) \in \text{Des}(p)$ a descent datum, the morphism $\tilde{\xi} := (\pi_1, \xi): X \otimes_B E \rightarrow X \otimes X$ is an equivalence relation (the equivalence relation associated to the descent datum $(X, f; \xi)$);*
2. *If p is a regular epimorphism, then it is an effective descent morphism if and only if for each $(X, f; \xi) \in \text{Des}(p)$, the equivalence relation $\tilde{\xi}$ is effective.*

Proof. See [13, Theorem 3.7 (c)] ■

⁴See e.g. [8] for basic concepts and definitions on monads, and algebras for a monad.

We say that an equivalence relation $R \hookrightarrow X \otimes X$ comes from a descent datum over $f : X \rightarrow E$ for $p : E \rightarrow B$ if it is isomorphic to $\tilde{\xi}$, for some $(X, f; \xi) \in \text{Des}(p)$. The following appears to be folklore.

Theorem 4.8. *Let $(\pi_1, \pi_2) : R \hookrightarrow X \otimes X$ be an equivalence relation. Then R comes from a descent datum over $f : X \rightarrow E$ for $p : E \rightarrow B$ if, and only if,*

$$\begin{array}{ccc} R & \xrightarrow{\pi_1} & X \\ f\pi_2 \downarrow & & \downarrow pf \\ E & \xrightarrow[p]{} & B \end{array} \quad (\text{P})$$

is a diagram of pullback. If that is the case, $\pi_2 : R \rightarrow X$ is the descent datum whose associated equivalence relation is R .

Proof. This follows immediately from the equivalence between the category $\text{Des}(p)$ introduced above and the category of discrete fibrations over the kernel pair of p (see [13, Section 5.A]). ■

We thus refer to R as a *descent equivalence relation* if R satisfies the above conditions relative to some f and p .

Corollary 4.9. *A regular category $\mathbf{C}$ is almost Barr-exact if, and only if, all descent equivalence relations are effective.*

4.3 Local coequivalences and descent

In this section we establish the connection between the logical property of LCoSP and descent. Recall from 2.1 that we identify a formula $\alpha(\bar{p})$ with the object in $\text{Alg}(L)_{fp}^{op}$ dual to the finitely presented algebra $\mathcal{F}_L(\bar{p})/\alpha(\bar{p})$ and that derivations $\beta(\bar{q}) \vdash_L \alpha(\bar{\theta}/\bar{p})$ represent morphisms from β to α in $\text{Alg}(L)_{fp}^{op}$.

To understand the connection, first we lay out all necessary elements, and relate them dually to their meaning in the previous section:

1. We consider $\alpha(\bar{p})$ and $\beta(\bar{p}, \bar{q})$ and a morphism $\beta(\bar{p}, \bar{q}) \vdash_L \alpha(\bar{p})$ in $\text{Alg}(L)_{fp}^{op}$ (read $p : E \rightarrow B$).
2. Morphisms to β correspond to $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \beta(\bar{p}, \bar{q})$ (read $f : X \rightarrow E$).
3. For any such morphism, one can compute the pullback of $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \alpha(\bar{p})$ along $\beta(\bar{p}, \bar{q}) \vdash_L \alpha(\bar{p})$ (read $X \otimes_B E$). This comes equipped with two projections:

$$\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{p}, \bar{q}_1/\bar{q}, \bar{r}) \text{ and } \gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \beta(\bar{p}, \bar{q}_2/\bar{q}).$$

4. A *descent datum* over $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \beta(\bar{p}, \bar{q})$ for $\beta(\bar{p}, \bar{q}) \vdash_L \alpha(\bar{p})$ is therefore given by a *connecting morphism*

$$\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{\mu}/\bar{p}, \bar{v}/\bar{q}, \bar{\xi}/\bar{r}),$$

(read as $X \otimes_B E \rightarrow X$), making the diagrams (1-3) of (D) commute.

Note that in light of the previous point, we can see that *local transition terms* have the precise shape necessary to be connecting morphisms. We now prove that they satisfy the necessary conditions:

Proposition 4.10. Let L be normal modal logic, $\alpha(\bar{p})$ a formula, $\beta(\bar{p}, \bar{q}) \in \text{Ext}(\alpha)$, and $\gamma(\bar{p}, \bar{q}, \bar{r}) \in \text{Ext}(\beta)$. To give a morphism from $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2)$ to $\gamma(\bar{p}, \bar{q}, \bar{r})$ in $\text{Alg}(L)_{fp}^{op}$ that makes the diagram (1) of (D) commute, is equivalent to giving a sequence $\bar{\xi} = (\xi_r(\bar{p}, \bar{q}_1, \bar{q}_2, \bar{r}) : r \in \bar{r})$ that satisfies $(\gamma\text{-compatibility})$ of Definition 4.1. Moreover:

- a) the diagram (2) of (D) commutes if and only if $\bar{\xi}$ satisfies (Identity) of Definition 4.1;
- b) the diagram (3) of (D) commutes if and only if $\bar{\xi}$ satisfies (Cocycle) of Definition 4.1.

Proof. As we said before, a morphism from $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2)$ to $\gamma(\bar{p}, \bar{q}, \bar{r})$ in $\text{Alg}(L)_{fp}^{op}$ is specified by sequences of formulas $\bar{\mu} = (\mu_p : p \in \bar{p})$, $\bar{v} = (v_q : q \in \bar{q})$ and $\bar{\xi} = (\xi_r : r \in \bar{r})$ in the variables $\bar{p}, \bar{q}_1, \bar{q}_2, \bar{r}$. Commutativity of (1) establishes that the projection $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \beta(\bar{p}, \bar{q}_2/\bar{q})$ coincides with the composition of $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{\mu}/\bar{p}, \bar{v}/\bar{q}, \bar{\xi}/\bar{r})$ with $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \beta(\bar{p}, \bar{q})$; the latter is $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \beta(\bar{\mu}/\bar{p}, \bar{v}/\bar{q})$ (composition is simply the composition of substitutions). The two morphisms coincide if and only if, up to $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2)$, the substitution $\bar{\mu}$ is equivalent to $\bar{p}$ and the substitution $\bar{v}$ is equivalent to $\bar{q}_2$; we can therefore simplify the specification of the initial morphism (as it presents the same morphism) in the following equivalent way

$$\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{p}, \bar{q}_2/\bar{q}, \bar{\xi}/\bar{r}),$$

which is exactly (γ -compatibility).

Moreover, the diagram (2) commutes if and only if the composition of $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \gamma(\bar{p}, \bar{q}/\bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}/\bar{q}_2)$ (read as $(1_x, f): X \rightarrow X \otimes_B E$) and our morphism $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{p}, \bar{q}_2/\bar{q}, \bar{\xi}/\bar{r})$, namely $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \gamma(\bar{p}, \bar{q}, \bar{\xi}(\bar{p}, \bar{q}/\bar{q}_1, \bar{q}/\bar{q}_2, \bar{r})/\bar{r})$, is the identity; this is equivalent to say that

$$\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L r \leftrightarrow \xi_r(\bar{p}, \bar{q}/\bar{q}_1, \bar{q}/\bar{q}_2, \bar{r})$$

for each $r \in \bar{r}$, which is exactly (Identity). The proof of b) is similar (observe that $(X \otimes_B E) \otimes_B E$ is $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \wedge \beta(\bar{p}, \bar{q}_3)$, with obvious projections). ■

Recall that the local coequivalence relation associated to γ and $\bar{\xi}$, is given explicitly by:

$$\rho_{\gamma, \bar{\xi}} = \gamma(\bar{p}_1, \bar{q}_1, \bar{r}_1) \wedge \gamma(\bar{p}_2, \bar{q}_2, \bar{r}_2) \wedge \bigwedge_{p \in \bar{p}} (p_1 \leftrightarrow p_2) \wedge \bigwedge_{r \in \bar{r}} (\xi_r(\bar{p}_1, \bar{q}_1, \bar{q}_2, \bar{r}_1) \leftrightarrow r_2).$$

Proposition 4.11. The finitely presented algebra $\mathcal{F}_L(\bar{p}_1, \bar{q}_1, \bar{r}_1, \bar{p}_2, \bar{q}_2, \bar{r}_2)/\rho_{\gamma, \bar{\xi}}$ is dual to the equivalence relation associated to the descent datum $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{p}, \bar{q}_2/\bar{q}, \bar{\xi}/\bar{r})$ over $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \beta(\bar{p}, \bar{q})$ for $\beta(\bar{p}, \bar{q}) \vdash_L \alpha(\bar{p})$.

Proof. Observe that the diagram (P) commutes. Indeed, the composition of $\rho_{\bar{\xi}, \gamma} \vdash_L \gamma(\bar{p}_1/\bar{p}, \bar{q}_1/\bar{q}, \bar{r}_1/\bar{r})$ and $\gamma(\bar{p}, \bar{q}, \bar{r}) \vdash_L \alpha(\bar{p})$, which is $\rho_{\bar{\xi}, \gamma} \vdash_L \alpha(\bar{p}_1/\bar{p})$, is equal to the composition of $\rho_{\bar{\xi}, \gamma} \vdash_L \beta(\bar{p}_2/\bar{p}, \bar{q}_2/\bar{q})$ and $\beta(\bar{p}, \bar{q}) \vdash_L \alpha(\bar{p})$, which is $\rho_{\bar{\xi}, \gamma} \vdash_L \alpha(\bar{p}_2/\bar{p})$: this is because $\rho_{\bar{\xi}, \gamma} \vdash_L p_1 \leftrightarrow p_2$ for all $p \in \bar{p}$. By the universal property of the pullback, there exists a unique morphism from $\rho_{\bar{\xi}, \gamma}$ to $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2)$, making the projections commute; it is straightforward to verify that such a morphism is

$$\rho_{\gamma, \bar{\xi}}(\bar{p}_1, \bar{q}_1, \bar{r}_1, \bar{p}_2, \bar{q}_2, \bar{r}_2) \vdash_L \gamma(\bar{p}_1/\bar{p}, \bar{q}_1/\bar{q}, \bar{r}_1/\bar{r}) \wedge \beta(\bar{p}_1/\bar{p}, \bar{q}_2/\bar{q}). \quad (\text{Univ})$$

(it is indeed a derivation, since $\rho_{\bar{\xi}, \gamma} \vdash_L \bigwedge_{p \in \bar{p}} (p_1 \leftrightarrow p_2)$ and $\gamma(\bar{p}_2, \bar{q}_2, \bar{r}_2) \vdash_L \beta(\bar{p}_2, \bar{q}_2)$, using substitution). By Theorem 4.8, the claim follows once we prove that the morphism (Univ) is an isomorphism and that, up to (Univ), the second projection $\rho_{\bar{\xi}, \gamma} \vdash_L \gamma(\bar{p}_2/\bar{p}, \bar{q}_2/\bar{q}, \bar{r}_2/\bar{r})$ is the descent datum $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{p}, \bar{q}_2/\bar{q}, \bar{\xi}/\bar{r})$. To prove that (Univ) is an isomorphism, it suffices to observe that

$$\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \rho_{\gamma, \bar{\xi}}(\bar{p}/\bar{p}_1, \bar{q}_1/\bar{r}_1, \bar{p}/\bar{p}_2, \bar{q}_2/\bar{\xi}/\bar{r}_2) \quad (\text{Univ}^*)$$

is its inverse: the above is a derivation by (γ -compatibility); the composite $(\text{Univ} \circ \text{Univ}^*)$ sends each propositional letter to itself, and is therefore trivial; whilst the only non-trivial substitutions of the composition $(\text{Univ}^* \circ \text{Univ})$ are the following: $\bar{p}_2$ is sent to $\bar{p}_1$, which ρ proves equivalent to $\bar{p}_2$, and $\bar{r}_2$ to $\xi(\bar{p}_1, \bar{q}_1, \bar{q}_2, \bar{r}_1)$ which ρ proves equivalent to $\bar{r}_2$.

Now observe that the composition of (Univ^*) with the second projection $\rho_{\bar{\xi}, \gamma} \vdash_L \gamma(\bar{p}_2/\bar{p}, \bar{q}_2/\bar{q}, \bar{r}_2/\bar{r})$ is exactly the descent datum $\gamma(\bar{p}, \bar{q}_1, \bar{r}) \wedge \beta(\bar{p}, \bar{q}_2) \vdash_L \gamma(\bar{p}, \bar{q}_2/\bar{q}, \bar{\xi}/\bar{r})$. ■

Theorem 4.12. *For L a logic with the Beth property, the following are equivalent:*

1. *L has the local coequivalence separation property.*
2. *In $\text{Alg}(L)_{fp}^{op}$ all descent equivalence relations are effective.*

If L has $\exists_{\text{MIP}}$ and the Beth property, then this is equivalent to $\text{Alg}(L)_{fp}^{op}$ being almost Barr-exact.

Proof. The fact that local coequivalences being separating coincides with descent equivalence relations being effective follows from Proposition 4.11. By Proposition 2.3, $\exists_{\text{MIP}}$ corresponds to regularity of the category, and hence in such a setting, the category will be almost Barr-exact. ■

5 Coequivalence separation in S5

In this final section we will focus on the case of the modal logic system S5. We begin by outlining a representation of limits of finite S5 frames, originally due to [2] (see also [11]) which will be convenient for our purposes. We will establish a criterion for equivalence relations in S5 to be effective. From this we will obtain an easy proof that S5 fails to have CoSP. We will moreover show that LCoSP does hold.

5.1 Representation

Recall that a map $f: (X, R) \rightarrow (Y, S)$ between Kripke frames is called a *p-morphism* if (1) whenever xRy then $f(x)Sf(y)$ and (2) whenever $f(x)Sy$ then there is some $x' \in X$ such that xRx' and $f(x') = y$.

Definition 5.1. We denote by S5Fr the category of S5-frames with p-morphisms, by $\text{S5Fr}_{l.f}$ the full subcategory of locally finite S5-frames, and by S5Fr_f the full subcategory of S5 finite frames.

It is obvious to see that a map $f: (X, R) \rightarrow (Y, S)$ between simple (finite) S5-frames is a p-morphism if and only if it is surjective. Thus the category of finite and simple S5-frames is definitionally equivalent to the category $\mathbf{FinSet}_{\neq \emptyset}^s$ of non-empty finite sets and surjective functions between them. By dualising the universal algebraic idea of representing an algebra by its subdirectly irreducible algebras (in this case, all simple algebras), we can identify a locally finite S5 frame with a *family* $\mathcal{F}$ of non-empty finite sets. Under such an identification we have the following easy facts:

Proposition 5.2. Let $\mathcal{F}$ and $\mathcal{G}$ belong to $\text{S5Fr}_{l.f}$ (or S5Fr_f):

1. A p-morphism $f: \mathcal{F} \rightarrow \mathcal{G}$ is equivalent to a *function* $f: \mathcal{F} \rightarrow \mathcal{G}$ and, a *family of surjective functions*, indexed on $\mathcal{F}$, $\{f_X: X \rightarrow f(X) \mid X \in \mathcal{F}\}$.
2. The subobjects of $\mathcal{G}$ are precisely the images of injective functions $f: \mathcal{F} \rightarrow \mathcal{G}$, with all f_X bijective.

For simplicity, the morphism will be denoted just by $f: \mathcal{F} \rightarrow \mathcal{G}$. The category $\text{S5Fr}_{l.f}$ is then an instance of the general construction associating to a category $\mathbf{C}$ the *category of families* $\text{Fam}(\mathbf{C})$, with $\mathbf{C} = \mathbf{FinSet}_{\neq \emptyset}^s$; moreover, S5Fr_f is equivalent to the full subcategory $\text{Fam}_f(\mathbf{C})$ of finite families (see [3]). From now on, we will use the notation of families to refer to the objects and the morphisms of $\text{S5Fr}_{l.f}$ and S5Fr_f .

Definition 5.3. Let X and Y be non-empty finite sets. A *bisimulation* from X to Y is a relation $A \subseteq X \times Y$ such that for each $x \in X$, there exists $y \in Y$ such that $(x, y) \in A$; and for each $y \in Y$, there exists $x \in X$ such that $(x, y) \in A$.

In other words, the restrictions of the two projections $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$ to A are both surjective functions. We denote by $B(X, Y)$ the set of bisimulations from X to Y . Bisimulations provide a nice way to describe pullbacks in $S5Fr_{l,f}$ and $S5Fr_f$.⁵ Consider the following diagram

$$\begin{array}{ccc} & \mathcal{W} & \\ & \downarrow q & \\ \mathcal{F} & \xrightarrow{p} & \mathcal{G} \end{array}$$

in either $S5Fr_{l,f}$ or $S5Fr_f$.

Definition 5.4. We say that $X \in \mathcal{W}$ and $U \in \mathcal{F}$ are (q, p) -compatible if $q(X) = p(U)$. In such a case we say that $D \in B(X, U)$ is (q, p) -admissible if, for all $(x, u) \in D$, we have $q_X(x) = p_U(u)$. Define $\text{Bis}(q, p)$ as the disjoint union

$$\text{Bis}(q, p) := \bigcup \{D \in B(X, U) \mid D \text{ is } (q, p)\text{-admissible and } X, U \text{ are } (q, p)\text{-compatible}\}.$$

From now on, $\text{Bis}(q, p)$ will be seen as an object of $S5Fr_{l,f}$ (each $D \in \text{Bis}(q, p)$ is a non-empty finite set, being a bisimulation from some $X \in \mathcal{W}$ to some $U \in \mathcal{F}$) and of $S5Fr_f$, in case both $\mathcal{W}$ and $\mathcal{F}$ are finite. Consider $D \in \text{Bis}(q, p)$, say $D \in B(X, U)$ (q, p) -admissible with $X \in \mathcal{W}$ and $U \in \mathcal{F}$ (q, p) -compatible. Set $\pi_1(D) := X$ and $\pi_2(D) := U$ and let

$$(\pi_1)_D : D \rightarrow \pi_1(D) = X \quad \text{and} \quad (\pi_2)_D : D \rightarrow \pi_2(D) = U$$

be the corresponding projections. Since $(\pi_1)_D$ and $(\pi_2)_D$ are surjective functions, D being a bisimulation, the data above define a pair of p-morphisms $\pi_1 : \text{Bis}(q, p) \rightarrow \mathcal{W}$ and $\pi_2 : \text{Bis}(q, p) \rightarrow \mathcal{F}$ (by Proposition 5.2.1.).

Proposition 5.5. The diagram

$$\begin{array}{ccc} \text{Bis}(q, p) & \xrightarrow{\pi_1} & \mathcal{W} \\ \pi_2 \downarrow & & \downarrow q \\ \mathcal{F} & \xrightarrow{p} & \mathcal{G} \end{array}$$

is a pullback in $S5Fr_{l,f}$ (and in $S5Fr_f$, if $\mathcal{W}$, $\mathcal{F}$ and $\mathcal{G}$ are finite).

Proof. Consider p-morphisms $f_1 : \mathcal{V} \rightarrow \mathcal{W}$ and $f_2 : \mathcal{V} \rightarrow \mathcal{F}$ such that $qf_1 = pf_2$. For each $Y \in \mathcal{V}$, we have $f_1(Y) \in \mathcal{W}$ and $f_2(Y) \in \mathcal{F}$, and the pair of surjective functions $(f_i)_Y : Y \rightarrow f_i(Y)$ induce a unique function $Y \rightarrow f_1(Y) \times f_2(Y)$; call $f(Y) \subseteq f_1(Y) \times f_2(Y)$ its image. Note that because $(f_1)_Y$ and $(f_2)_Y$ are surjective, we have that $f(Y)$ is a bisimulation from $f_1(Y)$ to $f_2(Y)$. Moreover, by $qf_1 = pf_2$, we have that $f_1(Y)$ and $f_2(Y)$ are (q, p) -compatible and that $f(Y)$ is (q, p) -admissible; in other words, $f(Y) \in \text{Bis}(q, p)$. Let $f_Y : Y \rightarrow f(Y)$ be the restriction of the $Y \rightarrow f_1(Y) \times f_2(Y)$ above on the codomain. By Proposition 5.2.1., the data above define a p-morphism $f : \mathcal{V} \rightarrow \text{Bis}(q, p)$, whose uniqueness is easily verified. ■

⁵This description goes back to [2], and [11].

Notation 5.1. When $\mathcal{G}$ is the terminal object, i.e. the family $\mathbf{1} := \{\{*\}\}$, then $\text{Bis}(q, p)$ is just the disjoint union, varying $X \in \mathcal{W}$ and $U \in \mathcal{F}$, of the sets $\text{B}(X, U)$ ((q, p) -compatibility and (q, p) -admissibility trivialize) it represents the product of $\mathcal{W}$ and $\mathcal{F}$. We then denote $\text{Bis}(q, p)$ by $\text{Bis}(\mathcal{W}, \mathcal{F})$. Moreover, if $\mathcal{W} = \mathcal{F}$, we simply write $\text{Bis}(\mathcal{W})$.

We will need a number of special operations on bisimulations.

Definition 5.6. Let $\mathcal{W}$ be a family, $X, Y, Z \in \mathcal{W}$, $A \in \text{B}(X, Y)$ and $B \in \text{B}(Y, Z)$. We say that $P \in \text{B}(A, B)$ is a *bisimulation of composable pairs* if for each $((x, y), (y', z)) \in P$ we have that $y = y'$. If P is a bisimulation of composable pairs, we denote by $\text{Comp}(P) = \{(x, z) \in X \times Z : \exists y \in Y, ((x, y), (y, z)) \in P\}$.

Proposition 5.7 (Closure properties of bisimulations). Given $\mathcal{W} \in \text{S5Fr}_{l.f}$ (or S5Fr_f), and $X, Y, Z \in \mathcal{W}$, we have:

1. For each $X \in \mathcal{W}$, the relation $\Delta_X = \{(x, x) \in X \times X : x \in X\}$ is in $\text{B}(X, X)$.
2. For each $A \in \text{B}(X, Y)$, we have $A^\dagger := \{(y, x) \in Y \times X \mid (x, y) \in A\}$ and $A^\dagger \in \text{B}(Y, X)$.
3. If $A \in \text{B}(X, Y)$ and $B \in \text{B}(Y, Z)$, the relation $AB := \{(x, z) \in X \times Z \mid \exists y \in Y ((x, y) \in A \text{ and } (y, z) \in B)\}$ is in $\text{B}(X, Z)$;
4. If $A \in \text{B}(X, Y)$ and $B \in \text{B}(Y, Z)$, and $P \in \text{B}(A, B)$, where P is a bisimulation of composable pairs, then $\text{Comp}(P) \in \text{B}(X, Z)$.

Proof. We show 4., the other points having similarly easy proofs. If $x \in X$, then because A is a bisimulation, there is some $y \in Y$ such that $(x, y) \in A$; since P is a bisimulation, there is then some $(y', z) \in B$ such that $((x, y), (y', z)) \in P$, which forces $y = y'$ (because P is a bisimulation of composable pairs). We found $z \in Z$ such that $(x, z) \in \text{Comp}(P)$. The other condition is similarly verified. ■

Remark 5.1. Note that the composition law above is stronger than simple composition of bisimulations. Given $X, Y, Z \in \mathcal{W}$, $A \in \text{B}(X, Y)$ and $B \in \text{B}(Y, Z)$ and $A, B \in \mathcal{E}$, then $AB = \{(x, z) : \exists y, (x, y) \in A \text{ and } (y, z) \in B\}$ is $\text{Comp}(A * B)$ where $A * B = \{((x, y), (y, z)) : (x, y) \in A, (y, z) \in B\}$, which is a bisimulation of composable pairs, since A, B are bisimulations. We note that the weaker condition, which we refer to as (*W-Composition*), that for $A, B \in \mathcal{E}$ we have $AB \in \mathcal{E}$ will often be sufficient in the proofs below.

We also note the following basic fact (see e.g. [18]):

Proposition 5.8. For each $\mathcal{W} \in \text{S5Fr}_{l.f}$, and $X, Y \in \mathcal{W}$ we have that $\text{B}(X, Y)$ forms a monoid: given $A, A' \in \text{B}(X, Y)$, if $A \subseteq A'$, then $AB \subseteq A'B$ and $CA \subseteq CA'$, whenever the compositions are defined. Moreover, we have $\Delta_X \subseteq AA^\dagger$ and $\Delta_Y \subseteq A^\dagger A$.

Recall from Proposition 5.2.2. that the lattice $\text{Sub}(\mathcal{G})$ of subobjects of $\mathcal{G}$ is (isomorphic to) the lattice of subsets $\mathcal{F} \subseteq \mathcal{G}$. Equivalence relations over $\mathcal{W}$ (see Definition 3.2) are thus special objects of $\text{Sub}(\text{Bis}(\mathcal{W}))$.

Definition 5.9. We say that a subobject $\mathcal{E} \in \text{Sub}(\text{Bis}(\mathcal{W}))$ is *equivalential* if:

1. (Identity) For each $X \in \mathcal{W}$, $\Delta_X \in \mathcal{E}$;
2. (Dagger) Whenever $X, Y \in \mathcal{W}$ and $A \in \text{B}(X, Y)$ and $A \in \mathcal{E}$, then $A^\dagger \in \mathcal{E}$;
3. (Composition) Whenever $X, Y, Z \in \mathcal{W}$, $A \in \text{B}(X, Y)$ and $B \in \text{B}(Y, Z)$, and $A, B \in \mathcal{E}$, and $P \in \text{B}(A, B)$ is a bisimulation of composable pairs, then $\text{Comp}(P) \in \mathcal{E}$.

Theorem 5.10. Let $\mathcal{W} \in \text{S5Fr}_{l.f}$. The following are equivalent for $\mathcal{E} \in \text{Sub}(\text{Bis}(\mathcal{W}))$:

1. $\mathcal{E}$ is an equivalential subobject of $\text{Bis}(\mathcal{W})$;
2. $\mathcal{E}$ is an equivalence relation on $\mathcal{W}$.

Proof. Assume that $\mathcal{E}$ is a subobject of $\text{Bis}(\mathcal{W})$. We show that:

- a) $\mathcal{E}$ satisfies (Identity) if and only if it satisfies internal reflexivity;
- b) $\mathcal{E}$ satisfies (Dagger) if and only if it satisfies internal symmetry;
- c) $\mathcal{E}$ satisfies (Composition) if and only if it satisfies internal transitivity.

We prove c), since a) and b) can be verified in a similar, albeit easier, manner. Recall that the categorical product $\mathcal{W} \otimes \mathcal{W}$ (represented by $\text{Bis}(\mathcal{W})$, with projections $\pi_i: \text{Bis}(\mathcal{W}) \rightarrow \mathcal{W}$) is an equivalence relation over $\mathcal{W}$ (it is the maximal one, meaning that all the others are subobjects of $\mathcal{W} \otimes \mathcal{W}$). Consider what we denoted by $(\mathcal{W} \otimes \mathcal{W}) \otimes_{\mathcal{W}} (\mathcal{W} \otimes \mathcal{W})$, namely the pullback of $\pi_2: \text{Bis}(\mathcal{W}) \rightarrow \mathcal{W}$ along $\pi_1: \text{Bis}(\mathcal{W}) \rightarrow \mathcal{W}$; it is represented by $\text{Bis}(\pi_2, \pi_1)$, with projections $\kappa_i: \text{Bis}(\pi_2, \pi_1) \rightarrow \text{Bis}(\mathcal{W})$. Unraveling the definitions of (π_2, π_1) -compatibility and (π_2, π_1) -admissibility, we obtain that $\text{Bis}(\pi_2, \pi_1)$ is the disjoint union, varying $X, Y, Z \in \mathcal{W}$, $A \in \text{B}(X, Y)$ and $B \in \text{B}(Y, Z)$, of the sets of bisimulations $P \in \text{B}(A, B)$ of composable pairs; then, the projection κ_1 sends P to A and $(\kappa_1)_P: P \rightarrow A$ sends $((x, y), (y, z))$ to (x, y) (κ_2 acts similarly). By Proposition 5.2.1., the family of surjective functions $P \rightarrow \text{Comp}(P)$, sending $((x, y), (y, z))$ to (x, z) , defines a p-morphism $\text{Comp}: \text{Bis}(\pi_2, \pi_1) \rightarrow \text{Bis}(\mathcal{W})$. It is straightforward to show that $\pi_1 \text{Comp} = \pi_1 \kappa_1$ and $\pi_2 \text{Comp} = \pi_2 \kappa_2$, proving that Comp is the morphism witnessing internal transitivity for the maximal relation $\mathcal{W} \otimes \mathcal{W}$ (as required by Definition 3.2 of categorical equivalence relation). For any other subobject $\mathcal{E} \in \text{Sub}(\mathcal{W} \otimes \mathcal{W})$, the pullback $\mathcal{E} \otimes_{\mathcal{W}} \mathcal{E}$ is a subobject of $\text{Bis}(\pi_2, \pi_1)$; it is given by those bisimulations $P \in \text{B}(A, B)$ of composable pairs for which $A, B \in \mathcal{E}$. Then, $\mathcal{E}$ satisfies internal transitivity if and only if the restriction of Comp to $\mathcal{E} \otimes_{\mathcal{W}} \mathcal{E}$ factorizes through $\mathcal{E}$ (see Remark 3.1); but this is exactly (Composition) of Definition 5.9 of equivalential subobject. ■

Let X, Y be non-empty finite sets. Note that in addition to the closure properties from Proposition 5.7, $\text{B}(X, Y)$ is closed under *non-empty*⁶ unions; moreover the unions are compatible with the composition relation, meaning that $A(B \cup B') = AB \cup AB'$ and $(C \cup C')A = CA \cup C'A$, whenever the compositions make sense. This observation allows us to characterize effectiveness of equivalence relations (see Definition 3.3):

Proposition 5.11. Let $\mathcal{E}$ be an equivalential subobject of $\text{Bis}(\mathcal{W})$. Then $\mathcal{E}$, as equivalence relation, is effective if, and only if, it is

1. (downward closed) If $X, Y \in \mathcal{W}$, $A, B \in \text{B}(X, Y)$, $A \in \mathcal{E}$ and $B \subseteq A$, then $B \in \mathcal{E}$;
2. (closed under non-empty unions) If $X, Y \in \mathcal{W}$, $A, B \in \text{B}(X, Y)$ and $A, B \in \mathcal{E}$, then $A \cup B \in \mathcal{E}$.

Proof. Consider a p-morphism $f: \mathcal{W} \rightarrow \mathcal{F}$. The pullback $\text{Bis}(f, f)$ of f along itself is a subobject of $\text{Bis}(\mathcal{W})$; by definition, whenever $X, Y \in \mathcal{W}$ and $A \in \text{B}(X, Y)$, we have

$$A \in \text{Bis}(f, f) \iff f(X) = f(Y) \text{ and } (\forall (x, y) \in A)(f_X(x) = f_Y(y))$$

It is then straightforward to prove that $\text{Bis}(f, f)$ is downward closed and closed under non-empty unions, given that belonging to this pullback corresponds to a condition on pairs (x, y) .

⁶By our definition, the empty relation is not a bisimulation; and the empty union of bisimulations would need to be the empty relation.

Vice versa, let $\mathcal{E}$ be an equivalential subobject of $\text{Bis}(\mathcal{W})$ and assume that it is downward closed and closed under non-empty unions (these assumptions will be used only in the very last part of the proof). Given $X, Y \in \mathcal{W}$, consider the family $\text{B}(X, Y) \cap \mathcal{E}$ of the bisimulations from X to Y that belong to $\mathcal{E}$. Call X and Y $\mathcal{E}$ -connected if $\text{B}(X, Y) \cap \mathcal{E}$ is non-empty. Being $\mathcal{E}$ -connected is an equivalence relation over the set $\mathcal{W}$ (by (Identity), (Dagger) and (W-Composition)); call $\mathcal{W}/\mathcal{E}$ the set of its equivalence classes. If $K \in \mathcal{W}/\mathcal{E}$ is one of such classes, then we can consider the disjoint union $\{(x, X) \mid X \in K, x \in X\}$ and say that (x, X) is equivalent to (y, Y) if there is $A \in \text{B}(X, Y) \cap \mathcal{E}$ such that $(x, y) \in A$ (it is indeed an equivalence relation, by (Identity), (Dagger) and (W-Composition)); call $K_{\mathcal{E}}$ the resulting quotient.

Consider the family $\mathcal{W}_{\mathcal{E}} := \{K_{\mathcal{E}} \mid K \in \mathcal{W}/\mathcal{E}\}$ and $f : \mathcal{W} \rightarrow \mathcal{W}_{\mathcal{E}}$ sending $X \in \mathcal{W}$ to $K_{\mathcal{E}}$, if $K \in \mathcal{W}/\mathcal{E}$ and $X \in K$, with $f_X : X \rightarrow f(X) = K_{\mathcal{E}}$ sending $x \in X$ to the equivalence class of (x, X) . Each f_X , for $X \in \mathcal{W}$, is surjective: consider an element in $K_{\mathcal{E}}$, say the class of (y, Y) , with $Y \in K$ and $y \in Y$; since $X \in K$, then X and Y are $\mathcal{E}$ -connected, hence we can find $A \in \text{B}(X, Y) \cap \mathcal{E}$ and, A being a bisimulation, $x \in X$ such that $(x, y) \in A$. Then $f_X(x)$ is the class of (x, X) , which is equal to the class of (y, Y) . The fact that all f_X are surjective proves at the same time that $\mathcal{W}_{\mathcal{E}}$ belongs to $\text{S5Fr}_{l,f}$ (it is a family of non-empty finite sets) and that f defines a p-morphism.⁷ We can now compare $\mathcal{E}$ with the kernel pair $\text{Bis}(f, f)$. Consider $X, Y \in \mathcal{W}$ and $A \in \text{B}(X, Y)$. If $A \in \mathcal{E}$, then $A \in \text{B}(X, Y) \cap \mathcal{E}$, hence $f(X) = f(Y)$ (X and Y are $\mathcal{E}$ -connected), and, given $(x, y) \in A$, we have $f_X(x) = f_Y(y)$. Vice versa, assume that $f(X) = f(Y)$ and that $f_X(x) = f_Y(y)$ for each $(x, y) \in A$. The former condition says that X and Y are $\mathcal{E}$ -connected, i.e. that $\text{B}(X, Y) \cap \mathcal{E}$ is non-empty; we can then consider its greatest element $M \in \text{B}(X, Y) \cap \mathcal{E}$, by closure under non-empty unions (take the union of all the bisimulations in $\text{B}(X, Y) \cap \mathcal{E}$). The latter says that, for each $(x, y) \in A$, there exists $B \in \text{B}(X, Y) \cap \mathcal{E}$ such that $(x, y) \in B$; in other words, $A \subseteq M$. We conclude that $A \in \mathcal{E}$, by downward closure. ■

Theorem 5.12. *The system S5 lacks the CoSP⁸.*

Proof. In Example 3.14 we provided a coequivalence relation which is not separating. In light of the characterization of effectiveness, we can this easily: the equivalence relation associated to the formula fails to be closed under non-empty unions. ■

5.2 Local coequivalences in S5

We now turn our attention to equivalence relations arising from descent data. For that purpose, we begin by rephrasing 4.8 in our setting. Let $f : \mathcal{W} \rightarrow \mathcal{F}$ be a morphism in $\text{S5Fr}_{l,f}$. Then, given $X, Y \in \mathcal{W}$ and $A \in \text{B}(X, Y)$, we have a bisimulation $A_f \in \text{B}(X, f(Y))$ defined as

$$A_f := \{(x, f_Y(y)) \mid (x, y) \in A\}.$$

We have an obvious surjective function $A \rightarrow A_f$. These data define a p-morphism $\text{Bis}(\mathcal{W}) \rightarrow \text{Bis}(\mathcal{W}, \mathcal{F})$. Observe that we can *push* $(-)_f$ to the right: if A and B are composable, then $(AB)_f = AB_f$.

Proposition 5.13. Let $\mathcal{E}$ be an equivalential subobject of $\text{Bis}(\mathcal{W})$. Then $\mathcal{E}$ comes from a descent data over $f : \mathcal{W} \rightarrow \mathcal{F}$ for $p : \mathcal{F} \rightarrow \mathcal{G}$ if and only if:

1. for each $A \in \mathcal{E}$, the function $A \rightarrow A_f$ is bijective and $A_f \in \text{Bis}(pf, p)$;

⁷It can be proven that $f : \mathcal{W} \rightarrow \mathcal{W}_{\mathcal{E}}$ is the coequalizer of $\mathcal{E} \rightrightarrows \mathcal{W}$.

⁸It should be noted that Theorem 5.12 likewise follows from the results of [6]: for locally tabular logics L , if L^∞ fails to have the CoSP, then L will likewise fail to have it, where L^∞ is a natural infinitary analogue of L (see [7]), whenever the counterexample is given by finite frames. As a consequence of the main result of that paper, S5^∞ fails to have that property, and the counterexample is finite, from which the result for S5 follows.

2. for each $D \in \text{Bis}(pf, p)$, there exists a unique $A \in \mathcal{E}$ such that $A_f = D$.

Proof. Recall that $\text{Bis}(pf, p)$ is the pullback of pf along p and it is a subobject of $\text{Bis}(\mathcal{W}, \mathcal{F})$. The fact that, for all $A \in \mathcal{E}$, we have $A_f \in \text{Bis}(pf, p)$ means that $(pf)\pi_1 = p(f\pi_2)$; then $A \rightarrow A_f$ is the map $\mathcal{E} \rightarrow \text{Bis}(pf, p)$ induced by the universal property. The other conditions establish bijectivity of such a map. The fact that this is equivalent with $\mathcal{E}$ arising from descent data then follows from Theorem 4.8. $\blacksquare$

Definition 5.14 (Lift of bisimulation). Given $\mathcal{E}$ an equivalential subobject of $\text{Bis}(\mathcal{W})$ arising as descent data, and $D \in \text{Bis}(pf, p)$, i.e. $D \in \text{B}(X, U)$ is (pf, p) -admissible, for some $X \in \mathcal{W}$ and $U \in \mathcal{F}$ a (pf, p) -compatible pair. We denote by $\text{Lift}(D)$ the *lift of D* the bisimulation uniquely determined by $\mathcal{E}$ relative to D , as in Proposition 5.13.2.: this means that $\text{Lift}(D) \in \mathcal{E}$ and $(\text{Lift}(D))_f = D$.

Given $X \in \mathcal{W}$ and $U \in \mathcal{F}$ (pf, p) -compatible, i.e. such that $pf(X) = p(U)$, there exists the greatest (pf, p) -admissible bisimulation in $\text{B}(X, U)$, namely

$$D_{X,U} := \{(x, u) \in X \times U \mid p_{f(X)}f_X(x) = p_U(u)\}.$$

Observe that $\text{Lift}(D_{X,U}) \in \text{B}(X, Z)$, where Z is some element of $\mathcal{W}$ such that $f(Z) = U$.

With these observations we are ready for the main result of this section and of this paper:

Theorem 5.15. *In $\text{S5Fr}_{l,f}$ and S5Fr_f all descent equivalence relations are effective.*

Proof. Let $\mathcal{E}$ be an equivalential subobject of $\text{Bis}(\mathcal{W})$, for some $\mathcal{W}$ in $\text{S5Fr}_{l,f}$, and assume that $\mathcal{E}$ comes from a descent data over some $f : \mathcal{W} \rightarrow \mathcal{F}$ for some $p : \mathcal{F} \rightarrow \mathcal{G}$. We prove that $\mathcal{E}$ is effective.

We use the characterization of effectiveness in Proposition 5.11. Let $X, Y \in \mathcal{W}$. We first prove that, if $\text{B}(X, Y) \cap \mathcal{E}$ is non-empty, then it has a greatest element. Take $A \in \text{B}(X, Y) \cap \mathcal{E}$. By Proposition 5.13.1., since $A \in \mathcal{E}$, we have that $A_f \in \text{Bis}(pf, p)$; this means that X and $f(Y)$ are (pf, p) -compatible and that $A_f \in \text{B}(X, f(Y))$ is (pf, p) -admissible. By (pf, p) -compatibility, we can consider $D_{X,f(Y)}$ and $D_{Y,f(Y)}$, together with $\text{Lift}(D_{X,f(Y)})$ and $\text{Lift}(D_{Y,f(Y)})$ determined by $\mathcal{E}$. We prove that $A^\dagger \text{Lift}(D_{X,f(Y)}) = \text{Lift}(D_{Y,f(Y)})$: by uniqueness of Proposition 5.13.2., since both $A^\dagger \text{Lift}(D_{X,f(Y)})$ and $\text{Lift}(D_{Y,f(Y)})$ belong to $\mathcal{E}$ (by (Dagger) and (W-Composition)), it suffices to prove that $(A^\dagger \text{Lift}(D_{X,f(Y)}))_f = (\text{Lift}(D_{Y,f(Y)}))_f$, which is equivalent to $A^\dagger D_{X,f(Y)} = D_{Y,f(Y)}$, by pushing $(-)_f$ to the right. Obviously, the left-to-right inclusion holds, since $A^\dagger D_{X,f(Y)} \in \text{B}(Y, f(Y))$ is (pf, p) -admissible and $D_{Y,f(Y)}$ is the greatest (pf, p) -admissible bisimulation in $\text{B}(Y, f(Y))$. For the other inclusion, take $(y, u) \in D_{Y,f(Y)}$, i.e. such that $p_{f(Y)}f_Y(y) = p_{f(Y)}(u)$. Using the fact that A is a bisimulation from X to Y , we can find $x \in X$ such that $(x, y) \in A$. Since A_f is (pf, p) -admissible, we have that $p_{f(X)}f_X(x) = p_{f(Y)}f_Y(y)$. As a consequence, $(x, u) \in D_{X,f(Y)}$. Composing, we have that $(y, u) \in A^\dagger D_{X,f(Y)}$. Putting everything together, since $\Delta_X \subseteq \text{Lift}(D_{X,f(Y)})(\text{Lift}(D_{X,f(Y)}))^\dagger$,

$$\begin{aligned} A &\subseteq \text{Lift}(D_{X,f(Y)})(\text{Lift}(D_{X,f(Y)}))^\dagger A = \\ &= \text{Lift}(D_{X,f(Y)})(A^\dagger \text{Lift}(D_{X,f(Y)}))^\dagger = \text{Lift}(D_{X,f(Y)})(\text{Lift}(D_{Y,f(Y)}))^\dagger. \end{aligned}$$

We proved that, if $\text{B}(X, Y) \cap \mathcal{E}$ is non-empty, then $\text{Lift}(D_{X,f(Y)})(\text{Lift}(D_{Y,f(Y)}))^\dagger$ is its greatest element.

We now prove that $\mathcal{E}$ is downward closed. Let $X, Y \in \mathcal{W}$ and let $A, B \in \text{B}(X, Y)$ such that $B \subseteq A$; assume that $A \in \mathcal{E}$. We want to prove that $B \in \mathcal{E}$. Consider M greatest element of $\text{B}(X, Y) \cap \mathcal{E}$ (it exists for what we said before, $\text{B}(X, Y) \cap \mathcal{E}$ being non-empty). Observe that

$$M \subseteq MM^\dagger B \subseteq MM^\dagger M \subseteq M. \quad (*)$$

The first inclusion follows from the fact that $\Delta_Y \subseteq M^\dagger B$, since $B \subseteq A \subseteq M$ ($A \in \mathbf{B}(X, Y) \cap \mathcal{E}$ and M is its greatest element); by $B \subseteq M$, we also have the second inclusion (the inclusion order is compatible with the composition); the last inclusion follows from the fact that $MM^\dagger M \in \mathbf{B}(X, Y) \cap \mathcal{E}$ (by (Dagger) and (W-Composition)) and that M is its greatest element. Observe that, by the (pf, p) -admissibility of A_f , we also have that $B_f \in \mathbf{B}(X, f(Y))$ is (pf, p) -admissible, since $B \subseteq A$, hence $B_f \subseteq A_f$; in other words, $B_f \in \mathbf{Bis}(pf, p)$. Consider $\text{Lift}(B_f) \in \mathcal{E}$. Applying $(-)_f$ to $(*)$ and pushing it to the right, we obtain $MM^\dagger B_f = M_f$. However, we also have $(MM^\dagger \text{Lift}(B_f))_f = MM^\dagger B_f$, since $(\text{Lift}(B_f))_f = B_f$. From the uniqueness part of Proposition 5.13.2., since both $MM^\dagger \text{Lift}(B_f)$ and M belong to $\mathcal{E}$, it follows that $MM^\dagger \text{Lift}(B_f) = M$. As a consequence, $\text{Lift}(B_f) \subseteq M$. We can then conclude that $B = \text{Lift}(B_f) \in \mathcal{E}$: they are both subsets of M , and both $B \rightarrow B_f$ and $\text{Lift}(B_f) \rightarrow B_f$ are the restrictions of $M \rightarrow M_f$, which is bijective by Proposition 5.13.1., since $M \in \mathcal{E}$.

Closure under non-empty unions is now easy. Let $X, Y \in \mathcal{W}$ and let $A, B \in \mathbf{B}(X, Y)$; assume that $A, B \in \mathcal{E}$. We want to prove that $A \cup B \in \mathcal{E}$. Consider M greatest element of $\mathbf{B}(X, Y) \cap \mathcal{E}$. Then, by $A \subseteq M$ and $B \subseteq M$, we have that $A \cup B \subseteq M$. Using downward closure, since $A \cup B \in \mathbf{B}(X, Y)$, we have that $A \cup B \in \mathcal{E}$. ■

Theorem 5.16. *The categories $\mathbf{S5Fr}_{l,f}$ and $\mathbf{S5Fr}_f$ are almost Barr-exact.*

Proof. Recall that both $\mathbf{S5Fr}_{l,f}$ and $\mathbf{S5Fr}_f$ are regular categories (see e.g., [9, Lemma 5.12.i])). By Corollary 4.9, almost Barr exactness corresponds to regularity together with coequivalence relations arising from descent data being effective. Thus the result follows from Theorem 5.15. ■

In [7], to a normal modal logic L (with the finite model property) is associated an infinitary extension L^∞ , whose algebraic models are given by profinite L -algebras (inverse limits of finite L -algebras). The category of profinite L -algebras (and morphisms preserving arbitrary meets and joins) is dual to the category $\mathbf{LFr}_{l,f}$ of locally finite Kripke frames validating L (and p -morphisms). The same bridge theorems connecting syntactic properties of a logic L and categorical properties of $\mathbf{Alg}(L)_{fp}^{op}$ hold for L^∞ and $\mathbf{LFr}_{l,f}$.

Theorem 5.17. *The logics $\mathbf{S5}$ and $\mathbf{S5}^\infty$ have the LCoSP.*

Proof. It is well known that $\mathbf{S5}$ enjoys $\exists_{\text{MIP}}$ (since it has Craig interpolation [15], and is locally tabular). Thus by Theorem 4.12, for it to have LCoSP is equivalent for $\mathbf{Alg}(L)_{fp}^{op}$ to be almost Barr-exact. This is precisely what we showed in Theorem 5.16. ■

Theorem 5.16 suggests that modal and intuitionistic logics open a new class of interesting examples of almost Barr-exact categories, which are neither locally cartesian categories, nor Barr-exact, which might be attractive for the purposes of descent theory.

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Resolving Asynchronous Distributed Knowledge

Philippe Balbiani
IRIT, CNRS—INPT—UT
philippe.balbani@irit.fr

Hans van Ditmarsch
IRIT, CNRS—INPT—UT
hansvanditmarsch@gmail.com

Clara Lerouvillos
IRIT, CNRS—INPT—UT
IHPST, Paris 1 Panthéon Sorbonne
clara.lerouvillos@irit.fr

There are by now various epistemic modal logics with intersection modalities for distributed knowledge and intersection update modalities for dynamic phenomena like agents sharing (all their) information, agents receiving information from other agents, and full information protocols. One of those is the logic of *Resolving Distributed Knowledge*, by Ågotnes and Wang. It has *distributed knowledge* modalities for arbitrary subsets of the set of all agents and it also has so-called *resolution* modalities for arbitrary subsets of agents sharing their knowledge. In that logic, the agents not involved in the knowledge sharing are aware of the agents sharing knowledge, agents are memory-less, and the kind of dynamics represents synchronous updates, where there is common awareness of the global clock. In contrast, in this contribution we present a logic for *Resolving Asynchronous Distributed Knowledge*. It is an asynchronous generalization of the synchronous logic of resolving distributed knowledge. The logical semantics is history-based: truth is not only with respect to a given world in a model, but also with respect to a given history of prior resolutions, of which each individual agent can only observe a part. In particular, an agent is unaware of resolutions for groups of agents not including her. As is to be expected, this comes with many technical complications, for example concerning the axiomatization. The synchronous axioms relating resolution to distributed knowledge are now invalid. The modelling advantages of such an asynchronous novel logic, for distributed computing and similar areas, are however substantial and a major asset.

1 Introduction

If Anne (a) knows p and Bill (b) knows $p \rightarrow q$ then neither knows q but they still have *distributed knowledge* of q . If they were to *share* their information, they would both know q . Distributed knowledge modalities are *intersection modalities*: p is distributed knowledge for a and b in a world w , iff p is true in all worlds v which are indistinguishable for a and for b from the actual world w . In this case, distributed knowledge of p becomes common knowledge of p by sharing. However, as is well known, propositions may be distributed knowledge even when the agents cannot share this information. For another scenario, if p is true but a does not know this, and b knows that, then a and b have distributed knowledge of the former, as it is true in all worlds which are indistinguishable for a and for b . But when b informs a of p , or even of p and that she does not know p , agent a learns that p which makes her ignorance false. Modalities for sharing information are *intersection updates* (intersection update modalities): in a world w proposition q is true after a and b share their knowledge iff q is true in the model wherein we have replaced the indistinguishability relations for a and b by the intersection of these two relations. Let now Cath (c) enter the scene and let us replay the first of the above scenarios, where all three agents a, b, c are initially aware that a knows p and b knows $p \rightarrow q$. Again Anne and Bill share their knowledge. What does Cath learn from this? That depends. If we assume synchrony, then Cath learns that Anne and Bill share their knowledge, so afterwards Cath knows that Anne and Bill now both know q . Also, Anne and Bill know that Cath knows this. But if we assume asynchrony, Anne and Bill may have shared their knowledge without Cath learning that. In which case Cath still considers it possible that Anne

does not know q . But also Anne and Bill now face more uncertainty. For example, if after Anne and Bill share their knowledge, Anne and Cath share their knowledge, Cath now knows that Anne knows p . But Bill, who was not involved in the second sharing, does not know that. And so on. Various logics have been proposed to formalize distributed knowledge and sharing knowledge and their interaction, and that should be seen as incorporating synchrony. Instead, we propose an asynchronous logic to formalize distributed knowledge and sharing knowledge and their interaction.

The remainder of this introduction is a succinct survey of modal logics with intersection modalities, modal logics with intersection updates, and modal logical approaches to asynchrony.

Intersection modalities. If we consider a modal language with modalities $\Box_a$ and $\Box_b$, interpreted on Kripke models with binary relations R_a and R_b , resp., the problem of intersection modalities boils down to the observation that adding a novel modality (suggestively named) $\Box_{a \cup b} p$ is true in a world w iff p is true in all worlds v which are R_a -accessible *or* R_b -accessible does not create problems. This is so because in such a case, $R_{a \cup b} = R_a \cup R_b$ and one simply has that $\Box_{a \cup b} p$ is equivalent to $\Box_a p \wedge \Box_b p$. Whereas adding a modality $\Box_{a \cap b}$ such that $\Box_{a \cap b} p$ is true in a world w iff p is true in all worlds v which are R_a -accessible *and* R_b -accessible creates problems. One now has that $R_{a \cap b} = R_a \cap R_b$ and we can no longer define $\Box_{a \cap b} p$ with the other modalities, the canonical model is not of the right kind. One way to address this is to expand the logical language with *nominals* [46, 47] which led to the development of *hybrid logics* [13, 4] and related logics [29, 30, 49]. Another way to address this led to the development of modal logics with intersection modalities, where we highlight Propositional Dynamic Logic (**PDL**) with intersection [19, 33], and epistemic logics with distributed knowledge [24, 32, 2], although there are many further approaches such as Boolean Modal Logic [26, 27], and knowledge representation logics [43, 51].

Propositional dynamic logic. In **PDL** we generalize from modalities $\Box_a$ to modalities $\Box_\alpha$ where α is a program and that are interpreted in models with recursively defined relations R_α , with the same basic programs $a, b, \dots$ as above but apart from $\cup$ operations of sequential execution and arbitrary iteration, as well as another basic program called test [34]. By also adding intersection $\alpha \cap \beta$ of programs to **PDL**, we obtain **IPDL** [19, 33]. Here, $\alpha \cap \beta$ represents parallel execution of programs α and β as $R_{\alpha \cap \beta} = R_\alpha \cap R_\beta$, so again, similar to $R_a \cap R_b$ above. Various works involving the complexity [38, 39, 40] and axiomatization [6] of **IPDL** have appeared.

Distributed knowledge. The notion of *distributed knowledge* is rooted in sociology, economics, and philosophy [35, 36, 50] under different terms; [35] is an admirable pamphlet against centralized planning and in favour of distributed planning (and authority), [36] calls it ‘impersonal knowledge’, and [50] ‘undiscovered public knowledge’. The earliest epistemic logical source is [31] (the later journal version [32] also gives a logical semantics), wherein the notion was called ‘implicit knowledge’, followed on the heels by [44] who proposes an axiomatization (without claiming completeness) in a temporal epistemic logic. In fact their non-temporal fragment is the, yet somewhat later, complete axiomatization of [24, 37], where these publications achieved completeness by different methods. All such works interpret modalities $\Box_a$ not on models with arbitrary relations R_a but on models with equivalence relations. Among the many issues of further interest concerning distributed knowledge, particularly in view of representing multiple agents sharing knowledge given uncertainty about their own and each other’s knowledge, is a certain discrepancy between syntactic and semantic intuitions of distributed knowledge, and what sharing knowledge actually means. Such issues were already discussed in the original [24, 37] and continue to be investigated in more recent times [25].

Intersection updates. After the history of intersection modalities we now proceed with the history of intersection updates, the history of *sharing* information. Given a Kripke model with two (typically) equivalence relations R_a and R_b , the *intersection update modality* $\Box_{a \cap b}^{\text{update}}$ is such that $\Box_{a \cap b}^{\text{update}} p$ is true

in a world w of a model M iff p is true in the same world w but in the *updated* model $M^{a \cap b}$ wherein the relations R_a and R_b are replaced by the relation $R_a \cap R_b$. That is all. Note that when we interpret the intersection modality we (may) *change the point of evaluation* in a given model but we *do not change the model* wherein we evaluate the formula bound by the modality, whereas if we interpret the intersection update modality we *do not change the point of evaluation* but we (may) *change the model* wherein we evaluate the formula bound by the modality. Intersection update modalities are therefore interpreted as updates of Kripke models and they can indeed be seen as a further development in *dynamic epistemic logics*, such as public announcement logic [48] and action model logic [8] (for references see [22, 42]), although these are updates with very different properties.

The first paper involving the intersection update to our knowledge was [12], in a mixed setting of epistemic and inquisitive logic. The intersection update here is the *resolve* action, encoding the answer to a question thus resolving an issue (intersecting an epistemic relation with an issue relation). On the heels of [12] were a number of related publications [9, 15, 14, 28, 7]. Here, [9] introduces intersection updates on plausibility structures whereas [7] further generalizes the inquisitive and epistemic setting of [12]. Independently, slightly later again, [1] focuses on intersection updates (called *resolution*) and distributed knowledge for arbitrary sets of agents and their interaction. Further developments introduce uncertainty even among agents who have shared information [10, 11], which also relates to a different strand of research coming out of distributed computing, and intersection updates incarnate the execution of *communication graphs* or *communication patterns* [54, 16, 17, 52, 53]. The operation of pooling or basic intersection in [18] corresponds to a intersection (update) modality similar to what we call resolution here; this updates models with plausibility relations. An attempt to combine distributed knowledge and intersection updates into a single modality for *sharing knowledge* is [5].

Asynchrony. Intersection modalities like distributed knowledge and intersection updates such as resolution and communication patterns assume synchrony (and even so in distributed computing, were a *round* of asynchronous events represents a snapshot recording time). Things become more complex with (full) asynchrony, in the absence of a global clock. This was already addressed in [32], and ‘full information protocols’ in distributed computing [41] correspond to intersection updates. In dynamic epistemic logics such asynchrony requires a history-based semantics [45, 20]. Even with more restricted information exchange such as in gossip protocols this leads to far more challenging axiomatizations (and higher complexities) [23, 21, 3].

Overview of content. In Section 2 we review the logic of resolving (synchronous) distributed knowledge. In Section 3 we introduce the logic of resolving asynchronous distributed knowledge, and compare it to the logic of synchronous distributed knowledge. In Section 4 we define an infinitary axiomatization for the logic of resolving asynchronous distributed knowledge and prove its completeness. Section 5 lists further research, on redundant resolutions, derivable rules, expressivity, and common knowledge.

2 Resolving distributed knowledge

We briefly present Ågotnes and Wang’s logic for resolving distributed knowledge [1]. Let A be a finite non-empty set of *agents*, and P a countable set of propositional variables (atoms). The language $\mathcal{L}_{DR}$ is defined by $\varphi ::= p \mid \top \mid (\varphi \wedge \varphi) \mid \neg\varphi \mid D_B\varphi \mid R_B\varphi$, where $p \in P$ and $B \subseteq A$. The sublanguage $\mathcal{L}_D$ with only modalities $D_B\varphi$ is the language of distributed knowledge. For $\Box_a$ we now write K_a , and it is defined by abbreviation as $D_{\{a\}}$. Formula $R_B\varphi$ is read as ‘after resolution for group of agents B , φ (is true)’.

The structures are multi-agent *epistemic models* $(W, \sim, V)$ for the set A of agents (where $(W, \sim)$ is a multi-agent *frame*). We can see $\sim$ as a set $\{\sim_a\}_{a \in A}$ of binary *indistinguishability relations* (or as a

(taut)	all instantiations of propositional tautologies	
(K _D)	$D_B(\varphi \rightarrow \psi) \rightarrow (D_B\varphi \rightarrow D_B\psi)$	
(T _D)	$D_B\varphi \rightarrow \varphi$	
(5 _D)	$\neg D_B\varphi \rightarrow D_B\neg D_B\varphi$	
(DG)	$D_B\varphi \rightarrow D_C\varphi$	if $B \subseteq C$
(RA)	$R_B p \leftrightarrow p$	
(RN)	$R_B\neg\varphi \leftrightarrow \neg R_B\varphi$	
(RC)	$R_B(\varphi \wedge \psi) \leftrightarrow (R_B\varphi \wedge R_B\psi)$	
(RD1)	$R_B D_C\varphi \leftrightarrow D_{B \cup C} R_B\varphi$	when $B \cap C \neq \emptyset$
(RD2)	$R_B D_C\varphi \leftrightarrow D_C R_B\varphi$	when $B \cap C = \emptyset$
(NecD)	From φ , infer $D_B\varphi$	
(NecR)	From φ , infer $R_B\varphi$	
(MP)	From φ and $\varphi \rightarrow \psi$, infer ψ	

Table 1: Axiomatisation **RD** for the logic of resolving distributed knowledge

function mapping each agent to such a relation $\sim_a$). Valuation V is a function from the set of atoms to the powerset of W mapping each atom to the subset of worlds where it is true. We write $\sim_B := \bigcap_{a \in B} \sim_a$ for the epistemic relation of a group B (so that $\sim_\emptyset = W \times W$).

If $M = (W, \sim, V)$ and $B \subseteq A$ then *updated model* $M^B := (W, \sim^B, V)$ is the result of resolution for group B , where $\sim_a^B := \bigcap_{b \in B} \sim_b$ if $a \in B$ and $\sim_a^B := \sim_a$ otherwise. In updated model M^B the relations for the agents $a \in B$ have been updated from $\sim_a$ to $\sim_B$. Hence, in M^B we have that $\sim_B^B = \sim_a^B$ for all $a \in B$, unlike in M . Given these notations, we have that $\sim_C^B = \sim_C$ if $B \cap C = \emptyset$ and that $\sim_C^B = \sim_{B \cup C}$ if $B \cap C \neq \emptyset$. Note that $M^a = M$ and that $M^\emptyset = M$. For $M^{\{a_1, \dots, a_n\}}$ we write $M^{a_1 \dots a_n}$, and for $(M^B)^C$ we write $M^{B.C}$.

Let a vector $\vec{G}$ represent a sequence $B_1 \dots B_n$, where $B_i \subseteq A$ for $1 \leq i \leq n$ —the empty sequence is ε . By $|\vec{G}|$ we denote the length of $\vec{G}$, and $a \notin \vec{G}$ means that $a \notin B$ for any $B \subseteq A$ such that B occurs in $\vec{G}$. We write $\sim_a^{\vec{G}}$ for the accessibility relation of an agent a in $M^{\vec{G}}$.

We now present the semantics. The satisfaction relation $\models$ is defined by induction on $\varphi \in \mathcal{L}_{DR}$, where $p \in P$ and $G \subseteq A$.

$M, w \models p$	iff	$w \in V(p)$
$M, w \models \top$	iff	true
$M, w \models \neg\varphi$	iff	$M, w \not\models \varphi$
$M, w \models \varphi \wedge \psi$	iff	$M, w \models \varphi$ and $M, w \models \psi$
$M, w \models D_B\varphi$	iff	$M, v \models \varphi$ for all $v \in W$ such that $w \sim_B v$
$M, w \models R_B\varphi$	iff	$M^B, w \models \varphi$

A formula $\varphi \in \mathcal{L}_{DR}$ is *valid* iff for all models $M = (W, \sim, V)$ and for all $w \in W$, $M, w \models \varphi$.

The axiomatization **RD** for the logic of resolving distributed knowledge in Table 1 extends that of the logic of distributed knowledge with reduction axioms for resolution and a derivation rule for necessitation of resolution. Completeness of this axiomatization is shown by reducing $\mathcal{L}_{DR}$ -formulas to $\mathcal{L}_D$ -formulas, by which we mean that any formula with resolution and distributed knowledge modalities is provably (and semantically) equivalent to a formula without resolution modalities. It can be shown that *Replacement of Equivalents* (RE: from $\varphi \leftrightarrow \psi$ derive $\chi[p/\varphi] \leftrightarrow \chi[p/\psi]$, where $\chi[p/\varphi]$ is uniform substitution of p in χ by φ) is derivable in **RD**. This is needed to reduce formulas of form $R_G R_H \varphi$ where $G \neq H$, as there is no axiom of shape $R_G R_H \varphi \leftrightarrow \dots$.

3 Resolving asynchronous distributed knowledge

We now propose a logical framework for resolving *asynchronous* distributed knowledge. The logical language $\mathcal{L}_{DR}$ is the same, whereas accessibility relations are now encoding asynchronous distributed knowledge. Given asynchrony, we propose a history-based semantics. That is, instead of interpreting formulas φ in pointed models (M, w) , where such an M could be an updated $M^{\vec{G}}$, we now wish to interpret formulas in pointed models $(M, w, \vec{G})$, where the resolution sequence explicitly remains at our disposition. This is necessary, because what an agent knows now may be different when she was involved in the last resolution in $\vec{G}$ than from when she was not. In order to compare pairs $(w, \vec{G})$ and $(v, \vec{H})$ we not only need the indistinguishability relation $\sim_a$ between worlds w and v but also a novel *resolution relation* $\approx_a$ between resolution sequences $\vec{G}$ and $\vec{H}$. Connecting both relations is the *view* by agent a of sequence $\vec{G}$, denoted $\text{see}_a(\vec{G})$, that is the set of agents B with whom a has shared the equivalence relations (e.g., recalling the introduction, after resolution ab , she has shared it with b as the new $\sim_a$ is now $\sim_{a \cap b}$). The set $\text{see}_a(\vec{G})$ determines the knowledge gained by agent a after history $\vec{G}$ —or after any other history $\vec{H}$ that she cannot distinguish from $\vec{G}$. Let us begin by defining the resolution relation and the view, and show some relevant results relating them.

Resolution relation. Let $a \in A$, and $\vec{G}, \vec{H} \in \mathcal{P}(A)^*$ be given. The *resolution relation* $\approx_a$ between resolution sequences is the equivalence closure of:

$$\begin{array}{lll} \varepsilon \approx_a \varepsilon & \text{iff} & \text{true} \\ \vec{G}.B \approx_a \vec{H} & \text{iff} & \vec{G} \approx_a \vec{H} \quad \text{when } a \notin B \\ \vec{G}.B \approx_a \vec{H}.B & \text{iff} & \vec{G} \approx_b \vec{H} \text{ for all } b \in B \quad \text{when } a \in B \end{array}$$

We further define $\approx_B := \bigcap_{a \in B} \approx_a$. Note that $\approx_\emptyset$ is the universal relation.

View. The *view* $\text{see}_a(\vec{G})$ of agent a of a resolution sequence $\vec{G}$ is the set $B \subseteq A$ of agents such that given any model $M = (W, \sim, V)$, agent a 's current relation $\sim_a^{\vec{G}}$ is $\bigcap_{b \in B} \sim_b$. Let $B \subseteq A$, then:

- $\text{see}_B(\varepsilon) := B$;
- $\text{see}_B(\vec{G}.C) := \text{see}_{B \cup C}(\vec{G})$ when $B \cap C \neq \emptyset$;
- $\text{see}_B(\vec{G}.C) := \text{see}_B(\vec{G})$ when $B \cap C = \emptyset$.

For $\text{see}_{\{a\}}(\vec{G})$ we write $\text{see}_a(\vec{G})$. Note that $\text{see}_B(\vec{G}) = C$ does not mean that the view of all agents in B is C , but only that the union of the views of all agents in B is C . It is easy to see that $\text{see}_B(\vec{G}) = \bigcup_{b \in B} \text{see}_b(\vec{G})$ (induction on the length of $\vec{G}$).

Before proceeding with the definition of the semantics, we state some properties of the resolution relation and the view, that will later prove useful. Straightforward proofs have been omitted.

Lemma 3.1. *If $a \notin \vec{I}$, then $\vec{G} \approx_a \vec{H}.\vec{I}$ if, and only if, $\vec{G} \approx_a \vec{H}$.*

Similarly, if $\vec{I} \in \mathcal{P}(A \setminus B)^*$, then $\vec{G} \approx_B \vec{H}.\vec{I}$ if, and only if, $\vec{G} \approx_B \vec{H}$.

Lemma 3.2. *$\varepsilon \approx_B \vec{G}$ if, and only if, $\vec{G} \in \mathcal{P}(A \setminus B)^*$.*

Consequently, we also have as a corollary that whenever $B \cap C = \emptyset$, then $C \approx_B \vec{G}$ iff $\vec{G} \in \mathcal{P}(A \setminus B)^*$.

Lemma 3.3. *If $a \in B$, then $\vec{G}.B \approx_a \vec{H}$ iff there are $\vec{H}_1, \vec{H}_2 \in \mathcal{P}(A)^*$ such that $\vec{H} = \vec{H}_1.B.\vec{H}_2$ with $\vec{G} \approx_B \vec{H}_1$ and $a \notin \vec{H}_2$.*

Lemma 3.4. *If $B \cap C \neq \emptyset$, then $C \approx_B \vec{H}$ iff there are $\vec{H}_1, \vec{H}_2 \in \mathcal{P}(A)^*$ such that $\vec{H} = \vec{H}_1.C.\vec{H}_2$ with $\vec{H}_1 \in \mathcal{P}(A \setminus (B \cup C))^*$ and $\vec{H}_2 \in \mathcal{P}(A \setminus B)^*$.*

Lemma 3.5. *If $\vec{G} \approx_{\text{see}_B(\vec{H})} \vec{I}$ and $\vec{H} \approx_B \vec{J}$, then $\vec{G}.\vec{H} \approx_B \vec{I}.\vec{J}$.*

Proof. The proof proceeds by induction on the length of $\vec{H}$. If $\vec{H} = \varepsilon$, suppose $\vec{G} \approx_B \vec{I}$ (because $\text{see}_B(\varepsilon) = B$) and $\varepsilon \approx_B \vec{J}$ by Lemma 3.1, $\varepsilon \approx_B \vec{J} \Rightarrow \vec{J} \in \mathcal{P}(A \setminus B)$ and then $\vec{G} \approx_B \vec{I}$ implies $\varepsilon.\vec{G} = \vec{G} \approx_B \vec{I}.\vec{J}$. If $\vec{H} = \vec{H}'.C$, suppose $\vec{G} \approx_{\text{see}_B(\vec{H}.C)} \vec{I}$ and $\vec{H}'.C \approx_B \vec{J}$. We now distinguish case $B \cap C = \emptyset$ from case $B \cap C \neq \emptyset$.

- If $B \cap C = \emptyset$, $\text{see}_B(\vec{H}.C) = \text{see}_B(\vec{H}')$ and $\vec{H}'.C \approx_B \vec{J} \Rightarrow \vec{H}' \approx_B \vec{J}$ (Corollary 3.1) so, by inductive hypothesis, $\vec{G}.\vec{H}' \approx_B \vec{I}.\vec{J}$. Hence $\vec{G}.\vec{H}.C \approx_B \vec{I}.\vec{J}$ by Corollary 3.1 again.
- If $B \cap C \neq \emptyset$, then $\text{see}_B(\vec{H}.C) = \text{see}_{B \cup C}(\vec{H}')$. Note that since $B \subseteq B \cup C$, $\vec{G} \approx_{\text{see}_{B \cup C}(\vec{H}')} \vec{I}$ implies $\vec{G} \approx_{\text{see}_B(\vec{H}')} \vec{I}$ (and respectively for C). Let $a \in B \cap C$. By Lemma 3.3, $\vec{H}'.C \approx_a \vec{J}$ implies $\vec{J} = \vec{J}_1.C.\vec{J}_2$ where $\vec{H}' \approx_C \vec{J}_1$ and $a \notin \vec{J}_2$. Then, we have $\vec{G} \approx_{\text{see}_C(\vec{H}')} \vec{I}$ and $\vec{H}' \approx_C \vec{J}_1$ so, by induction hypothesis, $\vec{G}.\vec{H}' \approx_C \vec{I}.\vec{J}_1$. Hence $\vec{G}.\vec{H}.C \approx_a \vec{I}.\vec{J}_1.C.\vec{J}_2 = \vec{I}.\vec{J}$. Let now $a \in B \setminus C$. Then, $\vec{H}.C \approx_a \vec{J}$ iff $\vec{H}' \approx_a \vec{J}$, and we have $\vec{G} \approx_{\text{see}_a(\vec{H}')} \vec{I}$ (for $\text{see}_a(\vec{H}.C) = \text{see}_a(\vec{H}')$). So, by induction hypothesis, $\vec{G}.\vec{H}' \approx_a \vec{I}.\vec{J}$ and hence $\vec{G}.\vec{H}.C \approx_a \vec{I}.\vec{J}$. Therefore $\vec{G}.\vec{H}.C \approx_B \vec{I}.\vec{J}$

□

Lemma 3.6. *$\vec{G} \approx_a \vec{H}$ implies $\text{see}_a(\vec{G}) = \text{see}_a(\vec{H})$.*

Proof. The proof is by induction on the length of $\vec{G}$.

Base case. We have: $\varepsilon \approx_a \vec{H}$ implies $a \notin \vec{H}$ (Lemma 3.2) so $\text{see}_a(\varepsilon) = \{a\} = \text{see}_a(\vec{H})$.

Induction case. Now consider $\vec{G}.B$. We distinguish $a \notin B$ from $a \in B$. If $a \notin B$ we have: $\vec{G}.B \approx_a \vec{H}$ iff (by definition) $\vec{G} \approx_a \vec{H}$, which implies (induction) $\text{see}_a(\vec{G}) = \text{see}_a(\vec{H})$, so $\text{see}_a(\vec{G}.B) = \text{see}_a(\vec{H})$. If now $a \in B$ the resolution sequence compared with must have shape $\vec{H}.B.\vec{I}$ where $a \notin \vec{I}$ (Lemma 3.3) so that: $\vec{G}.B \approx_a \vec{H}.B.\vec{I}$ iff $\vec{G}.B \approx_a \vec{H}.B$ (Lemma 3.1), which implies $\vec{G} \approx_b \vec{H}$ for all $b \in B$. This implies (induction) $\text{see}_b(\vec{G}) = \text{see}_b(\vec{H})$ for all $b \in B$, which implies $\bigcup_{b \in B} \text{see}_b(\vec{G}) = \bigcup_{b \in B} \text{see}_b(\vec{H})$ so (by definition) $\text{see}_a(\vec{G}.B) = \text{see}_a(\vec{H}.B) = \text{see}_a(\vec{H}.B.\vec{I})$. □

Consequently, we also have that $\vec{G} \approx_B \vec{H}$ implies $\text{see}_B(\vec{G}) = \text{see}_B(\vec{H})$. From [5] we further recall that for all models $M = (S, \sim, V)$, $\sim_{\vec{G}} = \sim_{\text{see}_B(\vec{G})}$. In other words, we might as well have stated that $\vec{G} \approx_B \vec{H}$ implies $\sim_{\vec{G}} = \sim_{\vec{H}}$, or that $\vec{G} \approx_B \vec{H}$ implies $\sim_{\text{see}_B(\vec{G})} = \sim_{\text{see}_B(\vec{H})}$, and will quote Lemma 3.6 in such cases.

However, $\text{see}_a(\vec{G}) = \text{see}_a(\vec{H})$ does not imply $\vec{G} \approx_a \vec{H}$. Typical counterexamples are that $\text{see}_a(\vec{G}.a) = \text{see}_a(\vec{G})$ whereas $\vec{G}.a \not\approx_a \vec{G}$ and $\text{see}_a(ab.ab) = \text{see}_a(ab)$ whereas $ab.ab \not\approx_a ab$.

We proceed to define the semantics.

Definition 3.7 (Semantics). By induction on $\varphi \in \mathcal{L}_{DR}$, where $p \in P$, $B \subseteq A$ and $\vec{G} \in \mathcal{P}(A)^*$.

$$\begin{array}{ll}
M, w, \vec{G} \models p & \text{iff } w \in V(p) \\
M, w, \vec{G} \models \top & \text{iff true} \\
M, w, \vec{G} \models \neg \varphi & \text{iff } M, w, \vec{G} \not\models \varphi \\
M, w, \vec{G} \models \varphi \wedge \psi & \text{iff } M, w, \vec{G} \models \varphi \text{ and } M, w, \vec{G} \models \psi \\
M, w, \vec{G} \models D_B \varphi & \text{iff } M, v, \vec{H} \models \varphi \text{ for all } v \in W, \vec{H} \in \mathcal{P}(A)^* \text{ such that } w \sim_B^{\vec{G}} v \text{ and } \vec{G} \approx_B \vec{H} \\
M, w, \vec{G} \models R_B \varphi & \text{iff } M, w, \vec{G}.B \models \varphi
\end{array}$$

As not uncommon in history-based semantics, two notions of validity emerge. A formula $\varphi \in \mathcal{L}_{DR}$ is ε -valid if $M, w, \varepsilon \models \varphi$ for all models $M = (W, \sim, V)$ and for all $w \in W$. A formula $\varphi \in \mathcal{L}_{DR}$ is $*$ -valid (or

always valid) if $M, w, \vec{G} \models \varphi$ for all $M = (W, \sim, V)$, $w \in W$, and all $\vec{G} \in \mathcal{P}(A)^*$. We should acknowledge that we are uncertain if ε -validity and $*$ -validity correspond. This is a question left for future research.

This asynchronous semantics looks very much like the synchronous semantics of the previous section, except for the clause for distributed knowledge. Let us be explicit about some differences and correspondences. First, Example 3.8 shows that $M, w, \vec{G} \models \varphi$ is not equivalent to (in the synchronous semantics) $M^{\vec{G}}, w \models \varphi$. So there is a real difference.

Example 3.8. Consider three agents a, b, c and model M and updated M^{ab} as in Figure 3.8. Instead of naming worlds by w, v , etcetera, we name them with their valuation, where $\bar{p}$ denotes $\neg p$. Reflexive arrows are omitted. We also assume symmetry and transitivity.

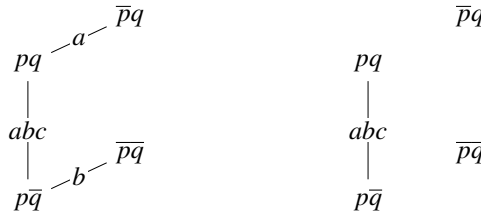


Figure 1: An epistemic model M (on the left) and its update M^{ab} (on the right) with three agents.

We now have that $M^{ab}, pq, \varepsilon \models K_c K_a p$ but not that $M, pq, ab \models K_c K_a p$. Agent c is unaware of agents a and b resolving their knowledge in model M : resolution ab is indistinguishable from the empty sequence ε for her. She therefore does not know agent a has learnt the truth about p as a consequence of this resolution. Since $M, pq, \varepsilon \not\models K_a p$ and $ab \approx_c \varepsilon$, therefore $M, pq, ab \not\models K_c K_a p$. And therefore also $M, pq, \varepsilon \not\models R_{ab} K_c K_a p$. Now consider the prior synchronous semantics. Then $M^{ab}, pq \models K_c K_a p$ and therefore $M, pq \models R_{ab} K_c K_a p$.

But we can also look at this in another way, rather illustrating a correspondence. Let us for a vanishing moment consider a *synchronous* distributed knowledge modality $\mathbf{D}_B \varphi$, interpreted as follows on our history-based models, wherein we only have replaced $\vec{G} \approx_B \vec{H}$ by $\vec{G} = \vec{H}$.

$$M, w, \vec{G} \models \mathbf{D}_B \varphi \quad \text{iff} \quad M, v, \vec{H} \models \varphi \text{ for all } v \in W, \vec{H} \in \mathcal{P}(A)^* \text{ such that } w \sim_B^{\vec{G}} v \text{ and } \vec{G} = \vec{H}$$

Now write $M, w, \vec{G} \models \varphi$ anywhere for the synchronous $M^{\vec{G}}, w \models \varphi$ (while replacing all D_B by $\mathbf{D}_B$). This embeds the synchronous semantics into the asynchronous semantics. So, with respect to the above example, indeed, $M, pq, ab \not\models K_c K_a p$, but on the other hand we now have $M, pq, ab \models \mathbf{K}_c \mathbf{K}_a p$, which after all corresponds to $M^{ab}, pq \models K_c K_a p$. Isn't that neat?

However, let us now go back to one language and two different semantics again. A further maybe somewhat curious observation is that resolution R_B has the same semantics either way, only the interpretation of distributed knowledge D_B is different synchronously and asynchronously. Despite the identical semantics, with asynchrony, resolution R_B encodes *partial synchronization* for group of agents B without any agents not in B being aware of that, whereas, with synchrony, resolution R_B encodes *full synchronization* for all agents however with aspects of *partial observation*: group of agents B jointly learn (all) each other's knowledge whereas all agents not in B learn that, but not what the agents in B learn. The agents not in B only partially observed the resolution.

Preparing the ground for the asynchronous axiomatization presented in the next section, let us review what parts of the synchronous axiomatization remain valid and what are now invalid. It is fairly simple.

All axioms and rules of **RD** remain valid (or validity preserving), except (RD1) and (RD2); and maybe (NecR). If ε -valid and $*$ -valid were the same (the open question), then we would have necessitation of resolution. (It is easy to see why: assume $\models \varphi$ and (NecR). Given arbitrary (M, w) , from the first we get that $M, w, \varepsilon \models \varphi$, and from that and the second $M, w, \varepsilon \models R_B \varphi$, so that $M, w, B \models \varphi$. We therefore easily show that φ is $*$ -valid by induction on the length of resolution sequences.)

Example 3.9. Model M in Figure 3.9 provides a counterexample to (RD1), and model M' below provides a counterexample to (RD2), where we have again named worlds with valuations of atoms.



Figure 2: Epistemic models M (on the left) and M' (on the right) with four agents.

We have that $M, pq, \varepsilon \models D_{abc} R_{ab} \neg K_a q$ but $M, pq, \varepsilon \not\models R_{ab} D_{bc} \neg K_a q$ —after resolution ab , agents b and c can imagine a has further shared knowledge with d , thereby learning that q . Therefore (RD1) is invalid.

Looking at M' , we have that $M', pq, \varepsilon \models K_a R_{bc} K_b p$ but $M', pq, \varepsilon \not\models R_{bc} K_a K_b p$ because a is not aware of b and c sharing their knowledge so she does not know b has learnt that p . Therefore (RD2) is invalid.

On the other hand, there are now novel validities involving distributed knowledge and resolution.

Lemma 3.10.

- (1) If $B \cap C \neq \emptyset$, then $\models R_B D_C \varphi \rightarrow D_{B \cup C} R_{B, \vec{I}} \varphi$ for all $\vec{I} \in \mathcal{P}(A \setminus C)^*$.
- (2) If $B \cap C \neq \emptyset$, then $\models D_{B \cup C} R_{B, \vec{I}} \varphi$ for all $\vec{I} \in \mathcal{P}(A \setminus C)^*$ implies $\models R_B D_C \varphi$.
- (3) If $B \cap C = \emptyset$, then $\models R_B D_C \varphi \leftrightarrow D_C \varphi$.

We recall once more, comparing to the first and second items jointly, (RD1) $\models R_B D_C \varphi \leftrightarrow D_{B \cup C} R_B \varphi$ when $B \cap C \neq \emptyset$, and, comparing to the third item (RD2) $\models R_B D_C \varphi \leftrightarrow D_C R_B \varphi$ when $B \cap C = \emptyset$. The proofs of the above are fairly straightforward but are omitted, as we will present a complete axiomatization **RAD** in the next section. Then, the third item is (in one direction) an instantiation of the later axiom (RD) for the case that $\vec{G} = B$ and $\vec{H} = \varepsilon$, whereas for the first item we have $\vec{G} = B$ as well as $\vec{H} = B, \vec{I}$. The second item instantiates derivation rule (RDI) of **RAD**.

Even for fairly simple (‘small’) models given a set of agents A , and even when everyone’s view is A (for all a , $\text{see}_a(\vec{G}) = A$) there is no bound to uncertainty caused by asynchrony (see the example below). This is different from synchrony, where once everyone’s view is A , this is commonly known. Two notes: with synchrony, we can also have arbitrary higher-order uncertainty, but at the price of large models (with enchainned equivalence classes for different agents); and with asynchrony, we can also have common knowledge that everyone’s view is A , namely after resolution with A (full synchronization).

Example 3.11. Let $|A| \geq 3$. Consider a resolution sequence $\vec{G}$ without $\emptyset$ and without singleton sets (irrelevant), and without A . Suppose towards a contradiction that there is bound to the length $|\vec{G}|$ of $\vec{G}$ after which a further resolution is no longer informative. Let $I \subseteq A$ with $1 < |I| < |A|$. Below we define a (unique) model M and a (unique) formula $\psi \in \mathcal{L}_{DR}$ that distinguishes $\vec{G}$ from $\vec{G}.I$. Consider the formula

$$\varphi := \bigwedge_{a \in A} K_a p \wedge \widehat{K}_a \wedge_{b \in A \setminus \{a\}} \neg(K_b p \vee K_b \neg p)$$

and a model M consisting of $2n + 1$ states namely $\{s^1\} \cup \{s_a^2 \mid a \in A\} \cup \{t_a^2 \mid a \in A\}$, where the relations $\sim_a$ are the reflexive closure of the conditions: for all $a, b \in A$ with $a \neq b$, $s_a^2 \sim_b t_a^2$, and for all $a \in A$, $s_a^2 \sim_a s^1$, and where valuation $V(p) = \{s^1\} \cup \{s_a^2 \mid a \in A\}$. We now have that $M, s^1, \varepsilon \models \varphi$ (namely, already in standard epistemic logic, $M, s^1 \models \varphi$). Let Σ be the sequence of length $|\vec{G}|$ of agents $a \in A$ such that the last is a member of I but not of B preceding I , the before last is a member of B but not of B' preceding B , and so on. Let the first B in $\vec{G}.I$ apart from the selected member a in Σ also contain an agent $b \neq a$. Now consider

$$\psi := K_\Sigma(K_a K_b p \wedge K_b K_a p)$$

where K_Σ abbreviates the stack $K_x \dots K_y$ listing all the members of Σ . Then $M, s^1, \vec{G} \not\models \psi$ whereas $M, s^1, \vec{G}.I \models \psi$.

For four agents a, b, c, d , (so-called ‘windmill’) model M is depicted in Figure 3. In such a model we have that, for example, $M, s^1, ab.bc \models K_c(K_a K_b p \wedge K_b K_a p)$ but $M, s^1, ab \not\models K_c(K_a K_b p \wedge K_b K_a p)$. For an example where everyone’s view is A but uncertainty still remains, consider the sequence of resolutions $abc.bcd.abd$ after which all four agents have accessibility relation $\sim_A$. We however have $M, s^1, abc.bcd.abd \not\models K_c K_a K_d p$, but now $M, s^1, abc.bcd.abd.abc \models K_c K_a K_d p$.

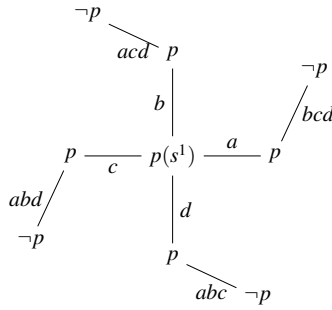


Figure 3: ‘Windmill’ model M with four agents.

4 Axiomatization

In this section we show soundness and completeness of the axiomatization **RAD**, that is composed of the axioms and rules given in Table 2. That $\varphi \in \mathcal{L}_{DR}$ is derivable in **RAD** is denoted $\vdash \varphi$. Axiomatization **RAD** contains an infinitary derivation rule (RDI) using admissible forms, defined below. As we have an infinitary derivation rule, in the completeness part of the proof we proceed by maximal consistent theories, also defined below, instead of the usual maximal consistent sets. Showing completeness involves ‘unravelling’ a canonical model, in order to get it into the right shape. Before we proceed, let us comment on the differences between axiomatizations **RD** and **RAD**. First, as $R_\varepsilon \varphi$ is φ by definition, the **RD** axioms (K_D) ... (DG) involving distributed knowledge are derivable from (in fact, instantiations of) the **RAD** axioms (RK_D) ... (RDG). Second, as resolutions $R_{\vec{G}}$ for a singleton sequence $\vec{G}$ are simply R_B for some $B \subseteq A$, we can also equate the **RD** axioms (RA), (RN) and (RC) with the similarly named **RAD** axioms (where the **RAD** axiom $R\top$ is derivable in **RD** using (NecR)). So the only difference is that the **RD** axioms (RD1) and (RD2) are missing (and they are not theorems, because we have shown through

(taut)	all instantiations of propositional tautologies	
(RK _D)	$R_{\vec{G}}D_B(\varphi \rightarrow \psi) \rightarrow R_{\vec{G}}(D_B\varphi \rightarrow D_B\psi)$	
(RT _D)	$R_{\vec{G}}D_B\varphi \rightarrow R_{\vec{G}}\varphi$	
(R5 _D)	$R_{\vec{G}}\neg D_B\varphi \rightarrow R_{\vec{G}}D_B\neg D_B\varphi$	
(RDG)	$R_{\vec{G}}D_B\varphi \rightarrow R_{\vec{G}}D_C\varphi$	if $B \subseteq C$
(RA)	$R_{\vec{G}}p \leftrightarrow p$	
(R $\top$)	$R_{\vec{G}}\top$	
(RN)	$R_{\vec{G}}\neg\varphi \leftrightarrow \neg R_{\vec{G}}\varphi$	
(RC)	$R_{\vec{G}}(\varphi \wedge \psi) \leftrightarrow (R_{\vec{G}}\varphi \wedge R_{\vec{G}}\psi)$	
(RD)	$R_{\vec{G}}D_B\varphi \rightarrow D_{see_B(\vec{G})}R_{\vec{H}}\varphi$	for all $\vec{H} \approx_B \vec{G}$
(NecD)	From φ , infer $D_B\varphi$	
(MP)	From φ and $\varphi \rightarrow \psi$, infer ψ	
(RDI)	From $\alpha(D_{see_B(\vec{G})}R_{\vec{H}}\varphi)$ for all $\vec{H} \approx_B \vec{G}$, infer $\alpha(R_{\vec{G}}D_B\varphi)$	

Table 2: Axiomatization **RAD** for the logic of resolving asynchronous distributed knowledge

counterexample that they are invalid), instead of which we now have axiom (RD) and rule (RDI) (that allow to derive the validities listed in Lemma 3.10). Finally, **RD** has (NecR) but not **RAD**, where an open question is whether (NecR) is derivable in **RAD**. Let us now proceed.

An *admissible form* α is defined by $\alpha ::= \sharp \mid (\varphi \rightarrow \alpha) \mid D_B\alpha$ where $\varphi \in \mathcal{L}_{DR}$ and $B \subseteq A$. The set of admissible forms is denoted *AForm*. An admissible form contains a unique occurrence of $\sharp$. For $\alpha \in AForm$ and $\varphi \in \mathcal{L}_{DR}$, $\alpha(\varphi)$ is the formula obtained by replacing $\sharp$ in α by φ .

Lemma 4.1. *If $\vdash \varphi \rightarrow \psi$, then $\vdash \alpha(\varphi) \rightarrow \alpha(\psi)$.*

Proof. The proof proceeds by straightforward induction on α . □

Theorem 4.2 (Soundness). *If $\vdash \varphi$ then $\models \varphi$.*

Proof. It needs to be shown that all axioms are valid and all rules preserve validity. As for the axioms, their validity is obvious, except for (RD). Let us then show $\models R_{\vec{G}}D_B\varphi \rightarrow D_{see_B(\vec{G})}R_{\vec{H}}\varphi$ for all $\vec{H} \in \mathcal{P}(A)^*$ such that $\vec{G} \approx_B \vec{H}$. Suppose there are $\vec{G}, \vec{H}, B$ and $M = (W, \sim, V)$, $w \in W$ such that $M, w, \varepsilon \models R_{\vec{G}}D_B\varphi$ and $M, w, \varepsilon \not\models D_{see_B(\vec{G})}R_{\vec{H}}\varphi$, where $\vec{G} \approx_B \vec{H}$. Then, there are $v \in W$ and $\vec{I} \in \mathcal{P}(A)^*$ such that $w \sim_{see_B(\vec{G})} v$ i.e. $w \sim_{\vec{G}} v$, $\varepsilon \approx_{see_B(\vec{G})} \vec{I}$ and $M, v, \vec{I} \not\models R_{\vec{H}}\varphi$, so $v, \vec{I}, \vec{H} \not\models \varphi$. Since $\varepsilon \approx_{see_B(\vec{G})} \vec{I}$ and $\vec{G} \approx_B \vec{H}$, by Lemma 3.5, $\varepsilon.\vec{G} = \vec{G} \approx_B \vec{I}.\vec{H}$. Hence from $M, v, \vec{I}, \vec{H} \not\models \varphi$ we can conclude $M, w, \vec{G} \not\models D_B\varphi$ i.e. $M, w, \varepsilon \not\models R_{\vec{G}}D_B\varphi$. This contradicts the hypothesis.

We now turn to the rules. That (MP) and (Nec) preserve validity can be standardly shown. Let us consider (RDI). For clarity, we only consider the case where $\alpha = \sharp$, whilst others can be treated similarly. Suppose $\models D_{see_B(\vec{G})}R_{\vec{H}}\varphi$ for all $\vec{H} \in \mathcal{P}(A)^*$ such that $\vec{G} \approx_B \vec{H}$. Suppose also, towards a contradiction, $\not\models R_{\vec{G}}D_B\varphi$. Then there are $M = (W, \sim, V)$, $w \in W$ such that $M, w, \varepsilon \not\models R_{\vec{G}}D_B\varphi$, i.e. $M, w, \vec{G} \not\models D_B\varphi$. Hence, there are $v \in W, \vec{H} \in \mathcal{P}(A)^*$ such that $w \sim_{\vec{G}} v$, $\vec{G} \approx_B \vec{H}$ and $M, v, \vec{H} \not\models \varphi$, so $M, v, \varepsilon \not\models R_{\vec{H}}\varphi$. But now, since $w \sim_{\vec{G}} v$, also $w \sim_{see_B(\vec{G})} v$. Moreover, $\varepsilon \approx_{see_B(\vec{G})} \varepsilon$ by definition. Hence, from $M, v, \varepsilon \not\models R_{\vec{H}}\varphi$ we get $M, w, \varepsilon \not\models D_{see_B(\vec{G})}R_{\vec{H}}\varphi$, where $\vec{G} \approx_B \vec{H}$. This contradicts the hypothesis. Therefore, $\models R_{\vec{G}}D_B\varphi$. □

Standard frames and semi-standard frames. Instead of models wherein $\sim_B$ is equal to $\bigcap_{b \in B} \sim_b$ (and even by definition) we need models wherein $\sim_B$ may be a proper subset of $\bigcap_{b \in B} \sim_b$. The first we

name *standard models*, based on *standard frames*, whereas the second are *semi-standard models* based on *semi-standard frames*. More precisely, a semi-standard frame is a structure $(W, \sim)$ where W is a non-empty set and, for all $B \subseteq A$, $\sim_B$ is an equivalence relation on W such that for all $B, C \subseteq A$, if $B \subseteq C$ then $\sim_C \subseteq \sim_B$. The completeness proofs involving distributed knowledge often involve ‘unravelling’ a canonical model based on a semi-standard frame into one that is based on a standard frame with the same information content [24, 1]. In our completeness proof we employ the results relating standard and semi-standard frames obtained in [5, Section 6] and in particular [5, Proposition 6.10] that the validities on standard frames and semi-standard frames correspond. Here, we merely adapt this to our setting, where the result to use is that from any model $M = (W, \sim, V)$ based on a semi-standard frame we can construct a model $M' = (W', \sim', V')$ based on a standard frame; in M' the worlds consist of pairs (w, f) where $w \in W$ and f is a function of the set $\mathcal{F}$ of such functions of type $\mathcal{P}(A) \times A \rightarrow \mathcal{P}(W)$. One can then show that (i) if $(w, f) \sim_B^{\vec{G}} (v, g)$, then $w \sim_B^{\vec{G}} v$, and that (ii) if $w \sim_B^{\vec{G}} v$, then there is $g \in \mathcal{F}$ such that $(w, f) \sim_B^{\vec{G}} (v, g)$, for all $\vec{G} \in \mathcal{P}(A)^*$.

Providing a procedure to transform each model based on a semi-standard frame into another modally equivalent model that is based on a standard frame requires the following notions. For all $B \subseteq A$ and for all $w \in W$, $[w]_B$ is the equivalence class of w modulo $\sim_B$. For all $X, Y \in \mathcal{P}(W)$, let $X + Y = (X \setminus Y) \cup (Y \setminus X)$ ¹. We recall that $\mathcal{F}$ is the set of all functions of type $f : \mathcal{P}(A) \times A \rightarrow \mathcal{P}(W)$.

Let now $M = (W, \sim, V)$ be a model based on a semi-standard frame. We define the model $M' = (W', \sim', V')$ by $W' := W \times \mathcal{F}$, $(w, f) \in V'(p)$ iff $w \in V(p)$ and for all $B \subseteq A$, $\sim'_B$ is the binary relation on W' such that for all $(w, f), (v, g) \in W'$, $(w, f) \sim'_B (v, g)$ iff, for all $C \subseteq A$, the two following conditions hold:

- $[w]_C + \sum_{a \in C} f(C, a) = [v]_C + \sum_{a \in C} g(C, a)$
- if $a \in B \cap C$, then $f(C, a) = g(C, a)$ for all $a \in A$.

Obviously, all $\sim'_B$ are equivalence relations, so $(W', \sim')$ is a frame. We now show that it is a standard frame.

Lemma 4.3. *The frame $(W', \sim')$ is semi-standard.*

Proof. Let $B, B' \subseteq A$. Suppose $B \subseteq B'$. Suppose $\sim'_{B'} \not\subseteq \sim'_B$. Hence, there exist $(w, f), (v, g) \in W'$ such that $(w, f) \sim'_{B'} (v, g)$ and $(w, f) \not\sim'_B (v, g)$. Thus, either there exists $C \subseteq A$ such that $[w]_C + \sum_{a \in C} f(C, a) \neq [v]_C + \sum_{a \in C} g(C, a)$, or there exist $C \subseteq A$ and $a \in B \cap C$ such that $f(C, a) \neq g(C, a)$. In the former case, since $(w, f) \sim'_{B'} (v, g)$, therefore $[w]_C + \sum_{a \in C} f(C, a) = [v]_C + \sum_{a \in C} g(C, a)$: a contradiction. In the latter case, since $B \subseteq B'$, also $a \in B' \cap C$. Since $(w, f) \sim'_{B'} (v, g)$, therefore $f(C, a) = g(C, a)$: a contradiction. Therefore $\sim'_{B'} \subseteq \sim'_B$. $\square$

Lemma 4.4. *The semi-standard frame $(W', \sim')$ is standard.*

Proof. Let $B, B' \subseteq A$. Suppose $\sim'_{B \cup B'} \not\supseteq \sim'_B \cap \sim'_{B'}$. Hence, there exist $(w, f), (v, g) \in W'$ such that $(w, f) \not\sim'_{B \cup B'} (v, g)$, $(w, f) \sim'_B (v, g)$ and $(w, f) \sim'_{B'} (v, g)$. Thus, for all $C \subseteq A$, $[w]_C + \sum_{b \in C} f(C, b) = [v]_C + \sum_{b \in C} g(C, b)$. Moreover, for all $C \subseteq A$ and for all $b \in B \cap C$, $f(C, b) = g(C, b)$ and for all $C \subseteq A$ and for all $b \in B' \cap C$, $f(C, b) = g(C, b)$. Since $(w, f) \not\sim'_{B \cup B'} (v, g)$, there exist $E \subseteq A$ and $a \in (B \cup B') \cap E$ such that $f(E, a) \neq g(E, a)$. But, since $a \in B \cup B'$, either $a \in B$, or $a \in B'$. In the former case, since for all $C \subseteq A$ and for all $b \in B \cap C$, $f(C, b) = g(C, b)$, therefore $f(E, a) = g(E, a)$: a contradiction. In the latter case, since for all $C \subseteq A$ and for all $b \in B' \cap C$, $f(C, b) = g(C, b)$, therefore $f(E, a) = g(E, a)$: a contradiction. Therefore $\sim'_{B \cup B'} \supseteq \sim'_B \cap \sim'_{B'}$. $\square$

¹Note that $(\mathcal{P}(W), \emptyset, W, +, \cap)$ is a Boolean ring.

Lemma 4.5. *For all $w, v \in W, f, g \in \mathcal{F}$ and $B \subseteq A$, if $(w, f) \sim'_B (v, g)$, then $w \sim_B v$.*

Proof. Let $B \subseteq A$ and $(w, f), (v, g) \in W'$ be such that $(w, f) \sim'_B (v, g)$. Hence, for all $C \subseteq A$, $[w]_C + \sum_{a \in C} f(C, a) = [v]_C + \sum_{a \in C} g(C, a)$. Moreover, for all $a \in B \cap C$, $f(C, a) = g(C, a)$. Thus, for $C = B$ we get $[w]_B + \sum_{a \in B} f(B, a) = [v]_B + \sum_{a \in B} g(B, a)$. Moreover, for all $a \in B$, $f(B, a) = g(B, a)$. Consequently, $[w]_B = [v]_B$. Hence, $w \sim_B v$. $\square$

Corollary 4.6. *For all $w, v \in W, f, g \in \mathcal{F}$ and $\vec{G} \in \mathcal{P}(A)^*, B \subseteq A$, if $(w, f) \sim'^{\vec{G}}_B (v, g)$, then $w \sim^{\vec{G}}_B v$.*

Lemma 4.7. *For all $w, v \in W, f \in \mathcal{F}$ and $B \subseteq A$, if $w \sim_B v$, then there is $g \in \mathcal{F}$ such that $(w, f) \sim'_B (v, g)$.*

Proof. Let $w, v \in W$. Suppose $w \sim_B v$, i.e. $[w]_B = [v]_B$. Since $(W', \sim')$ is semi-standard, for all $C \subseteq A$, if $C \subseteq B$, then $w \sim_C v$ and $[w]_C = [v]_C$. To construct $g : \mathcal{P}(A) \times A \rightarrow \mathcal{P}(W)$, consider an enumeration $\{a_1, a_2, \dots, a_k\}$ of the agents in A . For all $C \subseteq A$ and for all $a \in A$, we define $g(C, a)$ as follows:

- if $a \in B \cap C$ then $g(C, a) := f(C, a)$
- if $a \in C \setminus B$, let $1 \leq i \leq k$ be such that $a = a_i$. Now, if $i = \min\{1 \leq j \leq k \mid a_j \in C \setminus B\}$, then $g(C, a) := [w]_C + [v]_C + \sum_{b \in C \setminus B} f(C, b)$; otherwise $g(C, a) := \emptyset$.
- if $a \notin C$ then $g(C, a) := \emptyset$.

The reader may easily verify that $(w, f) \sim'_B (v, g)$. $\square$

Corollary 4.8. *For all $w, v \in W, f \in \mathcal{F}$ and $\vec{G} \in \mathcal{P}(A)^*, B \subseteq A$, if $w \sim^{\vec{G}}_B v$, then there is $g \in \mathcal{F}$ such that $(w, f) \sim'^{\vec{G}}_B (v, g)$.*

We can now prove the following lemma that allows one to transform any semi-standard model into a standard model while preserving the satisfaction relation.

Lemma 4.9. *Let semi-standard $M = (W, \sim, V)$ be given and standard $M' = (W', \sim', V')$ be constructed from M as above. For all $\varphi \in \mathcal{L}_{DR}$, $w \in W$, $f \in \mathcal{F}$ and $\vec{G} \in \mathcal{P}(A)^*$: $M, w, \vec{G} \models \varphi$ iff $M', (w, f), \vec{G} \models \varphi$.*

Proof. The proof proceeds by induction on φ . Let $w \in W, f \in \mathcal{F}$ and $\vec{G} \in \mathcal{P}(A)^*$.

- If $\varphi = p$, then obviously $M, w, \vec{G} \models p \Leftrightarrow M', (w, f), \vec{G} \models p$.
- If $\varphi = \top$, then obviously $M, w, \vec{G} \models \top$ and $M', (w, f), \vec{G} \models \top$.
- If $\varphi = \neg\psi$, $\varphi = \psi \wedge \psi$ or $\varphi = R_B\psi$, we conclude by applying the inductive hypothesis. This is straightforward.
- If $\varphi = D_B\psi$, suppose $M, w, \vec{G} \models D_B\psi$ and $M', (w, f), \vec{G} \not\models D_B\psi$. Then, there are $(v, g) \in W'$ and $\vec{H} \in \mathcal{P}(A)^*$ such that $(w, f) \sim'^{\vec{G}}_B (v, g)$, $\vec{G} \approx_B \vec{H}$ and $M', (v, g), \vec{H} \not\models \psi$. By induction hypothesis, then $M, v, \vec{H} \not\models \psi$. Since $(w, f) \sim'^{\vec{G}}_B (v, g)$, by Corollary 4.6, $w \sim^{\vec{G}}_B v$. Moreover, $\vec{G} \approx_B \vec{H}$. Therefore $M, w, \vec{G} \not\models D_B\psi$. This contradicts the hypothesis.

Suppose now $M, w, \vec{G} \not\models D_B\psi$ and $M', (w, f), \vec{G} \models D_B\psi$. Then, there are $v \in W, \vec{H} \in \mathcal{P}(A)^*$ such that $w \sim^{\vec{G}}_B v$, $\vec{G} \approx_B \vec{H}$ and $M, v, \vec{H} \not\models \psi$. Now, by Corollary 4.8, there is $g \in \mathcal{F}$ such that $(w, f) \sim'^{\vec{G}}_B (v, g)$. By induction hypothesis, from $M, v, \vec{H} \not\models \psi$ we get $M', (v, g), \vec{H} \not\models \psi$, where $(w, f) \sim'^{\vec{G}}_B (v, g)$ and $\vec{G} \approx_B \vec{H}$. Hence $M', (w, f), \vec{G} \not\models D_B\psi$: a contradiction. $\square$

Proposition 4.10. *For all $\varphi \in \mathcal{L}_{DR}$, if φ is valid on standard frames then φ is valid on semi-standard frames.*

Proof. By Lemma 4.9. $\square$

We now move to the proof of the completeness of **RAD**. This requires further terminology.

A theory T is a set of formulas $\varphi \in \mathcal{L}_{DR}$ such that T contains all formulas derivable in **RAD**; T is closed under (MP) and T is closed under (RDI). A theory T is *consistent* if it does not contain $\perp$. Furthermore, T is *maximal consistent* if T is consistent and, for all theories T' , if $T \subsetneq T'$, then T' is not consistent. Note that the only inconsistent theory is the theory containing all formulas. For T a theory, $\chi \in \mathcal{L}_{DR}$ and $B \subseteq A$ we define $T + \chi := \{\varphi \mid \chi \rightarrow \varphi \in T\}$ and $D_B T := \{\varphi \mid D_B \varphi \in T\}$.

Lemma 4.11. *Let T be a theory, $\chi \in \mathcal{L}_{DR}$ and $B \subseteq A$. Then, $T + \chi$ and $D_B T$ are also theories. Moreover, $T \subseteq T + \chi$ and $\chi \in T + \chi$. Finally, if $\neg\chi \notin T$, $T + \chi$ is consistent, and if $\chi \notin T$, $T + \neg\chi$ is consistent.*

Proof. This is standard. $\square$

Lemma 4.12. *Let $\vec{G} \in \mathcal{P}(A)^*$, $B \subseteq A$ and $\varphi \in \mathcal{L}_{DR}$. If T is a theory and $\alpha(R_{\vec{G}} D_B \varphi) \notin T$, then there is $\vec{H} \in \mathcal{P}(A)$ such that $\vec{G} \approx_B \vec{H}$ and $\alpha(D_{see_B(\vec{G})} R_{\vec{H}} \varphi) \notin T$.*

Proof. This is straightforward, since T is closed under (RDI). $\square$

Lemma 4.13 (Lindenbaum's Lemma). *If T is a consistent theory, then there is a maximal consistent theory Σ such that $T \subseteq \Sigma$.*

Proof. Let T be a consistent theory and $\{\varphi_0, \varphi_1, \dots\}$ an enumeration of formulas in $\mathcal{L}_{DR}$. For all $k \in \mathbb{N}$, we define the theory T_k as follows (from the construction and Lemma 4.11 it follows that these T_k are theories, because T is a theory):

$$T_0 := T$$

$$T_{k+1} := \begin{cases} T_k & \text{if } \neg\varphi_k \in T_k \\ T_k + \varphi_k & \text{if } \neg\varphi_k \notin T_k \text{ and } \varphi_k \text{ does not have the shape } \neg\alpha(R_{\vec{G}} D_B \psi) \\ T_k + \neg\alpha(D_{see_B(\vec{G})} R_{\vec{H}} \psi) & \text{if } \neg\varphi_k \notin T_k \text{ and } \varphi_k \text{ does have the shape } \neg\alpha(R_{\vec{G}} D_B \psi), \\ & \text{for some } \vec{H} \approx_B \vec{G} \text{ such that } \alpha(D_{see_B(\vec{G})} R_{\vec{H}} \psi) \notin T_k \end{cases}$$

Note that in the third case of the inductive step, such a sequence $\vec{H}$ is known to exist by Lemma 4.12.

Let now $\Sigma := \bigcup_{k \geq 0} T_k$. It needs to be shown that Σ is a maximal consistent theory. First note that, by construction, for all $\varphi \in \mathcal{L}_{DR}$, either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$.

We first show that Σ is a theory. That Σ contains **RAD** and is closed by modus ponens is straightforward. Suppose now there are $\vec{G} \in \mathcal{P}(A)^*$, $B \in \mathcal{P}(A)$, $\varphi \in \mathcal{L}_{DR}$ and $\alpha \in AForm$ such that $\alpha(D_{see_B(\vec{G})} R_{\vec{H}} \varphi) \in \Sigma$ for all $\vec{H} \approx_B \vec{G}$. Suppose also $R_{\vec{G}} D_B \varphi \notin \Sigma$. Then $\neg R_{\vec{G}} D_B \varphi \in \Sigma$. Let $k \in \mathbb{N}$ such that $\varphi_k = \neg R_{\vec{G}} D_B \varphi$. Then, since $\varphi_k \in \Sigma$, $\neg\varphi_k \notin \Sigma$ so in particular $\neg\varphi_k \notin T_k$. Since T_k is a theory and $R_{\vec{G}} D_B \varphi \notin T_k$, by Lemma 4.12, there is $\vec{H} \approx_B \vec{G}$ such that $\alpha(D_{see_B(\vec{G})} R_{\vec{H}} \varphi) \notin T_k$. Hence, by construction, $T_{k+1} = T_k + \neg\alpha(D_{see_B(\vec{G})} R_{\vec{H}} \varphi)$. Therefore $\alpha(D_{see_B(\vec{G})} R_{\vec{H}} \varphi) \notin \Sigma$, which contradicts the hypothesis.

Furthermore, Σ is consistent because so is each T_k . We now show that Σ is maximal consistent. Let Δ be a theory strictly containing Σ . Then, there is $\varphi_k \in \mathcal{L}_{DR}$ such that $\varphi_k \notin \Sigma$ and $\varphi_k \in \Delta$. Since $\varphi_k \notin \Sigma$, $\neg\varphi_k \in \Sigma \subset \Delta$. Hence, $\neg\varphi_k \in \Delta$, so Δ is not consistent. $\square$

We now (re)introduce the relation $\sim_B$. Let Σ, Δ be two maximal consistent theories, $B \subseteq A$. Then $\Sigma \sim_B \Delta$ iff $D_B \Sigma \subseteq \Delta$. Note that by definition $\sim_{B \cup C} \subseteq \sim_B \cap \sim_C$, but equality is not guaranteed.

Lemma 4.14 (Existence Lemma). *Let Σ be a maximal consistent theory, $B \subseteq A$, $\varphi \in \mathcal{L}_{DR}$. If $\hat{D}_B \varphi \in \Sigma$, then there is a maximal consistent theory Δ such that $\Sigma \sim_B \Delta$ and $\varphi \in \Delta$.*

Proof. Suppose $\hat{D}_B \varphi \in \Sigma$. Then $\neg D_B \neg \varphi \in \Sigma$ so $D_B \neg \varphi \notin \Sigma$. Hence $\neg \varphi \notin D_B \Sigma$. So, by Lemma 4.11, $D_B \Sigma + \varphi$ is consistent. Now, by Lindenbaum's Lemma, we can extend $D_B \Sigma + \varphi$ to a maximal consistent theory Δ such that $D_B \Sigma + \varphi \subseteq \Delta$. By Lemma 4.11 again, $D_B \Sigma \subseteq D_B \Sigma + \varphi$ so $D_B \Sigma \subseteq \Delta$. Hence $\Sigma \sim_B \Delta$. Moreover, $\varphi \in D_B \Sigma + \varphi$, so $\varphi \in \Delta$. $\square$

Definition 4.15 (Canonical model). The *canonical model* $M = (W, \{\sim_B\}_{B \subseteq A}, V)$ is defined by

- $W = \{\Sigma \mid \Sigma \text{ is a maximal consistent theory}\}$;
- $\Sigma \sim_B \Delta$ if, and only if, $D_B \Sigma \subseteq \Delta$;
- $V(p) = \{\Sigma \in W \mid p \in \Sigma\}$.

Notice the canonical model is defined on a semi-standard frame.

Lemma 4.16 (Truth Lemma). *For all $\varphi \in \mathcal{L}_{DR}$, $\vec{G} \in \mathcal{P}(A)^*$ and $\Sigma \in W$: $R_{\vec{G}} \varphi \in \Sigma$ iff $\Sigma, \vec{G} \models \varphi$.*

Proof. The proof proceeds by induction on φ .

- If $\varphi = p$: by (RA) we get: $R_{\vec{G}} p \in \Sigma \Leftrightarrow p \in \Sigma \Leftrightarrow \Sigma, \vec{G} \models p$.
- If $\varphi = \top$: $\Sigma, \vec{G} \models \top$ and, by (RT), $R_{\vec{G}} \top \in \Sigma$.
- If $\varphi = \neg \psi$: by (RN) we get: $R_{\vec{G}} \neg \psi \in \Sigma \Leftrightarrow \neg R_{\vec{G}} \psi \in \Sigma \Leftrightarrow R_{\vec{G}} \psi \notin \Sigma \stackrel{(IH)}{\Leftrightarrow} \Sigma, \vec{G} \not\models \psi \Leftrightarrow \Sigma, \vec{G} \models \neg \psi$.
- If $\varphi = \psi \wedge \chi$: by (RC) we get $R_{\vec{G}}(\psi \wedge \chi) \in \Sigma \Leftrightarrow R_{\vec{G}} \psi \wedge R_{\vec{G}} \chi \in \Sigma \Leftrightarrow R_{\vec{G}} \psi \in \Sigma$ and $R_{\vec{G}} \chi \in \Sigma \stackrel{(IH)}{\Leftrightarrow} \Sigma, \vec{G} \models \psi$ and $\Sigma, \vec{G} \models \chi \Leftrightarrow \Sigma, \vec{G} \models \psi \wedge \chi$.
- If $\varphi = R_B \psi$: $R_{\vec{G}} R_B \psi \in \Sigma \Leftrightarrow R_{\vec{G}.B} \psi \in \Sigma \stackrel{(IH)}{\Leftrightarrow} \Sigma, \vec{G}.B \models \psi \Leftrightarrow \Sigma, \vec{G} \models R_B \psi$.
- If $\varphi = D_B \psi$: suppose first $R_{\vec{G}} D_B \psi \in \Sigma$ and $\Sigma, \vec{G} \not\models D_B \psi$. Then there are $\Delta, \vec{H}$ such that $\Sigma \sim_B^{\vec{G}} \Delta$, $\vec{G} \approx_B \vec{H}$ and $\Delta, \vec{H} \not\models \psi$. By induction hypothesis, then, $R_{\vec{H}} \psi \notin \Delta$. Now, since $R_{\vec{G}} D_B \psi \in \Sigma$ and, by (RD) $R_{\vec{G}} D_B \psi \rightarrow D_{\text{see}_B(\vec{G})} R_{\vec{H}} \psi \in \Sigma$ (because $\vec{G} \approx_B \vec{H}$), so $D_{\text{see}_B(\vec{G})} R_{\vec{H}} \psi \in \Sigma$. Moreover, since $\Sigma \sim_B^{\vec{G}} \Delta$, also $\Sigma \sim_{\text{see}_B(\vec{G})} \Delta$, so $D_{\text{see}_B(\vec{G})} \Sigma \subseteq \Delta$. Then $R_{\vec{H}} \psi \in \Delta$, which contradicts $R_{\vec{H}} \psi \notin \Delta$.
Suppose now $R_{\vec{G}} D_B \psi \notin \Sigma$ and $\Sigma, \vec{G} \models D_B \psi$. Since $R_{\vec{G}} D_B \psi \notin \Sigma$ and Σ is closed under (RDI), there is $\vec{H} \in \mathcal{P}(A)^*$ such that $\vec{G} \approx_B \vec{H}$ and $D_{\text{see}_B(\vec{G})} R_{\vec{H}} \psi \notin \Sigma$. Then, $\neg D_{\text{see}_B(\vec{G})} R_{\vec{H}} \psi \in \Sigma$, so $\hat{D}_{\text{see}_B(\vec{G})} \neg R_{\vec{H}} \psi \in \Sigma$. By the Existence Lemma, there is a maximal consistent theory Δ such that $\Sigma \sim_{\text{see}_B(\vec{G})} \Delta$ and $\neg R_{\vec{H}} \psi \in \Delta$, so $R_{\vec{H}} \psi \notin \Delta$. By induction hypothesis then, $\Delta, \vec{H} \not\models \psi$. But since $\Sigma \sim_{\text{see}_B(\vec{G})} \Delta$, also $\Sigma \sim_B^{\vec{G}} \Delta$ and $\vec{G} \approx_B \vec{H}$, so $\Delta, \vec{H} \not\models \psi$ implies $\Sigma, \vec{G} \not\models D_B \psi$, which contradicts the hypothesis.

$\square$

Theorem 4.17 (Completeness). *For all $\varphi \in \mathcal{L}_{DR}$, if $\models \varphi$, then $\vdash \varphi$.*

Proof. Let $\varphi \in \mathcal{L}_{DR}$. We show the contrapositive: if $\not\models \varphi$, then $\not\vdash \varphi$. Suppose then $\not\models \varphi$. Then $\varphi \notin \mathbf{RAD}$ so $\mathbf{RAD} + \neg \varphi$ is consistent: we extend it to a maximal consistent theory Σ , by Lindenbaum's Lemma. Therefore, $\neg \varphi = R_{\varepsilon} \neg \varphi \in \Sigma$. By the Truth Lemma, $\Sigma, \varepsilon \models \neg \varphi$ so $\Sigma, \varepsilon \not\models \varphi$. Hence φ is not valid on semi-standard frames. By Proposition 4.10, then, φ is not valid on standard frames. Therefore $\not\vdash \varphi$. $\square$

5 Conclusion and further research

We presented a logic of resolving asynchronous distributed knowledge, compared it to the logic of resolving (synchronous) distributed knowledge known from the literature, and provided an infinitary axiomatization for our asynchronous logic. There are a fair number of open questions about this novel logic. (i) We defined two notions of validity, with respect to the empty resolution sequence and with respect to arbitrary resolution sequences, but we do not know whether these define the same set of validities. (ii) We gave an infinitary axiomatization, but we have no proof that a finitary axiomatization does not exist. (iii) Is necessitation of resolution validity preserving (resp. an admissible derivation rule)? (iv) Is satisfiability decidable, and if so, what is the complexity? (v) Given the embedding of the synchronous into the asynchronous semantics, what is the expressivity hierarchy comparing the language fragments with synchronous and with asynchronous distributed knowledge, and with or without resolution? It seems fairly straightforward to show that resolving asynchronous distributed knowledge is more expressive than resolving synchronous distributed knowledge, or at least on the level of so-called update expressivity that describes relations between pointed epistemic models instead of properties of pointed epistemic models in the case of formula expressivity. Concerning the latter, clearly, a sequence of two resolutions $ab.bc$ does not correspond to a single resolution B for some $B \subseteq A$, wherein all agents in B are equally well informed. A more interesting question is whether resolving asynchronous distributed knowledge is more expressive than logical semantics of synchronous distributed knowledge with more involved dynamics than resolution, such as [10, 11, 16]. For example, a sequence of resolutions $ab.bc$ corresponds to a single communication graph (as in [16]) namely where b and c receive information from a, b, c whereas a only receives information from a, b , and, for a more involved example, the uncertainty of an agent d between resolutions ab and $ab.bc$ can be simulated as non-public information exchange as in [11]. We wish to investigate that in the future. (vi) Can we determine when a resolution in a given sequence is redundant (because uninformative), such as (immediately) repeating the same resolution? (vii) We wish to extend the language and semantics with common knowledge.

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The Logic of Data Access and Data Exchanges

Alexandru Baltag

University of Amsterdam, ILLC
Amsterdam, Netherlands

thealexandrubaltag@gmail.com

Sonja Smets

University of Amsterdam, ILLC
Amsterdam, Netherlands

s.j.l.smets@uva.nl

We investigate a new logic that extends Dynamic Epistemic Logic (DEL), by combining standard epistemic modalities $K_a\phi$ and $K_A\phi$ for (individual and distributed) *propositional knowledge* with operators $K_a^\phi x$, $K_A^\phi x$ denoting (*conditional*) *non-propositional knowledge of a number* (in which an agent a or a group A have knowledge of the value of some variable x , if given additional information ϕ). We also generalize these operators, by considering formulas $|x|_A^\phi \leq N$ (for any natural number N), saying that: *conditional on ϕ , A can narrow down the possible values of variable x to at most N possibilities*. In order to name and compare such hypothetical values, we extend the logic further with *definite descriptions based on minimization operators*: $\mu_N x_A^\phi$ denotes the *least of the N possible values of x* (according to some fixed order $\leq$) that are considered possible by group A (given condition ϕ). On this static base, we consider DEL-style extensions with *dynamic modalities for general ‘data-exchange events’* (covering private and public propositional announcements, but also secret hacking of a private database, or public sharing of one’s data via open-source repositories, etc). In such scenarios, whole ‘chunks’ of information may be exchanged or modified: once access to a given source is gained, all the ‘data’ stored at that specific location becomes available. We give complete axiomatizations for the resulting logics, and prove their decidability and co-expressivity.

1 Introduction

Information may come in both propositional form (Boolean variables) and non-propositional form (e.g. numbers, names, addresses, pictures, videos, etc). Such *data* can be encoded as *values of local variables* (e.g. a cryptographic key or a password), that are stored and processed in certain *locations* or ‘sites’ (e.g., websites, folders, databases, etc), and can be retrieved when these sites are accessed as ‘sources’. Each such source can be thought of as an *agent* (either because it actually is the knowledge base of a natural or artificial agent, or because we think of it as an abstract ‘agent’ possessing exactly the information that is stored at it).¹ Such agents either ‘own’ their data, or else have gained access to them from other sources: we say that the agents ‘know’ the values of these variables, and some of them can also modify these values. More complex data may have an *extended location*, the information being *distributed* among a number of agents: one can recover the value of such complex variables only by accessing several sources.

To deal with non-propositional information, multi-agent epistemic logic has been extended in recent years with “knowing-what” operators K_ax or K_Ax for (individual or distributed) *knowledge of (the value of) some variable x* (in addition to the traditional modalities $K_a\phi$ and $K_A\phi$ for individual or distributed knowledge of a proposition ϕ). This work traces back to Plaza [22] and subsequent investigations by [18, 25, 26, 27, 20, 16, 21, 3, 12] on formalizing ‘knowledge de re’. While Plaza axiomatized the static logic of ‘knowing what’, the extension with dynamic operators $[\phi]\psi$ for (propositional) public announcements $!\phi$ was only later axiomatized by Wang and Fan [27]. To ‘pre-encode’ this dynamics using DEL-style reduction axioms [9], these authors needed a *conditional version of ‘knowing what’*: $K_a^\phi x$ means that a knows the value of x given the information that proposition ϕ was the case.

¹The same piece of data can be stored/copied at multiple locations, and each location can store multiple pieces of data.

In recent work [12], we extended this setting with operators $K_A^\phi x$ for *conditional distributed knowledge of the value of x by a group A* . This was motivated by the need to deal with a more complex dynamics, going beyond propositional announcements or other traditional DEL-events [9, 7, 5, 17, 6, 8, 23]. In a *data-exchange event*, agents may gain access to ‘sources’ (i.e., to other agents’ knowledge bases), in which case they can be assumed to instantly ‘read’ (and copy in their own knowledge base) *all* the information stored at those sources. Such events were previously considered in our work [10, 11], and in fact they subsume many other forms of non-propositional informational dynamics that were previously considered, e.g. “tell us all you know” in [2] (and with a different, interrogative interpretation in [24, 14]), in-group sharing [2, 15, 19, 4], and ‘resolution’ [1]. Other examples of data-exchange events covered in [12] include private changing of the value of a variable (e.g. one’s password), parallel data-sharing within different subgroups (e.g., in a poster session), suspected hacking of a private database, private detection of such hacking, etc. As noted in [10], when an agent a gains access to another agent b ’s knowledge base, her new state of knowledge matches the *distributed knowledge* of the group $\{a, b\}$. So on the side of propositional knowledge, we needed operators $K_A\phi$ for *distributed knowledge² within group A* . A similar move was also necessary on the side of “knowing what”: operators K_Ax , expressing that *the group A has distributed knowledge of the value of x* ; in fact, to pre-encode public-announcement dynamics we needed the *conditional version* $K_A^\phi x$ of these operators. Note that this type of conditional knowledge may *not* imply knowledge of the ‘real’ value of x (in the actual world), but only the fact that the agent or group *can uniquely determine a hypothetical value* of x (applicable only to worlds satisfying ϕ). To axiomatize these notions, in the presence of equality of values $x = y$, we were lead to introduce *definite description terms* x_A^ϕ that can *explicitly refer to such hypothetical values*: x_a^ϕ denotes (in the actual world) the (unique) value that x would have according to agent a if ϕ were true. Using this, we were able to provide in [12] a complete axiomatization for the corresponding dynamic-epistemic logic. But for this, we had to *restrict* the dynamics to ‘semi-public’ events (a class that does *not* include e.g. secret hacking). On the other hand, in [11] we axiomatized a logic with *unrestricted dynamics*, covering ‘arbitrary’ data-exchange events (including complicated hacking scenarios). But this was based on a *restricted static logic base*, that did *not* include non-propositional knowledge operators, but only the traditional epistemic modalities $K_a\phi$ and $K_A\phi$, as well as (a major generalization of) common knowledge $C_A\phi$ (in the form of polyadic conditional-epistemic group modalities $C_A^e\phi_1 \dots \phi_n$, indexed by data-exchange events e). There were *no explicit variables x* in the logic, so no assertions $x = y$ or $Px_1 \dots x_n$ talking about the properties of specific pieces of (non-propositional) data, as well as *no* “knowledge-what” operators K_ax , K_Ax or $K_A^\phi x$. And, because of these limitations, the class of data-exchange events was still inherently restricted: e.g., there was no event of privately changing the value (such as one’s password), and no events involving preconditions of the form K_ax (for instance, no “conditional hacking”, in which it is common knowledge that an agent a may be hacking another agent b ’s database if and only if she came to know b ’s password).

In this paper, we overcome the limitations present in the frameworks of both [11] and [12], by providing a common generalization, while also greatly increasing the expressivity of the logic. More specifically, we generalize the operators $K_A^\phi x$ to expressions $|x|_A^\phi \leq N$ (for any natural number N), saying that: *if given additional information ϕ , the group A can narrow down the possible values of variable x to at most N possibilities* (for any natural number N). In other words, the group has distributed knowledge that, if ϕ is the case then the value of x belongs to a given list of N possible values.³ This is a useful addition when dealing with cryptographic protocols. Indeed, if N is ‘small enough’ and x is say another

²The notation $D_A\phi$ is typically used for distributed knowledge, but we prefer here $K_A\phi$ because we think of this as capturing a natural notion of (virtual) group knowledge.

³Conditional knowledge $K_A^\phi x$ of the value can now be recovered as a special case (for $N = 1$).

agent b 's communication key or private password, then any intruder a having the capability $|x|_a^\varphi \leq N$ will be able to hack b 's communication (by simply trying out all the N possible values), as soon as she may receive the information φ . The same applies to a group A of hackers if they can collectively narrow down the possibilities, i.e., have the capability $|x|_A^\varphi \leq N$. As done in [12] for $K_A^\varphi x$, whenever we have $|x|_a \leq N$ we introduce *definite descriptions denoting each of the N hypothetical values of x* . For this, we assume given a salient *total order* $\leq$ on the set of values. This is useful in numerical applications, but it also allows us to convert cardinality statements $|x|_a \leq N$ into ordinal descriptions $1_N.x_A^\varphi, 2_N.x_A^\varphi$ etc, denoting “the first (or the second, etc) of the $\leq N$ possible values of x according to A given φ ”.⁴ To capture the properties of the total order, we include in our language *order statements* $x \leq y$ (besides equality of values $x = y$ and possibly other predicates $Px_1 \dots x_n$). We obtain a very expressive dynamic-epistemic logic, and we are able to completely axiomatize it and prove its decidability. Our proofs make an innovative use of known methods. For the static logic, we first use filtration to obtain a quasi-model: this is not a ‘real’ model, but just a syntactic construction, in which *variables have no values* (and the group relations are non-standard, as they are *not intersections* of the individual relations). We then follow the usual method for dealing with distributed knowledge, by unraveling the model into an infinite tree and redefining the relations to make them standard. However, new complications ensue due to the variables, and especially due to the combination of equality $x = y$ and non-propositional knowledge $K_A x$; the problems are in fact compounded by the presence of formulas for order $x \leq y$ and narrowing down $|x|_A^\varphi \leq N$. Here, we use our definite descriptions to ensure that the local properties of variables x are preserved and continuously transmitted at far-away nodes of the tree.⁵ Defining the total order on values over the tree requires a use of the Order Extension Principle, and hence of the Axiom of Choice. Finally, the completeness for the dynamic logic uses DEL-style reduction axioms; but again, the details are quite tantalizing. Even the soundness of some reduction laws (e.g., the ‘Cutoff-Min. Change’ axiom) is not at all obvious!

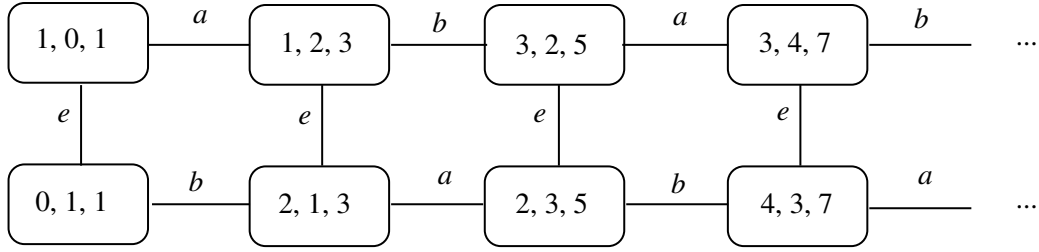
We should mention a self-imposed limitation: due to the page-limit, we chose *not* to include common knowledge operators $C_A \varphi$, whose dynamics introduces another source of complexity, requiring additional technical developments and proofs. We leave this step for the journal-version of this paper.

2 Motivating Examples

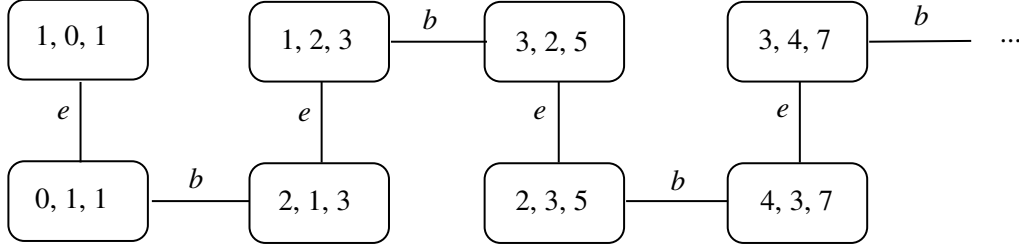
Example 1 Alice and Bob are each given a number $x(a), x(b) \in \mathbb{N} = \{0, 1, 2, \dots\}$, while the value of the sum $x(e) := x(a) + x(b)$ is stored in a closed envelope e . It is common knowledge that: (1) *each of the two sees his/her own number, but neither of them can see the others' number, nor the sum $x(e)$* ; (2) *one of the two numbers x^a, x^b is the immediate successor of the other* (i.e. either $x(a) = x(b) + 1$ or $x(b) = x(a) + 1$). We represent this situation using an *infinite Kripke model*, where states are triplets of numbers $(x(a), x(b), x(e))$, and the accessibility relations $\sim_a$ and $\sim_b$ encode the agents' uncertainty. We also have an accessibility relation $\sim_e$ for the ‘envelope’ (treated as an abstract ‘agent’), encoding the information contained in it. The formula $K_a x(a) \wedge K_b x(b)$, saying that *agents know their numbers*, is valid on this model, while $K_e x(e)$ says that the sum $x(e)$ is stored in the envelope. On the other hand, agents a and b can *together* figure out the sum: we have $K_{\{a,b\}} x(e)$, i.e. the group $A = \{a, b\}$ has “*distributed*” knowledge of $x(e)$. The formulas $|x(a)|_b \leq 2, |x(b)|_a \leq 2, |x(e)|_a \leq 2$ and $|x(e)|_b \leq 2$ say that the live agents can *narrow down the possibilities* (for the other's number, as well as for the sum) *to at most two*. To *name* these values, we write e.g. $1_2.x(e)_a$ for the *first* (i.e., the *least*) of the *two* possible values of the sum $x(e)$ according to Alice, $2_2.x(e)_a$ for the *second* least (=largest) of the *two*, etc. The picture is:

⁴We can define all these using a *cutoff-minimization construct* $\mu_N x_A^\varphi$, which is the same as $1_N.x_A^\varphi$: it denotes *the least of the $\leq N$ possible values of x (according to A given φ)*.

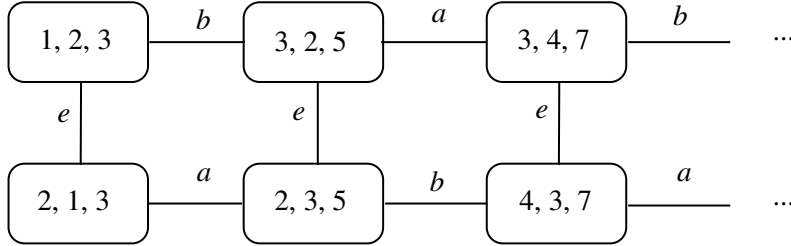
⁵In effect, the ‘values’ of variables x “hitch a ride” on the back of various group epistemic transitions $\rightarrow_A$ in the tree, and their properties are transmitted by passing them to *other* (A -preserved) terms.



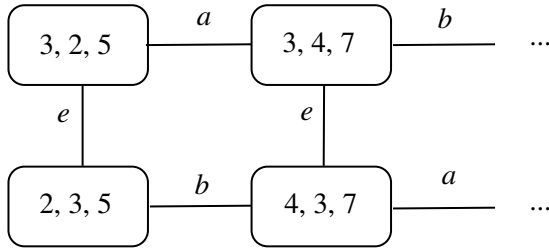
Example 1, continued Next, it is common knowledge that Alice ‘hacks’ Bob’s database. This is a ‘semi-public’ event $!(a : b)$, which updates the model by replacing $\sim_a$ with the intersection $\sim_a \cap \sim_b$:



Example 2: an alternative scenario Start in the initial situation from Example 1. This time no hacking is allowed, but Alice and Bob are asked: “Do you know each other’s number”? They both answer (truthfully, publicly and simultaneously): “I don’t know”. This is a *public announcement* $!(\neg K_a x(b) \wedge \neg K_b x(a))$. The updated model is obtained by deleting states (0, 1, 1) and (1, 0, 1) from the initial model:



If asked the same question again, they again answer “I don’t know”, which deletes (2, 1, 3) and (1, 2, 3):



If asked for the 3rd time, Alice says “I know Bob’s number” and Bob says “I don’t know”. The announcement $!(K_a x(b) \wedge \neg K_b x(a))$ deletes all states but (2, 3, 5): so we have $x(a) = 2, x(b) = 3, x(e) = 5$.

Example 3 Starting again in the initial situation from Example 1, agents are publicly announced that $x(a) < x(b)$. After that, Alice knows Bob’s number, i.e. $[(x(a) < x(b))]K_a x(b)$. We can reason hypothetically about this scenario in the original model, without updating it, using conditional knowledge $K_a^{x(a) < x(b)} x(b)$ of the (hypothetical) value of $x(b)$ given the condition $x_a < x_b$. We can also name this hypothetical value using a conditionalized version of the “least-of- N ” description introduced above, e.g. writing $1_1.x(b)_a^{x(a) < x(b)}$ for the unique (=least out of only one) value of $x(b)$ that is considered possible by

Alice conditional on $x(a) < x(b)$. But note that, when considering a condition that is *known to be false* (e.g., $x(a) = x(b)$), there are *no such hypothetical values*: so the best we can do is to interpret the ‘least’ value as the ‘infimum’ $1_1.x(b)_a^{x(a)=x(b)} = \inf \emptyset = \infty$. This means that we are working in $\mathbb{N}^\top := \mathbb{N} \cup \{\infty\}$.

Example 4: Back in time Imagine now that our story starts **earlier**, at a time when each ‘agent’ (Alice a , Bob b , and the envelope e) only ‘knows’ their own number (but not yet the correlations between numbers). The model is just $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, with $(x(a), x(b), x(c)) \sim_a (x'(a), x'(b), x'(c))$ iff $x(a) = x'(a)$, and similarly for b and e . Next, the correlations are publicly announced $!((x(a) = x(b) + 1 \vee x(b) = x(a) + 1) \wedge x(e) = x(a) + x(b))$, and after that the semi-public hacking $!(a:b)$ happens. Can we check that Alice will know $x(e)$ after this scenario *without updating the model* $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$? For this, we will need an operator $K_{a,b}^{(x(a)=x(b)+1 \vee x(b)=x(a)+1) \wedge (x(e)=x(a)+x(b))} x(e)$ for *conditional distributed knowledge of a value*.

To formalize the examples, we need the following ingredients: *equality* $x = y$; *order* $x < y$; *functions* $x = y + 1$, $z = x + y$; *(conditional) (distributed) knowledge of a number* $K_A^\phi x$; a way to express than an agent/group can *(conditionally) narrow down to N possible values* ($|x|_A^\phi \leq N$), and to *name these values in increasing order*: $1_N.x_A^\phi$, $2_N.x_A^\phi$, etc; *public announcements* $!\phi$, *semi-public hacking* $!(a:b)$, etc.

3 Syntax and Semantics of LDA and LDAE

Vocabularies: static and dynamic. A *static vocabulary* $\mathcal{V} = (\mathcal{A}, V, Prop, Pred, Funct, ar, \varepsilon)$ consists of the following:⁶ a finite set $\mathcal{A}$ of *agents* $a, b, c, \dots$, also called ‘locations’ (e.g., websites, databases, processors, etc), where data are stored and processed; a set V of *basic variables* $v, v', v'', \dots$; a set $Prop$ of *atomic propositions* $p, q, \dots$; a set $Pred$ of *predicate symbols* $P, Q, \dots$, including *equality* ($=$) and an *order predicate* $\leq$; a set $Funct$ of *function symbols* $F, G, \dots$, including two *constants* $\perp, \top$; an *arity map* $ar : Pred \cup Funct \rightarrow \mathbb{N}$, sending predicate symbols $P \in Pred$ and function symbols $F \in Funct$ to natural numbers $ar(P), ar(F) \in \mathbb{N}$, with $ar(\top) = ar(\perp) = 0$ and $ar(=) = ar(\leq) = 2$; an auxiliary symbol $\varepsilon \notin \mathcal{A} \cup V \cup Prop \cup Pred \cup Funct$, denoting ‘the environment’ (e.g., the ‘envelope’ in our example).

A *dynamic vocabulary* $(\mathcal{V}, \mathcal{E})$ is a pair consisting of a static vocabulary $\mathcal{V}$ and a *countable set* $\mathcal{E}$ of ‘event names’ (denoted by $e, e', f, \dots$), that includes some special symbols $!, ?, \tau$.

Groups Given the static vocabulary $\mathcal{V}$, a *group* $A \subseteq \mathcal{A}$ is any non-empty set of agents in $\mathcal{A}$. We use capital letters $A, B, \dots$ to denote groups.

Ordered Value Domains A *value domain* (for a given vocabulary $\mathcal{V}$) is a first-order model $\mathbf{D} = (D, I)$ for $\mathcal{V}$, consisting of: a *domain of ‘values’* D , of cardinality $|D| > 1$; and an *interpretation function* I , mapping each functional symbol F of arity n into a function $I(F) = F_{\mathbf{D}} : D^n \rightarrow D$ (so that the constants $\perp$ and $\top$ are interpreted as values $\perp_{\mathbf{D}}, \top_{\mathbf{D}} \in D$), and each relational symbol P of arity n into a set $I(P) = P_{\mathbf{D}} \subseteq D^n$ of n -tuples of values in D , subject to the following conditions: the equality symbol $=$ is interpreted as the *identity relation* $=_{\mathbf{D}}$ on D ; and the order symbol $\leq$ is interpreted as *some total order* $\leq_{\mathbf{D}} \subseteq D \times D$ on the set D having $\perp_{\mathbf{D}}$ as its *bottom element* and $\top_{\mathbf{D}}$ as its *top element* (i.e., $\perp_{\mathbf{D}} \leq_{\mathbf{D}} d \leq_{\mathbf{D}} \top_{\mathbf{D}}$ for all $d \in D$).

Minimization Given an ordered value domain $\mathbf{D} = (D, I)$, it is easy to see that *every finite non-empty subset* $D' \subseteq D$ has a *unique minimum* $\min D' \in D'$, s.t. $\min D' \leq_{\mathbf{D}} d'$ for all $d' \in D'$.

Notation: ‘Cutoff Minimization’ We can also consider, for any natural number $N \geq 1$, an ‘ N -cutoff’ version of minimization $\min_N$, defined on *all sets* $D' \subseteq D$ (including the infinite ones), by putting:

$$\begin{aligned} \min_N D' &:= \top_{\mathbf{D}} \text{ (‘trivial’)}, & \text{when } D' = \emptyset, \\ \min_N D' &:= \min D', & \text{when } 1 \leq |D'| \leq N, \text{ and} \\ \min_N D' &:= \perp_{\mathbf{D}} \text{ (‘undefined’)}, & \text{otherwise (for } |D'| > N). \end{aligned}$$

⁶All the sets $(\mathcal{A}, V, Prop, Pred, Funct, \mathcal{E})$ in a vocabulary are assumed to be mutually disjoint.

Notation: ‘Cutoff’ n^{th} element For all sets $D' \subseteq D$ and all $1 \leq n \leq N$, we can now introduce a notation $n_N(D')$ for “the n^{th} element of the $\leq N$ elements” of D , by putting:

$$1_N(D') := \min_N D', \quad (n+1)_N(D') := \min_{N-1} (D' - \{n_N(D')\}) \quad \text{for } 1 \leq n < N.$$

Epistemic State Models An *epistemic state model* over a value domain $\mathbf{D} = (D, I)$ is a tuple $\mathbf{M} = (S, \sim, \underline{\bullet}(\bullet))$, where: (i) S is a set of *states* (or ‘possible worlds’), typically denoted by $s, w, \dots$; (ii) $\sim: \mathcal{A} \rightarrow \mathcal{P}(S \times S)$ maps agents $a \in \mathcal{A}$ to *equivalence relations* $\sim_a \subseteq S \times S$, called ‘*indistinguishability*’ relations; (iii) $\underline{\bullet}(\bullet): S \times (V \cup \text{Prop}) \rightarrow D$ is an *assignment map* (‘valuation’), mapping pairs $(s, v) \in S \times V$ into arbitrary values $\underline{s}(v) \in D$, and mapping pairs $(s, p) \in S \times \text{Prop}$ into values $\underline{s}(p) \in \{\perp_{\mathbf{D}}, \top_{\mathbf{D}}\}$.

Group indistinguishability Given a state model $\mathbf{M} = (S, \sim, \underline{\bullet}(\bullet))$, we define *group indistinguishability relations* $\sim_A := \bigcap_{a \in A} \sim_a$ on S (for all groups $A \subseteq \mathcal{A}$) by taking intersections.

Syntax of LDAE For a fixed dynamic vocabulary $(\mathcal{V}, \mathcal{E})$, the (dynamic) *Logic of (group) Data Access & Exchange (LDAE)* has a syntax consisting of: (1) a set $\text{Var} := \text{Var}(\mathcal{V})$ of compound variables (or ‘terms’) x ; (2) a set $\text{Fml} := \text{Fml}(\mathcal{V})$ of formulas ϕ ; (3) a set of ‘syntactic’ event models; (4) a set *Events* of data-exchange events. These components are simultaneously defined by mutual recursion, as follows:

(1) & (2) **Variables and Formulas** The sets Var and Fml are given by the clauses:

$$\begin{array}{lcl} x & ::= & v \mid F(\bar{x}) \mid \phi \rightarrow x \mid x \mid \mu_N x_A^\phi \mid \mathbf{e}(x) \\ \phi & ::= & p \mid P\bar{x} \mid \phi \rightarrow \phi \mid K_A \phi \mid [\mathbf{e}] \phi \end{array}$$

where: $v \in V$ are basic variables; $p \in \text{Prop}$ are atoms; $\bar{x} = (x_1, \dots, x_n)$ are tuples of terms; $P \in \text{Pred}$ are n -ary predicate symbols; F are n -ary function symbols; $a \in \mathcal{A}$ are agents; $A \subseteq \mathcal{A}$ are groups; $N \geq 1$ are integers; and the *events* $\mathbf{e}$ are technically “pointed event models” $(\mathbf{E}, e)$, as defined below.

(3) & (4) **Event Models and Events** An *event model* $\mathbf{E} = (E, \sim, \underline{\bullet}(\bullet))$ consists of: (i) a finite set $E \subseteq \mathcal{E}$ of event names; (ii) an equivalence relation $\sim_a \subseteq E \times E$ (*agent a ’s indistinguishability over events*) for each $a \in \mathcal{A}$; (iii) a *change map* $\underline{e}(\bullet): \text{Prop} \cup V \cup \mathcal{A} \cup \{\varepsilon\} \rightarrow \text{Fml} \cup \text{Var} \cup \mathcal{P}(\mathcal{A})$ for each event $e \in E$, mapping ε to a formula $\underline{e}(\varepsilon) \in \text{Fml}$, atoms $p \in \text{Prop}$ to formulas $\underline{e}(p) \in \text{Fml}$, basic variables $v \in V$ to terms $\underline{e}(v) \in \text{Var}$, and agents $a \in \mathcal{A}$ to groups $\underline{e}(a) \subseteq \mathcal{A}$. We require these items to satisfy *two conditions*:

1. *Self-Access* (agents access their own database): $a \in \underline{e}(a)$;
2. *Known Access* (agents know their sources): $e \sim_a f$ implies $\underline{e}(a) = \underline{f}(a)$;

As mentioned, a *data-exchange event* is just a ‘pointed’ event model $\mathbf{e} = (\mathbf{E}, e)$, i.e. a pair of an event model $\mathbf{E} = (E, \sim, \underline{\bullet}(\bullet))$ and an event name $e \in E$. We denote by *Events* the set of data-exchange events.

The Static Logic LDA The ‘static’ fragment of our logic, called the (static) *Logic of (group) Data Access (LDA)*, consists of all formulas of LDAE that are built without the use of dynamic operators $[\mathbf{e}] \phi$ or $\mathbf{e}(x)$.

Precondition and Postconditions The *precondition* of any event $\mathbf{e} = (\mathbf{E}, e)$ is the formula $\text{pre}_{\mathbf{e}} := \underline{e}(\varepsilon)$, which intuitively gives the *event’s condition of possibility*: $\mathbf{e}$ can only happen in states satisfying its precondition $\text{pre}_{\mathbf{e}}$. The *event $\mathbf{e}$ ’s postcondition for $p \in \text{Prop}$* is the formula $\text{post}_{\mathbf{e}}(p) := \underline{e}(p)$; similarly, the *event’s postcondition for $v \in V$* is the term $\text{post}_{\mathbf{e}}(v) := \underline{e}(v)$. Intuitively, the postconditions determine the way an event changes the values of atoms and variables: the new value of variable v (or atom p) after event $\mathbf{e}$ coincides with the ‘old’ value of the formula $\text{post}_{\mathbf{e}}(p)$ (or the term $\text{post}_{\mathbf{e}}(v)$) before the event.

(Extended) Access Map Event $\mathbf{e}$ ’s *access map* is the restriction of $\underline{e}$ to $\mathcal{A}$, specifying for each agent $a \in \mathcal{A}$ the group $\underline{e}(a) \subseteq \mathcal{A}$ of all sources/agents (whose locations/databases are) accessed by agent a during the event. We can also extend the access map to groups $A \subseteq \mathcal{A}$, by putting $\underline{e}(A) := \bigcup \{\underline{e}(a) : a \in A\}$, for the set of sources that are distributedly accessible to the group A during event e .

Subexpression-complexity An *expression* α is any formula φ , term x or event $\mathbf{e}$ of *LDAE*. The *sub-expression complexity order* $<$ is the least transitive relation s.t.: (1) every formula is $>$ its subformulas, and also $>$ all terms and events occurring in it; (2) every term is $>$ its subterms, and also $>$ all formulas and events in it; (3) every event $\mathbf{e} = (\mathbf{E}, e)$ is $>$ all preconditions and postconditions of the form pref , $\underline{f}(p)$ and $\underline{f}(v)$ with $f \in E$, $p \in \text{Prop}$ and $v \in V$. It is easy to see that $<$ is a *well-founded partial order*.

Semantics We simultaneously define three semantic notions on state models $\mathbf{M} = (S, \sim, \bullet(\bullet))$:

1. a *satisfaction relation* $s \models_{\mathbf{M}} \varphi$ between states $s \in S$ (in any state model $\mathbf{M}$) and formulas $\varphi \in Fml$;
2. an *extended assignment/valuation function* from $S \times (Var \cup Fml)$ to D , mapping *(state, variable)*-pairs $(s, x) \in S \times Var$ into arbitrary values $s(x)_{\mathbf{M}} \in D$, and mapping *(state, formula)*-pairs $(s, \varphi) \in S \times Fml$ into extreme values $s(\varphi)_{\mathbf{M}} \in \{\perp_{\mathbf{D}}, \top_{\mathbf{D}}\}$.
3. a *product update operation*, mapping state models $\mathbf{M} = (S, \sim, \bullet(\bullet))$ and event models $\mathbf{E} = (E, \sim, \bullet(\bullet))$ into *updated state models* $\mathbf{M} \otimes \mathbf{E} = (S \otimes E, \sim, \bullet(\bullet))$ (over the same value domain $\mathbf{D}$).

For this definition, we need an auxiliary notation: for variables $x \in Var$, groups A , formulas φ and states $s \in S$, the *set of possible values of x at state s according to group A conditional on φ* is

$$(x_A^\varphi)_{s, \mathbf{M}} := \{w(x)_{\mathbf{M}} : w \sim_A s, w \models_{\mathbf{M}} \varphi\}.$$

With this notation, we define our three semantic notions by mutual recursion:

$s \models_{\mathbf{M}} p$	iff	$\underline{s}(p) = \top_{\mathbf{D}}$
$s \models_{\mathbf{M}} Px_1 \dots x_n$	iff	$(s(x_1)_{\mathbf{M}}, \dots, s(x_n)_{\mathbf{M}}) \in I(P)$
$s \models_{\mathbf{M}} \varphi \rightarrow \psi$	iff	$s \models_{\mathbf{M}} \varphi$ implies $s \models_{\mathbf{M}} \psi$
$s \models_{\mathbf{M}} K_A \varphi$	iff	$w \models_{\mathbf{M}} \varphi$ for all $w \sim_A s$
$s \models_{\mathbf{M}} [\mathbf{e}] \varphi$	iff	$(s, e) \in S \otimes E$ implies $(s, e) \models_{\mathbf{M} \otimes \mathbf{E}} \varphi$, where $\mathbf{e} = (\mathbf{E}, e)$.
$s(v)_{\mathbf{M}}$	=	$\underline{s}(v)$ as given in the model $\mathbf{M}$
$s(\varphi \rightarrow x y)_{\mathbf{M}}$	=	$s(x)_{\mathbf{M}}$ if $s \models_{\mathbf{M}} \varphi$, and
$s(\varphi \rightarrow x y)_{\mathbf{M}}$	=	$s(y)_{\mathbf{M}}$ if $s \not\models_{\mathbf{M}} \varphi$
$s(F(x_1, \dots, x_n))_{\mathbf{M}}$	=	$(I(F))(s(x_1)_{\mathbf{M}}, \dots, s(x_n)_{\mathbf{M}})$
$s(\mu_N x_A^\varphi)_{\mathbf{M}}$	=	$\min_N \text{Val}(x_A^\varphi)_{s, \mathbf{M}}$
$s(\mathbf{e}(x))_{\mathbf{M}}$	=	$(s, e)(x)_{\mathbf{M} \otimes \mathbf{E}}$ if $\mathbf{e} = (\mathbf{E}, e)$ is s.t. $(s, e) \in S \otimes E$, and
$s(\mathbf{e}(x))_{\mathbf{M}}$	=	$\top_{\mathbf{D}}$ otherwise.
$s(\varphi)_{\mathbf{M}}$	=	$\top_{\mathbf{D}}$ if $s \models_{\mathbf{M}} \varphi$, and
$s(\varphi)_{\mathbf{M}}$	=	$\perp_{\mathbf{D}}$ if $s \not\models_{\mathbf{M}} \varphi$.
$\mathbf{M} = (S, \sim, \bullet(\bullet)), \mathbf{E} = (E, \sim, \bullet(\bullet))$	$\mapsto$	$\mathbf{M} \otimes \mathbf{E} = (S \otimes E, \sim, \bullet(\bullet))$, where:
$S \otimes E$	=	$\{(s, e) \in S \times E \mid s \models_{\mathbf{M}} \text{pre}_{\mathbf{e}}\}$
$(s, e) \sim_a (s', e')$	iff	$s \sim_{\underline{e}(a)} s'$ and $e \sim_a e'$
$\underline{(s, e)}(p)$	=	$s(\text{post}_{\mathbf{e}}(p))_{\mathbf{M}}$
$\underline{(s, e)}(v)$	=	$s(\text{post}_{\mathbf{e}}(v))_{\mathbf{M}}$

Extended assignment on sets of expressions We can ‘lift’ the assignment map to the level of *sets* of variables $X \subseteq Var$ and *sets* of formulas $\Phi \subseteq Fml$, by putting: $s(X)_{\mathbf{M}} := \{s(x)_{\mathbf{M}} : x \in X\}$; $s(\Phi)_{\mathbf{M}} := \{s(\varphi)_{\mathbf{M}} : \varphi \in \Phi\}$. Whenever the model is understood, we skip the subscript $\mathbf{M}$, writing simply $s \models \varphi$, $s(x)$, $s(X)$ and $s(\Phi)$. For sets of formulas Φ , we also write $s \models \Phi$ whenever $s \models \varphi$ for all $\varphi \in \Phi$.

Abbreviations. We define the usual Boolean connectives $\text{true} := (\top = \top)$, $\text{false} := (\top = \perp)$, $\neg \varphi := (\varphi \rightarrow \text{false})$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \leftrightarrow \psi$, as well as the dual existential (Diamond) modalities $\langle K_A \rangle \varphi :=$

$\neg K_A \neg \varphi$, $\langle K_A^\theta \rangle \varphi := \neg K_A^\theta \neg \varphi$, $\langle e \rangle \varphi := \neg[e] \neg \varphi$. We also use the following abbreviations, for variables $x, y \in Var$, finite sets of variables $X, Y \subseteq Var$ and natural numbers n, N with $1 \leq n \leq N$:

$$\begin{aligned} x \in Y &:= \bigvee \{x = y : y \in Y\}, \quad ?_\varphi := \varphi \rightarrow \top \mid \perp, \quad \varphi \rightarrow x := \varphi \rightarrow x \mid \top, \\ K_A^\theta \varphi &:= K_A(\theta \rightarrow \varphi), \quad K_A^\theta x := K_A^\theta(x = \mu_1 x_A^\theta), \quad K_A x := K_A^{true} x, \\ 1_N.x_A^\theta &:= \mu_N x_A^\theta, \quad (n+1)_N.x_A^\theta := \mu_{N-1} x_A^{\theta \wedge x > n_N.x_A^\theta}, \\ |x|_A^\theta \leq 0 &:= K_A \neg \theta, \quad |x|_A^\theta \leq n := K_A^\theta(x \in Var^n(x_A^\theta)), \quad \text{where } Var^n(x_A^\theta) := \{i_n.x_A^\theta : 1 \leq i \leq n\}, \\ |x|_A^\theta = 0 &:= |x|_A^\theta \leq 0, \quad |x|_A^\theta > n := |x|_A^\theta \not\leq n, \\ |x|_A^\theta = (n+1) &:= |x|_A^\theta > n \wedge |x|_A^\theta \leq (n+1) \text{ for } n \geq 0, \\ \min\{x\} &:= x, \quad \min(X \cup \{y\}) := (\min X \leq y) \rightarrow (\min X) \mid y. \end{aligned}$$

Intuitively, $x \in Y$ says that x currently takes the same value as some term in Y . The term $?_\varphi$ is a Boolean variable, taking value $\top_D$ if φ is true, and value $\perp_D$ otherwise. The operator $\varphi \rightarrow x$ is the term analogue of material implication: it takes the value of x if φ is true, and value $\top_D$ otherwise. Next, ‘*knowledge of value*’ is definable in our logic, in both conditional and unconditional forms, for groups and individuals, via the abbreviations $K_A^\theta x$, $K_A x$, $K_A^\theta \bar{x}$, $K_A \bar{x}$. Note that, for the empty tuple $\lambda = ()$, we have $K_A^\theta \lambda = \bigwedge \emptyset = \top$. The term $n.x_A^\theta$ denotes the n^{th} value of x (in $\leq_D$ -order) considered possible by A , conditional on θ . The expressions $|x|_A^\theta = n$, $|x|_A^\theta \leq n$ and $|x|_A^\theta > n$ refer to the cardinality of the set of possible values of x , according to group A , given θ . Finally, $\min X$ denotes the minimum value of all terms in X .

Distributed Location For any expression, i.e., any term, formula or event $\alpha \in Var \cup Fml \cup Events$, its (distributed) location $\mathcal{L}(\alpha) \subseteq \mathcal{A} \cup \{\varepsilon\}$ is given by the following recursive clauses:

$$\begin{aligned} \mathcal{L}(p) &= \mathcal{L}(v) = \{\varepsilon\}, & \mathcal{L}(K_A \varphi) &= \mathcal{L}(\mu_N x_A^\varphi) = A, \\ \mathcal{L}(Px_1 \dots x_n) &= \mathcal{L}(F(x_1, \dots, x_n)) = \bigcup \{\mathcal{L}(x_i) : 1 \leq i \leq n\}, \\ \mathcal{L}(\varphi \rightarrow \psi) &= \mathcal{L}(\varphi) \cup \mathcal{L}(\psi), & \mathcal{L}(\varphi \rightarrow x \mid y) &= \mathcal{L}(x) \cup \mathcal{L}(\varphi) \cup \mathcal{L}(y), \\ \mathcal{L}(e) &= \mathcal{L}(pre_e), & \mathcal{L}([e]\varphi) &= \mathcal{L}(e) \cup \underline{e}(\mathcal{L}(\varphi)), & \mathcal{L}(e(x)) &= \mathcal{L}(e) \cup \underline{e}(\mathcal{L}(x)), \end{aligned}$$

where we used the extended access map $\underline{e}(A)$. In particular, this gives us that $\mathcal{L}(\perp) = \mathcal{L}(\top) = \mathcal{L}(true) = \mathcal{L}(false) = \emptyset$, $\mathcal{L}(\neg \varphi) = \mathcal{L}(\varphi)$, $\mathcal{L}(\varphi \wedge \psi) = \mathcal{L}(\varphi) \cup \mathcal{L}(\psi)$ and $\mathcal{L}(K_A^\theta x) = A$.

We can also extend the location map to finite sets of terms $X \subseteq Var$, by putting:

$$\mathcal{L}(X) := \bigcup_{x \in X} \mathcal{L}(x).$$

Locality The values of terms or propositions having a distributed location within a group $A \subseteq \mathcal{A}$ are distributed knowledge among group A ’s members, as shown by the results below.

Proposition 3.1. (Preservation) *Let s, w be states in a model $\mathbf{M}$ s.t. $s \sim_A w$ for some group $A \subseteq \mathcal{A}$, let φ be a static formula s.t. $\mathcal{L}(\varphi) \subseteq A$, and let $x \in Var$ be a static term s.t. $\mathcal{L}(x) \subseteq A$. Then we have:*

1. $s \models_{\mathbf{M}} \varphi$ iff $w \models_{\mathbf{M}} \varphi$;
2. $s(x)_{\mathbf{M}} = w(x)_{\mathbf{M}}$.

Corollary 3.2. (Local Knowledge) *For formulas φ and terms x , we have the following validities:*

$$\models \varphi \rightarrow K_A \varphi, \text{ whenever } \mathcal{L}(\varphi) \subseteq A; \qquad \models K_A x, \text{ whenever } \mathcal{L}(x) \subseteq A.$$

3.1 Examples of Data-Exchange Events and Event Models

In ‘semi-public’ event models (with only one event), it is common knowledge who can read whose data.

Public Announcements For every formula φ , the event $!\varphi = (\mathbf{E}_{!\varphi}, !)$ of publicly announcing φ has an event model $\mathbf{E}_{!\varphi} = (\{!\}, \sim, \bullet(\bullet))$ whose only event name is the special symbol $!$, with identity $\sim_a = \{(!, !)\}$ as accessibility relation for every agent, and where we put $!(p) = p$, $!(v) = v$, $!(a) = \{a\}$ and $!(\varepsilon) = \varphi$, hence $pre_{!\varphi} = \varphi$. This event encodes the dynamics of truthful public announcements [22]: the updated model $\mathbf{M} \otimes \mathbf{E}_{!\varphi}$ is (isomorphic to the one) obtained by deleting all the φ -worlds from the original model $\mathbf{M}$ (and keeping everything else the same).

Public and Semi-Public Sharing (“Tell Us All You Know”) For groups $A, B \subseteq \mathcal{A}$, $!(A : B) = (\mathbf{E}_{!(A:B)}, !)$ is a semi-public event whose model $\mathbf{E}_{!(A:B)} = (\{!\}, \sim, \bullet(\bullet))$ has the same structure as $\mathbf{E}_{!\varphi}$, except for the access map and the precondition, which are given by putting: $!(a) = B \cup \{a\}$ for all $a \in A$; $!(a) = \{a\}$ for $a \notin A$; and $!(\varepsilon) = true$ (so the precondition $pre_{!(A:B)} = true$ is tautological). In this action, it is common knowledge that all agents in A gain access to the databases of all agents in B . Its effect is to replace all the relations $\sim_a$ (with $a \in A$) from the original model $\mathbf{M}$ by the new relations $\sim_a^{!(A:B)} := \sim_a \cap \sim_B$ in the updated model $\mathbf{M} \otimes \mathbf{E}_{!(A:B)}$, while keeping everything else unchanged (including the relations $\sim_a$ with $a \notin A$). A special case is semi-public sharing $!(a : b)$ from b to a , obtained by taking $A = \{a\}$ and $B = \{b\}$: it is common that b shares all his knowledge with a . Another special case is A -public sharing $!B := !(\mathcal{A} : B)$, which is obtained by taking $A = \mathcal{A}$: all agents in B publicly share all their information. An even more special case is b -public sharing $!b := !\{b\} = !(\mathcal{A} : \{b\})$, obtained by taking $A = \{a\}$ for some designated agent a (and $B = \mathcal{A}$): agent a publicly “tells all she knows”.⁷

Sharing between Multiple Groups For groups $A_1, B_1, \dots, A_n, B_n$, the event $!(A_1 : B_1, \dots, A_n : B_n)$ is a semi-public event, whose model $\mathbf{E}_{!(A_1:B_1, \dots, A_n:B_n)} = (\{!\}, \sim, \bullet(\bullet))$ has the same structure as $\mathbf{E}_{!(A:B)}$, except that the access map is given by $!(a) = \{a\} \cup \bigcup \{B_i : i \leq n, a \in A_i\}$ for all a : it is common knowledge that all agents in each group A_i gain access to the databases of all agents in the corresponding group B_i .

Sharing within Groups For groups $A_1, \dots, A_n \subseteq \mathcal{A}$, the event of parallel sharing within different groups $!(A_1, \dots, A_n) = !(A_1 : A_1, \dots, A_n : A_n)$ is the special case of $!(A_1 : B_1, \dots, A_n : B_n)$ where $B_i = A_i$. In this action, it is common knowledge that all agents in each group A_i simultaneously share all their knowledge with all other agents in the same group A_i . A very special case is $n = 1$, which represents the resolution action $!(A)$: it is common knowledge that all the agents in A share their information with each other.⁸

Public Hacking $!H_{a:b}$ (as in the *WikiLeaks* case). It is common knowledge that agent a ‘hacks’ agent b ’s database, using her knowledge of b ’s password (represented by some variable v_b), and makes all b ’s data public. Everybody gets to see all b ’s data (including the value of his password v_b), but only a and b knew the password beforehand. The model is the same as for public sharing $!b$, except for the precondition, which is $pre_{!} := K_a v_b \wedge K_b v_b$: this event happens only if a knew b ’s password v_b before the event.

Semi-public Hacking This time, it is common knowledge that agent a hacks agent b ’s database (using her prior knowledge of his password v_b) and can read all his data; but the others cannot read these data. The event model is similar to the one for semi-public sharing $!(a : b)$ (and in particular, the access map is the same), except that the precondition is $K_a v_b \wedge K_b v_b$ (as in the case of public hacking $!H_{a:b}$).

Semi-public Change of Password $!(v_a := F(v_a, v'_a))$. It is common knowledge that a changes her password v_a to a value $F(v_a, v'_a)$, that is a function of her current password v_a and of another (secret) local

⁷This corresponds to a dynamics introduced and axiomatized in this form in [2], though technically isomorphic to the action of public resolution of an issue/question [24].

⁸The special case $!(\mathcal{A})$ was introduced in [2], and axiomatized in different contexts in [15, 19, 4]. The general case $!(\mathbf{G})$ was studied (as “group resolution”) in [1].

variable v'_a . The encryption function F is common knowledge, but the values of the former password v_a and of the other secret number v'_a are known (hopefully) only by a . The event model is similar for $!\phi$, except that the precondition is $pre_! = K_a v_a \wedge K_a v'_a$, and the postcondition for v_a is $\underline{!}(v_a) = F(v_a, v'_a)$.

Using larger event models, we can represent various forms of *private and semi-private data-exchanges*.

Secret Hacking Agent a may be hacking b 's database iff she got hold of b 's password v_b . Only the hacker (a) knows whether or not she succeeded to get v_b . We represent this event $SH_{a:b} = (\mathbf{E}, !)$ using a model $\mathbf{E} = (\{!, \tau\}, \sim, \underline{\bullet}(\bullet))$ with two events $!$ (for *successful hacking*) and τ (*unsuccessful hacking*). The preconditions are $pre_! := K_a v_b \wedge K_b v_b$ and $pre_\tau := \neg K_a v_b \wedge K_b v_b$. The access maps are $\underline{!}(a) := \{a, b\}$, $\underline{\tau}(a) = \{a\}$, and $\underline{!}(c) = \underline{\tau}(c) = \{c\}$ for all $c \neq a$; all postconditions are given by identity; a 's accessibility is the identity relation, and the others' relations are the universal relation.

Conditional Change of Password $!(K_a |v_a|_b^{true} \leq N / v_a := F(v_a, v'_a))$. It is common knowledge that a changes her password v_a (as in the semi-public change) *only iff she knows that b has succeeded to narrow down her possible passwords to N possibilities*. This is an event $(\mathbf{E}, !)$, whose event model $\mathbf{E} = (\{!, ?\}, \sim, \underline{\bullet}(\bullet))$ has two event names $!$ (for *changing a 's password*) and $?$ (for *no change*). The preconditions are $pre_! := K_a |v_a|_b^{true} \leq N$ and $pre_? := \neg K_a |v_a|_b^{true} \leq N$. The postconditions for v_a are $\underline{!}(v_a) := F(v_a, v'_a)$ and $\underline{?}(v_a) := v_a$, while all other postconditions and access map are given by identity: nobody gains access to others' databases (since the hacking is prevented by password-change). Finally, agent a 's accessibility is the identity relation, while all others' accessibility is the universal relation.

Secret Detection of Hacking We can modify the secret hacking example $SH_{a:b}$ to allow the possibility that b might *detect* a 's hacking attack (so that he might come to know that he is being hacked). Only b knows whether he actually detects an attack. This event $(\mathbf{E}, !)$ has a model $\mathbf{E} = \{!, ?, \tau\}$ with *three* event names $!$ (detected hacking), $?$ (successful, undetected hacking) and τ (unsuccessful hacking). The access maps for $!$ and τ are as in the model for $SH_{a:b}$, while the access map for $?$ is the same as for $!$; and the same goes for $pre_!$, pre_τ and $pre_?$. Besides loops for all agents, we also have $! \sim_a ?$ (i.e. a doesn't know whether his hacking is detected or not) and $? \sim_b \tau$ (b can't distinguish between undetected hacking and unsuccessful hacking), while $\sim_c$ is the universal relation for all outsiders $c \neq a, b$.

4 Axiomatization and Decidability

We first look at the *static fragment LDA*: its proof system **LDA** is in Table 1.

Proposition 4.1. *The following theorems are provable in the system **LDA**:*

1. (Propositional Introspection) $K_A \phi \rightarrow K_A K_A \phi$; $\neg K_A \phi \rightarrow K_A \neg K_A \phi$;
2. (Term Introspection) $K_A x$, for $\mathcal{L}(x) \subseteq A$;
3. (Group Monotonicity) $K_A \phi \rightarrow K_B \phi$, for $A \subseteq B$.

Theorem 4.2. (Soundness, Completeness and Decidability of **LDA**) *The proof system **LDA** is complete for the static fragment LDA. Moreover, the static logic LDA is decidable.*

Proof. *Completeness* is shown in Section 5, using Prop 4.1. *Soundness* is trivial, except for the 'Reaching the Minimum' axiom, whose soundness we sketch here. Suppose that $s \models \phi \wedge K_A^\phi(x \in Y)$ for some $Y \subseteq \text{Var}$ s.t. $|Y| \leq N$ and $\mathcal{L}(Y) \subseteq A$. To show that $s \models \langle K_A^\phi \rangle (\mu_N x_A^\phi = x)$, we first prove the following:

Claim: $\text{Val}(x_A^\phi)_{s, \mathbf{M}} \subseteq s(Y)_{\mathbf{M}}$.

To show this, let $d \in \text{Val}(x_A^\phi)_{s, \mathbf{M}}$, i.e. there is some $w \sim_A s$ s.t. $w \models \phi$ and $w(x) = d$. Since $s \models K_A^\phi(x \in Y)$, we infer that $w \models (x \in Y)$, hence $w(x) \in w(Y)$. But, given that $w \sim_A s$ and $\mathcal{L}(Y) \subseteq A$, we have that $w(Y) = s(Y)$ (by Proposition 3.1). So we obtain $d = w(x) \in w(Y) = s(Y)$. Since this holds for any $d \in \text{Val}(x_A^\phi)_{s, \mathbf{M}}$, we conclude that $\text{Val}(x_A^\phi)_{s, \mathbf{M}} \subseteq s(Y)$, thus establishing our Claim.

(I)	Axioms and rules of classical propositional logic (CPL):
Modus Ponens rule	& all instances of CPL axioms in the language of <i>LDAE</i>
(II)	Axioms for equality and order:
(Indiscernability)	$\bar{x} = \bar{y} \rightarrow (P\bar{x}\bar{z} \leftrightarrow P\bar{y}\bar{z})$
(Functionality)	$\bar{x} = \bar{y} \rightarrow F(\bar{x}) = F(\bar{y})$
(Definition by Cases)	$\varphi \rightarrow (x = (\varphi \rightarrow x y)), \quad \neg\varphi \rightarrow (y = (\varphi \rightarrow x y))$
(Transitivity)	$(x \leq y \wedge y \leq z) \rightarrow x \leq z$
(Anti-symmetry)	$(x \leq y \wedge y \leq x) \rightarrow x = y$
(Totality)	$x \leq y \vee y \leq x$
(Top and Bottom)	$\perp \leq x \leq \top$
(Non-trivial Constants)	$\top \neq \perp$
(III)	Axioms and rules for (distributed) knowledge:
(Necessitation)	From φ , infer $K_A \varphi$
(Distribution)	$K_A(\varphi \rightarrow \psi) \rightarrow (K_A \varphi \rightarrow K_A \psi)$
(Veracity)	$K_A \varphi \rightarrow \varphi$
(Local Knowledge)	$\varphi \rightarrow K_A \varphi, \quad \text{for } \mathcal{L}(\varphi) \subseteq A$
(IV)	Axioms for (cutoff) minimum value:
(Lower Bound)	$\varphi \rightarrow \mu_N x_A^\varphi \leq x$
(Trivial Minimum)	$K_A \neg\varphi \rightarrow \mu_N x_A^\varphi = \top$
(Undefined Minimum)	$ x _A^\varphi > N \rightarrow \mu_N x_A^\varphi = \perp$
(Reaching the Minimum)	$(\varphi \wedge K_A^\varphi(x \in Y)) \rightarrow \langle K_A^\varphi \rangle (\mu_N x_A^\varphi = x), \quad \text{for } Y \leq N \text{ s.t. } \mathcal{L}(Y) \subseteq A$

Table 1: The proof system **LDA** for the static fragment, using abbreviations $K_A^\theta \varphi$, $\langle K_A^\theta \rangle \varphi$, $x \in Y$, $|x|_A^\varphi > N$.

Using the above Claim and the fact that $|Y| \leq N$, we have that $|Val(x_A^\varphi)_{s,M}| \leq N$. Since we also have that $s(x) \in Val(x_A^\varphi)_{s,M} \neq \emptyset$ (since $s \models \varphi$), we obtain that $s(\mu_N x_A^\varphi) = \min_N Val(x_A^\varphi)_{s,M} = \min Val(x_A^\varphi)_{s,M} \in Val(x_A^\varphi)_{s,M}$ (by the semantic clause for μ_N and the definition of $\min_N$), and hence $s(\mu_N x_A^\varphi) = w(x)$ for some $w \sim_A s$ with $w \models \varphi$. Applying again Proposition 3.1, we have $w(\mu_N x_A^\varphi) = s(\mu_N x_A^\varphi) = w(x)$ (since $\mathcal{L}(\mu_N x_A^\varphi) = A$). This, together with $w \sim_A s$ and $w \models \varphi$, yields $s \models \langle K_A^\varphi \rangle (\mu_N x_A^\varphi = x)$, as desired. $\square$

As for the full *dynamic logic LDAE*, we need a few more notations and results to state our axioms.

Possible Values after an Event Recall that $Val(x_A^\varphi)_{s,M} = \{w(x)_M : w \sim_A s, w \models_M \varphi\}$ is the set of possible values at state s according to A given φ . The following (easily checked) result gives us a characterization of the corresponding set of possible values of x (according to A given φ) *after an event e*:

Lemma 4.3. $Val(x_A^\varphi)_{(s,e),M \otimes E} = \bigcup_{f \sim_A e} Val(\mathbf{f}(x)_{\underline{f}(A)}^{(f)\varphi})_{s,M}$

Counting the Possible Values after an Event We want a formula expressing the fact that *there will be at most N possible values of x* (according to A given φ) *after the event e*. Moreover, we want to express this *at the current state* (before the event). Put now $Var_e^N(x_A^\varphi) := \{n_N \mathbf{f}(x)_{\underline{f}(A)}^{(f)\varphi} : f \sim_A e, n \leq N\}$.

Semantically, given Lemma 4.3, it should be clear that, *if $|Val(\mathbf{f}(x)_{\underline{f}(A)}^{(f)\varphi})_{s,M}| \leq N$ holds for all events $f \sim_A e$, then $Val(x_A^\varphi)_{(s,e),M \otimes E} \subseteq s(Var_e^N(x_A^\varphi))_M$* . However, *this inclusion might be strict*: it can happen that *not all of the values in $s(Var_e^N(x_A^\varphi))_M$ are possible values of x according to A (given φ , after the event e)*. The problem is that whenever $|Val(\mathbf{f}(x)_{\underline{f}(A)}^{(f)\varphi})_{s,M}| < N$ for some $f \sim_A e$, we get a possibly 'fake' x -value $N_N \mathbf{f}(x)_{\underline{f}(A)}^{(f)\varphi} = \top_D$. So, for any $z \in Var_e^N(x_A^\varphi)$, its condition of possibility is given by the formula:

$$\Diamond z := \left(z = \top \rightarrow \bigvee_{f \sim_A e} \langle K_{f(A)}^{(f)\varphi} \rangle \mathbf{f}(x) = \top \right).$$

Thus, for any subset $Z \subseteq \text{Var}_e^N(x_A^\varphi)$, the formula

$$|Z|^\Diamond \leq N := \bigvee_{Y \subseteq Z, |Y| \leq N} \bigwedge_{z \in Z} \left(\Diamond z \rightarrow \bigvee_{y \in Y} z = y \right)$$

says that the number of possible values of x in Z (according to A given φ) is at most N . Finally, by applying this to the whole set $Z = \text{Var}_e^N(x_A^\varphi)$ above, we obtain the desired formula:

Lemma 4.4. *Let s be a state in a state model $\mathbf{M}$, let $\mathbf{e}$ be an event in an event model $\mathbf{E}$ with $s \models_{\mathbf{M}} \text{pre}_{\mathbf{e}}$, and let x, A, φ be s.t. $|\text{Val}(f(x)_{\underline{f}(A)}^{(f)\varphi})_{s, \mathbf{M}}| \leq N$ holds for every $f \sim_A e$. Then we have the equivalences:*

$$s \models_{\mathbf{M}} |\text{Var}_e^N(x_A^\varphi)|^\Diamond \leq N \text{ iff } |\text{Val}(x_A^\varphi)_{(s, \mathbf{e}), \mathbf{M} \otimes \mathbf{E}}| \leq N \text{ iff } (s, e) \models_{\mathbf{M} \otimes \mathbf{E}} |x_A^\varphi| \leq N.$$

Using these notations, the proof system **LDAE** for our full dynamic logic is given in Table 2.

(I)	Static Axioms and rules of LDA
All LDA rules & all	LDA axiom schemas extended to formulas of the full language LDAE
(II)	Reduction axioms and rules for prop. formulas:
([e]-Necessitation)	From φ , infer $[e]\varphi$
(Change of Facts)	$[e]p \leftrightarrow (\text{pre}_{\mathbf{e}} \rightarrow \text{post}_{\mathbf{e}}(p))$
(Change of Properties)	$[e]P\bar{x} \leftrightarrow (\text{pre}_{\mathbf{e}} \rightarrow \text{Pe}(\bar{x}))$
(Distributivity)	$[e](\varphi \rightarrow \psi) \leftrightarrow ([e]\varphi \rightarrow [e]\psi)$
(Knowledge Update)	$[e]K_A \varphi \leftrightarrow (\text{pre}_{\mathbf{e}} \rightarrow \bigwedge \{K_{\underline{e}(A)}[\mathbf{f}]\varphi : f \sim_A e\})$
(III)	Reduction axioms for data terms:
(Basic Value Change)	$\mathbf{e}(v) = (\text{pre}_{\mathbf{e}} \rightarrow \text{post}_{\mathbf{e}}(v))$
(Functional Change)	$\mathbf{e}(F(\bar{x})) = (\text{pre}_{\mathbf{e}} \rightarrow F(\mathbf{e}(\bar{x})))$
(Change of Cases)	$\mathbf{e}(\varphi \rightarrow x y) = ([e]\varphi \rightarrow \mathbf{e}(x) \mathbf{e}(y))$
(Cutoff-Min. Change)	$\mathbf{e}(\mu_N x_A^\varphi) = \left(\text{pre}_{\mathbf{e}} \rightarrow \left(\text{Var}_e^N(x_A^\varphi) ^\Diamond \leq N \rightarrow \min \{ \mu_N \mathbf{f}(x)_{\underline{e}(A)}^{(f)\varphi} : f \sim_A e \} \perp \right) \right)$

Table 2: The proof system **LDAE**, where $\mathbf{e} = (\mathbf{E}, e), \mathbf{f} = (\mathbf{E}, f) \in \text{Events}$.

Proposition 4.5. *The following derived reduction laws are provable in the system **LDAE**:*

- (Impossible Change) $[e]\text{false} \leftrightarrow \neg \text{pre}_e$; and $[e]\text{true} \leftrightarrow \text{true}$;
- (Negation & Conjunction Reduction) $[e]\neg\varphi \leftrightarrow (\text{pre}_e \rightarrow \neg[e]\varphi)$; and $[e](\varphi \wedge \psi) \leftrightarrow ([e]\varphi \wedge [e]\psi)$;
- (Preservation of Constants) $\mathbf{e}(\top) = \top$; and $\mathbf{e}(\perp) = (\text{pre}_e \rightarrow \perp)$;
- (Minimum Reduction) $\mathbf{e}(\min X) = \min \{ \mathbf{e}(x) : x \in X \}$.

Theorem 4.6. (Soundness, Completeness, Expressivity and Decidability of **LDAE**) *The proof system **LDAE** in Table 2 is sound and complete for the dynamic logic **LDAE**. Moreover, **LDAE** is provably co-expressive with its static fragment **LDA**, and thus it is decidable.*

Proof. *Completeness* is shown in Section 6. *Soundness* is trivial, except for the ‘Cutoff-Min’ reduction axiom, which we sketch here. Let $\mathbf{M}$ be any state model, and s be any state. We consider two cases:

Case 1: $s \not\models \text{pre}_{\mathbf{e}}$. In this case, both terms of the equality claimed in the axiom take value $\top_{\mathbf{S}}$ at s in $\mathbf{M}$.

Case 2: $s \models \text{pre}_{\mathbf{e}}$. In this case, $s(\mathbf{e}(\mu_N x_A^\varphi))_{\mathbf{M}} = (s, e)(\mu_N x_A^\varphi)_{\mathbf{M} \otimes \mathbf{E}} = \min_N (\text{Val}(x_A^\varphi)_{(s, e), \mathbf{M} \otimes \mathbf{E}})$, and there are two subcases to consider:

Subcase (2A): there exists some $f^0 \in E$ s.t. $f^0 \sim_A e$ and $|Val(\mathbf{f}^0(x)_{\underline{f^0(A)}}^{\langle \mathbf{f}^0 \rangle \varphi})_{s, \mathbf{M}}| > N$. Then we have by definition that $s(\mu_N \mathbf{f}^0(x)_{\underline{e(A)}}^{\langle \mathbf{f}^0 \rangle \varphi}) = \perp_{\mathbf{D}}$ (given that $f_0 \sim_A e$ implies $\underline{f^0(A)} = \underline{e(A)}$), and thus $\min\{\mu_N \mathbf{f}(x)_{\underline{e(A)}}^{\langle \mathbf{f} \rangle \varphi} : f \sim_A e\}$ takes value $\perp_{\mathbf{D}}$ at s . So the right-hand side term of the equality in the 'Cutoff-Min' reduction axiom takes value $\perp_{\mathbf{D}}$ at s (regardless of whether we have $s \models_{\mathbf{M}} |Var_e^N(x_A^\varphi)|^\diamond \leq N$ or not). On the other hand, the left-hand side also evaluates to $\perp_{\mathbf{D}}$ at s (since by Lemma 4.3, $|Val(\mathbf{f}^0(x)_{\underline{f^0(A)}}^{\langle \mathbf{f}^0 \rangle \varphi})_{s, \mathbf{M}}| > N$ implies that $|Val(x_A^\varphi)_{(s, e), \mathbf{M} \otimes \mathbf{E}}| > N$, hence $(s, e)(\mu_N x_A^\varphi)_{\mathbf{M} \otimes \mathbf{E}} = \min_N (Val(x_A^\varphi)_{(s, e), \mathbf{M} \otimes \mathbf{E}}) = \perp_{\mathbf{D}}$), as desired.

Subcase (2B): we have $|Val(\mathbf{f}(x)_{\underline{f(A)}}^{\langle \mathbf{f} \rangle \varphi})_{s, \mathbf{M}}| \leq N$ for all $f \in E$ s.t. $f \sim_A e$. We are in the conditions of Lemma 4.4, and there are again *two subcases* to consider:

Subcase (2B1): $s \not\models_{\mathbf{M}} |Var_e^N(x_A^\varphi)|^\diamond \leq N$. In this case, we can use Lemma 4.4 to check that both sides of the equality in the 'Cutoff-Min' reduction axiom evaluate to $\perp_{\mathbf{D}}$ at s .

Subcase (2B2): $s \models_{\mathbf{M}} |Var_e^N(x_A^\varphi)|^\diamond \leq N$. In this case, we can use Lemma 4.3 to show that $s(\mathbf{e}(\mu_N x_A^\varphi))_{\mathbf{M}} = (s, e)(\mu_N x_A^\varphi)_{\mathbf{M} \otimes \mathbf{E}} = \min_N (Val(x_A^\varphi)_{(s, e), \mathbf{M} \otimes \mathbf{E}}) = \min_N \left(\bigcup_{f \sim_A e} Val(\mathbf{f}(x)_{\underline{f(x)}}^{\langle \mathbf{f} \rangle \varphi})_{s, \mathbf{M}} \right)$. So the left-hand side of the equality in the 'Cutoff-Min' axiom evaluates at state s (in $\mathbf{M}$) to $\min_{f \sim_A e} \min_N Val(\mathbf{f}(x)_{\underline{f(x)}}^{\langle \mathbf{f} \rangle \varphi})_{s, \mathbf{M}}$. On the other hand, we can use Lemma 4.4 to check that the right-hand side of the equality in the 'Cutoff-Min' reduction axiom evaluates to the same expression at s . $\square$

5 Completeness and Decidability Proofs for LDA

Throughout this section, we fix a formula $\varphi_0 \in Fml$. We prove Theorem 4.2 by the method of *quasi-models*. But, to obtain an appropriate analogue of Fischer-Ladner closure, we need two auxiliary notions:

Restricted Vocabulary For any finite set $\Sigma \subseteq Fml$, the Σ -restricted vocabulary $\mathcal{V}_\Sigma := (\mathcal{A}_\Sigma, V_\Sigma, Prop_\Sigma, Pred_\Sigma, Funct_\Sigma, ar_\Sigma, \varepsilon)$ is formed as follows: $\mathcal{A}_\Sigma$ is the set of agents occurring (inside terms or modalities in formulas) in Σ ; V_Σ is the set of basic variables occurring (as subterms of any term) in formulas of Σ ; $Prop_\Sigma := Prop \cap \Sigma$ is the set of atomic propositions in Σ ; $Pred_\Sigma$ is the set of predicates occurring in (formulas of) Σ ; $Funct_\Sigma$ is the set of function symbols occurring in (formulas of) Σ ; ar_Σ is the restriction of ar to $Pred_\Sigma \cup Funct_\Sigma$. Clearly, if Σ is finite, then all the sets in $\mathcal{V}_\Sigma$ are finite.

The Σ -Restricted Set of Terms For any finite set of formulas $\Sigma \subseteq Fml$, the Σ -restricted set of terms Var_Σ is the smallest set of terms satisfying the following closure conditions: Var_Σ contains $\perp$ and $\top$, as well as all terms occurring in any formula of Σ (hence $V_\Sigma \subseteq Var_\Sigma$); Var_Σ is closed under subterms; if $i_N.x_A^\varphi \in Var_\Sigma$ for some $i \leq N$, then $j_N.x_A^\varphi \in Var_\Sigma$ for all $j \leq N$. Note that Var_Σ is only a *finite subset* of the (typically infinite) set $Var(\mathcal{V}_\Sigma)$ of terms of the language $LDA(\mathcal{V}_\Sigma)$.

We now proceed to introduce the appropriate notions of Fisher-Ladner Closure, syntactic types, and special sets of types called quasi-models.

Fisher-Ladner Closure Given now our fixed formula φ_0 , the *closure* of φ_0 is the smallest set of formulas $\Sigma = \Sigma(\varphi_0)$ satisfying the following closure conditions: $\varphi_0 \in \Sigma$; $true, false \in \Sigma$; Σ is closed under subformulas and single negations $\sim \varphi$; if $P \in Pred_\Sigma$ has arity $ar(P) = n$ and $\bar{x} = (x_1, \dots, x_n)$ is an n -tuple with all $x_1, \dots, x_n \in Var_\Sigma$, then $P\bar{x} \in \Sigma$; if $(K_A \varphi) \in \Sigma$, θ is a subformula of $\sim \varphi$ and $B \subseteq \mathcal{A}_\Sigma$, then $(K_B \theta) \in \Sigma$ (hence also $\langle K_A \rangle \theta \in \Sigma$); if $(\varphi \rightarrow x|y) \in Var_\Sigma$, then $\varphi \in \Sigma$; if $x, y \in Var_\Sigma$, then $(x = y), (x \leq y) \in \Sigma$; if $(\mu_N x_A^\varphi), z \in Var_\Sigma$ and $Y \subseteq Var_\Sigma$, then $K_A^\varphi(z \in Y), \langle K_A^\varphi \rangle(z \in Y) \in \Sigma$.

Types Let Σ be the closure of φ_0 . A Σ -type is a subset of Σ with the following properties:

1. for every $\varphi \in \Sigma$: $(\sim \varphi) \in \Delta$ iff $\varphi \notin \Delta$;

2. for every $(\varphi \wedge \psi) \in \Sigma$: $(\varphi \wedge \psi) \in \Delta$ iff $\varphi \in \Delta$ and $\psi \in \Delta$;
3. for every $P\bar{x}\bar{z} \in \Sigma$: if $(\bar{x} = \bar{y}), P\bar{x}\bar{z} \in \Delta$ then $P\bar{y}\bar{z} \in \Delta$;
4. for $F(\bar{x}), F(\bar{y}) \in \text{Var}_\Sigma$: if $(\bar{x} = \bar{y}) \in \Delta$, then $(F(\bar{x}) = F(\bar{y})) \in \Delta$;
5. if $(x \leq y), (y \leq z) \in \Delta$, then $(x \leq z) \in \Delta$;
6. if $(x \leq y), (y \leq x) \in \Delta$, then $(x = y) \in \Delta$;
7. either for all $x, y \in \text{Var}_\Sigma$, we have either $(x \leq y) \in \Delta$ or $(y \leq x) \in \Delta$;
8. for all $x \in \text{Var}_\Sigma$, we have $(\perp \leq x), (x \leq \top) \in \Delta$;
9. $(\top \neq \perp) \in \Delta$;
10. for every $(\varphi \rightarrow x|y) \in \Sigma$: $\varphi \in \Delta$ implies $(x = (\varphi \rightarrow x|y)) \in \Delta$; and $\varphi \notin \Delta$ implies $(y = (\varphi \rightarrow x|y)) \in \Delta$;
11. if $(K_A \varphi) \in \Delta$ then $\varphi \in \Delta$;
12. for all $\mu_N x_A^\varphi \in \text{Var}_\Sigma$: $(\mu_N x_A^\varphi \leq x) \in \Delta$;
13. if $\varphi, K_A^\varphi(\mu_N x_A^\varphi \neq x) \in \Delta$ and $Y \subseteq \text{Var}_\Sigma$ is s.t. $\mathcal{L}(Y) \subseteq A$ and $|Y| \leq N$, then $\langle K_A^\varphi \rangle(x \notin Y) \in \Delta$;
14. for all $\mu_N x_A^\varphi \in \text{Var}_\Sigma$: if $(K_A \neg \varphi) \in \Delta$, then $(\mu_N x_A^\varphi = \top) \in \Delta$;
15. for all $\mu_N x_A^\varphi \in \text{Var}_\Sigma$: if $(|x|_A^\varphi > N) \in \Delta$, then $(\mu_N x_A^\varphi = \perp) \in \Delta$.

Observation Types are closed under modus ponens: if Δ is a type and $(\varphi \rightarrow \psi), \varphi \in \Delta$, then $\psi \in \Delta$.

Accessibility relations on types For types Δ, Δ' and group $A \subseteq \mathcal{A}_\Sigma$, we put:

$$\Delta \sim_A \Delta' \text{ iff } \varphi \in \Delta \Leftrightarrow \varphi \in \Delta' \text{ for all } \varphi \in \Sigma \text{ s.t. } \mathcal{L}(\varphi) \subseteq A.$$

Proposition 5.1. The relations $\sim_A$ are equivalence relations on types, satisfying Group Monotonicity: $\Delta \sim_A \Delta'$ and $B \subseteq A$ imply $\Delta \sim_B \Delta'$.

Proof. This follows directly from the definition of relations $\sim_A$ on types. $\square$

Proposition 5.2. If $\Delta \sim_A \Delta'$ and $(K_A \varphi) \in \Delta$, then $\varphi \in \Delta'$.

Proof. Since $\mathcal{L}(K_A \varphi) = A$ and $\Delta \sim_A \Delta'$, we use the definition of $\sim_A$ on types and the fact that $(K_A \varphi) \in \Delta$ to infer that $(K_A \varphi) \in \Delta'$. This together with condition 11 on types, gives us the desired conclusion. $\square$

Hat notation. For every type Δ , we put $\hat{\Delta} := \bigwedge \Delta$ for the conjunction of all formulas in Δ .

Proposition 5.3. Given types Δ and Λ , if $\hat{\Delta} \wedge \langle K_A \rangle \hat{\Lambda}$ is consistent, then $\Delta \sim_A \Lambda$.

Proof. Let Δ and Λ be types as above, and suppose towards a contradiction that we have $\Delta \not\sim_A \Lambda$. Then there must exist $\varphi \in \Sigma$ s.t. $\mathcal{L}(\varphi) \subseteq A$ and $\varphi \in \Delta$, but $(\sim \varphi) \in \Lambda$. From this together with the assumption that $\hat{\Delta} \wedge \langle K_A \rangle \hat{\Lambda}$ is consistent, we infer that $\varphi \wedge \langle K_A \rangle \neg \varphi$ is consistent, and thus $\varphi \wedge \neg K_A \varphi$ is consistent. But this contradicts the fact that $\vdash \varphi \rightarrow K_A \varphi$ is an **LDA**-theorem for formulas φ with $\mathcal{L}(\varphi) \subseteq A$. $\square$

Quasi-Models A quasi-model for φ_0 is a set S of types over Σ , with the following two properties: (*) $\varphi_0 \in \Delta_0$ for some type $\Delta_0 \in S$; (**) if $\langle K_A \rangle \varphi \in \Delta \in S$, then there is some $\Delta' \in S$ with $\Delta \sim_A \Delta'$ and $\varphi \in \Delta'$.

Proposition 5.4. If $\Delta \in S$ is a type in a quasi-model S , then we have the following:

- (1) if $K_A(\varphi \rightarrow \psi) \in \Delta$ and $(K_A \varphi) \in \Delta$, then $K_A \psi \in \Delta$;
- (2) for every $(K_A \varphi) \in \Sigma$ s.t. $\mathcal{L}(\varphi) \subseteq A$, if $\varphi \in \Delta$ then $(K_A \varphi) \in \Delta$;
- (3) if $(K_A \varphi) \in \Delta$ and $B \subseteq A$, then $(K_B \varphi) \in \Delta$.

The Σ -canonical quasi-model A Σ -theory is any maximally consistent subset of Σ . We denote by S_Σ the set of all Σ -theories. We will show that S_Σ is a (finite) “canonical” quasi-model for Σ .

Lemma 5.5. *Every Σ -theory is a Σ -type.*

Proof. This is easy to check, using the axioms of **LDA** and Proposition 4.1. $\square$

Lemma 5.6. *For every $\Delta \in S_\Sigma$, if $\langle K_A \rangle \varphi \in \Delta$ then there is some $\Delta' \in S_\Sigma$ with $\Delta \sim_A \Delta'$ and $\varphi \in \Delta'$.*

Proof. Put $\Delta_A := \{\theta : \theta \in \Delta \text{ s.t. } \mathcal{L}(\theta) \subseteq A\} \cup \{\sim \theta : \theta \in (\Sigma - \Delta) \text{ s.t. } \mathcal{L}(\theta) \subseteq A\}$.

Claim: $\Delta_A \cup \{\varphi\}$ is consistent wrt the system **LDA**.

Proof of Claim: Suppose not. Then we have $\vdash \widehat{\Delta_A} \rightarrow \sim \varphi$. Applying Necessitation and Distribution, we obtain $\vdash K_A \widehat{\Delta_A} \rightarrow K_A \sim \varphi$. On the other hand, by inspecting the structure of Δ_A , it is easy to see that $\mathcal{L}(\Delta_A) \subseteq A$, so by Strong Introspection (Proposition 4.1) we have $\vdash \widehat{\Delta_A} \rightarrow K_A \widehat{\Delta_A}$. Putting these together, we get $\vdash \widehat{\Delta_A} \rightarrow K_A \sim \varphi$. Since $\Delta_A \subseteq \Delta$ and Δ is closed under Σ -consequences, we have $(K_A \sim \varphi) \in \Delta$. But this contradicts the assumption that $\langle K_A \rangle \varphi \in \Delta$ (given the consistency of Δ).

Using our Claim and the standard Lindenbaum Lemma, we get that $\Delta_A \cup \{\varphi\}$ has a Σ -maximally consistent extension $\Delta' \in S'_\Sigma$. So we have $\varphi \in \Delta'$ and $\Delta_A \subseteq \Delta'$, which implies that $\Delta \sim_A \Delta'$. $\square$

Proposition 5.7. *If $\varphi_0 \in \Sigma$ is consistent, then there exists a quasi-model for φ_0 .*

Proof. Take the set S_Σ of all Σ -theories. By the Lindenbaum Lemma, there exists a maximally consistent subset $\Delta_0 \in S_\Sigma$, such that $\varphi_0 \in \Delta_0$. By Lemmas 5.5 and 5.6, S_Σ is a quasi-model for φ_0 . $\square$

Corollary 5.8. *If φ_0 is satisfiable then there exists a quasi-model for φ_0 .*

The hard part is to prove the *converse* of this:

Proposition 5.9. *If S is a quasi-model for φ_0 , then φ_0 is satisfiable.⁹*

The rest of this section is dedicated to the proof of Proposition 5.9.

Unravelling: the tree of histories Let us fix a quasi-model S , a formula φ_0 and a type $\Delta_0 \in S$ with $\varphi_0 \in \Delta_0$. We will construct a model for φ_0 , based on an unravelling of S around Δ_0 . A *history* is a finite sequence $h = (\Delta_0, A^1, \Delta_1, \dots, A^n, \Delta_n)$ of any length $n \geq 0$, where $\Delta_1, \dots, \Delta_n \in S$ are types and $A^1, \dots, A^n \subseteq \mathcal{A}_\Sigma$ are groups, such that we have $\Delta_{i-1} \sim_{A^i} \Delta_i$ for all $i = 1, n$. Let H be the set of all histories. We denote by $\text{last}(h) := \Delta_n$ the last state in history h , and by $\rightarrow_A$ the natural *forward one-step relation* on histories in H , given by putting: $h \rightarrow_A h'$ iff $h' = (h, A, \Delta')$ (with $\text{last}(h) \sim_A \Delta' = \text{last}(h')$). We denote by $\leftarrow_A$ the *backward one-step relation*, defined as the converse of the forward relation: $h \leftarrow_A h'$ iff $h' \rightarrow_A h$. The one-step relations structure H into a *tree rooted at Δ_0* (with the immediate successor relation given by $h \rightarrow h'$ iff $h \rightarrow_A h'$ for some group A). In particular, we have *the tree property*: every two nodes h, h' of the tree are connected by a unique non-redundant path $h = h_0 \leftarrow_{A^1} h_1 \leftarrow_{A^2} \dots \leftarrow_{A^i} h_i \rightarrow_{A^{i+1}} \dots \rightarrow_{A^n} h_n = h'$ (-in which neighboring nodes are immediate successors, in one order or another, and no nodes are repeated).

Epistemic relations on histories To make this tree into a model for our restricted vocabulary $\mathcal{V}_\Sigma$, we define our *single-agent indistinguishability relations* $\sim_a \subseteq H \times H$ on histories, by putting

$$\sim_a := \left(\bigcup_{A \ni a} \rightarrow_A \cup \bigcup_{A \ni a} \leftarrow_A \right)^*,$$

⁹Note that *this would immediately give us the completeness and decidability of **LDA*** (by Proposition 5.9, Proposition 5.7 and the soundness of the system **LDA**, as well as the finiteness of the closure $\Sigma = \Sigma(\varphi_0)$, and the decidability of checking that a subset of Σ is a quasi-model).

where $\leftarrow_A$ is the converse of $\rightarrow_A$, the unions range over groups $A \subseteq \mathcal{A}_\Sigma$ s.t. $a \in A$, and R^* is the reflexive-transitive closure of R . Since our goal is to build a standard model, the *group indistinguishability relations* $\sim_A \subseteq H \times H$ are taken to be simply the *intersections* $\sim_A := \bigcap_{a \in A} \sim_a$ of all the individual relations.

It is useful to give more concrete characterizations of the relations $\sim_A$ (and $\sim_a$) on histories:

Lemma 5.10. *The following are equivalent, for $A \subseteq \mathcal{A}_\Sigma$ and histories $h, h' \in H$:*

1. $h \sim_A h'$;
2. $A \subseteq A^i$, for all groups A^i that appear as transition labels on the non-redundant path from h to h' .

Proof. Use the definitions of $\sim_a$ and $\sim_A = \bigcap_{a \in A} \sim_a$ on H , and the uniqueness of non-redundant path. $\square$

Lemma 5.11. *If $h \sim_A h'$, then $\text{last}(h) \sim_A \text{last}(h')$.*

Proof. The proof is by *induction on the length N of the non-redundant path* from h to h' .

For the *base case* $h = h'$, the conclusion follows trivially (given that $\sim_A$ are equivalence relations).

Inductive case: Suppose the non-redundant path from h and h' has length $N + 1$, and let us look at the last transition on this path. Given Lemma 5.10, this transition can be either of the form $h_N \rightarrow_{A^N} h_{N+1} = h'$, or of the form $h_N \leftarrow_{A^N} h_{N+1} = h'$, with $A^N \supseteq A$. By definition of $\rightarrow_A$ on histories, we have either $h' = (h_N, A^N, \text{last}(h'))$ or $h_N = (h', A^N, \text{last}(h_N))$, with $\text{last}(h_N) \sim_{A^N} \text{last}(h')$ in both cases. By Monotonicity (Proposition 5.1) and the fact that $A \subseteq A^N$, we obtain $\text{last}(h_N) \sim_A \text{last}(h')$. On the other hand, we also have $\text{last}(h) \sim_A \text{last}(h_N)$ (-since the non-redundant path from h to h_N has length N , so by the induction hypothesis the pair (h, h_N) satisfies the conclusion of our Lemma, with h' replaced by h_N). Putting these two together (and using the transitivity of $\sim_A$), we conclude that $\text{last}(h) \sim_A \text{last}(h')$, as desired. $\square$

Lemma 5.12. *If $(K_A \varphi) \in \text{last}(h)$ and $h \sim_A h'$, then $\varphi \in \text{last}(h')$.*

Proof. By Lemma 5.11, $h \sim_A h'$ implies $\text{last}(h) \sim_A \text{last}(h')$. This, together with $K_A \varphi \in \text{last}(h)$, implies by Proposition 5.2 that $\varphi \in \text{last}(h')$, as desired. $\square$

Lemma 5.13. (Diamond Lemma) *If $(K_A \varphi) \in \text{last}(h)$, then there exists some $h' \sim_A h$ s.t. $\varphi \in \text{last}(h')$.*

Proof. By Lemma 5.6, there exists some type $\Delta' \sim_A \text{last}(h)$ s.t. $\varphi \in \Delta'$. Take $h' = (h, A, \Delta')$. By Lemma 5.10 we have $h' \sim_A h$, and obviously $\varphi \in \Delta' = \text{last}(h')$, as desired. $\square$

Corollary 5.14. *If $(K_A \varphi) \in \Sigma$ and $h \in H$, then: $(K_A \varphi) \in \text{last}(h)$ iff we have $\varphi \in \text{last}(h')$ for all $h' \sim_A h$.*

The Value Domain: a Quotient Construction The *set of values* D of our model will be a quotient of the Cartesian product $H \times \text{Var}_\Sigma$. We define an *equivalence relation* $\approx$ on pairs (*history, variable*) in $H \times \text{Var}_\Sigma$ (telling us *when two such pairs represent the same value*), as well as a *total preorder* $\lesssim$ on these pairs in $H \times \text{Var}_\Sigma$ (telling us *when the value of a pair is at most equal to another pair's value*). Then we take our canonical set of objects D to be *the quotient of $H \times \text{Var}_\Sigma$ with respect to $\approx$* , while the preorder $\lesssim$ induces our desired *total order* $\leq$ on the quotient D .

For this, we first introduce another *equivalence relation* $\sim$ on $H \times \text{Var}_\Sigma$ (representing *identity of objects at a given node*), and another partial preorder $\lesssim$ on $\sim$ on $H \times \text{Var}_\Sigma$ (representing the *order relation on values at a given node*). This is given by putting:

$$\begin{aligned} (h, x) \sim (h', x') & \text{ iff } h = h' \text{ and } (x = x') \in \text{last}(h), \\ (h, x) \lesssim (h', x') & \text{ iff } h = h' \text{ and } (x \leq x') \in \text{last}(h). \end{aligned}$$

Second, we define (*forward and backward*) *one-step relations* $\rightarrow_ =$ and $\rightarrow_ \leq$ on pairs in $H \times \text{Var}_\Sigma$:

$$(h, x) \rightarrow_ = (h', x') \text{ iff } \exists y \in \text{Var}_\Sigma \exists A \supseteq \mathcal{L}(y) \text{ s.t. } h \rightarrow_A h', (x = y) \in \text{last}(h) \text{ \& } (y = x') \in \text{last}(h');$$

$$\begin{aligned}
(h, x) \leftarrow_& (h', x') \text{ iff } \exists y \in \text{Var}_\Sigma \exists A \supseteq \mathcal{L}(y) \text{ s.t. } h \leftarrow_A h', (x = y) \in \text{last}(h) \text{ \& } (y = x') \in \text{last}(h'); \\
(h, x) \rightarrow_\leq (h', x') \text{ iff } \exists y \in \text{Var}_\Sigma \exists A \supseteq \mathcal{L}(y) \text{ s.t. } h \rightarrow_A h', (x \leq y) \in \text{last}(h) \text{ \& } (y \leq x') \in \text{last}(h'); \\
(h, x) \leftarrow_\leq (h', x') \text{ iff } \exists y \in \text{Var}_\Sigma \exists A \supseteq \mathcal{L}(y) \text{ s.t. } h \leftarrow_A h', (x \leq y) \in \text{last}(h) \text{ \& } (y \leq x') \in \text{last}(h').
\end{aligned}$$

Note that $\leftarrow_&$ is just the converse of $\rightarrow_\leq$, but $\leftarrow_\leq$ is *not* the converse of $\rightarrow_\leq$.

Value Identity and Order Finally, we define our main equivalence relation $\approx$ and our total preorder $\lesssim$ on pairs $(\text{history}, \text{variable})$ in $H \times \text{Var}_\Sigma$, by putting: $\approx := (\sim \cup \rightarrow_\leq \cup \leftarrow_\leq)^*$; and $\lesssim := (\lesssim \cup \rightarrow_\leq \cup \leftarrow_\leq)^*$, where R^* is the reflexive-transitive closure of a relation R . It is useful to have a more concrete characterization of $\approx$ and $\lesssim$, in terms of the non-redundant path from h to h' :

Lemma 5.15. (“Path Lemma”) *Let $(h, x), (h', x') \in H \times \text{Var}_\Sigma$, and let $h = h_0 \leftarrow h_1 \leftarrow \dots \leftarrow h_i \rightarrow \dots \rightarrow h_n = h'$ be the non-redundant path from h to h' . Then the following are equivalent:*

- $(h, x) \approx (h', x')$;
- *either $(h, x) \sim (h', x')$ (if $n = 0$), or else there exist $x_0, x_1, \dots, x_i, \dots, x_n \in \text{Var}_\Sigma$ s.t. we have: $(h, x) = (h_0, x_0) \leftarrow_& (h_1, x_1) \leftarrow_& (h_2, x_2) \leftarrow_& \dots \leftarrow_& (h_i, x_i) \rightarrow_\leq \dots \rightarrow_\leq (h_{n-1}, x_{n-1}) \rightarrow_\leq (h_n, x_n) = (h', x')$.*

Similarly, the following are equivalent:

- $(h, x) \lesssim (h', x')$;
- *either $(h, x) \lesssim (h', x')$ (if $n = 0$), or else there exist $x_0, x_1, \dots, x_i, \dots, x_n \in \text{Var}_\Sigma$ s.t. we have: $(h, x) = (h_0, x_0) \leftarrow_\leq (h_1, x_1) \leftarrow_\leq (h_2, x_2) \leftarrow_\leq \dots \leftarrow_\leq (h_i, x_i) \rightarrow_\leq \dots \rightarrow_\leq (h_{n-1}, x_{n-1}) \rightarrow_\leq (h_n, x_n) = (h', x')$.*

Corollary 5.16. *If $h, h' \in H$ and $x \in \text{Var}_\Sigma$ are s.t. $h \sim_A h'$ and $\mathcal{L}(x) \subseteq A$, then $(h, x) \approx (h', x)$.*

Proof. Let $h = h_0 \leftarrow_{A^1} h_1 \leftarrow_{A^2} \dots \leftarrow_{A^i} h_i \rightarrow_{A^{i+1}} \dots \rightarrow_{A^n} h_n = h'$ be the non-redundant path from h to h' . Since $h \sim_A h'$, we know that $A \subseteq A_k$ for all k (by Lemma 5.10). Using this together with $\mathcal{L}(x) \subseteq A$ (and the definition of the relation $(h, x) \rightarrow_\leq (h', x')$ on history-variable pairs), we obtain $(h, x) \leftarrow_& (h_1, x) \leftarrow_& \dots \leftarrow_& (h_i, x) \rightarrow_\leq \dots \rightarrow_\leq (h_n, x) = (h', x)$. By the Path Lemma, we have $(h, x) \approx (h', x)$. $\square$

Corollary 5.17. *Let $x, x' \in \text{Var}_\Sigma$ and $h, h' \in H$, and suppose that the non-redundant path from h to h' is of the form $h = h_0 \rightarrow_A h_1 \rightarrow_A \dots \rightarrow_A h_i \rightarrow_A \dots \rightarrow_A h_n = h'$. Then we have $(h, x) \approx (h', x')$ iff there exists $y \in \text{Var}_\Sigma$ with $\mathcal{L}(y) \subseteq A$, $(x = y) \in \text{last}(h)$ and $(y = x') \in \text{last}(h')$.*

From Quasi-Model to Model We are first defining our *first-order data model* $\mathbf{D} = (D, I)$ for the restricted vocabulary $\mathcal{V}_\Sigma$: as announced, the *set of ‘values’* D is the quotient

$$D := (H \times \text{Var}_\Sigma) / \approx = \{[h, x] : (h, x) \in H \times \text{Var}_\Sigma\},$$

where $[h, x]$ denotes the equivalence class of (h, x) modulo $\approx$, defined by

$$[h, x] := \{(h', x') \in H \times \text{Var}_\Sigma : (h, x) \approx (h', x')\} \text{ (for any given pair } (h, x) \in H \times \text{Var}_\Sigma).$$

The partial preorder $\lesssim$ on pairs $(h, x) \in H \times \text{Var}_\Sigma$ induces a *partial order on the equivalence classes* $[h, x] \in D$, which in its turn can be extended to some *total order on D* , that we will denote by $\leq_{\mathbf{D}}$.¹⁰

The *interpretation function* I will map n -ary functional symbols $F \in \text{Funct}_\Sigma$ into n -ary functions $I(F) : D^n \rightarrow D$ given by: $I(F)([h, x_1], \dots, [h, x_n]) := [h, F(x_1, \dots, x_n)]$ if $F(x_1, \dots, x_n) \in \text{Var}_\Sigma$; and $I(f)([h, x_1], \dots, [h, x_n]) := \perp_{\mathbf{D}}$, otherwise; it will also map n -ary predicate symbols $P \in \text{Pred}_\Sigma \setminus \{\leq\}$ into n -ary relations $I(P) \subseteq D^n$ given by: $I(P) := \{([h, x_1], \dots, [h, x_n]) \in D^n : h \in H, \bar{x} = (x_1, \dots, x_n) \in \text{Var}_\Sigma^n \text{ s.t. } P\bar{x} \in \text{last}(h)\}$; while the interpretation $I(\leq)$ will be just the total order $\leq_{\mathbf{D}}$ constructed above.

¹⁰Though not unique, such a total order $\leq_{\mathbf{D}}$ exists by the Order-Extension Principle, a well-known consequence of the Axiom of Choice.

The Model Finally, our epistemic state model $\mathbf{M} = (H, \sim, \underline{\bullet}, \underline{\bullet}(\bullet))$ is given by taking: as set of states, the set H of all histories; the indistinguishability relations $\sim_a \subseteq H \times H$ are as defined above on histories¹¹; the valuation/assignment map $\underline{\bullet}(\bullet) : H \times (V_\Sigma \cup Prop_\Sigma) \rightarrow D$ is given by putting: $\underline{h}(v) := [h, v]$ for $v \in V_\Sigma$; and $\underline{h}(p) := \top_{\mathbf{D}}$ iff $p \in last(h)$ (-else, $\underline{h}(p) := \perp_{\mathbf{D}}$) for $p \in Prop_\Sigma$.

Lemma 5.18. (“Interpretation Lemma”) *The interpretation I is well-defined, i.e. we have the following:*

1. $(h, \bar{x}) \approx (h', \bar{y})$ implies $(h, F(\bar{x})) \approx (h', F(\bar{y}))$;
2. $(h, \bar{x}) \approx (h', \bar{y})$ implies that: $(P\bar{x}\bar{z}) \in last(h)$ iff $(P\bar{y}\bar{z}) \in last(h')$;
3. $I(E)$ is really the identity relation on D .

Proof. Induction on the length of the non-redundant path from h to h' , using our conditions on types. $\square$

Lemma 5.19. (“Preservation of Values”) *If $h \sim_A h'$ and $x \in Var_\Sigma$ is s.t. $\mathcal{L}(x) \subseteq A$, then $[h, x] = [h', x]$.*

Proof. This follows immediately from Corollary 5.16 and the definition of $[h, x]$. $\square$

The next results use the following notation, for $h \in H$, $x \in Var_\Sigma$, $\varphi \in \Sigma$ and $A \subseteq \mathcal{A}$:

$$Val_h(x_A^\varphi) := \{[h', x] \in D \mid h' \sim_A h, \varphi \in last(h')\}$$

Lemma 5.20. (“Trivial-Minimum Lemma”) *If $\mu_N x_A^\varphi \in Var_\Sigma$ and $h \in H$ are s.t. $Val_h(x_A^\varphi) = \emptyset$, then $[h, \mu_N x_A^\varphi] = \top_{\mathbf{D}}$.*

Proof. First, we prove an auxiliary Claim: $(K_A \neg \varphi) \in last(h)$. To show this, suppose towards a contradiction that $(K_A \neg \varphi) \notin last(h)$. This implies that $\langle K_A \rangle \varphi \in last(h)$ (given the closure conditions on types and the fact that $\mu_N x_A^\varphi \in Var_\Sigma$ implies $\langle K_A \rangle \varphi \in \Sigma$). By the Diamond Lemma 5.13, there exists some $h' \sim_A h$ with $\varphi \in last(h')$, and thus $[h', x] \in Val_h(x_A^\varphi)$. But this contradicts the assumption that $Val_h(x_A^\varphi) = \emptyset$.

Using now the above Claim, and applying condition (14) on types, we obtain that $(\mu_N x_A^\varphi = \top) \in last(h)$, i.e. $[h, \mu_N x_A^\varphi] = \top_{\mathbf{D}}$, as desired. $\square$

Lemma 5.21. (“Trivial-Value Lemma”) *If $[h, \mu_N x_A^\varphi] = \top_{\mathbf{D}}$, then $Val_h(x_A^\varphi) \subseteq \{\top_{\mathbf{D}}\}$.*

Proof. From $[h, \mu_N x_A^\varphi] = \top_{\mathbf{D}}$, we obtain that $(h, \mu_N x_A^\varphi) \approx (h, \top)$. By the Path Lemma 5.15, we must have $(\mu_N x_A^\varphi = \top) \in last(h)$. Suppose now (towards a contradiction) that $Val_h(x_A^\varphi) \not\subseteq \{\top_{\mathbf{D}}\}$, i.e. there exists some $h' \sim_A h$ with $\varphi, (x \neq \top) \in last(h')$. By Lemma 5.11, $h \sim_A h'$ implies that $last(h) \sim_A last(h')$. From this, together with the fact that $(\mu_N x_A^\varphi = \top) \in last(h)$ and that $\mathcal{L}(\mu_N x_A^\varphi = \top) = A$, we derive that $(\mu_N x_A^\varphi = \top) \in last(h')$ (by the definition of $\sim_A$ on types). Using condition 12 on types, we get that $(\top \leq x) \in last(h')$, which together with conditions 8 and 6 on types, gives us that $(x = \top) \in last(h')$, contradicting the above assumption that $(x \neq \top) \in last(h')$. $\square$

Lemma 5.22. (“Undefined-Minimum Lemma”) *Let $\mu_N x_A^\varphi \in Var_\Sigma$ be s.t. $[h, \mu_N x_A^\varphi] = \perp_{\mathbf{D}}$. Then we have either $\perp_{\mathbf{D}} \in Val_h(x_A^\varphi)$ or else $|Val_h(x_A^\varphi)| > N$*

Proof. Assume towards a contradiction that $[h, \mu_N x_A^\varphi] = \perp_{\mathbf{D}}$, but $\perp_{\mathbf{D}} \notin Val_h(x_A^\varphi)$; i.e.: $(\mu_N x_A^\varphi = \perp) \in last(h)$, but $(x \neq \perp) \in last(h')$ for all $h' \sim_A h$ with $\varphi \in last(h')$. By Corollary 5.14, $K_A^\varphi(x \neq \perp) \in last(h)$. This, together with $K_A(\mu_N x_A^\varphi = \perp) \in last(h)$ (which follows from $(\mu_N x_A^\varphi = \perp) \in last(h)$, by Proposition 5.4(2), and with the closure conditions on Σ and $\mathcal{L}(\mu_N x_A^\varphi = \perp) = A$), yield $K_A^\varphi(\mu_N x_A^\varphi \neq x) \in last(h)$.

To show that $|Val_h(x_A^\varphi)| > N$, we will construct a sequence

$$h_0 \rightarrow_A h_1 \rightarrow_A \dots \rightarrow_A h_n \rightarrow_A \dots \rightarrow_A h_N, \text{ with } h_n \sim_A h \text{ and } \varphi \in last(h_n) \text{ for all } n,$$

¹¹Note that this is meant to be a (standard) model, so the group indistinguishability relations $\sim_A$ are just the intersections $\bigcap_{a \in A} \sim_a$, thus coinciding with the general relations $\sim_A$ introduced above.

together with a sequence of N ‘witnesses’ $y_1, \dots, y_n, \dots, y_N \in \text{Var}_\Sigma$, with $\mathcal{L}(y_n) \subseteq A$ for all n .

The construction is by recursion on $n \leq N$. For the base step ($n = 0$), note that $(\langle K_A \rangle \varphi) \in \text{last}(h)$ (since otherwise we’d have $(\mu_N x_A^\varphi = \top) \in \text{last}(h)$ by condition (14) on types, contradicting the fact that $(\mu_N x_A^\varphi = \perp) \in \text{last}(h)$, given also condition (9) on types). By condition (**) on quasi-models, there exists $\Delta_0 \in S$ s.t. $\text{last}(h) \sim_A \Delta_0$ and $\varphi \in \Delta_0$. Take now $h_0 := (h, A, \Delta_0)$, which fulfills the desired specifications.

For the step n of the induction (with $1 \leq n \leq N$, assuming given h_{n-1} and $y_1, \dots, y_{n-1}$ with the above properties), we first take the n^{th} witness y_n to be any term in Var_Σ with $\mathcal{L}(y_n) \subseteq A$ and $(x = y_n) \in \text{last}(h_{n-1})$, if such a term exists; and otherwise, we just put $y_n := \perp$. Next, we note that, by the induction hypothesis, we have $h_{n-1} \sim_A h$ and $\varphi \in \text{last}(h_{n-1})$, hence $\text{last}(h_{n-1}) \sim_A \text{last}(h)$. Together with the fact that $K_A^\varphi(\mu_N x_A^\varphi \neq x) \in \text{last}(h)$, this gives us that $\varphi, K_A^\varphi(\mu_N x_A^\varphi \neq x) \in \text{last}(h)$. By condition (13) on types (applied to $Y := \{y_1, \dots, y_n\}$, where note that $\mathcal{L}(Y) \subseteq A$ and $|Y| \leq N$), we obtain $\langle K_A^\varphi \rangle \bigwedge_{1 \leq i \leq n} (x \neq y_i) \in \text{last}(h_{n-1})$. Using again clause (**) on quasi-models, there exists some $\Delta_n \in S$ s.t. $\text{last}(h_{n-1}) \sim_A \Delta_n$ and $\varphi, \bigwedge_{1 \leq i \leq n} (x \neq y_i) \in \Delta_n$. Take now $h_n := (h_{n-1}, A, \Delta_n)$, which obviously fulfills the desired specifications.

Given this construction, it is clear that $[h_n, x] \in \text{Val}_h(x_A^\varphi)$ for all $0 \leq n \leq N$. We will prove that *all these $N + 1$ values are distinct* (which immediately gives us that $|\text{Val}_h(x_A^\varphi)| > N$, as desired):

Let $1 \leq n \leq N$. It is enough to show that $[h_n, x] \neq [h_m, x]$ for all $m < n$. Suppose not, i.e. assume towards a contradiction that we have $(h_n, x) \approx (h_m, x)$ for some $m < n$. Given that the non-redundant path from h_m to h_n has the shape $h_m \rightarrow_A h_{m+1} \rightarrow_A \dots \rightarrow_A h_n$, we can apply Corollary 5.17 to conclude that there exists some $y \in \text{Var}_\Sigma$ with $\mathcal{L}(y) \subseteq A$, $(y = x) \in \text{last}(h_m)$ and $(y = x) \in \text{last}(h_n)$. On the other hand, we also have by construction that $(x = y_{m+1}) \in \text{last}(h_m)$ and $(x \neq y_{m+1}) \in \text{last}(h_n)$ (since $m + 1 \leq n$). Putting all these together and using our conditions on equality, we obtain that $(y = y_{m+1}) \in \text{last}(h_m)$ and $(y \neq y_{m+1}) \in \text{last}(h_n)$. But this contradicts the fact that $\text{last}(h_m) \sim_A \text{last}(h_n)$ (since $h_m \sim_A h_n$ by construction), given that $\mathcal{L}(y = y_{m+1}) \subseteq A$ (and given the definition of the relation $\sim_A$ on types). $\square$

Lemma 5.23. (“Least-of- N Values Lemma”) *If $\mu_N x_A^\varphi \in \text{Var}_\Sigma$ is s.t. $[h, \mu_N x_A^\varphi] \neq \perp_{\mathbf{D}}, \top_{\mathbf{D}}$, then:*

1. $(|x|_A^\varphi \leq N) \in \text{last}(h)$;
2. $|\text{Val}_h(x_A^\varphi)| \leq N$;
3. $[h, \mu_N x_A^\varphi] = \min \text{Val}_h(x_A^\varphi)$; i.e., there exists some history $h_0 \in H$, satisfying: $h_0 \sim_A h$; $\varphi \in \text{last}(h_0)$; and $[h, \mu_N x_A^\varphi] = [h_0, x] \leq_{\mathbf{D}} [h', x]$ for all $[h', x] \in \text{Val}_h(x_A^\varphi)$.

Proof. For part 1, from $[h, \mu_N x_A^\varphi] \neq \perp_{\mathbf{D}}$ we obtain that $(\mu_N x_A^\varphi \neq \perp) \in \text{last}(h)$, so by condition (15) on types we have $(|x|_A^\varphi \leq N) \in \text{last}(h)$, as desired.

For part 2, we use part 1 and Lemma 5.12, to obtain that $(\bigvee_{1 \leq i \leq N} x = i_N \cdot x_A^\varphi) \in \text{last}(h')$ holds for all $[h', x] \in \text{Val}_h(x_A^\varphi)$; hence, for every $[h', x] \in \text{Val}_h(x_A^\varphi)$ there exists $i \in \{1, \dots, N\}$ s.t. $[h', x] = [h', i_N \cdot x_A^\varphi] = [h, i_N \cdot x_A^\varphi]$ (where the last equality follows by the Corollary 5.16 from the fact that $\mathcal{L}(i_N \cdot x_A^\varphi) = A$ together with $h' \sim_A h$). Thus, we have that $\text{Val}_h(x_A^\varphi) \subseteq \{[h, i_N \cdot x_A^\varphi] \mid 1 \leq i \leq N\}$, which implies that $|\text{Val}_h(x_A^\varphi)| \leq N$.

For part 3, we use the assumption that $[h, \mu_N x_A^\varphi] \neq \top_{\mathbf{D}}$ and Lemma 5.20 to obtain $\text{Val}_h(x_A^\varphi) \neq \emptyset$; i.e., there exists $h_1 \sim_A h$ s.t. $\varphi \in \text{last}(h_1)$. Putting this together with the fact that $(|x|_A^\varphi \leq N) \in \text{last}(h_1)$ (which follows from part 1 and $h_1 \sim_A h$, using $\mathcal{L}(|x|_A^\varphi \leq N) = A$ and Lemma 5.11), and applying condition (13) on types (with $Y := \text{Var}^N(x_A^\varphi) = \{i_N \cdot x_A^\varphi \mid 1 \leq i \leq N\}$, while recalling that by definition $|x|_A^\varphi \leq N := K_A^\varphi(x \in \text{Var}^N(x_A^\varphi))$), we conclude that $\langle K_A^\varphi \rangle (x = \mu_N x_A^\varphi) \in \text{last}(h_1)$. By the Diamond Lemma 5.13, there exists some $h_0 \sim_A h_1 \sim_A h$ with $\varphi, (x = \mu_N x_A^\varphi) \in \text{last}(h_0)$, hence $[h_0, x] = [h_0, \mu_N x_A^\varphi] = [h, \mu_N x_A^\varphi]$ (with the last equality due to $h_0 \sim_A h$ and $\mathcal{L}(\mu_N x_A^\varphi) = A$, by Lemma 5.19). Finally, to prove that $[h, \mu_N x_A^\varphi] = \min \text{Val}_h(x_A^\varphi)$, let $[h', x] \in \text{Val}_h(x_A^\varphi)$, i.e., $h' \sim_A h$ with $\varphi \in \text{last}(h')$, and we need to show that $[h, \mu_N x_A^\varphi] \leq_{\mathbf{D}} [h', x]$. By condition (12) on types, we have $(\mu_N x_A^\varphi \leq x) \in \text{last}(h')$, hence $[h', \mu_N x_A^\varphi] \leq_{\mathbf{D}} [h', x]$. Since $h \sim_A h'$ and $\mathcal{L}(\mu_N x_A^\varphi) = A$, Lemma 5.19 gives us that $[h, \mu_N x_A^\varphi] = [h', \mu_N x_A^\varphi] \leq_{\mathbf{D}} [h', x]$, as desired. $\square$

Lemma 5.24. (“Truth Lemma”) *For every $\varphi \in \Sigma$ and $x \in \text{Var}_\Sigma$, the following hold for all $h \in H$:*

1. $h \models_{\mathbf{M}} \varphi$ iff $\varphi \in \text{last}(h)$;
2. $\underline{h}(x)_{\mathbf{M}} = [h, x]$.

Proof. We prove 1.&2. for all expressions $\alpha \in \Sigma \cup \text{Var}_\Sigma$ by induction on sub-expression complexity.

(i) *Predicative Atoms:* $\varphi = P\bar{x}$, with $\bar{x} = (x_1, \dots, x_n)$. For $P \in \text{Pred} \setminus \{\leq\}$, we have the equivalences: $h \models P\bar{x}$ iff $h(\bar{x}) \in I(P)$ iff (by the induction hypothesis for claim (2)) $([h, x_1], \dots, [h, x_n]) \in I(P)$ iff $\exists h' \in H$ s.t. $((h, \bar{x}) \approx (h', \bar{x}) \& (P\bar{x}) \in \text{last}(h'))$ iff $(P\bar{x}) \in \text{last}(h)$ (by Lemma 5.18). For $\leq$, we have the equivalences: $h \models x \leq y$ iff $(h, x) \lesssim (h, y)$ iff $(h, x) \lesssim (h, y)$ iff $[h, x] \leq_{\mathbf{D}} [h, y]$ (by the Path Lemma).

(ii) *Propositional atoms:* $p \in \text{Prop}_\Sigma$. By definition, $h \models p$ iff $\underline{h}(p) = \top_{\mathbf{D}}$ iff $p \in \text{last}(h)$.

(iii) *Basic variables:* $v \in V_\Sigma$. By definition, $h(v)_{\mathbf{M}} = \underline{h}(v) = [h, v]$.

(iv) *Boolean Case:* $\varphi \rightarrow \psi$. These is trivial, using conditions 1 and 2 on types.

(v) *K_A -modal Case:* $(K_A\varphi) \in \Sigma$. From *left-to-right*: assume (towards a contradiction) that $h \models K_A\varphi$ but $(K_A\varphi) \notin \text{last}(h)$. By condition 1 on types, we have $(\langle K_A \rangle \sim \varphi) \in \text{last}(h)$, and so by the property (**) of quasi-models, there exists a type $\Delta' \in S$ s.t. $\text{last}(h) \sim_A \Delta'$ and $(\sim \varphi) \in \Delta'$. Take $h' := (h, A, \Delta')$: this is a well-defined history in H , satisfying $h \sim_A h'$ and $\text{last}(h') = \Delta'$. From $h \models K_A\varphi$ we infer that $h' \models \varphi$, and so by the induction hypothesis we have $\varphi \in \text{last}(h') = \Delta'$, in contradiction to $(\sim \varphi) \in \Delta'$.

For the *right-to-left* direction: we assume that $(K_A\varphi) \in \text{last}(h)$, and we have to prove that $h \models K_A\varphi$. For this, let $h' \in H$ be s.t. $h \sim_A h'$, and we need to show that $h' \models \varphi$. From $h \sim_A h'$ we obtain that $\text{last}(h) \sim_A \text{last}(h')$ (by Lemma 5.11), which together with $h \models K_A\varphi$ gives us $\varphi \in \text{last}(h')$ (by Proposition 5.2). Applying the induction hypothesis, we conclude that $h' \models \varphi$, as desired.

(vi) *Terms defined by cases $\varphi \rightarrow x|y$.* Given $(\varphi \rightarrow x|y) \in \Sigma$, we have $\varphi \in \Sigma$, so we have that either $\varphi \in \text{last}(h)$, or else $(\sim \varphi) \in \text{last}(h)$. In the first case, we have $(\varphi \rightarrow x|y = x) \in \text{last}(h)$ (by condition 10 on types), hence $[h, \varphi \rightarrow x|y] = [h, x]$; and on the other hand, $\varphi \in \text{last}(h)$ yields $h \models \varphi$ (by the induction hypothesis for Claim 1), so by definition we have $\underline{h}(\varphi \rightarrow x|y)_{\mathbf{M}} = \underline{h}(x)_{\mathbf{M}}$, and by the induction hypothesis for Claim 2 we have $\underline{h}(x)_{\mathbf{M}} = [h, x]$, thus obtaining $\underline{h}(\varphi \rightarrow x|y)_{\mathbf{M}} = \underline{h}(x)_{\mathbf{M}} = [h, x] = [h, \varphi \rightarrow x|y]$, as desired. The second case is similar: from $(\sim \varphi) \in \text{last}(h)$ we get $(\varphi \rightarrow x|y = y) \in \text{last}(h)$ (by condition 10 on types), hence $[h, x|y] = [h, y]$; and on the other hand, $(\sim \varphi) \in \text{last}(h)$ implies $\varphi \notin \text{last}(h)$ (by the consistency of types), which by the induction hypothesis for Claim 1 yields $h \not\models \varphi$, so by definition we have $\underline{h}(\varphi \rightarrow x|y)_{\mathbf{M}} = \underline{h}(y)_{\mathbf{M}}$, and by the induction hypothesis for Claim 2 we have $\underline{h}(y)_{\mathbf{M}} = [h, y]$, thus obtaining $\underline{h}(\varphi \rightarrow x|y)_{\mathbf{M}} = \underline{h}(y)_{\mathbf{M}} = [h, y] = [h, \varphi \rightarrow x|y]$, as desired.

(vii) *Functional terms $F(\bar{x}) \in \text{Var}_\Sigma$ for some $\bar{x} = (x_1, \dots, x_n)$.* We assume by the induction hypothesis that $\underline{h}(\bar{x})_{\mathbf{M}} = [h, \bar{x}]$, and using the semantics of functional terms and the definition of $I(F)$ in our history model, we obtain $\underline{h}(F(\bar{x}))_{\mathbf{M}} = (I(F))(h(\bar{x})) = (I(F))[h, \bar{x}] = [h, F(\bar{x})]$, as desired.

(viii) *Least-value terms $\mu_N x_A^\varphi \in \text{Var}_\Sigma$.* Using the notation $\text{Val}_h(x_A^\varphi) := \{[h', x] \in D : h' \sim_A h, \varphi \in \text{last}(h')\}$ and the induction hypothesis for Claims (1) and (2), we have $\text{Val}_h(x_A^\varphi) = \text{Val}(x_A^\varphi)_{h, \mathbf{M}}$, where $\text{Val}(x_A^\varphi)_{h, \mathbf{M}} := \{\underline{h}(x)_{\mathbf{M}} : h' \sim_A h, h' \models_{\mathbf{M}} \varphi\}$ is the notation from Section 3. We distinguish *three subcases*:

Case 1: $[h, \mu_N x_A^\varphi] = \top_{\mathbf{D}}$. By the Trivial-Value Lemma 5.21, we have $\text{Val}_h(x_A^\varphi) \subseteq \{\top_{\mathbf{D}}\}$, and hence $\min_N \text{Val}_h(x_A^\varphi) = \top_{\mathbf{D}}$ (since by definition we have $\min_N \emptyset = \min_N \{\top_{\mathbf{D}}\} = \top_{\mathbf{D}}$). Given this, we obtain that $\underline{h}(\mu_N x_A^\varphi)_{\mathbf{M}} = \min_N \text{Val}(x_A^\varphi)_{h, \mathbf{M}} = \min_N \text{Val}_h(x_A^\varphi) = \top_{\mathbf{D}} = [h, \mu_N x_A^\varphi]$, as desired.

Case 2: $[h, \mu_N x_A^\varphi] = \perp_{\mathbf{D}}$. By the Undefined-Minimum Lemma 5.22, we have either $\perp_{\mathbf{D}} \in \text{Val}_h(x_A^\varphi)$ or else $|\text{Val}_h(x_A^\varphi)| > N$. In both cases, we get $\min_N \text{Val}_h(x_A^\varphi) = \perp_{\mathbf{D}}$ by definition, so we obtain $\underline{h}(\mu_N x_A^\varphi)_{\mathbf{M}} = \min_N \text{Val}(x_A^\varphi)_{h, \mathbf{M}} = \min_N \text{Val}_h(x_A^\varphi) = \perp_{\mathbf{D}} = [h, \mu_N x_A^\varphi]$, as desired.

Case 3: $[h, \mu_N x_A^\varphi] \neq \top_{\mathbf{D}}, \perp_{\mathbf{D}}$. By Lemma 5.23, we have both $|\text{Val}_h(x_A^\varphi)| \leq N$ and $[h, \mu_N x_A^\varphi] = \min \text{Val}_h(x_A^\varphi)$, i.e., $[h, \mu_N x_A^\varphi] = \min_N \text{Val}_h(x_A^\varphi)$, thus $\underline{h}(\mu_N x_A^\varphi)_{\mathbf{M}} = \min_N \text{Val}(x_A^\varphi)_{h, \mathbf{M}} = \min_N \text{Val}_h(x_A^\varphi) = [h, \mu_N x_A^\varphi]$, as desired. $\square$

Proof of Proposition 5.9: If S is a quasi-model for φ_0 , then by applying claim 1 of the Truth Lemma 5.24 to φ_0 and to the history $h_0 := (\Delta_0)$, we conclude that $h_0 \models \varphi_0$ in our history-based model $\mathbf{M}$ above.

6 Completeness and Decidability Proofs for LDAE

In this section we prove Theorem 4.6. We first establish our co-expressivity result: *LDAE and LDA are provably co-expressive*. For this, we need a few preliminary notions and results.

Reducible expressions A formula θ in LDAE is said to be *reducible* if it is provably equivalent to a 'static' formula; i.e., if there exists some formula θ' in the static fragment LDA s.t. $\vdash \theta \leftrightarrow \theta'$ is provable in LDAE. A term $x \in \text{Var}_{\text{LDAE}}$ is *reducible* if it is provably equal to a static term; i.e., if there exists some $x' \in \text{Var}_{\text{LDA}}$ s.t. $\vdash x = x'$ is provable in LDAE. Finally, an event $\mathbf{e} = (\mathbf{E}, e)$ is *reducible* if all preconditions pre_f and all post-conditions of the form $\underline{e}(p)$ or $\underline{e}(v)$ are reducible (for all $f \in E$, $p \in \text{Prop}$ and $v \in V$).

Lemma 6.1. (Replacement of Equivalents) *Suppose that $\vdash \varphi \leftrightarrow \varphi'$, $\vdash \theta \leftrightarrow \theta'$, $\vdash x = x'$, $\vdash y = y'$ and $\vdash x_i = x'_i$ (for all $i = 1, n$) are provable in LDAE. Then the following are also provable in LDAE:*

1. $\vdash \varphi \rightarrow x y = \varphi \rightarrow x' y'$	5. $\vdash (\varphi \rightarrow \theta) \leftrightarrow (\varphi' \rightarrow \theta')$
2. $\vdash F(x_1, \dots, x_n) = F(x'_1, \dots, x'_n)$	6. $\vdash K_A \varphi \leftrightarrow K_A \varphi'$
3. $\vdash \mu_N x_A^\varphi = \mu_N (x')_A^{\varphi'}$	7. $\vdash [\mathbf{e}] \varphi \leftrightarrow [\mathbf{e}] \varphi'$
4. $\vdash Px_1 \dots x_n \leftrightarrow Px'_1 \dots x'_n$	8. $\vdash \mathbf{e}(x) = \mathbf{e}(x')$

We first prove a preliminary “one-step reduction” result:

Lemma 6.2. *Let $\mathbf{e}$ be any reducible event. Then, for every 'static' formula θ in LDA and every 'static' term $x \in \text{Var}_{\text{LDA}}$, the formula $[\mathbf{e}]\theta$ and the term $\mathbf{e}(x)$ are also reducible.*

Proof. Since $\mathbf{e} = (\mathbf{E}, e)$ is reducible, its precondition pre_e is also reducible. Let us fix some static formula ρ s.t. $\vdash \text{pre}_e \leftrightarrow \rho$ is provable in LDAE. We'll prove both claims simultaneously, by induction on sub-expression complexity (-and we'll make liberal use of Lemma 6.1, without explicitly mentioning it):

For $\theta := p$: since e is reducible, there exists some static formula θ' s.t. $\vdash \underline{e}(p) \leftrightarrow \theta'$. Using the Change of Facts axiom, we can see that $[\mathbf{e}]p$ is provably equivalent to $\rho \rightarrow \theta'$.

For $\theta := Px_1 \dots x_n$: by the induction hypothesis, for all $i = 1, n$ we have $\vdash \mathbf{e}(x_i) = x'_i$ for some static terms $x'_1, \dots, x'_n \in \text{Var}_{\text{LDA}}$. Using also the Indiscernability axiom and the Property Change reduction axiom, $[\mathbf{e}]\theta$ is provably equivalent to $\text{pre}_e \rightarrow Px'_1 \dots x'_n$, and hence also to $\rho \rightarrow Px'_1 \dots x'_n$.

For $\theta := \varphi \rightarrow \psi$ is similar: by induction, there exist static formulas φ_e and ψ_e s.t. $\vdash [\mathbf{e}]\varphi \leftrightarrow \varphi_e$ and $\vdash [\mathbf{e}]\psi \leftrightarrow \psi_e$ are provable. Using the $[\mathbf{e}]$ -Distributivity axiom, we obtain $\vdash [\mathbf{e}]\theta \leftrightarrow (\varphi_e \rightarrow \psi_e)$.

For $\theta := K_A \varphi$: by induction, for each $f \in E$ there exists some static formula φ_f s.t. $\vdash [\mathbf{f}]\varphi \leftrightarrow \varphi_f$. Putting this together with the Knowledge Update axiom, we get $\vdash [\mathbf{e}]\theta \leftrightarrow \bigwedge \{\rho \rightarrow K_{\underline{e}(A)} \varphi_f : f \sim_A e\}$.

For $x := v$ (basic variable): since e is reducible, there exists some static term x' s.t. $\vdash \underline{e}(v) = x'$. Using this and the Value Change axiom, we obtain that $\vdash \mathbf{e}(x) = (\rho \rightarrow x' | \top)$, so $\mathbf{e}(x)$ is reducible.

For $x := F(x_1, \dots, x_n)$: by the induction hypothesis, there exist static terms $x'_1, \dots, x'_n$ s.t. $\vdash \mathbf{e}(x_i) = x'_i$ for all $i \leq n$. Using the Functional Change reduction axiom, we obtain $\vdash \mathbf{e}(x) = (\rho \rightarrow F(x'_1, \dots, x'_n))$.

For $x := \varphi \rightarrow y|z$: by the induction hypothesis, there exist static terms y', z' and static formula φ' , s.t. $\vdash \mathbf{e}(y) = y'$, $\vdash \mathbf{e}(z) = z'$ and $\vdash [\mathbf{e}]\varphi \leftrightarrow \varphi'$ are provable. Using these, as well as the Case Change reduction axiom, we obtain $\vdash \mathbf{e}(x) = (\varphi' \rightarrow y'|z')$, and so $\mathbf{e}(x)$ is reducible.

For $x := \mu_N y_A^\varphi$: first note that by definition, the reducibility of $\mathbf{e} = (\mathbf{E}, e)$ implies the reducibility of all $\mathbf{f} = (\mathbf{E}, f)$ with $f \in E$. By the induction hypothesis and Proposition 4.5, for every $f \in E$ there exist a

static formula φ_f and a static term y_f , s.t. $\vdash \langle \mathbf{f} \rangle \varphi \leftrightarrow \varphi_f$ and $\vdash \mathbf{f}(y) = y_f$. Thus, if we put

$$\eta := \{n_N(y_f)_{\underline{f}(A)}^{\varphi_f} : f \sim_A e, n \leq N\}^\diamond \leq N,$$

then $\vdash \eta \leftrightarrow \text{Val}_e^N(x_A \varphi)^\diamond \leq N$ is provable in **LDAE**. Using the Min. Val. Change axiom, $\mathbf{e}(\mu_N y_A^\varphi)$ is provably equal to $\rho \rightarrow (\eta \rightarrow \min \{\mu_N(y_f)_{\underline{f}(A)}^{\varphi_f} : f \sim_A e\} | \perp)$. $\square$

Using this, we can establish our full reduction result:

Lemma 6.3. (Co-expressivity of **LDAE** and **LDA**) *All expressions (formulas, terms and events) α of **LDAE** are reducible. As a consequence, **LDAE** and **LDA** have the same expressive power.*

Proof. Induction on the subexpression-complexity of the expression α . The base cases $\alpha := p \in \text{Prop}$ and $\alpha := v \in V$ are trivial. The cases $\alpha := \varphi \rightarrow x|y$, $\alpha := F(\bar{x})$, $\alpha := \mu_N x_A^\varphi$, $\alpha := P\bar{x}$, $\alpha := \varphi \rightarrow \theta$ and $\alpha := K_A \varphi$ are straightforward: use the induction hypothesis and parts 1-6 of Lemma 6.1.

For $\alpha := [\mathbf{e}]\varphi$: by induction, $\mathbf{e}$ and φ are reducible, so there exists static φ' s.t. $\vdash \varphi \leftrightarrow \varphi'$. By Lemma 6.1.7, $\vdash \alpha \leftrightarrow [\mathbf{e}]\varphi'$, and by Lemma 6.2 $[\mathbf{e}]\varphi'$ is reducible (since $\mathbf{e}$ is reducible and φ' is static), so $\vdash [\mathbf{e}]\varphi' \leftrightarrow \varphi''$ for some static φ'' . Using transitivity of equivalence, we get $\vdash \alpha \leftrightarrow \varphi''$, so α is reducible.

For $\alpha := \mathbf{e}(x)$: by induction, $\mathbf{e}$ and x are reducible, so there exists a static term x' s.t. $\vdash x = x'$. By Lemma 6.1.8, we have $\vdash \alpha = \mathbf{e}(x')$, and by Lemma 6.2 $\mathbf{e}(x')$ is reducible (since $\mathbf{e}$ is reducible and x' is static), so $\vdash \mathbf{e}(x') = x''$ for some static term x'' . Using transitivity of equality, we obtain $\vdash \alpha = x''$.

Finally, the case $\alpha := \mathbf{e} = (\mathbf{E}, e)$ is straightforward: for all $f \in E$, $p \in \text{Prop}$ and $v \in V$, the preconditions pre_f and postconditions $\underline{f}(p)$ and $\underline{f}(v)$ are $< \mathbf{e}$ in the subexpression complexity order, so they are all reducible (by the induction hypothesis), and hence $\mathbf{e}$ is also reducible (by definition). $\square$

Proof of Theorem 4.6: Completeness follows immediately from Lemma 6.3, the soundness of **LDAE**, and the completeness of the system **LDA** (Theorem 4.2); decidability follows from the decidability of the static logic **LDA** (Theorem 4.2) and the co-expressivity of **LDA** and **LDAE** (Lemma 6.3).

7 Conclusions and Future Work

In this paper we axiomatize a decidable but richly expressive logic, that deals with a large class of data-exchange events, while capturing group knowledge of both propositional and non-propositional data, as well as a group's ability to narrow down the values of a variable to finitely many possibilities.

There are a number of things still left to do. As already mentioned, one can add common knowledge operators $C_A \varphi$, but pre-encoding their dynamics requires a generalization to polyadic conditionals $C_A^e \varphi$ as in [11]. We leave this for a journal version, where we also plan to explore the relationships of our formalism with the Logic of Functional Dependence [13] and Graded (Multi)Modal Logic.

Our axioms are somewhat complicated due to the fact that we did *not* succeed to prove FMP (Finite Model Property) for our logics. This would have allowed us to replace cutoff-minimization $\mu_N x_A^\varphi$ by plain minimization $\min x_A^\varphi$ (over all A -possible x -values given φ), which would greatly simplify our axioms. On the other hand, we *don't* have a counterexample to FMP, so this issue is still an open question.

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Labelled Sequent Calculi for Propositional Team Logics

Fausto Barbero
University of Helsinki

Marianna Girlando
University of Amsterdam,
University of Southern Denmark

Valentin Müller
University of Bern

Fan Yang
Utrecht University

Team semantics is a general framework where formulas are not interpreted with respect to a single point of evaluation, but with respect to sets of such points. Team semantics is used in dependence logic, to reason about dependencies between variables, and in inquisitive logic, to formalize the meaning of questions. We provide sound and complete labelled sequent calculi for four logics based on team semantics: basic inquisitive logic, propositional intuitionistic dependence logic, and their respective extensions with tensor disjunction. For technical reasons, we restrict ourselves to languages with finitely many propositional atoms. The rules of weakening, contraction and cut are shown to be admissible in each of our calculi. In the last part of the paper, we present terminating proof search procedures for variants of our proof systems, in which labels have a simplified structure.

1 Introduction

In this paper, we introduce labelled sequent calculi for a family of propositional logics based on team semantics. *Team semantics* is a generalization of Tarskian semantics where formulas are evaluated over sets of evaluation points (called *teams*) rather than single evaluation points. It was introduced by Hodges [15] to characterize dependence and independence between variables. This framework was later systematically developed in *dependence logic* [26]. *Inquisitive semantics* [10] also uses the framework to formalize the meaning of interrogative sentences in natural language. Team semantics has found applications in diverse areas, such as database theory (e.g., [14]), formal semantics of natural language [3, 7], social choice theory [24] and quantum information theory [1, 2, 11].

Most prominent logics based on team semantics are *downward closed*, i.e., their satisfaction relation is preserved under taking subteams. In this work, we focus on this class of logics, and in particular we consider basic inquisitive logic (InqB, from [10]), propositional intuitionistic dependence logic (PID, from [29]) and their respective extensions with tensor disjunction $\vee$ (InqB $_{\vee}$ and PID $_{\vee}$). All these logics include operators for intuitionistic implication $\rightarrow$ and global (or inquisitive) disjunction $\vee$. The logics PID and PID $_{\vee}$ additionally include dependence atoms. A Hilbert-style system for the logic InqB was presented in [10], while natural deduction systems for propositional dependence logic (which does not have an implication) and its variants (including PID, PID $_{\vee}$ and InqB $_{\vee}$) were introduced in [7, 29].

While the semantic properties of downward closed team logics are well understood in the literature, their proof theory has not been fully investigated. In particular, designing analytic proof systems for these logics remains a challenging task. All the calculi in the literature are generalizations of the Gentzen-style sequent formalism. For instance, Frittella *et al.* [13] introduced a multi-type display calculus for InqB in the style of Belnap [5], and Anttila, Iemhoff and Yang [4] developed a sequent calculus for InqB $_{\vee}$ employing a form of deep inference. Both systems enrich the structure of Gentzen sequents.

An alternative approach is to enrich the language of the calculus instead, which is the approach we take in this paper for our labelled sequent calculi for the logics InqB, InqB $_{\vee}$, PID and PID $_{\vee}$. The calculi we present for these logics are *modular*, meaning that by adding rules they can be adapted to capture

logics built up from different sets of connectives or atoms. For instance, by extending our calculi for InqB and $\text{InqB}_\vee$ with two natural rules for dependence atoms, we obtain calculi for PID and $\text{PID}_\vee$. The modularity feature sets our proof systems apart from the structured calculi presented in [4, 13].

Labelled sequent calculi for the logic InqB (which does not include the tensor disjunction $\vee$) have already been presented by Chen and Ma [6] and later by Müller [17, Chapter 3]¹ as well as by Litak and Sano [16, Remark 4]. Our calculi for the four logics under consideration have a similar design as the ones presented in [6, 17], i.e., we also use labels to represent arbitrary teams, singleton teams as well as certain set-theoretic operations on them. Our labelled calculus for InqB , however, is simpler than the ones from [6] and [17, Chapter 3], since it does not include labels representing intersections. In fact, we will use such labels only for the logics involving tensor disjunction (i.e., for $\text{InqB}_\vee$ and $\text{PID}_\vee$).

A key technical challenge in designing calculi for logics with tensor disjunction is its different behaviour on classical and general formulas, which makes it difficult to define labelled rules for $\text{InqB}_\vee$ and $\text{PID}_\vee$ using the standard techniques. To address this difficulty, we restrict our attention to languages containing only *finitely many* propositional variables. This assumption (which does not affect validity in $\text{InqB}_\vee$ or $\text{PID}_\vee$) restricts the semantics to finitely many valuations and teams, thus allowing us to introduce a new order rule (*fin*) that provides enough labels to describe arbitrary valuations and teams. Thanks to this rule, we can simulate the behaviour of the tensor over classical formulas.

Using syntactic arguments, we prove that all our calculi enjoy cut-admissibility. This property, together with the admissibility of additional rules capturing key semantic properties, allows us to establish the completeness of our proof systems. We then present a family of *terminating* labelled calculi for all the logics under consideration. These calculi are obtained by simplifying the structure of labels in our systems. Their construction crucially relies on the assumption that the language has finitely many propositional variables. Our terminating calculus for InqB is similar to the ones introduced by Litak and Sano [16] for basic inquisitive logic and for bounded versions of inquisitive predicate logic. We extend this approach to cover $\text{InqB}_\vee$, PID and $\text{PID}_\vee$, and provide an explicit proof search algorithm for all the calculi.

The paper is structured as follows. In Section 2, we recall the semantics and axiom systems of the logics we consider, and in Section 3, we introduce our labelled calculi. In Section 4, we discuss the structural properties of the calculi, and in Section 5, we establish their completeness. In Section 6, finally, we present an explicit proof search algorithm for terminating variants of our calculi.

2 Preliminaries

We start by recalling some basic notions. For further details, see [8, 10, 29]. Throughout this paper, we will assume a non-empty set P of *propositional variables* (or *atoms*). For reasons that will become apparent later on, we assume that P is *finite*.² Elements of P are denoted by the letters p and q (possibly with subscripts). The *language of downward-closed team logic*, notation $\mathcal{L}$, is given by the grammar

$$\varphi ::= p \mid \perp \mid =(p) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \wp \varphi \mid \varphi \rightarrow \varphi \quad (p \in P).$$

We also define $\neg\varphi := \varphi \rightarrow \perp$ and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. We will refer to $\vee$, $\wp$ and $\rightarrow$ as *tensor disjunction*, *global disjunction* and *intuitionistic implication*, respectively. In the literature on dependence logic, expressions of the form $=(p)$ are known as *constancy dependence atoms* and used to

¹In [17], Müller also presents labelled sequent calculi for *intuitionistic inquisitive logic* [9] and *inquisitive Kripke logics* [8, Chapter 6]. The calculi for the latter were subsequently extended to various systems of *inquisitive modal logic* (see [18, 19]).

²Strictly speaking, this assumption is only needed in our treatment of systems involving tensor disjunction (i.e., for the logics $\text{InqB}_\vee$ and $\text{PID}_\vee$ introduced below). For simplicity, however, we also make this assumption for logics without tensor.

Table 1: Four team-based logics.

Logic	Connectives	Hilbert System
Basic propositional inquisitive logic (InqB)	$\perp, \wedge, \vee, \rightarrow$	HInqB
Propositional intuitionistic dependence logic (PID)	$\perp, \wedge, \vee, \rightarrow, =(\cdot)$	HPID
Propositional inquisitive logic with tensor (InqB _∨)	$\perp, \wedge, \vee, \rightarrow, \vee$	HInqB _∨
Propositional intuitionistic dependence logic with tensor (PID _∨)	$\perp, \wedge, \vee, \rightarrow, =(\cdot), \vee$	HPID _∨

express that p has a fixed value [29, p. 564]. Dependence atoms with multiple arguments can be defined from constancy atoms by putting $=(q_1, \dots, q_n, p) := (=(q_1) \wedge \dots \wedge =(q_n)) \rightarrow =(p)$. Intuitively, $=(q_1, \dots, q_n, p)$ expresses that the value of p *depends* on the values of $q_1, \dots, q_n$. In the literature on inquisitive logic, the global disjunction is also known as *inquisitive disjunction* and used to form alternative questions. Thus, in this framework, $\varphi \vee \psi$ is interpreted as the question *whether φ or ψ* [8, Sect. 2.2].

Formulas of our language are evaluated over *teams* (a.k.a. *information states*). A team over P is a set of valuations $v : P \rightarrow \{0, 1\}$. The set of all valuations is given by $\mathcal{V}(P) := \{v \mid v : P \rightarrow \{0, 1\}\}$. Note that, since P is finite, $\mathcal{V}(P)$ is also finite, and each team $T \subseteq \mathcal{V}(P)$ contains at most $|\mathcal{V}(P)| = 2^{|P|}$ elements.

Definition 2.1 (Satisfaction). Let $T \subseteq \mathcal{V}(P)$ be a team. The *satisfaction relation* $\models$ is defined as follows:

- $T \models p$ iff $v(p) = 1$ for all $v \in T$,
- $T \models \perp$ iff $T = \emptyset$,
- $T \models =(p)$ iff $u(p) = v(p)$ for all $u, v \in T$,
- $T \models \varphi \wedge \psi$ iff $T \models \varphi$ and $T \models \psi$,
- $T \models \varphi \vee \psi$ iff there are $R, S \subseteq T$ with $T = R \cup S$ such that $R \models \varphi$ and $S \models \psi$,
- $T \models \varphi \vee \psi$ iff $T \models \varphi$ or $T \models \psi$,
- $T \models \varphi \rightarrow \psi$ iff for all $S \subseteq T$, if $S \models \varphi$, then $S \models \psi$.

It is easy to show that all $\mathcal{L}$ -formulas satisfy the *empty team property* and *downward closure*.

Downward closure: For all teams $T, S \subseteq \mathcal{V}(P)$, if $T \models \varphi$ and $S \subseteq T$, then $S \models \varphi$.

Empty team property: $\emptyset \models \varphi$.

A formula $\varphi \in \mathcal{L}$ is said to be *classical*, if it does not contain any occurrences of the connectives $\vee$ and $=(\cdot)$. Classical formulas will be denoted by the letters α, β, γ . We also write $\mathcal{L}_c$ for the set of all classical formulas. Classical formulas satisfy both *union closure* and *flatness* (or *truth-conditionality*).

Union closure: For all teams $T, S \subseteq \mathcal{V}(P)$, if $S \models \alpha$ and $T \models \alpha$, then $S \cup T \models \alpha$.

Flatness (or truth-conditionality): For every team $T \subseteq \mathcal{V}(P)$, $T \models \alpha$ iff $\{v\} \models \alpha$ for all $v \in T$.

The first column of Table 1 contains four team-based logics: basic inquisitive logic (InqB), propositional intuitionistic dependence logic (PID), and their respective extensions with tensor disjunction (InqB_∨ and PID_∨). The second column specifies the corresponding fragment of the language $\mathcal{L}$. So, for example, the language of InqB consists of all formulas built up from the variables in P by means of the connectives $\perp, \wedge, \vee, \rightarrow$, and the language of PID consists of all formulas built up from the variables by means of the connectives $\perp, \wedge, \vee, \rightarrow, =(\cdot)$. The language associated with one of our four logics L will also be denoted by $\mathcal{L}(L)$. Observe that, in particular, $\mathcal{L}(\text{PID}_{\vee})$ coincides with the full language $\mathcal{L}$.

The last column of Table 1 contains the names of four Hilbert-style systems. The axiom schemes of these systems are presented in the upper section of Figure 1. Note that, in the schemes (S), (DN), (Elim), we require α to range over classical formulas only. The lower section of Figure 1 contains the definitions of our Hilbert calculi. For example, HInqB_∨ consists of the basic axioms, the tensor axioms and the rule of modus ponens. The Hilbert system associated with a logic L will also be denoted by $\mathcal{H}(L)$.

Basic Axioms		
(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$	(A2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	(A3) $(\varphi \wedge \psi) \rightarrow \varphi, (\varphi \wedge \psi) \rightarrow \psi$
(A4) $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$	(A5) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$	(A6) $\varphi \rightarrow (\varphi \vee \psi), \psi \rightarrow (\varphi \vee \psi)$
(A7) $\perp \rightarrow \varphi$	(S) $(\alpha \rightarrow (\varphi \vee \psi)) \rightarrow ((\alpha \rightarrow \varphi) \vee (\alpha \rightarrow \psi))$	(DN) $\neg\neg\alpha \rightarrow \alpha$
Tensor Axioms		
(Intro) $\varphi \rightarrow (\varphi \vee \psi), \psi \rightarrow (\varphi \vee \psi)$	(Elim) $(\varphi \rightarrow \alpha) \rightarrow ((\psi \rightarrow \alpha) \rightarrow ((\varphi \vee \psi) \rightarrow \alpha))$	(Mon) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \vee \chi) \rightarrow (\psi \vee \chi))$
(Com) $(\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$	(Dis) $(\varphi \vee (\psi \vee \chi)) \rightarrow ((\varphi \vee \psi) \vee (\varphi \vee \chi))$	
Constancy Axiom (C): $\neg(p) \leftrightarrow (p \vee \neg p)$ (for $p \in \mathcal{P}$)		
<i>Side condition:</i> In (S), (DN) and (Elim), we require $\alpha \in \mathcal{L}_c$ to be a classical formula.		
Hilbert-Style Systems		
HInqB : basic axioms;	HInqB _∨ : basic axioms and tensor axioms;	
HPID : basic axioms and constancy axiom;	HPID _∨ : basic axioms, tensor axioms and constancy axiom.	
All systems also include <i>modus ponens</i> : from φ and $\varphi \rightarrow \psi$, infer ψ .		

Figure 1: Hilbert-style axiomatizations.

Let now L be any of the logics from Table 1 and let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(L)$. We write $\Gamma \vdash \varphi$ and say that φ is *provable* from Γ in $\mathcal{H}(L)$, if there exists a sequence $\psi_1, \dots, \psi_n$ of formulas from $\mathcal{L}(L)$ such that $\varphi = \psi_n$ and every ψ_i with $1 \leq i \leq n$ is either an axiom of $\mathcal{H}(L)$ or an element of Γ , or it can be obtained from two formulas occurring earlier in the sequence by modus ponens. Moreover, we write $\Gamma \models_L \varphi$ and say that φ is *entailed* by Γ in L , if for every team $T \subseteq \mathcal{V}(\mathcal{P})$, $T \models \psi$ for all $\psi \in \Gamma$ implies $T \models \varphi$.

Theorem 2.2 (Soundness and Completeness). *Let L be any of the logics from Table 1. For every set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(L)$, it is the case that $\Gamma \vdash \varphi$ holds in $\mathcal{H}(L)$ if and only if $\Gamma \models_L \varphi$ holds in L .*

Proof. See [8, 10] for (essentially) the completeness proof for HInqB, [29] for (essentially) that of HPID, and [7] for (essentially) that of HInqB_∨. Completeness of HPID_∨ follows directly from these results. $\square$

3 Labelled Sequent Calculi

We will now introduce labelled sequent calculi for each of the logics presented in Table 1. The labels used in our calculi are built up from two infinite sets of *team variables*, denoted by $\mathfrak{S}$ and $\mathfrak{A}$. The variables from $\mathfrak{S}$ range over *singleton teams* and the variables from $\mathfrak{A}$ range over teams of *arbitrary size*. Elements of $\mathfrak{S}$ are denoted by the letters u, v, w , and elements of $\mathfrak{A}$ are denoted by the letters x, y, z .

Definition 3.1 (Labels). The sets of labels Lab and $\text{Lab}_\cap$ are generated by the following two grammars:

$$\begin{aligned}
 (\text{Lab}) \quad \pi &::= v \mid x \mid \emptyset \mid \pi \cup \pi & (v \in \mathfrak{S} \text{ and } x \in \mathfrak{A}). \\
 (\text{Lab}_\cap) \quad \pi &::= v \mid x \mid \emptyset \mid \pi \cup \pi \mid \pi \cap \pi
 \end{aligned}$$

The set of labels Lab will be used in our calculi for InqB and PID, and the set of labels $\text{Lab}_\cap$ will be used in our calculi for InqB_∨ and PID_∨. Arbitrarily complex labels from any of the sets Lab and $\text{Lab}_\cap$ will be denoted by the meta-variables π, σ, τ . Intuitively, $\pi \cup \sigma$ represents the *union* and $\pi \cap \sigma$ represents the *intersection* of the teams denoted by π and σ . The constant $\emptyset$ stands for the *empty team*.

Definition 3.2. An *interpretation* is a function $I : \mathfrak{S} \cup \mathfrak{A} \rightarrow \mathcal{P}(\mathcal{V}(\mathcal{P}))$ that assigns, to each $v \in \mathfrak{S}$, a singleton team $I(v) \subseteq \mathcal{V}(\mathcal{P})$, and to each $x \in \mathfrak{A}$, an arbitrary team $I(x) \subseteq \mathcal{V}(\mathcal{P})$. An interpretation I is extended to arbitrary labels by defining $I(\emptyset) := \emptyset$, $I(\pi \cup \sigma) := I(\pi) \cup I(\sigma)$ and $I(\pi \cap \sigma) := I(\pi) \cap I(\sigma)$.

Initial Sequents (Axioms)	
$\text{ax} \frac{}{v : p, \Gamma \Rightarrow \Delta, v : p}$	$\text{ax}_{\perp} \frac{}{v : \perp, \Gamma \Rightarrow \Delta}$
$\text{ax}_{\emptyset} \frac{}{v \subseteq \emptyset, \Gamma \Rightarrow \Delta}$	
Basic Logical Rules	
$\text{at}_L \frac{v : p, v \subseteq \pi, \pi : p, \Gamma \Rightarrow \Delta}{v \subseteq \pi, \pi : p, \Gamma \Rightarrow \Delta}$	$\text{at}_R \frac{v \subseteq \pi, \Gamma \Rightarrow \Delta, v : p}{\Gamma \Rightarrow \Delta, \pi : p} (\ddagger)$
$\perp_L \frac{v : \perp, v \subseteq \pi, \pi : \perp, \Gamma \Rightarrow \Delta}{v \subseteq \pi, \pi : \perp, \Gamma \Rightarrow \Delta}$	$\perp_R \frac{v \subseteq \pi, \Gamma \Rightarrow \Delta, v : \perp}{\Gamma \Rightarrow \Delta, \pi : \perp} (\ddagger)$
$\wedge_L \frac{\pi : \varphi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}$	$\wedge_R \frac{\Gamma \Rightarrow \Delta, \pi : \varphi \quad \Gamma \Rightarrow \Delta, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \wedge \psi}$
$\vee_L \frac{\pi : \varphi, \Gamma \Rightarrow \Delta \quad \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta}$	$\vee_R \frac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi}$
$\rightarrow_L \frac{\pi \subseteq \sigma, \sigma : \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta, \pi : \varphi \quad \pi \subseteq \sigma, \sigma : \varphi \rightarrow \psi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi \subseteq \sigma, \sigma : \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}$	$\rightarrow_R \frac{x \subseteq \pi, x : \varphi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \rightarrow \psi} (§)$
Tensor Rules	
$\vee_L \frac{\pi \subseteq x \cup y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta} (\#)$	$\vee_R \frac{\pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \sigma : \varphi \quad \pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \tau : \psi}{\pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi}$
Constancy Rules	
$=_L \frac{u \subseteq \pi, v \subseteq \pi, u : p, v : p, \pi : =(p), \Gamma \Rightarrow \Delta \quad u \subseteq \pi, v \subseteq \pi, \pi : =(p), \Gamma \Rightarrow \Delta, u : p, v : p}{u \subseteq \pi, v \subseteq \pi, \pi : =(p), \Gamma \Rightarrow \Delta}$	$=_R \frac{u \subseteq \pi, v \subseteq \pi, u : p, \Gamma \Rightarrow \Delta, v : p}{\Gamma \Rightarrow \Delta, \pi : =(p)} (*)$
Basic Order Rules	
$\text{tr} \frac{\pi \subseteq \tau, \pi \subseteq \sigma, \sigma \subseteq \tau, \Gamma \Rightarrow \Delta}{\pi \subseteq \sigma, \sigma \subseteq \tau, \Gamma \Rightarrow \Delta}$	$\text{sg} \frac{\pi \subseteq \emptyset, \pi \subseteq v, \Gamma \Rightarrow \Delta \quad v \subseteq \pi, \pi \subseteq v, \Gamma \Rightarrow \Delta}{\pi \subseteq v, \Gamma \Rightarrow \Delta}$
$\text{cd} \frac{v \subseteq \pi, v \subseteq \pi \cup \sigma, \Gamma \Rightarrow \Delta \quad v \subseteq \sigma, v \subseteq \pi \cup \sigma, \Gamma \Rightarrow \Delta}{v \subseteq \pi \cup \sigma, \Gamma \Rightarrow \Delta}$	$\text{ul} \frac{\pi \cup \sigma \subseteq \tau, \pi \subseteq \tau, \sigma \subseteq \tau, \Gamma \Rightarrow \Delta}{\pi \subseteq \tau, \sigma \subseteq \tau, \Gamma \Rightarrow \Delta}$
$\text{rf} \frac{\pi \subseteq \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$	
Special Order Rules	
$\text{dis} \frac{\pi \subseteq (\pi \cap \sigma) \cup (\pi \cap \tau), \pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta}{\pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta}$	$\text{il} \frac{\pi_1 \cap \pi_2 \subseteq \pi_i, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (i = 1, 2)$
$\text{ir} \frac{\pi \subseteq \sigma \cap \tau, \pi \subseteq \sigma, \pi \subseteq \tau, \Gamma \Rightarrow \Delta}{\pi \subseteq \sigma, \pi \subseteq \tau, \Gamma \Rightarrow \Delta}$	
$\text{fin} \frac{\pi = \emptyset, \Gamma \Rightarrow \Delta \quad \pi = v_1 \cup \dots \cup v_n, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (\bowtie), \text{ for } n = 2^{ P }$	
<i>Side conditions:</i> The symbol $(\ddagger)$ indicates that $v \in \mathfrak{S}$ has to be fresh and π has to satisfy $\pi \notin \mathfrak{S}$. The symbol $(§)$ indicates that $x \in \mathfrak{A}$ must be fresh. In the rule marked with $(\#)$, we require $x, y \in \mathfrak{A}$ to be fresh and distinct. The symbol $(*)$ indicates that $u, v \in \mathfrak{S}$ have to be fresh and distinct. In the rule marked with $(\bowtie)$, we require $v_1, \dots, v_n$ to be a collection of $n = 2^{ P }$ pairwise distinct fresh variables from $\mathfrak{S}$. Moreover, the notation “$\sigma = \tau$” is used as an abbreviation for the pair of relational atoms “$\sigma \subseteq \tau, \tau \subseteq \sigma$”.	
Labelled Sequent Calculi	
• GlnqB : initial sequents, basic logical rules and basic order rules; • GPID : initial sequents, basic logical rules, constancy rules and basic order rules; • $\text{GlnqB}_{\vee}$: initial sequents and all of the above rules except for the constancy rules; • $\text{GPID}_{\vee}$: initial sequents and all of the above rules.	

Figure 2: Labelled sequent calculi.

A *labelled formula* is an expression of the form $\pi : \varphi$, where π is a label and φ is a formula. Labelled formulas internalise the satisfaction relation into our calculi, so $\pi : \varphi$ may be read as “The team π satisfies the formula φ ”. A *relational atom* is an expression of the form $\pi \subseteq \sigma$, where π and σ are labels, meaning that π is a subset of σ . A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ is a finite multiset of labelled formulas and relational atoms and Δ is a finite multiset containing only labelled formulas.

We say that a labelled formula $\pi : \varphi$ is *satisfied* by an interpretation I if it holds that $I(\pi) \models \varphi$. And a relational atom $\pi \subseteq \sigma$ is satisfied by I if $I(\pi) \subseteq I(\sigma)$. A sequent $\Gamma \Rightarrow \Delta$ is *valid* if for every interpretation I , it is the case that, if I satisfies all elements of Γ , then I satisfies at least one element of Δ .

The axioms and rules of our labelled sequent calculi are presented in the upper section of Figure 2.

Note that, in the rule *fin*, we use the notation “ $\sigma = \tau$ ” as an abbreviation for the pair of atoms “ $\sigma \subseteq \tau, \tau \subseteq \sigma$ ”. Furthermore, certain rules come with additional side conditions. For example, $v \in \mathfrak{S}$ has to be a fresh variable in all rules marked with $(\ddagger)$, and $x \in \mathfrak{A}$ has to be fresh in the rule marked with $(\S)$. Here, a variable is said to be *fresh*, if it does not occur in the conclusion of the respective rule application. The fresh variables introduced by the rules $\text{at}_R, \perp_R, \rightarrow_R, \vee_L, =_R$ and *fin* are also referred to as the *eigenvariables* of these rules.

The rules of our proof systems are inspired by the G3-style approach to labelled calculi as detailed, e.g., in [21] for the case of modal logic. However, the richer semantics of team-based logics requires non-trivial adjustments and the addition of several more order rules than in the modal case. The basic logical rules, the tensor rules and the constancy rules simply mirror the satisfaction conditions from Definition 2.1. The basic and special order rules, on the other hand, formalize various set-theoretic properties of teams. The rule *fin* plays a somewhat special role here, as it is directly linked to the assumption that P is finite. Intuitively, this rule expresses the fact that every team π is either empty or composed of $n = 2^{|P|}$ not necessarily distinct valuations. Note that, in applications of *fin*, we do not require the union operators in the label $v_1 \cup \dots \cup v_n$ to be grouped in any specific way. So, for example, in the case $|P| = 2$, the sequents $\pi = ((v_1 \cup v_2) \cup v_3) \cup v_4, \Gamma \Rightarrow \Delta$ and $\pi = (v_1 \cup v_2) \cup (v_3 \cup v_4), \Gamma \Rightarrow \Delta$ would both be correct instances of the right premise of *fin*.

The lower section of Figure 2 contains the definitions of our calculi. For example, the calculus GPID consists of the initial sequents, the basic logical rules, the constancy rules and the basic order rules. Given any of the logics L from Table 1, we will write $\mathcal{G}(L)$ for the corresponding calculus from Figure 2.

Definition 3.3. Let L be one of the logics from Table 1 and let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(L)$ be a set of formulas. We will write $\Gamma \vdash \varphi$ and say that φ is *provable* from Γ in the calculus $\mathcal{G}(L)$, if there exists a finite subset $\Delta \subseteq \Gamma$ and a variable $x \in \mathfrak{A}$ such that $x : \Delta \Rightarrow x : \varphi$ is derivable in $\mathcal{G}(L)$, where $(x : \Delta) := \{x : \psi \mid \psi \in \Delta\}$.

In each of the axioms and rules from Figure 2, we call Γ and Δ the *left context* and the *right context*, respectively. An occurrence of an expression in an axiom or in the conclusion of a rule is called *principal*, if it does not belong to the context. To ensure that contraction is admissible, we adopt the *closure condition* from [12, 20–22]: if an instance of a rule of one of our calculi contains two principal occurrences of a relational atom A in the conclusion, then also the contracted instance of the rule (in which the two occurrences of A are replaced by a single one) is taken to be part of the corresponding calculus. For example, in the special case where $\pi = \sigma$, we permit not only the instance of the rule *ul* shown on the left-hand side below, but also the contracted instance displayed on the right-hand side:

$$\text{ul} \frac{\pi \cup \pi \subseteq \tau, \pi \subseteq \tau, \pi \subseteq \tau, \Gamma \Rightarrow \Delta}{\pi \subseteq \tau, \pi \subseteq \tau, \Gamma \Rightarrow \Delta} \quad \text{ul} \frac{\pi \cup \pi \subseteq \tau, \pi \subseteq \tau, \Gamma \Rightarrow \Delta}{\pi \subseteq \tau, \Gamma \Rightarrow \Delta}$$

The following lemma contains several sequents that are derivable in our calculi. Observe that sequent (1) corresponds to *downward closure* and sequents (2) and (3) correspond to the *empty team property*.

Lemma 3.4 (Generalized Initial Sequents). *Let L be any of the logics from Table 1 and let $\mathcal{G}(L)$ be the corresponding labelled sequent calculus. All sequents of the following forms are derivable in $\mathcal{G}(L)$:*

1. $\pi \subseteq \sigma, \sigma : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi;$
2. $\pi \subseteq \emptyset, \Gamma \Rightarrow \Delta, \pi : \varphi;$
3. $\pi \subseteq \sigma, \sigma : \perp, \Gamma \Rightarrow \Delta, \pi : \varphi;$
4. $\pi : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi.$

Proof. By induction on φ . Most cases are treated in the same way as in [17, Lemmas 3.2.1–3.2.2]. $\square$

4 Structural Properties

We now examine the structural properties of our labelled calculi. Specifically, we will prove the admissibility of weakening and contraction, the invertibility of all rules, and the admissibility of the cut rule. In

$$\begin{array}{c}
(u/v) \frac{\Gamma \Rightarrow \Delta}{\Gamma(u/v) \Rightarrow \Delta(u/v)} (\ddagger) \quad (\pi/x) \frac{\Gamma \Rightarrow \Delta}{\Gamma(\pi/x) \Rightarrow \Delta(\pi/x)} (\ddagger) \quad w_L \frac{\Gamma \Rightarrow \Delta}{E, \Gamma \Rightarrow \Delta} (§) \quad w_R \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \pi : \varphi} \\
c_L \frac{E, E, \Gamma \Rightarrow \Delta}{E, \Gamma \Rightarrow \Delta} (§) \quad c_R \frac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \varphi}{\Gamma \Rightarrow \Delta, \pi : \varphi} \quad \text{cut} \frac{\Gamma \Rightarrow \Delta, \pi : \varphi \quad \pi : \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}
\end{array}$$

Side conditions: In the rules marked with $(\ddagger)$, we require that u and v are variables from $\mathfrak{S}$ and that x is a variable from $\mathfrak{A}$, while π can be an arbitrarily complex label. The symbol $(§)$ indicates that E may be either a relational atom or a labelled formula.

Figure 3: The substitution rules and the structural rules of weakening, contraction and cut.

$$\begin{array}{c}
\text{aux}_1 \frac{\pi \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \Gamma \Rightarrow \Delta}{\pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta} \quad \text{aux}_2 \frac{\pi \subseteq (\tau \cup \sigma) \cup (\tau' \cup \sigma'), \Gamma \Rightarrow \Delta}{\pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta} \quad \text{mp} \frac{\Gamma \Rightarrow x : \varphi \quad \Delta \Rightarrow x : \varphi \rightarrow \psi}{\Gamma, \Delta \Rightarrow x : \psi} \quad \text{flat} \frac{v \subseteq \pi, \Gamma \Rightarrow \Delta, v : \alpha}{\Gamma \Rightarrow \Delta, \pi : \alpha} (\ddagger)
\end{array}$$

$(\ddagger)$ means that α has to be a classical formula and v has to be a fresh variable from $\mathfrak{S}$.

Figure 4: Further admissible rules.

Section 5, we will use these results to establish the completeness of our proof systems. In what follows, let L be any of the logics from Table 1 and let $\mathcal{G}(L)$ be the associated sequent calculus from Figure 2.

We first recall some basic definitions. By the *length of a branch* in a derivation, we mean the number of sequents occurring in the branch. The *height of a derivation* $\mathcal{D}$ is the length of one of its longest branches, minus one. A rule R is *height-preserving admissible* (or *hp-admissible*) in the calculus $\mathcal{G}(L)$, if whenever each premise of R is derivable by a derivation of height at most n , then the conclusion of R is derivable by a derivation of height at most n . If derivability of the premises implies derivability of the conclusion, but the height is not preserved, R is just said to be *admissible*. A rule R is *height-preserving invertible* (or *hp-invertible*) in $\mathcal{G}(L)$, if whenever the conclusion of R is derivable by a derivation of height at most n , then each premise of R is derivable, with the same bound on height [25, Def. 3.4.4].

Given any variable $s \in \mathfrak{S} \cup \mathfrak{A}$ and labels $\pi, \sigma \in \text{Lab}_\cap$, we write $\sigma(\pi/s)$ for the result of substituting π for all occurrences of s in σ . The notation $\Gamma(\pi/s)$ stands for the result of substituting π for s in all labels that occur in a multiset Γ . By the *substitution rules*, we mean the first two rules displayed in Figure 3.

Lemma 4.1. *The substitution rules are hp-admissible in $\mathcal{G}(L)$.*

The admissibility of both substitution rules is established simultaneously, by induction on the height of a derivation for the premise. Further details are provided in [17, Prop. 3.2.6]. Using this result, we can now prove the admissibility of the weakening rules, the contraction rules and the cut rule (see Figure 3).

Theorem 4.2 (Structural properties). *The calculus $\mathcal{G}(L)$ has the following structural properties:*

1. *The weakening rules are hp-admissible.*
2. *All rules of $\mathcal{G}(L)$ are hp-invertible.*
3. *The contraction rules are hp-admissible.*
4. *The cut rule is admissible in $\mathcal{G}(L)$.*

The proof of this theorem follows the same structure as the standard admissibility proofs for structural rules in G3-style systems (see [21, 23]). Further details are in Appendix A, and similar proofs are in [17].

5 Completeness

To establish the completeness of our proof systems, we will now prove that each of the labelled sequent calculi from Figure 2 is complete with respect to the corresponding Hilbert-style system from Figure 1. For technical reasons, we first prove the admissibility of the rules aux_1 and aux_2 presented in Figure 4.

Lemma 5.1. *The rules aux_1 and aux_2 are admissible in each of the labelled calculi from Figure 2.*

A proof is provided in Appendix B. Next, we need to establish the admissibility of the rules mp and flat (see Figure 4). The rule mp corresponds to *modus ponens* and the rule flat accounts for the *flatness* of classical formulas: if a classical formula α is true under every valuation v in a team, then it is also satisfied by this team (see Section 2). For the calculi not involving tensor (i.e., for GlnqB and GPID), the admissibility of the rule flat can simply be established by induction on the structure of α (cf. [17, Prop. 3.2.14]). For the other two calculi (i.e., for $\text{GlnqB}_\vee$ and $\text{GPID}_\vee$), we first need the following lemma.

Lemma 5.2. *Let L be $\text{lnqB}_\vee$ or $\text{PID}_\vee$, let $\mathcal{G}(L)$ be the corresponding labelled calculus and let $\alpha \in \mathcal{L}(L)$ be a classical formula. All sequents of the form $\pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \alpha$ are derivable in $\mathcal{G}(L)$.*

Proof. Let L be one of $\text{lnqB}_\vee$ and $\text{PID}_\vee$, let $\mathcal{G}(L)$ be the associated labelled calculus and let $\alpha \in \mathcal{L}(L)$ be classical. The proof is by induction on the structure of α . We only consider the following cases.

Case 1: $\alpha = p$ for some variable $p \in P$. In this case, the desired sequent can be derived as follows:

$$\text{cd} \frac{\frac{\text{Lemma 3.4 (1)}}{\dots, v \subseteq \pi, \pi : p, \Gamma \Rightarrow \Delta, v : p} \quad \frac{\text{Lemma 3.4 (1)}}{\dots, v \subseteq \sigma, \sigma : p, \Gamma \Rightarrow \Delta, v : p}}{v \subseteq \pi \cup \sigma, \pi : p, \sigma : p, \Gamma \Rightarrow \Delta, v : p} \text{at}_R \frac{}{\pi : p, \sigma : p, \Gamma \Rightarrow \Delta, \pi \cup \sigma : p}$$

Case 2: $\alpha = \beta \vee \gamma$, where β and γ are classical. Using the admissible rules aux_1 and aux_2 from Figure 4, we now construct the following derivation:

$$\begin{array}{c} \frac{\frac{\text{By ind. hyp.}}{\dots, x_1 : \beta, y_1 : \beta \Rightarrow \dots, x_1 \cup y_1 : \beta} \quad \frac{\text{By ind. hyp.}}{\dots, x_2 : \gamma, y_2 : \gamma \Rightarrow \dots, x_2 \cup y_2 : \gamma}}{\dots, \pi \cup \sigma \subseteq (x_1 \cup y_1) \cup (x_2 \cup y_2), x_1 : \beta, x_2 : \gamma, y_1 : \beta, y_2 : \gamma, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \beta \vee \gamma} \vee_R \\ \text{ul} \frac{}{\pi \subseteq (x_1 \cup y_1) \cup (x_2 \cup y_2), \sigma \subseteq (x_1 \cup y_1) \cup (x_2 \cup y_2), x_1 : \beta, x_2 : \gamma, y_1 : \beta, y_2 : \gamma, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \beta \vee \gamma} \\ \text{aux}_1 \frac{}{\pi \subseteq x_1 \cup x_2, \sigma \subseteq (x_1 \cup y_1) \cup (x_2 \cup y_2), x_1 : \beta, x_2 : \gamma, y_1 : \beta, y_2 : \gamma, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \beta \vee \gamma} \\ \text{aux}_2 \frac{}{\pi \subseteq x_1 \cup x_2, \sigma \subseteq y_1 \cup y_2, x_1 : \beta, x_2 : \gamma, y_1 : \beta, y_2 : \gamma, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \beta \vee \gamma} \\ \vee_L \frac{}{\pi \subseteq x_1 \cup x_2, x_1 : \beta, x_2 : \gamma, \sigma : \beta \vee \gamma, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \beta \vee \gamma} \\ \vee_L \frac{}{\pi : \beta \vee \gamma, \sigma : \beta \vee \gamma, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \beta \vee \gamma} \end{array}$$

Case 3: $\alpha = \beta \rightarrow \gamma$, where β and γ are classical. We can now construct the derivation

$$\begin{array}{c} \frac{\frac{\text{Lemma 3.4 (1)}}{x \cap \pi \subseteq x, \dots, x : \beta \Rightarrow \dots, x \cap \pi : \beta} \quad \frac{\text{Lemma 3.4 (1)}}{\dots, x \subseteq \pi \cup \sigma, x \cap \pi : \gamma, x \cap \sigma : \gamma \Rightarrow \dots, x : \gamma}}{\dots, x : \beta \Rightarrow \dots, x \cap \pi : \beta} \text{il} \frac{}{\dots, x \subseteq \pi \cup \sigma, x : \beta, \pi : \beta \rightarrow \gamma, \sigma : \beta \rightarrow \gamma, \Gamma \Rightarrow \Delta, x : \gamma} \rightarrow_L \\ \frac{}{\dots, x : \beta \Rightarrow \dots, x \cap \pi : \beta} \text{il} \frac{}{\dots, x \subseteq \pi \cup \sigma, x : \beta, \pi : \beta \rightarrow \gamma, \sigma : \beta \rightarrow \gamma, \Gamma \Rightarrow \Delta, x : \gamma} \rightarrow_R \\ \frac{}{\pi : \beta \rightarrow \gamma, \sigma : \beta \rightarrow \gamma, \Gamma \Rightarrow \Delta, \pi \cup \sigma : \beta \rightarrow \gamma} \end{array}$$

where the subderivation $\mathcal{D}$ is of the form

$$\text{cut} \frac{\frac{\text{By ind. hyp.}}{\dots, x \cap \pi : \gamma, x \cap \sigma : \gamma \Rightarrow \dots, (x \cap \pi) \cup (x \cap \sigma) : \gamma} \quad \frac{\text{Lemma 3.4 (1)}}{x \subseteq (x \cap \pi) \cup (x \cap \sigma), (x \cap \pi) \cup (x \cap \sigma) : \gamma \Rightarrow x : \gamma}}{\dots, x \subseteq (x \cap \pi) \cup (x \cap \sigma), x \cap \pi : \gamma, x \cap \sigma : \gamma \Rightarrow \dots, x : \gamma} \text{dis} \frac{}{\dots, x \subseteq \pi \cup \sigma, x \cap \pi : \gamma, x \cap \sigma : \gamma \Rightarrow \dots, x : \gamma}$$

The case $\alpha = \perp$ can be treated in the same way as case 1. The case $\alpha = \beta \wedge \gamma$ is straightforward. $\square$

We are now ready to prove the desired admissibility results.

Lemma 5.3. *Let L be any of the logics from Table 1 and let $\mathcal{G}(L)$ be the corresponding labelled sequent calculus. The modus ponens rule mp and the flatness rule flat are admissible in $\mathcal{G}(L)$.*

Proof. To prove the admissibility of mp, suppose that $\Gamma \Rightarrow x : \varphi$ and $\Delta \Rightarrow x : \varphi \rightarrow \psi$ are both derivable in $\mathcal{G}(L)$ by proof trees $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. By the invertibility of $\rightarrow_R$, the admissibility of substitution and an application of rf, we may transform $\mathcal{D}_2$ into a derivation $\mathcal{D}_2'$ for $x : \varphi, \Delta \Rightarrow x : \psi$. But then, by applying the cut rule to $\mathcal{D}_1$ and $\mathcal{D}_2'$, we also obtain a derivation for $\Gamma, \Delta \Rightarrow x : \psi$, as desired.

Next, we prove the admissibility of the rule flat. If $L \in \{\text{InqB}, \text{PID}\}$, the proof proceeds by an easy induction on the structure of α , which is $\vee$ -free. Proofs can be found in [17, Prop. 3.2.14] and [19, Prop. 4.6]. Suppose now that $L \in \{\text{InqB}_\vee, \text{PID}_\vee\}$. Furthermore, consider an arbitrary infinite sequence $u_1, u_2, u_3, \dots$ of variables from $\mathfrak{S}$. For every $k \geq 1$, let $\theta(k)$ be the label defined by $\theta(k) := u_1 \cup \dots \cup u_k$, where the union operator is assumed to be *left-associative*. So, for example, $\theta(4)$ stands for $((u_1 \cup u_2) \cup u_3) \cup u_4$. Using induction on $k \geq 1$, we first prove the admissibility of the following restricted rule:

$$\text{flat}[k] \frac{v \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha}{\Gamma \Rightarrow \Delta, \theta(k) : \alpha}$$

where α must be a classical formula and $v \in \mathfrak{S}$ must be fresh. For the base case, let $k = 1$ and suppose that $v \subseteq u_1, \Gamma \Rightarrow \Delta, v : \alpha$ is derivable in $\mathcal{G}(L)$. Using the admissibility of substitution, this yields a derivation for $u_1 \subseteq u_1, \Gamma \Rightarrow \Delta, u_1 : \alpha$. By an application of rf, it now follows that $\Gamma \Rightarrow \Delta, u_1 : \alpha$ is derivable.

For the inductive step, let $k > 1$ and suppose that the premise $v \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha$ of the rule flat[k] is derivable in the system $\mathcal{G}(L)$ by a proof tree $\mathcal{D}$. Using the admissibility of substitution, weakening, contraction and cut, we may then derive the conclusion of flat[k] in the following way:

$$\begin{array}{c} \mathcal{D} \\ \text{w}_L \frac{v \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha}{v \subseteq \theta(k), v \subseteq u_k, u_k \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha} \\ \text{tr} \frac{v \subseteq u_k, u_k \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha}{v \subseteq u_k, \Gamma \Rightarrow \Delta, v : \alpha} \\ \text{ur} \frac{(u_k/v) \frac{v \subseteq u_k, \Gamma \Rightarrow \Delta, v : \alpha}{u_k \subseteq u_k, \Gamma \Rightarrow \Delta, u_k : \alpha}}{\Gamma \Rightarrow \Delta, u_k : \alpha} \\ \text{rf} \frac{\Gamma \Rightarrow \Delta, u_k : \alpha}{\Gamma \Rightarrow \Delta, u_k : \alpha} \\ \text{cut} \frac{\Gamma \Rightarrow \Delta, u_k : \alpha}{\Gamma \Rightarrow \Delta, u_k : \alpha} \end{array} \quad \begin{array}{c} \mathcal{D} \\ \text{w}_L \frac{v \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha}{v \subseteq \theta(k), v \subseteq \theta(k-1), \theta(k-1) \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha} \\ \text{tr} \frac{v \subseteq \theta(k-1), \theta(k-1) \subseteq \theta(k), \Gamma \Rightarrow \Delta, v : \alpha}{v \subseteq \theta(k-1), \Gamma \Rightarrow \Delta, v : \alpha} \\ \text{ur} \frac{v \subseteq \theta(k-1), \Gamma \Rightarrow \Delta, v : \alpha}{\Gamma \Rightarrow \Delta, \theta(k-1) : \alpha} \\ \text{IH} \frac{\Gamma \Rightarrow \Delta, \theta(k-1) : \alpha}{\Gamma \Rightarrow \Delta, \theta(k-1) : \alpha} \\ \text{cut} \frac{\Gamma \Rightarrow \Delta, \theta(k-1) : \alpha}{\Gamma \Rightarrow \Delta, \theta(k-1) : \alpha} \end{array} \quad \begin{array}{c} \text{Lemma 5.2} \\ \overline{\theta(k-1) : \alpha}, \overline{u_k : \alpha} \Rightarrow \overline{\theta(k) : \alpha} \\ \text{cut} \frac{\Gamma \Rightarrow \Delta, \theta(k-1) : \alpha}{\Gamma \Rightarrow \Delta, \theta(k-1) : \alpha} \end{array}$$

where “IH” denotes an application of the induction hypothesis. This concludes the induction. In order to prove the admissibility of the unrestricted rule flat, suppose now that the sequent $v \subseteq \pi, \Gamma \Rightarrow \Delta, v : \alpha$ is derivable in $\mathcal{G}(L)$ by a proof tree $\mathcal{D}$, where $\pi \in \text{Lab}_\cap$ is an arbitrary label and $v \in \mathfrak{S}$ is a fresh variable not occurring in $\Gamma \Rightarrow \Delta, \pi : \alpha$. Moreover, let $n := 2^{|\text{P}|}$ and let $u_1, \dots, u_n$ be a collection of n pairwise distinct variables with $v \notin \{u_1, \dots, u_n\}$ such that $u_1, \dots, u_n$ also do not occur in $\Gamma \Rightarrow \Delta, \pi : \alpha$. Finally, let $\xi(n)$ be the label given by $\xi(n) := u_1 \cup \dots \cup u_n$, where the union operator is again assumed to be left-associative. Using the admissibility of flat[n], we may then derive the conclusion of flat as follows:

$$\begin{array}{c} \mathcal{D} \\ \text{w}_L \frac{v \subseteq \pi, \Gamma \Rightarrow \Delta, v : \alpha}{v \subseteq \pi, v \subseteq \xi(n), \pi = \xi(n), \Gamma \Rightarrow \Delta, v : \alpha} \\ \text{tr} \frac{v \subseteq \xi(n), \pi = \xi(n), \Gamma \Rightarrow \Delta, v : \alpha}{\pi = \xi(n), \Gamma \Rightarrow \Delta, \xi(n) : \alpha} \\ \text{flat}[n] \frac{\pi = \xi(n), \Gamma \Rightarrow \Delta, \xi(n) : \alpha}{\pi = \xi(n), \Gamma \Rightarrow \Delta, \xi(n) : \alpha} \\ \text{cut} \frac{\pi = \xi(n), \Gamma \Rightarrow \Delta, \xi(n) : \alpha}{\pi = \xi(n), \Gamma \Rightarrow \Delta, \xi(n) : \alpha} \end{array} \quad \begin{array}{c} \text{Lemma 3.4 (1)} \\ \overline{\pi = \xi(n)}, \overline{\xi(n) : \alpha} \Rightarrow \overline{\pi : \alpha} \\ \text{cut} \frac{\pi = \xi(n), \Gamma \Rightarrow \Delta, \xi(n) : \alpha}{\pi = \xi(n), \Gamma \Rightarrow \Delta, \pi : \alpha} \end{array}$$

where the notation “ $\sigma = \tau$ ” is used as an abbreviation for the pair of relational atoms “ $\sigma \subseteq \tau, \tau \subseteq \sigma$ ”. $\square$

In the case $L \in \{\text{InqB}_\vee, \text{PID}_\vee\}$, our admissibility proof for flat crucially uses the rule fin and thus relies on the assumption that P is finite. Using the admissibility of the rules mp and flat, we now prove that each of our calculi is complete with respect to the corresponding Hilbert-style system from Figure 1.

Lemma 5.4. *Let L be any of the logics from Table 1, let $\mathcal{H}(L)$ be the Hilbert-style system and let $\mathcal{G}(L)$ be the labelled sequent calculus for L . If $\Gamma \vdash \varphi$ holds in $\mathcal{H}(L)$, then $\Gamma \vdash \varphi$ holds in $\mathcal{G}(L)$.*

Proof. By induction on the length of a proof for $\Gamma \vdash \varphi$ in the Hilbert system $\mathcal{H}(L)$. The inductive step is easy, since we already know that modus ponens is admissible in $\mathcal{G}(L)$ (Lemma 5.3). Hence, it suffices to show that all axioms of $\mathcal{H}(L)$ are provable in $\mathcal{G}(L)$. A few cases are considered in Appendix C. $\square$

Theorem 5.5 (Soundness and Completeness). *Let L be any of the logics from Table 1. The labelled sequent calculus $\mathcal{G}(L)$ is sound and complete with respect to L . That is, for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(L)$, it is the case that $\Gamma \vdash \varphi$ holds in $\mathcal{G}(L)$ if and only if $\Gamma \models_L \varphi$ holds in L .*

Proof. For the soundness direction, one may first use induction on the structure of a derivation to show that every sequent derivable in $\mathcal{G}(L)$ is valid. Soundness then follows as an immediate consequence (cf. [17, Prop. 3.3.4]). The completeness part follows directly from Theorem 2.2 and Lemma 5.4. $\square$

6 Proof Search

We now want to describe a terminating proof search algorithm for the logics presented in Section 2. To avoid certain complications arising from the complex syntax of the labels presented in Definition 3.1, we first introduce an alternative class of labelled sequent calculi for our logics, in which labels are defined as *sets of variables*.³ This eliminates the need for relational atoms and order rules, which simplifies the proof of termination significantly. Throughout this section, we will assume an infinite set $\mathfrak{V}$ of variables ranging over propositional *valuations*. Elements of $\mathfrak{V}$ are denoted by the meta-variables u and v .

Definition 6.1 (Labels). A *label* is a finite set of variables $\pi \subseteq \mathfrak{V}$.

A label of the form $\pi = \{v_1, \dots, v_k\}$ stands for the finite team consisting of the valuations represented by $v_1, \dots, v_k$ (which do not need to be distinct). Arbitrary labels are denoted by π, σ, τ . A *split* of a label π is a pair of labels (σ, τ) such that $\pi = \sigma \cup \tau$. The set of all splits of π is denoted by $S(\pi)$, so $S(\pi) := \{(\sigma, \tau) \mid \pi = \sigma \cup \tau\}$. A *labelled formula* is an expression of the form $\pi : \varphi$, where $\pi \subseteq \mathfrak{V}$ is a label and $\varphi \in \mathcal{L}$ is a formula. If $\pi = \{v\}$ is a singleton label, then we write $v : \varphi$ instead of $\pi : \varphi$. As before, $\pi : \varphi$ may be read as “ φ is satisfied by the team π ”. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of labelled formulas (thus, sequents do *not* contain relational atoms).

Definition 6.2 (Interpretation of Labels). An *interpretation* is a function $I : \mathfrak{V} \rightarrow \mathcal{V}(P)$ that assigns, to each variable $v \in \mathfrak{V}$, a valuation $I(v) \in \mathcal{V}(P)$. Such a function I is extended to labels by putting $I(\emptyset) := \emptyset$ as well as $I(\pi) := \{I(v_1), \dots, I(v_k)\}$ for every label $\pi = \{v_1, \dots, v_k\}$ with $k \geq 1$.

A labelled formula $\pi : \varphi$ is *satisfied* by an interpretation I , if $I(\pi) \models \varphi$. And a sequent $\Gamma \Rightarrow \Delta$ is *valid*, if for each interpretation I , we have: if I satisfies all elements of Γ , then I satisfies at least one element of Δ . The axioms and rules of our terminating calculi are presented in Figure 5. Note that, in the figure, we use square brackets to denote collections of premises. So, for example, an application of the rule at_R with principal formula $\pi : p$ now has exactly $|\pi|$ premises—namely, one premise for each variable $v \in \pi$. In the special case $\pi = \emptyset$, we treat at_R , $\perp_R$ and $=_R$ as *initial sequents* (i.e., rules with zero premises). Some of our rules are also subject to additional side conditions, which are indicated on the right-hand side of the “inference line”. For instance, the pair (σ, τ) must be a split of π in applications of $\vee_R$. The rules at_L , $\perp_L$, $\rightarrow_L$, $\vee_R$, $=_L$ are called *cumulative*, since their principal formulas are repeated in their premises.

³A very similar calculus for InqB was already presented by Litak and Sano [16, Remark 4]. In fact, their system essentially coincides with the calculus for InqB introduced in Definition 6.3 below. The only noteworthy differences are that Litak and Sano use slightly different axioms and require labels to be non-empty.

$$\begin{array}{c}
\text{ax} \frac{}{v : p, \Gamma \Rightarrow \Delta, v : p} \quad \text{ax}_{\perp} \frac{}{v : \perp, \Gamma \Rightarrow \Delta} \quad \text{at}_L \frac{v : p, \pi : p, \Gamma \Rightarrow \Delta}{\pi : p, \Gamma \Rightarrow \Delta} (v \in \pi) \quad \text{at}_R \frac{[\Gamma \Rightarrow \Delta, v : p]_{v \in \pi}}{\Gamma \Rightarrow \Delta, \pi : p} (|\pi| \neq 1) \\
\perp_L \frac{v : \perp, \pi : \perp, \Gamma \Rightarrow \Delta}{\pi : \perp, \Gamma \Rightarrow \Delta} (v \in \pi) \quad \perp_R \frac{[\Gamma \Rightarrow \Delta, v : \perp]_{v \in \pi}}{\Gamma \Rightarrow \Delta, \pi : \perp} (|\pi| \neq 1) \quad \wedge_L \frac{\pi : \varphi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \wedge_R \frac{\Gamma \Rightarrow \Delta, \pi : \varphi \quad \Gamma \Rightarrow \Delta, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \wedge \psi} \\
\vee_L \frac{\pi : \varphi, \Gamma \Rightarrow \Delta \quad \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta} \quad \vee_R \frac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi} \quad \rightarrow_L \frac{\pi : \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta, \sigma : \varphi}{\pi : \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta} \quad \rightarrow_R \frac{\pi : \varphi \rightarrow \psi, \sigma : \psi, \Gamma \Rightarrow \Delta}{\sigma \subseteq \pi} (\sigma \subseteq \pi) \\
\rightarrow_R \frac{[\sigma : \varphi, \Gamma \Rightarrow \Delta, \sigma : \psi]_{\sigma \subseteq \pi}}{\Gamma \Rightarrow \Delta, \pi : \varphi \rightarrow \psi} \quad \vee_L \frac{[\sigma : \varphi, \tau : \psi, \Gamma \Rightarrow \Delta]_{(\sigma, \tau) \in S(\pi)}}{\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta} \quad \vee_R \frac{\Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \sigma : \varphi \quad \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \tau : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi} (\pi = \sigma \cup \tau) \\
=_L \frac{u : p, v : p, \pi : =(p), \Gamma \Rightarrow \Delta \quad \pi : =(p), \Gamma \Rightarrow \Delta, u : p, v : p}{\pi : =(p), \Gamma \Rightarrow \Delta} (u, v \in \pi) \quad =_R \frac{[u : p, \Gamma \Rightarrow \Delta, v : p]_{(u, v) \in \pi \times \pi}}{\Gamma \Rightarrow \Delta, \pi : =(p)}
\end{array}$$

Figure 5: Axioms and rules of our terminating labelled sequent calculi.

Definition 6.3. Let L be any of the logics from Table 1 and let C be the corresponding set of connectives (see the second column of Table 1). We write $\mathcal{G}^*(L)$ for the labelled sequent calculus consisting of the initial sequents (ax and $\text{ax}_{\perp}$), the rules for atoms (at_L and at_R) and the left and right rules for the connectives in C (see Figure 5).⁴ For any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(L)$, we also write $\Gamma \vdash \varphi$ and say that φ is *provable* from Γ in $\mathcal{G}^*(L)$, if there exists a finite subset $\Delta \subseteq \Gamma$ and a label $\pi \subseteq \mathfrak{V}$ with $|\pi| = 2^{|\mathcal{P}|}$ such that the sequent $\pi : \Delta \Rightarrow \pi : \varphi$ is derivable in $\mathcal{G}^*(L)$, where $(\pi : \Delta) := \{\pi : \psi \mid \psi \in \Delta\}$.

Lemma 6.4 (Soundness). *Let L be one of the logics from Table 1. The calculus $\mathcal{G}^*(L)$ is sound with respect to L . That is, for every $\Gamma \cup \{\varphi\} \in \mathcal{L}(L)$, if $\Gamma \vdash \varphi$ holds in $\mathcal{G}^*(L)$, then $\Gamma \models_L \varphi$ holds in L .*

Proof. Using induction on the structure of a derivation, one can first show that every sequent derivable in $\mathcal{G}^*(L)$ is valid. Suppose now that $\Gamma \vdash \varphi$ holds in $\mathcal{G}^*(L)$, i.e., there exists a finite subset $\Delta \subseteq \Gamma$ and a label $\pi = \{v_1, \dots, v_n\}$ with $n = 2^{|\mathcal{P}|}$ such that $\pi : \Delta \Rightarrow \pi : \varphi$ is derivable in $\mathcal{G}^*(L)$. By what was said above, $\pi : \Delta \Rightarrow \pi : \varphi$ must then be valid. Let $T \subseteq \mathcal{V}(\mathcal{P})$ be an arbitrary team and assume that $T \models \psi$ for all $\psi \in \Gamma$. If $T = \emptyset$, then we trivially have $T \models \varphi$. Thus, let us suppose that $T \neq \emptyset$. Since $|\mathcal{V}(\mathcal{P})| = 2^{|\mathcal{P}|}$, we know that T is of the form $T = \{u_1, \dots, u_k\}$ for some k with $1 \leq k \leq 2^{|\mathcal{P}|}$. Let now I be the interpretation given by $I(v_i) := u_i$ for all i with $1 \leq i \leq k$, and $I(v_i) := u_1$ for all i with $k < i \leq 2^{|\mathcal{P}|}$. Then, clearly, we have $I(\pi) = T$. Hence, from $T \models \Gamma$ and $\Delta \subseteq \Gamma$, we obtain $I(\pi) \models \Delta$. By the validity of $\pi : \Delta \Rightarrow \pi : \varphi$, this yields $I(\pi) \models \varphi$ and thus $T \models \varphi$. Since T was an arbitrary team with $T \models \Gamma$, this shows that $\Gamma \models_L \varphi$. $\square$

Throughout the rest of this section, let L be any of the logics from Table 1. We will present an algorithm that performs root-first proof search in the calculus $\mathcal{G}^*(L)$. Our algorithm will construct a proof search tree for a given sequent by applying the rules of $\mathcal{G}^*(L)$ bottom-up to the topmost sequents in the tree. The rules will be applied exhaustively, as all possibilities need to be explored, but non-redundantly, to avoid repetitions. As a consequence, the search terminates within a finite number of steps, yielding either a proof or a *saturated branch* (Lemma 6.7). Intuitively, a branch is saturated if its topmost sequent is not an axiom and no further rules can be non-redundantly applied to it. We will then show that a saturated branch can always be used to define an interpretation that falsifies every sequent in the branch (Lemma 6.9). This will allow us to establish the completeness of our calculi. We start by introducing some terminology.

⁴So, for example, the calculus $\mathcal{G}^*(\text{InqB})$ consists of the initial sequents, the rules for atoms and the rules for $\perp, \wedge, \vee, \rightarrow$. The calculus $\mathcal{G}^*(\text{PID}_{\vee})$, on the other hand, consists of the initial sequents, the rules for atoms and the rules for $\perp, \wedge, \vee, \rightarrow, =(\cdot), \vee$.

Algorithm 1: An excerpt from the procedure $\text{PROOFSEARCH}(L)$ **Input:** A finite set of formulas $\Gamma \subseteq \mathcal{L}(L)$ and a formula $\varphi \in \mathcal{L}(L)$.**Output:** If $\Gamma \vdash \varphi$ holds in $\mathcal{G}^*(L)$, then the procedure outputs “success”. Otherwise, the procedure outputs “failure”.

```

1: Select a label  $\pi$  with  $|\pi| = 2^{|\mathcal{P}|}$  and initialize  $\mathfrak{T}$  as the proof search tree consisting only of the root node  $\pi : \Gamma \Rightarrow \pi : \varphi$ ;
2: while true do
3:   % root-first applications of  $\rightarrow_R$ 
4:   while  $\mathfrak{T}$  has an open branch  $B$  with a topmost sequent of the form  $\Pi \Rightarrow \Delta, \sigma : \psi \rightarrow \chi$  do
5:     | extend  $\mathfrak{T}$  by writing, for every subset  $\tau \subseteq \sigma$ , the sequent  $\tau : \psi, \Pi \Rightarrow \Delta, \tau : \chi$  on top of  $B$ ;
6:   % root-first applications of  $\rightarrow_L$ 
7:   while  $\mathfrak{T}$  has an open branch  $B$  with  $\sigma : \psi \rightarrow \chi \in L(B)$  and there exists some  $\tau \subseteq \sigma$  such that  $(\tau : \psi) \notin \downarrow\Delta_B$  and  $(\tau : \chi) \notin \downarrow\Gamma_B$  do
8:     | extend  $\mathfrak{T}$  by writing both  $L(B) \Rightarrow R(B), \tau : \psi$  and  $\tau : \chi, L(B) \Rightarrow R(B)$  on top of  $B$ ;
9:   % root-first applications of  $\vee_R$ 
10:  while  $\mathfrak{T}$  has an open branch  $B$  with  $(\sigma : \psi \vee \chi) \in R(B)$  and there exists some  $(\tau_1, \tau_2) \in S(\sigma)$  such that  $(\tau_1 : \psi) \notin \downarrow\Delta_B$  and  $(\tau_2 : \chi) \notin \downarrow\Delta_B$  do
11:    | extend  $\mathfrak{T}$  by writing both  $L(B) \Rightarrow R(B), \tau_1 : \psi$  and  $L(B) \Rightarrow R(B), \tau_2 : \chi$  on top of  $B$ ;
12:  % root-first applications of  $\vee_L$ 
13:  while  $\mathfrak{T}$  has an open branch  $B$  with  $(\sigma : \psi \vee \chi) \in L(B)$  and  $\exists u, v \in \sigma$  s.t.  $[(u : p) \notin \downarrow\Gamma_B \text{ or } (v : p) \notin \downarrow\Gamma_B]$  and  $[(u : p) \notin \downarrow\Delta_B \text{ or } (v : p) \notin \downarrow\Delta_B]$  do
14:    | extend  $\mathfrak{T}$  by writing both  $u : p, v : p, L(B) \Rightarrow R(B)$  and  $L(B) \Rightarrow R(B), u : p, v : p$  on top of  $B$ ;
15:  % ... add here similar while-loops for the other rules ...
16:  if  $\mathfrak{T}$  is closed then return “success”; %  $\mathfrak{T}$  is now a derivation for the root sequent
17:  if  $\mathfrak{T}$  has a saturated branch  $B$  then return “failure”; %  $B$  determines a counter-interpretation for the root sequent

```

Definition 6.5 (Proof Search Tree). A *proof search tree* is a finite tree of sequents $\mathfrak{T}$ such that (1) the root node of $\mathfrak{T}$ is of the form $\pi : \Gamma \Rightarrow \pi : \varphi$ for some label π with $|\pi| = 2^{|\mathcal{P}|}$, and (2) $\mathfrak{T}$ is built up from root-first applications of the rules of $\mathcal{G}^*(L)$. A *branch* in a proof search tree $\mathfrak{T}$ is a finite sequence B of consecutive sequents in $\mathfrak{T}$ such that the first sequent in B is the root node of $\mathfrak{T}$ and the last sequent in B is one of the leaf nodes of $\mathfrak{T}$. The last sequent in a branch B is also referred to as the *topmost sequent* of B .

A branch B is *closed* if the topmost sequent of B is an instance of an axiom (note that sequents of the form $\Gamma \Rightarrow \Delta, \emptyset : p$ or $\Gamma \Rightarrow \Delta, \emptyset : \perp$ or $\Gamma \Rightarrow \Delta, \emptyset : \equiv(p)$ now also count as axioms). A branch that is not closed is *open*. A proof search tree $\mathfrak{T}$ is closed if every branch in $\mathfrak{T}$ is closed. Given a branch B of the form $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n$, we also define $\downarrow\Gamma_B := \bigcup_{1 \leq i \leq n} \Gamma_i$ and $\downarrow\Delta_B := \bigcup_{1 \leq i \leq n} \Delta_i$, so $\downarrow\Gamma_B$ is the union of all the *antecedents* and $\downarrow\Delta_B$ is the union of all the *succedents* of the sequents in B . Moreover, we write $L(B) := \Gamma_n$ for the antecedent and $R(B) := \Delta_n$ for the succedent of the *topmost sequent* of B .

Definition 6.6 (Saturated Branch). A branch B is *saturated*, if B is open and each of the following holds:

1. for $\alpha \in \mathcal{P} \cup \{\perp\}$: if $(\pi : \alpha) \in \downarrow\Gamma_B$, then also $(v : \alpha) \in \downarrow\Gamma_B$ for all $v \in \pi$;
2. for $\alpha \in \mathcal{P} \cup \{\perp\}$: if $(\pi : \alpha) \in \downarrow\Delta_B$ and $\pi \neq \emptyset$, then $(v : \alpha) \in \downarrow\Delta_B$ for some $v \in \pi$;
3. if $(\pi : \varphi \wedge \psi) \in \downarrow\Gamma_B$, then both $(\pi : \varphi) \in \downarrow\Gamma_B$ and $(\pi : \psi) \in \downarrow\Gamma_B$;
4. if $(\pi : \varphi \wedge \psi) \in \downarrow\Delta_B$, then $(\pi : \varphi) \in \downarrow\Delta_B$ or $(\pi : \psi) \in \downarrow\Delta_B$;
5. if $(\pi : \varphi \vee \psi) \in \downarrow\Gamma_B$, then $(\pi : \varphi) \in \downarrow\Gamma_B$ or $(\pi : \psi) \in \downarrow\Gamma_B$;
6. if $(\pi : \varphi \vee \psi) \in \downarrow\Delta_B$, then both $(\pi : \varphi) \in \downarrow\Delta_B$ and $(\pi : \psi) \in \downarrow\Delta_B$;
7. if $(\pi : \varphi \rightarrow \psi) \in \downarrow\Gamma_B$ and $\sigma \subseteq \pi$, then $(\sigma : \varphi) \in \downarrow\Delta_B$ or $(\sigma : \psi) \in \downarrow\Gamma_B$;
8. if $(\pi : \varphi \rightarrow \psi) \in \downarrow\Delta_B$, then $(\sigma : \varphi) \in \downarrow\Gamma_B$ and $(\sigma : \psi) \in \downarrow\Delta_B$ for some $\sigma \subseteq \pi$;
9. if $(\pi : \varphi \vee \psi) \in \downarrow\Gamma_B$, then $(\sigma : \varphi) \in \downarrow\Gamma_B$ and $(\tau : \psi) \in \downarrow\Gamma_B$ for some split $(\sigma, \tau) \in S(\pi)$;
10. if $(\pi : \varphi \vee \psi) \in \downarrow\Delta_B$ and $(\sigma, \tau) \in S(\pi)$, then $(\sigma : \varphi) \in \downarrow\Delta_B$ or $(\tau : \psi) \in \downarrow\Delta_B$;
11. if $(\pi : \equiv(p)) \in \downarrow\Gamma_B$ and $u, v \in \pi$, then $[(u : p) \in \downarrow\Gamma_B, (v : p) \in \downarrow\Gamma_B]$ or $[(u : p) \in \downarrow\Delta_B, (v : p) \in \downarrow\Delta_B]$;
12. if $(\pi : \equiv(p)) \in \downarrow\Delta_B$ and $\pi \neq \emptyset$, then $(u : p) \in \downarrow\Gamma_B$ and $(v : p) \in \downarrow\Delta_B$ for some $(u, v) \in \pi \times \pi$.

We are now ready to describe the desired proof search procedure for the calculus $\mathcal{G}^*(L)$. An excerpt from the pseudocode for this procedure is presented in Algorithm 1. Given a finite set $\Gamma \subseteq \mathcal{L}(L)$ and a formula $\varphi \in \mathcal{L}(L)$ as input, our algorithm first initializes $\mathfrak{T}$ as the proof search tree consisting only of the root node $\pi : \Gamma \Rightarrow \pi : \varphi$, where π is a label with $|\pi| = 2^{|\mathcal{P}|}$ (line 1). Afterwards, the procedure successively extends $\mathfrak{T}$ by applying the rules of $\mathcal{G}^*(L)$ root-first in all possible ways to the topmost sequents in $\mathfrak{T}$. The corresponding while-loops for $\rightarrow_R, \rightarrow_L, \vee_R, \vee_L$ are displayed in lines 3–14 (the while-loops for the other

rules are similar). In applications of the cumulative rules at_L , $\perp_L$, $\rightarrow_L$, $\vee_R$, $=_L$, we use a suitable loop-checking mechanism to prevent the algorithm from producing infinitely many copies of “equivalent” rule applications. For example, before performing a root-first application of $\vee_R$ with principal formula $\sigma : \psi \vee \chi$ for a split $(\tau_1, \tau_2) \in S(\sigma)$ in a branch B , we first check whether $(\tau_1 : \psi) \in \downarrow\Delta_B$ or $(\tau_2 : \chi) \in \downarrow\Delta_B$. If one of these holds, the application is taken to be redundant and not performed in B . Otherwise, the rule is applied in the usual way (lines 10–11). The corresponding loop-checking conditions for $\rightarrow_L$ and $=_L$ are displayed in lines 7 and 13, respectively. The expansion of $\mathfrak{T}$ is stopped, if either $\mathfrak{T}$ becomes closed or a saturated branch is found (lines 16–17). In the first case, the procedure outputs “success”, as $\mathfrak{T}$ is now a derivation for the root sequent. In the second case, the output is “failure”, since a saturated branch will always allow us to construct an interpretation that falsifies the root sequent (see Lemma 6.9).

Lemma 6.7. *Our proof search procedure for $\mathcal{G}^*(L)$ terminates on every input (Γ, φ) , where Γ is finite.*

Proof. Suppose that the algorithm is executed on some finite input (Γ, φ) , so it successively constructs a proof search tree $\mathfrak{T}$ whose root sequent is of the form $\pi : \Gamma \Rightarrow \pi : \varphi$, where $|\pi| = 2^{|\mathcal{P}|}$. By inspection of the rules from Figure 5, we can see that each labelled formula $\sigma : \psi$ that occurs in $\mathfrak{T}$ satisfies the following two conditions: (1) ψ is a subformula of some formula in $\Gamma \cup \{\varphi\}$, and (2) σ is a subset of the label π . Since there are only finitely many such labelled formulas (and since we do not allow redundant applications of cumulative rules), it follows that only a finite number of rule applications is possible in the tree. Hence, every branch becomes either closed or saturated after a finite number of steps. $\square$

We will now prove that, from a saturated branch B , one can always extract an interpretation I_B such that all elements of $\downarrow\Gamma_B$ are satisfied and all elements of $\downarrow\Delta_B$ are not satisfied by I_B . This shows that our proof search procedure is in fact *exhaustive*: if the algorithm does not find a proof for the root sequent $\pi : \Gamma \Rightarrow \pi : \varphi$ of the search tree, then this sequent can always be falsified by some interpretation.

Definition 6.8. Let B be a saturated branch. The interpretation $I_B : \mathfrak{V} \rightarrow \mathcal{V}(\mathcal{P})$ is defined as follows: for every $p \in \mathcal{P}$ and every $v \in \mathfrak{V}$, we put $I_B(v)(p) := 1$, if $(v : p) \in \downarrow\Gamma_B$, and we put $I_B(v)(p) := 0$ otherwise.

Lemma 6.9. *Let B be a saturated branch. For all $\varphi \in \mathcal{L}(L)$ and all labels $\pi \subseteq \mathfrak{V}$, the following holds:*

- if $(\pi : \varphi) \in \downarrow\Gamma_B$, then $I_B(\pi) \models \varphi$,
- if $(\pi : \varphi) \in \downarrow\Delta_B$, then $I_B(\pi) \not\models \varphi$.

Proof. Both claims are proved simultaneously by induction on φ . We only consider the following cases.

Case 1: $\varphi = p$ for some $p \in \mathcal{P}$. For the first part, assume that $(\pi : p) \in \downarrow\Gamma_B$ and let $u \in I_B(\pi)$ be an arbitrary valuation. By definition of $I_B(\pi)$, there must be some variable $v \in \pi$ such that $u = I_B(v)$. From $(\pi : p) \in \downarrow\Gamma_B$ and $v \in \pi$, it now follows $(v : p) \in \downarrow\Gamma_B$ by Definition 6.6 (1). Hence, by definition of I_B , we obtain $I_B(v)(p) = 1$ and thus $u(p) = 1$. Since $u \in I_B(\pi)$ was arbitrary, this shows that $I_B(\pi) \models p$.

For the second part, suppose that $(\pi : p) \in \downarrow\Delta_B$. Since B is open, we must have $\pi \neq \emptyset$ (otherwise, the topmost sequent of B would be an instance of at_R with zero premises and therefore an initial sequent). Thus, by Definition 6.6 (2), there is some $v \in \pi$ such that $(v : p) \in \downarrow\Delta_B$. As a consequence, we must have $(v : p) \notin \downarrow\Gamma_B$: otherwise, B would contain an instance of ax , which contradicts the assumption that B is open. By definition of I_B , it now follows $I_B(v)(p) = 0$. Since $I_B(v) \in I_B(\pi)$, this yields $I_B(\pi) \not\models p$.

Case 2: $\varphi = \psi \rightarrow \chi$. For the first part, suppose for a contradiction that $(\pi : \psi \rightarrow \chi) \in \downarrow\Gamma_B$ and $I_B(\pi) \not\models \psi \rightarrow \chi$, i.e., there is some team $T \subseteq I_B(\pi)$ with $T \models \psi$ and $T \not\models \chi$. Since $I_B(\pi)$ is finite, we can write T in the form $T = \{I_B(v_1), \dots, I_B(v_n)\}$ for some variables $v_1, \dots, v_n \in \pi$ with $n \geq 0$. Let σ be the label $\sigma := \{v_1, \dots, v_n\}$. We then have $T = I_B(\sigma)$ and $\sigma \subseteq \pi$. From $(\pi : \psi \rightarrow \chi) \in \downarrow\Gamma_B$ and $\sigma \subseteq \pi$, we obtain $(\sigma : \psi) \in \downarrow\Delta_B$ or $(\sigma : \chi) \in \downarrow\Gamma_B$ by Definition 6.6 (7). By the induction hypothesis, this yields $I_B(\sigma) \not\models \psi$ or $I_B(\sigma) \models \chi$, so it follows $T \not\models \psi$ or $T \models \chi$. But this is a contradiction to the fact that $T \models \psi$ and $T \not\models \chi$.

For the second part, assume that $(\pi : \psi \rightarrow \chi) \in \downarrow \Delta_B$. By Definition 6.6 (8), we then have $(\sigma : \psi) \in \downarrow \Gamma_B$ and $(\sigma : \chi) \in \downarrow \Delta_B$ for some $\sigma \subseteq \pi$. By the induction hypothesis, this implies $I_B(\sigma) \models \psi$ and $I_B(\sigma) \not\models \chi$. Moreover, since $\sigma \subseteq \pi$, we must have $I_B(\sigma) \subseteq I_B(\pi)$. Hence, it follows $I_B(\pi) \not\models \psi \rightarrow \chi$.

Case 3: $\phi = \psi \vee \chi$. For the first part, suppose $(\pi : \psi \vee \chi) \in \downarrow \Gamma_B$. By Definition 6.6 (9), we then have $(\sigma : \psi) \in \downarrow \Gamma_B$ and $(\tau : \chi) \in \downarrow \Gamma_B$ for some σ, τ with $\pi = \sigma \cup \tau$. By the induction hypothesis, this yields $I_B(\sigma) \models \psi$ and $I_B(\tau) \models \chi$. And since $\pi = \sigma \cup \tau$, we have $I_B(\pi) = I_B(\sigma) \cup I_B(\tau)$, so $I_B(\pi) \models \psi \vee \chi$.

For the second part, assume $(\pi : \psi \vee \chi) \in \downarrow \Delta_B$. Moreover, suppose for a contradiction that $I_B(\pi) \models \psi \vee \chi$, i.e., there are teams S and T with $I_B(\pi) = S \cup T$ such that $S \models \psi$ and $T \models \chi$. Since $I_B(\pi) = S \cup T$, there must be some labels σ and τ with $\pi = \sigma \cup \tau$ such that $S = I_B(\sigma)$ and $T = I_B(\tau)$. For these labels, we have $I_B(\sigma) \models \psi$ and $I_B(\tau) \models \chi$. Moreover, since $(\pi : \psi \vee \chi) \in \downarrow \Delta_B$ and $\pi = \sigma \cup \tau$, it follows $(\sigma : \psi) \in \downarrow \Delta_B$ or $(\tau : \chi) \in \downarrow \Delta_B$ by Definition 6.6 (10). By the induction hypothesis, this yields $I_B(\sigma) \not\models \psi$ or $I_B(\tau) \not\models \chi$. But this is a contradiction to the fact that $I_B(\sigma) \models \psi$ and $I_B(\tau) \models \chi$. Therefore, $I_B(\pi) \not\models \psi \vee \chi$.

Case 4: ϕ is of the form $=(p)$. For the first part, suppose for a contradiction that $(\pi : =(p)) \in \downarrow \Gamma_B$ and $I_B(\pi) \not\models =(p)$, i.e., there are valuations $u_1, u_2 \in I_B(\pi)$ such that $u_1(p) \neq u_2(p)$. Since $u_1, u_2 \in I_B(\pi)$, there must be variables $v_1, v_2 \in \pi$ with $u_1 = I_B(v_1)$ and $u_2 = I_B(v_2)$, so it follows $I_B(v_1)(p) \neq I_B(v_2)(p)$. Since $(\pi : =(p)) \in \downarrow \Gamma_B$ and $v_1, v_2 \in \pi$, we know by Definition 6.6 (11) that one of the following holds: (1) $(v_1 : p) \in \downarrow \Gamma_B$ and $(v_2 : p) \in \downarrow \Gamma_B$, or (2) $(v_1 : p) \in \downarrow \Delta_B$ and $(v_2 : p) \in \downarrow \Delta_B$. In either case, it follows $I_B(v_1)(p) = I_B(v_2)(p)$ by the induction hypothesis. But this is a contradiction to $I_B(v_1)(p) \neq I_B(v_2)(p)$.

For the second part, assume $(\pi : =(p)) \in \downarrow \Delta_B$. Since B is open, we must have $\pi \neq \emptyset$ (otherwise, the topmost sequent of B would be an instance of $=_R$ with zero premises). Thus, by Definition 6.6 (12), we have $(u : p) \in \downarrow \Gamma_B$ and $(v : p) \in \downarrow \Delta_B$ for some $u, v \in \pi$. By the induction hypothesis, this yields $I_B(u)(p) \neq I_B(v)(p)$. And since $u, v \in \pi$, we have $I_B(u), I_B(v) \in I_B(\pi)$, so it follows $I_B(\pi) \not\models =(p)$. $\square$

Theorem 6.10 (Completeness). *The calculus $\mathcal{G}^*(L)$ is sound and complete with respect to L . That is, for any finite subset $\Gamma \cup \{\phi\} \subseteq \mathcal{L}(L)$, it is the case that $\Gamma \vdash \phi$ holds in $\mathcal{G}^*(L)$ iff $\Gamma \models_L \phi$ holds in L .*

Proof. The soundness of $\mathcal{G}^*(L)$ has been established in Lemma 6.4. For the completeness direction, let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}(L)$ be finite and suppose that $\Gamma \not\vdash \phi$. Let $\mathfrak{T}$ be the tree constructed by our proof search algorithm for the input (Γ, ϕ) , so the root of $\mathfrak{T}$ is of the form $\pi : \Gamma \Rightarrow \pi : \phi$, where $|\pi| = 2^{|\mathcal{P}|}$. Since $\Gamma \not\vdash \phi$, we know that $\mathfrak{T}$ cannot be closed, so it must have a saturated branch B . Let I_B be the interpretation determined by B . By Lemma 6.9, all elements of $\downarrow \Gamma_B$ are satisfied and all elements of $\downarrow \Delta_B$ are not satisfied by I_B . Thus, in particular, we have $I_B(\pi) \models \psi$ for all $\psi \in \Gamma$ and $I_B(\pi) \not\models \phi$. Therefore, $\Gamma \not\models_L \phi$. $\square$

7 Conclusion

We presented two kinds of labelled sequent calculi for a family of team-based propositional logics. We first introduced the systems $\mathcal{G}(L)$, whose labels are strings of symbols representing arbitrary teams, their unions and intersections. The order rules of these calculi explicitly reflect the semantic reasoning on teams. We then defined the systems $\mathcal{G}^*(L)$, whose labels are finite sets of variables. This simpler framework made it possible to provide a terminating proof search procedure for the calculi.

The proof systems $\mathcal{G}^*(L)$ have the clear advantage of inducing a decision algorithm for the logics under consideration. The calculi $\mathcal{G}(L)$, on the other hand, seem to have two other advantages. First, each rule of $\mathcal{G}(L)$ has a fixed number of premises, whereas in $\mathcal{G}^*(L)$, the number of premises of certain rules depends on the size of the label associated with the principal formula. Thus, at least for the tensor-free systems, proofs in $\mathcal{G}(L)$ can be significantly shorter than proofs in $\mathcal{G}^*(L)$. Second, in the calculi $\mathcal{G}(L)$, one may use variables to reason about arbitrary teams (or arbitrary subsets of a given team), which more

closely mirrors the actual mathematical reasoning one would employ to show that a certain formula is satisfied by a team or that all formulas of a certain form are valid in the corresponding logic.

For future work, we intend to further simplify our calculi by circumventing the need for intersection labels, and by extending our proof systems for $\text{InqB}_\vee$ and $\text{PID}_\vee$ to languages with infinitely many atomic propositions. Moreover, we plan to develop labelled calculi for broader families of team-based logics. A primary goal is to find proof systems for downward-closed team logics *without* the intuitionistic implication $\rightarrow$, most notably for propositional dependence logic [29]. For such logics, only natural deduction systems [29] and deep-inference-style sequent calculi [4] are known. We also intend to investigate team logics with the union closure property [28, 30], for which no sequent-style proof systems have been developed up to now. Another natural step would be to extend our calculi to logics based on modal or intuitionistic variants of team semantics [9, 27]. Finally, we plan to use our calculi to establish meta-theoretical properties, such as syntactic interpolation, for the logics under consideration.

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A Proof of Theorem 4.2

1. *The weakening rules are hp-admissible in $\mathcal{G}(L)$.* This is proved by an easy induction on the height of a derivation for the premise $\Gamma \Rightarrow \Delta$ of the respective rule. More details can be found in [17, Prop. 3.2.7].

2. *All rules of $\mathcal{G}(L)$ are hp-invertible.* The hp-invertibility of the basic and special order rules follows immediately from the hp-admissibility of weakening. The same holds for the rules at_L , $\perp_L$, $\rightarrow_L$, $\vee_R$, $=_L$. For all other rules, we reason by induction on the height of a derivation for the conclusion of the rule. We only show the invertibility of the rule $\vee_L$ (the other cases are similar). Let $\mathcal{D}$ be a derivation for the sequent $\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta$, and let n be the height of $\mathcal{D}$.

If $n = 0$, then $\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta$ must be an axiom. Consequently, $\pi : \varphi \vee \psi$ cannot be principal, as it is not atomic. Thus, for all variables $x, y \in \mathfrak{A}$, the sequent $\pi \subseteq x \cup y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta$ is also an axiom.

If $n \geq 1$, we distinguish cases according to the last rule R applied in $\mathcal{D}$. If $\pi : \varphi \vee \psi$ is not principal in R , we apply the induction hypothesis to the premises of R (possibly with a renaming of eigenvariables), and apply R again. If $\pi : \varphi \vee \psi$ is principal, then R is an application of $\vee_L$. The premise of $\vee_L$ is of the form $\pi \subseteq x' \cup y', x' : \varphi, y' : \psi, \Gamma \Rightarrow \Delta$, where $x', y' \in \mathfrak{A}$ are fresh. This sequent is derivable by a derivation of height $n - 1$. By hp-substitution, we obtain a derivation of height $\leq n$ for $\pi \subseteq x \cup y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta$.

3. *The contraction rules are hp-admissible in $\mathcal{G}(L)$.* The hp-admissibility of both contraction rules is proved simultaneously, by induction on the height of a derivation $\mathcal{D}$ for the premise, and by distinguishing cases according to the last rule applied in $\mathcal{D}$. Let $\Gamma \Rightarrow \Delta$ be a sequent that contains a duplication of some relational atom or labelled formula E in the antecedent or in the succedent. Let $\mathcal{D}$ be a derivation for $\Gamma \Rightarrow \Delta$ in $\mathcal{G}(L)$ and let n be the height of $\mathcal{D}$. Using induction on n , we show that also the contracted version of $\Gamma \Rightarrow \Delta$ (in which the two occurrences of E are replaced by a single one) is derivable by a proof tree of height $\leq n$. If $\mathcal{D}$ is of height $n = 0$, then $\Gamma \Rightarrow \Delta$ is an axiom, so the contracted version of $\Gamma \Rightarrow \Delta$ is also an axiom and therefore derivable by a proof tree of height $n = 0$.

Suppose now that $n \geq 1$. If E is not principal in the last rule R applied in $\mathcal{D}$, we apply the induction hypothesis to the premises of R , and then apply R again. And if E is principal in R , we distinguish two cases.

Case 1: E is a relational atom. If only one of the two occurrences of E is principal in the last rule application R in $\mathcal{D}$, then we use the same argument as above. And if both occurrences of E are principal in R , then we use the *closure condition* introduced in Section 3.

Case 2: E is a labelled formula. In this case, we use the hp-invertibility of the logical rules of $\mathcal{G}(L)$. For example, if $\mathcal{D}$ ends with an application of $\vee_L$ for which E is principal, the premise of the rule is of the form $\pi \subseteq x \cup y, x : \varphi, y : \psi, \pi : \varphi \vee \psi, \Sigma \Rightarrow \Delta$, for some fresh $x, y \in \mathfrak{A}$. The height of the derivation of this sequent is $n - 1$. By hp-invertibility and hp-substitution, we obtain a derivation of height $\leq n - 1$ for the sequent $\pi \subseteq x \cup y, \pi \subseteq x \cup y, x : \varphi, x : \varphi, y : \psi, y : \psi, \Sigma \Rightarrow \Delta$. Using three applications of the inductive hypothesis (which do not increase the height) we obtain a derivation of height $\leq n - 1$ for $\pi \subseteq x \cup y, x : \varphi, y : \psi, \Sigma \Rightarrow \Delta$. An application of $\vee_L$ now yields a derivation of height $\leq n$ for $\pi : \varphi \vee \psi, \Sigma \Rightarrow \Delta$.

4. *The cut rule is admissible in $\mathcal{G}(L)$.* The proof works in the standard way (see, e.g., [23, Theorem 2.4.3]) and proceeds by induction on two parameters: the *height* of a cut rule application and the *rank* of the cut formula, which is a measure of the complexity of labelled formulas.⁵ The *height* of a cut rule application is the sum of the heights of the two subderivations ending with the premises of the cut. Our measure of the complexity of a labelled formula $\pi : \varphi$ takes into account both the complexity of φ and the complexity of the label π (this is needed to deal with cuts involving the rules $\text{at}_L, \text{at}_R, \perp_L, \perp_R$).

Formally, the *degree of a label* $\pi \in \text{Lab}_\cap$, notation $dg(\pi)$, is defined by setting $dg(\pi) := 0$, if $\pi \in \mathfrak{S}$, and $dg(\pi) := 1$, if $\pi \notin \mathfrak{S}$. The *degree of a formula* $\varphi \in \mathcal{L}(L)$, notation $dg(\varphi)$, is defined to be the number of occurrences of the connectives $\perp, \wedge, \vee, \rightarrow, =(\cdot), \vee$ in φ . The *rank of a labelled formula* is the pair of natural numbers given by $rk(\pi : \varphi) := (dg(\varphi), dg(\pi))$. As in [18], we assume that ranks of labelled formulas are ordered *lexicographically*. That is, we write $rk(\pi : \varphi) < rk(\sigma : \psi)$, if we either have $dg(\varphi) < dg(\psi)$, or we have both $dg(\varphi) = dg(\psi)$ and $dg(\pi) < dg(\sigma)$. It is easy to verify that, if π is a label with $\pi \notin \mathfrak{S}$, then $rk(v : \varphi) < rk(\pi : \varphi)$ for all variables $v \in \mathfrak{S}$ and all formulas $\varphi \in \mathcal{L}(L)$.

The cut-admissibility proof proceeds by a main induction on the rank of the cut formula and a subinduction on the height of the cut. We distinguish three main cases, depending on the two premises of the cut: (i) one of the two premises of the cut is an axiom; (ii) neither of the two premises is an axiom and the cut formula is not principal on one side; and (iii) neither of the two premises of the cut is an axiom and the cut formula is principal on both sides. We show only two subcases of case (iii); the other cases follow quite standardly, and a discussion for several of them can be found in [17, Theorem 3.2.13].

First, suppose that the cut formula is a propositional atom. In this case, the cut must be of the form

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\text{cut} \frac{\text{at}_R \frac{v \subseteq \pi, \Gamma \Rightarrow \Delta, v : p}{\Gamma \Rightarrow \Delta, \pi : p} \quad \text{at}_L \frac{u : p, u \subseteq \pi, \pi : p, \Pi \Rightarrow \Sigma}{u \subseteq \pi, \pi : p, \Pi \Rightarrow \Sigma}}{u \subseteq \pi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}}$$

where π is a label with $\pi \notin \mathfrak{S}$ and v is a fresh variable from $\mathfrak{S}$. Using the hp-admissibility of substitution and contraction, this application of the cut rule is converted into

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\text{cut} \frac{(u/v) \frac{v \subseteq \pi, \Gamma \Rightarrow \Delta, v : p}{u \subseteq \pi, \Gamma \Rightarrow \Delta, u : p} \quad \text{at}_R \frac{v \subseteq \pi, \Gamma \Rightarrow \Delta, v : p}{\Gamma \Rightarrow \Delta, \pi : p} \quad \text{cut} \frac{u : p, u \subseteq \pi, \pi : p, \Pi \Rightarrow \Sigma}{u : p, u \subseteq \pi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}}{c_L, c_R \frac{u \subseteq \pi, u \subseteq \pi, \Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Sigma}{u \subseteq \pi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}}}$$

where c_L, c_R denote multiple applications of the contraction rules. The new cut on $\pi : p$ is of smaller height than the original one, and the cut on $u : p$ is of smaller rank (since $u \in \mathfrak{S}$ and $\pi \notin \mathfrak{S}$). Next, suppose that the cut formula is of the form $\pi : \varphi \vee \psi$. Our derivation then has the following shape:

⁵By the *cut formula*, we mean the labelled formula $\pi : \varphi$ displayed in the presentation of the cut rule in Figure 3.

$$\begin{array}{c}
\mathcal{D}'_1 \quad \mathcal{D}''_1 \quad \mathcal{D}_2 \\
\frac{\pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \sigma : \varphi}{\pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi} \quad \frac{\pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \tau : \psi}{\pi \subseteq \sigma \cup \tau, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \quad \frac{\pi \subseteq x \cup y, x : \varphi, y : \psi, \Pi \Rightarrow \Sigma}{\pi : \varphi \vee \psi, \Pi \Rightarrow \Sigma} \\
\text{cut}
\end{array}$$

Without loss of generality, we may assume that the eigenvariables x and y do not occur in the labels σ and τ (if this condition is not satisfied, we simply apply the hp-admissibility of substitution). We construct the following derivation, where $\mathcal{E}$, the subderivation for the sequent $\pi \subseteq \sigma \cup \tau, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \tau : \psi$, is constructed in a similar way as the subderivation for $\pi \subseteq \sigma \cup \tau, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \sigma : \varphi$ displayed below:

$$\begin{array}{c}
\mathcal{D}'_1 \quad \mathcal{D}_2 \quad \mathcal{D}_2 \\
\frac{\pi \subseteq \sigma \cup \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \sigma : \varphi}{\pi \subseteq \sigma \cup \tau, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \sigma : \varphi} \quad \frac{\pi \subseteq x \cup y, x : \varphi, y : \psi, \Pi \Rightarrow \Sigma}{\pi : \varphi \vee \psi, \Pi \Rightarrow \Sigma} \quad \frac{(\tau/y) \pi \subseteq x \cup y, x : \varphi, y : \psi, \Pi \Rightarrow \Sigma}{(\sigma/x) \pi \subseteq \sigma \cup \tau, \sigma : \varphi, \tau : \psi, \Pi \Rightarrow \Sigma} \\
\text{cut} \quad \text{cut} \quad \text{cut} \\
\frac{\pi \subseteq \sigma \cup \tau, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \tau : \psi}{\pi \subseteq \sigma \cup \tau, \pi \subseteq \sigma \cup \tau, \Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma} \quad \frac{\pi \subseteq \sigma \cup \tau, \pi \subseteq \sigma \cup \tau, \Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma}{\pi \subseteq \sigma \cup \tau, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \\
\text{cut} \quad \text{c}_L, \text{c}_R
\end{array}$$

The cut on $\pi : \varphi \vee \psi$ has smaller height than the original one, and is thus justified by the inductive hypothesis. Another such cut is contained in the subderivation $\mathcal{E}$. The cut on $\sigma : \varphi$ and the one on $\tau : \psi$ have smaller rank than the original cut, and are also justified by the inductive hypothesis. $\square$

B Proof of Lemma 5.1

Let L be any of the logics from Table 1 and let $\mathcal{G}(L)$ be the corresponding labelled calculus from Figure 2. We only prove the admissibility of the rule aux_1 (the admissibility proof for aux_2 works in the same way). Suppose that the premise $\pi \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \Gamma \Rightarrow \Delta$ is derivable in $\mathcal{G}(L)$. Using the admissibility of weakening, we can then derive the conclusion of aux_1 in the following way:

$$\begin{array}{c}
\frac{\pi \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \Gamma \Rightarrow \Delta}{\dots, \pi \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \Gamma \Rightarrow \Delta} \text{w}_L \\
\text{tr} \quad \frac{\dots, \pi \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \Gamma \Rightarrow \Delta}{\dots, \sigma \cup \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta} \\
\text{ul} \quad \frac{\dots, \sigma \cup \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta}{\dots, \sigma \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta} \\
\text{tr} \quad \frac{\dots, \sigma \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta}{\dots, \sigma \cup \tau \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta} \\
\text{ur} \quad \frac{\dots, \sigma \cup \tau \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta}{\dots, \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta} \\
\text{tr} \quad \frac{\dots, \sigma' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta}{\sigma' \subseteq \sigma' \cup \tau', \sigma' \cup \tau' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta} \\
\text{ur} \quad \frac{\sigma' \subseteq \sigma' \cup \tau', \sigma' \cup \tau' \subseteq (\sigma \cup \tau) \cup (\sigma' \cup \tau'), \pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta}{\pi \subseteq \sigma \cup \sigma', \Gamma \Rightarrow \Delta}
\end{array}$$

C Supplement to the Proof of Lemma 5.4

We need to show that all axioms of the Hilbert system $\mathcal{H}(L)$ are provable in the labelled calculus $\mathcal{G}(L)$ (see Definition 3.3). For the sake of brevity, we only consider the case $L = \text{PID}_\vee$ (the other cases are treated in the same way). Showing the provability of the schemes (A1)–(A7), (Intro), (Dis), (C) is straightforward. The scheme (S) can be derived in the same way as in [17, Lemma 3.3.5] and [18, p. 137]. And for the schemes (DN), (Com), (Mon), (Elim), we can construct the following derivations in $\text{GPID}_\vee$:

$$\begin{array}{c}
\text{Lemma 3.4 (2)} \quad \text{Lemma 3.4 (1)} \\
\frac{z \subseteq \emptyset, \dots \Rightarrow v : \alpha, z : \perp}{\rightarrow_R} \quad \frac{v \subseteq z, \dots, z : \alpha \Rightarrow v : \alpha, z : \perp}{\rightarrow_L} \\
\frac{z \subseteq v, v \subseteq y, y \subseteq x, y : \neg \alpha, z : \alpha \Rightarrow v : \alpha, z : \perp}{\rightarrow_R} \quad \frac{v \subseteq y, y \subseteq x, y : \neg \alpha \Rightarrow v : \alpha, v : \neg \alpha}{\rightarrow_L} \quad \text{ax}_\perp \quad \frac{v : \perp, \dots \Rightarrow v : \alpha}{\rightarrow_R} \\
\text{flat} \quad \frac{v \subseteq y, y \subseteq x, y : \neg \alpha \Rightarrow v : \alpha}{\rightarrow_R} \quad \frac{y \subseteq x, y : \neg \alpha \Rightarrow y : \alpha}{\rightarrow_R} \\
\Rightarrow x : \neg \alpha \rightarrow \alpha
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\text{Lemma 3.4 (4)}}{\dots, z_2 : \Psi \Rightarrow \dots, z_2 : \Psi} \quad \frac{\text{Lemma 3.4 (4)}}{\dots, z_1 : \Phi \Rightarrow \dots, z_1 : \Phi}}{\vee_R \dots, y \subseteq z_2 \cup z_1, z_1 : \Phi, z_2 : \Psi \Rightarrow y : \Psi \vee \Phi} \\
\text{tr} \frac{\dots, z_1 \cup z_2 \subseteq z_2 \cup z_1, y \subseteq z_1 \cup z_2, z_1 : \Phi, z_2 : \Psi \Rightarrow y : \Psi \vee \Phi}{\dots, z_1 \subseteq z_2 \cup z_1, z_2 \subseteq z_2 \cup z_1, y \subseteq z_1 \cup z_2, z_1 : \Phi, z_2 : \Psi \Rightarrow y : \Psi \vee \Phi} \\
\text{ul} \frac{\dots, z_1 \subseteq z_2 \cup z_1, z_2 \subseteq z_2 \cup z_1, y \subseteq z_1 \cup z_2, z_1 : \Phi, z_2 : \Psi \Rightarrow y : \Psi \vee \Phi}{\dots, z_2 \subseteq z_2 \cup z_1, y \subseteq z_1 \cup z_2, y \subseteq x, z_1 : \Phi, z_2 : \Psi \Rightarrow y : \Psi \vee \Phi} \\
\text{ur} \frac{\dots, z_2 \subseteq z_2 \cup z_1, y \subseteq z_1 \cup z_2, y \subseteq x, z_1 : \Phi, z_2 : \Psi \Rightarrow y : \Psi \vee \Phi}{\vee_L \frac{y \subseteq z_1 \cup z_2, y \subseteq x, z_1 : \Phi, z_2 : \Psi \Rightarrow y : \Psi \vee \Phi}{\rightarrow_R \frac{y \subseteq x, y : \Phi \vee \Psi \Rightarrow y : \Psi \vee \Phi}{\Rightarrow x : (\Phi \vee \Psi) \rightarrow (\Psi \vee \Phi)}}} \\
\\
\frac{\frac{\text{Lemma 3.4 (1)}}{z \cap z_1 \subseteq z_1, \dots, z_1 : \Phi \Rightarrow \dots, z \cap z_1 : \Phi}}{\rightarrow_L \dots, z_1 : \Phi \Rightarrow \dots, z \cap z_1 : \Phi} \quad \frac{\frac{\text{Lemma 3.4 (4)}}{\dots, z \cap z_1 : \Psi \Rightarrow \dots, z \cap z_1 : \Psi} \quad \frac{\text{Lemma 3.4 (1)}}{z \cap z_2 \subseteq z_2, \dots, z_2 : \chi \Rightarrow \dots, z \cap z_2 : \chi}}{\vee_R \dots, z \subseteq (z \cap z_1) \cup (z \cap z_2), z \cap z_1 : \Psi, z_2 : \chi \Rightarrow z : \Psi \vee \chi} \\
\text{dis} \frac{\dots, z \subseteq (z \cap z_1) \cup (z \cap z_2), z \cap z_1 : \Psi, z_2 : \chi \Rightarrow z : \Psi \vee \chi}{\dots, z \subseteq z_1 \cup z_2, z \cap z_1 : \Psi, z_2 : \chi \Rightarrow z : \Psi \vee \chi} \\
\text{tr} \frac{\dots, z \cap z_1 \subseteq y, z \subseteq z_1 \cup z_2, y : \Phi \rightarrow \Psi, z_1 : \Phi, z_2 : \chi \Rightarrow z : \Psi \vee \chi}{z \cap z_1 \subseteq z, z \subseteq z_1 \cup z_2, z \subseteq y, y \subseteq x, y : \Phi \rightarrow \Psi, z_1 : \Phi, z_2 : \chi \Rightarrow z : \Psi \vee \chi} \\
\text{il} \frac{z \subseteq z_1 \cup z_2, z \subseteq y, y \subseteq x, y : \Phi \rightarrow \Psi, z_1 : \Phi, z_2 : \chi \Rightarrow z : \Psi \vee \chi}{\vee_L \frac{z \subseteq y, y \subseteq x, y : \Phi \rightarrow \Psi, z : \Phi \vee \chi \Rightarrow z : \Psi \vee \chi}{\rightarrow_R \frac{y \subseteq x, y : \Phi \rightarrow \Psi \Rightarrow y : (\Phi \vee \chi) \rightarrow (\Psi \vee \chi)}{\rightarrow_R \Rightarrow x : (\Phi \rightarrow \Psi) \rightarrow ((\Phi \vee \chi) \rightarrow (\Psi \vee \chi))}}} \\
\\
\frac{\frac{\text{Lemma 3.4 (1)}}{\dots, v \subseteq x_1, x_1 : \Phi \Rightarrow \dots, v : \Phi} \quad \frac{\text{Lemma 3.4 (4)}}{\dots, v : \alpha \Rightarrow v : \alpha}}{\rightarrow_L \dots, v \subseteq x_1, v \subseteq y, y : \Phi \rightarrow \alpha, x_1 : \Phi \Rightarrow v : \alpha} \quad \frac{\frac{\text{Lemma 3.4 (1)}}{\dots, v \subseteq x_2, x_2 : \Psi \Rightarrow \dots, v : \Psi} \quad \frac{\text{Lemma 3.4 (4)}}{\dots, v : \alpha \Rightarrow v : \alpha}}{\rightarrow_L \dots, v \subseteq x_2, v \subseteq z, z : \Psi \rightarrow \alpha, x_2 : \Psi \Rightarrow v : \alpha} \\
\text{cd} \frac{\dots, v \subseteq x_1, v \subseteq y, y : \Phi \rightarrow \alpha, x_1 : \Phi \Rightarrow v : \alpha \quad \dots, v \subseteq x_2, v \subseteq z, z : \Psi \rightarrow \alpha, x_2 : \Psi \Rightarrow v : \alpha}{\text{tr} \dots, v \subseteq x_1 \cup x_2, v \subseteq z, z \subseteq y, y : \Phi \rightarrow \alpha, z : \Psi \rightarrow \alpha, x_1 : \Phi, x_2 : \Psi \Rightarrow v : \alpha} \\
\text{tr} \frac{\dots, v \subseteq x_1 \cup x_2, v \subseteq z, z \subseteq y, y : \Phi \rightarrow \alpha, z : \Psi \rightarrow \alpha, x_1 : \Phi, x_2 : \Psi \Rightarrow v : \alpha}{\dots, v \subseteq x_1 \cup x_2, v \subseteq x_0, x_0 \subseteq z, z \subseteq y, y : \Phi \rightarrow \alpha, z : \Psi \rightarrow \alpha, x_1 : \Phi, x_2 : \Psi \Rightarrow v : \alpha} \\
\text{tr} \frac{\dots, v \subseteq x_1 \cup x_2, v \subseteq x_0, x_0 \subseteq z, z \subseteq y, y : \Phi \rightarrow \alpha, z : \Psi \rightarrow \alpha, x_1 : \Phi, x_2 : \Psi \Rightarrow v : \alpha}{\vee_L \frac{v \subseteq x_0, x_0 \subseteq z, z \subseteq y, y \subseteq x, y : \Phi \rightarrow \alpha, z : \Psi \rightarrow \alpha, x_0 : \Phi \vee \Psi \Rightarrow v : \alpha}{\text{flat} \frac{x_0 \subseteq z, z \subseteq y, y \subseteq x, y : \Phi \rightarrow \alpha, z : \Psi \rightarrow \alpha, x_0 : \Phi \vee \Psi \Rightarrow x_0 : \alpha}{\rightarrow_R \frac{z \subseteq y, y \subseteq x, y : \Phi \rightarrow \alpha, z : \Psi \rightarrow \alpha \Rightarrow z : (\Phi \vee \Psi) \rightarrow \alpha}{\rightarrow_R \frac{y \subseteq x, y : \Phi \rightarrow \alpha \Rightarrow y : (\Psi \rightarrow \alpha) \rightarrow ((\Phi \vee \Psi) \rightarrow \alpha)}{\rightarrow_R \Rightarrow x : (\Phi \rightarrow \alpha) \rightarrow ((\Psi \rightarrow \alpha) \rightarrow ((\Phi \vee \Psi) \rightarrow \alpha))}}}}}
\end{array}$$

Note that, in the derivations for (DN) and (Elim), we also use the admissible rule flat from Figure 4.

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Belief Contraction in Dynamic Epistemic Logic

Gaia Belardinelli

Department of Philosophy
Stanford University
Stanford, USA
gaiabel@stanford.edu

Snow Zhang

Department of Philosophy
University of California, Berkeley
Berkeley, USA
snowzhang@berkeley.edu

Dynamic epistemic logic (DEL) represents belief change via model transformations induced by epistemic events. Its standard formulation (Baltag, Moss, Solecki, 1998) provides a natural account of belief expansion through the elimination of possibilities, but it cannot model belief contraction about factual propositions. A classic response enriches Kripke models with plausibility orderings, representing contraction as an update that promotes certain possibilities over others. We show that this approach has expressive limitations. In particular, the approach cannot model belief that violates positive introspection and contraction dynamics in response to a *hedged* public announcement that φ *might* be false. Motivated by these considerations, we introduce a mechanism for belief contraction defined directly on standard Kripke models, without any constraints on the doxastic accessibility relation. We show that it satisfies some of the standard properties of belief contraction but not others, study the conditions under which contraction may be unsuccessful, and provide a sound and complete axiomatization of the logic via reduction axioms. We also define a more general dynamic logic that is an extension of standard DEL and accommodates belief contractions due to events such as private or semi-private announcements, and provide a complete and sound axiomatization of the general logic.

1 Introduction

Dynamic epistemic logic (DEL) is a family of logics that model multi-agent belief change by incorporating information revealed by epistemic events—such as public or private announcements—into agents’ epistemic states. In standard DEL [6], this incorporation proceeds by *eliminating* all possibilities that are incompatible with the disclosed information. As a result, standard DEL can’t straightforwardly model announcements that make agents reconsider possibilities they previously ruled out as impossible, as required for belief contraction. A popular solution is to distinguish “hard” from “soft” updates; while hard updates involve *eliminating* possibilities, soft updates involve *promoting* possibilities in terms of their comparative *plausibility* [12, 8]. Then, belief contraction about a factual proposition p consists in coming to judge that a $\neg p$ -possibility is at least as plausible as the most plausible p -possibility, thereby losing the belief that p holds.

While the plausibility-based approach can model belief contraction [8, 23], it has certain expressive limitations. First, since plausibility orderings are transitive, the framework validates positive introspection for beliefs (axiom 4), and so can’t model agents who are mistaken about their own beliefs. Second, the approach can’t accommodate belief contractions due to *hedged public announcements*—announcements that p *might be false*. Intuitively, such an announcement could cause an agent who initially believes p to suspend judgment on p without gaining any new (factual) beliefs. We show that there is a precise sense in which such dynamics *can’t* be modeled in the plausibility framework.

These limitations motivate the search for a new model of belief contraction. We offer such a model in this paper. We introduce a dynamic operation of belief contraction on standard, multi-agent Kripke

models of beliefs, with no assumptions on the doxastic accessibility relations. Roughly, a hedged announcement that φ *might be false* triggers a contraction on φ , which involves adding $\neg\varphi$ -possibilities to an agent's belief set if she believes φ , and doing nothing otherwise. We define a sound and complete logic for this contraction modality, which we call *hedged public announcement logic* (HPAL), and prove that this update satisfies a number of natural properties of belief contraction. We then define a more general dynamic logic GDEL that conservatively extends standard DEL to also accommodate belief contractions due to events such as private or semi-private announcements. We show that it contains HPAL and that its updates are always a simulation of a refinement of an initial Kripke model. Finally, we provide a complete and sound axiomatization of the general logic.

Related literature. The logic GDEL can be viewed as a natural extension of standard DEL. As contraction is a form of information-loss, our model is also related to logics of *forgetting* and simulation modal logic [18, 19, 22], as well as to the fragment of graph modifier logic that adds edges to Kripke models [5]. Our theory however contrasts with the theory of belief contraction as studied in AGM [1]; our contraction operator does not satisfy many of the AGM postulates. This is to be expected, as dynamic contraction involves transformations that could change the truth-values of epistemic formulas, which is not considered in AGM. Philosophically, the problem of belief contraction due to hedged public announcement is related to the problem of evaluating indicative conditionals that have modal antecedents, which remains an open problem [32, 27, 10].

The proofs of main theorems can be found in the Appendix.

2 Standard DEL framework

We begin by reviewing the basics of DEL and fix notations.¹ Let At be a countable set of atomic formulas and Ag be a finite set of agents. Let $\mathcal{L}$ be the *doxastic language* generated from At by the following grammar: $\varphi := \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B_a\varphi$, where $p \in At$ and $a \in Ag$. We write $\perp$ for $\neg\top$ and $\tilde{B}_a\varphi$ for $\neg B_a\neg\varphi$. A *literal* is an atomic formula p or its negation $\neg p$. Let $\bigwedge_{\varphi \in S} \varphi = \top$ if S is empty.

We adopt the standard semantics of epistemic logic, where a Kripke model is a triple $\mathcal{M} = (W, R, V)$ with $W \neq \emptyset$ a set of worlds, $R_a \subseteq (W \times W)$ an accessibility relation defined for all $a \in Ag$, and $V : At \rightarrow \wp(W)$ is a valuation function. We use notation $R_a(w) = \{v \in W : (w, v) \in R_a\}$ to represent *the set of possible worlds compatible with agent a 's belief at w* .

The satisfaction relation between a Kripke model $\mathcal{M} = (W, R, V)$ with world $w \in W$ and a formula $\varphi \in \mathcal{L}$ is defined as standard, where in particular the semantics of the belief modality is given by:

$$\mathcal{M}, w \models B_a\varphi \quad \text{iff} \quad \mathcal{M}, v \models \varphi \text{ for all } v \in R_a(w).$$

A formula φ is *valid in a model* $\mathcal{M}$ (notation: $\mathcal{M} \models \varphi$), if for all worlds w in $\mathcal{M}$, $\mathcal{M}, w \models \varphi$. A formula φ is *valid in a class of models* $\mathfrak{L}$ (notation: $\models_{\mathfrak{L}} \varphi$), if for all models $\mathcal{M}$ in $\mathfrak{L}$, $\mathcal{M} \models \varphi$.

A *standard event model* is a triple $\mathcal{E} = (E, Q, pre)$ where $E \neq \emptyset$ is a finite set of events, $Q_a \subseteq (E \times E)$ is an accessibility relation between events, for all $a \in Ag$, and $pre : E \rightarrow \mathcal{L}$ a precondition function.² We use notation $Q_a(e) = \{f \in E : (e, f) \in Q_a\}$ for the set of events that are considered possible by a at e .

For a Kripke model $\mathcal{M} = (W, R, V)$ and a standard event model $\mathcal{E} = (E, Q, pre)$, the *standard product update* of $\mathcal{M}$ with $\mathcal{E}$ is the Kripke model $\mathcal{M} \otimes \mathcal{E} = (W^{\mathcal{E}}, R^{\mathcal{E}}, V^{\mathcal{E}})$ where $W^{\mathcal{E}} = \{(w, e) \in W \times E : w \models$

¹For presentation purposes, in this section we only introduce the basic doxastic language without dynamic modalities. We introduce the full language of DEL in the last section, where we also use it. For an introduction to DEL, see [7].

²In the most general form of DEL, the precondition function assigns to each event a formula from the dynamic language (i.e. $\mathcal{L}$ extended with dynamic formulas). We keep it simple here and consider this general DEL form in the last section.

$pre(e)\}$, $R_a^\mathcal{E}(w, e) = \{(v, f) \in W^\mathcal{E} : v \in R_a(w) \text{ and } f \in Q_a(e)\}$, and $V^\mathcal{E}(p) = \{(w, e) \in W^\mathcal{E} : w \in V(p)\}$, for all $p \in At$. Informally, a world (w, e) represents *the state of affairs at w after e occurs*. We call these event models and product updates *standard* to distinguish them from the framework introduced later.

It is well known that standard DEL has undesirable consequences when agents are presented with information that contradicts their beliefs [12, 26]. For a concrete illustration:

Example 2.1. Alice believes that she locked her office door before she left. She runs into Bob, who tells her that *her office door was open*. Alice trusts Bob and revises her beliefs accordingly.

Let p denote the proposition “Alice’s office door is open”. Figure 1 illustrates that, if we model Alice’s belief revision using a standard product update, then she will end up having inconsistent beliefs.



Figure 1: **Left:** The Kripke model $\mathcal{M} = (W, R, V)$ representing Alice’s initial belief state. An edge from a world v to a world w means $w \in R_a(v)$. We have $\mathcal{M} \models B_a \neg p \wedge \neg B_a p$. **Middle:** The standard event model $\mathcal{E} = (E, Q, pre)$ representing Bob’s announcement. The event e contains its own precondition $pre(e) = p$. The loop on e means $e \in Q_a(e)$. **Right:** The Kripke model $\mathcal{M} \otimes \mathcal{E} = (W^\mathcal{E}, R^\mathcal{E}, V^\mathcal{E})$ representing Alice’s belief state after the update, where $R_a^\mathcal{E}((v, e)) = \emptyset$ and so e.g. $\mathcal{M} \otimes \mathcal{E} \models B_a p \wedge B_a \neg p$.

3 Plausibility models

The “standard diagnosis” of the problem of modeling belief contraction in standard DEL is that we need a richer view of beliefs [12]. In particular, we need a framework where an agent believes not just any proposition entailed by her information, but those that are true at the ‘best’ or ‘most plausible’ possibilities compatible with her information [8, 12]. I believe that I am not dreaming. It is not *impossible* that I am, but those possibilities are less *plausible* than those in which I am awake. Similarly, one interpretation of Alice’s initial doxastic situation is that she believes that her office is closed, not because she has completely ruled out the possibility that it is open, but only that she judges that possibility to be relatively implausible. And it is this relative plausibility judgment that gets revised by Bob’s testimony.

This idea is formalized by modeling belief in terms of plausibility orderings over possible worlds. Formally, a *plausibility order on a set S* is a connected, reflexive and transitive relation $\leq \subseteq S \times S$ such that every non-empty subset $S' \subseteq S$ has at least one $\leq$ -minimal element, i.e. $Min_{\leq}(S') = \{w \in S' : \forall v \in S', w \leq v\} \neq \emptyset$.³ A *plausibility model* is a Kripke model $(W, \leq, V)$ with $\leq_a$ a plausibility order on W , for all agents $a \in Ag$. We write $\prec_a$ for its asymmetric component, i.e. $w \prec_a v$ if $w \leq_a v$ and $v \not\leq_a w$. The semantics of $\mathcal{L}$ over plausibility models is standard for propositional formulas, but belief is interpreted differently from above, capturing that the agent believes what is the case at the most plausible worlds:

$$\mathcal{M}, w \models B_a \varphi \text{ iff } \mathcal{M}, v \models \varphi \text{ for all } v \in Min_{\leq_a}(W).$$

A *plausibility event model* is an event model $\mathcal{E} = (E, \leq, pre)$ with $\leq_a$ a plausibility order on E for all $a \in Ag$. We write $\prec_a$ for its asymmetric component. As in standard DEL, event models update plausibility models via product update. The *plausibility-update* of $\mathcal{M} = (W, \leq, V)$ with $\mathcal{E}$ is the plausibility model $\mathcal{M} * \mathcal{E} = (W^\mathcal{E}, \leq^\mathcal{E}, V^\mathcal{E})$ where $W^\mathcal{E}$ and $V^\mathcal{E}$ are defined as in standard product updates and $\leq^\mathcal{E}$ is given

³Note that we assume the plausibility order to be *connected*. This is more restrictive than usual [8], but it is justified here as our framework only models belief and omits knowledge.

by $(w, e) \preceq_a^\mathcal{E} (v, f)$ iff $e <_a f$, or $e \equiv_a f$ and $w \preceq_a v$. Figure 2 shows how to model Example 2.1 via a plausibility update. Alice's change of mind is captured as a *soft* update where no world is ruled out as impossible, but their relative plausibility is switched.

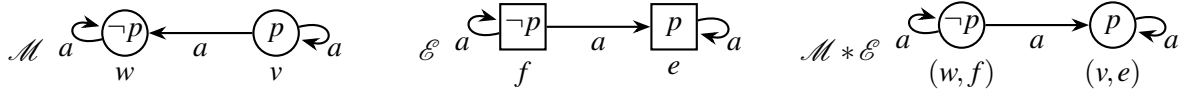


Figure 2: **Left:** The plausibility model $\mathcal{M} = (W, \preceq, V)$ representing Alice's initial belief state. An edge from v to w means $w \preceq_a v$. We have $\mathcal{M} \models B_a \neg p \wedge \neg B_a p$. **Middle:** The plausibility event model $\mathcal{E} = (E, \leq, pre)$ capturing a soft update that p holds. An edge from f to e means $e \leq_a f$. **Right:** The plausibility model $\mathcal{M} * \mathcal{E}$ representing Alice's belief state after updating, with $\mathcal{M} * \mathcal{E} \models B_a p \wedge \neg B_a \neg p$.

3.1 Limitations of plausibility models

One limitation of the plausibility model of beliefs is that it validates axiom 4: $B_a \varphi \rightarrow B_a B_a \varphi$. For this reason, it cannot model agents who are mistaken about their own beliefs. Such cases, however, seem possible: an employer might profess that they believe in gender equality while consistently favoring one gender group in hiring. One way of describing this employer is that they in fact believe that candidates from one gender group are more qualified, even though they (falsely) believe that they don't believe that.

Another limitation of the approach, mentioned above, concerns belief contraction that does not involve adopting any new factual beliefs. A variant of Example 2.1 illustrates what we have in mind:

Example 3.1. As before, Alice believes that she locked the door to her office before she left. She runs into Bob, who tells her that *her office door might be open*. Given Bob's testimony, Alice comes to suspend judgment on whether she locked the door to her office before she left.

Let p denote the proposition *Alice's office door is open*. Alice's belief states before and after the update can be modeled by $\mathcal{M}$ and $\mathcal{N}$ in Figure 3, respectively.



Figure 3: **Left:** The plausibility model $\mathcal{M} = (W, \preceq, V)$ representing Alice's initial belief state, where $\mathcal{M} \models B_a \neg p \wedge \neg B_a p$. Conventions for edges are as in Fig. 2. **Right:** The plausibility model $\mathcal{N} = (W, \preceq^\mathcal{N}, V)$ representing Alice's belief state after updating on Bob's testimony, where $\mathcal{N} \models \neg B_a p \wedge \neg B_a \neg p$.

However, it turns out that this update *cannot* be modeled as a plausibility update. More precisely, let $\mathcal{K} \equiv \mathcal{K}'$ denote modal equivalence between two models $\mathcal{K}$ and $\mathcal{K}'$ with respect to $\mathcal{L}$.⁴ Then:

Theorem 3.2. *There is no plausibility event model $\mathcal{E}$ such that $\mathcal{M} * \mathcal{E} \equiv \mathcal{N}$.*

Proof. Let $\mathcal{M} = (W, \preceq, V)$ and $\mathcal{N} = (W, \preceq^\mathcal{N}, V)$ be as depicted in Figure 3. Suppose towards a contradiction that there is a plausibility event model $\mathcal{E} = \{E, \leq, pre\}$ such that $\mathcal{M} * \mathcal{E} = (W^\mathcal{E}, \preceq^\mathcal{E}, V^\mathcal{E})$ is a plausibility model that is modally equivalent to $\mathcal{N}$. We claim that, for any $(v, f) \in W^\mathcal{E}$, there exists a $(v, f') \neq (v, f)$ such that $(v, f') \prec_a^\mathcal{E} (v, f)$. It follows then that $\mathcal{M} * \mathcal{E}$ contains an infinite descending

⁴Recall that two Kripke (or plausibility) models $\mathcal{K} = (W, R, V)$ and $\mathcal{K}' = (W', R', V')$ are *modally equivalent* with respect to $\mathcal{L}$ if for every formula $\varphi \in \mathcal{L}$ and every state $w \in W$, there exists a state $w' \in W'$ such that $\mathcal{K}, w \models \varphi$ iff $\mathcal{K}', w' \models \varphi$, and conversely, for every $w' \in W'$ there exists a state $w \in W$ with the same property.

chain, which contradicts the assumption that it is a plausibility model. Let $(v, f) \in W^{\mathcal{E}}$. By assumption, $(\mathcal{M} * \mathcal{E}, (v, f))$ is modally equivalent to $(\mathcal{N}, v)$. Since $(\mathcal{N}, v) \models \neg B_a p$, we have $(\mathcal{M} * \mathcal{E}, (v, f)) \models \neg B_a p$. So there exists $(w, e) \in W^{\mathcal{E}}$ such that $(w, e) \preceq_a^{\mathcal{E}} (v, f)$. For this (w, e) , it must be that $(\mathcal{M} * \mathcal{E}, (w, e))$ is modally equivalent to $(\mathcal{N}, w)$ (since it can't be modally equivalent to $(\mathcal{N}, v)$). Since $\mathcal{N}, w \models B_a p$, there exists $(v, f') \in W^{\mathcal{E}}$ such that $(v, f') \preceq_a^{\mathcal{E}} (w, e)$. Moreover, since $v \not\leq_a w$, we must have $f' <_a e \leq_a f$. So $f' <_a f$ and therefore $(v, f') \prec_a^{\mathcal{E}} (v, f)$. $\square$

Note that if two models are not modally equivalent with respect to $\mathcal{L}$, then they are not modally equivalent with respect to any extension of $\mathcal{L}$. So it follows from Theorem 3.2 that there is no plausibility event model $\mathcal{E}$ such that $\mathcal{M} * \mathcal{E}$ is modally equivalent to $\mathcal{N}$ with respect to $\mathcal{L}$ enriched with, e.g. modalities for conditional beliefs, safe beliefs or dynamic modalities. Relatedly, since modal equivalence is generally assumed to be a necessary condition for an adequate notion of bisimilarity, it also follows that there is no plausibility event model $\mathcal{E}$ such that $\mathcal{M} * \mathcal{E}$ is bisimilar to $\mathcal{N}$ with respect to any adequate notion of bisimilarity (see e.g. [16, 2, 3] for different notions of bisimilarity for plausibility structures).

4 A new model for belief contraction

Theorem 3.2 shows that plausibility updates can't weaken an agent's belief in a way that makes her consider the two worlds equiplausible. We now introduce a framework for belief contraction that can achieve that. In general, belief contraction can be caused by many different kinds of epistemic events. In this section, we focus on a simple case of belief contraction due to *hedged public announcements*, like Bob's testimony that Alice's door *might* be open.⁵ Formally, we represent the effects of such announcements by the dynamic modality $[\div \varphi] \psi$. Intuitively, it says that ψ is the case after the hedged public announcement that φ *might be false*. Let the *contraction language* $\mathcal{L}^{\div}$ be the language generated by the grammar of $\mathcal{L}$ extended with the following clauses for the dynamic modality and a universal modality: $\varphi := [\div \varphi] \varphi \mid \forall \varphi$. We define $\exists \varphi := \neg \forall \neg \varphi$.

A standard (non-hedged) public announcement that φ *is false* is modeled as eliminating all the φ -possibilities from all agents' doxastic spaces [29]. A natural thought, then, is that a hedged public announcement that φ *might be false* should add all the $\neg \varphi$ -possibilities to all agents' doxastic spaces. However, this is not quite right. Adding all such possibilities to all agents' doxastic spaces may force those who already consider $\neg \varphi$ possible to update their beliefs and start considering additional $\neg \varphi$ -possibilities, thereby losing more beliefs than is warranted by the announcement. This suggests that, unlike standard public announcements, the epistemic effects of a hedged announcement that φ might be false depend on the agents' initial beliefs, and in particular on whether they believe φ prior to the announcement. For this section, we assume that this is the only relevant difference: if an agent already considers $\neg \varphi$ possible, then the announcement φ *might be false* has no effects on her beliefs; if the agent believes φ , then the announcement leads her to consider all $\neg \varphi$ -worlds as possible.

Definition 4.1 (Contraction on φ). Given a Kripke model $\mathcal{M} = (W, R, V)$ and a formula $\varphi \in \mathcal{L}^{\div}$. Let $\mathcal{M}^{\div \varphi} = (W, R^{\div \varphi}, V)$ be defined by, for all $a \in Ag$ and all $w \in W$:

$$R_a^{\div \varphi}(w) = \begin{cases} R_a(w) \cup \{v \in W : \mathcal{M}, v \models \neg \varphi\} & \text{if } \mathcal{M}, w \models B_a \varphi \\ R_a(w) & \text{otherwise} \end{cases}$$

⁵The name *hedged public announcement* reflects the public and non-committal ("hedged") character of the announcement, which opens up possibilities rather than restricting them—much like epistemic modals in so-called "hedged assertions" [13].

Call $\mathcal{M}^{\div \varphi}$ the model obtained from $\mathcal{M}$ by contracting on φ . Note that it is always the case that $R_a(w) \subseteq R_a^{\div \varphi}(w)$. This means that a hedged public announcement always *extends* the initial Kripke model. The semantics of $\mathcal{L}^{\div}$ is given by the semantics of $\mathcal{L}$ over Kripke models extended with these clauses:

$$\begin{aligned} \mathcal{M}, w \models [\div \varphi] \psi & \text{ iff } \mathcal{M}^{\div \varphi}, w \models \psi; \\ \mathcal{M}, w \models \forall \varphi & \text{ iff for all } v \in W, \mathcal{M}, v \models \varphi. \end{aligned}$$

Consider again Example 3.1. In the present framework, the effect of Bob's announcement that *Alice's office door might be open* can be modeled by contracting on the proposition *Alice's office door is closed* ($\neg p$), which amounts to adding world v to Alice's set of doxastic possibilities at w and v . See Figure 4.



Figure 4: **Left:** The Kripke model $\mathcal{M}$ (same as Figure 1), representing Alice's initial belief state. **Right:** The Kripke model $\mathcal{M}^{\div \neg p}$ representing Alice's belief state after updating on Bob's hedged announcement, obtained by contracting on $\neg p$. We have $\mathcal{M}^{\div \neg p} \models \neg B_a \neg p \wedge \neg B_a p$, so Alice lost a belief in $\neg p$.

One natural question is what is the logic of the contraction update as defined above. The next theorem shows that such logic is given by the system presented in Table 1.

CL	all classical propositional tautologies
$\forall K$	$\forall(\varphi \rightarrow \psi) \rightarrow (\forall \varphi \rightarrow \forall \psi)$
$\forall T$	$\forall \varphi \rightarrow \varphi$
$\forall 5$	$\neg \forall \varphi \rightarrow \forall \neg \forall \varphi$
$\forall B$	$\forall \varphi \rightarrow B_a \varphi$
BK	$B_a(\varphi \rightarrow \psi) \rightarrow (B_a \varphi \rightarrow B_a \psi)$
A1	$[\div \varphi] p \leftrightarrow p$
A2	$[\div \varphi] \neg \psi \leftrightarrow \neg [\div \varphi] \psi$
A3	$[\div \varphi] (\psi \wedge \chi) \leftrightarrow [\div \varphi] \psi \wedge [\div \varphi] \chi$
A4	$[\div \varphi] \forall \psi \leftrightarrow \forall [\div \varphi] \psi$
A5	$[\div \varphi] B_a \psi \leftrightarrow (B_a [\div \varphi] \psi \wedge (B_a \varphi \rightarrow \forall (\neg \varphi \rightarrow [\div \varphi] \psi)))$
MP	From φ and $\varphi \rightarrow \psi$, infer ψ
RE	From $\varphi \leftrightarrow \psi$, infer $\chi[\varphi/p] \leftrightarrow \chi[\psi/p]$
NEC	Necessitation rules for all modalities ⁶

Table 1: The Theory of Hedged Public Announcement Logic (HPAL)

Theorem 4.2. *The dynamic logic of contraction updates is completely axiomatized by HPAL.*

Proof sketch. Soundness follows by straightforward validity arguments. For completeness, given the reduction axioms, every formula with dynamic modalities is provably equivalent in HPAL to a formula without. So the completeness of HPAL follows from the completeness of the basic doxastic logic with the universal modality [14, Theorem 7.2]. $\square$

Besides axioms and rules of inference of the basic modal logic K with the universal modality (which is an S5 modality), the theory contains inference rules and reduction axioms for the contraction modality that

⁶Notice that necessitation for the belief modality is redundant, as it is derivable from $\forall B$ and $\forall$ -necessitation.

parallel the reduction axioms in Public Announcement Logic (PAL) [29, 20, 31]. Axiom A5 is specific to our logic of contraction. It says that after contraction on φ (viz. a hedged announcement that φ might be false), an agent believes ψ iff she doesn't believe φ prior to the announcement and she believes that ψ holds after the announcement, or she believes φ prior to the announcement and at any $\neg\varphi$ -worlds, ψ holds after the announcement.

5 Preservation and Moorean formulas

A well-known phenomenon in public announcement logic is that some formulas become false after they are announced. As a result, the formula is not believed after the announcement. In PAL, formulas that become false or are not believed after an announcement are both called *unsuccessful formulas*. Let $[\!|\varphi|\!]$ be the dynamic modality “after the announcement of φ ” in PAL.⁷ There are thus two closely related notions of *success* in PAL: (i) what is announced is true after its announcement ($[\!|\varphi|\!]\varphi$ is valid); (ii) what is announced is believed after its announcement ($[\!|\varphi|\!]B_a\varphi$ is valid for all $a \in Ag$). For clarity, we will say φ is *preserved in PAL* if it satisfies (i); φ is *successful in PAL* if it satisfies (ii). It is well-known that, in PAL, any formula that satisfies (i) also satisfies (ii), though the converse does not hold.⁸ In this section, we focus on developing a notion of preservation for HPAL that is analogous to (i) for PAL.

A natural starting point is to say that φ is *preserved* if it is true after the hedged announcement that φ might be true, viz. $[\div\neg\varphi]\varphi$ is valid. But this isn't quite right: while a public announcement that φ is true *eliminates* all $\neg\varphi$ -possibilities, a hedged public announcement that φ *might* be true doesn't eliminate any possibilities, and so any possibility where φ is false before the announcement remains in the model (though they may satisfy φ in the new model after the announcement). This observation suggests the following alternative definition of preservation for hedged public announcements (let $\mathcal{L}$ be a class of models that validates logic **L**):

Definition 5.1 (Preservation). φ is *preserved* (in the logic **L**) if $\models_{\mathcal{L}} \varphi \rightarrow [\div\neg\varphi]\varphi$

Not all formulas in $\mathcal{L}^\div$ are preserved in HPAL. Consider Example 3.1 again. Suppose Bob tells Alice instead that *she might have a false belief that her office door is closed*, viz. it might be true that $p \wedge B_a\neg p$. Figure 5 shows that this hedged announcement is not preserved.



Figure 5: Alice's belief update as a contraction on $\neg(p \wedge B_a\neg p)$. The formula $p \wedge B_a\neg p$ is true at v before the update (on the left) but false at v after the update (on the right). Hence, $p \wedge B_a\neg p$ is not preserved.

More generally, say φ is *self-refuting* (in the logic **L**) if $\models_{\mathcal{L}} \varphi \rightarrow [\div\neg\varphi]\neg\varphi$. Clearly, φ is not preserved if it is self-refuting. The above example is a special case of the following result:

⁷In PAL, the formula $[\!|\varphi|\!]\psi$ intuitively says ψ is true after eliminating all $\neg\varphi$ -possibilities from the model; see [20, Chapter 4] and references therein for more on PAL.

⁸The proof that (i) implies (ii) is analogous to the “only if”-direction of [20, Proposition 4.32], though here the modality is belief rather than knowledge, and in particular it does not necessarily satisfy the T-axiom, which is the reason why the other direction does not hold in general. For example, suppose $Ag = \{a\}$, let $\varphi := p \wedge B_a\neg p \wedge \neg B_a\perp$, and consider a Kripke model and a world w in it where such φ is true. Then, after the public announcement of φ , which eliminates all possibilities where φ is not true and so also all worlds that were accessible for a at w , a will have inconsistent beliefs at w and believe (trivially) that φ is true. Hence, $[\!|\varphi|\!]B_a\varphi$ is valid. On the other hand, $[\!|\varphi|\!]\varphi$ is invalid and in particular $[\!|\varphi|\!]\neg B_a\perp$ is invalid.

Proposition 5.2. *The formulas $p \wedge \exists B_a \neg p$ and $p \wedge B_a \neg p$ are self-refuting in HPAL.*⁹

Proof. Let $\varphi := p \wedge \exists B_a \neg p$. Suppose $\mathcal{M}, w \models p \wedge \exists B_a \neg p$. Then $\mathcal{M} \models \exists B_a \neg p$ and so $\mathcal{M}^{\div \neg \varphi} = \mathcal{M}^{\div \neg p}$. Thus $\mathcal{M}^{\div \neg \varphi} \models \forall \tilde{B}_a p$ and so $\mathcal{M}^{\div \neg \varphi}, w \models \neg \varphi$. Similarly, let $\psi := p \wedge B_a \neg p$. Suppose $\mathcal{M}, w \models \psi$. Since $B_a \neg p \models B_a \neg \psi$, $\mathcal{M}, w \models B_a \neg \psi$. So $w \in R_a^{\div \neg \psi}(w)$. Since $\mathcal{M}, w \models p$, $\mathcal{M}^{\div \neg \psi}, w \models \tilde{B}_a p$. Given that $\tilde{B}_a p \models \neg \psi$, $\mathcal{M}^{\div \neg \psi}, w \models \neg \psi$. $\square$

So which formulas are preserved? Recall that a formula φ is *existential* if it is built only from literals, $\wedge$, $\vee$ and $\tilde{B}_a$; φ is *universal* if it is built only from literals, $\wedge$, $\vee$ and B_a . It's well-known that universal formulas remain true after being (standardly) publicly announced, in any normal modal logic [20]. Their dual—existential formulas—are always preserved by hedged public announcements:

Theorem 5.3. *If $\varphi \in \mathcal{L}^{\div}$ is equivalent in HPAL to an existential formula, then it is preserved in HPAL.*

Proof. Suppose φ is equivalent in HPAL to an existential formula. Then φ is preserved under model extension [4]. Since $\mathcal{M}^{\div \neg \varphi}$ extends $\mathcal{M}$, if $\mathcal{M}, w \models \varphi$, then $\mathcal{M}^{\div \neg \varphi}, w \models \varphi$, i.e. $\mathcal{M}, w \models \varphi \rightarrow [\div \neg \varphi] \varphi$. $\square$

One upshot of Theorem 5.3 is that, while the “strong” Moorean sentence $p \wedge B_a \neg p$ is never preserved, the “weak” Moorean sentence $p \wedge \neg B_a p$, which is equivalent to an existential formula, is preserved.

So being logically equivalent to an existential formula is sufficient for being preserved. But it is not necessary. In particular, the non-existential formula $p \wedge B_a p$ (agent a has a true belief that p) is preserved.

Proposition 5.4. *The formula $p \wedge B_a p$ is preserved in HPAL.*

Proof. Let $\mathcal{M} = (W, R, V)$ be a Kripke model with $\mathcal{M}, w \models p \wedge B_a p$. As $R_a^{\div \neg (p \wedge B_a p)}(w) \subseteq R_a(w) \cup \{v \in W : \mathcal{M}, v \models p \wedge B_a p\} \subseteq \{v \in W : \mathcal{M}, v \models p\}$ and p is preserved, then $\mathcal{M}^{\div \neg (p \wedge B_a p)}, w \models p \wedge B_a p$. $\square$

It would be nice to have a complete characterization of the set of preserved formulas, which we leave as an open question. But there is a partial result. Let's consider *knowledge* instead of *belief* for a moment, and restrict our attention to single-agent models. The standard logic for knowledge is **S5**, and extending HPAL with **S5** axioms implies that all formulas in $\mathcal{L}$ are preserved.

Theorem 5.5. *Suppose $|Ag| = 1$. Then every formula in $\mathcal{L}$ is preserved in HPAL extended with **S5**.*

Proof sketch. In **S5**, if $\mathcal{M}, w \models \varphi$ with $\varphi \in \mathcal{L}$, then every world in the equivalence class of w satisfies $\tilde{B}_a \varphi$ and so contracting on $\neg \varphi$ does not change relations inside those classes. Then, $(\mathcal{M}, w)$ and $(\mathcal{M}^{\div \neg \varphi}, w)$ are bisimilar, and since $\mathcal{L}$ is bisimulation-invariant, φ is preserved. $\square$

Note that model $\mathcal{M}$ in Figure 5 validates **KD45**, so the preservation result doesn't hold in these settings.¹⁰

6 AGM-style contraction principles

In the AGM literature, belief contraction is modeled as a static operation on the set of formulas, representing the agent's belief base, and is characterized by a set of axioms. As [9] points out, many of those axioms do not naturally hold in the dynamic setting where belief contraction induces model transformations. In this section, we give an overview of which of the AGM axioms hold/fail for our contraction operator. Following [30], we render AGM statements that express that the agent believes χ after contracting by φ , by the formula $[\div \varphi] B_a \chi$. The modal version of the AGM axioms for contraction can then be expressed by the following formulas, where $\varphi, \psi, \chi \in \mathcal{L}^{\div}$:

⁹Note that the Moorean sentence $p \wedge B_a \neg p$ is unsatisfiable in **S5**, while $p \wedge \exists B_a \neg p$ is satisfiable even if belief is factive.

¹⁰As an anonymous referee pointed out, the result does not generalize to the multi-agent setting.

- **Closure.** $[\div \varphi](B_a(\psi \rightarrow \chi) \rightarrow (B_a\psi \rightarrow B_a\chi))$.
- **Success.** $\exists \neg \varphi \rightarrow [\div \varphi] \neg B_a\varphi$.
- **Inclusion.** $[\div \varphi]B_a\psi \rightarrow B_a\psi$.
- **Vacuity.** $\neg B_a\varphi \rightarrow (\chi \leftrightarrow [\div \varphi]\chi)$.
- **Extensionality.** If $\varphi \leftrightarrow \psi$ then $\models [\div \varphi]\chi \leftrightarrow [\div \psi]\chi$.
- **Conjunctive Inclusion.** $[\div(\varphi \wedge \psi)] \neg B_a\varphi \rightarrow ([\div(\varphi \wedge \psi)]B_a\chi \rightarrow [\div \varphi]B_a\chi)$.
- **Conjunctive Overlap.** $([\div \varphi]B_a\chi \wedge [\div \psi]B_a\chi) \rightarrow [\div(\varphi \wedge \psi)]B_a\chi$.
- **Consistency.** $\neg B_a\perp \rightarrow [\div \varphi] \neg B_a\perp$.

From the list above we omitted the axiom called **Recovery** (expansion by φ after contraction by φ restores the agent's original beliefs), as it involves an expansion operation which we do not focus on.¹¹ We added to this list a natural property called **Consistency**, which is discussed in [30, 26].

Some of these properties are valid in HPAL. In particular, **Closure** and **Extensionality** hold unrestrictedly in our semantic framework. **Consistency** also holds, as the contraction operation never removes edges. For the other properties, the situation is more mixed, as the next two subsections show.

6.1 Success

Recall that φ is successful in PAL if it is believed after it is announced: $[\div \varphi]B_a\varphi$ is valid for all $a \in Ag$. A natural analogue of this notion in our contraction settings is that φ is *not believed* after a hedged announcement that it might be false, viz. $[\div \varphi] \neg B_a\varphi$ for all $a \in Ag$. The only caveat concerns the case where φ is valid in the model. In that case, since there are no $\neg\varphi$ -possibilities, contracting on φ leaves the model unchanged and so may leave agents' beliefs in φ unchanged.¹² This illustrates why **Success** is given by the conditional $\exists \neg \varphi \rightarrow [\div \varphi] \neg B_a\varphi$. Say that a formula φ is *successful* if it satisfies this property.

As with preservation, not all formulas are successful. For instance, the hedged announcement that $p \wedge B_a \neg p$ might be true in Figure 5 is unsuccessful: after updating on Bob's announcement that she might have a false belief that $\neg p$, by contracting on $\neg(p \wedge B_a \neg p)$, Alice will come to believe that she believes neither p nor $\neg p$, and so she will believe that she doesn't have a false belief that p (i.e. $\mathcal{M}^{\div \neg(p \wedge B_a \neg p)} \models B_a \neg(p \wedge B_a \neg p)$). In PAL, if a formula remains true after being announced, then it is believed after the announcement [20, Proposition 4.32]. A similar result holds for HPAL.

Proposition 6.1. *In HPAL, if $\neg\varphi \in \mathcal{L}^{\div}$ is preserved, then φ is successful.*

Proof. Let $\mathcal{M} = (W, R, V)$ and suppose $\mathcal{M}, w \models \exists \neg \varphi$. Assume that $\neg\varphi$ is preserved. We show that $\mathcal{M}, w \models [\div \varphi] \neg B_a\varphi$. There are two cases. First, suppose that $\mathcal{M}, w \models \neg B_a\varphi$. Then, there is $v \in R_a(w) = R_a^{\div \varphi}(w)$ such that $\mathcal{M}, v \models \neg\varphi$. As $\neg\varphi$ is preserved, $\mathcal{M}, v \models [\div \varphi] \neg\varphi$ and so $\mathcal{M}, w \models [\div \varphi] \neg B_a\varphi$. Second, suppose $\mathcal{M}, w \models B_a\varphi$. Since $\mathcal{M}, w \models \exists \neg \varphi$, there is a $v \in W$ with $\mathcal{M}, v \models \neg\varphi$ and $v \in R_a^{\div \varphi}(w)$. By preservation of $\neg\varphi$, $\mathcal{M}, v \models [\div \varphi] \neg\varphi$ and so $\mathcal{M}, w \models [\div \varphi] \neg B_a\varphi$. $\square$

Given that existential formulas are preserved (Theorem 5.3), we then obtain the following corollary, which gives a sufficient condition for a formula to be successful:

¹¹Notice, however, that **Recovery** does not hold in general in this framework: while the contraction operator $[\div \varphi]$ can informally be viewed as a dual of the public announcement operator $[\div \varphi]$, it is not a formal dual, in the sense that contraction by φ followed by expansion by φ does not in general recover the original model (whether we model expansion by world- or arrow-eliminations). We explore this issue in a longer version of the paper.

¹²This mirrors the AGM constraint that beliefs in validities cannot be given up [1, 25].

Corollary 6.2. *If $\varphi \in \mathcal{L}^\div$ is equivalent in HPAL to a universal formula, then φ is successful in HPAL.*

Proof. If φ is equivalent to a universal formula, $\neg\varphi$ is equivalent to an existential formula. By Theorem 5.3 existential formulas are preserved, and by Prop. 6.1 if $\neg\varphi$ is preserved, then φ is successful. $\square$

6.2 Other properties

The previous section showed that **Success** is not valid in HPAL. However, by Corollary 6.2, all universal formulas are successful and in particular all propositional formulas are successful. The same holds for **Inclusion**, **Conjunctive Inclusion** and **Conjunctive Overlap**. These postulates fail for reasons similar to the failure of **Success**. For instance, in Figure 4, after Bob’s hedged announcement, Alice comes to believe that she doesn’t believe p —a belief that she didn’t have before, which is a failure of **Inclusion** ($\mathcal{M}, w \models \neg B_a \neg B_a \neg p \wedge [\div \neg p] B_a \neg B_a \neg p$). This is to be expected: on the dynamic approach, the truths of doxastic formulas can be influenced by hedged announcements. However, all three postulates hold if the believed formula is propositional. Let $\mathfrak{K}$ be the class of all Kripke models.

Proposition 6.3. *The following are valid in HPAL, where χ and λ are propositional and $\varphi, \psi \in \mathcal{L}^\div$:*

1. **Propositional Inclusion.** $[\div \varphi] B_a \chi \rightarrow B_a \chi$.
2. **Propositional Conjunctive Inclusion.** $[\div \lambda \wedge \psi] \neg B_a \lambda \rightarrow ([\div \lambda \wedge \psi] B_a \chi \rightarrow [\div \lambda] B_a \chi)$.
3. **Propositional Conjunctive Overlap.** $([\div \varphi] B_a \chi \wedge [\div \psi] B_a \chi) \rightarrow [\div \varphi \wedge \psi] B_a \chi$.

Proof sketch. **Propositional Inclusion** follows from $R_a(w) \subseteq R_a^{\div \varphi}(w)$ and from propositional truth being unaffected by contraction. **Propositional Conjunctive Inclusion** and **Overlap** follow because contracting on a conjunction introduces no alternatives beyond those required to give up one of its conjuncts. $\square$

Finally, while **Vacuity** is also among the properties that do not hold unrestrictedly, the following strong version holds:

- **Strong Vacuity.** $\bigwedge_{a \in Ag} \forall \neg B_a \varphi \rightarrow (\chi \leftrightarrow [\div \varphi] \chi)$.

This principle strengthens the standard **Vacuity** principle listed above by requiring that *no agent* believes the contracted formula *at any world in the model*. Such restrictions reflect the multi-agent and public nature of our contraction operation. They are not required in classic AGM theory, where beliefs are propositional, nor in dynamic doxastic logic, which is single-agent [1, 30].

Proposition 6.4. *Strong Vacuity is a theorem of HPAL.*

Proof. Let $\varphi, \chi \in \mathcal{L}^\div$ and consider a Kripke model $\mathcal{M} = (W, R, V)$ and a world w in it. Assume that $\mathcal{M}, w \models \forall \neg B_a \varphi$, for all $a \in Ag$, i.e., $\mathcal{M}, u \models \neg B_a \varphi$ for all $u \in W$ and $a \in Ag$. Then $R_a(u) = R_a^{\div \varphi}(u)$ for all worlds u and all agents a , i.e., $\mathcal{M} = \mathcal{M}^{\div \varphi}$ and so $\mathcal{M}, w \models \chi$ iff $\mathcal{M}^{\div \varphi}, w \models \chi$, that is, $\mathcal{M}, w \models \chi$ iff $\mathcal{M}, w \models [\div \varphi] \chi$. $\square$

7 Generalized DEL

In this section, we introduce a generalized DEL framework that can model belief contraction resulting from different kinds of announcements, like hedged private announcements.

Let the *DEL language* $\mathcal{L}_{DEL}$ be the language generated by the grammar of the doxastic language $\mathcal{L}$ extended with the following clauses $\varphi ::= [\mathcal{E}, e] \varphi \mid \forall \varphi$, where $\mathcal{E}$ is a generalized event model (introduced below) and e is an event in it. The formula $[\mathcal{E}, e] \varphi$ reads “after $(\mathcal{E}, e)$ happens, φ is the case”.

Definition 7.1 (Generalized event model). A *generalized event model* is a tuple $\mathcal{E} = (E, Q, Q^+, pre)$ where E and Q_a are defined as in standard event models, $Q_a^+ \subseteq E \times E$ is an accessibility relation between events defined for all agents $a \in Ag$, and $pre : E \rightarrow \mathcal{L}_{DEL}$ is a precondition function.

A generalized event model is a standard event model extended with additional accessibility relations Q_a^+ , for each agent $a \in Ag$, and in which preconditions can be formulas of the dynamic language $\mathcal{L}_{DEL}$. As before, we use notation $Q_a^+(e) = \{f \in E : (e, f) \in Q_a^+\}$. Intuitively, both $Q_a(e)$ and $Q_a^+(e)$ represent events that are accessible, and thus considered possible, by agent a at event e . The difference, which will become formally clear with the product update definition below, is that $Q_a^+(e)$ contains alternatives that are *introduced by event e as possible for agent a* (irrespective of a 's prior beliefs), while $Q_a(e)$ is the standard accessibility relation containing possibilities that are not *ruled out for agent a by event e* .

The distinction is illustrated by the event model $\mathcal{E}$ in Figure 6, which is the event model corresponding to Example 3.1. Intuitively, f represents the event *Bob tells Alice her office door might be open and it is actually closed*, and h represents the event *Bob tells Alice her office door might be open and it is actually open*. For Alice, Bob's announcement doesn't rule out either f or h , but it introduces h as possible. Formally, this means $Q_a(f) = Q_a(h) = \{f, h\}$, while $Q_a^+(f) = Q_a^+(h) = \{h\}$.

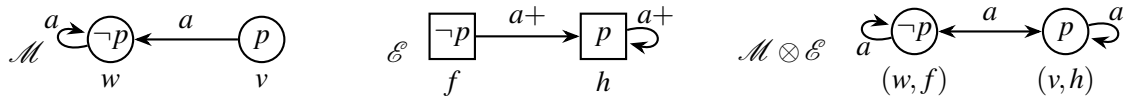


Figure 6: **Left:** The Kripke model $\mathcal{M}$ representing Alice's initial belief states. **Middle:** The generalized event model $\mathcal{E} = (E, Q, Q^+, pre)$ representing Bob's announcement to Alice that p might hold. An edge from an event f to an event h labeled by $+a$ means that $h \in Q_a^+(f)$. We omit the Q -edges since $Q_a(e) = E$ for all $e \in E$. **Right:** The Kripke model $\mathcal{M} \otimes \mathcal{E} = (W^\mathcal{E}, R^\mathcal{E}, V^\mathcal{E})$ representing Alice's belief states after Bob's announcement to Alice.

Given this interpretation, the product update is then defined as the following:

Definition 7.2 (Generalized product update). Given a Kripke model $\mathcal{M} = (W, R, V)$ and a generalized event model $\mathcal{E} = (E, Q, Q^+, pre)$, their product update $\mathcal{M} \otimes \mathcal{E}$ is defined as $\mathcal{M} \otimes \mathcal{E} = (W^\mathcal{E}, R^\mathcal{E}, V^\mathcal{E})$ where $W^\mathcal{E}$ and $V^\mathcal{E}$ are defined as in standard product updates (cf. Section 2) and $R^\mathcal{E}$ is given by:

$$(v, f) \in R_a^\mathcal{E}(w, e) \text{ iff either } f \in Q_a^+(e), \text{ or } v \in R_a(w) \text{ and } f \in Q_a(e).$$

This product update behaves like a standard product update in eliminating possibilities (cf. Section 2), but additionally allows to expand the set of accessible worlds by adding all worlds (v, f) where f is a newly considered possibility. In particular, the first disjunct allows agents to start considering worlds paired with events in Q_a^+ , regardless of the original accessibility relation.

The semantics of the dynamic modality is standard [20]:

$$\mathcal{M}, w \models [\mathcal{E}, e]\varphi \quad \text{iff} \quad \text{if } \mathcal{M}, w \models pre(e) \text{ then } \mathcal{M} \otimes \mathcal{E}, (w, e) \models \varphi.$$

The following definition shows that generalized DEL can express contraction as induced by hedged public announcements.

Definition 7.3 (Event model for contraction on φ). Let $\varphi \in \mathcal{L}_{DEL}$. An *event model for contraction on φ* is a generalized event model $\mathcal{E}(\div \varphi) = (E, Q, Q^+, pre)$ where

1. $E = \{\varphi \wedge \bigwedge_{a \in A} \neg B_a \varphi \wedge \bigwedge_{a \in Ag \setminus A} B_a \varphi : A \subseteq Ag\} \cup \{\neg \varphi \wedge \bigwedge_{a \in A} \neg B_a \varphi \wedge \bigwedge_{a \in Ag \setminus A} B_a \varphi : A \subseteq Ag\}.$

2. For all $a \in \text{Ag}$ and $e \in E$, $Q_a(e) = E$ and $Q_a^+(e) = \begin{cases} \{f \in E : \text{pre}(f) \models \neg\varphi\} & \text{if } \text{pre}(e) \models B_a\varphi; \\ \emptyset & \text{otherwise.} \end{cases}$
3. For all $e \in E$, $\text{pre}(e) = e$.

Event model $\mathcal{E}(\div\varphi)$ contains events for all possible configurations of the truth value of φ and which agents believe φ . The relation Q_a is universal, capturing that the announcement reveals nothing about which event occurred. The relation Q_a^+ captures contraction: agents who believed φ come to consider $\neg\varphi$ -events possible, while the others are unaffected.

Proposition 7.4. For any Kripke model $\mathcal{M}$ and formula $\varphi \in \mathcal{L}$, $\mathcal{M}^{\div\varphi}$ is isomorphic to $\mathcal{M} \otimes \mathcal{E}(\div\varphi)$.

Proof sketch. Both updates preserve all worlds of $\mathcal{M}$, so there is an isomorphism between the updated models: as $R_a(w) \subseteq R_a^{\div\varphi}(w)$ and $Q_a(e) = E$ for all $a \in \text{Ag}, e \in E$, all relations in $\mathcal{M}$ are preserved by both updates. Also, the only relations added in both cases are from $B_a\varphi$ -worlds to $\neg\varphi$ -worlds. $\square$

Next we consider an example involving hedged private announcement:

Example 7.5. Alice just told Clark that she locked her office door before she left. Later, while Clark is evidently distracted, Bob privately tells Alice that her office door might actually be open. Alice comes to suspend judgment on whether she locked the door, while Clark does not notice this exchange, continuing to believe that Alice's office door is closed and that Alice believes that.

We model Bob's private announcement using the generalized event model $\mathcal{E}$ as described below:

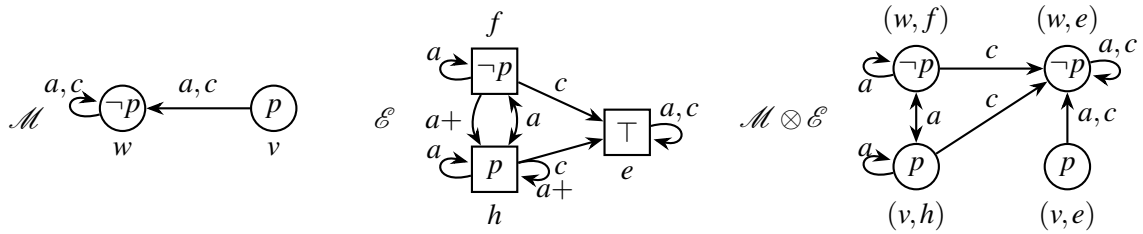


Figure 7: **Left:** The Kripke model $\mathcal{M}$ representing Alice and Clark's initial belief states. We have $\mathcal{M} \models B_a\neg p \wedge B_c\neg p \wedge B_cB_a\neg p$. **Middle:** The generalized event model $\mathcal{E} = (E, Q, Q^+, \text{pre})$ representing Bob's private announcement to Alice that p might hold. An edge from an event f to an event h labeled by $+a$ means that $h \in Q_a^+(f)$. If an edge's label does not contain $+$, then the edge is a Q -relation, e.g. $e \in Q_c(f)$. **Right:** The Kripke model $\mathcal{M} \otimes \mathcal{E} = (W^\mathcal{E}, R^\mathcal{E}, V^\mathcal{E})$ representing Alice and Clark's belief states after Bob's private announcement to Alice.

We now look into what is the logic of generalized updates. First, we define the composition of two generalized event models and then show that it is equivalent to the sequential updates of the two.

Definition 7.6 (Composition). Given two generalized event models $\mathcal{E} = (E^\mathcal{E}, Q^\mathcal{E}, Q^{+\mathcal{E}}, \text{pre}^\mathcal{E})$ and $\mathcal{F} = (E^\mathcal{F}, Q^\mathcal{F}, Q^{+\mathcal{F}}, \text{pre}^\mathcal{F})$, then their composition is $\mathcal{E} \circ \mathcal{F} = (E, Q, Q^+, \text{pre})$ where

- $E = E^\mathcal{E} \times E^\mathcal{F}$;
- $(e', f') \in Q_a((e, f))$ iff $e' \in Q_a^\mathcal{E}(e)$ and $f' \in Q_a^\mathcal{F}(f)$;
- $(e', f') \in Q_a^+((e, f))$ iff $f' \in Q_a^{+\mathcal{F}}(f)$, or $f' \in Q_a^\mathcal{F}(f)$ and $e' \in Q_a^{+\mathcal{E}}(e)$;
- $\text{pre}((e, f)) = \text{pre}^\mathcal{E}(e) \wedge [\mathcal{E}, e] \text{pre}^\mathcal{F}(f)$.

CL	all classical propositional tautologies
$\forall K$	$\forall(\varphi \rightarrow \psi) \rightarrow (\forall\varphi \rightarrow \forall\psi)$
$\forall T$	$\forall\varphi \rightarrow \varphi$
$\forall 5$	$\neg\forall\varphi \rightarrow \forall\neg\varphi$
$\forall B$	$\forall\varphi \rightarrow B_a\varphi$
BK	$B_a(\varphi \rightarrow \psi) \rightarrow (B_a\varphi \rightarrow B_a\psi)$
A1	$[\mathcal{E}, e]p \leftrightarrow (pre(e) \rightarrow p)$
A2	$[\mathcal{E}, e]\neg\varphi \leftrightarrow (pre(e) \rightarrow \neg[\mathcal{E}, e]\varphi)$
A3	$[\mathcal{E}, e](\varphi \wedge \psi) \leftrightarrow ([\mathcal{E}, e]\varphi \wedge [\mathcal{E}, e]\psi)$
A4	$[\mathcal{E}, e]\forall\varphi \leftrightarrow (pre(e) \rightarrow \bigwedge_{f \in E} \forall[\mathcal{E}, f]\varphi)$
A5	$[\mathcal{E}, e]B_a\varphi \leftrightarrow (pre(e) \rightarrow (\bigwedge_{f \in Q_a^+(e)} \forall[\mathcal{E}, f]\varphi \wedge \bigwedge_{f \in Q_a(e)} B_a[\mathcal{E}, f]\varphi))$
A6	$[\mathcal{E}, e][\mathcal{F}, f]\varphi \leftrightarrow [\mathcal{E} \circ \mathcal{F}, (e, f)]\varphi$
MP and necessitation rules for all modalities	

Table 2: The Theory GDEL

The following shows that this is given by the system presented in Table 2.¹³

Theorem 7.7. *The dynamic logic of generalized updates is completely axiomatized by GDEL.*

Proof sketch. The proof strategy is analogous to that used for Theorem 4.2, using the composition axiom A6 instead of the inference rule RE. $\square$

The theory GDEL contains the static axioms of HPAL, together with the inference rules for all modalities. Additionally, it contains reduction axioms for event models analogous to those of standard DEL. Axiom A5 captures the effect of generalized updates on belief via the relations Q_a and Q_a^+ .

The next theorem shows that generalized updates always produce a model that is a simulation of a refinement of the initial model (refinement and simulation are defined standardly [14], but see Appendix). As noted in [15, 18], refinements decrease uncertainty by eliminating possibilities, while simulations capture increases in uncertainty by adding possibilities. Generalized updates thus can be seen as updates where agents may rule out possibilities while starting to consider new ones, as in belief revision.

Theorem 7.8. *For any Kripke model $\mathcal{M}$ and generalized event model $\mathcal{E}$, $\mathcal{M} \otimes \mathcal{E}$ is a simulation of a refinement of $\mathcal{M}$.*

Proof sketch. The idea is to split a generalized update into two steps. First ignore the Q_a^+ edges and obtain a standard DEL update, which gives a refinement of the initial model [15]. The full generalized update then only adds edges to this model, so it is a simulation of that refinement. $\square$

Before concluding this section, we discuss one limitation of the contraction operation as defined in Section 4, and how the generalized product update helps to overcome it.

Example 7.9. Alice and Clark walk down the hallway. Alice believes that she closed her office door ($\neg p$) and that there will be a department meeting in the afternoon (q). Clark, however, is uncertain about both questions, though he believes that Alice believes the truth no matter what it is, and so does Alice. They then encounter Bob, who tells them that Alice's office door might be open.

¹³Note that, unlike the axiomatization of HPAL, the axiomatization of GDEL includes a composition axiom (A6) but without RE as a valid rule of inference. We conjecture that we can omit A6 and include RE as a valid inference rule, though we leave the verification of this conjecture to future work.

Intuitively, given Bob's announcement, Clark will come to think that, if Alice initially believes that her office door is closed, then she'll come to suspend judgment about whether her office door is open, but she will remain confident about whether there will be a department meeting in the afternoon. So Alice's and Clark's beliefs should be modeled by $\mathcal{N}$ in Figure 8.

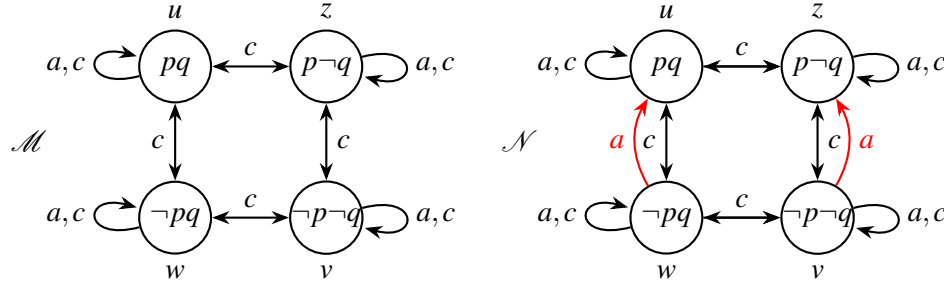


Figure 8: **Left:** The Kripke model $\mathcal{M}$ representing Alice and Clark's initial belief states. We have $\mathcal{M} \models B_c(B_ap \vee B_a\neg p) \wedge B_c(B_aq \vee B_a\neg q)$. **Right:** The Kripke model $\mathcal{N}$ representing Alice and Clark's belief states after Bob's hedged announcement, where $\mathcal{N}, w \models \neg B_c(B_ap \vee B_a\neg p) \wedge B_c(B_aq \vee B_a\neg q)$.

Proposition 7.10. *There is no formula $\varphi \in \mathcal{L}^\div$ such that $\mathcal{M}^\div \varphi \equiv \mathcal{N}$.*

Proof. Suppose towards a contradiction that $\mathcal{M}^\div \varphi \equiv \mathcal{N}$ for some $\varphi \in \mathcal{L}^\div$. Since every world in $\mathcal{M}$ satisfies a distinct set of propositional formulas and contraction preserves propositional valuation, it follows that for all $x \in W$, $(\mathcal{M}^\div \varphi, x) \equiv (\mathcal{N}, x)$. Since $u \notin R_a(w)$ but $u \in R_a^\div \varphi(w)$, we have $\mathcal{M}, w \models B_a \varphi$ and $\mathcal{M}, u \models \neg \varphi$. Similarly, $\mathcal{M}, v \models B_a \varphi$ and $\mathcal{M}, z \models \neg \varphi$. By the definition of contraction, $\{x \in W : \mathcal{M}, x \models \neg \varphi\} \subseteq R_a^\div \varphi(w)$. So $z \in R_a^\div \varphi(w)$. So $\mathcal{M}^\div \varphi, w \models \neg B_a q$. Since $\mathcal{N}, w \models B_a q$, this contradicts the assumption that $(\mathcal{M}^\div \varphi, w)$ and $(\mathcal{N}, w)$ are modally equivalent. $\square$

One way of addressing this problem within the HPAL framework is to restrict the set of worlds that Alice starts considering to a subset of p -possibilities, e.g. those that are compatible with her background knowledge or “entrenched” beliefs. We can also model these dynamics using GDEL. Consider $\mathcal{E} = (E, Q, Q^+, pre)$ depicted as in Figure 9, which captures three aspects of the update: (i) Bob's announcement doesn't eliminate any possibilities (so the Q -relations are universal); (ii) if Alice believes p , then Bob's announcement doesn't make her consider new possibilities ($Q_a^+(e) = Q_a^+(g) = \emptyset$) (iii) if Alice believes $\neg p$, then Bob's announcement makes her consider p possible without revising her beliefs about q ($Q_a^+(f) = \{e\}$ and $Q_a^+(h) = \{g\}$). It's not hard to check that $\mathcal{M} \otimes \mathcal{E} = \mathcal{N}$.

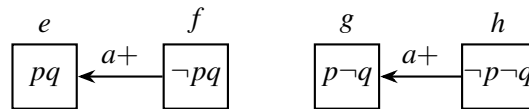


Figure 9: The generalized event model $\mathcal{E} = (E, Q, Q^+, pre)$ representing Bob's announcement to Alice and Clark. As before, we omit the Q -edges, since $Q_i(x) = E$ for all events $x \in E$ and agents $i \in \{a, c\}$. Each event's precondition is given by the conjunction of the literals listed inside the event. So for example event e has precondition $p \wedge q$.

8 Conclusion and future work

In this paper, we introduced a logic for belief contraction due to hedged public announcements, and then a generalized DEL theory for belief expansion and contraction. There are many questions left open for future work, such as: the full characterization of successful formulas for HPAL, the interaction between contraction and expansion operators, properties of a revision operator defined in terms of their combinations via the Levi identity [25], and the iterative properties of GDEL. Of special importance are the closure properties of GDEL. While updates with standard event models and plausibility event models preserve natural properties of accessibility relations, such as transitivity and Euclideaness, this is not true of generalized product updates, even if both the Q - and the Q^+ -relations in the generalized event model are equivalence relations. This has the undesirable consequence that an agent who had full introspection of her own beliefs prior to the update may have higher-order uncertainties about her own beliefs after the update. While it is possible to enforce preservation of introspection in our framework, it is an open question for which class of generalized event models this can be guaranteed.

In addition to the above questions, we also plan to investigate how our model compares with alternative models of belief contraction [28, 24] as well as other models of information loss, such as forgetting and awareness growth [17, 11], and the dynamic approach to interpreting conditionals with modal antecedents. In particular, one may ask how expressive our generalized updates are with respect to plausibility updates. In one sense, we can transform any distinguished Kripke model $\mathcal{K} = (W, R, V)$ into any Kripke model $\mathcal{K}' = (W', R', V')$ with $W = W'$ and $V = V'$ by adding the edges in $\mathcal{K}'$ without keeping any edges in $\mathcal{K}$.¹⁴ To this extent, our framework can emulate the belief dynamics that can be modeled in the plausibility framework, though we'll leave a detailed comparison to future works.

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A Proofs

Throughout, by *logical equivalence* we mean provable in **K**. We let $\llbracket \varphi \rrbracket = \{w \in W : \mathcal{M}, w \models \varphi\}$, and $\llbracket \varphi \rrbracket^{\div \Psi} = \{w \in W^{\div \Psi} : \mathcal{M}^{\div \Psi}, w \models \varphi\}$.

Definition A.1 (Bisimulation, refinement, simulation). Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be two Kripke models. A *bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ is a non-empty relation $Z \subseteq W \times W'$ such that for all $(w, w') \in Z$ and $a \in \text{Ag}$:

$$(\text{Atom}): w \in V(p) \text{ iff } w' \in V'(p), \text{ for all } p \in \text{At};$$

¹⁴Recall that $\mathcal{K} = (W, R, V)$ is *distinguished* if for every world w there is a unique formula φ such that $\mathcal{M}, w \models \varphi$. The simulation strategy is similar to the strategy used in [21] to show that in DEL with postconditions, one can transform any Kripke model in any other Kripke model. The strategy there is also to get rid of the structure of the initial Kripke model and then reconstruct it by means of carefully devised event models with postconditions.

(Forth): If $v \in R_a(w)$ then there exists $v' \in W'$ such that $v' \in R'_a(w')$ and $(v, v') \in Z$;

(Back): If $v' \in R'_a(w')$ then there exists $v \in W$ such that $v \in R_a(w)$ and $(v, v') \in Z$;

When a bisimulation exists between two models, we say that they are *bisimilar*. A relation that satisfies Atom and Back is a *refinement*. When a refinement exists between $\mathcal{M}$ and $\mathcal{M}'$, we say that $\mathcal{M}'$ is a *refinement* of $\mathcal{M}$. A relation that satisfies Atom and Forth is a *simulation*. When a simulation exists between $\mathcal{M}$ and $\mathcal{M}'$, we say that $\mathcal{M}'$ is a *simulation* of $\mathcal{M}$.

Proof of Theorem 4.2. For soundness, the only non-trivial axiom is A5, namely $[\div \phi]B_a\psi \leftrightarrow (B_a[\div \phi]\psi \wedge (B_a\phi \rightarrow \forall(\neg\phi \rightarrow [\div \phi]\psi)))$. Suppose $\mathcal{M}, w \models [\div \phi]B_a\psi$. We claim that $\mathcal{M}, w \models B_a[\div \phi]\psi$. Let $v \in R_a(w)$. Then $v \in R_a^{\div \phi}(w)$. By assumption, $\mathcal{M}^{\div \phi}, v \models B_a\psi$. So $\mathcal{M}^{\div \phi}, v \models \psi$. Thus $\mathcal{M}, v \models [\div \phi]\psi$. So $\mathcal{M}, w \models B_a[\div \phi]\psi$. Next, we claim that, if $\mathcal{M}, w \models B_a\phi$ and $\mathcal{M}, w \models [\div \phi]B_a\psi$, then for every $v \in W$, $\mathcal{M}, v \models \neg\phi \rightarrow [\div \phi]\psi$. Let $v \in W$. Suppose $\mathcal{M}, v \models \neg\phi$. Since $\mathcal{M}, w \models B_a\phi$, $v \in R^{\div \phi}(w)$. Since $\mathcal{M}, w \models [\div \phi]B_a\psi$, $\mathcal{M}^{\div \phi}, v \models \psi$. So $\mathcal{M}, v \models [\div \phi]\psi$.

Conversely, suppose $\mathcal{M}, w \models B_a[\div \phi]\psi \wedge (B_a\phi \rightarrow \forall(\neg\phi \rightarrow [\div \phi]\psi))$. There are two cases, either $\mathcal{M}, w \models \neg B_a\phi$ or $\mathcal{M}, w \models B_a\phi$. Suppose $\mathcal{M}, w \models \neg B_a\phi$. Let $v \in R^{\div \phi}(w)$. Then $v \in R_a(w)$. Since $\mathcal{M}, w \models B_a[\div \phi]\psi$, we have $\mathcal{M}, v \models [\div \phi]\psi$. So $\mathcal{M}^{\div \phi}, v \models \psi$. Thus $\mathcal{M}, w \models [\div \phi]B_a\psi$. Now suppose $\mathcal{M}, w \models B_a\phi$. So $\mathcal{M}, w \models \forall(\neg\phi \rightarrow [\div \phi]\psi)$. Let $v \in R^{\div \phi}(w)$. Either $v \in R_a(w)$ or $\mathcal{M}, v \models \neg\phi$. In the first case, by the assumption that $\mathcal{M}, w \models B_a[\div \phi]\psi$, we have $\mathcal{M}^{\div \phi}, v \models \psi$. In the second case, since $\mathcal{M}, w \models \forall(\neg\phi \rightarrow [\div \phi]\psi)$, we have $\mathcal{M}, v \models [\div \phi]\psi$ or equivalently $\mathcal{M}^{\div \phi}, v \models \psi$. So $\mathcal{M}, w \models [\div \phi]B_a\psi$.

We now show that RE is validity preserving. Suppose $\models \phi \leftrightarrow \psi$. We proceed by induction on the complexity of the environment χ . If χ is a propositional variable $p \in At$, then $\chi[\phi/p] = \phi$ and $\chi[\psi/p] = \psi$. So $\models \chi[\phi/p] \leftrightarrow \chi[\psi/p]$ by assumption. The cases of negation and conjunction follow from propositional logic. Suppose $\models \chi[\phi/p] \leftrightarrow \chi[\psi/p]$. Then

$$\begin{aligned} \mathcal{M}, w \models B_a\chi[\phi/p] & \text{ iff } \text{for all } v \in R_a(w) \mathcal{M}, v \models \chi[\phi/p] \\ & \text{ iff } \text{for all } v \in R_a(w) \mathcal{M}, v \models \chi[\psi/p] \\ & \text{ iff } \mathcal{M}, w \models B_a\chi[\psi/p] \end{aligned}$$

A similar argument shows that $\models \forall\chi[\phi/p] \leftrightarrow \forall\chi[\psi/p]$. Let $\lambda \in \mathcal{L}^{\div}$. Then

$$\begin{aligned} \mathcal{M}, w \models [\div \lambda]\chi[\phi/p] & \text{ iff } \mathcal{M}^{\div \lambda}, w \models \chi[\phi/p] \\ & \text{ iff } \mathcal{M}^{\div \lambda}, w \models \chi[\psi/p] \\ & \text{ iff } \mathcal{M}, w \models [\div \lambda]\chi[\psi/p] \end{aligned}$$

Lastly, for any Kripke model $\mathcal{M} = (W, R, V)$, we claim that $\mathcal{M}^{\div \chi[\phi/p]} = \mathcal{M}^{\div \chi[\psi/p]}$. It suffices to show that $R_a^{\div \chi[\phi/p]}(w) = R_a^{\div \chi[\psi/p]}(w)$ for all $w \in W$ and $a \in Ag$. Suppose $\mathcal{M}, w \models \neg B_a\chi[\phi/p]$. Then by the inductive hypothesis, $\mathcal{M}, w \models \neg B_a\chi[\psi/p]$ and so $R_a^{\div \chi[\phi/p]}(w) = R_a(w) = R_a^{\div \chi[\psi/p]}(w)$. Suppose $\mathcal{M}, w \models B_a\chi[\phi/p]$. Then by IH, $\mathcal{M}, w \models B_a\chi[\psi/p]$ and

$$\begin{aligned} R_a^{\div \chi[\phi/p]}(w) &= R_a(w) \cup \{v \in W : \mathcal{M}, v \models \neg\chi[\phi/p]\} \\ &= R_a(w) \cup \{v \in W : \mathcal{M}, v \models \neg\chi[\psi/p]\} \\ &= R_a^{\div \chi[\psi/p]}(w) \end{aligned}$$

So $\mathcal{M}^{\div \chi[\phi/p]} = \mathcal{M}^{\div \chi[\psi/p]}$. Thus, for any Kripke model $\mathcal{M}$ and world w ,

$$\begin{aligned}
\mathcal{M}, w \models [\div \chi[\varphi/p]]\lambda & \text{ iff } \mathcal{M}^{\div \chi[\varphi/p]}, w \models \lambda \\
& \text{ iff } \mathcal{M}^{\div \chi[\psi/p]}, w \models \lambda \\
& \text{ iff } \mathcal{M}, w \models [\div \chi[\psi/p]]\lambda
\end{aligned}$$

Given the reduction axioms, every formula with dynamic modalities is provably equivalent in HPAL to a formula without dynamic modalities. So the completeness of HPAL follows from the completeness of the basic doxastic logic with universal modalities. $\square$

Proof of Theorem 5.5. Suppose $Ag = \{a\}$. Let $\varphi \in \mathcal{L}$. Let $\mathcal{M} = (W, R, V)$ be a Kripke model that validates **S5**. Suppose $\mathcal{M}, w \models \varphi$. Since R_a is an equivalence relation, for all $v \in R_a(w)$, $\mathcal{M}, v \models \tilde{B}_a \varphi$. Then $R_a(v) = R_a^{\div \neg \varphi}(v)$ for all $v \in R_a(w)$. Consider $Z = \{(v, v) : v \in R_a(w)\}$. We claim that Z is a bisimulation between $(\mathcal{M}, w)$ and $(\mathcal{M}^{\div \neg \varphi}, w)$. Clearly the two worlds satisfy the same proposition atoms. Let $(v, v) \in Z$. Then $v \in R_a(w)$. Suppose $u \in R_a(v)$. Since R_a is an equivalence relation, we have $u \in R_a(w)$ and so $(u, u) \in Z$. Moreover, since $R_a(v) = R_a(w) = R_a^{\div \neg \varphi}(w) = R_a^{\div \neg \varphi}(v)$, we have $u \in R_a^{\div \neg \varphi}(v)$. Similarly, suppose $u \in R_a^{\div \neg \varphi}(v)$. Then $u \in R_a^{\div \neg \varphi}(w) = R_a(w) = R_a(v)$. So Z is a bisimulation between $(\mathcal{M}, w)$ and $(\mathcal{M}^{\div \neg \varphi}, w)$. Thus, $\mathcal{M}^{\div \neg \varphi}, w \models \varphi$. $\square$

Proof of Proposition 6.3. Let $\varphi, \psi, \chi, \lambda \in \mathcal{L}^{\div}$ with χ, λ propositional and consider a Kripke model $\mathcal{M} = (W, R, V)$ and a world w in it. Notationally, let $\mathcal{M}^{\div \varphi} = (W^{\div \varphi}, R^{\div \varphi}, V^{\div \varphi})$, $\llbracket \psi \rrbracket = \{w \in W : \mathcal{M}, w \models \psi\}$ and $\llbracket \psi \rrbracket^{\div \varphi} = \{w \in W : \mathcal{M}^{\div \varphi}, w \models \psi\}$.

1. *Propositional Inclusion:* Assume that $\mathcal{M}, w \models [\div \varphi]B_a \chi$. Then $\mathcal{M}^{\div \varphi}, w \models B_a \chi$. So $R_a^{\div \varphi}(w) \subseteq \llbracket \chi \rrbracket^{\div \varphi}$. By definition of contraction $R_a(w) \subseteq R_a^{\div \varphi}(w) \subseteq \llbracket \chi \rrbracket^{\div \varphi}$. As χ is propositional, $\llbracket \chi \rrbracket = \llbracket \chi \rrbracket^{\div \varphi}$. Hence, $\mathcal{M}, w \models B_a \chi$.
2. *Propositional Conjunctive Inclusion:* Suppose $\mathcal{M}, w \models [\div \lambda \wedge \psi] \neg B_a \lambda$ and $\mathcal{M}, w \models [\div \lambda \wedge \psi] B_a \chi$. There are two cases: either $\mathcal{M}, w \models \neg B_a(\lambda \wedge \psi)$ or $\mathcal{M}, w \models B_a(\lambda \wedge \psi)$. Suppose the first. Then $R_a^{\div \lambda \wedge \psi}(w) = R_a(w)$. Since $\mathcal{M}, w \models [\div \lambda \wedge \psi] \neg B_a \lambda$, there exists $v \in R_a^{\div \lambda \wedge \psi}(w)$ with $\mathcal{M}^{\div \lambda \wedge \psi}, v \models \neg \lambda$. As λ is propositional, then $\mathcal{M}, v \models \neg \lambda$, and since $R_a^{\div \lambda \wedge \psi}(w) = R_a(w)$ then $\mathcal{M}, w \models \neg B_a \lambda$. So $R_a^{\div \lambda}(w) = R_a(w)$. Hence, $R_a^{\div \lambda \wedge \psi}(w) = R_a^{\div \lambda}(w)$. So $\mathcal{M}, w \models [\div \lambda \wedge \psi] B_a \chi$ implies $R_a^{\div \lambda}(w) = R_a^{\div \lambda \wedge \psi}(w) \subseteq \llbracket \chi \rrbracket^{\div \lambda \wedge \psi} = \llbracket \chi \rrbracket^{\div \lambda}$, where the last identity holds as χ is propositional. Hence, $\mathcal{M}, w \models [\div \lambda] B_a \chi$. Now suppose the second case holds. Then $\mathcal{M}, w \models B_a \lambda$, and so $R_a^{\div \lambda}(w) = R_a(w) \cup \llbracket \neg \lambda \rrbracket \subseteq R_a(w) \cup \llbracket \neg \lambda \vee \neg \psi \rrbracket = R_a^{\div (\lambda \wedge \psi)}(w)$. As by assumption $\mathcal{M}, w \models [\div \lambda \wedge \psi] B_a \chi$, then $R_a^{\div (\lambda \wedge \psi)}(w) \subseteq \llbracket \chi \rrbracket^{\div (\lambda \wedge \psi)} = \llbracket \chi \rrbracket^{\div \lambda}$. Hence, $R_a^{\div \lambda}(w) \subseteq \llbracket \chi \rrbracket^{\div \lambda}$ and so $\mathcal{M}, w \models [\div \lambda] B_a \chi$.
3. *Propositional Conjunctive Overlap.* Suppose that $\mathcal{M}, w \models [\div \varphi] B_a \chi \wedge [\div \psi] B_a \chi$. Then $R_a^{\div \varphi}(w) \subseteq \llbracket \chi \rrbracket^{\div \varphi}$ and $R_a^{\div \psi}(w) \subseteq \llbracket \chi \rrbracket^{\div \psi}$. As χ is propositional, $\llbracket \chi \rrbracket^{\div \varphi} = \llbracket \chi \rrbracket^{\div \psi} = \llbracket \chi \rrbracket^{\div \varphi \wedge \psi}$, and so $(R_a^{\div \varphi}(w) \cup R_a^{\div \psi}(w)) \subseteq \llbracket \chi \rrbracket^{\div \varphi \wedge \psi}$. Two cases: either $\mathcal{M}, w \models B_a(\varphi \wedge \psi)$ or not. If the first, then $\mathcal{M}, w \models B_a \varphi$ and $\mathcal{M}, w \models B_a \psi$. So $R_a^{\div \varphi}(w) = R_a(w) \cup \llbracket \neg \varphi \rrbracket$ and $R_a^{\div \psi}(w) = R_a(w) \cup \llbracket \neg \psi \rrbracket$ and $R_a^{\div \varphi \wedge \psi}(w) = R_a(w) \cup \llbracket \neg \varphi \vee \neg \psi \rrbracket = R_a(w) \cup \llbracket \neg \varphi \rrbracket \cup \llbracket \neg \psi \rrbracket$. Then, $R_a^{\div \varphi \wedge \psi}(w) = R_a^{\div \varphi}(w) \cup R_a^{\div \psi}(w)$, and so $R_a^{\div \varphi \wedge \psi}(w) \subseteq \llbracket \chi \rrbracket^{\div \varphi \wedge \psi}$. Hence, $\mathcal{M}, w \models [\div \varphi \wedge \psi] B_a \chi$. Suppose instead that $\mathcal{M}, w \models \neg B_a(\varphi \wedge \psi)$. Then, $R_a^{\div \varphi \wedge \psi}(w) = R_a(w)$, and since by initial assumption $R_a^{\div \varphi}(w) \subseteq \llbracket \chi \rrbracket^{\div \varphi \wedge \psi}$ and by definition of contraction $R_a(w) \subseteq R_a^{\div \varphi}(w)$, then $R_a(w) \subseteq \llbracket \chi \rrbracket^{\div \varphi \wedge \psi}$. Hence, $\mathcal{M}, w \models [\div \varphi \wedge \psi] B_a \chi$. $\square$

Proof of Proposition 7.4. Let $\mathcal{M} = (W, R, V)$ be a Kripke model. Notationally, we let $\mathcal{E}(\div \varphi) = (E, Q, Q^+, pre)$, $\mathcal{M} \otimes \mathcal{E}(\div \varphi) = (W^{\mathcal{E}(\div \varphi)}, R^{\mathcal{E}(\div \varphi)}, V^{\mathcal{E}(\div \varphi)})$ and $\mathcal{M}^{\div \varphi} = (W^{\div \varphi}, R^{\div \varphi}, V^{\div \varphi})$. The function $h: W^{\mathcal{E}(\div \varphi)} \rightarrow W^{\div \varphi}$, defined by $h((w, e)) = w$, for all $w \in W$, is a bijection: first, notice that, by

definition of $\mathcal{E}(\div \varphi)$, for each world $w \in W$ there is exactly one event $e \in E$ such that $\mathcal{M}, w \models \text{pre}(e)$, and so $(w, e) \in W^{\mathcal{E}(\div \varphi)}$ iff $w \in W$. Then notice that, by definition of $\div \varphi$ -update, $W^{\div \varphi} = W$, and so $(w, e) \in W^{\mathcal{E}(\div \varphi)}$ iff $w \in W^{\div \varphi}$. We now show that h preserves atomic valuations and accessibility relations. The atomic valuations are clearly preserved, as $V^{\mathcal{E}(\div \varphi)}(p) = V(p) = V^{\div \varphi}(p)$ for all $p \in \text{At}$. For the accessibility relations, let $a \in \text{Ag}$ and $(w, e) \in W^{\mathcal{E}(\div \varphi)}$. We want to show that $(v, f) \in R_a^{\mathcal{E}(\div \varphi)}(w, e)$ iff $v \in R_a^{\div \varphi}(w)$. We consider two cases: either $\mathcal{M}, w \not\models B_a \varphi$, or $\mathcal{M}, w \models B_a \varphi$. Suppose $\mathcal{M}, w \not\models B_a \varphi$. Then $\mathcal{E}(\div \varphi)$ is such that $Q_a(e) = E$ and $Q_a^+(e) = \emptyset$, and $\mathcal{M}^{\div \varphi}$ is such that $R_a(w) = R_a^{\div \varphi}(w)$. So we have that $(v, f) \in R_a^{\mathcal{E}(\div \varphi)}(w, e)$ iff (by definition of generalized product update) $v \in R_a(w)$ iff (by definition of $\div \varphi$ -update) $v \in R_a^{\div \varphi}(w)$, as we wanted to show. Suppose instead $\mathcal{M}, w \models B_a \varphi$. Assume $(v, f) \in R_a^{\mathcal{E}(\div \varphi)}(w, e)$. Then either $v \in R_a(w)$ and $f \in Q_a(e)$, or $f \in Q_a^+(e)$. If $v \in R_a(w)$, then $v \in R_a^{\div \varphi}(w)$, as desired. If instead $f \in Q_a^+(e)$, then $\mathcal{M}, v \models \neg \varphi$ and so $v \in R_a^{\div \varphi}(w)$, as desired. For the other direction, assume $v \in R_a^{\div \varphi}(w)$. Then either $v \in R_a(w)$ or $\mathcal{M}, v \models \neg \varphi$. If $v \in R_a(w)$, then since $Q_a(e) = E$, $(v, f) \in R_a^{\mathcal{E}(\div \varphi)}(w, e)$. If instead $\mathcal{M}, v \models \neg \varphi$, then by $(v, f) \in W^{\mathcal{E}(\div \varphi)}$, $\text{pre}(f) \models \neg \varphi$ by definition of $\mathcal{E}(\div \varphi)$. Since $\mathcal{M}, w \models B_a \varphi$ then $Q_a^+(e) = \{g \in E : \text{pre}(g) \models \neg \varphi\}$, and so $(v, f) \in R_a^{\mathcal{E}(\div \varphi)}(w, e)$. Hence in all cases we have the desired. $\square$

Proof of Theorem 7.7. It is straightforward that the inference rules preserve validity. We check the soundness of A4 and A5. The soundness of the other axioms is immediate. Let $\mathcal{M} = (W, R, V)$ be a Kripke model, let $\mathcal{E} = (E, Q, Q^+, \text{pre})$ be a generalized event model, and let $\mathcal{M} \otimes \mathcal{E} = (W^{\mathcal{E}}, R^{\mathcal{E}}, V^{\mathcal{E}})$.

- A4: Note that, if $\mathcal{M}, w \not\models \text{pre}(e)$, then the biconditional holds automatically at $(\mathcal{M}, w)$. Suppose $\mathcal{M}, w \models \text{pre}(e)$ and $\mathcal{M}, w \models [\mathcal{E}, e] \forall \varphi$. Then $\mathcal{M} \otimes \mathcal{E}, (w, e) \models \forall \varphi$, i.e. for all $(v, f) \in W^{\mathcal{E}}$, $\mathcal{M} \otimes \mathcal{E}, (v, f) \models \varphi$. In other words, for all $f \in E$ and $v \in W$ such that $\mathcal{M}, v \models \text{pre}(f)$, $\mathcal{M}, v \models [\mathcal{E}, f] \varphi$. So $\mathcal{M}, w \models \bigwedge_{f \in E} \forall [\mathcal{E}, f] \varphi$.

Conversely, suppose $\mathcal{M}, w \models \text{pre}(e)$ but $\mathcal{M}, w \not\models \neg [\mathcal{E}, e] \forall \varphi$, i.e. $\mathcal{M} \otimes \mathcal{E}, (w, e) \not\models \forall \varphi$. So there is $(v, f) \in W^{\mathcal{E}}$ such that $\mathcal{M} \otimes \mathcal{E}, (v, f) \not\models \varphi$. It follows that $\mathcal{M}, v \models \neg [\mathcal{E}, f] \varphi$ and so $\mathcal{M}, w \models \neg \bigwedge_{f \in E} \forall [\mathcal{E}, f] \varphi$.

- A5: Suppose $\mathcal{M}, w \models [\mathcal{E}, e] B_a \varphi \wedge \text{pre}(e)$. Let $f \in Q_a^+(e)$. Then for any $v \in W$, if $\mathcal{M}, v \models \text{pre}(f)$, we have $(v, f) \in R_a^{\mathcal{E}}((w, e))$ and so by assumption $\mathcal{M}^{\mathcal{E}}, (v, f) \models \varphi$. Thus $\mathcal{M}, w \models \bigwedge_{f \in Q_a^+(e)} \forall [\mathcal{E}, f] \varphi$. Let $f \in Q_a(e)$ and $v \in R_a(w)$. Then $(v, f) \in R_a^{\mathcal{E}}((w, e))$ and so $\mathcal{M}^{\mathcal{E}}, (v, f) \models \varphi$. Thus $\mathcal{M}, v \models [\mathcal{E}, f] \varphi$. So $\mathcal{M}, w \models \bigwedge_{f \in Q_a(e)} B_a [\mathcal{E}, f] \varphi$.

Conversely, suppose $\mathcal{M}, w \models \text{pre}(e) \rightarrow \bigwedge_{f \in Q_a^+(e)} \forall [\mathcal{E}, f] \varphi \wedge \bigwedge_{f \in Q_a(e)} B_a [\mathcal{E}, f] \varphi$ and let $(v, f) \in R_a^{\mathcal{E}}((w, e))$. Either $f \in Q_a^+(e)$ or $f \in Q_a(e)$ and $v \in R_a(w)$. In the first case, since $\mathcal{M}, w \models \forall [\mathcal{E}, f] \varphi$, we have $\mathcal{M}, v \models [\mathcal{E}, f] \varphi$. In the second case, since $\mathcal{M}, w \models B_a [\mathcal{E}, f] \varphi$, we have $\mathcal{M}, v \models [\mathcal{E}, f] \varphi$. So either way, $\mathcal{M}^{\mathcal{E}}, (v, f) \models \varphi$ and so $\mathcal{M}^{\mathcal{E}}, (w, e) \models B_a \varphi$.

- A6: The proof mainly requires showing that there exists an isomorphism between $(\mathcal{M} \otimes \mathcal{E}) \otimes \mathcal{F} = (W^{\mathcal{E} \circ \mathcal{F}}, R^{\mathcal{E} \circ \mathcal{F}}, V^{\mathcal{E} \circ \mathcal{F}})$ and $\mathcal{M} \otimes (\mathcal{E} \circ \mathcal{F}) = (W^{\mathcal{E} \circ \mathcal{F}}, R^{\mathcal{E} \circ \mathcal{F}}, V^{\mathcal{E} \circ \mathcal{F}})$. By invariance of truth under isomorphism, this will imply the desired. We use notation $\mathcal{M} \otimes \mathcal{E} = (W^{\mathcal{E}}, R^{\mathcal{E}}, V^{\mathcal{E}})$, and $\mathcal{E} = (E^{\mathcal{E}}, Q^{\mathcal{E}}, Q^{+\mathcal{E}}, \text{pre}^{\mathcal{E}})$ and $\mathcal{F} = (E^{\mathcal{F}}, Q^{\mathcal{F}}, Q^{+\mathcal{F}}, \text{pre}^{\mathcal{F}})$.

To show the existence of the isomorphism, let $h : W^{\mathcal{E} \circ \mathcal{F}} \rightarrow W^{\mathcal{E} \circ \mathcal{F}}$ be defined by $h((w, e), f) = (w, (e, f))$. Clearly, h is a bijection: $((w, e), f) \in W^{\mathcal{E} \circ \mathcal{F}}$ iff $\mathcal{M}, w \models \text{pre}^{\mathcal{E}}(e)$ and $\mathcal{M} \otimes \mathcal{E}, (w, e) \models \text{pre}^{\mathcal{F}}(f)$, iff $\mathcal{M}, w \models \text{pre}^{\mathcal{E}}(e) \wedge [\mathcal{E}, e] \text{pre}^{\mathcal{F}}(f)$, iff $(w, (e, f)) \in W^{\mathcal{E} \circ \mathcal{F}}$. It also preserves valuations, since for every atom $p \in \text{At}$, $((w, e), f) \in V^{\mathcal{E} \circ \mathcal{F}}(p)$ iff $w \in V(p)$ iff $(w, (e, f)) \in V^{\mathcal{E} \circ \mathcal{F}}(p)$. Finally, h preserves accessibility relations. Let $((w', e'), f') \in R_a^{\mathcal{E} \circ \mathcal{F}}((w, e), f)$. We want to show that $(w', (e', f')) \in R_a^{\mathcal{E} \circ \mathcal{F}}((w, (e, f)))$. Let $((w', e'), f') \in R_a^{\mathcal{E} \circ \mathcal{F}}((w, (e, f)))$. Two cases: either $f' \in$

$Q_a^{+\mathcal{F}}(f)$ or $(w', e') \in R_a^\mathcal{E}((w, e))$ and $f' \in Q_a^\mathcal{F}(f)$. If $f' \in Q_a^{+\mathcal{F}}(f)$ then $(e', f') \in Q_a^{+\mathcal{E} \circ \mathcal{F}}(e, f)$ and so $(w', (e', f')) \in R_a^{\mathcal{E} \circ \mathcal{F}}((w, (e, f)))$. If $(w', e') \in R_a^\mathcal{E}((w, e))$ and $f' \in Q_a^\mathcal{F}(f)$, then again two cases, either $e' \in Q_a^{+\mathcal{E}}(e)$, or $w' \in R_a(w)$ and $e' \in Q_a^\mathcal{E}(e)$. If $e' \in Q_a^{+\mathcal{E}}(e)$ then by $f' \in Q_a^\mathcal{F}(f)$ and definition of composition, $(e', f') \in Q_a^{+\mathcal{E} \circ \mathcal{F}}((e, f))$, which implies $(w', (e', f')) \in R_a^{\mathcal{E} \circ \mathcal{F}}((w, (e, f)))$. If $w' \in R_a(w)$ and $e' \in Q_a^\mathcal{E}(e)$, then by $f' \in Q_a^\mathcal{F}(f)$ and definition of composition we have $(e', f') \in Q_a^{\mathcal{E} \circ \mathcal{F}}((e, f))$, and so $(w', (e', f')) \in R_a^{\mathcal{E} \circ \mathcal{F}}((w, (e, f)))$. Hence, in all cases the desired holds. The other direction (that is, if $(w', (e', f')) \in R_a^{\mathcal{E} \circ \mathcal{F}}((w, (e, f)))$ then $((w', e'), f') \in R_a^\mathcal{F}((w, e), f)$) follows by a similar straightforward application of the definition of composition and generalized product update.

Thus, h is an isomorphism. By invariance of truth under isomorphisms, for every $((w, e), f) \in W^{\mathcal{E} \circ \mathcal{F}}$ and every $\varphi \in \mathcal{L}_{\text{DEL}}$, we have $((\mathcal{M} \otimes \mathcal{E}) \otimes \mathcal{F}, ((w, e), f)) \models \varphi$ iff $(\mathcal{M} \otimes (\mathcal{E} \circ \mathcal{F}), (w, (e, f))) \models \varphi$. It remains only to connect this with the truth conditions for the dynamic modalities. Let $w \in W$. If $\mathcal{M}, w \not\models \text{pre}^\mathcal{E}(e)$, then both $[\mathcal{E}, e][\mathcal{F}, f]\varphi$ and $[\mathcal{E} \circ \mathcal{F}, (e, f)]\varphi$ are true at w vacuously, since $\text{pre}^{\mathcal{E} \circ \mathcal{F}}(e, f) = \text{pre}^\mathcal{E}(e) \wedge [\mathcal{E}, e]\text{pre}^\mathcal{F}(f)$. If $\mathcal{M}, w \models \text{pre}^\mathcal{E}(e)$ but $\mathcal{M} \otimes \mathcal{E}, (w, e) \not\models \text{pre}^\mathcal{F}(f)$, then again $[\mathcal{E}, e][\mathcal{F}, f]\varphi$ is true at w vacuously, and so is $[\mathcal{E} \circ \mathcal{F}, (e, f)]\varphi$, since $\mathcal{M}, w \not\models \text{pre}^{\mathcal{E} \circ \mathcal{F}}(e, f)$. Finally, if $\mathcal{M}, w \models \text{pre}^\mathcal{E}(e)$ and $\mathcal{M} \otimes \mathcal{E}, (w, e) \models \text{pre}^\mathcal{F}(f)$, then both product points $((w, e), f)$ and $(w, (e, f))$ exist, and the desired equivalence follows from the isomorphism above. Hence, in all cases, $\mathcal{M}, w \models [\mathcal{E}, e][\mathcal{F}, f]\varphi$ iff $\mathcal{M}, w \models [\mathcal{E} \circ \mathcal{F}, (e, f)]\varphi$. Since $\mathcal{M}$ and w were arbitrary, the equivalence is valid.

Completeness of GDEL follows from the completeness of K with the universal modality by standard reduction arguments. $\square$

Proof of Theorem 7.8. Let $\mathcal{M} = (W, R, V)$ be a Kripke model and $\mathcal{E} = (E, Q, Q^+, \text{pre})$ be a generalized event model. Notationally, let $\mathcal{M} \otimes \mathcal{E} = (W^\mathcal{E}, R^\mathcal{E}, V^\mathcal{E})$. Our proof strategy is to define a submodel of $\mathcal{M} \otimes \mathcal{E}$ that serves as the intermediate refinement step, and then show that $\mathcal{M} \otimes \mathcal{E}$ is a simulation of that model. Let $(W^{\mathcal{E}^-}, R^{\mathcal{E}^-}, V^{\mathcal{E}^-})$ be a Kripke model such that $W^{\mathcal{E}^-} = W^\mathcal{E}$, $R_a^{\mathcal{E}^-}(w, e) = \{(v, f) \in W^\mathcal{E} : v \in R_a(w), f \in Q_a(e)\}$, for all $a \in \text{Ag}$, and $V^{\mathcal{E}^-} = V^\mathcal{E}$. This is a submodel of $\mathcal{M} \otimes \mathcal{E}$, only missing the edges between worlds that the Q^+ relation added. It is then clear that we recover the full $R^\mathcal{E}$ by taking the union of $R^{\mathcal{E}^-}$ with the set containing those edges, that is $R_a^\mathcal{E}(w, e) = R_a^{\mathcal{E}^-}(w, e) \cup \{(v, f) \in W^\mathcal{E} : f \in Q_a^+(e)\}$, for all $a \in \text{Ag}$ and $(w, e) \in W^\mathcal{E}$. Now, $(W^{\mathcal{E}^-}, R^{\mathcal{E}^-}, V^{\mathcal{E}^-})$ is a standard DEL update, and it is a well known result that it is a refinement of $\mathcal{M}$ [15]. We now show that $\mathcal{M} \otimes \mathcal{E}$ is a simulation of $\mathcal{M} \otimes \mathcal{E}^-$, from which we can conclude that $\mathcal{M} \otimes \mathcal{E}$ is a simulation of a refinement of $\mathcal{M}$. Let $Z \subseteq W^{\mathcal{E}^-} \times W^\mathcal{E}$ be such that $((w, e), (w, e)) \in Z$. As $R^\mathcal{E}$ only adds edges to $R^{\mathcal{E}^-}$ without removing any, it clearly holds that if $(v, f) \in R_a^{\mathcal{E}^-}(w, e)$, then $(v, f) \in R_a^\mathcal{E}(w, e)$, and since $((v, f), (v, f)) \in Z$ then Z is a simulation. $\square$

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Modal Measurable Logics via a Modal Loomis-Sikorski Representation Theorem

Nick Bezhanishvili

University of Amsterdam
Amsterdam, The Netherlands
n.bezhanishvili@uva.nl

Jim de Groot*

University of Bern
Bern, Switzerland
jim.degroot@unibe.ch

Lawrence S. Moss

Indiana University
Bloomington, IN, USA
lmoss@iu.edu

We investigate a modal extension of the infinitary classical logic with countable meets and joins, formulated with an eye toward measure-theoretic work in dynamical systems and in point-free ergodic theory. We define a modal formalism in this language, which we call modal measurable logics. We also introduce a Kripke-like semantics for these logics in measurable spaces taking a designated modal σ -ideal into consideration. Using a restriction of Jónsson-Tarski duality and a modal extension of the Loomis-Sikorski theorem, we prove completeness of modal measurable logics with respect to this new semantics.

1 Introduction

In (modal) logic, representation theorems are often used to bridge the gap between algebraic and geometric (frame-based) semantics. This allows one to transfer algebraic completeness of a logic to a completeness theorem for its frames. For instance, the celebrated Stone representation theorem for Boolean algebras yields completeness of classical propositional logic [3]. Similarly, Priestley’s representation theorem of distributive lattices yields completeness of positive logic with respect to its poset semantics [4], and Esakia’s representation of Heyting algebras [5, 6] yields completeness of intuitionistic logic with respect to intuitionistic Kripke frames [3]. Representation theorems for modal logics are often based on an existing representation for the underlying propositional logic, such as the Jónsson–Tarski representation of modal algebras which gives rise to completeness of modal logic with respect to Kripke semantics [2, 3]. In this paper we carry on with this tradition.

The aim of this paper is to develop a modal formalism for reasoning about relational structures based on measurable spaces. Alternatively, such structures can be viewed as Kripke frames equipped with a measurable space structure. Specifically, we work with a modal language with countable meets and joins. Ideally, these should be interpreted as intersections and unions; we investigate to what extent this is feasible when aiming for a completeness result.

As a starting point, we consider the countably infinite analogue K_σ of the classical modal logic K , obtained by extending the classical propositional logic with countable joins and meets with a unary modal operator $\Diamond$ that distributes over countable joins. We call the algebras corresponding to K_σ σ -modal algebras. These arise from extending σ -Boolean algebras with a countable-join-preserving operator. Our first contribution is to prove a Jónsson–Tarski representation theorem for σ -modal algebras, which yields order-topological completeness for σ -modal logics.

However, in the order-topological semantics of σ -modal logics, countable meets and joins are interpreted as the *interiors* of intersections and the *closures* of unions, respectively. While interesting,

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these structures do not form concrete σ -algebras in which countable meets and joins are interpreted as set-theoretic intersections and unions themselves.

To remedy this, we prove a modal version of the celebrated Loomis-Sikorski theorem, which was proven (independently) by Loomis [17] and Sikorski [22] in the 1940s. Recall that this theorem states that every “abstract” σ -Boolean algebra (that is, every Boolean algebra in which countable joins and meets exist) is a quotient of a “concrete” σ -algebra of sets (where the meets and joins are intersections and unions) modulo a σ -ideal. This provides a concrete representation of σ -Boolean algebras. Our modal Loomis-Sikorski theorem is one of the main technical contributions of the paper.

It turns out that to prove the modal Loomis-Sikorski theorem, we need to extend the logic with a special infinitary rule, which we call the *Infinite Descending Chain* rule (IDC). Similar rules are not uncommon in the realm of infinitary modal logics, and one can for instance be found in [8]. We call our logic *Modal Loomis-Sikorski Logic* MLS, and refer to the corresponding algebras as *measurable σ -modal algebras*. We prove:

Modal Loomis-Sikorski theorem: Every measurable σ -modal algebra is the quotient of a “concrete” σ -modal algebra by a modal σ -ideal.

Based on this result, we develop a “measurable” semantics for Modal Loomis-Sikorski Logic based on *modal measurable spaces* with a modal σ -ideal N of marked sets that we think of as “designated null-sets”. A modal measurable space consists of a measurable space (X, Σ) and a relation R on X such that $\diamond$ (the modal operator corresponding to R) maps measurable sets to measurable sets. Modal formulas are then interpreted in the modal algebra $(\Sigma, \diamond)$ quotiented by N .

Quotienting by N seems unavoidable, both for technical reasons (in view of the Loomis-Sikorski theorem) and for conceptual ones. The ideal N will typically consist of the null-sets of some probability measure, which is also the perspective adopted by Scott [21]. We note that a similar move was advocated by Lando [14, 15, 16], Fernández-Duque [7] and Pavlov [19]. Indeed, [19] states that “One can argue that an object of the right category of spaces in measure theory is not a set equipped with a σ -algebra of measurable sets, but rather a set S equipped with a σ -algebra M of measurable sets and a σ -ideal N of negligible sets, i.e., sets of measure 0.” Moreover, Jamneshan and Tao write in [12] that the quotient algebra should be viewed “as a ‘point-free’ or ‘pointless’ abstraction of [a measurable space] X in which the null-sets have been ‘deleted’.” They further observe: “This ‘point-free’ approach to ergodic theory seems particularly well suited for studying actions of uncountable (discrete) groups, as by deleting the null-sets in advance, one can avoid to a large extent the standard difficulty that an uncountable union of null-sets is null.” A similar approach appears in G. Bezhanishvili and Fernández-Duque [1].

Using our modal version of the Loomis-Sikorski theorem, we prove the main completeness result of this paper:

Completeness theorem: Every axiomatic extension of Modal Loomis-Sikorski Logic is sound and complete with respect to a class of marked modal measurable spaces.

With our work, we hope to lay the foundation for a modal logic approach to measure-based dynamical systems and (point-free) ergodic theory. As a first attempt, we give an example of a modal measurable space used in the analysis of measure-preserving transformations in dynamical systems.

Related work One line of work connects modal logic in its probabilistic variant to reasoning about uncertainty in games. This includes early work on the formalisation of Harsanyi type spaces due to Heifetz and Samet [11] (where measure theory first appears in a prominent way), and later papers which

use a probabilistic modal logic, such as Moss and Viglizzo [18]. Neither of these papers take up questions of completeness (as we do). Some papers which do obtain completeness include Goldblatt [10] and Zhou [23]. What all of the above papers have in common is that their modalities are quantitative: the sentences are of the form $B_p\varphi$, where $0 \leq p \leq 1$, which can be read as saying that the probability of (or “belief” in) the sentence φ is at least p . In the current paper, we investigate the use of modal operators to study a more dynamic intuition, rather than a quantitative one.

Another line of loosely related work concerns the combination of modal logic and topology, such as [1, 16, 21]. This type of work differs from ours because it often uses a topological interpretation of the modalities, following a long line of work on topological interpretations of modal logic, while we do not use the measurable space structure in this way.

Yet another set of papers of potential relevance here are ones which call on dualities related to measure theory, such as [8, 13]. The main difference with our work is the semantic formalism: in [8, 13] models are given by *Markov processes*. These are coalgebras of the probability measure functor, and they might be thought of as probabilistic analogues of Kripke frames. That is, they are structures where the accessibility relation comes with probabilistic information and the state set is “mainly just a set.” Our setting, on the other hand, considers a measurable space with an accessibility relation that is “mainly just a relation.” (What we are saying here is not strictly correct: there are connections between the set-theoretic and measure-theoretic aspects, both our paper and other work.) Now in moving closer to our ultimate goal, one would like both a measure-theoretic space and a probabilistic transition relation. We view the work in this paper as a necessary step towards this goal.

2 Stone duality and Jónsson-Tarski duality

For the results in this section we refer to e.g. [2, 3, 9]. A *Stone space* is a compact Hausdorff space (X, τ) whose clopen subsets form a basis for the topology τ . Write **Stone** for the category of Stone spaces and continuous functions, and **BA** for the category of Boolean algebras and (Boolean) homomorphisms.

The collection $\text{Clp}(X, \tau)$ of clopen subsets of a Stone space forms a Boolean algebra. Moreover, if $f : (X, \tau) \rightarrow (Y, \sigma)$ is a continuous function between Stone spaces, then $f^{-1} : \text{Clp}(Y, \sigma) \rightarrow \text{Clp}(X, \tau)$ is a Boolean homomorphism. This gives rise to a contravariant functor

$$\text{Clp} : \mathbf{Stone} \rightarrow \mathbf{BA}.$$

Conversely, every Boolean algebra B gives rise to a Stone space $\text{Uf}(B) := (X_B, \tau_B)$ where X_B is the set of ultrafilters of B and τ_B is the topology on X_B generated by the sets of the form $\theta(b) := \{u \in X_B \mid b \in u\}$, where b ranges over B . Furthermore, if $h : A \rightarrow B$ is a Boolean homomorphism, then $h^{-1} : (X_B, \tau_B) \rightarrow (X_A, \tau_A)$ is a continuous function. This yields a contravariant functor

$$\text{Uf} : \mathbf{BA} \rightarrow \mathbf{Stone}.$$

Theorem 2.1 (Stone duality). *Clp and Uf establish a dual equivalence $\mathbf{Stone} \cong^{\text{op}} \mathbf{BA}$.*

A σ -Boolean algebra is a Boolean algebra with countable meets and countable joins, and a σ -homomorphism is a homomorphism that preserves them. Write $\sigma\mathbf{BA}$ for the category of σ -Boolean algebras and σ -homomorphisms. A Boolean algebra B is a σ -Boolean algebra if and only if its dual Stone space $\text{Uf}(B)$ satisfies:

$$\text{if } \{a_n \mid n \in \omega\} \text{ is a countable set of clopen sets, then } \overline{\bigcup_{n \in \omega} a_n} \text{ is clopen.} \quad (1)$$

(Here, if Y is a subset of a topological space, then $\overline{Y}$ denotes its closure.) In Stone spaces where (1) holds, we can endow the set of clopens with a σ -BA structure: for a clopen a , we take $\neg a$ to be the set-theoretic complement of a , and for each countable sequence $(a_n)_{n \in \omega}$ of clopens we define

$$\bigvee_{n \in \omega} a_n = \overline{\bigcup_{n \in \omega} a_n} \quad \bigwedge_{n \in \omega} a_n = \text{int} \left(\bigcap_{n \in \omega} a_n \right) \quad (2)$$

Define a σ -continuous function between σ -Stone spaces to be a function whose dual is a σ -homomorphism, and write σStone for the category of σ -Stone spaces and σ -continuous functions. Then [9, Section 39]:

Theorem 2.2. *Stone duality restricts to a dual equivalence $\sigma\text{Stone} \equiv^{\text{op}} \sigma\text{BA}$.*

Stone duality for Boolean algebras can be extended to a duality for modal algebras. By a *modal algebra* we mean a pair $(B, \Diamond)$ consisting of a Boolean algebra B and a unary operator $\Diamond$ on B that preserves finite joins. (Here and always we include the empty join, with $\bigvee \emptyset = \perp$.) With the expected notion of homomorphism, these form the category $\mathbf{MA}$ of modal algebras.

Dually, we work with *modal spaces*. These are structures of the form (X, τ, R) such that (X, τ) is a Stone space, R is a point-closed relation (i.e. $R[x]$ is closed in (X, τ) for each $x \in X$), and $\Diamond a := \{x \in X \mid R[x] \cap a \neq \emptyset\}$ is clopen whenever a is clopen. When the topological structure is viewed as a collection of admissible sets, modal spaces are often referred to as *descriptive (general) frames* [2, Section 5.5].

With bounded continuous morphisms, modal spaces comprise the category $\mathbf{MSpace}$. The functor Clp can be extended to a functor

$$\text{Clp} : \mathbf{MSpace} \rightarrow \mathbf{MA}$$

by sending a modal space (X, τ, R) to the modal algebra $(\text{Clp}(X, \tau), \Diamond)$. Conversely, if $(B, \Diamond)$ is a modal algebra then $(X_B, \tau_B, R_\Diamond)$ is a modal space, where $uR_\Diamond v$ iff $(b \in v \text{ implies } \Diamond b \in u \text{ for all } b \in B)$. With the same action on morphisms, the functor Uf extends to

$$\text{Uf} : \mathbf{MA} \rightarrow \mathbf{MSpace}.$$

Theorem 2.3 (Jónsson-Tarski duality). *Clp and Uf establish a dual equivalence $\mathbf{MSpace} \equiv^{\text{op}} \mathbf{MA}$.*

In particular, we note that the map $\theta_B : (B, \Diamond) \rightarrow (\text{Clp}(X_B, \tau_B), \Diamond)$, given by $\theta_B(b) := \{u \in X_B \mid b \in u\}$, is an isomorphism for every modal algebra $(B, \Diamond)$.

3 The σ -modal logic and the modal Loomis-Sikorski logic

The obvious “ σ -analogue” of normal modal logic is the extension with countable meets and joins and a diamond that distributes over countable joins. This yields what we call *σ -modal logic*. However, in order to prove a modal analogue of the Loomis-Sikorski theorem, and obtain completeness with respect to modal measurable spaces modulo an ideal of null-sets, we need to stipulate an additional rule. Adding this to σ -modal logic yields what we call *Modal Loomis-Sikorski logic*.

Definition 3.1. The *modal measurable language* is the collection $\mathcal{L}_\sigma$ of formulas defined by the grammar

$$\varphi ::= p \mid \neg \varphi \mid \bigvee_{i \in \omega} \varphi_i \mid \Diamond \varphi$$

where p ranges over a countably infinite set Prop of proposition letters. We abbreviate $\varphi \vee \psi := \bigvee(\varphi, \psi, \psi, \psi, \dots)$ and $\perp := p \vee \neg p$, and define $\top, \rightarrow, \wedge, \wedge$ and $\Box$ as expected.

Definition 3.2. Let $Ax \subseteq \mathcal{L}_\sigma$ be a set of axioms. The σ -modal logic $K_\sigma \oplus Ax$ is the least set of $\mathcal{L}_\sigma$ -formulas derivable from the following axioms and rules:

- All substitution instances of classical tautologies (allowing for countable conjunction and disjunction) and all substitution instances of formulas in Ax (this includes the infinite De Morgan laws);
- The join rule: for any countable sequence $(\varphi_n)_{n \in \omega}$ and formula ψ :

$$\frac{\varphi_n \rightarrow \psi \quad n \in \omega}{(\bigvee_{n \in \omega} \varphi_n) \rightarrow \psi}$$

- The modal axiom and rule:

$$\Diamond(\bigvee_{n \in \omega} \varphi_n) \leftrightarrow \bigvee_{n \in \omega} \Diamond \varphi_n \quad \frac{\neg \varphi}{\neg \Diamond \varphi}$$

- Modus ponens:

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$$

Modal Loomis-Sikorski logic $MLS \oplus Ax$ is the set of $\mathcal{L}_\sigma$ -formulas derivable from the axioms and rules above together with the *infinite descending chain rule* (IDC):

$$\frac{\neg \bigwedge_{n \in \omega} \varphi_n \quad \varphi_{n+1} \rightarrow \varphi_n \text{ for all } n \in \omega}{\neg \bigwedge_{n \in \omega} \Diamond \varphi_n}. \quad (\text{IDC})$$

A formula φ is called *derivable* in K_σ or MLS if it can be obtained from the axioms and rules of the respective logics. If φ is derivable in K_σ or MLS then we write $\vdash_{K_\sigma} \varphi$ or $\vdash_{MLS} \varphi$, respectively.

In either logic, we get $\varphi_k \rightarrow \bigvee_{n \in \omega} \varphi_n$ as a substitution instance of a classical tautology, as well as the following dual axioms and rules:

$$(\bigwedge_{n \in \omega} \varphi_n) \rightarrow \varphi_k \quad (k \in \omega), \quad \Box(\bigwedge_{n \in \omega} \varphi_n) \leftrightarrow \bigwedge_{n \in \omega} \Box \varphi_n, \quad \frac{\psi \rightarrow \varphi_n \quad n \in \omega}{\psi \rightarrow (\bigwedge_{n \in \omega} \varphi_n)}, \quad \frac{\varphi}{\Box \varphi}.$$

Although the use of diamond as a primitive connective provided intuition for some of the proofs, using boxes as primitives may yield more natural axioms and rules.

4 Algebraic semantics for modal σ -logics

Next, we present the algebraic semantics of K_σ and MLS . Recall that a σ -Boolean algebra is a Boolean algebra that has countable meets and countable joins. We extend this to include an additional operator, $\Diamond$.

Definition 4.1. A σ -modal algebra is a pair $(B, \Diamond)$ consisting of a σ -Boolean algebra B and a unary operator $\Diamond : B \rightarrow B$ that satisfies

$$\Diamond \perp = \perp \quad \text{and} \quad \bigvee_{n \in \omega} \Diamond b_n = \Diamond \bigvee_{n \in \omega} b_n \quad (3)$$

for any countable sequence $(b_n)_{n \in \omega}$ of elements in B .

A *measurable σ -modal algebra* is a σ -modal algebra $(B, \Diamond)$ that additionally satisfies the following condition: for every countable descending chain $(b_n)_{n \in \omega}$ (i.e. every countable collection of elements in B such that $b_0 \geq b_1 \geq b_2 \geq \dots$),

$$\bigwedge_{n \in \omega} b_n = \perp \quad \text{implies} \quad \bigwedge_{n \in \omega} \Diamond b_n = \perp. \quad (4)$$

This rule is motivated by the desire for a modal Loomis-Sikorski theorem: it dually corresponds to $\Diamond$ sending Baire null-sets to Baire null-sets (Theorem 7.3), which is a property used in Lemma 8.3 en route to proving the Loomis-Sikorski theorem (Theorem 8.5).

A formula φ is *valid* in a σ -modal algebra $(B, \Diamond)$, notation: $(B, \Diamond) \models \varphi$, if it evaluates to $\top$ under every assignment of elements in B to the proposition letters. We write $\sigma\mathbf{MA} \models \varphi$ if φ is valid in every σ -modal algebra, and $\mathbf{MMA} \models \varphi$ if φ is valid in every measurable σ -modal algebra. A set $Ax \subseteq \mathcal{L}_\sigma$ of axioms is valid in a σ -modal algebra $(B, \Diamond)$ if all formulas in Ax are valid.

Remark 4.2. The condition in (4) which we require in a measurable σ -modal algebra may be reformulated to emphasize a connection to compactness. For all sequences (b_n) , not necessarily descending,

$$\bigwedge_{\text{finite } F \subseteq \omega} \Diamond \bigwedge_{n \in F} b_n > \perp \quad \text{implies} \quad \bigwedge_{n \in \omega} b_n > \perp$$

It is also possible to reformulate the (IDC) rule in the logic in a parallel way.

Recall that a σ -ideal of a σ -Boolean algebra B is a nonempty set closed downward in the order and also under countable joins. We call two elements $a, b \in B$ *equivalent modulo I* , and write $a \equiv_I b$, if their symmetric difference $a \Delta b := (a \wedge \neg b) \vee (b \wedge \neg a)$ is in I , and denote the equivalence class of b by $[b]$. Then we can equip $B/I = \{[b] \mid b \in B\}$ with a σ -Boolean algebra structure by defining $\perp^I = [\perp]$, $\neg^I[b] = [\neg b]$ and $\bigvee_{n \in \omega}^I [b_n] = [\bigvee_{n \in \omega} b_n]$. This extends to σ -modal algebras as follows.

Definition 4.3. Let $(B, \Diamond)$ be a σ -modal algebra. A *modal σ -ideal* is a σ -ideal I of B such that $b \in I$ implies $\Diamond b \in I$ for all $b \in B$. We define the *quotient* of $(B, \Diamond)$ by I by

$$(B, \Diamond)/I := (B/I, \Diamond^I)$$

where $\Diamond^I[b] = [\Diamond b]$.

The next lemma allows us to show that the definition of $\Diamond^I[b]$ is well defined.

Lemma 4.4. Let $(B, \Diamond)$ be a σ -modal algebra and $I \subseteq B$ a modal σ -ideal. Then $a \equiv_I b$ implies $\Diamond a \equiv_I \Diamond b$ for all $a, b \in B$.

Proof. First, we note

$$\Diamond a \leq \Diamond(a \vee b) = \Diamond((a \vee b) \wedge (\neg b \vee b)) = \Diamond((a \wedge \neg b) \vee b) = \Diamond(a \wedge \neg b) \vee \Diamond b.$$

This implies $\Diamond a \wedge \neg \Diamond b \leq \Diamond(a \wedge \neg b)$. Similarly, we can derive $\Diamond b \wedge \neg \Diamond a \leq \Diamond(b \wedge \neg a)$. Therefore,

$$(\Diamond a) \Delta (\Diamond b) = (\Diamond a \wedge \neg \Diamond b) \vee (\Diamond b \wedge \neg \Diamond a) \leq \Diamond(a \wedge \neg b) \vee \Diamond(b \wedge \neg a) = \Diamond(a \Delta b).$$

Now if $a \equiv_I b$ then $a \Delta b \in I$ so by definition $\Diamond(a \Delta b) \in I$. Since I is downward closed, we find $(\Diamond a) \Delta (\Diamond b) \in I$, as desired. $\square$

Proposition 4.5. If $(B, \Diamond)$ is a σ -modal algebra and $I \subseteq B$ a modal σ -ideal, then $(B, \Diamond)/I$ is a σ -modal algebra again.

Proof. We know from Lemma 4.4 that $a \equiv_I b$ implies $\Diamond a \equiv_I \Diamond b$, so $\Diamond^I$ is well defined. Additionally, it follows immediately from the definitions that the equations in (3) are satisfied, showing that $(B, \Diamond)/I$ is a σ -modal algebra. $\square$

Recall that a formula φ is valid on a σ -modal algebra $(B, \Diamond)$ if it evaluates to $\top$ under every evaluation of the proposition letters. A rule ρ is valid if for every assignment of the proposition letters: if all formulas in the antecedent of ρ evaluate to $\top$, then the consequent also evaluates to $\top$.

Theorem 4.6. *Let $Ax \subseteq \mathcal{L}_\sigma$ be a set of axioms (formulas).*

1. $K_\sigma \oplus Ax$ is sound and complete with respect to the class of σ -modal algebras that validate Ax .
2. $MLS \oplus Ax$ is sound and complete with respect to the class of measurable σ -modal algebras that validate Ax .

Proof. Soundness of all these logics follows from the fact that all the axioms and rules are valid in every (measurable) σ -modal algebra $(B, \Diamond)$. For the propositional axioms and rules this follows from the fact that B is a σ -Boolean algebra, and the modal axioms and rules correspond precisely to those for the algebras. For example, it is easy to check that the Infinite Descending Chain rule (IDC) corresponds to Equation (4).

For completeness, we use a standard Lindenbaum-Tarski construction; this is enabled by Proposition 4.5. Let L be the set of formulas modulo provable equivalence, i.e. $\varphi \sim_L \psi$ iff $L \vdash \varphi \leftrightarrow \psi$. The usual argument shows that every non-theorem of $L = K_\sigma \oplus Ax$ is refuted on this algebra. It remains to show that this algebra is a $K_\sigma \oplus Ax$ -algebra, which can be verified using a standard argument.

Finally, for $MLS \oplus Ax$ we need to show that the Lindenbaum-Tarski algebra validates $MLS \oplus Ax$, in particular, that it validates (IDC). However, this also follows from general considerations. Note that given a logic L , all the rules of L are admissible in it [20]. Also, in any variety corresponding to a logic L , the Lindenbaum-Tarski algebra of L validates a rule ρ if and only if ρ is admissible in L [20, Theorems 1.3.21 and 1.4.5]. Thus, the Lindenbaum-Tarski algebra of $MLS \oplus Ax$ validates (IDC) and, as above, it also validates Ax , which finishes the proof. $\square$

5 Modal measurable spaces

We turn to the semantic structures in which we will interpret $\mathcal{L}_\sigma$. We use measurable spaces with an additional relation R , where propositional connectives are interpreted in the measurable space as usual and the modal operator $\Diamond$ is interpreted via R . On the one hand, such a semantic structure may be viewed as a modal analogue of a measurable space. On the other hand it is a Kripke frame with a measurable space structure.

We prove that such structures provide a sound semantics for K_σ , and characterise precisely for which such spaces the infinite descending chain rule (IDC) is valid. In Section 6 we take the key step of this paper by extending this semantics to incorporate an ideal of designated null-sets.

Definition 5.1. A *modal measurable space* is a tuple $\mathfrak{F} = (X, R, \Sigma)$ consisting of a measurable space (X, Σ) and a relation R such that for every $a \in \Sigma$ we have $\Diamond a := \{x \in X \mid R[x] \cap a \neq \emptyset\} \in \Sigma$.

A *modal measurable model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ consisting of a modal measurable space $\mathfrak{F}$ and a valuation $V : \text{Prop} \rightarrow \Sigma$ that assigns to each proposition letter p a measurable subset of X . The interpretation of an $\mathcal{L}_\sigma$ -formula φ in $\mathfrak{M}$ is defined recursively via

$$\llbracket p \rrbracket^{\mathfrak{M}} = V(p), \quad \llbracket \neg \varphi \rrbracket^{\mathfrak{M}} = X \setminus \llbracket \varphi \rrbracket^{\mathfrak{M}}, \quad \llbracket \bigvee_{n \in \omega} \varphi_n \rrbracket^{\mathfrak{M}} = \bigcup_{n \in \omega} \llbracket \varphi_n \rrbracket^{\mathfrak{M}}, \quad \llbracket \Diamond \varphi \rrbracket^{\mathfrak{M}} = \Diamond \llbracket \varphi \rrbracket^{\mathfrak{M}}.$$

A formula φ is *valid* in a modal measurable space $\mathfrak{F} = (X, R, \Sigma)$ if $\llbracket \varphi \rrbracket^{\mathfrak{M}} = X$ for every modal measurable model of the form $\mathfrak{M} = (\mathfrak{F}, V)$. We write $\text{MMS} \models \varphi$ if φ is valid on all modal measurable spaces.

We clearly have:

Lemma 5.2. *Let $\mathfrak{M}$ be a measurable Kripke model. Then $\llbracket \varphi \rrbracket$ is measurable for each formula φ .*

The complex algebra of a modal measurable space is defined as expected:

Definition 5.3. The *complex algebra* of a modal measurable space $\mathfrak{F} = (X, R, \Sigma)$ is $\mathfrak{F}^+ := (\Sigma, \diamond)$.

The semantic interpretation of φ corresponds to the algebraic interpretation in $\mathfrak{F}^+$, viewing an assignment V as an assignment for $\mathfrak{F}^+$. The following proposition entails that K_σ is sound with respect to modal measurable spaces by showing that the complex algebra of every modal measurable space is a σ -modal algebra.

Lemma 5.4. *Let $\mathfrak{F} = (X, R, \Sigma)$ be a modal measurable space. Then $\mathfrak{F}^+$ is a σ -modal algebra.*

Proof. We know that Σ is a σ -Boolean algebra, so we only verify the equations from (3). To this end, compute $\diamond \emptyset = \{x \in X \mid R[x] \cap \emptyset \neq \emptyset\} = \emptyset$ and

$$\diamond \bigcup_{n \in \omega} b_n = \{x \in X \mid R[x] \cap (\bigcup_{n \in \omega} b_n) \neq \emptyset\} = \{x \in X \mid R[x] \cap b_n \neq \emptyset \text{ for some } n \in \omega\} = \bigcup_{n \in \omega} \diamond b_n.$$

It follows that $\mathfrak{F}^+$ is indeed a σ -modal algebra. $\square$

The following result characterises the modal measurable spaces which validate the rule (IDC). It is not needed in what follows.

Definition 5.5.

1. Let (X, Σ) be a measurable space. We call a set $D \subseteq X$ *measurably compact* if for every countable measurable cover $C = \{c_n \mid n \in \omega\}$ of X there exists a finite subset $C' \subseteq C$ that covers D .
2. A modal measurable space (X, R, Σ) is called *pointwise measurably compact* if the set $R[x]$ is measurably compact for each $x \in X$.

Lemma 5.6. *Let $\mathfrak{F} = (X, R, \Sigma)$ be a modal measurable space. Then $\mathfrak{F}^+$ is a measurable σ -modal algebra if and only if $\mathfrak{F}$ is pointwise measurably compact.*

Proof. Suppose that $\mathfrak{F}$ is pointwise measurably compact and let $(a_n)_{n \in \omega}$ be a countable sequence of elements of $\mathfrak{F}^+$ such that $a_n \geq a_{n+1}$ for all $n \in \omega$, and $\bigcap_{n \in \omega} a_n = \emptyset$. We need to show that $\bigcap_{n \in \omega} \diamond a_n = \emptyset$.

By assumption $\{X \setminus a_n \mid n \in \omega\}$ is a measurable cover of X , so for any $x \in X$ we can find a finite number $a_{n_1}, \dots, a_{n_k}$ of a_n 's such that $R[x] \subseteq (X \setminus a_{n_1}) \cup \dots \cup (X \setminus a_{n_k})$. It follows that $R[x] \cap a_{n_1} \cap \dots \cap a_{n_k} = \emptyset$. Since the a_n form a descending chain, we can take $m := \min\{n_1, \dots, n_k\}$ to get $R[x] \cap a_m = \emptyset$. This proves $x \notin \diamond a_m$, hence $x \notin \bigcap_{n \in \omega} \diamond a_n$. Since $x \in X$ was arbitrary, we conclude $\bigcap_{n \in \omega} \diamond a_n = \emptyset$, as desired.

For the converse, suppose there exists some countable measurable cover C of X and a world $x \in X$ such that $R[x]$ cannot be covered by any finite subset of C . Write $C = \{c_n \mid n \in \omega\}$. Define $a_n := X \setminus (c_0 \cup \dots \cup c_{n-1})$. Then $a_n \supseteq a_{n+1}$ for all $n \in \omega$ and $\bigcap_{n \in \omega} a_n = X \setminus \bigcup_{n \in \omega} c_n = X \setminus X = \emptyset$, so the premisses of (IDC) are valid. But for every $n \in \omega$ we have $R[x] \not\subseteq c_0 \cup \dots \cup c_{n-1}$, so $R[x] \cap a_n \neq \emptyset$, hence $x \in \diamond a_n$. Therefore $x \in \bigcap_{n \in \omega} \diamond a_n \neq \emptyset$, showing that (IDC) is not valid on $\mathfrak{F}^+$. $\square$

Example 5.7. Here is an example of a modal measurable space that is not pointwise measurably compact. Consider $\mathfrak{F} = (X, R, \Sigma)$ with $X = \omega$, i.e. the set of natural numbers, $R[x] = \omega$ for all $x \in X$ and $\Sigma = \mathcal{P}(\omega)$. Then $\mathfrak{F}$ is a modal measurable space, but it is not pointwise measurably compact. Indeed, $C = \{\{n\} \mid n \in \omega\}$ is a measurable cover of X but for any $x \in X$, since $R[x] = \omega$, there is no finite subcover of C covering $R[x]$.

Theorem 5.8. *The logic K_σ is sound with respect to the class of all modal measurable spaces. The logic MLS is sound with respect to the class of pointwise measurably compact modal measurable spaces.*

Proof. This follows from Lemmas 5.4 and 5.6 and the fact that the algebraic semantics is sound with respect to the two logics. $\square$

We end this section with the following open problem:

Problem 5.9. Are K_σ and MLS complete with respect to suitable classes of modal measurable spaces? If not, can we axiomatise the logics of (pointwise measurably compact) modal measurable spaces?

6 Marked modal measurable spaces

We ended the previous section with the open problem of axiomatising the logic of modal measurable spaces. In this section we introduce *marked* modal measurable spaces, which can be considered as “point-free analogues” of modal measurable spaces. Marked modal measurable spaces extend modal measurable spaces with a designated σ -ideal of (marked) null-sets. Their use is closely connected to a point-free approach to ergodic theory, see e.g. [12, 19]. We introduce a new semantics on these spaces and prove in Section 8 that MLS is sound and complete with respect to this semantics.

Definition 6.1. A *marked modal measurable space* or *mmm-space* is a tuple (X, R, Σ, N) where

- (X, R, Σ) is a modal measurable space; and
- N is a modal σ -ideal on $\mathfrak{F}^+ = (\Sigma, \diamond)$ such that for all countable collections of measurable sets $(a_n)_{n \in \omega}$,

$$\bigcap_{n \in \omega} a_n \in N \quad \text{implies} \quad \bigcap_{n \in \omega} \diamond a_n \in N.$$

The *complex algebra* of an mmm-model is the σ -modal algebra $\mathfrak{F}^\dagger := (\Sigma, \diamond)/N$.

It follows from Lemma 5.4 and Proposition 4.5 that $\mathfrak{F}^\dagger$ is a σ -modal algebra.

Example 6.2. Let (X, O) be a topological space, let Σ be the induced Borel σ -algebra, and let μ be a probability measure on Σ . Let $f : X \rightarrow X$ be a *measure-preserving* transformation, as in dynamical systems and ergodic theory, i.e. $f^{-1}(b) \in \Sigma$ and $\mu f^{-1}(b) = \mu(b)$ for all $b \in \Sigma$. Let R be the graph of f , i.e. xRy iff $f(x) = y$, and let N be the collection of null-sets. Then $\diamond a = f^{-1}(a)$ for any measurable set a . Furthermore, for any sequence $(a_n)_{n \in \omega}$ of measurable sets whose intersection is in N , we have $\bigcap \diamond a_n = \bigcap f^{-1}(a_n) = f^{-1}(\bigcap a_n) \in N$ because f is measure-preserving. Therefore (X, R, Σ, N) is an mmm-space.

As a concrete example, we may consider (X, O) to be the unit interval with the Euclidean topology, μ the Lebesgue measure and $f(x)$ the Bernoulli map $f(x) = 2x \bmod 1$.

Definition 6.3. An *mmm-model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ consisting of an mmm-space $\mathfrak{F} = (X, R, \Sigma, N)$ and a valuation $V : \text{Prop} \rightarrow \Sigma$. The *interpretation* of an $\mathcal{L}_\sigma$ -formula φ in $\mathfrak{M}$ is the equivalence class of $\llbracket \varphi \rrbracket$ in the complex algebra $\mathfrak{F}^\dagger$, which we denote by $\langle \varphi \rangle^{\mathfrak{M}}$ for the class of mmm-spaces.

A formula φ is said to be *valid* on an mmm-space $\mathfrak{F}$ if $\langle \varphi \rangle^{\mathfrak{M}}$ is the top element of $\mathfrak{F}^\dagger$ for every mmm-model $\mathfrak{M} = (\mathfrak{F}, V)$ based on $\mathfrak{F}$. In other words, φ is valid on $\mathfrak{F}$ if $X \setminus \llbracket \varphi \rrbracket^{\mathfrak{M}} \in N$ for every mmm-model $\mathfrak{M}$ based on $\mathfrak{F}$. For a set Ax , we write $\text{MMM}(Ax)$ for class of mmm-spaces validating all formulas in Ax .

Theorem 6.4. *For any $Ax \subseteq \mathcal{L}_\sigma$, the logic $\text{MLS} \oplus Ax$ is sound with respect to $\text{MMM}(Ax)$.*

Proof. Let $\mathfrak{F} = (X, R, \Sigma, N)$ be an mmm-space that validates Ax . Then its complex algebra $\mathfrak{F}^\dagger$ is a σ -modal algebra by Proposition 4.5 and it validates Ax by assumption. Moreover, since $\bigcap_{n \in \omega} a_n \in N$ implies $\bigcap_{n \in \omega} \diamond a_n \in N$, the complex algebra also satisfies (IDC). Therefore $\mathfrak{F}^\dagger$ is a measurable σ -modal algebra that validates Ax . Soundness now follows from the algebraic soundness given in Theorem 4.6(2). $\square$

7 Jónsson-Tarski duality for (measurable) σ -modal algebras

We restrict Jónsson-Tarski duality (see Section 2) to a duality for σ -modal algebras and measurable σ -modal algebras. In order to prove completeness of MLS in Section 8, we only need this restriction on objects, so we focus on restricting the object part of Jónsson-Tarski duality.

Definition 7.1. Let (X, τ) be a Stone space. A *basic null-set* is a set of the form $\bigcap_{n \in \omega} a_n$, where $(a_n)_{n \in \omega}$ is a sequence of clopen sets of (X, τ) such that $\text{int}(\bigcap_{n \in \omega} a_n) = \emptyset$. A *Baire null-set* is a subset of X that can be covered by countably many basic null-sets.

Definition 7.2. A σ -modal space is a modal space $\mathbb{X} = (X, \tau, R)$ such that (X, τ) is a σ -Stone space and such that for any countable sequence $(a_n)_{n \in \omega}$ of clopen sets,

$$\diamond \left(\overline{\bigcup_{n \in \omega} a_n} \right) = \overline{\diamond \left(\bigcup_{n \in \omega} a_n \right)}. \quad (5)$$

A *measurable σ -modal space* is a σ -modal space that additionally satisfies that for every Baire null-set S , the set

$$\diamond S := \{x \in X \mid R[x] \cap S \neq \emptyset\}$$

is a Baire null-set too.

We will show that these properties correspond to the Jónsson-Tarski duals being (measurable) σ -modal algebras. In the next result, we recall that the σ -Boolean algebra structure in the dual space $\mathbb{X}$ of a Boolean (or modal) algebra B is given by (2).

Theorem 7.3. Let $(B, \diamond)$ be a modal algebra and $\mathbb{X} = (X, \tau, R)$ its dual modal space. Then

1. $(B, \diamond)$ is a σ -modal algebra if and only if $\mathbb{X}$ is a σ -modal space;
2. $(B, \diamond)$ is a measurable σ -modal algebra if and only if $\mathbb{X}$ is a measurable σ -modal space.

Proof. (1) Suppose $(B, \diamond)$ is a σ -modal algebra. Then we know from Theorem 2.2 that (X, τ) is a σ -Stone space. Furthermore, for any countable sequence $(b_n)_{n \in \omega}$ of elements in B we have

$$\bigvee_{n \in \omega} \diamond b_n = \diamond \bigvee_{n \in \omega} b_n$$

so it follows from Jónsson-Tarski duality that for any sequence $(a_n)_{n \in \omega}$ of elements in $\text{Clp}(\mathbb{X})$ we have

$$\overline{\bigcup_{n \in \omega} \diamond a_n} = \diamond \overline{\bigcup_{n \in \omega} a_n}$$

By definition of $\diamond$ we have $\diamond \bigcup_{n \in \omega} a_n = \bigcup_{n \in \omega} \diamond a_n$, which entails the desired equality $\diamond(\overline{\bigcup_{n \in \omega} a_n}) = \overline{\bigcup_{n \in \omega} \diamond a_n}$.

For the converse, assume that (X, τ, R) satisfies (5). By assumption (X, τ) is a σ -Stone space and $(B, \diamond)$ is a modal algebra, so we know that B is a σ -Boolean algebra and that $\diamond \perp = \perp$. It remains

to show that diamonds distribute over countable joins. Since $(B, \diamond) \cong (\text{Clp}(\mathbb{X}), \diamond)$ is a modal algebra isomorphism and every such isomorphism is automatically a σ -modal algebra isomorphism, it suffices to show that the dual Boolean algebra of (X, τ, R) satisfies

$$\overline{\bigcup_{n \in \omega} \diamond a_n} = \diamond \overline{\bigcup_{n \in \omega} a_n}$$

for every sequence $(a_n)_{n \in \omega}$ of elements in $\text{Clp}(\mathbb{X})$. In light of (5), it suffices to show $\overline{\bigcup_{n \in \omega} \diamond a_n} = \diamond \bigcup_{n \in \omega} a_n$. But this follows from the fact that $\bigcup_{n \in \omega} \diamond a_n = \diamond \bigcup_{n \in \omega} a_n$, which is an immediate consequence of the definition of $\diamond$.

(2) Suppose that $(B, \diamond)$ is a measurable σ -modal algebra. Then using the isomorphism $(B, \diamond) \cong (\text{Clp}(\mathbb{X}), \diamond)$, we know that for any countable descending chain $a_0 \supseteq a_1 \supseteq a_2 \supseteq \dots$ of clopen subsets of $\mathbb{X}$,

$$\text{int}\left(\bigcap_{n \in \omega} a_n\right) = \emptyset \quad \text{implies} \quad \text{int}\left(\bigcap_{n \in \omega} \diamond a_n\right) = \emptyset.$$

Let S be a Baire null-set. Then $S \subseteq \bigcup_{n \in \omega} S_n$ for some collection $(S_n)_{n \in \omega}$ of basic null-sets. Since $\diamond S \subseteq \diamond(\bigcup_{n \in \omega} S_n) = \bigcup_{n \in \omega} \diamond S_n$ and the countable union of Baire null-sets is a Baire null-set (by definition), it suffices to show that $\diamond S_n$ is a Baire null-set for any basic null-set S_n .

So let $S = \bigcap_{n \in \omega} a_n$ be a basic null-set, where $(a_n)_{n \in \omega}$ is a sequence of clopen subsets of $\mathbb{X}$ such that $\text{int}(\bigcap_{n \in \omega} a_n) = \emptyset$. Defining $a'_n := a_0 \cap \dots \cap a_n$ yields a countable descending chain such that $\bigcap_{n \in \omega} a'_n = \bigcap_{n \in \omega} a_n$, hence $\text{int}(\bigcap_{n \in \omega} a'_n) = \emptyset$. This implies $\text{int}(\bigcap_{n \in \omega} \diamond a'_n) = \emptyset$. If we can prove that

$$\bigcap_{n \in \omega} \diamond a'_n = \diamond S \tag{6}$$

then we are done. To this end, let $x \in \diamond S$. Then there is some y such that xRy and $y \in S$. This implies $y \in a'_n$, hence $x \in \diamond a'_n$, for all $n \in \omega$. Therefore $x \in \bigcap_{n \in \omega} \diamond a'_n$. For the converse, suppose $x \in \bigcap_{n \in \omega} \diamond a'_n$. Then using the fact that the a'_n are decreasing, one can show that $\{R[x]\} \cup \{a'_n \mid n \in \omega\}$ has the finite intersection property. So by compactness there exists some $y \in R[x] \cap \bigcap_{n \in \omega} a'_n = R[x] \cap S$, so $x \in \diamond S$. This proves (6).

Conversely, suppose $\diamond S$ is a Baire null-set for any Baire null-set S . We show that $(B, \diamond)$ is a measurable σ -modal algebra by checking (4). Let $(a_n)_{n \in \omega}$ be a countable descending chain of clopen sets such that $\text{int}(\bigcap_{n \in \omega} a_n) = \emptyset$. Then $\bigcap_{n \in \omega} a_n$ is a basic null-set by definition. The assumption now gives that $\diamond(\bigcap_{n \in \omega} a_n)$ is a Baire null-set. Using (6) we then conclude that $\bigcap_{n \in \omega} \diamond a_n$ is a Baire null-set, as desired. $\square$

Remark 7.4. It follows immediately from the definitions that every Baire null-set in a Stone space is a meager set, but it is unclear whether the converse holds. We leave this as an open question.

8 Completeness via a modal Loomis-Sikorski representation

While Theorem 7.3 provides a duality for the algebraic semantics of both K_σ and MLS, it does not yield completeness for either with respect to (marked) modal measurable spaces. This is because countable meets and joins in (measurable) σ -modal spaces need not be given by intersections and unions. In order to obtain a completeness result for MLS with respect to mmm-spaces, we prove a modal analogue of the Loomis-Sikorski theorem which states that every measurable modal algebra can be obtained as the quotient of a modal measurable space by a modal σ -ideal.

Definition 8.1. Let (X, R) be a Kripke frame. A *modal σ -field* on (X, R) is a collection $\mathcal{F} \subseteq \mathcal{P}X$ that contains X and is closed under countable unions, complements and $\diamond$ (hence also contains $\emptyset$ and is closed under countable intersections).

We note that a modal measurable space (Definition 5.1) can be viewed as a triple (X, R, Σ) consisting of a Kripke frame (X, R) together with a modal σ -field Σ on (X, R) .

The arbitrary intersection of modal σ -fields on a Kripke frame is again a modal σ -field. Therefore we can define the modal σ -field on (X, R) generated by some collection C of subsets of X as the intersection of all modal σ -fields containing C .

Definition 8.2. Let $\mathbb{X} = (X, \tau, R)$ be a σ -modal space. Then we write $\text{Baire}(\mathbb{X})$ for the modal σ -field on (X, R) generated by the clopen subsets of (X, τ) . We call sets in $\text{Baire}(\mathbb{X})$ *modal Baire sets*.

In analogy with the non-modal case (see e.g. [9, Chapter 40]) we find that the modal Baire null-sets, i.e. sets of a Stone space that are both modal Baire sets and Baire null-sets, form a modal σ -ideal. This only works for measurable σ -modal spaces, because we need the fact that $\diamond S$ is a Baire null-set whenever S is a Baire null-set. As a consequence, the modal analogue of the Loomis-Sikorski theorem (Theorem 8.5) only applies to measurable σ -modal algebras.

Lemma 8.3. Let $\mathbb{X} = (X, \tau, R)$ be a measurable σ -modal space. Then the collection $\mathcal{M}$ of modal Baire null-sets of (X, τ) forms a modal σ -ideal of $\text{Baire}(\mathbb{X})$.

Proof. It follows from the definition of a Baire null-set that $\mathcal{M}$ is a σ -ideal, and it is closed under $\diamond$ because by the definition of a measurable σ -modal space, $\diamond S$ is a Baire null-set for any Baire null-set S . $\square$

Lemma 8.4. Every modal Baire set of a measurable σ -modal space is equivalent to a unique clopen set modulo the modal σ -ideal $\mathcal{M}$ of modal Baire null-sets.

Proof. Let $\mathbb{X} = (X, \tau, R)$ be a measurable σ -modal space and $\mathcal{M}$ the modal σ -ideal of modal Baire null-sets. Let $\mathcal{F}$ be the class of subsets of X that are equivalent to a clopen subset modulo $\mathcal{M}$. We claim that $\mathcal{F}$ is a modal σ -field on (X, R) that includes $\text{Baire}(\mathbb{X})$.

To this end, first we note that every clopen subset a of $\mathbb{X}$ is in $\mathcal{F}$ because $a \Delta a = \emptyset \in \mathcal{M}$. To see that $\mathcal{F}$ is closed under complementation, let $S \in \mathcal{F}$ and suppose a is a clopen set such that $S \equiv_{\mathcal{M}} a$, i.e. $S \Delta a \in \mathcal{M}$. Then

$$(X \setminus S) \Delta (X \setminus a) = S \Delta a \in \mathcal{M}$$

and since $X \setminus a$ is clopen, we get $X \setminus S \in \mathcal{F}$.

Next, let $(S_n)_{n \in \omega}$ be a sequence of sets in $\mathcal{F}$, and let $(a_n)_{n \in \omega}$ be a sequence of clopen sets such that $S_n \equiv_{\mathcal{M}} a_n$ for each $n \in \omega$. Then we have

$$\left(\bigcup_{n \in \omega} S_n \right) \Delta \left(\bigcup_{n \in \omega} a_n \right) \subseteq \bigcup_{n \in \omega} (S_n \Delta a_n) \in \mathcal{M},$$

so $\bigcup_{n \in \omega} S_n \equiv_{\mathcal{M}} \bigcup_{n \in \omega} a_n$. Now let $a := \overline{\bigcup_{n \in \omega} a_n}$. Then a is clopen because $\mathbb{X}$ is a σ -modal space. Moreover, we have

$$a \Delta \left(\bigcup_{n \in \omega} a_n \right) = a \setminus \left(\bigcup_{n \in \omega} a_n \right) = a \cap \bigcap_{n \in \omega} (X \setminus a_n),$$

which is in $\mathcal{M}$ because

$$\text{int} \left(a \cap \bigcap_{n \in \omega} (X \setminus a_n) \right) = \text{int}(a) \cap \text{int} \left(\bigcap_{n \in \omega} X \setminus a_n \right) = \text{int}(a) \cap X \setminus \overline{\left(\bigcup_{n \in \omega} a_n \right)} = \text{int}(a) \cap (X \setminus a) = \emptyset.$$

So we have $\bigcup_{n \in \omega} S_n \equiv_{\mathcal{M}} \bigcup_{n \in \omega} a_n \equiv_{\mathcal{M}} a$. Hence $\bigcup_{n \in \omega} S_n \in \mathcal{F}$.

Finally, if $S \in \mathcal{F}$ and $S \equiv_{\mathcal{M}} a$, where a is clopen, then Lemma 4.4 entails $\Diamond S \equiv_{\mathcal{M}} \Diamond a$. Since $\Diamond a$ is clopen, it follows that $\Diamond S \in \mathcal{F}$. Therefore $\mathcal{F}$ is a modal σ -field on (X, R) that contains the clopen subsets, hence $\text{Baire}(\mathbb{X}) \subseteq \mathcal{F}$. This shows that every modal Baire set is equivalent modulo $\mathcal{M}$ to a clopen subset.

For uniqueness, suppose that the modal Baire set S is equivalent modulo $\mathcal{M}$ to each of the clopen sets a and b . Then $a \equiv_{\mathcal{M}} b$, so $a \Delta b \in \mathcal{M}$. Then $a \Delta b$ is open because a and b are clopen, and since every Baire null-set is in particular meager we can use [9, Theorem 28] (which is a corollary of the Baire Category Theorem) to conclude $a \Delta b = \emptyset$. Therefore $a = b$. $\square$

We can now prove a modal analogue of the Loomis-Sikorski theorem [9, Chapter 40, Corollary 1].

Theorem 8.5. *Every measurable σ -modal algebra is isomorphic to a modal σ -field modulo a modal σ -ideal.*

Proof. Let $(B, \Diamond)$ be a σ -modal algebra, $\mathbb{X} = (X, \tau, R)$ its dual σ -modal space, and $\text{Baire}(\mathbb{X})$ the modal σ -field of modal Baire sets of $\mathbb{X}$. Let f be the function that takes each $S \in \text{Baire}(\mathbb{X})$ to the unique clopen set a to which S is equivalent. Then f is a surjective function.

We claim that f is a σ -homomorphism. Clearly $f(X) = X$ and $f(\emptyset) = \emptyset$. If $S \in \text{Baire}(\mathbb{X})$ and $S \equiv_{\mathcal{M}} a$ then we know $(X \setminus S) \equiv_{\mathcal{M}} (X \setminus a)$. This implies $f(X \setminus S) = X \setminus a = X \setminus f(S)$, so f preserves negations. Similarly, if $(S_n)_{n \in \omega}$ is a countable sequence of modal Baire sets and $f(S_n) = a_n$ for each $n \in \omega$, then we know that

$$f\left(\bigcup_{n \in \omega} S_n\right) = \overline{\bigcup_{n \in \omega} a_n} = \bigcap_{n \in \omega} a_n = \bigcap_{n \in \omega} f(S_n).$$

Finally, we check that f commutes with $\Diamond$. Let $\Diamond S \in \text{Baire}(\mathbb{X})$, and let a be the unique clopen set such that $S \equiv_{\mathcal{M}} a$. Then $f(S) = a$. Moreover, by Lemma 8.3, we have $f(\Diamond S) = \Diamond a$ as well: $S \equiv_{\mathcal{M}} a$ implies that $\Diamond S \equiv_{\mathcal{M}} \Diamond a$, and this just means that $f(\Diamond S) = \Diamond a$. So $f(\Diamond S) = \Diamond a = \Diamond f(S)$. All in all, f is a surjective modal σ -homomorphism. So $(B, \Diamond) \cong \text{Baire}(\mathbb{X})/\mathcal{M}$. $\square$

This sets us up to prove completeness of $\text{MLS} \oplus \text{Ax}$ with respect to the class of mmm-spaces validating all axioms in Ax .

Theorem 8.6. *The logic $\text{MLS} \oplus \text{Ax}$ is sound and complete with respect to the class of mmm-spaces validating Ax . That is, is for every formula φ we have*

$$\text{MLS} \oplus \text{Ax} \vdash \varphi \quad \text{iff} \quad \text{MMM}(\text{Ax}) \Vdash \varphi.$$

Proof. We proved soundness in Theorem 6.4. For completeness, suppose φ is a formula such that $\text{MLS} \oplus \text{Ax} \not\vdash \varphi$. Then by Theorem 4.6 there exists a measurable σ -modal algebra $(B, \Diamond)$ such that $(B, \Diamond) \Vdash \text{Ax}$ and $(B, \Diamond) \not\vdash \varphi$. By the proof of the Modal Loomis-Sikorski Theorem 8.5 there exists a modal measurable space (X, R, Σ) and a modal σ -ideal $\mathcal{M}$ that is closed under $\Diamond$ such that $(\Sigma, \Diamond)/\mathcal{M}$ is isomorphic to $(B, \Diamond)$. This implies that $\mathfrak{F} = (X, R, \Sigma, \mathcal{M})$ is an mmm-space such that $\mathfrak{F} \not\vdash \varphi$, as desired. $\square$

9 Conclusion

In this paper we introduced a new semantics and logic for reasoning about relational structures based on measurable spaces. Our main method is a modal version of the Loomis-Sikorski theorem, which yields a completeness result for these infinitary logics with respect to the new semantics.

We view this work as a first step in developing a modal logical framework for reasoning about measure-based dynamical systems and (point-free) ergodic theory. This naturally gives rise to many avenues for future research, the first of which was already mentioned in Problem 5.9 above.

Second, as mentioned in the introduction, our framework considers the standard Kripke semantics but based on measurable spaces. A natural next step would be to make the relations probabilistic and, more generally, to incorporate probability explicitly into the framework. Orthogonal to this, it would be interesting to investigate whether the logic and semantics benefit a coalgebraic perspective.

Finally, the framework used in this paper could be generalised to other algebraic structures arising in classical logic, such as distributive lattices and (modal) Heyting algebras. This leads to the question of whether there are Loomis–Sikorski type theorems for distributive lattices, (modal) Heyting algebras, and what the corresponding logics, semantics, and completeness results would be. Ultimately, we would like to develop constructive measure theory based on σ -Heyting algebras, with an appropriate analogue of the Loomis–Sikorski theorem.

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A Semantics for Belief in Simplicial Complexes

Adam Bjorndahl

Carnegie Mellon University
Pittsburgh, PA
abjorn@cmu.edu

Philip Sink

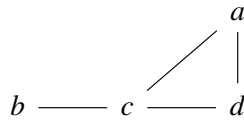
Carnegie Mellon University
Pittsburgh, PA
psink@andrew.cmu.edu

We provide novel semantics for belief using simplicial complexes. In our framework, belief is a KD45 modality that satisfies “knowledge implies belief” (“If you know ϕ , then you believe ϕ ”); in addition, we adopt the (standard) assumption that each facet in our simplicial models contains exactly one vertex for each agent. No existing model of belief in simplicial complexes that we are aware of is able to satisfy all of these conditions without trivializing belief to coincide with knowledge. We establish a truth-preserving correspondence between our simplicial framework and standard relational models for knowledge and belief; this involves, notably, proving that all relational models can be simulated using proper relational models, a result of independent interest. Finally, we apply these results to provide a simple axiomatization.

1 Introduction

Interpreting multi-agent epistemic logic using simplicial complexes is a recent and thriving area of research. [2, 6, 8, 7, 11, 9, 10, 4] Our goal in this paper is to extend this approach to accommodate not only knowledge but also *belief*; this turns out to involve some subtleties, the resolutions of which are interesting in their own right.

To set the stage, we begin by briefly reviewing the basic notion of a simplicial complex and the intended epistemic interpretation. Informally, a *simplex* can be pictured as a multi-dimensional triangle; that is, a collection of vertices where every vertex is connected to every other vertex by an edge. So with three vertices we obtain a standard triangle, four gives a tetrahedron, and in general, n many vertices gives an $(n - 1)$ -dimensional triangle. A *simplicial complex* can be thought of as a collection of simplices with some perhaps sharing vertices, edges, or larger faces. The simple example below depicts a 1-dimensional simplex (bc) and a 2-dimensional simplex (acd) glued together at a single vertex (c):



Formally, a **simplicial complex** is a set S closed under subsets; that is, if $X \subseteq Y \in S$, then $X \in S$. Intuitively, S is the set of all triangles in the simplicial complex. Closure under subsets corresponds to the idea that any triangle (of any dimension) contains all of the lower-dimensional triangles that make it up; for example, a 2-dimensional triangle contains its edges (which are 1-dimensional triangles), and these edges contain their endpoint vertices (which are 0-dimensional triangles). Accordingly, every element $X \in S$ is called a *face*, the elements in such an X are called *vertices*, and those faces $X \in S$ that are maximal with respect to set inclusion are called *facets*. For example, the diagram above corresponds to the simplicial complex

$$S = \{\{b, c\}, \{a, c, d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{b\}, \{a\}, \{c\}, \{d\}, \emptyset\},$$

which has facets $\{b, c\}$ and $\{a, c, d\}$.

The basic idea behind interpreting epistemic logic using simplicial complexes is to associate vertices with agents and to view facets as analogous to possible worlds [6, 9]. Intuitively, a vertex associated to agent a represents a particular “perspective” of a , with the simplicial structure encoding which agent perspectives are epistemically compatible with which others. Thus, whereas in standard relational semantics a ’s knowledge is given by quantifying over those worlds that a “considers possible” (via an accessibility relation), in the simplicial setting this quantification is instead over those facets that share a vertex associated to agent a . We formalize all the details in Section 2.

As noted, the goal of this paper is to generalize this semantics to provide a model for multi-agent belief instead of (or in addition to) knowledge. A key obstacle to this endeavor is that simplicial semantics hardcodes *factivity* ($K\phi \rightarrow \phi$) in a way that relational semantics does not. Naturally, models for belief ought not validate the corresponding principle ($B\phi \rightarrow \phi$), since agents can have false beliefs.

In relational semantics, knowledge is typically weakened to belief by passing to a subset of those worlds considered possible. More precisely, if $R(w)$ denotes the set of all worlds epistemically accessible from w , then a reasonable candidate for a relation Q representing *doxastic* accessibility (i.e., beliefs) is one where $Q \subseteq R$ (so also $Q(w) \subseteq R(w)$). This ensures that knowledge implies belief ($K\phi \rightarrow B\phi$) and allows agents to potentially believe falsehoods even though they cannot know falsehoods (because Q need not be reflexive even though R is assumed to be).

Somewhat surprisingly, the obvious analogous approach does not work out in the simplicial setting. Suppose our model begins with a simplicial complex S in the background (representing knowledge, as usual), and adds a subcomplex $S' \subseteq S$ to represent the beliefs of each agent—so to determine an agent’s beliefs we would only quantify over facets in S' . Then, as desired, for a given facet X in S , it might be the case that ϕ is true at X yet at all facets X' in S' which are a -accessible from X , ϕ is false. This produces a counterexample to factivity for belief. However, at all facets X' in S' , every agent *will* have factive beliefs, and as a consequence this model does not allow agents to consider it possible that any *other* agents have false beliefs; this is formalized in Proposition 2.1.

Our solution to this issue is to enrich the model with multiple “belief” subcomplexes—one for each agent. This solves the problem outlined above, and leads naturally to the question of how these enriched simplicial models are related to standard relational models for knowledge and belief. This question, too, is not entirely straightforward, since the standard translation between simplicial and relational frameworks requires a “properness” condition that turns out to be quite restrictive when it comes to representing beliefs. We discuss this translation, and its restrictiveness, in Section 3, and then establish in Section 4 a general result that allows one to simulate any model with a proper model, effectively dissolving the restriction. This result is of independent interest and as such we attempt to present the details of that section in a relatively self-contained way.

We are not the first to propose a representation of belief using simplicial complexes. Both “Knowledge and Simplicial Complexes” [6] as well as “Simplicial Belief” [2] tackle the same idea. While each of these approaches is interesting in its own right, they differ substantially from our models both in formal implementation and in the overall logic of belief that they validate. We outline these frameworks and discuss their relationship to our models in Section 6.

The rest of the paper is organized as follows. In Section 2, we provide the core definitions, introduce our new model, and prove that in this framework multiple complexes are needed to allow the right kind of inter-agent uncertainty. In Section 3 we establish a translation between standard relational models for knowledge and belief and our simplicial representation, highlighting the role of the properness condition, which is thoroughly analyzed in Section 4. Section 5 applies the translation results to prove soundness and completeness. Section 6 concludes with a discussion of related and future work.

2 Semantics for Belief

The literature on simplicial semantics divides roughly into two approaches to defining valuations for propositional atoms. Loosely speaking, the first approach assigns truth values directly to the facets [10, 8, 11], while the second “vertex-based” approach assigns them (in a partial way) to the vertices and then “lifts” them to facets. [4, 6, 9, 7] While we feel that the latter approach is more interesting from a logical and epistemic perspective, the former is much more technically straightforward and will allow us to better illustrate the specific novelty of the belief models we wish to present in this work, so that is the approach we adopt here.

Let Ag be a finite set of agents. A **simplicial frame** is a tuple (N, V, S) where N is a set of *nodes*, $V : N \rightarrow Ag$ assigns an agent to each node (this is sometimes called the *coloring function*), and $S \subseteq 2^N$ is a simplicial complex. Let $\mathcal{F}(S)$ denote the set of facets of S . As in much of the previous literature [6, 9], we henceforth restrict attention to simplicial frames that satisfy the **uniquely colored facets (UCF)** condition: for each $X \in \mathcal{F}(S)$ and each $a \in Ag$, there is exactly one node $n \in X$ such that $V(n) = a$. Note that this implies that all facets of S contain exactly $|Ag|$ nodes, one for each agent.

Let $\mathfrak{P}$ be a countable set of propositional atoms. A **simplicial model** $\mathcal{M}$ is a simplicial frame (N, V, S) together with a *valuation function* $L : \mathfrak{P} \rightarrow 2^{\mathcal{F}(S)}$; intuitively, $L(P)$ tells us which facets P is true at. In this setting, the language $\mathcal{L}_K(Ag)$ recursively defined by

$$\varphi ::= P \mid \perp \mid \varphi \rightarrow \psi \mid K_a \varphi,$$

where $P \in \mathfrak{P}$ and $a \in Ag$, can be interpreted in simplicial models as follows:

$$\begin{aligned} \mathcal{M}, X &\models P \text{ iff } X \in L(P) \\ \mathcal{M}, X &\not\models \perp \\ \mathcal{M}, X &\models \varphi \rightarrow \psi \text{ iff } \mathcal{M}, X \models \varphi \text{ implies } \mathcal{M}, X \models \psi \\ \mathcal{M}, X &\models K_a \varphi \text{ iff } \forall Y \in \mathcal{F}(S) \text{ if } \pi_a(Y) = \pi_a(X) \text{ then } \mathcal{M}, Y \models \varphi, \end{aligned}$$

where $\pi_a(Z) = V^{-1}(a) \cap Z$, that is, π_a maps each facet to the unique a -colored vertex it contains. The other Boolean connectives are defined in the usual way.

This is the standard simplicial semantics for multi-agent knowledge: agent $a \in Ag$ knows φ at facet X just in case φ is true at all facets that share an a -colored node with X . Note that the factivity of knowledge follows immediately: if $X \models K_a \varphi$, then since trivially $\pi_a(X) = \pi_a(X)$, the semantic clause for K_a forces us to conclude that $X \models \varphi$.

To incorporate belief—which of course we do not want to be factive—we add in additional simplicial complexes. A **simplicial belief model** is a tuple $(N, V, S, \{S_a\}_{a \in Ag}, L)$, where (N, V, S, L) is a simplicial model and each S_a is a nonempty simplicial complex contained in S ; we further assume that each complex S_a satisfies the UCF condition. Note that this guarantees that $\mathcal{F}(S_a) \subseteq \mathcal{F}(S)$. Expand the language recursively to include unary belief modalities B_a for each agent $a \in Ag$ and call the result $\mathcal{L}_{KB}(Ag)$. Moreover, to guarantee consistency of beliefs (i.e., that $\neg B_a \perp$ is valid), we also assume that for every a -colored node n , there is $X \in \mathcal{F}(S_a)$ such that $n \in X$. Given a simplicial belief model $\mathcal{M}$, we can then enrich the semantics with the following clause:

$$\mathcal{M}, X \models B_a \varphi \text{ iff } \forall Y \in \mathcal{F}(S_a) \text{ if } \pi_a(Y) = \pi_a(X) \text{ then } \mathcal{M}, Y \models \varphi.$$

So this is just like the clause for knowledge, except for each agent a we quantify only over facets in S_a rather than all the facets of S . Because of this, belief is not in general forced to be factive, since at any $X \in \mathcal{F}(S) \setminus \mathcal{F}(S_a)$ we can have $X \models B_a P$ but $X \not\models P$ (for example, if $L(P) = \mathcal{F}(S_a)$).

As discussed in Section 1, a natural question that arises is why we need a different simplicial complex S_a for each agent a . After all, simplicial models for multi-agent knowledge make do with just one. The issue is made clear in the following proposition.

Proposition 2.1. *Let $\mathcal{M}$ be a simplicial belief model with $S_a = S_b$. Then $\mathcal{M} \models B_a(B_b\varphi \rightarrow \varphi)$.*

Proof. Let $X \in \mathcal{F}(S)$, and consider any $Y \in \mathcal{F}(S_a)$ such that $\pi_a(Y) = \pi_a(X)$. If $Y \models B_b\varphi$, then by definition for all $Z \in \mathcal{F}(S_b)$ such that $\pi_b(Z) = \pi_b(Y)$ we must have $Z \models \varphi$. Since $\mathcal{F}(S_b) = \mathcal{F}(S_a)$ and $Y \in \mathcal{F}(S_a)$ by assumption, we know that $Y \in \mathcal{F}(S_b)$, so in fact we must have $Y \models \varphi$. This shows that $Y \models B_b\varphi \rightarrow \varphi$ and thus $X \models B_a(B_b\varphi \rightarrow \varphi)$, as desired. $\square$

In other words, when S_a and S_b coincide, agent a becomes incapable of considering agent b fallible with respect to their beliefs (and, of course, vice-versa). This is not an issue for multi-agent knowledge because knowledge is not fallible at all (so, naturally, agents know this).

3 Connection to Relational Semantics

It is well-known [6] that a broad class of relational models for multi-agent knowledge can be transformed into simplicial models in a truth-preserving manner. We briefly review this transformation here in order to illustrate how it works—and where it fails—when belief is added into the mix.

Recall that a **relational model (over Ag)** is a tuple $\mathcal{N} = (W, (R_a)_{a \in Ag}, v)$ where W is a nonempty set of *possible worlds*, each R_a is a binary relation on W called an *accessibility relation*, and $v : \mathfrak{P} \rightarrow 2^W$ is a *valuation*. Semantics are given in the usual way:

$$\begin{aligned} \mathcal{N}, w &\models P \text{ iff } w \in v(P) \\ \mathcal{N}, w &\not\models \perp \\ \mathcal{N}, w &\models \varphi \rightarrow \psi \text{ iff } \mathcal{N}, w \models \varphi \text{ implies } \mathcal{N}, w \models \psi \\ \mathcal{N}, w &\models K_a\varphi \text{ iff } \forall w' \in R_a(w) (\mathcal{N}, w' \models \varphi), \end{aligned}$$

where $R_a(w) = \{w' \in W : wR_aw'\}$. A **relational frame** is a model without the valuation.

Except where otherwise noted, we restrict our attention to relational models where each R_a is an equivalence relation—such models validate factivity ($K_a\varphi \rightarrow \varphi$), positive introspection ($K_a\varphi \rightarrow K_aK_a\varphi$), and negative introspection ($\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$)—and refer to these as **relational models for introspective knowledge**. Since these principles are also validated by all simplicial models, we must impose them on relational structures in order to have any hope of a truth-preserving transformation. We write $[w]_a$ to denote the equivalence class of w with respect to R_a .

Given a relational model $\mathcal{N}$ as above, we define a simplicial frame $(N_{\mathcal{N}}, V_{\mathcal{N}}, S_{\mathcal{N}})$ as follows:

- $N_{\mathcal{N}} = \{([w]_a, a) : w \in W, a \in Ag\}$
- $V_{\mathcal{N}}([w]_a, a) = a$
- $S_{\mathcal{N}} = \{F \subseteq N_{\mathcal{N}} : \bigcap_{n \in F} e(n) \neq \emptyset\}$,

where $e : N_{\mathcal{N}} \rightarrow 2^W$ is projection to the first coordinate (returning the equivalence class associated with each node). Intuitively, each equivalence class $[w]_a$ captures a certain epistemic “perspective” of agent a , and thus we take our nodes to be all such equivalence classes (labelled by the agent in question). The simplicial structure is then determined by grouping precisely those nodes that correspond to equivalence classes with nonempty intersection, since these are the perspectives that can all obtain simultaneously (i.e., at the worlds in the intersection). Thus, each face of $S_{\mathcal{N}}$ corresponds to nonempty intersection of equivalence classes from distinct agents.

Proposition 3.1. *The simplicial frame $(N_{\mathcal{N}}, V_{\mathcal{N}}, S_{\mathcal{N}})$ satisfies UCF.*

Proof. We must show that each facet contains exactly one node for each agent. First observe that no face in $S_{\mathcal{N}}$ can contain more than one a -colored node, since if $([w]_a, a)$ and $([w']_a, a)$ are distinct, then $[w]_a$ and $[w']_a$ must also be distinct and thus have empty intersection.

Now suppose that for some $a \in \text{Ag}$, $F \in S_{\mathcal{N}}$ does not contain an a -colored node. By definition, there exists some $w \in \bigcap_{n \in F} e(n)$; it follows immediately that $[w]_a \cap \bigcap_{n \in F} e(n) \neq \emptyset$, and thus $F \cup \{([w]_a, a)\} \in S_{\mathcal{N}}$. This shows that F is not a facet (since it's not maximal), which in turn implies that any facet must contain an a -colored node for every $a \in \text{Ag}$. $\square$

The latter half of the proof above suggests a natural way to associate worlds in $\mathcal{N}$ with facets in the simplicial frame we have constructed: namely, define $f: W \rightarrow \mathcal{F}(S_{\mathcal{N}})$ by

$$f(w) = \{([w]_a, a) : a \in \text{Ag}\}.$$

It is not hard to see that f is surjective, but unfortunately, it may not be injective: $f(w) = f(w')$ precisely when, for all $a \in \text{Ag}$, $[w]_a = [w']_a$. In this case there is no clear way to define a valuation on $\mathcal{F}(S_{\mathcal{N}})$ that “simulates” the valuation in $\mathcal{N}$, since two worlds that disagree about the truth value of P , say, may be associated to the same facet.

This motivates the following definition that has become standard in the literature [6]: a relational model is called **proper** if, for all $w \in W$, $|\bigcap_{a \in \text{Ag}} [w]_a| = 1$.¹ From this it easily follows that f is injective. Define $L_{\mathcal{N}}(P) = f[v(P)] = \{f(w) : w \in v(P)\}$ and let $M_{\mathcal{N}}$ be the simplicial model $(N_{\mathcal{N}}, V_{\mathcal{N}}, S_{\mathcal{N}}, L_{\mathcal{N}})$. We then have:

Theorem 3.2. *Let $\mathcal{N}$ be a proper relational model. For all formulas $\varphi \in \mathcal{L}_K(\text{Ag})$, for all $w \in W$,*

$$\mathcal{N}, w \models \varphi \text{ iff } M_{\mathcal{N}}, f(w) \models \varphi.$$

Proof. Induction on formulas. Let $P \in \mathfrak{P}$. First suppose $\mathcal{N}, w \models P$; then $w \in v(P)$, so $f(w) \in f[v(P)] = L_{\mathcal{N}}(P)$, which yields $M_{\mathcal{N}}, f(w) \models P$, as desired. Conversely, suppose $M_{\mathcal{N}}, f(w) \models P$; then $f(w) \in L_{\mathcal{N}}(P) = f[v(P)]$. Because f is injective, we must then have $w \in v(P)$, which yields $\mathcal{N}, w \models P$ as desired.

The case for $\perp$ is trivial.

Now suppose inductively that for all $w \in W$, we have that $\mathcal{N}, w \models \varphi$ iff $M_{\mathcal{N}}, f(w) \models \varphi$ and $\mathcal{N}, w \models \psi$ iff $M_{\mathcal{N}}, f(w) \models \psi$. We wish to show that this equivalence holds also for $\varphi \rightarrow \psi$ and for $K_a \varphi$. For the former, observe that $\mathcal{N}, w \models \varphi \rightarrow \psi$ if and only if $\mathcal{N}, w \models \varphi$ implies $\mathcal{N}, w \models \psi$, which (by the inductive hypothesis) is true if and only if $M_{\mathcal{N}}, f(w) \models \varphi$ implies $M_{\mathcal{N}}, f(w) \models \psi$. And of course this is true if and only if $M_{\mathcal{N}}, f(w) \models \varphi \rightarrow \psi$, as desired.

At last we turn to the knowledge modalities. Suppose $\mathcal{N}, w \models K_a \varphi$. Fix arbitrary $Y \in \mathcal{F}(S_{\mathcal{N}})$ such that $\pi_a(Y) = \pi_a(f(w))$. By definition of f , $\pi_a(Y) = \pi_a(\{([w]_a, a) : a \in \text{Ag}\}) = ([w]_a, a)$. Because f is surjective, we can find $w' \in W$ such that $f(w') = Y$. So $([w']_a, a) = \pi_a(f(w')) = \pi_a(Y) = ([w]_a, a)$, and therefore $[w']_a = [w]_a$, meaning $w' \in [w]_a$. We know from our initial supposition that for all $w' \in [w]_a$, $\mathcal{N}, w' \models \varphi$. So, by the inductive hypothesis, we have $M_{\mathcal{N}}, f(w') \models \varphi$, hence $M_{\mathcal{N}}, Y \models \varphi$; since Y was chosen arbitrarily, this yields $M_{\mathcal{N}}, f(w) \models K_a \varphi$.

Conversely, suppose $M_{\mathcal{N}}, f(w) \models K_a \varphi$. By definition, for all $Y \in \mathcal{F}(S_{\mathcal{N}})$ such that $\pi_a(Y) = \pi_a(f(w))$, $M_{\mathcal{N}}, Y \models \varphi$. Let $w' \in [w]_a$. Then, since $[w']_a = [w]_a$, we must have $\pi_a(f(w)) = \pi_a(f(w'))$, so

¹This definition of properness differs from standard presentations when the relations are not equivalence relations, as has also been noted in [12].

$\mathcal{M}_{\mathcal{N}}, f(w') \models \varphi$. By the inductive hypothesis, then, $\mathcal{N}, w' \models \varphi$, and since w' was chosen arbitrarily, we conclude $\mathcal{N}, w \models K_a \varphi$. □

The existing literature has largely taken properness to be a technical condition, introduced in order to facilitate translations from relational to simplicial models. [6, 4, 10, 9] Whether it is a *reasonable* restriction, in the sense of being epistemically motivated or defensible, is debatable (and actively debated by the authors of this paper!).² What is less contentious is that once we incorporate belief into the models, properness is far too strong a requirement. A concrete example will help to illustrate this.

A **relational model for introspective knowledge and belief (over Ag)** is just a relational model for introspective knowledge supplemented with new relations $Q_a \subseteq R_a$, one for each $a \in Ag$; we require that each Q_a be serial and constant on R_a -equivalence classes: that is, if $wR_a w'$ then $Q_a(w) = Q_a(w')$. These new relations are used to interpret belief modalities in the language $\mathcal{L}_{KB}(Ag)$ in the usual way, namely:

$$w \models B_a \varphi \text{ iff } \forall w' \in Q_a(w) (w' \models \varphi).$$

Under these semantics, the properties of Q_a listed above ensure that our models validate knowledge implies belief ($K_a \varphi \rightarrow B_a \varphi$), consistency of belief ($B_a \varphi \rightarrow \neg B_a \neg \varphi$), strong positive introspection for belief ($B_a \varphi \rightarrow K_a B_a \varphi$), and strong negative introspection for belief ($\neg B_a \varphi \rightarrow K_a \neg B_a \varphi$).

Consider the model depicted in Figure 1 for three agents $Ag = \{a, b, c\}$; the red, blue, and green arrows display the relations Q_a, Q_b , and Q_c , respectively. In particular, at all worlds w , agent a considers only w_1 and w_2 possible, agent b is sure that the true world is w_1 , and agent c is sure that the true world is w_2 . Note that we have not explicitly depicted the relations R_a, R_b , or R_c , but because each of these must be an equivalence relation containing the corresponding belief relation, there is only one possibility: they are all the complete relation. Thus, the corresponding relational model for introspective knowledge is not proper and so cannot be translated into a simplicial model.

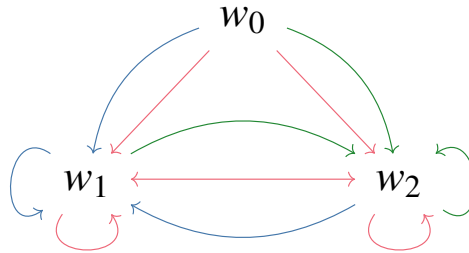


Figure 1: A 3-agent relational frame for introspective knowledge and belief

However, there does not seem to be anything trivial or redundant about this model of belief; as such, an inability to translate it into the simplicial setting is a substantive restriction. Fortunately, while this model itself cannot be so translated, there does exist an equivalent (in the sense of bisimilarity) model that can be. This follows from the results of the next section.

²One could argue that it eliminates a kind of “redundancy of perspectives”, though this depends on a further philosophical supposition that every relevant fact about a world is captured in some agent’s perspective.

4 Properness

The results of this section are relevant outside the context of the simplicial semantics for belief that is the central project of this paper; as such, we present them here first for simplicial models generally and only subsequently extend them to simplicial belief models.

Let $\mathcal{N} = (W, (R_a)_{a \in Ag}, v)$ be a relational model with $Ag = \{1, \dots, n\}$; in this section we do *not* assume that the relations R_i are equivalence relations (or have any properties). As defined above, $\mathcal{N}$ is called **proper** if it contains no pair of distinct points $w, w' \in W$ such that for all $i \in Ag$, wR_iw' . As noted, this properness assumption has been deployed repeatedly in the literature to facilitate translation from relational models to simplicial models [6, 4, 10, 9], but to the best of our knowledge there has been no systematic study of just how restrictive it is. In this section we show that in fact properness is not restrictive at all: *every* relational model is equivalent (via bisimulation) to a proper relational model.³

We begin with the case where W is finite; say $W = \{w_1, \dots, w_m\}$. We will construct a new relational model $\tilde{\mathcal{N}} = (\tilde{W}, (\tilde{R}_i)_{i=1}^n, \tilde{v})$, prove that it is proper, and exhibit a surjective, bounded morphism from $\tilde{\mathcal{N}}$ to $\mathcal{N}$.⁴

Set $\tilde{W} = W \times W$; it is helpful to picture this new state space as consisting of m disjoint copies of the original set W , namely,

$$\tilde{W} = W \times \{w_1\} \cup \dots \cup W \times \{w_m\},$$

so the point $(w_j, w_k) \in \tilde{W}$ may be thought of as the j th element of the k th copy of W . Accordingly, we define

$$\tilde{v}(P) = \{(w_j, w_k) \in \tilde{W} : w_j \in v(P)\};$$

in other words, the primitive propositions true at (w_j, w_k) in $\tilde{\mathcal{N}}$ are precisely those that are true at w_j in $\mathcal{N}$.

Lastly we must define the relations, and we do so with the idea of preserving this correspondence in truth between $(\tilde{\mathcal{N}}, (w_j, w_k))$ and $(\mathcal{N}, w_j)$ (indeed, we will ultimately show that projection to the first coordinate is the promised bounded morphism). For $i > 1$, define

$$(w_j, w_k)\tilde{R}_i(w_{j'}, w_{k'}) \text{ iff } k = k' \text{ and } w_j R_i w_{j'}.$$

Notice that if we imposed this same definition for $i = 1$, then $\tilde{\mathcal{N}}$ would simply be the disjoint union of m copies of $\mathcal{N}$. Instead, we will define $\tilde{R}_1$ using a different partition of $\tilde{W}$: let $=_m$ denote equality modulo m , and for each $\ell \in \{0, \dots, m-1\}$, let

$$\tilde{W}_\ell = \{(w_j, w_k) \in \tilde{W} : k - j =_m \ell\},$$

Thus, $\tilde{W}_0 = \{(w_1, w_1), (w_2, w_2), \dots, (w_m, w_m)\}$ is the “diagonal”, $\tilde{W}_1 = \{(w_1, w_2), (w_2, w_3), \dots, (w_m, w_1)\}$, and so on. It is then easy to check the following.

Proposition 4.1. *The collection $\{\tilde{W}_\ell : 0 \leq \ell \leq m-1\}$ partitions $\tilde{W}$. Moreover, for each $\tilde{W}_\ell$, projection to the first component is a bijection between $\tilde{W}_\ell$ and W .*

³The only other formal discussion of this topic that we are aware of occurs in [10], where the translation of a particular “canonical” Kripke model, which is not itself proper, is conjectured to be bisimilar to a proper Kripke model using a broad “unwinding” method. Unfortunately, this technique would not, in general, preserve properties like transitivity and symmetry, limiting its usefulness in the context of simplicial semantics, where these properties are typically necessary.

⁴For a review of bounded morphisms, see for example [1].

Proof. Suppose $(w_j, w_k) \in \tilde{W}_\ell \cap \tilde{W}_{\ell'}$. Then $\ell =_m k - j =_m \ell'$. Since $\ell, \ell' \in \{0, \dots, m-1\}$, we have that $\ell = \ell'$. This shows that the $\tilde{W}_\ell$ are mutually disjoint. Moreover, given any $(w_j, w_k) \in \tilde{W}$, clearly $(w_j, w_k) \in \tilde{W}_\ell$ for $\ell =_m k - j$; thus, $\{\tilde{W}_\ell : 0 \leq \ell \leq m-1\}$ partitions $\tilde{W}$.

Define $\rho : \tilde{W} \rightarrow W$ by $\rho(w_j, w_k) = w_j$, namely, projection to the first component. Let $(w_j, w_k), (w_{j'}, w_{k'}) \in \tilde{W}_\ell$, and suppose $\rho(w_j, w_k) = \rho(w_{j'}, w_{k'})$, so $w_j = w_{j'}$ (which means that $j = j'$). Then, since $k - j =_m \ell =_m k' - j'$, we have that $k = k'$ and so $w_k = w_{k'}$. This shows ρ is injective when restricted to any $\tilde{W}_\ell$. Next, given $w_j \in W$, let $k \in \{1, \dots, m\}$ be such that $k =_m j + \ell$; then by definition $(w_j, w_k) \in \tilde{W}_\ell$ and $\rho(w_j, w_k) = w_j$, which shows that ρ is surjective when restricted to any $\tilde{W}_\ell$. $\square$

Having established this new way of breaking $\tilde{W}$ into copies of W , we define

$$(w_j, w_k) \tilde{R}_1(w_{j'}, w_{k'}) \text{ iff } k - j =_m k' - j' \text{ and } w_j R_1 w_{j'}.$$

So, like the other relations, $\tilde{R}_1$ is also m disjoint copies of R_1 , but skewed across a different partition: one copy on each set $\tilde{W}_\ell$.

Proposition 4.2. $\tilde{\mathcal{N}}$ is proper.

Proof. Suppose $(w_j, w_k) \tilde{R}_i(w_{j'}, w_{k'})$ for each $i \in \{1, \dots, n\}$. Taking $i = 1$, we know that $k - j =_m k' - j'$. Taking $i > 1$, we know that $k = k'$. So, $j = j'$, and therefore $(w_j, w_k) = (w_{j'}, w_{k'})$. $\square$

Proposition 4.3. Projection to the first coordinate is a surjective, bounded morphism from $\tilde{\mathcal{N}}$ to $\mathcal{N}$.

Proof. We have already seen that ρ is surjective, and it is clear that ρ also preserves the truth values of atomic formulas. To show ρ is a bounded morphism, we need to establish the “back” and “forth” conditions for the relations R_i , for which the only interesting case is $i = 1$.

First, the “back” condition: let $(w_j, w_k) \in \tilde{W}$ and $w_{j'} \in W$, and suppose $\rho(w_j, w_k) R_1 w_{j'}$. Fix ℓ such that $(w_j, w_k) \in \tilde{W}_\ell$. Choose $k' \in \{1, \dots, m\}$ such that $k' =_m j' + \ell$, and consider $(w_{j'}, w_{k'})$. We have that $k' - j' =_m \ell$, so $(w_{j'}, w_{k'}) \in \tilde{W}_\ell$. It follows that $(w_j, w_k) \tilde{R}_1(w_{j'}, w_{k'})$ and of course $\rho(w_{j'}, w_{k'}) = w_{j'}$, as desired.

For the “forth” condition, suppose that $(w_j, w_k) \tilde{R}_1(w_{j'}, w_{k'})$; then it is immediate from the definition of $\tilde{R}_1$ that $w_j R_1 w_{j'}$. $\square$

The construction for countable W is quite analogous, and perhaps even more straightforward since there is no need to appeal to modular arithmetic. In this case we have a bijection $g : W \rightarrow \mathbb{Z}$. Define $\tilde{\mathcal{N}} = (\tilde{W}, (\tilde{R}_i)_{i=1}^n, \tilde{v})$ by setting

- $\tilde{W} = W \times W$;
- $\tilde{v}(P) = \{(x, y) \in \tilde{W} : x \in v(P)\}$;
- for $i > 1$,

$$(x, y) \tilde{R}_i(x', y') \text{ iff } y = y' \text{ and } x R_i x',$$

and

$$(x, y) \tilde{R}_1(x', y') \text{ iff } g(y) - g(x) = g(y') - g(x') \text{ and } x R_1 x'.$$

Once again, we have:

Proposition 4.4. $\tilde{\mathcal{N}}$ is proper, and projection to the first coordinate is a surjective, bounded morphism from $\tilde{\mathcal{N}}$ to $\mathcal{N}$.

Proof. For each $\ell \in \mathbb{Z}$, define $\tilde{W}_\ell = \{(x, y) \in \tilde{W} : g(y) - g(x) = \ell\}$. Analogous to the above, it is easy to see that the collection $\{\tilde{W}_\ell : \ell \in \mathbb{Z}\}$ is a partition of $\tilde{W}$, and moreover that $\rho : \tilde{W}_\ell \rightarrow W$ is a bijection for each ℓ .

The proof of properness is also completely parallel to the finite case: suppose $(x, y)\tilde{R}_i(x', y')$ for each $i \in \{1, \dots, n\}$. Taking $i = 1$, we know that $g(y) - g(x) = g(y') - g(x')$; taking $i > 1$, we know that $y = y'$. It follows that $g(x) = g(x')$, so $x = x'$ and $(x, y) = (x', y')$.

It is clear as before that ρ is surjective and preserves the truth values of atomic formulas. Thus, to show that it is a bounded morphism, we just need to establish the “back” and “forth” conditions for the relations R_i ; once again, the only interesting case is $i = 1$.

As above, “forth” is immediate from the definition of $\tilde{R}_1$. For “back”, let $(x, y) \in \tilde{W}$ and suppose that xR_1x' . Let $y' = g^{-1}(g(y) - g(x) + g(x'))$; then $g(y') - g(x') = g(y) - g(x)$, so by definition we have $(x, y)\tilde{R}_1(x', y')$ and of course $\rho(x', y') = x'$, as desired. $\square$

This translation can also be extended to continuum-sized models in the obvious way: namely, by replacing $\mathbb{Z}$ with $\mathbb{R}$ in the proof above (literally every other part of the proof remains the same). This is useful, for example, if one wishes to apply the transformation to a canonical model, which is typically of size continuum.⁵

Finally, we observe that many properties of the relations R_i are preserved by this translation: for example, it is easy to see that if R_i is reflexive, symmetric, transitive, serial, or Euclidean, then $\tilde{R}_i$ is as well—in all cases the proof follows immediately from the fact that $\tilde{R}_i$ can be represented as a disjoint union of “copies” of R_i . Thus we have the following summary result:

Theorem 4.5. *Let $\mathcal{N}$ be a relational model. Then there is a relational model $\tilde{\mathcal{N}}$ which is proper and a surjective, bounded morphism from $\tilde{\mathcal{N}}$ to $\mathcal{N}$. Furthermore, if $\mathcal{N}$ is reflexive (respectively: symmetric, transitive, serial, Euclidean) then $\tilde{\mathcal{N}}$ is also reflexive (respectively: symmetric, transitive, serial, Euclidean).*

With that established, we return to our consideration of belief. Roughly speaking, by expanding this result to apply also to belief, we can translate the model in Figure 1 into a proper model that “simulates” it via an appropriate surjective, bounded morphism. This will yield the model depicted in Figure 2.

Because the relational model in Figure 2 is proper, we are able to transform it into a simplicial model using (a slight generalization of) Theorem 3.2. The result of this transformation is shown in Figure 3. In the next section we fill in the details—namely, generalizing Theorems 4.5 and 3.2 to relational models for knowledge *and* belief—and use these results to establish a novel completeness result.

Before turning to the technical details, it is worth returning to reflect on the properness condition in light of this example. Intuitively, the model shown in Figure 1 encodes a single perspective for each agent: the set of doxastically accessible worlds is constant for each of Q_a , Q_b , and Q_c , after all. However, with only one perspective for every agent, simplicial semantics only has the resources to produce a single facet, which is not enough to capture this scenario. As such, any technique for translating relational models into simplicial models must sometimes proliferate agential perspectives; indeed, Figure 2 has 9 worlds instead of 3 and intuitively includes 3 “perspectives” for each agent, each of which is a “copy” of that agent’s original, single perspective (up to formula satisfaction). In Section 6, we discuss how the more general notion of *simplicial sets* may be a useful framework for avoiding this redundancy of representation.

⁵For models of other cardinalities, we could again duplicate the proof given above if we had a structure like $\mathbb{Z}$ or $\mathbb{R}$ but with the needed cardinality. And indeed, such a structure is guaranteed to exist by the Löwenheim-Skolem theorem. In fact, we only really need to apply the Löwenheim-Skolem theorem to the group axioms, and not $Th(\mathbb{R})$; the proof above uses only the existence of additive inverses and the fact that $a - b = a - c$ implies $b = c$.

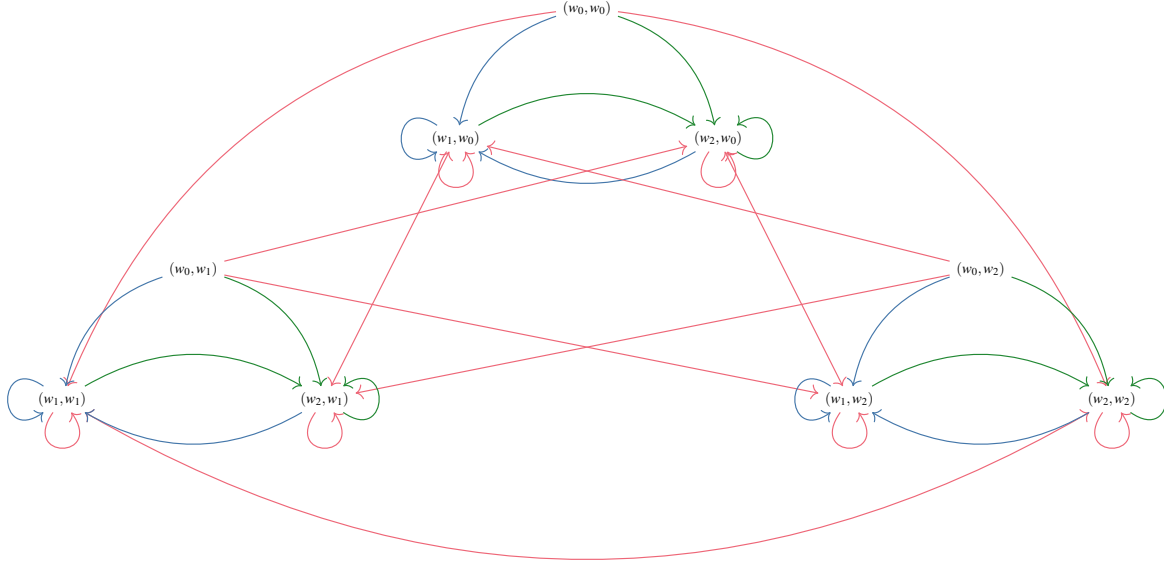


Figure 2: A translation of the model from Figure 1 into a proper model

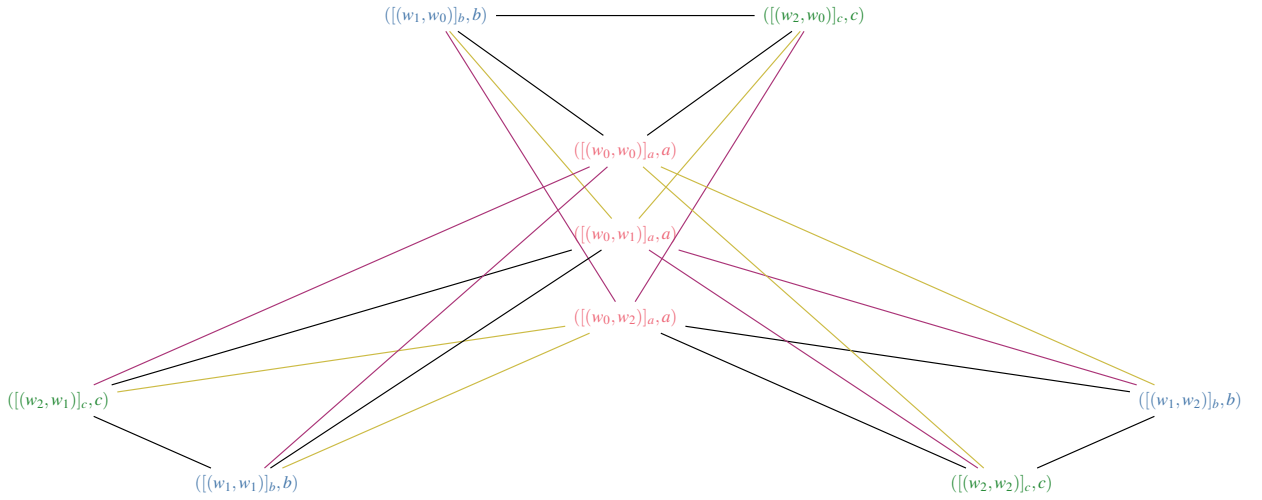


Figure 3: A translation of the model from Figure 2 into a simplicial model (violet facets belong to S_a and S_b , yellow facets belong to S_a and S_c , while black facets belong to only S)

5 Soundness and Completeness

Recall that relational models for introspective knowledge and belief validate knowledge implies belief ($K_a\phi \rightarrow B_a\phi$), strong positive introspection ($B_a\phi \rightarrow K_aB_a\phi$), and strong negative introspection ($\neg B_a\phi \rightarrow K_a\neg B_a\phi$). Let FULL be the axiom system obtained by adding these three axiom schemes to the system S5 for knowledge combined with KD45 for belief.⁶

Theorem 5.1. *FULL is sound and complete with respect to the class of all simplicial belief models.*

Proof. We prove the result in much the same way as has been done for comparable results in previous literature. [6, 9] Soundness is straightforward so we focus on completeness. We first generate a canonical relational model for introspective knowledge and belief, using maximal FULL-consistent sets in the usual way. Next we transform this canonical model into an equivalent, proper model using a generalization of Theorem 4.5. Finally, we translate this proper model into a simplicial belief model using essentially the same technique as in Section 3.

Let W be the collection of maximal FULL-consistent sets. For arbitrary $a \in Ag$, we define R_a as follows: wR_aw' if and only if for all formulas ϕ , if $K_a\phi \in w$, then $\phi \in w'$. Similarly, let Q_a be given by wQ_aw' if and only if for all formulas ϕ , if $B_a\phi \in w$, then $\phi \in w'$. Lastly, define $v(P) = \{w \in W : P \in w\}$. Set $\mathcal{N} = (W, (R_a)_{a \in Ag}, (Q_a)_{a \in Ag}, v)$.

It is well known that with the above definitions for relations, and the fact that the S5 axioms for knowledge are included in FULL, each R_a is an equivalence relation. Indeed, it is straightforward to check that $\mathcal{N}$ is a relational model for introspective knowledge and belief. For example, the fact that $K_a\phi \rightarrow B_a\phi \in \text{FULL}$ forces $Q_a \subseteq R_a$. We will show here in detail that each Q_a is constant on R_a -equivalence classes, and leave the remaining, easy checks to the reader. Suppose wR_aw' . Suppose further that $wQ_a u$, and $B_a\phi \in w'$. From strong positive introspection it follows that $K_aB_a\phi \in w'$. Since wR_aw' , we have that $w'R_aw$, and therefore $B_a\phi \in w$. Since $wQ_a u$, it follows that $\phi \in u$. Since ϕ was arbitrary, this shows that $w'Q_a u$, which suffices to show that Q_a is constant on R_a -equivalence classes.

It is easy to show, in the standard way, that $\mathcal{N}, w \models \phi$ if and only if $\phi \in w$. Thus $\mathcal{N}$ refutes every non-theorem of FULL. Our next step is to find a *proper* relational model for introspective knowledge and belief that does the same; for this we extend the results of Section 4.

Since our language is countable, it follows that W is of size continuum. Let $g: W \rightarrow \mathbb{R}$ be a bijection. Define $\tilde{W} = W^2$ and fix a distinguished $b \in Ag$; for all $a \neq b$, define

$$\begin{aligned} (w, u)\tilde{R}_a(w', u') &\text{ iff } u = u' \text{ and } wR_aw' \\ (w, u)\tilde{Q}_a(w', u') &\text{ iff } u = u' \text{ and } wQ_aw' \end{aligned}$$

and set

$$\begin{aligned} (w, u)\tilde{R}_b(w', u') &\text{ iff } g(u) - g(w) = g(u') - g(w) \text{ and } wR_bw' \\ (w, u)\tilde{Q}_b(w', u') &\text{ iff } g(u) - g(w) = g(u') - g(w) \text{ and } wQ_bw' \end{aligned}$$

Finally, define $\tilde{v}(P) = \{(x, y) \in \tilde{W} : x \in v(P)\}$, and let $\tilde{\mathcal{N}} = (\tilde{W}, (\tilde{R}_a)_{a \in Ag}, (\tilde{Q}_a)_{a \in Ag}, \tilde{v})$. Thus, just as in Section 4, we are making $|W|$ -many copies of the model $\mathcal{N}$, but for one special agent, b , we “skew” their accessibility relations (both R_b and Q_b) so they connect worlds across copies rather than within copies.

⁶For an overview of these standard axiom systems, see for example [1].

This guarantees that (except for reflexive edges), agent b 's accessibility relations never coincide with any other agent's, which in turn ensures the model is proper.

Since $Q_a \subseteq R_a$ for all $a \in \text{Ag}$, the definitions above guarantee that also $\tilde{Q}_a \subseteq \tilde{R}_a$ for all $a \in \text{Ag}$. It's also easy to check that for all $a \in \text{Ag}$, $\tilde{R}_a$ is an equivalence relation and $\tilde{Q}_a$ is serial and constant on $\tilde{R}_a$ -equivalence classes. Theorem 4.5 now guarantees $\tilde{\mathcal{N}}$ is proper (with respect to the relations $\tilde{R}_a$) and moreover, that the map $\rho : \tilde{W} \rightarrow W$ given by projection to the first coordinate is a surjective, bounded morphism (again, with respect to the $\tilde{R}_a$ relations). But it is easy to check that ρ is also a bounded morphism with respect to the $\tilde{Q}_a$ relations. It follows that every formula refuted in $\mathcal{N}$ is also refuted in $\tilde{\mathcal{N}}$, as desired.

The final step is to transform $\tilde{\mathcal{N}}$ into an equivalent simplicial belief model. Because $\tilde{\mathcal{N}}$ is proper and each $\tilde{R}_a$ is an equivalence relation, we can define $N_{S_{\tilde{\mathcal{N}}}}, V_{S_{\tilde{\mathcal{N}}}}, S_{S_{\tilde{\mathcal{N}}}}$, and $L_{S_{\tilde{\mathcal{N}}}}$ as we did above, in the lead-up to Theorem 3.2. What's missing are the belief subcomplexes, which we define as follows:

$$\mathcal{F}(S_{\tilde{\mathcal{N}},a}) = \{F \in \mathcal{F}(S_{\tilde{\mathcal{N}}}) : f^{-1}(F)\tilde{Q}_af^{-1}(F)\},$$

recalling that $f : \tilde{W} \rightarrow \mathcal{F}(S_{\tilde{\mathcal{N}}})$ is the bijective correspondence between worlds in $\tilde{\mathcal{N}}$ and facets in $S_{\tilde{\mathcal{N}}}$. Specifying the facets of $S_{\tilde{\mathcal{N}},a}$ uniquely determines the simplicial complex itself; it also guarantees that $S_{\tilde{\mathcal{N}},a}$ is a UCF subcomplex of $S_{\tilde{\mathcal{N}}}$. Moreover, fix some arbitrary a -colored node $n \in N_{S_{\tilde{\mathcal{N}}}}$. By definition of $N_{S_{\tilde{\mathcal{N}}}}$, $e(n) \neq \emptyset$. Fix $w \in e(n)$; because $\tilde{Q}_a$ is serial, there is w' such that $w\tilde{Q}_aw'$. It follows that $w'\tilde{Q}_aw'$ and $w' \in e(n)$. So, $n \in \{([w']_i, i)\}_{i \in \text{Ag}} \in \mathcal{F}(S_{\tilde{\mathcal{N}},a})$, as desired.

Of course, given any proper relational model for introspective knowledge and belief, $\mathcal{N}$, we can use the above definitions to produce a simplicial belief model $\mathcal{M}_{\mathcal{N}}$. We now extend Theorem 3.2 in this context to the full language including belief modalities.

Lemma 5.2. *Let $\mathcal{N}$ be a proper relational model for introspective knowledge and belief (over Ag). For all formulas $\varphi \in \mathcal{L}_{KB}(\text{Ag})$, for all $w \in W$,*

$$\mathcal{N}, w \models \varphi \text{ iff } \mathcal{M}_{\mathcal{N}}, f(w) \models \varphi.$$

Proof. The proof is identical to that of Theorem 3.2 except for the inductive step for the belief modalities, which must be added. So suppose inductively, as in that proof, that for all $w \in W$, $\mathcal{N}, w \models \varphi$ iff $\mathcal{M}_{\mathcal{N}}, f(w) \models \varphi$. We wish to show that for all $w \in W$, $\mathcal{N}, w \models B_a\varphi$ iff $\mathcal{M}_{\mathcal{N}}, f(w) \models B_a\varphi$.

Fix $w \in W$ and suppose $\mathcal{N}, w \models B_a\varphi$. Let $F \in \mathcal{F}(S_{\mathcal{N},a})$ be such that $\pi_a(F) = \pi_a(f(w))$. It follows that $f^{-1}(F)Q_af^{-1}(F)$ and $wR_af^{-1}(F)$. Because Q_a is constant on R_a equivalence classes, $Q_a(w) = Q_a(f^{-1}(F))$; thus, $wQ_af^{-1}(F)$, and so $\mathcal{N}, f^{-1}(F) \models \varphi$. By the inductive hypothesis, we have that $\mathcal{M}_{\mathcal{N}}, F \models \varphi$. It follows that $\mathcal{M}_{\mathcal{N}}, f(w) \models B_a\varphi$, as desired.

Suppose now that $\mathcal{M}_{\mathcal{N}}, f(w) \models B_a\varphi$. Let $u \in W$ be such that wQ_au . By the Euclidean property we have that uQ_au . It follows that $f(u) \in \mathcal{F}(S_{\mathcal{N},a})$. Moreover, wQ_au implies that wR_au , and so $\pi_a(f(u)) = \pi_a(f(w))$. It follows that $\mathcal{M}_{\mathcal{N}}, f(u) \models \varphi$. By the inductive hypothesis, we have that $\mathcal{N}, u \models \varphi$. Therefore, $\mathcal{N}, w \models B_a\varphi$, as desired. $\square$

We can now establish completeness. Suppose for simplicial belief models we have $\Gamma \models \varphi$, and suppose for contradiction that $\Gamma \not\models_{\text{FULL}} \varphi$. Let w be a maximal FULL-consistent set extending $\Gamma \cup \{\neg\varphi\}$ (obtained, as usual, via Lindenbaum's Lemma). Then $\mathcal{N}, w \models \Gamma \cup \{\neg\varphi\}$. It then follows from the above that (for any u), $\mathcal{M}_{\tilde{\mathcal{N}}}, f((w, u)) \models \Gamma \cup \{\neg\varphi\}$, a contradiction. $\square$

6 Related and Future Work

The simplicial representation of belief we introduce in this paper stands in contrast to two existing approaches. In “Knowledge and Simplicial Complexes” [6], standard UCF simplicial models (N, V, S, L) are augmented with idempotent functions $f_a: V^{-1}(a) \rightarrow V^{-1}(a)$, one for each agent $a \in Ag$. These functions are used to define belief via the following clause:

$$\mathcal{M}, X \models B_a \varphi \text{ iff } \forall Y \in \mathcal{F}(S) \text{ if } \pi_a(Y) = f_a(\pi_a(X)) \text{ then } \mathcal{M}, Y \models \varphi.$$

Loosely speaking, whereas a node $n \in V^{-1}(a)$ represents a “perspective” of agent a for the purposes of evaluating knowledge modalities at facets containing n , for the purposes of evaluating *belief* modalities at facets containing n , this semantics instead uses the node $f_a(n)$. In other words, agent a ’s beliefs at X are given by her knowledge at $f_a(\pi_a(X))$. Because each f_a is idempotent, it is easy to check that the B_a modalities satisfy the KD45 axioms.

However, this notion of belief has essentially no relationship to knowledge; in particular, $K_a \varphi \rightarrow B_a \varphi$ is not valid. A natural solution to this might be to strengthen the semantics so as to validate this formula. Unfortunately, this results in triviality, for it requires the following to hold for all $X \in \mathcal{F}(S)$:

$$\{Y \in \mathcal{F}(S) : f_a(\pi_a(X)) = \pi_a(Y)\} \subseteq \{Y \in \mathcal{F}(S) : \pi_a(X) = \pi_a(Y)\};$$

under the UCF assumption, this containment does not hold in general unless f_a is the identity function. In this case, of course, belief and knowledge coincide. Thus, this model for belief cannot satisfy both UCF and the validity of “knowledge implies belief” without trivializing belief.

We find a quite different approach to defining belief in simplicial complexes in “Simplicial Belief”. [2] The idea here depends on breaking the UCF assumption so that facets may contain one *or more* a -colored nodes, and then using the number of nodes a given facet contains to determine whether or not it is “accessible” for the purposes of evaluating belief. Specifically, given $X \in \mathcal{F}(S)$, let $m_a(X) := |\{n \in X : V(n) = a\}|$ count the number of a -colored nodes in X , and for any $X, Y \in \mathcal{F}(S)$, define

$$X \leq_a Y \text{ iff } m_a(X) \leq m_a(Y).$$

Write $X \sim_a Y$ if and only if $\exists n \in X \cap Y$ such that $V(n) = a$, and define $\leq_a = \leq_a \cap \sim_a$. These relations are used to define belief via the following clause:

$$\mathcal{M}, X \models B_a \varphi \text{ iff } \forall Y \in \mathcal{F}(S) \text{ if } Y \leq_a X \text{ then } \mathcal{M}, Y \models \varphi.$$

This says that if φ is true at all facets Y whose intersection with X contains an a -colored node, and such that the number of a -colored nodes in Y is bounded above by the number in X , then $B_a \varphi$ is true at X .⁷ The rough intuition here, as articulated in [2], is that facets containing more a -perspectives are less plausible (to agent a) than those containing fewer, since a is in some sense less certain of their own state of mind in the former.

It’s easy to see that this definition validates $K_a \varphi \rightarrow B_a \varphi$. And it is certainly interesting from a formal perspective to relax the UCF assumption and leverage this extra freedom to define belief. However, the epistemic interpretation of facets with multiple a -perspectives remains somewhat murky. Unfortunately,

⁷The paper also provides a stronger notion of belief, taking the belief relation to access worlds with a *minimum* number of perspectives, rather than all those with fewer than the starting world. The same result we show for the given definition also applies to this one.

if we re-impose the UCF condition, this of course implies that for all $X \in \mathcal{F}(S)$, $m_a(X) = 1$, making $\leq_a$ the total relation on $\mathcal{F}(S)$, hence $\leq_a = \sim_a$. In this case again we see that belief and knowledge coincide.

In contrast to these approaches, our framework provides a definition of belief in simplicial complexes satisfying *all* three of: (1) UCF, (2) validity of $K_a\phi \rightarrow B_a\phi$ is valid, and (3) $B_a\phi \rightarrow K_a\phi$ is *not* valid. This has a few further upshots. The usual techniques for translating frame models into simplicial models apply very cleanly (indeed, almost trivially) in our setting, yielding soundness and completeness with respect to the simple FULL axiom system (Theorem 5.1). The only wrinkle in the proof is the condition that the underlying relational frame be proper, which is essentially circumvented via Theorem 4.5.

This approach is similar to the “directed” approach taken in David Lehnher’s PhD thesis “Simplicial Structures for Epistemic Reasoning in Multi-agent Systems”. [12] In this thesis, a knowledge model $\mathcal{M}$ is augmented with a function $\rho: V \times S \rightarrow 2$, which specifies whether or not a face in S is considered “possible” from a node in V . It is assumed that if $\rho(n, F) = 1$, then there is $F' \in \mathcal{F}(S)$ such that $n \in F'$, and if $G \subseteq F$, $\rho(n, G) = 1$. The semantic clause for belief is given by $\mathcal{M}, X \models B_a\phi$ if and only if for all Y such that $\rho(\pi_a(X), Y) = 1$, $\mathcal{M}, Y \models \phi$.

Our approach makes a simplifying assumption to emphasize the key notion of belief subcomplexes, so we can explore their contribution more directly. Namely, we do not assume that variables are assigned to nodes, but rather to facets.⁸ Nevertheless, we can define the collection of subcomplexes and ρ interchangeably, at least for facets. For the first direction, $S_a := \{F \in \mathcal{F}(S) \mid \rho(\pi_a(F), F) = 1\}$. For the other direction, if $F \in \mathcal{F}(S)$, say that $\rho(\pi_a(F), F) = 1$ if and only if $F \in \mathcal{F}(S_a)$, and 0 otherwise.

There is another, more recent approach which attempts to model belief using chromatic hypergraphs. [5] Given N and V as above, a **chromatic hypergraph** C is a subset of $2^N \times 2^N$, called “hyper edges”, which are the stand-in for worlds. For each $(X, Y) \in C$, $X \cap Y = \emptyset$. Moreover, if $n, n' \in X \cup Y$, and if $V(n) = V(n')$, then $n = n'$. Hence, each ordered pair $(X, Y) \in C$ is uniquely colored across the union of X and Y . It is also assumed that, for each $n \in N$, there is a pair (X, Y) such that $n \in X \cup Y$, and $|X \cup Y| = |Ag|$. We can think of those pairs (X, Y) where $|X \cup Y| = |Ag|$ as the equivalent of “facets” for chromatic hypergraphs. And, the disjoint pair X and Y divide each “facet” into two pieces. This is how doxastic information is read off of a chromatic hypergraph. X is the set of nodes that consider the whole hyper edge (X, Y) “doxastically accessible” or possible. Consider two hyper edges (X, Y) and (X', Y') . If $n \in (X \cup Y) \cap X'$, then $V(n)$ considers (X', Y') possible from (X, Y) . While this approach is promising, it requires structural assumptions significantly more elaborate than the usual presentation of simplicial semantics. We hope then that our approach provides a simpler and more streamlined way of incorporating belief into simplicial semantics.

As mentioned above, the existing literature has largely taken properness as a technical condition, introduced in order to facilitate translations from relational to simplicial models. [6, 4, 10, 9] However, as noted at the end of Section 4, in order to “make” a model proper we have to multiply worlds in a way that produces substantial redundancy. As a result, the structures become harder to parse—both the relational models and their simplicial correlates. We suspect, for example, that few will find the simplicial model in Figure 4 as easy to understand as the relational model in Figure 1.

It is natural to wonder, then, if there is any way to avoid imposing properness altogether. One promising approach utilizes the notion of *simplicial sets*.⁹ Roughly speaking, simplicial sets generalize simplicial complexes by allowing us to count faces *with multiplicity*, similar to the generalization of a set to a multiset, or a graph to a multigraph. For instance, in a simplicial complex, the triangle $\{a, b, c\}$ is

⁸Future work in our belief semantics will re-introduce this assumption.

⁹The idea of using simplicial sets in the context of simplicial models was introduced in [11]. Recent work expanding this idea is contained in [3]

uniquely determined by the three nodes it contains, whereas in a simplicial set, there could be multiple triangles on the same underlying set of nodes $\{a, b, c\}$. Visually, one may picture multiple triangular “membranes” connecting these nodes. Informally, we may define a simplicial set as a “multiset of multisets closed under subsets”.

Using this structure, one can capture the model in Figure 1 with a simplicial set S together with subsimplicial sets S_a , S_b , and S_c , as depicted in Figure 4.

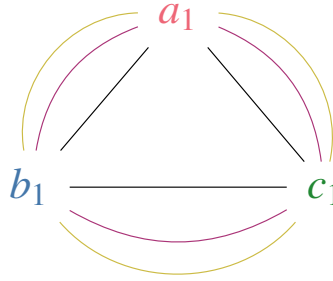


Figure 4: A translation of the model from Figure 1 into a simplicial set S together with subsimplicial sets S_a , S_b , and S_c (violet facets belong to S_a and S_b , yellow facets belong to S_a and S_c , while black facets belong to only S)

Intuitively, the node a_1 corresponds to the (unique!) perspective that agent a has in Figure 1, namely the worlds w_1 and w_2 . Similarly, b_1 corresponds to the view that only w_1 is possible, and c_1 corresponds to only seeing w_2 . But we now have the freedom to associate not one but three facets with this set of perspectives. The black facet corresponds with w_0 , the world accessible to none of the three agents. The violet facet belongs to S_a and S_b , but not S_c , corresponding to w_1 , the world accessible to agents a and b but not c . And similarly the yellow facet belongs to S_a and S_c , but not S_b , corresponding to w_2 . Thus we see that simplicial sets give us the resources to represent non-proper models without artificially proliferating perspectives; this provides a promising direction for future research in simplicial semantics and particularly in the representation of belief therein.

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Most Properties are Undecidable for Transitive Tense Logics

Qian Chen

The Tsinghua-UvA JRC for Logic
Department of Philosophy
Tsinghua University
Beijing, China
chenq21@mails.tsinghua.edu.cn

Tenyo Takahashi

Institute for Logic, Language and Computation
University of Amsterdam
Amsterdam, The Netherlands
t.takahashi@uva.nl

A logics' property is decidable in a class of logics if there exists an algorithm that decides whether a finitely axiomatizable logic in the class has the property. Many properties are undecidable for bimodal logics but decidable for linear tense logics, which leads to a general question on how the interactions of modalities affect the decidability of properties. In this paper, we study the decidability of properties for transitive tense logics and show that most properties are undecidable in the lattice NExtK_4 of transitive tense logics, including Kripke completeness, the finite model property, and decidability. Our proof method adapts Chagrov's approach of constructing a reduction from an undecidable problem of Minsky machines to the decision problem for logics' properties, yielding a general scheme of proving the undecidability of these properties.

1 Introduction

A central part of the study of modal logic is to determine whether a logic has a certain property, such as Kripke completeness, the finite model property (FMP), and decidability. From a global viewpoint, this gives rise to the following algorithmic problems: In a class of logics, is it decidable whether a logic has a certain property?¹ More precisely, a logics' property (or simply, property) P in a class $\mathcal{C}$ of logics is identified with a subclass of $\mathcal{C}$, and P is *decidable* if there exists an algorithm such that, for any finitely axiomatizable logic $L \in \mathcal{C}$, given by its finite axiomatization, the algorithm decides whether L has the property P .² We refer to [4, Section 17.6] and [33] for historical accounts. See Table 1 for a summary of the results discussed in this introduction.

Let K_n denote the least normal n -modal logic and $K = K_1$. For a normal modal logic L , let $\text{NExt}L$ be the lattice of all normal extensions of L . In the unimodal case, many properties have been shown to be undecidable in $\text{NExt}K$. Thomason [27] showed the undecidability of Kripke completeness, and a series of works by Chagrov and his co-authors [5, 6, 10, 8, 7] introduced a general method to show the undecidability of various properties, including the FMP, first-order definability, decidability, tabularity, and the coincidence with a fixed tabular logic. Kracht and Wolter [16] proved independently that decidability, the FMP, and tabularity are undecidable in $\text{NExt}K$ via the Thomason-Simulation [25, 26]. On the positive side, it is well-known that consistency is decidable for normal unimodal logics as the lattice $\text{NExt}K$ has only two coatoms [17] (see also [4, Section 17.2]). Recently, Takahashi [24] proved that the property of being a union-splitting of $\text{NExt}K$ and strict Kripke completeness are decidable in $\text{NExt}K$ (see also [23]).

If P is decidable in $\text{NExt}L$, then it is also decidable in $\text{NExt}L'$ for any extension L' of L with finitely many axioms (Proposition 2.2). On the other hand, a property that is undecidable in a given lattice $\mathcal{C}$ of

¹This should not be confused with the decidability of logics; a logic is *decidable* if its membership problem is decidable.

²We have to restrict to finitely axiomatizable logics because an input of an algorithm must be a finite object; see Subsection 2.2 and [4, Section 17.1] for more discussions.

logics may become decidable in a sublattice of $\mathcal{C}$. For example, in the lattice NExtK4 of transitive modal logics (recall that $\text{K4} = \text{K} \oplus \Box p \rightarrow \Box \Box p$), a sublattice of NExtK , every tabular logic has only finitely many immediate predecessors and all of them are tabular [3], so the coincidence with a fixed tabular logic is decidable in this setting [14, 21] (see also [4, Theorem 17.3]). Moreover, as far as we know, the decidability of tabularity in NExtK4 is still open [22], whereas it is undecidable in NExtK as mentioned above.

In this paper, we study the decision problem for properties for *tense logics*. Tense logics are normal bimodal logics extending the logic $\text{K}_t = \text{K}_2 \oplus \{p \rightarrow \Box \Diamond p, p \rightarrow \blacksquare \Diamond p\}$, where, following convention, we denote the two modalities $\Box$ and $\blacksquare$ with their duals $\Diamond$ and $\blacklozenge$. The intended meaning of $\Box$ and $\blacksquare$ are “always true in the future” and “always true in the past”, respectively. The least tense logic K_t was introduced in the 1960s by Prior [19, 20]. Philosophically, tense logics can be viewed as logics of time; for example, $\text{K4}_t = \text{K}_t \oplus \Diamond p \rightarrow \Diamond p$ is the logic of transitive time flows, and $\text{Lin}_t = \text{K4}_t \oplus (\Diamond \blacklozenge p \vee \blacklozenge \Diamond p \rightarrow p \vee \Diamond p \vee \blacklozenge p)$ is the logic of linear time flows (see also [33]). Algebraically, the modalities $\Box$ and $\blacklozenge$ are adjoint, in the sense that $\blacklozenge \phi \rightarrow \psi \in L$ if and only if $\phi \rightarrow \Box \psi \in L$ for any tense logic L . Thus, tense logics may also be viewed as “logics of adjointness,” and they serve as examples of bimodal logics in which two modalities interact in a natural way.

The bimodal logics $\text{K}_2 \subseteq \text{K}_t \subseteq \text{K4}_t \subseteq \text{Lin}_t$ form a chain in NExtK_2 . Many properties, including Kripke completeness, the FMP, decidability, and even consistency, are undecidable in NExtK_n for $n \geq 2$ [27]. On the other hand, the aforementioned properties are all decidable in NExtLin_t [30, 32]. These results indicate that the interactions of modalities significantly affect the decidability of properties, which naturally raises the question of whether these properties are decidable in NExtK_t or NExtK4_t . Chagrov and Shehtman [9] proved that tabularity, the coincidence with a fixed tabular tense logic, and consistency are undecidable in NExtK4_t (and thus also undecidable in NExtK_t), while the decidability of other properties, such as Kripke completeness, the FMP, and decidability, remained open.

Although K_t and K4_t are the minimal tense extensions of K and K4 respectively, the undecidability of properties in NExtK_t and NExtK4_t does not follow directly from the undecidability results for NExtK4 and NExtK . The minimal tense extension map $(\cdot)_t : \text{NExtK} \rightarrow \text{NExtK}_t$ is not injective [28], while it remains unknown whether $(\cdot)_t \upharpoonright \text{NExtK4}$ is injective [31, p. 158]. Moreover, Wolter [29] presented a modal logic $L \in \text{NExtK4}$ having the FMP whose minimal tense extension $L_t \in \text{NExtK4}_t$ is Kripke incomplete. It follows that the map $(\cdot)_t$ preserves neither Kripke completeness nor the FMP, while it remains open whether decidability is preserved (see [31, p. 132]). The interactions between tense modalities make lattices of tense logics completely different from those of unimodal logics (see [15, 11, 12, 13]).

In this paper, we provide a general criterion (Theorem 3.5) for a property to be undecidable in NExtK4_t . Most properties studied for logics fall into this criterion, including Kripke completeness, the FMP, and decidability; see Corollary 3.13 for a more comprehensive list of undecidable properties in NExtK4_t following from the theorem. It follows that these properties are also undecidable in NExtK_t .

Our proof adapts the method of [9], reducing an undecidable problem regarding Minsky machines to the decision problem for a property. *Minsky machines*, also called counter machines or register machines, are a type of mathematical model of computation as strong as Turing machines [18]. There exist a Minsky machine M and a configuration c_0 of M such that it is undecidable whether a given configuration of M is reachable from c_0 by computation of M (see, e.g., [4, Theorem 16.3]). Let us call this undecidable problem Q . Given a property P , we will find a logic $L \in \text{NExtK4}_t$ that has P and construct a computable reduction from Q to the decision problem for P as follows. For a configuration c , the reduction produces a logic $L(c) \in \text{NExtK4}_t$ such that:

1. if c is reachable from c_0 , then $L(c) = L$, which implies that $L(c)$ has P ;

2. if c is not reachable from c_0 , then $L(c)$ does not have P .

Thus, if we could decide whether a logic in NExtK4_t has P , we would be able to decide the problem Q : Given a configuration c , we compute the logic $L(c)$ and ask if it has the property P , the answer of which is also the answer to Q . Since Q is undecidable, it follows that P is undecidable.

Table 1 summarizes the results on the decidability of some major properties discussed so far. The entries marked with † are results established in this paper.

	Cons.	Tab.	Fixed Tab.	KC	FMP	Dec.
NExtK	✓	×	×	×	×	×
NExtK4	✓	?	✓	×	×	×
NExtK ₂	×	×	×	×	×	×
NExtK _t	×	×	×	×	×	×
NExtK4 _t	×	×	×	×	×	×
NExtLin _t	✓	?	✓	✓	✓	✓

Abbreviations. Cons.: consistency; Tab.: tabularity; Fixed Tab.: coincidence with a fixed tabular logic; KC: Kripke completeness; FMP: finite model property; Dec.: Decidability.

Table 1: (Un)decidability of logics' properties in lattices of unimodal and bimodal logics

This paper is organized as follows. Section 2 introduces preliminaries on tense logics and Minsky machines. In section 3, we prove the main theorem and apply it to show the undecidability of logics' properties. Finally, Section 4 concludes the paper with an overview of future work.

2 Preliminaries

2.1 Preliminaries of modal and tense logic

Recall that for each $n \in \mathbb{Z}^+$, the n -modal language $\mathcal{L}_n$ is obtained by adding the modalities $\Box_0, \dots, \Box_{n-1}$ to the propositional language. A *normal n -modal logic* is a set of $\mathcal{L}_n$ -formulas which contains all classical tautologies, K-axioms $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q) \in L$ for all $i < n$, and is closed under the rules Modus Ponens ($\varphi, \varphi \rightarrow \psi / \psi$), Substitution ($\varphi(p_1, \dots, p_n) / \varphi(\psi_1, \dots, \psi_n)$) and Necessitation ($\varphi / \Box_i \varphi$). Let K_n denote the minimal normal n -modal logic. We refer to 2-modal logics as *bimodal logics*.

A *tense logic* is a normal bimodal logic containing the axioms $p \rightarrow \Box_0 \Diamond_1 p$ and $p \rightarrow \Box_1 \Diamond_0 p$. For tense logics, we write $\mathcal{L}_t$ for the formal language, $\Box$ for $\Box_0$ and $\blacksquare$ for $\Box_1$. Let K_t denote the *minimal tense logic*. For each tense logic L , let $\text{NExt}L$ denote the lattice of all normal extensions of L . For every tense logic L and set of formulas Σ , let $L \oplus \Sigma$ denote the smallest tense logic containing $L \cup \Sigma$. We write $L \oplus \varphi$ for $L \oplus \{\varphi\}$. A tense logic L is *finitely axiomatizable* if $L = K_t \oplus \varphi$ for some $\varphi \in \mathcal{L}_t$. A tense logic L is *consistent* if $\perp \notin L$, and so the only inconsistent tense logic is $\mathcal{L}_t$.

A *general frame* is a triple $\mathbb{F} = (X, R, A)$ where X is a non-empty set, R a binary relation on X and A a subset of $\mathcal{P}(X)$ such that (i) $\emptyset \in A$, and (ii) A is closed under Boolean operations on $\mathcal{P}(X)$, $R[\cdot]$ and $R^{-1}[\cdot]$, where $R[Y] := \{x \in X : \exists y \in Y (Rxy)\}$ and $R^{-1}[Y] := \{x \in X : \exists y \in Y (Ryx)\}$ for all $Y \subseteq X$. A *Kripke frame* $\mathfrak{F}$ is a general frame of the form $(X, R, \mathcal{P}(X))$ and we simply write (X, R) . Let GFr , Fr , and Fin denote the classes of all general frames, Kripke frames, and finite Kripke frames, respectively.

A *model* is a pair $\mathfrak{M} = (\mathbb{F}, V)$ where $\mathbb{F} \in \text{GFr}$ and $V : \text{Prop} \rightarrow A$ a valuation in $\mathbb{F}$. V is extended to $V : \mathcal{L}_t \rightarrow A$ as usual: $V(\Diamond \varphi) = R[V(\varphi)]$ and $V(\Box \varphi) := X \setminus R^{-1}[X \setminus V(\varphi)]$. The expressions $\mathfrak{M}, x \models \varphi$, $\mathbb{F}, x \models \varphi$, $\mathbb{F} \models \varphi$ and $\mathbb{F} \models \Sigma$ are defined as usual. Note that $\mathfrak{M}, x \models \Diamond \varphi$ if and only if $\mathfrak{M}, y \models \varphi$ for some $y \in R^{-1}[x]$. For all sets $\Sigma \subseteq \mathcal{L}_t$ of formulas and classes $\mathcal{K} \subseteq \text{GFr}$ of general frames, let

$$\mathcal{K}(\Sigma) := \{\mathbb{F} \in \mathcal{K} : \mathbb{F} \models \Sigma\} \text{ and } \text{Log}(\mathcal{K}) := \{\varphi : \mathcal{K} \models \varphi\}.$$

For example, given a tense logic L , we write $\text{Fin}(L)$ for the class of all finite frames validating L . We call $\text{Log}(\mathcal{K})$ the *tense logic of* $\mathcal{K}$. Recall that a Kripke frame (X, R) is called *transitive* if R is transitive. Then $\text{K4}_t := \text{K}_t \oplus \Diamond \Diamond p \rightarrow \Diamond p$ is the tense logic of transitive frames. In other words, we have $\text{K4}_t = \text{Log}(\{\mathfrak{F} \in \text{Fr} : \mathfrak{F} \text{ is transitive}\})$. A tense logic L is *transitive* if L extends K4_t , i.e., $L \supseteq \text{K4}_t$.

Let us recall some properties of tense logics. Let L be a tense logic. Then (i) L is *Kripke complete*, if $L = \text{Log}(\text{Fr}(L))$; (ii) L has the *finite model property (FMP)*, if $L = \text{Log}(\text{Fin}(L))$; (iii) L is *tabular*, if $L = \text{Log}(\mathfrak{F})$ for some finite frame $\mathfrak{F}$; (iv) L is *canonical*, if $L = \text{Log}(\mathfrak{F}^L)$, where $\mathfrak{F}^L$ is the *canonical frame* for L ; (v) L is *first-order definable*, if $L = \text{Log}(\mathcal{K})$, where $\mathcal{K}$ is a class of Kripke frames defined by a set of first-order sentences. Moreover, we say that L is *locally tabular* if for each $n \in \omega$, L contains only finitely many non- L -equivalent formulas built up from the propositional variables $p_0, \dots, p_{n-1}$. Finally, we say that L is *decidable* if there is an algorithm that, given a formula φ , decides whether $\varphi \in L$. This *membership decidability* is a property for logics, and should not be confused with decidability of logics' properties; in particular, it is legitimate to say decidability, as a property, is decidable or not.

2.2 Decision problem for logics' properties

In this section, instead of the decidability of logics, we focus on the decidability of logics' properties. We refer to [4, Chapter 17] and [33] for a detailed introduction and survey of the decision problem for properties of modal logics. We will work in the tense (or more generally, bimodal) setting. We identify a property P in the lattice $\text{NExt}(L_0)$ with the set of logics in $\text{NExt}(L_0)$ that satisfy P , that is, $P = \{L \in \text{NExt}(L_0) : L \text{ satisfies } P\}$.

Definition 2.1. Let L_0 be a tense logic. A property P is *decidable* in $\text{NExt}(L_0)$ iff the set $\{\varphi : L_0 \oplus \varphi \in P\}$ is decidable.

Following convention, we restrict ourselves to finitely axiomatizable logics because an input for an algorithm must be a finite object. We do not consider all recursively axiomatizable logics, as Kuznetsov showed that otherwise the only decidable properties would be the trivial ones (see [4, Section 17.1]). Since most logics we encounter in practice are finitely axiomatizable, this is not a serious drawback. A finitely axiomatizable logic will be encoded by a finite set of formulas axiomatizing the logic, or equivalently, a single formula axiomatizing the logic.

Note that determining a property in a larger lattice of logics is at least as hard as in a smaller one, in the following sense.

Proposition 2.2. Let $L \in \text{NExtK}_n$ and L' be an extension of L with finitely many axioms. If a property P is undecidable in $\text{NExt}L'$, then it is undecidable in $\text{NExt}L$.

Proof. We may assume $L' = L \oplus \varphi$ for a formula φ . We prove the contrapositive. Let P be a decidable property in $\text{NExt}L$. Then, given a formula ψ , we can determine whether $L' \oplus \psi$ has P by asking whether $L \oplus \varphi \wedge \psi$ has P since the two logics are the same. $\square$

Minsky machines

The most commonly used method for proving the undecidability of a decision problem is to construct a computable reduction from another problem that is already known to be undecidable to the problem. In this paper, we will use an undecidable problem about Minsky machines. We recall the basics of Minsky machines in the rest of this section and refer to [4, Section 16.1] and [18] for more details; see also [4, Sections 16 and 17] for various applications of Minsky machines to obtain undecidability results.

A *Minsky machine* with two registers (also called a counter/register machine with two counters/registers) is a finite set of instructions acting on two registers. We will only use Minsky machines with two registers, so we simply call them Minsky machines. A Minsky machine has finitely many *states*. A *register* can store a natural number and is assumed to be unbounded. So, a situation of a Minsky machine is represented by a tuple $\langle s, n, m \rangle$, called a *configuration*, where s is the current state and n and m are the natural numbers on each register. An instruction operates on the state and one of the two registers: it increments the number in the register, or tests if the number in the register is zero and decrements it if not. More specifically, an instruction I has one of the following four forms:

- $I = t \rightarrow \langle t', 1, 0 \rangle$ means that I turns the state t into t' and increment the first register,
- $I = t \rightarrow \langle t', 0, 1 \rangle$ means that I turns the state t into t' and increment the second register,
- $I = t \rightarrow \langle t', -1, 0 \rangle (\langle t'', 0, 0 \rangle)$ means that I turns the state t into t' and decrements the first register if the number in the first register is non-zero, and turns the state t into t'' otherwise,
- $I = t \rightarrow \langle t', 0, -1 \rangle (\langle t'', 0, 0 \rangle)$ means that I turns the state t into t' and decrements the second register if the number in the second register is non-zero, and turns the state t into t'' otherwise.

For example, applying the instruction $I = s \rightarrow \langle s', -1, 0 \rangle (\langle s'', 0, 0 \rangle)$ to the configuration $\langle s, n, m \rangle$, we obtain the configuration $\langle s', n-1, m \rangle$ if $n \geq 1$ and the configuration $\langle s'', n, m \rangle$ if $n = 0$.

In this paper, Minsky machines are assumed to be *deterministic*, that is, for each state t there is at most one instruction that acts on the state t . For a Minsky machine M , we write $M : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle$ if, starting from the configuration $\langle s, n, m \rangle$, by applying the instructions in M , we can reach the configuration $\langle t, k, l \rangle$ in finitely many (possibly 0) steps. We drop M if it is clear from the context.

Example 2.3. Let $M = \{s \rightarrow \langle s, 1, 0 \rangle\}$. Then, for any $n, m \in \omega$,

$$\{\langle t, k, l \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\} = \{\langle s, n+i, m \rangle : i \in \omega\}.$$

We will use the following undecidable problem, which is called the *second configuration problem* in [4, Theorem 16.3].

Theorem 2.4. *There exist a Minsky machine M and a configuration $\langle s, n, m \rangle$ such that the reachability from $\langle s, n, m \rangle$ in M is undecidable, that is, the set $\{\langle t, k, l \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\}$ is undecidable.*

3 Undecidable properties in NExtK4_t

The aim of this section is to prove the general undecidability result Theorem 3.5, which implies the undecidability of various properties summarized in Corollary 3.13. The proof idea follows Chagrov's method of using Minsky machines [5, 6, 9] (see also [33]). Let M be a Minsky machine and $\langle s, n, m \rangle$ be a configuration of M such that the set $\{\langle t, k, l \rangle : M : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\}$ is undecidable, given by Theorem 2.4. Since the Minsky machine M is finite, we may assume that M contains t_0 many states, labeled as $0, \dots, t_0 - 1$. To state our main theorem, we introduce the following general frame that encodes the problem $\{\langle t, k, l \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\}$.

Definition 3.1. Let $\mathbb{F} = (W, R, A)$ be the general frame defined as follows:

- $W = \{\langle t, k, l \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\} \cup \{a_n, b_n, c_n : n < \omega\} \cup \{a', b', b''\}$;
- R is the transitive closure of the union of the following binary relations:
 - $\{(c_i, c_j) : j < i < \omega\}$;
 - $\{(a_i, a_j) : j < i < \omega\} \cup \{(a', a_0)\}$;
 - $\{(b_i, b_j) : j < i < \omega\} \cup \{(b', b_0), (b', b'')\}$;
 - $\{(\langle t, k, l \rangle, c_i), (\langle t, k, l \rangle, a_k), (\langle t, k, l \rangle, b_l), (\langle t, k, l \rangle, \langle t, k, l \rangle) : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\}$.
- $A = \{U \subseteq W : U \text{ is finite or cofinite on } \{c_i : i \in \omega\}\}$.

It is clear that A is closed under $\cap$, $W \setminus (\cdot)$, $R[\cdot]$ and $R^{-1}[\cdot]$, so $\mathbb{F}$ is well-defined as a general frame. The underlying Kripke frame (W, R) of $\mathbb{F}$ is depicted in Figure 1.

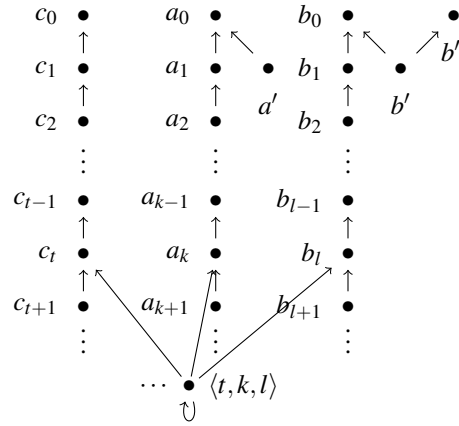


Figure 1: The underlying frame of $\mathbb{F}$

Before studying properties of the general frame $\mathbb{F}$, let us introduce a set of new modal operators $\Delta^{\leq n}$ and their duals $\nabla^{\leq n}$, which will play an important role in our proofs.

Definition 3.2. For each $n \in \omega$ and $\varphi, \psi \in \mathcal{L}_t$, we define the formula $\Delta^{\leq n} \varphi$ by:

$$\Delta^{\leq 0} \varphi = \varphi \text{ and } \Delta^{\leq k+1} \varphi = \Delta^{\leq k} \varphi \vee \Diamond \Delta^{\leq k} \varphi \vee \Diamond \Delta^{\leq k} \varphi.$$

As usual, we define the dual operator $\nabla^{\leq n}$ of $\Delta^{\leq n}$ by $\nabla^{\leq n} \varphi := \neg \Delta^{\leq n} \neg \varphi$.

For each Kripke frame $\mathfrak{F} = (X, R)$, let $R_{\#}$ be the binary relation $(= \cup R \cup R^{-1})$ on X , that is,

$$R_{\#} = \{(x, y) \in X \times X : x = y \text{ or } Rxy \text{ or } Ryx\}.$$

It follows that for all $x, y \in X$, if $y \in R_{\#}^n[x]$, then there exists a sequence $C = \langle x_i : i \leq k \rangle$ with $k < n$ such that (i) $x = x_0$, (ii) $y = x_k$, and (iii) $x_{i+1} \in R_{\#}[x_i]$ for all $i < k$. We call C a k -path from x to y .

Proposition 3.3. Let $\mathfrak{M} = (X, R, V)$ be a model, $x \in X$ and $\varphi \in \mathcal{L}_t$. Then for all $k \in \omega$,

$$\mathfrak{M}, x \models \Delta^{\leq k} \varphi \text{ if and only if } \mathfrak{M}, y \models \varphi \text{ for some } y \in R_{\#}^k[x].$$

Proof. By induction on k . □

Lemma 3.4. $\mathbb{F} \models \Delta^{\leq 6} p \rightarrow \Delta^{\leq 5} p$.

Proof. Any two points in W are connected by an $(R \cup R^{-1})$ -chain of length no more than 5. □

Now we state the main theorem. Let

$$\text{KC} = \{L \in \text{NExtK4}_t : L \text{ is Kripke complete}\}$$

and

$$\text{DEC} = \{L \in \text{NExtK4}_t : L \text{ is decidable}\}.$$

Theorem 3.5. *Let P be a property and $\alpha \in \mathcal{L}_t$ be a formula such that $\mathbb{F} \not\models \alpha$ and $\text{K4}_t \oplus \alpha \in P$ and $P \subseteq \text{KC} \cup \text{DEC}$. Then P is undecidable, i.e., the set $\{\varphi \in \mathcal{L}_t : \text{K4}_t \oplus \varphi \in P\}$ is undecidable.*

The rest of this section is dedicated to proving this theorem. We start by introducing formulas that define a point or a set of points in $\mathbb{F}$. Their meaning is summarized in Lemma 3.6. For each $n \in \omega$, we define the formulas φ_{c_n} , φ_{a_n} and φ_{b_n} as follows:

- $\varphi_{c_0} := \Box \perp \wedge \blacksquare \blacklozenge \top$ and $\varphi_{c_k} := \Diamond^k \varphi_{c_0} \wedge \neg \Diamond^{k+1} \varphi_{c_0}$;
- $\varphi_{a_0} := \Box \perp \wedge \blacklozenge (\blacksquare \perp \wedge \Box \blacklozenge^2 \top)$ and $\varphi_{a_k} := (\varphi_{a_0} \vee \Diamond \varphi_{a_0}) \wedge \Box \neg \varphi_{c_0} \wedge \blacklozenge \top \wedge \Diamond^k \varphi_{a_0} \wedge \neg \Diamond^{k+1} \varphi_{a_0}$;
- $\varphi_{b_0} := \Box \perp \wedge \blacklozenge \blacklozenge \top \wedge \blacklozenge \blacksquare \perp$ and $\varphi_{b_k} := (\varphi_{b_0} \vee \Diamond \varphi_{b_0}) \wedge \Box \neg \varphi_{c_0} \wedge \blacklozenge \top \wedge \Diamond^k \varphi_{b_0} \wedge \neg \Diamond^{k+1} \varphi_{b_0}$.

Note that these formulas are all variable-free. Intuitively, for each $x \in \{a, b, c\}$ and $n \in \omega$, the formula φ_{x_n} is designed to be true at exactly x_n in $\mathbb{F}$, regardless of valuations. Moreover, we define

- $\varphi_A := (\varphi_{a_0} \vee \Diamond \varphi_{a_0}) \wedge \Box \neg \varphi_{c_0} \wedge \blacklozenge \top$;
- $\varphi_B := (\varphi_{b_0} \vee \Diamond \varphi_{b_0}) \wedge \Box \neg \varphi_{c_0} \wedge \blacklozenge \top$.

Similarly, φ_A and φ_B are true exactly at $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$ in $\mathbb{F}$, respectively. To simulate the action of $+1$ and -1 on the two tapes of M , we define the following formulas:

- $\psi_A := \varphi_A \wedge p_A \wedge \neg \Diamond p_A$;
- $\psi_A^+ := \varphi_A \wedge \Diamond p_A \wedge \neg \Diamond \Diamond p_A$;
- $\psi_B := \varphi_B \wedge p_B \wedge \neg \Diamond p_B$;
- $\psi_B^+ := \varphi_B \wedge \Diamond p_B \wedge \neg \Diamond \Diamond p_B$,

where p_A, p_B are fresh variables. Intuitively, if ψ_A is true at some point, then the point must be a_i , where $i = \min\{j : a_j \models p_A\}$; then ψ_A^+ is true at the next point, namely, a_{i+1} . A similar intuition applies to ψ_B and ψ_B^+ as well. Moreover, a key syntactic observation is: if s is the substitution $[\Diamond^k \varphi_{a_0} / p_A, \Diamond^l \varphi_{b_0} / p_B]$ for some $k, l \in \omega$, then $s(\psi_A) = \varphi_{a_k}$, $s(\psi_A^+) = \varphi_{a_{k+1}}$, $s(\psi_B) = \varphi_{b_l}$, and $s(\psi_B^+) = \varphi_{b_{l+1}}$. This will be used in Lemma 3.7.

Finally, for each state t of M and formulas $\pi, \kappa \in \mathcal{L}_t$, we define:

$$\sigma(t, \pi, \kappa) := \Diamond \varphi_{c_t} \wedge \Box \neg \varphi_{c_{t+1}} \wedge \Diamond \pi \wedge \Box \neg (\Diamond \pi \wedge \Box \neg \varphi_{c_0}) \wedge \Diamond \kappa \wedge \Box \neg (\Diamond \kappa \wedge \Box \neg \varphi_{c_0}).$$

As we will see in Lemma 3.6, the formula $\sigma(t, \pi, \kappa)$ is true exactly at the point $\langle t, k, l \rangle$ if the formulas π and κ are true exactly at a_k and b_l , respectively.

Lemma 3.6. *For any $w \in W$, valuation V on $\mathbb{F}$, and $n \in \omega$, the following holds:*

- (1) $\mathbb{F}, w \models \varphi_{c_n}$ if and only if $w = c_n$.
- (2) $\mathbb{F}, w \models \varphi_{a_n}$ if and only if $w = a_n$.
- (3) $\mathbb{F}, w \models \varphi_{b_n}$ if and only if $w = b_n$.
- (4) $\mathbb{F}, w \models \varphi_A$ if and only if $w \in \{a_i : i \in \omega\}$.

- (5) $\mathbb{F}, w \models \varphi_B$ if and only if $w \in \{b_i : i \in \omega\}$.
- (6) $\mathbb{F}, V, w \models \psi_A$ if and only if $V(\psi_A) = \{a_i\} = \{w\}$ and $V(\psi_A^+) = \{a_{i+1}\}$ for some $i \in \omega$.
- (7) $\mathbb{F}, V, w \models \psi_A^+$ if and only if $V(\psi_A) = \{a_i\}$ and $V(\psi_A^+) = \{a_{i+1}\} = \{w\}$ for some $i \in \omega$.
- (8) $\mathbb{F}, V, w \models \psi_B$ if and only if $V(\psi_B) = \{b_i\} = \{w\}$ and $V(\psi_B^+) = \{b_{i+1}\}$ for some $i \in \omega$.
- (9) $\mathbb{F}, V, w \models \psi_B^+$ if and only if $V(\psi_B) = \{b_i\}$ and $V(\psi_B^+) = \{b_{i+1}\} = \{w\}$ for some $i \in \omega$.
- (10) If $V(\pi) = \{a_k\}$ and $V(\kappa) = \{b_l\}$, then $\mathbb{F}, V, w \models \sigma(t, \pi, \kappa)$ if and only if $w = \langle t, k, l \rangle$.

Proof. We only prove (1), (4), (6), and (10) and leave the rest to the readers. The proof of (1) proceeds by induction on n . Let $n = 0$. The right-to-left direction is clear. Suppose $\mathbb{F}, w \models \varphi_{c_0}$. Then $R[w] = \emptyset$ and $R^{-1}[u] \neq \emptyset$ for all $u \in R^{-1}[w]$, which entails $w = c_0$. Let $n > 0$. Suppose $\mathbb{F}, w \models \varphi_{c_n}$. By the induction hypothesis, $c_{n-1} \in R[w] \setminus R[R[w]]$, thus $w = c_n$, and (1) follows.

For (4), the right-to-left direction is straightforward. For the other direction, suppose $\mathbb{F}, w \models \varphi_A$. Then $\mathbb{F}, w \models \varphi_{a_0} \vee \Diamond \varphi_{a_0}$, which entails $w \in R^{-1}[a_0] \cup \{a_0\}$. Since $\mathbb{F}, w \models \Box \neg \varphi_{c_0} \wedge \Diamond \top$, we see that $w \notin \{\langle t, k, l \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\}$ and $w \neq a'$. Thus, $w \in \{a_i : i \in \omega\}$.

For (6), the right-to-left direction is again straightforward. For the other direction, suppose $\mathbb{F}, w \models \psi_A$. Take any $u \in W$ such that $\mathbb{F}, V, u \models \psi_A$. By (4) $u = a_i$ for some $i \in \omega$. By $\mathbb{F}, V, u \models p_A \wedge \neg \Diamond p_A$, we see that u is an R -maximal point in $V(p_A) \cap \{a_i : i \in \omega\}$. Since R is a linear order on $V(p_A) \cap \{a_i : i \in \omega\}$, we have $V(\psi_A) = \{a_i\} = \{w\}$. As $V(\Diamond p_A \wedge \neg \Diamond \Diamond p_A) \cap \{a_i : i \in \omega\} = \{a_{i+1}\}$, we have $V(\psi_A^+) = \{a_{i+1}\}$.

For (10), suppose $V(\pi) = \{a_k\}$ and $V(\kappa) = \{b_l\}$. Then clearly, $\mathbb{F}, V, \langle t, k, l \rangle \models \sigma(t, \pi, \kappa)$. For the other direction, suppose $\mathbb{F}, V, w \models \sigma(t, \pi, \kappa)$. Then, $\mathbb{F}, V, w \models \Diamond \varphi_{c_t} \wedge \Diamond \pi$, which entails $w = \langle t', k', l' \rangle$ for some $\langle t', k', l' \rangle$. By $\mathbb{F}, V, w \models \Diamond \varphi_{c_t} \wedge \Box \neg \varphi_{c_{t+1}}$, we have $w \in R^{-1}[c_t] \setminus R^{-1}[c_{t+1}]$ and so $t' = t$. Note that $\mathbb{F}, V, a_{k+1} \models \Diamond \pi \wedge \Box \neg \varphi_{c_0}$. By $\mathbb{F}, V, w \models \Diamond \pi \wedge \Box \neg (\Diamond \pi \wedge \Box \neg \varphi_{c_0})$, we see that $w \in R^{-1}[a_k] \setminus R^{-1}[a_{k+1}]$ and so $k' = k$. Similarly, we obtain $l = l'$. Thus, $w = \langle t, k, l \rangle$. $\square$

Next, we encode the behavior of the Minsky machine M by formulas. With each instruction I in M , we associate a formula AxI as follows.

- $AxI := \neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \psi_A, \psi_B) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t', \psi_A^+, \psi_B)$, if $I = t \rightarrow \langle t', 1, 0 \rangle$.
- $AxI := \neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \psi_A, \psi_B) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t', \psi_A, \psi_B^+)$, if $I = t \rightarrow \langle t', 0, 1 \rangle$.
- $AxI := [\neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \psi_A^+, \psi_B) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t', \psi_A, \psi_B)]$
 $\wedge [\neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_0}, \psi_B) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t'', \varphi_{a_0}, \psi_B)]$, if $I = t \rightarrow \langle t', -1, 0 \rangle (\langle t'', 0, 0 \rangle)$.
- $AxI := [\neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \psi_A, \psi_B^+) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t', \psi_A, \psi_B)]$
 $\wedge [\neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \psi_A, \varphi_{b_0}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t'', \psi_A, \varphi_{b_0})]$, if $I = t \rightarrow \langle t', 0, -1 \rangle (\langle t'', 0, 0 \rangle)$.

Each formula encodes the behavior of the corresponding instruction. Let $AxM := \bigwedge_{I \in M} AxI$. This is a well-defined formula since there are only finitely many instructions in M .

Lemma 3.7. *For each configuration $\langle t, k, l \rangle$, if $\langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle$, then*

$$\neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l}) \in K4_t \oplus AxM.$$

Proof. The proof proceeds by induction on the length of the computation of M . Consider the computation of the form $\langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle \rightarrow \langle t', k', l' \rangle$, where the last step is an application of I . As the induction hypothesis, we have

$$\neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l}) \in K4_t \oplus AxM.$$

We divide cases according to the shape of I .

Case (1): $I = t \rightarrow \langle t', 1, 0 \rangle$. It suffices to show that

$$\neg\alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l}) \rightarrow \neg\alpha \wedge \Delta^{\leq 5} \sigma(t', \varphi_{a_{k+1}}, \varphi_{b_l}) \in K4_t \oplus AxM.$$

This is clear by applying the substitution $[\Diamond^k \varphi_{a_0}/p_A, \Diamond^l \varphi_{b_0}/p_B]$ to AxI .

Case (2): $I = t \rightarrow \langle t', -1, 0 \rangle (\langle t'', 0, 0 \rangle)$. Suppose $k \neq 0$. Then by applying to AxI the substitution $[\Diamond^{k-1} \varphi_{a_0}/p_A, \Diamond^l \varphi_{b_0}/p_B]$, we have

$$\neg\alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l}) \rightarrow \neg\alpha \wedge \Delta^{\leq 5} \sigma(t', \varphi_{a_{k-1}}, \varphi_{b_l}) \in K4_t \oplus AxM.$$

If $k = 0$, then by applying to AxI the substitution $[\Diamond^l \varphi_{b_0}/p_B]$, we have

$$\neg\alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_0}, \varphi_{b_l}) \rightarrow \neg\alpha \wedge \Delta^{\leq 5} \sigma(t'', \varphi_{a_0}, \varphi_{b_l}) \in K4_t \oplus AxM.$$

The cases $I = t \rightarrow \langle t', 0, 1 \rangle$ and $I = t \rightarrow \langle t', 0, -1 \rangle (\langle t'', 0, 0 \rangle)$ follow similarly. $\square$

Lemma 3.8. $\mathbb{F} \models AxM$.

Proof. Take any instruction $I \in M$. Suppose that I has the form $t \rightarrow \langle t', 1, 0 \rangle$. Take any point w and any valuation V on $\mathbb{F}$ such that $\mathbb{F}, V, w \models \neg\alpha \wedge \Delta^{\leq 5} (\sigma(t, \psi_A, \psi_B))$. Then $\mathbb{F}, V, u \models \sigma(t, \psi_A, \psi_B)$ for some $u \in W$. Since $\mathbb{F}, V, u \models \Diamond \psi_A \wedge \Diamond \psi_B$, by Lemma 3.6 (6) and (8), there exists $k, l \in \omega$ such that $V(\psi_A) = \{a_k\}$ and $V(\psi_B) = \{b_l\}$. By Lemma 3.6 (10), $u = \langle t, k, l \rangle$ and so $\langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle$ by the definition of W . Then, since $I \in M$, we have $\langle s, n, m \rangle \rightsquigarrow \langle t', k+1, l \rangle$. By Lemma 3.6 (7), (8) and (10), $\mathbb{F}, V, \langle t', k+1, l \rangle \models \sigma(t', \psi_A^+, \psi_B)$. So, we have $\mathbb{F}, V, w \models \neg\alpha \wedge \Delta^{\leq 5} (\sigma(t', \psi_A^+, \psi_B))$. Thus, $\mathbb{F} \models AxI$. Similarly, $\mathbb{F} \models AxI$ for instructions $I \in M$ of other forms. Hence, $\mathbb{F} \models AxM$. $\square$

Now we construct the reduction, extending that in [9]. For each configuration $\langle t, k, l \rangle$, we define the logic

$$\begin{aligned} L(t, k, l) := & K4_t \oplus AxM \oplus (\neg\alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg\alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l})) \rightarrow \alpha \\ & \oplus (\neg\alpha \rightarrow \nabla^{\leq 5} (\varphi_{c_{t_0}} \rightarrow \varphi_0) \wedge \Delta^{\leq 5} \varphi_{c_{t_0}}) \end{aligned}$$

where

$$\varphi_0 = (\blacksquare(\blacksquare(p \rightarrow \blacksquare p) \rightarrow p) \rightarrow \blacksquare p) \wedge \blacksquare((\Box q \wedge \neg q) \rightarrow \blacklozenge(\Box^2 q \wedge \Diamond \neg q)) \wedge \blacklozenge \Box^{t_0+2} \perp$$

and p and q are fresh variables w.r.t. AxM and α . Note that the axioms of $L(t, k, l)$ are computable from a configuration $\langle t, k, l \rangle$. The subsequent lemmas show some properties of $L(t, k, l)$. Intuitively, the formula φ_0 is designed to be such that any extension of $K4_t \oplus \varphi_0$ is Kripke incomplete. More precisely, we have

Lemma 3.9. Let $\mathfrak{F} = (Y, S)$ be a Kripke frame such that $\mathfrak{F} \models K4_t$. Then, $\mathfrak{F}, y \not\models \varphi_0$ for any $y \in Y$.

Proof. Suppose that $\mathfrak{F}, y \models \varphi_0$ for a contradiction. Since $\mathfrak{F}, y \models \blacklozenge \Box^{t_0+2} \perp$, $\mathfrak{F}, b' \models \Box^{t_0+2} \perp$ for some irreflexive $b' \in S^{-1}[y]$. Now take any irreflexive point $z \in S^{-1}[y]$. Let V_z be a valuation on $\mathfrak{F}$ such that $V_z(q) = R[z]$. Then $\mathfrak{F}, z \models \Box q \wedge \neg q$. By $\mathfrak{F}, y \models \blacksquare((\Box q \wedge \neg q) \rightarrow \blacklozenge(\Box^2 q \wedge \Diamond \neg q))$, we have $\mathfrak{F}, z \models \blacklozenge(\Box^2 q \wedge \Diamond \neg q)$ and so $\mathfrak{F}, V_z, z' \models \Box^2 q \wedge \Diamond \neg q$ for some $z' \in S^{-1}[z]$. Then, z' is again irreflexive because otherwise $\mathfrak{F}, V_z, z \models q$, and $z' \in S^{-1}[y]$ by the transitivity. Thus, there exists an infinite S^{-1} -chain $\{z_i : i \in \omega\}$ of irreflexive points in $S^{-1}[y]$. Let V be a valuation on $\mathfrak{F}$ such that $V(p) = \{z_{2j} : j \in \omega\}$. Then we see that $\mathfrak{F}, V, y \not\models \blacksquare(\blacksquare(p \rightarrow \blacksquare p) \rightarrow p) \rightarrow \blacksquare p$, which contradicts $\mathfrak{F}, y \models \varphi_0$. $\square$

On the other hand, the following lemma holds:

Lemma 3.10. $\mathbb{F}, c_{t_0} \models \varphi_0$.

Proof. First, we show that $\mathbb{F}, c_{t_0} \models \blacksquare(\blacksquare(p \rightarrow \blacksquare p) \rightarrow p) \rightarrow \blacksquare p$. Take any valuation V on $\mathbb{F}$. Then $V(p) \in A$, so either $V(p) \cap \{c_i : i \in \omega\}$ is finite or $\{c_i : i < \omega\} \setminus V(p)$ is finite. Let $\mathfrak{M} = (\mathbb{F}, V)$. Suppose $\mathfrak{M}, c_{t_0} \not\models \blacksquare(\blacksquare(p \rightarrow \blacksquare p) \rightarrow p) \rightarrow \blacksquare p$ for a contradiction. Then $\mathfrak{M}, c_{t_0} \models \blacksquare(\blacksquare(p \rightarrow \blacksquare p) \rightarrow p)$ and $\mathfrak{M}, c_{t_0} \models \Diamond \neg p$. Thus, $R^{-1}[c_{t_0}] \setminus V(p) \neq \emptyset$, and for every $w \in R^{-1}[c_{t_0}] \setminus V(p)$, we have $\mathfrak{M}, w \models \Diamond(p \wedge \Diamond \neg p)$. Since $R^{-1}[c_{t_0}] = \{c_i : i > t_0\}$, it follows that both $\{c_i : i < \omega\} \cap V(p)$ and $\{c_i : i < \omega\} \setminus V(p)$ are infinite, which is a contradiction. Thus, $\mathbb{F}, c_{t_0} \models \blacksquare(\blacksquare(p \rightarrow \blacksquare p) \rightarrow p) \rightarrow \blacksquare p$.

To show that $\mathbb{F}, c_{t_0} \models \blacksquare((\Box q \wedge \neg q) \rightarrow \Diamond(\Box^2 q \wedge \Diamond \neg q))$. Take any point $w \in R^{-1}[c_{t_0}]$ and any valuation V in $\mathbb{F}$. Note that $R^{-1}[c_{t_0}] = \{c_i : i > t_0\}$ since M contains t_0 many states labeled as $0, \dots, t_0 - 1$. Then $w = c_i$ for some $i > t_0$. Let $\mathfrak{M} = (\mathbb{F}, V)$. Suppose $\mathfrak{M}, c_i \models \Box q \wedge \neg q$. Then $\mathfrak{M}, c_{i+1} \models \Box^2 q \wedge \Diamond \neg q$, which entails $\mathfrak{M}, c_i \models \Diamond(\Box^2 q \wedge \Diamond \neg q)$. Thus, $\mathbb{F}, c_i \models (\Box q \wedge \neg q) \rightarrow \Diamond(\Box^2 q \wedge \Diamond \neg q)$, and so $\mathbb{F}, c_{t_0} \models \blacksquare((\Box q \wedge \neg q) \rightarrow \Diamond(\Box^2 q \wedge \Diamond \neg q))$.

Finally, $\mathbb{F}, c_{t_0} \models \Diamond \Box^{t_0+2} \perp$ follows from $\mathbb{F}, c_{t_0+1} \models \Box^{t_0+2} \perp$. Thus, we conclude that $\mathbb{F}, c_{t_0} \models \varphi_0$. $\square$

The following two lemmas summarize how the reduction works.

Lemma 3.11. Let $\langle t, k, l \rangle$ be a configuration such that $\langle s, n, m \rangle \not\rightsquigarrow \langle t, k, l \rangle$. Then the following hold.

1. $\mathbb{F} \models L(t, k, l)$.
2. $L(t, k, l)$ is Kripke incomplete.
3. $L(t, k, l)$ is undecidable.

Proof. For (1), we already know from Lemma 3.8 that $\mathbb{F} \models AxM$. Since $\langle t, k, l \rangle \notin W$, by Lemma 3.6 (2), (3), and (10), we have $\mathbb{F} \models \neg \sigma(t, \varphi_{a_k}, \varphi_{b_l})$ and so $\mathbb{F} \models \neg \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l})$. Similarly, since $\langle s, n, m \rangle \rightsquigarrow \langle s, n, m \rangle$, we have $\mathbb{F}, \langle s, n, m \rangle \models \sigma(s, \varphi_{a_n}, \varphi_{b_m})$. By Lemma 3.4, $\mathbb{F} \models \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m})$. Take any $w \in W$ and valuation V in $\mathbb{F}$. If $\mathbb{F}, V, w \models \alpha$, then $\mathbb{F}, V, w \models \neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \wedge \neg \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l})$, i.e. $\mathbb{F}, V, w \models \neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l})$. Thus, $\mathbb{F} \models (\neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t, \varphi_{a_k}, \varphi_{b_l})) \rightarrow \alpha$. By Lemmas 3.10 and 3.6 (1), we have $\mathbb{F} \models \varphi_{c_{t_0}} \rightarrow \varphi_0$ and so $\mathbb{F} \models \nabla^{\leq 5} (\varphi_{c_{t_0}} \rightarrow \varphi_0)$. By Lemmas 3.4 and 3.6 (1), we have $\mathbb{F} \models \Delta^{\leq 5} \varphi_{c_{t_0}}$. Thus, $\mathbb{F} \models \nabla^{\leq 5} (\varphi_{c_{t_0}} \rightarrow \varphi_0) \wedge \Delta^{\leq 5} \varphi_{c_{t_0}}$. Hence, $\mathbb{F} \models L(t, k, l)$.

For (2), we first recall that $\mathbb{F} \not\models \alpha$ follows from the assumption. By (1), we obtain that $\mathbb{F} \models L(t, k, l)$ and so $\alpha \notin L(t, k, l)$. Thus, it suffices now to show that $\alpha \in \text{Log}(\text{Fr}(L(t, k, l)))$. Take any Kripke frame $\mathfrak{F} = (Y, S) \in \text{Fr}(L(t, k, l))$. Suppose $\mathfrak{F} \not\models \alpha$. Then $\mathfrak{F}, V, w \models \neg \alpha$ for some $w \in Y$ and valuation V on $\mathfrak{F}$. Since $\mathfrak{F} \models L(t, k, l)$, we have $\mathfrak{F}, w \models \neg \alpha \rightarrow \nabla^{\leq 5} (\varphi_{c_{t_0}} \rightarrow \varphi_0) \wedge \Delta^{\leq 5} \varphi_{c_{t_0}}$. Thus, for any valuation V' that agrees with V on the variables occurring in α , we have $\mathfrak{F}, V', w \models \neg \alpha$ and so $\mathfrak{F}, V', w \models \nabla^{\leq 5} (\varphi_{c_{t_0}} \rightarrow \varphi_0) \wedge \Delta^{\leq 5} \varphi_{c_{t_0}}$. Note that φ_0 and α have no common variable and $\varphi_{c_{t_0}}$ is variable-free. So, we have $\mathfrak{F}, w \models \nabla^{\leq 5} (\varphi_{c_{t_0}} \rightarrow \varphi_0) \wedge \Delta^{\leq 5} \varphi_{c_{t_0}}$. Since $\varphi_{c_{t_0}}$ is variable-free, there exists $u \in S_{\sharp}^5[w]$ such that $\mathfrak{F}, u \models \varphi_{c_{t_0}}$, and so $\mathfrak{F}, u \models \varphi_0$, which contradicts Lemma 3.9. Thus, $\mathfrak{F} \models \alpha$, and we conclude $\alpha \in \text{Log}(\text{Fr}(L(t, k, l)))$.

For (3), it suffices to show that the following holds for any configuration $\langle t', k', l' \rangle$,

$$(\dagger) \langle s, n, m \rangle \rightsquigarrow \langle t', k', l' \rangle \text{ if and only if } \neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t', \varphi_{a_{k'}}, \varphi_{b_{l'}}) \in L(t, k, l).$$

This yields a reduction from the problem $\{\langle t', k', l' \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t', k', l' \rangle\}$ to the decision problem for $L(t, k, l)$, which implies the undecidability of $L(t, k, l)$. The left-to-right direction of $(\dagger)$ follows from Lemma 3.7. Suppose $\langle s, n, m \rangle \not\rightsquigarrow \langle t', k', l' \rangle$. Since $\mathbb{F} \not\models \alpha$, similar to the proof of (1), we see that $\mathbb{F} \models \neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t', \varphi_{a_{k'}}, \varphi_{b_{l'}})$. By (1) $\mathbb{F} \models L(t, k, l)$, we have $\neg \alpha \wedge \Delta^{\leq 5} \sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg \alpha \wedge \Delta^{\leq 5} \sigma(t', \varphi_{a_{k'}}, \varphi_{b_{l'}}) \notin L(t, k, l)$, which concludes the other direction. $\square$

Lemma 3.12. *For each configuration $\langle t, k, l \rangle$, if $\langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle$ then $L(t, k, l) = K4_t \oplus \alpha$.*

Proof. Let $L = K4_t \oplus \alpha$. It is clear that $L(t, k, l) \subseteq L$. Suppose that $\langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle$. By Lemma 3.7, $\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l}) \in K4_t \oplus AxM$. Thus, we obtain $\alpha \in L(t, k, l)$, since $(\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \rightarrow \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l})) \rightarrow \alpha \in L(t, k, l)$, and thus $L(t, k, l) = L$. $\square$

Now we are ready to prove the main theorem.

Proof of Theorem 3.5. Let P be a property and $\alpha \in \mathcal{L}_t$ be a formula such that $\mathbb{F} \not\models \alpha$ and $K4_t \oplus \alpha \in P$ and $P \subseteq KC \cup DEC$. The reduction $\langle t, k, l \rangle \mapsto L(t, k, l)$ satisfies the following:

- If $\langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle$, then $L(t, k, l) = K4_t \oplus \alpha$ by Lemma 3.12, so $L(t, k, l) \in P$,
- If $\langle s, n, m \rangle \not\rightsquigarrow \langle t, k, l \rangle$, then $L(t, k, l) \notin KC \cup DEC$ by Lemma 3.11, so $L(t, k, l) \notin P$.

Thus, we obtain a reduction from the set $\{\langle t, k, l \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle\}$, which is undecidable, to the set $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \in P\}$, which is therefore also undecidable. $\square$

As a corollary of Theorem 3.5, we obtain the following undecidable properties in $NExtK4_t$.

Corollary 3.13. *The following sets are undecidable:*

1. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ is Kripke complete}\},$
2. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ is canonical}\},$
3. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ is first-order definable}\},$
4. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ has the FMP}\},$
5. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ is locally tabular}\},$
6. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ is tabular}\},$
7. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ is decidable}\},$
8. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi = L\},$ where L is an arbitrarily fixed tabular logic,
9. $\{\varphi \in \mathcal{L}_t : K4_t \oplus \varphi \text{ is consistent}\}.$

Proof. All properties in (1) - (7) are contained in $KC \cup DEC$. Since the logic $\text{Log}(\bullet) = K4_t \oplus \Box\bot \wedge \blacksquare\bot$ is tabular, it has all properties in (1) - (7). Also, it is clear that $\mathbb{F} \not\models \Box\bot \wedge \blacksquare\bot$. So, applying Theorem 3.5 with $\alpha = \Box\bot \wedge \blacksquare\bot$, we obtain the undecidability of (1) - (7). For (8), take any tabular tense logic L . It follows from [13, Theorem 3.9] that there exists a formula α such that $L = K4_t \oplus \alpha$ (this also follows from a general fact in universal algebra proved by Birkhoff [2]; see also [1, Theorem 5.27]), and every extension of L is again tabular. Since $\text{Log}(\mathbb{F})$ is non-tabular, we have $\mathbb{F} \not\models \alpha$. By Theorem 3.5, (8) is undecidable. For (9), it suffices to show the complement, namely, the inconsistency is undecidable. This follows from Theorem 3.5 by taking $\alpha = \bot$. $\square$

Remark 3.14. Since $K4_t = K_t \oplus \Diamond\Diamond p \rightarrow \Diamond p$, it follows from Proposition 2.2 that the properties mentioned in Corollary 3.13 are also undecidable in the lattice $NExtK_t$.

Remark 3.15. Recall that consistency is decidable in $NExtK$ because the lattice $NExtK$ has exactly two coatoms, namely, maximal consistent logics. The result that consistency is undecidable in $NExtK4_t$ aligns with fact that there are $2^{\aleph_0}$ many coatoms in the lattice $NExtK4_t$ [13].

4 Conclusion

The main motivation behind this paper is to understand how interactions of modalities affect the decidability of logics' properties. We have shown that most properties, as listed in Corollary 3.13, are undecidable in the lattice NExtK4_t . These results, together with the known ones (see Table 1), suggest that the interactions of modalities only make the decision problem for properties harder. We would conjecture that, for a property P such as Kripke completeness, the finite model property, or decidability, if P is undecidable in NExtL for a unimodal logic L , then it is also undecidable in NExtL_t . Note that this does not follow directly from the minimal tense extension map $(\cdot)_t : \text{NExtK} \rightarrow \text{NExtK}_t$, as the map may not preserve or reflect these properties, as discussed in the introduction.

Moreover, another observation from Table 1 is that a stronger base logic tends to make a property decidable. The undecidability results for NExtK4_t indicate that K4_t is not strong enough in this sense. We leave it for future research to analyze the decidability of logics' properties in the lattice of extensions of other, stronger or incomparable bimodal logics. For example, the tense logic S4_t of pre-orders, Grz_t of posets and the product logic $\text{S5} \times \text{S5}$ would be natural choices.

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Goldblatt-Thomason Theorem for Probability Logic

Somayeh Chopoghloo¹ Massoud Pourmahdian^{1,2} Reihane Zoghifard¹

¹School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran*

²Department of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran

s.chopoghloo@ipm.ir

pourmahd@ipm.ir

r.zoghi@gmail.com

Probability logic (PL) extends propositional logic with countably many probability operators, one for each rational number between 0 and 1. The formulas of this logic are interpreted over the class of Markov processes, i.e., structures of the form $\langle \Omega, \Sigma, T \rangle$, where $\langle \Omega, \Sigma \rangle$ is a measurable space and T is a Markov kernel.

The main contribution of this paper is the establishment of the Goldblatt-Thomason theorem for probability logic. As an application, we show that the class of Harsanyi type spaces is definable in PL. Moreover, we obtain some variants of the Goldblatt-Thomason theorem for specific subclasses of Markov processes.

1 Introduction

Probability logic aims to provide a formal framework for studying probabilistic systems, such as Markov processes. It supports reasoning about quantitative statements of the form ‘an event occurs with probability at least r ’ and has been widely applied in diverse areas, including theoretical computer science [6, 20], artificial intelligence [7], and economics [1, 12].

There are different syntactic frameworks for studying probabilistic systems, such as first-order syntax (see, e.g., [14]) which may provide a more expressive language, or modal syntax (see, e.g., [7, 12]) which is less expressive but decidable. Throughout this paper, we investigate the modal framework. Probability logic is a multimodal extension of propositional logic obtained by adding a family of modal operators $\{L_r \mid r \in \mathbb{Q} \cap [0, 1]\}$. The intended interpretation of a formula $L_r \phi$ is that ‘the probability of ϕ is at least r ’.

Markov processes provide a mathematically rigorous model for probabilistic state-based systems, particularly in theoretical computer science. A (labeled) Markov process consists of a measurable state space equipped with transition probability kernels (indexed by actions), thereby modeling systems that combine probabilistic evolution with observable interaction. Such structures serve, for example, as denotational models for probabilistic programming languages and as semantic foundations for verification, equivalence, abstraction, and compositional reasoning in stochastic systems. Formally, a Markov process is a structure $\langle \Omega, \Sigma, T \rangle$, where $\langle \Omega, \Sigma \rangle$ is a measurable space and $T : \Omega \times \Sigma \rightarrow [0, 1]$ is a Markov kernel.

In this paper, we investigate definability of classes of Markov processes by means of theories in probability logic. Our main result establishes a version of the Goldblatt-Thomason theorem for this setting. The classical theorem of Goldblatt and Thomason characterizes the modally definable elementary classes of Kripke frames in terms of certain frame constructions. It states that an elementary (first-order definable) class of Kripke frames is modally definable if and only if it is closed under disjoint unions,

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generated subframes, bounded morphic images, and reflects ultrafilter extensions. The original proof is algebraic in nature, relying on representation theory for modal algebras. It is further shown that a class of general frames—i.e., Kripke frames equipped with a Boolean algebra of admissible sets—is modally definable precisely when it is closed under disjoint unions, generated subframes, bounded morphic images, and both closed under and reflective of ultrafilter extensions. Subsequently, van Benthem provided a model-theoretic proof of the original Goldblatt-Thomason theorem [3].

Goldblatt-Thomason style characterizations have since been established for numerous extensions of modal and intuitionistic logics, including graded modal logic [24], intuitionistic logic [22, 23, 10, 19], and coalgebraic modal logic [17]. In particular, van Benthem studied definable classes of finite (transitive) frames [2]. He showed that for every finite transitive point-generated frame $\mathfrak{F}$ there exists a formula $\varphi_{\mathfrak{F}}$, known as a Jankov-Fine formula, such that whenever $\varphi_{\mathfrak{F}}$ is satisfiable on a frame $\mathfrak{G}$, there exists a surjective bounded morphism from a generated subframe of $\mathfrak{G}$ onto $\mathfrak{F}$. Variants of such Jankov-Fine constructions have been employed to characterize definable classes of finite structures in other logical settings, for example, in conditional logic over finite posets [8].

The study of definability in probability logic presents substantial additional difficulties. A central obstacle is the failure of compactness. The classical Goldblatt-Thomason theorem is formulated for elementary classes and can be weakened in terms of closure under ultrapowers. In contrast, satisfaction of probability formulas is not preserved under ultraproducts, and compactness fails accordingly. Our approach therefore follows the version for general frames and replaces closure under ultrapowers with ultrafilter extensions. However, defining suitable ultrafilter extensions for Markov processes is technically non-trivial. To address this issue, we adapt ideas from [15], working with Markov processes whose σ -algebras admit countable generating sets. Indeed, Kozen and coauthors [15] studied Stone duality for Markov processes. In order to analyze the corresponding algebraic structures, known as Aumann algebras, they introduced an ultrafilter construction on a particular class of Markov processes, termed Stone-Markov processes. They proved the duality between countable Aumann algebras and Stone-Markov Markov processes. Accordingly, we also consider the class of Markov processes with countably generated σ -algebras, and employ methods similar to those in [15] to define ultrafilter extensions of Markov processes.

In addition to the general case, we investigate definability over the class of finite Markov processes, obtaining a corresponding characterization for this important special case.

The outline of this article is as follows. In Section 2, a brief review of the basic notions and facts needed for other parts is given. In Section 3, model-theoretic constructions used in the Goldblatt-Thomason theorem are introduced. In Section 4, a version of the Goldblatt-Thomason theorem is established. Finally, in Section 5, a version of the Goldblatt-Thomason theorem for the class of finite Markov processes is given.

2 Preliminaries

In this section, we review the basic materials needed for the main results of this paper.

Let $\mathbb{P}$ be a countable set of propositional variables. The formulas of probability logic (PL) are defined recursively by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid L_r\varphi$$

where $p \in \mathbb{P}$ and $r \in \mathbb{Q}_0 := [0, 1] \cap \mathbb{Q}$. The Boolean connectives $\top$, $\perp$, $\rightarrow$, $\leftrightarrow$, and $\vee$ are defined in the usual way. For a formula φ , the expression $L_r\varphi$ is interpreted as ‘the probability of φ is at least r ’. Likewise,

$M_r\varphi$ is defined as an abbreviation for $L_{1-r}\neg\varphi$ and is interpreted as ‘the probability of φ is at most r ’. Moreover, for $r_1, \dots, r_k \in \mathbb{Q}_0$, we use the formula $L_{r_1 \dots r_k}\varphi$ as an abbreviation for $L_{r_1} \dots L_{r_k}\varphi$. Additionally, $L_r^0\varphi := \varphi$ and $L_r^n\varphi := \underbrace{L_r \dots L_r}_{n\text{-times}}\varphi$ for each $n > 0$ and $r \in \mathbb{Q}_0$.

Definition 2.1. Let $\langle \Omega, \Sigma \rangle$ be a measurable space. A function $T : \Omega \times \Sigma \rightarrow [0, 1]$ is called a *Markov kernel* (or *transition probability*) if the following conditions hold:

- for each $w \in \Omega$, the mapping $T(w, \cdot) : \Sigma \rightarrow [0, 1]$ is a probability measure on Σ ;
- for each $N \in \Sigma$, the mapping $T(\cdot, N) : \Omega \rightarrow [0, 1]$ is a measurable function on Ω .

The triple $\langle \Omega, \Sigma, T \rangle$ is then called a *Markov process* on the state space $\langle \Omega, \Sigma \rangle$. For convenience, we may write $T(w)(N)$ instead of $T(w, N)$.

Definition 2.2. A Markov process $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ is said (*countably*) *generated* by a (countable) subalgebra $\mathcal{B}$ of Σ if $\mathcal{B}$ is a (countable) Boolean algebra and $\Sigma := \sigma(\mathcal{B})$, the smallest σ -algebra containing $\mathcal{B}$. We always assume that $F_r(N) := \{w \in \Omega \mid T(w)(N) \geq r\} \in \mathcal{B}$, for each $N \in \mathcal{B}$ and $r \in \mathbb{Q}_0$. We also define $F_{r_1 \dots r_n}(N)$ as $F_{r_1} \dots F_{r_n}(N)$, for $r_1, \dots, r_n \in \mathbb{Q}_0$.

Convention 2.3. If we represent the Markov process $\mathfrak{P}$ as $\langle \Omega, \mathcal{B}, T \rangle$, then we mean that $\Sigma = \sigma(\mathcal{B})$.

Definition 2.4. A *Markov model*, or simply a *model*, over (or based on) a Markov process $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$ is a tuple $\mathfrak{M} = \langle \Omega, \mathcal{B}, T, v \rangle$ where $v : \mathbb{P} \rightarrow \mathcal{B}$ is a valuation function which assigns to every propositional variable $p \in \mathbb{P}$, a set $v(p) \in \mathcal{B}$.

Remark 2.5. Note that restricting the valuation to the generating subalgebra $\mathcal{B}$ is analogous to defining valuations on general Kripke frames, in which each propositional variable p is assigned a value in the augmented algebra associated with the frame. This restriction is necessary in order to adapt the method used in proving the Goldblatt–Thomason theorem for general frames to the setting of Markov processes.

Definition 2.6. The *satisfaction relation* for arbitrary formulas in a given model $\mathfrak{M} = \langle \Omega, \mathcal{B}, T, v \rangle$, is defined inductively as follows:

$$\begin{aligned} \mathfrak{M}, w \models p & \text{ iff } w \in v(p), \\ \mathfrak{M}, w \models \neg\varphi & \text{ iff } \mathfrak{M}, w \not\models \varphi, \\ \mathfrak{M}, w \models \varphi \wedge \psi & \text{ iff } \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi, \\ \mathfrak{M}, w \models L_r\varphi & \text{ iff } T(w, [[\varphi]]_{\mathfrak{M}}) \geq r, \text{ where } [[\varphi]]_{\mathfrak{M}} = \{w \in \Omega \mid \mathfrak{M}, w \models \varphi\}. \end{aligned}$$

A set Γ of formulas is satisfiable in the world w if $\mathfrak{M}, w \models \varphi$ for each $\varphi \in \Gamma$.

Note that by Convention 2.3, we have $[[\varphi]]_{\mathfrak{M}} \in \mathcal{B}$, for each formula φ .

Definition 2.7. A formula φ is *valid in a model* $\mathfrak{M} = \langle \Omega, \mathcal{B}, T, v \rangle$, denoted by $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, w \models \varphi$ for all $w \in \Omega$. Similarly, φ is *valid in a Markov process* $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$, denoted by $\mathfrak{P} \models \varphi$, if it is valid in every model based on $\mathfrak{P}$. We say that φ is *valid in a class* $\mathcal{C}$ of Markov processes, denoted by $\models_{\mathcal{C}} \varphi$, if it is valid in every element $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ of $\mathcal{C}$. For abbreviation, we may omit the subscript $\mathcal{C}$ and instead write $\models \varphi$ when it is the class of all Markov processes. A set of formulas Γ is satisfiable in the class $\mathcal{C}$ if there is a Markov process $\mathfrak{P} \in \mathcal{C}$ and a model $\mathfrak{M}$ based on $\mathfrak{P}$ such that $\mathfrak{M}, w \models \Gamma$ for some $w \in \Omega$. Likewise, Γ is finitely satisfiable in $\mathcal{C}$ if each finite subset $\Gamma' \subseteq \Gamma$ is satisfiable in $\mathcal{C}$.

Proposition 2.8. Let φ be a formula and $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ a Markov process. Then the following conditions are equivalent:

1. φ is valid in $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$.

2. φ is valid in $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$, for every subalgebra $\mathcal{B}$ of Σ with $\Sigma = \sigma(\mathcal{B})$.
3. φ is valid in every $\mathfrak{P}' = \langle \Omega, \Sigma', T' \rangle$, where Σ' is a countably generated σ -subalgebra of Σ and $T' = T \upharpoonright \Sigma'$.
4. φ is valid in every $\mathfrak{P}' = \langle \Omega, \mathcal{B}', T' \rangle$, where $\mathcal{B}'$ is a countable subalgebra of Σ and $T' = T \upharpoonright \sigma(\mathcal{B}')$.

Proof. The equivalence of Parts 1 and 2, and respectively, Parts 3 and 4 are obvious. To see that Part 4 implies Part 1, notice that if φ is not valid in $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$, then there exists a valuation $v : \mathbb{P} \rightarrow \Sigma$ such that in $\mathfrak{M} = \langle \mathfrak{P}, v \rangle$, we have that $\mathfrak{M}, w \not\models \varphi$, for some $w \in \Omega$. Now if we take $\mathcal{B}' = \{ \llbracket \theta \rrbracket_{\mathfrak{M}} \mid \theta \text{ is a formula} \}$, then φ is not valid in $\mathfrak{P}' = \langle \Omega, \mathcal{B}', T \upharpoonright \sigma(\mathcal{B}') \rangle$. Finally, It is clear that Part 2 implies Part 4. $\square$

Definition 2.9. A class of Markov processes $\mathcal{C}$ is definable in probability logic or *P-definable*, if there is a set of formulas Γ such that for any Markov process $\mathfrak{P}$ we have $\mathfrak{P} \in \mathcal{C}$ if and only if $\mathfrak{P} \models \Gamma$.

Definition 2.10. [15, Definition 10] Let $\langle \Omega, \mathcal{B}, T \rangle$ be a countably generated Markov process. We say that this process is a *Stone-Markov process* if $\mathcal{B}$ is a countable base of clopen sets for the topology generated by $\mathcal{B}$, named $\tau_{\mathcal{B}}$, that satisfies the Baire property (also called *topologically complete*).

Note that if $\langle \Omega, \tau_{\mathcal{B}} \rangle$ is a Polish space or a compact separable space, then $\langle \Omega, \mathcal{B}, T \rangle$ is a Stone-Markov process. This property is referred to as *saturation* in [15].

3 Some Model-Theoretic Concepts

In this section, we introduce several model-theoretic constructions that are needed for the proof of the Goldblatt–Thomason Theorem for probability logic.

We begin by defining the notion of a Markov sub-process.

Definition 3.1 (Markov sub-process). Let $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ be a Markov process, and let $\Omega' \in \Sigma$ be a measurable subset such that

$$T(w, \Omega') = 1 \quad \text{for all } w \in \Omega'.$$

Let $\Sigma' = \{N \cap \Omega' \mid N \in \Sigma\}$ be the induced σ -algebra on Ω' . Define the transition function $T' : \Omega' \times \Sigma' \rightarrow [0, 1]$ by

$$T'(w, N \cap \Omega') = T(w, N),$$

for all $w \in \Omega'$ and $N \in \Sigma$.

Lemma 3.2. *Under the assumptions of Definition 3.1:*

1. the triple $\mathfrak{P}' = \langle \Omega', \Sigma', T' \rangle$ is a Markov process;
2. if $\mathfrak{P}$ is countably generated, then so is $\mathfrak{P}'$.

Proof. 1. The main point is to show that T' is well defined. Let $N_1, N_2 \in \Sigma$ and $N_1 = N_2$. Since $T(w, \Omega') = 1$ for all $w \in \Omega'$, we have

$$T'(w, N_1 \cap \Omega') = T(w, N_1) = T(w, N_2) = T'(w, N_2 \cap \Omega'),$$

so T' is well defined. Standard arguments show that T' is a Markov kernel.

2. Suppose that $\mathcal{B}$ is a countable Boolean algebra generating Σ . Then

$$\mathcal{B}' = \{B \cap \Omega' \mid B \in \mathcal{B}\}$$

generates Σ' . Moreover, for each $r \in [0, 1]$ and $B \in \mathcal{B}$,

$$F_r(B \cap \Omega') = \{w \in \Omega' \mid T'(w, B \cap \Omega') \geq r\} = F_r(B) \cap \Omega'.$$

Thus $\mathfrak{P}'$ is countably generated. □

Once the notion of a sub-process has been established, we can define the corresponding notion of a sub-model.

Definition 3.3. Let $\mathfrak{M} = \langle \Omega, \Sigma, T, v \rangle$ be a Markov model. Suppose that $\Omega' \in \Sigma$ satisfies the conditions of Definition 3.1, and let $\mathfrak{P}' = \langle \Omega', \Sigma', T' \rangle$ be the corresponding sub-process. Define a valuation v' by setting

$$v'(p) = v(p) \cap \Omega' \quad \text{for each } p \in \mathbb{P}.$$

Then $\mathfrak{M}' = \langle \Omega', \Sigma', T', v' \rangle$ is called a *sub-model* of $\mathfrak{M}$.

Lemma 3.4. Let $\mathfrak{M}'$ be a sub-model of $\mathfrak{M}$. Then for every formula φ and every $w \in \Omega'$,

$$\mathfrak{M}, w \models \varphi \quad \text{iff} \quad \mathfrak{M}', w \models \varphi.$$

Corollary 3.5. Let φ be a formula. If $\mathfrak{P} \models \varphi$ and $\mathfrak{P}'$ is a sub-process of $\mathfrak{P}$, then $\mathfrak{P}' \models \varphi$.

Proof. Suppose not. Then there exists a valuation v' on $\mathfrak{P}'$ and a state $w \in \Omega'$ such that $\mathfrak{M}', w \not\models \varphi$. Extend v' to a valuation v on $\mathfrak{P}$ by setting $v(p) = v'(p)$ for all $p \in \mathbb{P}$. By the previous lemma, $\mathfrak{M}, w \not\models \varphi$, contradicting $\mathfrak{P} \models \varphi$. □

For a given model $\mathfrak{M}$ the following lemma introduces a sub-model $\mathfrak{M}'$ of $\mathfrak{M}$ which plays the same role as a *generated sub-model* in the context of the Kripke semantics.

Lemma 3.6. Suppose $\mathfrak{M} = \langle \Omega, \Sigma, T, v \rangle$ is a model, $w \in \Omega$ and Γ is a set of formulas such that $\mathfrak{M}, w \models L_1^n \varphi$ for each $\varphi \in \Gamma$ and $n \geq 0$. Then there exists a sub-model $\mathfrak{M}' = \langle \Omega', \Sigma', T', v' \rangle$ such that $w \in \Omega'$ and $\mathfrak{M}' \models \Gamma$.

Proof. Let $\Gamma = \{\varphi_0, \varphi_1, \dots\}$. So we have $\mathfrak{M}, w \models L_1^n \varphi_i$ for each $i, n \geq 0$. Therefore, $T(w, [[L_1^n \varphi_i]]_{\mathfrak{M}}) = 1$, for each $i, n \geq 0$. Put $\Omega_{i,n} = [[L_1^n \varphi_i]]_{\mathfrak{M}}$ and $\Omega' = \bigcap_{i,n} \Omega_{i,n}$. Then, $w \in \Omega'$ and Ω' is a measurable set. Moreover, for each $w' \in \Omega'$ and $i, n \geq 0$, we have $\mathfrak{M}, w' \models L_1^{n+1} \varphi_i$. Hence, $T(w', \Omega_{i,n}) = 1$, for each $i, n \geq 0$. This implies that $T(w', \Omega') = 1$. Now, if we take $\mathfrak{M}' = \langle \Omega', \Sigma', T', v' \rangle$ as a sub-model of $\mathfrak{M}$, then $\mathfrak{M}'$ satisfies the required claim. □

Definition 3.7. Let $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ be a Markov process. We say that a Markov process $\mathfrak{P}' = \langle \Omega', \Sigma', T' \rangle$ is an *event sub-process* of $\mathfrak{P}$ if Σ' is a σ -sub-algebra of Σ and $T' = T \upharpoonright_{\Sigma'}$ is a Markov kernel.

The following lemma is immediate by Proposition 2.8.

Lemma 3.8. Let $\mathfrak{P}'$ be an event sub-process of $\mathfrak{P}$. Then $\mathfrak{P} \models \varphi$ implies that $\mathfrak{P}' \models \varphi$, for any formula φ .

Next, we define disjoint unions of Markov processes.

Definition 3.9 (Disjoint union). 1. Let I be a non-empty index set, and for each $i \in I$ let $\mathfrak{P}_i = \langle \Omega_i, \Sigma_i, T_i \rangle$ be a Markov process. The *disjoint union* $\uplus_{i \in I} \mathfrak{P}_i$ is the Markov process $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ defined as follows:

- $\Omega = \biguplus_{i \in I} \Omega_i$;
- Σ is the σ -algebra generated by the algebra of all sets of the form $\biguplus_{i \in I} N_i$, where $N_i \in \Sigma_i$;
- For all $j \in I$ and $w \in \Omega_j$,

$$T(w, \biguplus_{i \in I} N_i) = T_j(w, N_j).$$

2. If $\mathfrak{M}_i = \langle \Omega_i, \Sigma_i, T_i, v_i \rangle$ is a Markov model for each $i \in I$, then the disjoint union $\biguplus_{i \in I} \mathfrak{M}_i$ is the Markov model $\mathfrak{M} = \langle \biguplus_{i \in I} \mathfrak{P}_i, v \rangle$, where

$$v(p) = \biguplus_{i \in I} v_i(p) \quad \text{for all } p \in \mathbb{P}.$$

Lemma 3.10. *The disjoint union $\mathfrak{P} = \biguplus_{i \in I} \mathfrak{P}_i$ and $\mathfrak{M} = \biguplus_{i \in I} \mathfrak{M}_i$ are respectively a Markov process and a Markov model.*

- Lemma 3.11.** 1. *Let $\{\mathfrak{M}_i\}_{i \in I}$ be a family of Markov models and let φ be a formula. Then for each $w \in \Omega_i$,*

$$\biguplus_{i \in I} \mathfrak{M}_i, w \models \varphi \quad \text{iff} \quad \mathfrak{M}_i, w \models \varphi.$$

2. *Let $\{\mathfrak{P}_i\}_{i \in I}$ be a family of Markov processes. Then for every formula φ ,*

$$\biguplus_{i \in I} \mathfrak{P}_i \models \varphi \quad \text{iff} \quad \mathfrak{P}_i \models \varphi \text{ for all } i \in I.$$

Next, the notion of zigzag morphisms are presented that gives rise to an important notion of similarity between two Markov processes.

Definition 3.12 (Zigzag morphisms). Let $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ and $\mathfrak{P}' = \langle \Omega', \Sigma', T' \rangle$ be two Markov processes. A surjective measurable function $f : \Omega \rightarrow \Omega'$ is called a *zigzag morphism* from $\mathfrak{P}$ to $\mathfrak{P}'$ if it satisfies the condition:

$$T(w, f^{-1}(N')) = T'(f(w), N'),$$

for each $w \in \Omega$ and $N' \in \Sigma'$.

A zigzag morphism from $\mathfrak{M} = \langle \mathfrak{P}, v \rangle$ to $\mathfrak{M}' = \langle \mathfrak{P}', v' \rangle$ is a zigzag morphism f from $\mathfrak{P}$ to $\mathfrak{P}'$ such that $v(p) = f^{-1}(v'(p))$, for each $p \in \mathbb{P}$.

Subsequently, the following preservation property can be derived.

Lemma 3.13. *Let $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ and $\mathfrak{P}' = \langle \Omega', \Sigma', T' \rangle$ be two Markov processes. If there exists a zigzag morphism between $\mathfrak{P}$ and $\mathfrak{P}'$, then, $\mathfrak{P} \models \varphi$ implies $\mathfrak{P}' \models \varphi$, for every formula φ .*

Proof. Assume that $\mathfrak{P} \models \varphi$. Let $\mathfrak{M} = \langle \mathfrak{P}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{P}', v' \rangle$ be arbitrary models, and let $f : \mathfrak{M} \rightarrow \mathfrak{M}'$ be a zigzag morphism. By the invariance of satisfaction under zigzag morphisms, for every world $w \in \Omega$ we have

$$\mathfrak{M}, w \models \varphi \text{ if and only if } \mathfrak{M}', f(w) \models \varphi.$$

Since $\mathfrak{P} \models \varphi$, it follows that $\mathfrak{M}, w \models \varphi$ for all $w \in \Omega$, and hence $\mathfrak{M}', w' \models \varphi$ for all $w' \in \Omega'$. Therefore, $\mathfrak{P}' \models \varphi$. $\square$

3.1 Ultrafilter Extensions

We now introduce ultrafilter extensions for countably generated Markov processes, a construction that parallels the classical ultrafilter extension of Kripke structures. Throughout this section, we assume familiarity with ultrafilters on Boolean algebras.

We recall from [18] the definition of the *ultrafilter limit* of a sequence $(r_w)_{w \in \Omega}$ of real numbers with respect to an ultrafilter $\mathfrak{u}$ over a Boolean algebra $\mathcal{B}$. This limit, denoted by $\lim_{\mathfrak{u}} r_w$, is defined as the unique real number $r \in [0, 1]$ such that

$$\{w \in \Omega \mid |r_w - r| < \varepsilon\} \in \mathfrak{u} \quad \text{for every } \varepsilon > 0.$$

Fact 3.14. *Let $(r_w)_{w \in \Omega}$ be a sequence of real numbers in $[0, 1]$, and let $\mathfrak{u}$ be an ultrafilter over Ω . Then the following properties hold:*

1. *If $\{w \in \Omega \mid r_w \geq r\} \in \mathfrak{u}$, then $\lim_{\mathfrak{u}} r_w \geq r$.*
2. *If $\lim_{\mathfrak{u}} r_w > r$, then $\{w \in \Omega \mid r_w > r\} \in \mathfrak{u}$.*
3. *$\lim_{\mathfrak{u}} r_w \geq r$ if and only if $\{w \in \Omega \mid r_w \geq s\} \in \mathfrak{u}$ for every $s < r$.*

Definition 3.15. Let $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$ be a countably generated Markov process. We define the structure $\langle \mathcal{U}^*, \mathcal{B}^*, K^* \rangle$ as follows:

- $\mathcal{U}^*$ is the set of all ultrafilters on $\mathcal{B}$;
- $\mathcal{B}^* := \{A^* \mid A \in \mathcal{B}\}$, where

$$A^* := \{\mathfrak{u} \in \mathcal{U}^* \mid A \in \mathfrak{u}\};$$

- $K^* : \mathcal{U}^* \times \mathcal{B}^* \rightarrow [0, 1]$ is defined by

$$K^*(\mathfrak{u}, A^*) := \lim_{\mathfrak{u}} T(w, A).$$

Lemma 3.16. *The triple $\langle \mathcal{U}^*, \mathcal{B}^*, K^* \rangle$ satisfies the following properties:*

1. *$\mathcal{B}^*$ is a countable Boolean algebra over $\mathcal{U}^*$. Moreover, $\mathcal{B}^*$ forms a basis of clopen sets for the compact Hausdorff Stone topology $\beta(\mathcal{U}^*)$;*
2. *for each $\mathfrak{u} \in \mathcal{U}^*$, the function $K^*(\mathfrak{u}, \cdot) : \mathcal{B}^* \rightarrow [0, 1]$ is a pre-measure.*

Proof. 1. For $A, B \in \mathcal{B}$, the following equalities hold:

$$A^* \cap B^* = (A \cap B)^*, \quad A^* \cup B^* = (A \cup B)^*, \quad (A^*)^c = (A^c)^*,$$

since elements of $\mathcal{U}^*$ are ultrafilters. The remaining claims follow from standard results in Stone duality; see, for example, [13].

2. It is straightforward to verify that $K^*(\mathfrak{u}, \cdot)$ is finitely additive. Now suppose $(B_n^*)_{n \in \mathbb{N}}$ is a decreasing sequence in $\mathcal{B}^*$ such that

$$\bigcap_{n=1}^{\infty} B_n^* = \emptyset.$$

Since each B_n^* is clopen in the compact space $\beta(\mathcal{U}^*)$, there exists $N \in \mathbb{N}$ such that $B_n^* = \emptyset$ for all $n > N$. Consequently,

$$K^*(\mathfrak{u}, B_n^*) = 0 \quad \text{for all } n > N,$$

and hence $\lim_{n \rightarrow \infty} K^*(\mathfrak{u}, B_n^*) = 0$.

□

By Carathéodory's Extension Theorem, the pre-measure K^* extends uniquely to a Markov kernel

$$T^* : \mathcal{U}^* \times \Sigma^* \rightarrow [0, 1],$$

where Σ^* is the Borel σ -algebra generated by $\mathcal{B}^*$.

Corollary 3.17. *Let $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$ be a countably generated Markov process. Then the structure*

$$\mathfrak{P}^* = \langle \mathcal{U}^*, \mathcal{B}^*, T^* \rangle$$

is a countably generated Stone–Markov process.

Proof. It suffices to verify that for each $N \in \Sigma^*$, the function

$$T^*(\cdot, N) : \mathcal{U}^* \rightarrow [0, 1]$$

is measurable. That is, for every $r \in \mathbb{Q}_0$,

$$\{u \in \mathcal{U}^* \mid T^*(u, N) \geq r\} \in \Sigma^*.$$

Without loss of generality, assume $N = B^*$ for some $B \in \mathcal{B}$. By Fact 3.14(3), we have

$$F_r(B^*) = \{u \in \mathcal{U}^* \mid T^*(u, B^*) \geq r\} = \bigcap_{s < r} (F_s(B))^*.$$

Since $(F_s(B))^* \in \Sigma^*$ for all $s < r$, it follows that $F_r(B^*) \in \Sigma^*$. □

Next, we isolate the key notion of nested Archimedean ultrafilters, which constitute the basic building blocks of ultrafilter extensions.

Definition 3.18. Let $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$ be a countably generated Markov process. An ultrafilter u on $\mathcal{B}$ is called a *nested Archimedean ultrafilter* if, for all $t_1, \dots, t_n, r \in \mathbb{Q}_0$ and all $A \in \mathcal{B}$, the following condition holds: whenever

$$F_{t_1 \dots t_n s}(A) \in u \quad \text{for all } s < r,$$

then

$$F_{t_1 \dots t_n r}(A) \in u.$$

Let $\mathcal{U}_{na}^*$ denote the set of all nested Archimedean ultrafilters. We show that $\mathcal{U}_{na}^*$ forms a G_δ -subspace of $\mathcal{U}^*$ such that $T^*(u, \mathcal{U}_{na}^*) = 1$, for each $u \in \mathcal{U}_{na}^*$.

Lemma 3.19. *The following properties hold.*

1. $\mathcal{U}_{na}^*$ forms a dense G_δ -subspace of $\mathcal{U}^*$;
2. $T^*(u, \mathcal{U}_{na}^*) = 1$, for each $u \in \mathcal{U}_{na}^*$;
3. $\{u \in \mathcal{U}_{na}^* \mid T^*(u, B^*) \geq r\} = (F_r(B))^* \cap \mathcal{U}_{na}^*$, for each $B \in \mathcal{B}$.

Proof. 1. It is shown in [15] that

$$\mathcal{U}_{na}^* = \bigcap_{B \in \mathcal{B}} \bigcap_{t_1, \dots, t_n, r \in \mathbb{Q}_0} \left(\bigcup_{s < r} (F_{t_1 \dots t_n s}^c(B))^* \cup (F_{t_1 \dots t_n r}(B))^* \right).$$

Since $\mathcal{O}_{t_1 \dots t_n r}(B) := \left(\bigcup_{s < r} (F_{t_1 \dots t_n s}^c(B))^* \cup (F_{t_1 \dots t_n r}(B))^* \right)$ is an open subset of $\mathcal{U}^*$, it follows that $\mathcal{U}_{na}^*$ is a G_δ -subspace of $\mathcal{U}^*$. Moreover, by Rasiowa–Sikorski Lemma [21], there exists an ultrafilter $\mathfrak{u} \in \mathcal{U}^*$ such that for $\emptyset \neq B' \in \mathcal{B}$ and $t_1, \dots, t_n, r \in \mathbb{Q}_0$, we have that $B' \in \mathfrak{u}$ and whenever

$$F_{t_1 \dots t_n s}(B) \in \mathfrak{u} \quad \text{for all } s < r,$$

then

$$F_{t_1 \dots t_n r}(B) \in \mathfrak{u}.$$

Therefore, $\mathfrak{u} \in \mathcal{O}_{t_1 \dots t_n r}(B) \cap (B')^*$. This implies that $\mathcal{O}_{t_1 \dots t_n r}(B)$ is dense and open. But, Since $\mathcal{U}^*$ is topologically complete, $\mathcal{U}_{na}^*$ is dense.

2. It follows from Item 1 that $\mathcal{U}_{na}^* \in \Sigma^*$. Now to show the claim, it suffices to prove that for each $\mathfrak{u} \in \mathcal{U}_{na}^*$,

$$T^*(\mathfrak{u}, \mathcal{U}^* \setminus \mathcal{U}_{na}^*) = 0.$$

Equivalently, we have to show that for each $B \in \mathcal{B}$ and $t_1, \dots, t_n, r \in \mathbb{Q}_0$,

$$\begin{aligned} & T^*(\mathfrak{u}, \bigcap_{s < r} (F_{t_1 \dots t_n s}(B))^* \cap (F_{t_1 \dots t_n r}^c(B))^*) \\ &= T^*(\mathfrak{u}, \bigcap_{s < r} (F_{t_1 \dots t_n s}(B))^*) - T^*(\mathfrak{u}, (F_{t_1 \dots t_n r}(B))^*) \\ &= \inf_{s < r} \{T^*(\mathfrak{u}, (F_{t_1 \dots t_n s}(B))^*)\} - T^*(\mathfrak{u}, (F_{t_1 \dots t_n r}(B))^*) \\ &= 0 \end{aligned}$$

Let $\beta := T^*(\mathfrak{u}, (F_{t_1 \dots t_n r}(B))^*)$. Then,

Claim: For each positive $\varepsilon \in \mathbb{Q}_0$, there exists $s < r$ such that

$$T^*(\mathfrak{u}, (F_{t_1 \dots t_n s}(B))^*) \leq \varepsilon + \beta.$$

Proof of claim: Suppose not. Then, for some ε , we have

$$\lim_{\mathfrak{u}} T(w, F_{t_1 \dots t_n s}(B)) > \varepsilon + \beta,$$

for each $s < r$. Now pick $\varepsilon' \in \mathbb{Q}_0$ such that $0 \leq \beta < \varepsilon' \leq \varepsilon + \beta$. Then, for each $s < r$,

$$\{w \in \Omega \mid T(w, F_{t_1 \dots t_n s}(B)) \geq \varepsilon'\} \in \mathfrak{u}.$$

Since $\mathfrak{u}$ is a nested Archimedean ultrafilter, it follows that

$$\{w \in \Omega \mid T(w, F_{t_1 \dots t_n r}(B)) \geq \varepsilon'\} \in \mathfrak{u}.$$

So,

$$\lim_{\mathfrak{u}} T(w, F_{t_1 \dots t_n r}(B)) = \beta \geq \varepsilon',$$

a contradiction.

3. This holds, since $(F_r(B))^* \cap \mathcal{U}_{na}^* = \bigcap_{s < r} ((F_s(B))^* \cap \mathcal{U}_{na}^*)$.

□

In light of the above lemma, we define the ultrafilter extensions.

Definition 3.20 (Ultrafilter extensions). Given a Markov process $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$, we define the *ultrafilter extension* of $\mathfrak{P}$ as $\mathfrak{Ue}(\mathfrak{P}) = \langle \mathcal{U}_{na}^*, \Sigma_{na}^*, T_{na}^* \rangle$ where

1. $\Sigma_{na}^* := \{N \cap \mathcal{U}_{na}^* \mid N \in \Sigma^*\}$;
2. $T_{na}^* : \mathcal{U}_{na}^* \times \Sigma_{na}^* \rightarrow [0, 1]$ such that for each $u \in \mathcal{U}_{na}^*$ and $N \in \Sigma^*$,

$$T_{na}^*(u, N \cap \mathcal{U}_{na}^*) = T^*(u, N).$$

Equivalently, $\mathfrak{Ue}(\mathfrak{P})$ is a sub-process of $\mathfrak{P}^* = \langle \mathcal{U}^*, \mathcal{B}^*, T^* \rangle$, introduced in Corollary 3.17. Notice that $\mathfrak{Ue}(\mathfrak{P}) = \langle \mathcal{U}_{na}^*, \mathcal{B}_{na}^*, T_{na}^* \rangle$ where $\mathcal{B}_{na}^* := \{B^* \cap \mathcal{U}_{na}^* \mid B \in \mathcal{B}\}$.

In the following lemma, we show that the validity of formulas is *reflected* by the ultrafilter extensions. To prove this, we first examine the following lemma.

Lemma 3.21. Let $\mathfrak{M} = \langle \Omega, \mathcal{B}, T, v \rangle$ be a model such that $\mathcal{B}$ is countable. Consider the model $\mathfrak{Ue}(\mathfrak{M}) = \langle \mathfrak{Ue}(\mathfrak{P}), v_{na}^* \rangle$ where $\mathfrak{Ue}(\mathfrak{P}) = \langle \mathcal{U}_{na}^*, \mathcal{B}_{na}^*, T_{na}^* \rangle$ and v_{na}^* is a valuation on $\mathfrak{Ue}(\mathfrak{P})$ defined as

$$v_{na}^*(p) := \{u \in \mathcal{U}_{na}^* \mid v(p) \in u\}, \text{ for each } p \in \mathbb{P}.$$

Then, the following conditions hold:

1. $v_{na}^*(p) = \mathcal{U}_{na}^* \cap (v(p))^* \in \mathcal{B}_{na}^*$, for each $p \in \mathbb{P}$.
2. For each formula φ and $u \in \mathcal{U}_{na}^*$, we have

$$\mathfrak{Ue}(\mathfrak{M}), u \models \varphi \text{ if and only if } \{w \in \Omega \mid \mathfrak{M}, w \models \varphi\} \in u.$$

$$\text{Therefore, } [[\varphi]]_{\mathfrak{Ue}(\mathfrak{M})} = \mathcal{U}_{na}^* \cap ([[\varphi]])_{\mathfrak{M}}^*.$$

Proof. 1. This follows immediately by definition.

2. This can be shown by induction on the complexity of formulas. The basic and induction steps for Boolean connectives are clear. We assume that the induction hypothesis holds for ψ . Now for $\varphi = L_r \psi$, we have

$$\begin{aligned} [[L_r \psi]]_{\mathfrak{Ue}(\mathfrak{M})} &= \{u \in \mathcal{U}_{na}^* \mid T_{na}^*(u, [[\psi]]_{\mathfrak{Ue}(\mathfrak{M})}) \geq r\} \\ &= \{u \in \mathcal{U}_{na}^* \mid T^*(u, \mathcal{U}_{na}^* \cap ([[\psi]])_{\mathfrak{M}}^*) \geq r\}, \text{ (by induction hypothesis for } \psi) \\ &= \{u \in \mathcal{U}_{na}^* \mid T^*(u, ([[\psi]])_{\mathfrak{M}}^*) \geq r\}, \text{ (by Definition 3.20 (2))} \\ &= \mathcal{U}_{na}^* \cap ([[L_r \psi]])_{\mathfrak{M}}^*, \text{ (by Lemma 3.19 (3))} \\ &= \mathcal{U}_{na}^* \cap ([[L_r \psi]])_{\mathfrak{M}}^*. \end{aligned}$$

□

Lemma 3.22. Suppose $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ is a countably generated Markov processes. Then, for each formula φ ,

$$\mathfrak{Ue}(\mathfrak{P}) \models \varphi \text{ implies } \mathfrak{P} \models \varphi.$$

Proof. Assume the above claim is not true for some formula φ . Then by Proposition 2.8 (2), there exists a countable subalgebra $\mathcal{B}$ of Σ such that $\langle \Omega, \mathcal{B}, T \rangle \not\models \varphi$. Hence, for some model $\mathfrak{M} = \langle \mathfrak{P}, v \rangle$ and $w \in \Omega$, we have that $\mathfrak{M}, w \not\models \varphi$. So $\emptyset \neq [[\neg \varphi]]_{\mathfrak{M}} \in \mathcal{B}$. Now consider an ultrafilter $u \in \mathcal{U}_{na}^*$ with $[[\neg \varphi]]_{\mathfrak{M}} \in u$, which exists by Rasiowa–Sikorski Lemma. Then on the basis of Lemma 3.21 (2), we have $\mathfrak{Ue}(\mathfrak{M}), u \models \varphi$, a contradiction. □

The proof of the above lemma easily yields the following corollary.

Corollary 3.23. *Under the condition of the above corollary, for each formula φ ,*

$$\mathfrak{P} \models \varphi \text{ implies } \langle \mathcal{U}_{na}^*, \mathcal{B}_{na}^*, T_{na}^* \rangle \models \varphi.$$

We can conclude this section by the following lemma which shows that for a given theory Γ , the class of all Markov processes in which Γ is valid is closed under the constructions introduced above.

Lemma 3.24. *Assume that Γ is a set of formulas and let $\mathcal{C}_\Gamma = \{\mathfrak{P} \mid \mathfrak{P} \models \Gamma\}$, then $\mathcal{C}_\Gamma$ is closed under Markov sub-processes, event sub-processes, disjoint unions, zigzag morphisms and reflected by ultrafilter extensions of countably generated Markov processes¹. Furthermore, $\mathfrak{P} \in \mathcal{C}$ if and only if every countably generated event sub-process of $\mathfrak{P}$ is in $\mathcal{C}$.*

Since the conditions stated in the above lemma are important conditions for proving the Goldblatt-Thomason theorem, we isolate those classes that satisfy the conditions of the above lemma.

Definition 3.25. Let $\mathcal{C}$ be a class of Markov processes. The class $\mathcal{C}$ is said to have the Goldblatt-Thomason property (GT-property for short) if it satisfies the conditions of Lemma 3.24.

4 The Goldblatt-Thomason Theorem

In this section we prove the Goldblatt-Thomason theorem for probability logic. Motivated by the same theorem in the context of basic modal logic with Kripke semantics we consider the ultrafilter extension as replacement of ultraproduct construction. This also rooted back to the original paper of Goldblatt and Thomason [9].

Recall that a class of Markov processes $\mathcal{C}$ is closed under ultrafilter extensions, if for every countably generated Markov process $\mathfrak{P}$, if $\mathfrak{P} \in \mathcal{C}$, then $\mathfrak{Ue}(\mathfrak{P}) \in \mathcal{C}$.

Theorem 4.1 (Goldblatt-Thomason Theorem). *Let $\mathcal{C}$ be a class of Markov processes that is closed under taking ultrafilter extensions. Then $\mathcal{C}$ is P-definable if and only if it has the GT-property.*

Proof. The implication from left to right follows directly from Lemma 3.24. For the converse direction, let $Th(\mathcal{C})$ denote the set of all formulas of an arbitrary countable language $\mathcal{L}$ that are valid in the class $\mathcal{C}$. We show that $Th(\mathcal{C})$ defines $\mathcal{C}$, i.e., for any Markov processes $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$, we have

$$\mathfrak{P} \in \mathcal{C} \text{ iff } \mathfrak{P} \models Th(\mathcal{C}).$$

By definition of $Th(\mathcal{C})$, the forward implication is immediate. Hence, assume that $\mathfrak{P} \models Th(\mathcal{C})$. By Proposition 2.8, we may assume without loss of generality that $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$, where $\mathcal{B}$ is a countable Boolean algebra and $\Sigma = \sigma(\mathcal{B})$.

Now consider the language $\mathcal{L}_\mathfrak{P}$ consisting of propositional variables $\{p_A \mid A \in \mathcal{B}\}$. Define a valuation $v_\mathfrak{P}$ on $\mathfrak{P}$ by setting $v_\mathfrak{P}(p_A) = A$, for each $A \in \mathcal{B}$. Let $\mathfrak{M} = \langle \mathfrak{P}, v_\mathfrak{P} \rangle$ and let $\Gamma = Th(\mathfrak{M})$ be the set of formulas valid in $\mathfrak{M}$. Observe that Γ contains the following formulas

$$\neg p_\emptyset \tag{1}$$

$$p_A \wedge p_B \leftrightarrow p_{A \cap B} \tag{2}$$

$$p_A \leftrightarrow \neg p_{A^c} \tag{3}$$

$$p_{F_r A} \leftrightarrow L_r p_A \tag{4}$$

¹i.e., if $\mathfrak{Ue}(\mathfrak{P}) \in \mathcal{C}_\Gamma$, then $\mathfrak{P} \in \mathcal{C}_\Gamma$.

for each $A, B \in \mathcal{B}$ and $r \in \mathbb{Q}_0$. Moreover, if $\varphi \in \Gamma$ then $L_1 \varphi \in \Gamma$.

Now for each $w \in \Omega$, define $\Gamma_w = \{\gamma \in \mathcal{L}_{\mathfrak{P}} \mid \mathfrak{M}, w \models \gamma\}$. Clearly, $Th(\mathcal{C}) \subseteq \Gamma \subseteq \Gamma_w$.

Claim 1: Γ_w is finitely satisfiable in $\mathcal{C}$.

Suppose otherwise. Then there exists a finite subset $\Delta \subseteq \Gamma_w$ that is not satisfiable in $\mathcal{C}$. By definition of $Th(\mathcal{C})$, this implies

$$\neg \bigwedge \Delta \in Th(\mathcal{C}) \subseteq \Gamma_w,$$

contradicting $\Delta \subseteq \Gamma_w$.

For each finite $\Delta \subseteq \Gamma_w$, let $\mathfrak{N}_\Delta = \langle \mathfrak{Q}_\Delta, \nu_\Delta \rangle$ be a model satisfying Δ with $\mathfrak{Q}_\Delta \in \mathcal{C}$. Define $\mathfrak{N}'_w = (\mathfrak{Q}'_w, \nu_w)$ where $\mathfrak{Q}'_w = \biguplus_{\Delta \subseteq \Gamma_w} \mathfrak{Q}_\Delta \in \mathcal{C}$ and $\nu_w = \biguplus_{\Delta \subseteq \Gamma_w} \nu_\Delta$. Let $\mathcal{B}_w = \{[[\varphi]]_{\mathfrak{N}_w} \mid \varphi \in \mathcal{L}_{\mathfrak{P}}\}$, and define $\mathfrak{N}_w = \mathfrak{N}'_w \upharpoonright_{\mathcal{B}_w}$. Then $\mathfrak{N}_w$ is an event sub-process based on $\mathfrak{Q}_w = (\mathfrak{Q}'_w) \upharpoonright_{\mathcal{B}_w} \in \mathcal{C}$. One can verify that $u_w = \{[[\varphi]]_{\mathfrak{N}_w} \mid \varphi \in \Gamma_w\}$ is a nested Archimedean ultrafilter over $\mathcal{B}_w$. Hence, $u_w \in \mathfrak{Ue}(\mathfrak{N}_w)$ and $\mathfrak{Ue}(\mathfrak{Q}_w) \in \mathcal{C}$, since $\mathcal{C}$ is closed under taking ultrafilter extensions. Moreover, by Lemma 3.21, $\mathfrak{Ue}(\mathfrak{N}_w), u_w \models \Gamma_w$. Applying Lemma 3.6 to Γ , we obtain a sub-model $\mathfrak{M}'_w = \langle \mathfrak{P}'_w, \nu'_w \rangle$ with $\mathfrak{P}'_w \in \mathcal{C}$ and $u_w \in \mathfrak{M}'_w$ such that $\mathfrak{M}'_w \models \Gamma$ and $\mathfrak{M}'_w, u_w \models \Gamma_w$. Define

$$\mathfrak{M}' = (\biguplus_{w \in \Omega} \mathfrak{M}_w) \upharpoonright_{\mathcal{B}'} = \langle \mathfrak{P}', \nu' \rangle,$$

and $\mathfrak{P}' = \langle \mathfrak{Q}', \mathcal{B}', T', \nu' \rangle$, where $\mathcal{B}' = \{[[\varphi]]_{\biguplus_{w \in \Omega} \mathfrak{N}_w} \mid \varphi \in \mathcal{L}_{\mathfrak{P}}\}$. Thus, $\mathfrak{Ue}(\mathfrak{P}') \in \mathcal{C}$.

Claim 2: There is a zigzag morphism from $\mathfrak{Ue}(\mathfrak{P}')$ to $\mathfrak{Ue}(\mathfrak{P})$.

Define $f : \mathfrak{Ue}(\mathfrak{P}') \rightarrow \mathfrak{Ue}(\mathfrak{P})$ by $f(u') = \{A \in \mathcal{B} \mid \mathfrak{Ue}(\mathfrak{M}'), u' \models p_A\}$, for each $u' \in \mathfrak{Q}'$.

1. $Im(f) \subseteq \mathfrak{Ue}(\mathfrak{P})$: By the definition of Γ and this fact that $p_A \rightarrow p_B \equiv \top$ for each $A \subseteq B \in \mathcal{B}$, it is easy to see that $f(u')$ is an ultrafilter over $\mathcal{B}$, for each $u' \in \mathfrak{Q}'$. To prove that it is a nested Archimedean ultrafilter, assume that

$$F_{t_1 \dots t_n s} A \in f(u') \quad \forall s < r.$$

Then,

$$\mathfrak{Ue}(\mathfrak{M}), u' \models L_{t_1 \dots t_n s} p_A \quad \forall s < r.$$

So we have $\mathfrak{Ue}(\mathfrak{M}), u' \models L_{t_1 \dots t_n r} p_A$ and by applying axiom 4 of Γ , $\mathfrak{Ue}(\mathfrak{M}), u' \models p_{F_{t_1 \dots t_n r} A}$. So $F_{t_1 \dots t_n r} A \in f(u')$.

2. f is surjective: Let u be a nested Archimedean ultrafilter over $\mathcal{B}$. Then, consider $\mathfrak{v} := \{[[p_A]]_{\mathfrak{M}'} \mid A \in u\}$. We show that $\mathfrak{v}$ is a nested Archimedean ultrafilter on $\mathcal{B}'$. Since all formulas of Γ are valid in $\mathfrak{M}'$, it follows that every formula φ in the language $\mathcal{L}_{\mathfrak{P}}$ is logically equivalent to a propositional variable p_A . Now since u is an ultrafilter over $\mathcal{B}$, either $[[p_A]]_{\mathfrak{M}'} \in \mathfrak{v}$ or $[[p_A]]_{\mathfrak{M}'}^c \in \mathfrak{v}$. Also, if $[[p_A]]_{\mathfrak{M}'} \in \mathfrak{v}$, $[[p_B]]_{\mathfrak{M}'} \in \mathfrak{v}$, then so $[[p_A]]_{\mathfrak{M}'} \cap [[p_B]]_{\mathfrak{M}'} = [[p_{A \cap B}]]_{\mathfrak{M}'} \in \mathfrak{v}$. Moreover, since $A \in u$ is non-empty, there exists $w \in A$. Therefore, $[[p_A]]_{\mathfrak{M}'} \in u_w = \{[[\varphi]]_{\mathfrak{N}_w} \mid \varphi \in \Gamma_w\}$. This implies $[[p_A]]_{\mathfrak{M}'} \neq \emptyset$. Hence $\emptyset \notin \mathfrak{v}$. Finally, to show that $\mathfrak{v}$ is a nested Archimedean ultrafilter, let $A \in u$ and

$$F_{t_1 \dots t_n s} [[p_A]]_{\mathfrak{M}'} \in \mathfrak{v} \quad \forall s < r.$$

Then

$$F_{t_1 \dots t_n s} [[p_A]]_{\mathfrak{M}'} = [[p_{F_{t_1 \dots t_n s} A}]]_{\mathfrak{M}'} \in \mathfrak{v} \quad \forall s < r.$$

Therefore,

$$F_{t_1 \dots t_n r} A \in u \quad \forall s < r.$$

But since u is a nested Archimedean ultrafilter, it follows that $F_{t_1 \dots t_n r} A \in u$. So, $[[p_{F_{t_1 \dots t_n r} A}]]_{\mathfrak{M}'} = F_{t_1 \dots t_n r} [[p_A]]_{\mathfrak{M}'} \in \mathfrak{v}$.

3. f is a measurable zigzag morphism: We have to verify that for $A^* = \{u \in \mathcal{U}_{na}^* \mid A \in u\}$, $f^{-1}(A^*)$ is measurable and

$$T_{na}^{I*}(u, f^{-1}(A^*)) = T_{na}^*(f(u), A^*) \quad (*)$$

for each $u \in \Omega'$. To see the measurability of f , notice that

$$f^{-1}(A^*) = \{u' \in \mathcal{U}_{na}^{I*} \mid [[p_A]]_{\mathfrak{M}'} \in u'\} \in \mathcal{B}_{na}^*.$$

On the other hand, to show $(*)$, assume for some $r \in \mathbb{Q}_0$, we have

$$T_{na}^{I*}(u, f^{-1}(A^*)) = \lim_{u'} T'(w', [[p_A]]_{\mathfrak{M}'}) \geq r.$$

As u' is a nested Archimedean ultrafilter and $[[p_{F_r A}]]_{\mathfrak{M}'} = [[L_r p_A]]_{\mathfrak{M}'}$, we have $[[p_{F_r A}]]_{\mathfrak{M}'} \in u'$. Therefore, $F_r A \in f(u')$. So $\mathfrak{Ue}(\mathfrak{M}), u' \models L_r p_A$ which implies that

$$T_{\mathcal{G}}(f(u'), [[p_A]]_{\mathfrak{Ue}(\mathfrak{M})}) = T_{\mathcal{G}}(f(u'), A^*) \geq r.$$

Conversely, one can show that if $T_{na}^*(f(u'), A^*) \geq r$, then so $T_{na}^{I*}(u, f^{-1}(A^*)) \geq r$. Thus, $(*)$ is achieved.

Since f is a zigzag morphism, $\mathfrak{Ue}(\mathfrak{P}) \in \mathcal{C}$. Moreover, as $\mathcal{C}$ reflects ultrafilter extensions, it follows that $\mathfrak{P} \in \mathcal{C}$. $\square$

4.1 Some Examples

Here in this subsection, we show that the class of Harsanyi type spaces [11] is an important instance for a P-definable class which is closed under ultrafilter extensions, and has the GT-property.

Example 4.2. For a Markov process $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ and $w \in \Omega$ let

$$[T(w)] = \{w' \in \Omega \mid T(w) = T(w')\}.$$

Recall that a Markov process $\mathfrak{P}$ is of *Harsanyi type* if $T(w, E) = 1$, for each $w \in \Omega$ and $E \in \Sigma$ with $[T(w)] \subseteq E$. Note that $[T(w)]$ is not necessarily in Σ , unless Σ is countably generated [25, Page 10]. In fact in case of countably generated Markov process $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$,

$$[T(w)] = \bigcap_{B \in \mathcal{B}} \{w' \in \Omega \mid T(w)(B) = T(w')(B)\} \quad (5)$$

$$= \bigcap_{B \in \mathcal{B}} \bigcap_{r \in \mathbb{Q}_0} \{w' \in \Omega \mid T(w)(B) \geq r \leftrightarrow T(w')(B) \geq r\}. \quad (6)$$

We denote the class of all Harsanyi type spaces by $\mathcal{Har}$.

Proposition 4.3. Assume that $\mathfrak{P}$ is a countably generated Harsanyi type space. Then $\mathfrak{P} \in \mathcal{Har}$ if and only if the following set of formulas is valid in $\mathfrak{P}$:

$$\Gamma_{har} = \{L_r p \rightarrow L_1 L_r p \mid r \in \mathbb{Q}_0\} \cup \{\neg L_r p \rightarrow L_1 \neg L_r p \mid r \in \mathbb{Q}_0\}.$$

Proof. It is known that, [25], these formulas are valid on any Harsanyi type space.

Now assume that $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$ is a countably generated Markov process which is not Harsanyi type. Hence, there is $w \in \Omega$ such that $T(w)([T(w)]) < 1$. Therefore, $T(w)([T(w)]^c) > 0$. Let $\Sigma = \sigma(\mathcal{B})$ where $\mathcal{B}$ is countable. By 5, we have

$$[T(w)]^c = \bigcup_{B \in \mathcal{B}} \bigcup_{r \in \mathbb{Q}_0} \{w' \in \Omega \mid T(w)(B) \geq r \leftrightarrow T(w')(B) \geq r\}^c.$$

Thus, there are $B \in \mathcal{B}$ and $r \in \mathbb{Q}_0$ such that either $T(w)(B) \geq r$, while $T(w)((F_r(B))^c) > 0$ or $T(w)(B) < r$, while $T(w)(F_r(B)) > 0$. In either of these cases, if we define a valuation v over $\mathfrak{P}$ as $v(p) = B$, then we turn $\mathfrak{P}$ into a model $\mathfrak{M}$ such that we have either

$$\mathfrak{M}, w \models L_r p \wedge \neg L_1 L_r p \quad \text{or} \quad \mathfrak{M}, w \models \neg L_r p \wedge L_1 \neg L_r p.$$

□

Proposition 4.4. *The class of all countably generated Harsanyi type spaces is closed under ultrafilter extensions.*

Proof. Assume that $\mathfrak{P} = \langle \Omega, \mathcal{B}, T \rangle$ is a Harsanyi type space with $\mathcal{B}$ countable. Let $u \in \mathcal{U}_{na}^*$. Then

$$[T_{na}^*(u)] = \bigcap_{B^* \in \mathcal{B}_{na}^*} \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u, B^*) = T_{na}^*(u', B^*)\} \quad (7)$$

$$= \bigcap_{B^* \in \mathcal{B}_{na}^*} \bigcap_{r \in \mathbb{Q}_0} \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u, B^*) \geq r \leftrightarrow T_{na}^*(u', B^*) \geq r\}. \quad (8)$$

Now for $B \in \mathcal{B}$ and $r \in \mathbb{Q}_0$ there are two cases to consider.

Case 1: $T_{na}^*(u, B^*) < r$. In this case

$$\begin{aligned} \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u, B^*) \geq r \leftrightarrow T_{na}^*(u', B^*) \geq r\} &= \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u', B^*) < r\} \\ &= \{u' \in \mathcal{U}_{na}^* \mid (F_r(B))^c \in u'\}. \end{aligned}$$

So

$$T_{na}^*(u, \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u', B^*) < r\}) = \lim_u T(w, (F_r(B))^c)$$

On the other hand $T_{na}^*(u, B^*) = \lim_u T(w, B) < r$. Therefore $\{w \in \Omega \mid T(w, B) < r\} \in u$. Notice that since $\mathfrak{P}$ is Harsanyi,

$$\{w \in \Omega \mid T(w, B) < r\} \subseteq \{w \in \Omega \mid T(w, (F_r(B))^c) = 1\} \in u.$$

Thus $\lim_u T(w, (F_r(B))^c) = 1$.

Case 2: $T_{na}^*(u, B^*) \geq r$. In this case we have

$$\begin{aligned} \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u, B^*) \geq r \leftrightarrow T_{na}^*(u', B^*) \geq r\} &= \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u', B^*) \geq r\} \\ &= \{u' \in \mathcal{U}_{na}^* \mid F_s(B) \in u', \forall s < r\} \\ &= \{u' \in \mathcal{U}_{na}^* \mid F_r(B) \in u'\}. \end{aligned}$$

We also have $T_{na}^*(u, B^*) = \lim_u T(w, B) > s$, for each $s < r$. So,

$$\{w \in \Omega \mid T(w, B) > s\} \subseteq \{w \in \Omega \mid T(w, B) \geq s\} \in u, \quad \forall s < r.$$

Hence $\{w \in \Omega \mid T(w, B) \geq r\} \in u$, since u is a nested Archimedean ultrafilter. Again this yields that $\{w \in \Omega \mid T(w, F_r(B)) = 1\} \in u$, and $\lim_u T(w, F_r(B)) = 1$. The above argument shows that $T_{na}^*(u, (F_r(B))^*) = 1$.

From both cases we conclude that

$$T_{na}^*(u, \{u' \in \mathcal{U}_{na}^* \mid T_{na}^*(u, B^*) \geq r \leftrightarrow T_{na}^*(u', B^*) \geq r\}) = 1,$$

for each $B \in \mathcal{B}$ and $r \in \mathbb{Q}_0$. Thus $T_{na}^*(u, [T_{na}^*(u)]) = 1$. □

Proposition 4.5. *The class $\mathcal{Har}$ has the GT-property.*

Proof. We only show that $\mathcal{Har}$ reflects the ultrafilter extensions, since the other properties can be easily shown. Assume that $\mathcal{Ue}(\mathfrak{P}) = \langle \mathcal{U}_{na}^*, \Sigma^*, T_{na}^* \rangle \in \mathcal{Har}$ for a countably generated $\mathfrak{P} = \langle \Omega, \Sigma, T \rangle$. Then $\mathcal{Ue}(\mathfrak{P}) \models \Gamma_{har}$. So, by Lemma 3.22, we have $\mathfrak{P} \models \Gamma_{har}$. But, since $\mathfrak{P}$ is countably generated by Lemma 4.3, we conclude that $\mathfrak{P} \in \mathcal{Har}$ $\square$

Corollary 4.6. *The class $\mathcal{Har}$ is P-definable.*

5 Some Variants of the Goldblatt-Thomason Theorem

In this section, we study an alternative version of the Goldblatt-Thomason theorem for the class of finite Markov processes with rational Markov kernels, denoted by $\text{FMP}_{\mathbb{Q}_0}$. This important subclass of Markov processes can be also seen as a natural generalization of finite Kripke structures for which there exists a version of the Goldblatt-Thomason theorem [2]. So it would be natural to investigate a variant of this theorem for $\text{FMP}_{\mathbb{Q}_0}$. So throughout this section, by a finite process we mean a Markov process $\mathfrak{P} = \langle \Omega, \mathcal{P}(\Omega), T \rangle$, where Ω is finite and $T(w, w') := T(w, \{w'\}) \in \mathbb{Q}_0$.

One can associate to any finite Markov process $\mathfrak{P} = \langle \Omega, \mathcal{P}(\Omega), T \rangle$ a finite directed graph (or Kripke frame) $(\Omega, R_{\mathfrak{P}})$, which enables us to translate certain model theoretic concepts of probability logic into the corresponding notions of basic modal logic.

Definition 5.1. Let $\mathfrak{P} = \langle \Omega, \mathcal{P}(\Omega), T \rangle$ be a finite Markov processes. Define a binary relation $R_{\mathfrak{P}} \subseteq \Omega \times \Omega$ as

$$wR_{\mathfrak{P}}w' \text{ if and only if } T(w, w') > 0.$$

Let $R_{\mathfrak{P}}(w) = \{w' \mid wR_{\mathfrak{P}}w'\}$. Then we have $T(w, R_{\mathfrak{P}}(w)) = 1$, for each $w \in \Omega$. Assume that $\bar{R}_{\mathfrak{P}}$ is the reflexive and transitive closure of $R_{\mathfrak{P}}$. Then for each $w' \in \bar{R}_{\mathfrak{P}}(w)$ we have $R_{\mathfrak{P}}(w') \subseteq \bar{R}_{\mathfrak{P}}(w)$. So, $T(w', \bar{R}_{\mathfrak{P}}(w)) = 1$, for each $w' \in \bar{R}_{\mathfrak{P}}(w)$. Hence one can consider the sub-processes

$$\mathfrak{P}_w = \langle \bar{R}_{\mathfrak{P}}(w), \mathcal{P}(\bar{R}_{\mathfrak{P}}(w)), T \upharpoonright_{\bar{R}_{\mathfrak{P}}(w)} \rangle.$$

$\mathfrak{P}_w$ serves as a point-generated sub-frame of $(\Omega, R_{\mathfrak{P}})$.

A Markov process $\mathfrak{P}$ is called a point-generated process if there is $w \in \Omega$ such that $\mathfrak{P} = \mathfrak{P}_w$.

Note that if $\mathfrak{P}_w$ is a point generated finite process, then $\mathfrak{P}_w \models \varphi$ if and only if $\mathfrak{P}_w, w \models L_1^k \varphi$ for some $k \geq 0$.

Recall that for directed graph $(\Omega, R_{\mathfrak{P}})$ and $w, v \in \Omega$ the *distance* between w, v , denoted by $d(w, v)$, is the length of the shortest path from w to v if there exists a path between them and ∞ if there is no such path. Then, for $n \geq 0$ by $R_{\mathfrak{P}}(w) \upharpoonright_n$ we mean the set of all $v \in \Omega$ with $d(w, v) \leq n$.

Below we define the notion of a local n -zigzag morphism. Before defining this notion notice that for two finite processes $\mathfrak{P} = \langle \Omega, \mathcal{P}(\Omega), T \rangle$ and $\mathfrak{P}' = \langle \Omega', \mathcal{P}(\Omega'), T' \rangle$ the function $f : \Omega \rightarrow \Omega'$ is a zigzag morphism if it is surjective and for each $z \in \Omega$ and $z' \in \Omega'$ we have

$$T(z, f^{-1}(z')) = T'(f(z), z'). \quad (9)$$

Remark 5.2. Also note that any finite Markov process $\mathfrak{P}$ is a zigzag image of the disjoint union of its point-generated sub-processes, i.e., there is a zigzag morphism $f : \uplus_{w \in \Omega} \mathfrak{P}_w \rightarrow \mathfrak{P}$, where for each $v \in \uplus_{w \in \Omega} \mathfrak{P}_w$, $f(v)$ is equal to its corresponding state in Ω .

Definition 5.3 (Local n -Zigzag). Let $\mathfrak{P}_w$ and $\mathfrak{P}'_{w'}$ be two finite point generated processes. A *local n -zigzag function*, for $n \geq 0$ is a function $f : \Omega \rightarrow \Omega'$ such that $R'(w') \upharpoonright_n \subseteq \text{Img}(f)$ and (9) holds for all $z \in R(w) \upharpoonright_n$ and $z' \in R'(w') \upharpoonright_n$.

Likewise a local n -zigzag morphism from $\mathfrak{M} = \langle \mathfrak{P}_w, v \rangle$ to $\mathfrak{M}' = \langle \mathfrak{P}'_{w'}, v' \rangle$ is a local n -zigzag morphism f from $\mathfrak{P}_w$ to $\mathfrak{P}'_{w'}$ such that $v(p) = f^{-1}(v'(p))$, for each $p \in \mathbb{P}$.

We recall that $\deg_p(\varphi) \in \mathbb{N}$ is defined inductively as follows:

- $\deg_p(q) = 0$, for $q \in \mathbb{P}$,
- $\deg_p(\neg\varphi) = \deg_p(\varphi)$,
- $\deg_p(\varphi \wedge \psi) = \max(\deg_p(\varphi), \deg_p(\psi))$,
- $\deg_p(L_r\varphi) = \deg_p(\varphi) + 1$.

Lemma 5.4. Assume that $\mathfrak{M}_w = \langle \Omega, \mathcal{P}(\Omega), T, v \rangle$ and $\mathfrak{M}'_{w'} = \langle \Omega', \mathcal{P}(\Omega'), T', v' \rangle$ are two point-generated models and $f : \Omega \rightarrow \Omega'$ is a local n -zigzag morphism. Then

$$\mathfrak{M}, w \models \psi \text{ if and only if } \mathfrak{M}', f(w) \models \psi,$$

for each ψ with $\deg_p(\psi) \leq n$.

Proof. We prove it by induction on n and complexity of ψ . We only prove the induction step for $L_r\theta$, assuming $\deg_p(\theta) \leq n$. Suppose that f is a local $(n+1)$ -zigzag morphism. Then by induction hypothesis, we have

$$\llbracket \theta \rrbracket_{\mathfrak{M}} \cap R(w) = f^{-1}(\llbracket \theta \rrbracket_{\mathfrak{M}'} \cap R'(f(w))),$$

since for each $z \in R(w)$, the function $f \upharpoonright_{\bar{R}(z)} : \bar{R}(z) \rightarrow \bar{R}'(f(z))$ is a local n -zigzag morphism between $\mathfrak{P}_z$ and $\mathfrak{P}'_{f(z)}$. Hence,

$$T(w, \llbracket \theta \rrbracket_{\mathfrak{M}}) = T(w, \llbracket \theta \rrbracket_{\mathfrak{M}} \cap R(w)) = T(f(w), \llbracket \theta \rrbracket_{\mathfrak{M}'} \cap R'(f(w))) = T(f(w), \llbracket \theta \rrbracket_{\mathfrak{M}'}).$$

But this implies

$$\mathfrak{M}, w \models L_r\theta \text{ if and only if } \mathfrak{M}', f(w) \models L_r\theta.$$

□

Below we prove a version of the Goldblatt-Thomason theorem for a class of finite Markov processes with rational probabilities. Before stating the theorem we isolate an important property which is necessary for a subclass $\mathcal{C}$ of $\text{FMP}_{\mathbb{Q}_0}$ to be definable.

Definition 5.5. Let $\mathcal{C}$ be a subclass of $\text{FMP}_{\mathbb{Q}_0}$. We say that $\mathcal{C}$ is closed under local zigzag morphism if for each point-generated process $\mathfrak{P}_w$ we have $\mathfrak{P}_w \in \mathcal{C}$ whenever for each $n > 0$ there is a point-generated process $\mathfrak{P}'_n \in \mathcal{C}$ where there is a local n -zigzag from $\mathfrak{P}'_n$ to $\mathfrak{P}$.

Proposition 5.6. Let Γ be a set of formulas and assume that $\mathcal{C}$ is the class of all finite Markov processes $\mathfrak{P}$ in which Γ is valid. Then $\mathcal{C}$ is closed under local zigzag morphisms.

Proof. Assume that $\mathfrak{P}_w = \langle \Omega_w, \mathcal{P}(\Omega_w), T \rangle$ is a point generated Markov process in $\text{FMP}_{\mathbb{Q}_0}$ such that it is an image of local zigzag morphisms. We have to show that $\mathfrak{P}_w \models \Gamma$. Assume not. Then for some $\varphi \in \Gamma$ we have $\mathfrak{P}_w \not\models \varphi$. Hence there exist $\mathfrak{M} = \langle \mathfrak{P}_w, v \rangle$ and $k \geq 0$ such that $\mathfrak{M}, w \not\models L_1^k \varphi$. Now take $n \geq \max(|\Omega_w|, \deg_p(L_1^k \varphi))$, $\mathfrak{P}_{w'} \in \mathcal{C}$ and $f : \mathfrak{P}_{w'} \rightarrow \mathfrak{P}_w$ a local n -zigzag morphism. By the choice of n , we conclude that f is a surjection. Moreover, we can define v' on $\mathfrak{P}_{w'}$ such that f is a local n -zigzag morphism from $\mathfrak{M}' = \langle \mathfrak{P}_{w'}, v' \rangle$ to $\mathfrak{M} = \langle \mathfrak{P}_w, v \rangle$. Hence in the light of Lemma 5.4, we have $\mathfrak{M}', w' \not\models L_1^k \varphi$. Therefore, $\mathfrak{P}_{w'} \not\models \varphi$, a contradiction. □

Theorem 5.7. Assume that $\mathcal{C}$ is a subclass of $\text{FMP}_{\mathbb{Q}_0}$. Then $\mathcal{C}$ is definable if and only if it is closed under generated sub-processes, finite disjoint unions, and local zigzag morphisms.

Proof. The nontrivial direction is the right-to-left one, so assume that $\mathfrak{P}$ is a finite process with $\mathfrak{P} \models \text{Th}(\mathcal{C})$. We have to show that $\mathfrak{P} \in \mathcal{C}$.

By Remark 5.2 and since $\mathcal{C}$ is closed under finite disjoint unions, without loss of generality, we can assume that $\mathfrak{P}$ is point-generated. Suppose $\Omega = \{w_0, w_1, \dots, w_m\}$ where $\mathfrak{P} = \mathfrak{P}_{w_0}$. Let $n > 0$ be arbitrary. Consider the language $\mathcal{L}_{\mathfrak{P}} = \{p_0, \dots, p_m\}$, and the valuation v on $\mathfrak{P}$ as $v(p_i) = \{w_i\}$, for each $1 \leq i \leq n$. Define the following formulas:

- i. p_0 .
- ii. $L_r p_i \wedge M_r p_i$, if $T(w_0, w_i) = r$.
- iii. $L_1^k(p_0 \vee \dots \vee p_m)$, for $0 \leq k \leq n$.
- iv. $L_1^k(p_i \rightarrow \neg p_j)$, for $0 \leq k \leq n$ and $i \neq j$.
- v. $L_1^k(p_i \rightarrow (L_r p_j \wedge M_r p_j))$, for $0 \leq k \leq n$ if $T(w_i, w_j) = r$.

While (i) and (ii) are satisfied in w_0 , the other ones are valid on $\mathfrak{M}$, and in particular, are satisfied in w_0 . Let φ_{n, w_0} be the conjunction of the above formulas. Clearly, $\varphi_{n, w_0} \in \text{Th}(\mathcal{C})$ and is satisfiable in $\mathcal{C}$, as otherwise $\neg \varphi_{n, w_0} \in \text{Th}(\mathcal{C})$, which is a contradiction. So there exists $\mathfrak{N}_n = \langle \mathfrak{P}'_n, v' \rangle$ with $\mathfrak{P}'_n = \langle \Omega', \mathcal{P}(\Omega'), T' \rangle \in \mathcal{C}$ such that $\mathfrak{N}_n, z_0 \models \varphi_{n, w_0}$ for some $z_0 \in \Omega'$. Again, without loss of generality, we can assume that $\mathfrak{P}'$ is point-generated with $\mathfrak{P}' = \mathfrak{P}'_{z_0}$.

Now define $f_n : \Omega' \rightarrow \Omega$ as $f_n(z) = w_i$ if $z \in R(v_0) \upharpoonright_n$ and $\mathfrak{N}_n, z \models p_i$. Otherwise let $f_n(z) = w_0$. We show that f_n is a local n -zigzag morphism.

- f_n is well-defined: assume that $z \in R(z_0) \upharpoonright_n$. Hence $d(v_0, z) = k \leq n$, and $\mathfrak{N}_n, v_0 \models L_1^k(p_0 \vee \dots \vee p_m)$. Therefore, this implies that $\mathfrak{N}_n, z \models (p_0 \vee \dots \vee p_m)$. Thus, there is $0 \leq i \leq m$ such that $\mathfrak{N}_n, z \models p_i$. Furthermore, for $i \neq j$ we have $\mathfrak{N}_n, z_0 \models L_1^k(p_i \rightarrow \neg p_j)$ which implies that $\mathfrak{N}_n, z \models p_i \rightarrow \neg p_j$. So we conclude that there is a unique i such that $f_n(z) = w_i$.
- $\text{Img}(f_n) \subseteq R(w_0) \upharpoonright_n$: Assume that $w \in \Omega$ and $d(w_0, w) = k \leq n$. So there are $w_{j_1}, \dots, w_{j_k} \in \Omega$ with $w_0 R w_{j_1} \dots R w_{j_k}$ and $w_{j_k} = w$. Suppose that $T(w_{j_i}, w_{j_{i+1}}) = r_i > 0$, for each $0 \leq i \leq k-1$. Since $\mathfrak{M}, w_0 \models L_{r_1} p_{j_1} \wedge L_1(p_{j_1} \rightarrow L_{r_2} p_{j_2})$, it follows that $\mathfrak{N}_n, v_0 \models L_{r_1} p_{j_1} \wedge L_1(p_{j_1} \rightarrow L_{r_2} p_{j_2})$. Hence, $T'(z_0, \llbracket p_{j_1} \rrbracket_{\mathfrak{N}_n}) = r_1 > 0$ and $T'(z_0, \llbracket p_{j_1} \rightarrow L_{r_2} p_{j_2} \rrbracket_{\mathfrak{N}_n}) = 1$. So there is $z_1 \in \Omega'$ such that $T'(z_0, z_1) > 0$ and $\mathfrak{N}_n, z_1 \models p_{j_1} \wedge L_{r_2} p_{j_2}$. Again, since $\mathfrak{M}, w_0 \models L_1^2(p_{j_2} \rightarrow L_{r_3} p_{j_3})$, it follows that $T'(z_1, \llbracket p_{j_2} \rightarrow L_{r_3} p_{j_3} \rrbracket_{\mathfrak{N}_n}) = 1$. Thus there is $z_2 \in \Omega'$ such that $\mathfrak{N}_n, z_2 \models p_{j_2}$ and $T'(z_1, z_2) > 0$. Continuing this method, one can find a sequence $z_0 R' z_1 \dots R' z_i$ such that $\mathfrak{N}_n, z_i \models p_{j_i}$ for each $0 < i \leq k$. So, $f_n(z_k) = w$.
- f_n is a local n -zigzag morphism: note that the measurability of f_n is automatic. So, we only need to show that

$$T'(z, f_n^{-1}(w)) = T(f_n(z), w),$$

for each $w \in R(w) \upharpoonright_n$ and $z \in R'(z_0) \upharpoonright_n$. Assume that for some $0 \leq i, j \leq m$ we have $\mathfrak{N}_n, z \models p_i$ and $\mathfrak{M}, w \models p_j$. Thus $f(z) = w_i$. If $T(w, w_i) = r$, then for each $0 \leq k \leq n$,

$$\mathfrak{M}, w_0 \models L_1^k(p_j \rightarrow (L_r p_i \wedge M_r p_i)).$$

So

$$\mathfrak{N}_n, z_0 \models L_1^k(p_j \rightarrow (L_r p_i \wedge M_r p_i)).$$

Hence,

$$T'(z, \llbracket p_i \rrbracket_{\mathfrak{N}_n}) = T'(z, f_n^{-1}(w)) = r.$$

Since $\mathfrak{P}'_n \in \mathcal{C}$ and $f_n : \mathfrak{P}'_n \rightarrow \mathfrak{P}$ is a local n -zigzag morphism, it follows that $\mathfrak{P} \in \mathcal{C}$, as required. $\square$

6 Conclusions and further studies

In this paper, we established a Goldblatt-Thomason-style theorem for probability logic, characterizing the classes of Markov processes definable in probability logic via suitable model-theoretic closure properties. We also obtained a corresponding result for finite Markov processes. These results connect probabilistic modal logic with the classical model-theoretic theory of definability.

A natural direction for future research is to extend these results to infinitary probability logics, which allow countable conjunctions and disjunctions and whose expressive power is closely related to the σ -additivity of probability measures. As shown in [5], important stochastic properties of Markov processes, such as stationarity, invariance, irreducibility, and recurrence, are expressible in this setting. This indicates that infinitary probability logics provide a more suitable framework for capturing genuinely measure-theoretic probabilistic phenomena. Accordingly, an important problem for future work is to establish a Goldblatt-Thomason-style theorem for infinitary probability logic and to characterize the associated model-theoretic closure conditions for definability.

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Inquisitive Action Logic

Ivano Ciardelli

University of Padua, Italy

ivano.ciardelli@unipd.it

We introduce *inquisitive action logic*, InqAL, a multi-agent modal logic for reasoning about action. While traditional approaches focus on what *properties* of the outcome an agent can *force*, InqAL also captures what *aspects* of the outcome an agent *determines* through their actions. As we argue, such claims of agentive determination are naturally analyzed as modal claims involving questions.

Technically, InqAL is a multi-agent extension of inquisitive neighborhood logic [12] based on concurrent game structures. With respect to statements, it is expressively equivalent to the individual-agent fragment of the socially friendly coalition logic recently proposed by Goranko and Enqvist [20].

We present an axiomatization of InqAL and prove completeness and decidability via the finite model property. Along the way, we establish a representation theorem for *actual effectivity functions*, associating to an agent the sets of outcomes corresponding to their possible actions; we give exact conditions under which a multi-agent neighborhood frame arises from a concurrent game structure.

1 Introduction

There is a long tradition of using modal logics to reason about actions, outcomes, and abilities (for surveys, see [1, 29]). This enterprise is motivated by philosophical, juridical, and computational concerns: logics of agency have been used, e.g., to spell out precise notions of responsibility [23, 3], and for the automated verification of software properties [16]. Logics in this tradition typically focus on the powers of agents (or coalitions) in an interactive situation, i.e., on the facts they are able to force through their actions. Something which is just as important, however, is what aspects of the world an agent *determines*, or *influences*, through their actions. For an illustration, imagine that Alice and Bob are creating clay animals together; Alice moulds an animal with the clay, and Bob paints it. While Alice works, Bob picks a color, without looking at what Alice is moulding. In this scenario, it is natural to say the following:

- (1) a. Alice determines the shape of the animal (but not its color).
- b. Bob determines the color of the animal (but not its shape).

These facts matter, among other things, for ascribing responsibility: suppose little Charlie asks his siblings to make a red cat for him; if what he gets is a blue cat, he should complain to Bob, not to Alice.

How can we analyze a statement like (1-a)? More precisely, what is the *thing* that Alice determines? Standardly, in modal logic, modalities apply to propositions. However, “the shape of the animal” does not denote a proposition; rather, it is associated with a set of propositions: that the animal is a dog, that the animal is a cat, etcetera. What Alice determines is not a specific one of these propositions, but *which* among these propositions is going to be actualized. In other words, what Alice determines is the answer to a question, namely, the question “what the shape is”. Determining, then, can be seen as a relation between an agent and a question. What does this relation amount to? Here is a natural idea: an agent can be said to determine a question if the answer to the question is not settled *a priori*, but it becomes settled once we fix the agent’s action. Implementing this idea takes us into the realm of *inquisitive modal logic*, a quickly growing research area that combines inquisitive logic and modal logic.

Inquisitive logic [10] is a framework that allows one to build logical systems including not only, as usual, formulas expressing propositions (e.g., *that the animal is a cat*), but also formulas expressing questions (e.g., *what the shape of the animal is*). One can then define inquisitive modalities that apply to such formulas (see, a.o., [7, 15, 8, 17, 18, 27, 28, 14, 25, 24, 12]). In our case, the idea is to capture agentive determination in terms of an inquisitive modality $\boxtimes$ indexed by agents which, when applied to formulas shape and color formalizing the questions *what is the shape* and *what the color*, delivers modal statements $\boxtimes_{\text{Alice}}\text{shape}$ and $\boxtimes_{\text{Bob}}\text{color}$ capturing, respectively, the claims in (1-a) and (1-b).

In this paper, we will develop this idea in detail. We will define an *inquisitive action logic*, InqAL, which allows us to analyze statements like (1-a) and (1-b) along the lines we described, but which also recovers the modal account of ability familiar from the classical work of Brown [5] and more recent work on coalition logic, CL [26]. Our logic is a multi-agent version of a recently developed system, *inquisitive neighborhood logic*, InqNL [9, 12]. InqNL is based on a binary modality $\Rightarrow$, acting semantically as a strict conditional quantifying over neighborhoods; analogously, our multi-agent extension InqAL contains one such modality $\Rightarrow_a$ for each agent a ; in terms of this modality, some derivative unary modalities (including the modality $\boxtimes_a$ mentioned above) can be defined. Besides being multi-agent, InqAL is geared specifically towards an interpretation in terms of action; as a consequence, while InqNL is interpreted over arbitrary neighborhood models, InqAL starts with more concrete structures known as *concurrent game structures* or *multi-agent transition systems* [2, 26, 19, 6], modeling processes in which a system transitions from a state to another based on the joint actions of multiple agents. Now, from such models one can extract a corresponding neighborhood model, where the neighborhoods for a given agent correspond to the actions available to them, each neighborhood encoding the range of outcomes that may possibly result if the agent performs that action. The neighborhood maps induced in this way are known in the literature as *actual effectivity functions* [6] (in contrast with the *effectivity functions* studied in coalition logic [26, 21], which are upward-closed sets of neighborhoods). The problem of characterizing which families of neighborhood maps are the actual effectivity functions of some concurrent game structure is currently open. In this paper, we settle this problem for the case in which we only consider single agents, and not proper coalitions. This result is interesting in its own right, but it also plays a key role in our study of InqAL, giving us a characterization of the relevant class of neighborhood models.

Our logic InqAL is also connected to *socially friendly coalition logic* (SFCL), an extension of coalition logic recently proposed by Goranko and Enqvist [20]. Whereas standard coalition logic focuses on what outcomes agents can *force* through their actions, SFCL also captures what outcomes agents can *enable* for others, while still forcing their own goals. These properties are crucial for the possibility of cooperation between agents (hence the name *socially friendly*). Technically, SFCL is a multi-agent extension of *instantial neighborhood logic*, INL [4]. As proved in [12], inquisitive neighborhood logic has, with respect to statements, the same expressive power as INL. In the same vein, our inquisitive action logic has, with respect to statements, the same expressive power as SFCL. Still, as we will see, the way modal claims are expressed in these logics is very different; in particular, the translation from InqAL to SFCL can lead to an exponential increase in the size of the formula.

The paper is structured as follows. In §2 we provide a characterization of those neighborhood frames that arise as actual effectivity functions from some concurrent game structure. In §3 we introduce InqNL_{act}, a multi-agent version of inquisitive neighborhood logic. In §4 we introduce InqAL, which interprets InqNL_{act} over concurrent game models, and discuss how this logic allows us to express interesting properties relating to what agents can force, enable, or determine in an interactive situation. In §5 we give an axiomatization of InqAL and prove completeness and decidability. In §6 we compare the expressive power of our logic with that of Socially Friendly Coalition Logic. §7 concludes.

2 Characterizing actual effectivity functions of agents

Concurrent game structures [2, 26, 6] model processes in which the transition from one stage to the next is determined by the simultaneous actions of multiple agents. The formal definition is as follows.

Definition 2.1. Let $\mathcal{A} = \{a_1, \dots, a_k\}$ be a finite set of agents. A *concurrent game structure* (or CGS for short) for $\mathcal{A}$ is a triple $\mathcal{S} = (W, \text{act}, \text{out})$ where:

- W is a non-empty set of *worlds*, representing stages of the process;
- act is a function which assigns to each agent a and world w a non-empty set $\text{act}(a, w)$ of actions. A function τ assigning to each agent a_i a corresponding action $\tau(a_i) \in \text{act}(a_i, w)$ is called an *action profile* at w ; the set of action profiles at w is denoted $\text{ActP}(w)$.
- out is a function that, given a world w and an action profile τ at w , returns a world $\text{out}(w, \tau)$; intuitively, $\text{out}(w, \tau)$ is the world that results from w when each agent a_i performs the action $\tau(a_i)$.

At a world w in a CGS, a range of outcomes are possible, depending on the choices of the agents, namely:

$$O(w) = \{\text{out}(w, \tau) \mid \tau \in \text{ActP}(w)\}$$

By picking a particular action, an agent typically narrows down the set of possible outcomes. The set of outcomes which are possible from w given that agent a performs action $\tau_a \in \text{act}(a, w)$ is:

$$O_a(w, \tau_a) = \{\text{out}(w, \tau) \mid \tau \in \text{ActP}, \tau(a) = \tau_a\}$$

We call this the *output set* of action τ_a for a at w . By collecting the output sets of all actions available to an agent, we obtain their *actual effectivity function*. More precisely, the actual effectivity function of agent a in a CGS $\mathcal{S}$ is the map $\Sigma_a^{\mathcal{S}} : W \rightarrow \wp\wp(W)$ given by:

$$\Sigma_a^{\mathcal{S}}(w) = \{O_a(w, \tau_a) \mid \tau_a \in \text{act}(a, w)\}$$

In this way, a CGS determines a corresponding multi-agent neighborhood frame $F_{\mathcal{S}} = (W, (\Sigma_a^{\mathcal{S}})_{a \in \mathcal{A}})$. But not just *any* neighborhood frame $F = (W, (\Sigma_a)_{a \in \mathcal{A}})$, where $\Sigma_a : W \rightarrow \wp\wp(W)$, can be a system of actual effectivity functions of some CGS. This raises a natural question: *which* neighborhood frames arise from some CGS? The following theorem provides an answer to this question.¹

Theorem 2.2. Let $\mathcal{A} = \{a_1, \dots, a_k\}$ be a finite set of agents with $k > 1$.² For a multi-agent neighborhood frame $F = (W, (\Sigma_a)_{a \in \mathcal{A}})$ the following are equivalent:

1. $F = F_{\mathcal{S}}$ for some concurrent game structure $\mathcal{S}$.
2. The following three conditions are satisfied for each world $w \in W$:
 - (a) *Existence of actions:* for all $a \in \mathcal{A}$, $\Sigma_a(w) \neq \emptyset$;
 - (b) *Uniform range:* for all $a, b \in \mathcal{A}$, $\bigcup \Sigma_a(w) = \bigcup \Sigma_b(w)$;
 - (c) *Independence:* for all $s_1 \in \Sigma_{a_1}(w), \dots, s_k \in \Sigma_{a_k}(w) : s_1 \cap \dots \cap s_k \neq \emptyset$.

¹A reviewer asks why it is difficult to extend this result from individual agents to coalitions. The reason is that coalitions can be related to each other by inclusion or overlap; in these cases, there are complex interplays between their effectivity functions.

²When $k = 1$, i.e., in the single-agent case, the characterization is trivial: as the reader can readily verify, a neighborhood frame $F = (W, \Sigma)$ is induced by a CGS if and only if $\Sigma(w)$ is a non-empty set of singletons for every $w \in W$. We therefore focus on the interesting multi-agent case with $k > 1$.

Proof. $1 \Rightarrow 2$ amounts to the claim that the actual effectivity functions of a CGS satisfy conditions (a)-(c) in 2. Condition (a) follows from the fact that $\text{act}(a, w)$ is non-empty by definition. Condition (b) is due to the observation that for every a , $\bigcup \Sigma_a^\mathcal{S}(w)$ coincides with the set of all possible outcomes at w , $O(w)$. Condition (c) is due to the fact that whenever we take neighborhoods $O_{a_i}(w, \tau_{a_i}) \in \Sigma_{a_i}^\mathcal{S}(w)$ for $i = 1, \dots, k$, the function τ defined by $\tau(a_i) = \tau_{a_i}$ is an action profile, and we have $\text{out}(w, \tau) \in O_{a_i}(w, \tau_{a_i})$ for each i .

To show $2 \Rightarrow 1$, suppose $F = (W, (\Sigma_a)_{a \in \mathcal{A}})$ is a neighborhood frame satisfying (a)-(c). Our aim is to define a CGS $\mathcal{S} = (W, \text{act}, \text{out})$ with $F_\mathcal{S} = F$, i.e., such that for each agent a , Σ_a coincides with the actual effectivity function $\Sigma_a^\mathcal{S}$ of agent a in $\mathcal{S}$.

For this, equip the set W with a group structure, so that $(W, e, *, (\cdot)^{-1})$ is a group. Also, fix a well-ordering of W , and for any non-empty $s \subseteq W$, let $\min(s)$ denote the least element of s according to this well-ordering.³ We define the actions available to an agent a at a world w as pairs consisting of a neighborhood for a , and a world:

$$\text{act}(a, w) = \{(s, v) \mid s \in \Sigma_a(w), v \in W\}$$

Condition (a) guarantees that this is a non-empty set, as required by the definition of a CGS.

Now given a world w , an action profile at w can be identified with a sequence $\tau = ((s_1, v_1), \dots, (s_k, v_k))$ where $s_i \in \Sigma_{a_i}(w)$ for $i \leq k$. We define the outcome of τ at w in the following way:

$$\text{out}(w, \tau) = \begin{cases} v_1 * \dots * v_k & \text{if } (v_1 * \dots * v_k) \in (s_1 \cap \dots \cap s_k) \\ \min(s_1 \cap \dots \cap s_k) & \text{otherwise} \end{cases}$$

Condition (c) guarantees that $s_1 \cap \dots \cap s_k$ is non-empty, so that $\min(s_1 \cap \dots \cap s_k)$ exists, which ensures that $\text{out}(w, \tau)$ is well-defined. Note further that, by definition, $\text{out}(w, \tau) \in s_1 \cap \dots \cap s_k$.

We claim that the CGS so defined induces F . To show this, the key step is to prove that for each agent a and action $(s, v) \in \text{act}(a, w)$, the set of outputs that may result from a performing (s, v) is simply s :

$$O_a(w, (s, v)) = s \tag{1}$$

The inclusion $\subseteq$ is clear: whenever τ is an action profile that includes the action (s, v) , the corresponding outcome $\text{out}(w, \tau)$ is bound to be in s by definition of out . To show the inclusion $\supseteq$, consider an arbitrary world $u \in s$. We need to show that there is an action profile τ including (s, v) whose outcome is u .

For simplicity, suppose $a = a_1$ and let us write (s_1, v_1) for (s, v) . Since $u \in s_1$ and $s_1 \in \Sigma_{a_1}(w)$, we have $u \in \bigcup \Sigma_{a_1}(w)$. By condition (b), for each $i > 1$ we have $u \in \bigcup \Sigma_{a_i}(w)$, so we can pick a neighborhood $s_i \in \Sigma_{a_i}(w)$ with $u \in s_i$. Now for $1 < i < k$, pick some worlds $v_i \in W$ arbitrarily. Finally, let v_k be the element $(v_1 * \dots * v_{k-1})^{-1} * u$. Now consider the action profile $\tau = ((s_1, v_1), \dots, (s_k, v_k))$. We have $v_1 * \dots * v_k = (v_1 * \dots * v_{k-1}) * (v_1 * \dots * v_{k-1})^{-1} * u = u$. By the way the sets s_i were chosen we know that $u \in s_1 \cap \dots \cap s_k$, and so by definition of out we have $\text{out}(w, \tau) = u$. Since the profile τ includes the action (s, v) , this shows that $u \in O_a(w, (s, v))$, as required to complete the proof of (1).

Finally, using (1), the actual effectivity function for agent a turns out to be

$$\Sigma_a^\mathcal{S}(w) = \{O_a(w, \tau_a) \mid \tau_a \in \text{act}(a, w)\} = \{O_a(w, (s, v)) \mid s \in \Sigma_a(w), v \in W\} = \{s \mid s \in \Sigma_a(w)\} = \Sigma_a(w)$$

which shows that indeed, $F_\mathcal{S} = F$, as desired. $\square$

³Given the axiom of choice, it is well-known that any set admits both a well-ordering and a group structure. No appeal to the axiom of choice is needed if W is finite or countably infinite, as will be the case in our completeness proof in Section 5.

3 Inquisitive multi-agent neighborhood logic

In this section we present a multi-agent version of inquisitive neighborhood logic, $\text{InqNL}_{\mathcal{A}}$ [12]. We call this logic $\text{InqNL}_{\mathcal{A}}$ where $\mathcal{A} = \{a_1, \dots, a_k\}$ is a finite set of agents. The proofs of the results in this section are the same as for InqNL ; we do not repeat them here, but include pointers to the results in [12].

Syntax. Given a finite set $\mathcal{P}$ of atoms, the set $\mathcal{L}_{\mathcal{A}}$ of $\text{InqNL}_{\mathcal{A}}$ -formulas is given by:

$$\varphi ::= p \mid \perp \mid (\varphi \wedge \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \Rightarrow_a \varphi)$$

where $p \in \mathcal{P}$ and $a \in \mathcal{A}$. As standard in inquisitive logic, we also use the following abbreviations: $\neg\varphi := (\varphi \rightarrow \perp)$; $\top := \neg\perp$; $(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$; $?\varphi := (\varphi \vee \neg\varphi)$. The connective $\vee$, called *inquisitive disjunction*, is regarded as a question-forming disjunction. Thus, e.g., the formula $?p$ (short for $p \vee \neg p$) is regarded as the question *whether or not p*. The operators $\Rightarrow_a$ are indexed versions of the binary modality $\Rightarrow \text{InqNL}$. Two unary modalities (read “window” and “kite”) are defined in terms of $\Rightarrow_a$ as follows:

$$\boxplus_a \varphi := (\top \Rightarrow_a \varphi) \quad \boxminus_a \varphi := \neg(\varphi \Rightarrow_a \perp)$$

An important fragment of the language is given by *declarative formulas* (or *declaratives*, for short), where inquisitive disjunction occurs only in the scope of a modal operator. More formally, the set $\mathcal{L}_{\mathcal{A}}^!$ of declaratives is defined as follows, where φ can be rewritten as any formula from $\mathcal{L}_{\mathcal{A}}$:

$$\alpha ::= p \mid \perp \mid (\alpha \wedge \alpha) \mid (\alpha \rightarrow \alpha) \mid (\varphi \Rightarrow_a \varphi)$$

Intuitively, declaratives stand for statements (as we will see, this is reflected in a key semantic property). We will use the meta-variables φ, ψ, χ for arbitrary formulas, and α, β, γ for declaratives.

The *modal depth* of a formula is defined as usual as the maximum number of nestings of modal operators in it. More precisely, we set: $\text{md}(p) = \text{md}(\perp) = 0$; $\text{md}(\varphi \wedge \psi) = \text{md}(\varphi \rightarrow \psi) = \text{md}(\varphi \vee \psi) = \max(\text{md}(\varphi), \text{md}(\psi))$; $\text{md}(\varphi \Rightarrow_a \psi) = \max(\text{md}(\varphi), \text{md}(\psi)) + 1$. The set of formulas in $\mathcal{L}_{\mathcal{A}}$ with modal depth up to n is denoted $\mathcal{L}_{\mathcal{A}}^n$; the set of declaratives with modal depth up to n is denoted $\mathcal{L}_{\mathcal{A}}^{!n}$.

Semantics. We interpret formulas relative to *multi-agent inhabited neighborhood models* (main-models, for short), which are triples $M = (W, (\Sigma_a)_{a \in \mathcal{A}}, V)$ where $W \neq \emptyset$ is a set of worlds, $\Sigma_a : W \rightarrow \wp \wp(W)$ is a map assigning to each $w \in W$ a set $\Sigma_a(w)$ of non-empty subsets of W (called the *neighborhoods* of w) and $V : \mathcal{P} \rightarrow \wp(W)$ is a valuation function. As customary in inquisitive logic, the semantics of $\text{InqNL}_{\mathcal{A}}$ is not given in terms of truth relative to possible worlds, but instead in terms of *support* relative to sets of possible worlds, called *information states* (or simply *states*).

Definition 3.1 (Support in main-models). Let $M = (W, (\Sigma_a)_{a \in \mathcal{A}}, V)$ be a main-model. The relation of support between information states $s \subseteq W$ and formulas $\varphi \in \mathcal{L}_{\mathcal{A}}$ is defined as follows:

- $M, s \models p \iff s \subseteq V(p)$
- $M, s \models \perp \iff s = \emptyset$
- $M, s \models \varphi \wedge \psi \iff M, s \models \varphi \text{ and } M, s \models \psi$
- $M, s \models \varphi \vee \psi \iff M, s \models \varphi \text{ or } M, s \models \psi$
- $M, s \models \varphi \rightarrow \psi \iff \forall t \subseteq s : M, t \models \varphi \text{ implies } M, t \models \psi$
- $M, s \models \varphi \Rightarrow_a \psi \iff \forall w \in s \forall t \in \Sigma_a(w) : M, t \models \varphi \text{ implies } M, t \models \psi$

All the clauses except for the last one are standard in inquisitive logic (see [10] for discussion), while the clause for the modality $\Rightarrow_a$ is a direct multi-agent adaptation of the one for $\Rightarrow$ in InqNL [12].

Entailment and equivalence are defined in the obvious way: a set of formulas Φ entails a formula ψ , denoted $\Phi \models_{\text{InqNL}_{\mathcal{A}}} \psi$, if for every main-model M and state s that supports all formulas in Φ , s also supports ψ . Two formulas ϕ, ψ are equivalent, denoted $\phi \equiv_{\text{InqNL}_{\mathcal{A}}} \psi$, if they are supported by the same states in every main-model. To ease notation, throughout this section we drop the subscript $\text{InqNL}_{\mathcal{A}}$.

As usual in inquisitive logic, support is persistent (if $M, s \models \phi$ and $t \subseteq s$, then $M, t \models \phi$), and the empty state trivially supports every formula. Moreover, we retrieve a notion of truth at a world w by defining it as support relative to the corresponding singleton $\{w\}$.

Definition 3.2 (Truth). ϕ is *true* at a world w of a main-model M , denoted $M, w \models \phi$, in case $M, \{w\} \models \phi$. The set of worlds at which ϕ is true in a model M is denoted $|\phi|_M$.

Some simple calculations show that the Boolean connectives have their usual truth-functional behavior (for instance, $M, w \models \neg\phi \iff M, w \not\models \phi$), and that modal formulas have the following truth conditions:

- $M, w \models (\phi \Rightarrow_a \psi) \iff \forall s \in \Sigma_a(w) : M, s \models \phi \text{ implies } M, s \models \psi$
- $M, w \models \boxplus\phi \iff \forall s \in \Sigma_a(w) : M, s \models \phi$
- $M, w \models \boxdot\phi \iff \exists s \in \Sigma_a(w) : M, s \models \phi$

Declaratives have a special semantic property: for them, support boils down to truth at each world. In fact, up to equivalence, declaratives are *exactly* the formulas for which this holds.

Definition 3.3 (Truth-conditionality). A formula $\phi \in \mathcal{L}_{\mathcal{A}}$ is *truth-conditional* if for every model M and state s we have $M, s \models \phi \iff \forall w \in s : M, w \models \phi$.

Proposition 3.4 (cf. Prop. 2.7 in [12]). *Every declarative $\alpha \in \mathcal{L}_{\mathcal{A}}^!$ is truth-conditional. Conversely, every truth-conditional formula of $\mathcal{L}_{\mathcal{A}}$ is equivalent to a declarative.*

Not all formulas of $\text{InqNL}_{\mathcal{A}}$ are truth-conditional. As a simple example, consider the formula $?p$ (which abbreviates $p \vee \neg p$). A simple calculation yields the following support conditions:

$$M, s \models ?p \iff p \text{ has the same truth value in all worlds in } s$$

Intuitively, $?p$ is supported at s if it is *settled* in s whether p is true or false. Clearly, $?p$ is supported at every singleton state, and so true at each world, in every model; yet, it is not supported by every state.

Obviously, formulas that are not truth-conditional cannot be equivalent to a declarative. However, any formula ϕ is equivalent to a finite inquisitive disjunction of declaratives, called the *resolutions* of ϕ .

Definition 3.5 (Resolutions). The set $\mathcal{R}(\phi)$ of *resolutions* of a formula $\phi \in \mathcal{L}_{\mathcal{A}}$ is defined as follows:

- $\mathcal{R}(\alpha) = \{\alpha\}$ if α is an atom, $\perp$, or a modal formula ($\phi \Rightarrow_a \psi$);
- $\mathcal{R}(\phi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\phi), \beta \in \mathcal{R}(\psi)\}$;
- $\mathcal{R}(\phi \vee \psi) = \mathcal{R}(\phi) \cup \mathcal{R}(\psi)$;
- $\mathcal{R}(\phi \rightarrow \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\phi)} (\alpha \rightarrow f(\alpha)) \mid f : \mathcal{R}(\phi) \rightarrow \mathcal{R}(\psi)\}$.

Note that $\mathcal{R}(\phi)$ is always a finite non-empty set of declaratives, and for any declarative α , $\mathcal{R}(\alpha) = \{\alpha\}$.

Proposition 3.6 (Normal form, cf. Prop. 2.9 in [12]). *For any $\phi \in \mathcal{L}_{\mathcal{A}}$, $\phi \equiv \bigvee \mathcal{R}(\phi)$.*

In terms of resolutions we may also define a further modal operator, $\Box_a$, which will play a role below:

$$\Box_a \varphi := \bigvee \{ \Box_a \alpha \mid \alpha \in \mathcal{R}(\varphi) \}$$

This operator has been considered in previous work on inquisitive modal logic [15, 7, 11].⁴ In particular, it has been argued to give a natural generalization of the standard knowledge modality of epistemic logic, allowing for a uniform analysis of knowledge ascriptions involving declarative complements (*knowing that*) and interrogative complements (*knowing whether/who/what*, etc).

Note that $\Box_a \varphi$ is a declarative. A simple calculation yields the following truth conditions:

$$\bullet \quad M, w \models \Box_a \varphi \iff M, \bigcup \Sigma_a(w) \models \varphi$$

To conclude, we mention a fact that will be useful later on: given that we start with a finite set of atoms, there are only finitely many formulas of a given modal depth, up to logical equivalence.

Proposition 3.7 (cf. Corollary 5.17 in [12]). *The quotient of $\mathcal{L}_{\mathcal{A}}^n$ under equivalence is finite.*

4 Inquisitive action logic

In this section we turn to the logic which is the focus of this paper: *inquisitive action logic*, InqAL. This logic is a special case of the multi-agent inquisitive neighborhood logic InqNL _{$\mathcal{A}$} introduced in the previous section, obtained by focusing on a particular class of neighborhood models: those whose neighborhood functions represent actual effectivity functions of some concurrent game structure. For simplicity, we focus on the multi-agent case, i.e., we assume from now on that $\mathcal{A}$ contains at least two agents; for some comments on the trivial, but somewhat special, single-agent case, see Footnote 7.

Definition 4.1. A *concurrent game model* (or *cg-model*, for short) is a pair $\mathcal{M} = (\mathcal{S}, V)$ consisting of a concurrent game structure $\mathcal{S} = (W, \text{act}, \text{out})$ and a valuation function $V : \mathcal{P} \rightarrow \wp(W)$.

A concurrent game model $\mathcal{M} = (\mathcal{S}, V)$ with $\mathcal{S} = (W, \text{act}, \text{out})$ determines a corresponding main-model $M_{\mathcal{M}} = (W, (\Sigma_a^{\mathcal{S}})_{a \in \mathcal{A}}, V)$, whose neighborhood functions are the actual effectivity functions of agents. We can thus interpret the formulas of $\mathcal{L}_{\mathcal{A}}$ relative to a cg-model, via its associated main-model.

Definition 4.2. Let $\mathcal{M}$ be a cg-model, s a state in $\mathcal{M}$, and $\varphi \in \mathcal{L}_{\mathcal{A}}$. We write $\mathcal{M}, s \models \varphi$ if $M_{\mathcal{M}}, s \models \varphi$.

Entailment and equivalence in InqAL, denoted $\models_{\text{InqAL}}$ and $\equiv_{\text{InqAL}}$ respectively, are defined in the same way as for InqNL _{$\mathcal{A}$} , but with respect to cg-models instead of main-models. Note that, being determined by a smaller class of models, the logic InqAL is an extension of InqNL _{$\mathcal{A}$} .

In the setting of concurrent game models, our modal formulas take on a specific significance. We will illustrate this by means of some examples (for the sake of space, we omit the simple calculations involved). First, consider a modal formula of the form $\Diamond_a \alpha$, where $\alpha \in \mathcal{L}_{\mathcal{A}}^1$. We have:

$$\mathcal{M}, w \models \Diamond_a \alpha \iff \exists \tau_a \in \text{act}(a, w) : O_a(w, \tau_a) \subseteq |\alpha|_{\mathcal{M}}$$

Thus, $\Diamond_a \alpha$ expresses the fact that agent a has an action available that guarantees the truth of α (i.e., if that action is executed, the outcome will make α true, regardless what the other agents do). In short, $\Diamond_a \alpha$ says that a can force α , recovering the standard semantics for ability familiar from [5] and [26].⁵

⁴In this previous work, $\Box$ is taken as primitive, while here we take it as a defined operator. Either choice has advantages. The advantage of our present choice is that having fewer primitives simplifies the completeness proof below.

⁵In fact, it is possible to show that the $\Diamond$ -fragment of our language is equi-expressive with the fragment of coalition logic that contains only modalities for individual agents. See §3 of [12] for an analogous result in the case of InqNL.

As a second example, consider the modal formula $(\alpha \Rightarrow_a \beta)$, where $\alpha, \beta \in \mathcal{L}_{\mathcal{A}}^!$. We have:

$$\mathcal{M}, w \models (\alpha \Rightarrow_a \beta) \iff \forall \tau_a \in \text{act}(a, w) : O_a(w, \tau_a) \subseteq |\alpha|_{\mathcal{M}} \text{ implies } O_a(w, \tau_a) \subseteq |\beta|_{\mathcal{M}}$$

What this formula expresses is that, in order to force α , agent a necessarily has to force β as well. By negating such statements, we can express the fact that the agent can force certain outcomes without precluding others. Consider, e.g., the formula $\neg(\alpha \Rightarrow_a \neg\beta)$. We have:

$$\mathcal{M}, w \models \neg(\alpha \Rightarrow_a \neg\beta) \iff \exists \tau_a \in \text{act}(a, w) : O_a(w, \tau_a) \subseteq |\alpha|_{\mathcal{M}} \text{ and } O_a(w, \tau_a) \cap |\beta|_{\mathcal{M}} \neq \emptyset$$

The formula expresses the fact that agent a can perform an action which forces α without precluding β . More generally, a formula of the form $\neg(\alpha \Rightarrow_a \neg\beta_1 \vee \dots \vee \neg\beta_n)$ expresses the fact that it is possible for a to force α without precluding any of $\beta_1, \dots, \beta_n$. In this way, the central properties that the Socially Friendly Coalition Logic of Goranko and Enqvist [20] is designed to capture can be expressed in InqAL.

Let us now consider formulas involving the modalities $\boxplus_a$ and $\boxminus_a$. First consider the case in which the argument is a declarative $\alpha \in \mathcal{L}_{\mathcal{A}}^!$. In this case, $\mathcal{R}(\alpha) = \{\alpha\}$, and so the two modalities coincide: by definition of $\boxminus_a$, we have $\boxminus_a \alpha = \boxplus_a \alpha$. Recall from Section 2 that, in the case of actual effectivity functions, the union of the neighborhoods simply coincides with the set $O(w)$ of all outcomes possible at w : for any agent a , $\bigcup \Sigma_a^{\mathcal{S}}(w) = O(w)$. Using this fact, we obtain the following:

$$\mathcal{M}, w \models \boxminus_a \alpha \iff \mathcal{M}, w \models \boxplus_a \alpha \iff O(w) \subseteq |\alpha|_{\mathcal{M}}$$

Thus, regardless of the agent a , the (identical) formulas $\boxplus_a \alpha$ and $\boxminus_a \alpha$ express the fact that α is *unavoidable*: it will be true at the next stage regardless what the agents do.

When the argument is not a declarative, however, the modalities $\boxminus_a$ and $\boxplus_a$ come apart. Let us illustrate this with the case of a polar question $?\alpha$, where $\alpha \in \mathcal{L}_{\mathcal{A}}^!$. Since $\mathcal{R}(?\alpha) = \{\alpha, \neg\alpha\}$, the formula $\boxminus_a ?\alpha$ amounts to $\boxplus_a \alpha \vee \boxplus_a \neg\alpha$, which says that either α is unavoidably true at the next stage, or it is unavoidably false. More formally:

$$\mathcal{M}, w \models \boxminus_a ?\alpha \iff \alpha \text{ has the same truth value in all worlds in } O(w)$$

Thus, $\boxminus_a ?\alpha$ says that whether α will be true at the next stage is *predetermined*, i.e., settled *a priori* regardless of the actions of the agents. By contrast, for the modal formula $\boxplus_a ?\alpha$ we have the following:

$$\mathcal{M}, w \models \boxplus_a ?\alpha \iff \forall \tau_a \in \text{act}(a, w) : \alpha \text{ has the same truth value in all worlds in } O_a(w, \tau_a)$$

Thus, what $\boxplus_a ?\alpha$ expresses is that once we fix the action of a , the truth value of α at the next stage is settled: the actions of other agents cannot affect it. These findings generalize: if φ denotes a question, then $\boxminus_a \varphi$ expresses the fact that φ is settled *a priori*, regardless of what the agents do, while $\boxplus_a \varphi$ expresses the fact that φ is settled once we fix the action of agent a .

We can now see how the analysis of agentive determination we suggested in the introduction can be formalized in InqAL. Our guiding idea was this: at a certain stage in a process, an agent a determines a question φ if (i) the answer to φ is not settled *a priori* but (ii) it becomes settled once we fix the action of agent a . In our logic, (i) is captured by $\neg\boxminus_a \varphi$, and (ii) by $\boxplus_a \varphi$. We can thus define a modality $\boxtimes_a$ capturing agentive determination in the following way:

$$\boxtimes_a \varphi := \neg\boxminus_a \varphi \wedge \boxplus_a \varphi$$

For instance, the claim that agent a determines whether α will be true at the next stage is expressed by $\boxtimes_a ?\alpha := \neg\boxminus_a ?\alpha \wedge \boxplus_a ?\alpha$, which says that the truth value of α is not predetermined (i.e., not the same

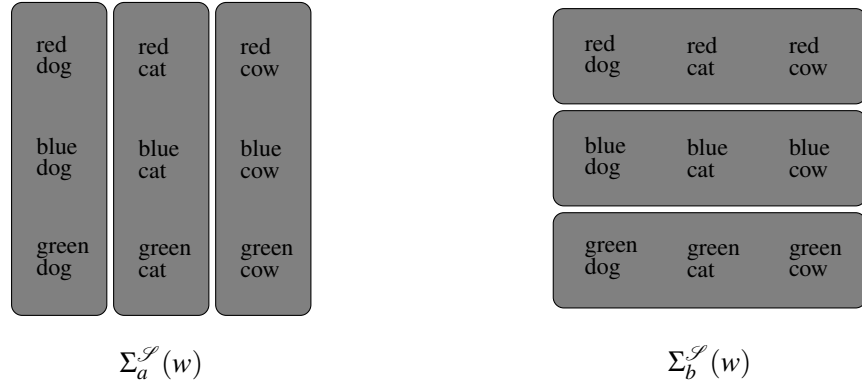


Figure 1: The neighborhoods (i.e., outcome sets) associated to Alice’s actions (left) and Bob’s action (right) in our toy example.

in all possible outcomes), but it is determined once we fix the action of a (i.e., it is constant within each neighborhood for a). This is, in my view, a very natural analysis of the determination claim.⁶

To illustrate this analysis further, consider the scenario from the introduction, where Alice (a) and Bob (b) are creating clay animals together. Let’s say that there are three shapes that Alice is able to mould—cat, dog, and cow—and three colors available to Bob—red, blue, and green. A simple modeling of the scenario is one where Alice has three actions available to her: τ_{cat} (mould a cat), τ_{dog} (mould a dog), and τ_{cow} (mould a cow); similarly, Bob has three actions: τ_{red} (paint red), τ_{blue} (paint blue), τ_{green} (paint green). Each joint action by Alice and Bob, for instance $\tau_{\text{cat}} \tau_{\text{red}}$, results in a specific outcome, for instance $\text{out}(w, \tau_{\text{cat}} \tau_{\text{red}}) = \text{red cat}$. In total, there are $3 \times 3 = 9$ possible outcomes (red cat, blue dog, etc.) which make up the total set $O(w)$. For each agent, fixing an action leaves us with only 3 possible outcomes; for instance, $O_a(w, \tau_{\text{cat}}) = \{\text{red cat}, \text{blue cat}, \text{green cat}\}$, while $O_b(w, \tau_{\text{red}}) = \{\text{red cat}, \text{red dog}, \text{red cow}\}$. The neighborhoods corresponding to these outcome sets for Alice and Bob are depicted in Figure 1.

Now we can consider a language with six propositional atoms, cat, dog, cow, red, blue, green, with the obvious interpretation (e.g., cat expresses the proposition “the animal is a cat”). Using inquisitive disjunction, we can build two formulas shape and color expressing, respectively, the questions “what the shape is” and “what the color is”:

$$\text{shape} := (\text{cat} \vee \text{dog} \vee \text{cow}) \quad \text{color} := (\text{red} \vee \text{blue} \vee \text{green})$$

An information state s in our model supports the question shape just in case in all worlds in s , the animal has the same shape; similarly, s supports color if in all worlds in s , the animal has the same color.

By embedding these questions under modalities we can now describe what aspects of the outcome Alice and Bob respectively determine through their actions: Alice determines the shape ($\boxtimes_a \text{shape}$) but not the color ($\neg \boxtimes_a \text{color}$), while Bob determines the color ($\boxtimes_b \text{color}$) but not the shape ($\neg \boxtimes_b \text{shape}$).

In order to see that $\boxtimes_a \text{shape}$ (i.e., $\neg \boxtimes_a \text{shape} \wedge \boxplus_a \text{shape}$) is true at our world w , we reason as follows. First, since the shape of the animal is not constant across all possible outcomes, we have $\mathcal{M}, O(w) \not\models \text{shape}$, which ensures $\mathcal{M}, w \models \neg \boxtimes_a \text{shape}$; this captures the fact that in our scenario, the shape is not

⁶Interestingly, exactly the same modal pattern, $\neg \boxtimes_a \phi \wedge \boxplus_a \phi$ is argued in inquisitive epistemic logic to capture the idea that an agent *wonders about*, or is *interested in*, a certain question [15]. It is striking that such seemingly different notions across different modal domains plausibly share the same logical structure. Revealing this common structure is one of the payoffs of the formal analysis of question-oriented modal notions made possible by inquisitive modal logic.

predetermined, but depends on the actions of the agents. Second, within each of the three outcome sets $O_a(w, \tau_{cat}), O_a(w, \tau_{dog}), O_a(w, \tau_{cow})$, corresponding to the three actions for a , the shape of the animal is constant; therefore, each of these sets supports shape, ensuring $\mathcal{M}, w \models \boxplus_a \text{shape}$; this captures the fact that once we fix Alice's action, the shape of the outcome is settled, regardless of what Bob does.

By contrast, $\boxtimes_a \text{color}$ (i.e., $\neg \Box_a \text{color} \wedge \boxplus_a \text{color}$) is false at w , since its second conjunct is false. Consider any outcome set for a , for instance $O_a(w, \tau_{cat})$. The color of the animal is not constant across this set, and so this set does not support color. Since not all the outcome sets for a (in fact, none of them) support color, $\mathcal{M}, w \not\models \boxplus_a \text{color}$. This reflects the fact even if we fix Alice's action, the color of the outcome is not settled—it depends in part (and in fact, in our case, completely) on what Bob does.

5 Axiomatization and decidability

We start by presenting an axiomatization of $\text{InqNL}_{\mathcal{A}}$, which is the basis for our axiomatization of InqAL .

The propositional axioms include instances of axioms for intuitionistic propositional logic, with $\vee$ in the role of intuitionistic disjunction, and all instances of the following, where $\alpha \in \mathcal{L}_{\mathcal{A}}^!$ and $\varphi, \psi \in \mathcal{L}_{\mathcal{A}}$:

- $\neg\neg\alpha \rightarrow \alpha$ (Declarative double negation)
- $(\alpha \rightarrow \varphi \vee \psi) \rightarrow (\alpha \rightarrow \varphi) \vee (\alpha \rightarrow \psi)$ (Split)

The modal axioms include all instances of the following schemata, capturing the behavior of the modalities $\Rightarrow_a$ as strict conditionals (see [12] for discussion of these axioms in the uni-modal case and [22] for a more general study of strict conditionals on an intuitionistic basis):

- $(\varphi \Rightarrow_a \psi) \wedge (\psi \Rightarrow_a \chi) \rightarrow (\varphi \Rightarrow_a \chi)$ (Transitivity)
- $(\varphi \Rightarrow_a \psi) \wedge (\varphi \Rightarrow_a \chi) \rightarrow (\varphi \Rightarrow_a (\psi \wedge \chi))$ (Right conjunction)
- $(\varphi \Rightarrow_a \chi) \wedge (\psi \Rightarrow_a \chi) \rightarrow ((\varphi \vee \psi) \Rightarrow_a \chi)$ (Left disjunction)

The inference rules are *modus ponens* ($\varphi, \varphi \rightarrow \psi / \psi$) and *conditional necessitation* ($\varphi \rightarrow \psi / \varphi \Rightarrow_a \psi$).

If $\psi \in \mathcal{L}_{\mathcal{A}}$ is derivable in this system we write $\vdash_N \psi$. For $\Phi, \Psi \subseteq \mathcal{L}_{\mathcal{A}}$, we write $\Phi \vdash_N \Psi$ in case for some finite $\Phi_0 \subseteq \Phi$ and $\Psi_0 \subseteq \Psi$ we have $\vdash_N \bigwedge \Phi_0 \rightarrow \bigvee \Psi_0$. Instead of $\{\varphi_1, \dots, \varphi_n\} \vdash_N \{\psi_1, \dots, \psi_m\}$ we write simply $\varphi_1, \dots, \varphi_n \vdash_N \psi_1, \dots, \psi_m$. We write $\varphi \dashv\vdash_N \psi$ if we have both $\varphi \vdash_N \psi$ and $\psi \vdash_N \varphi$.

The following soundness and strong completeness theorem generalizes the one for InqNL . The proof is the obvious multi-agent adaptation of the one for InqNL given in [12].

Theorem 5.1. *For all $\Phi \subseteq \mathcal{L}_{\mathcal{A}}$ and $\psi \in \mathcal{L}_{\mathcal{A}}$, $\Phi \models_{\text{InqNL}_{\mathcal{A}}} \psi \iff \Phi \vdash_N \psi$.*

To obtain a system for InqAL , we add three axiom schemata reflecting the properties of actual effectivity functions we identified in Theorem 2.2. For all declaratives α , all formulas $\varphi_1, \dots, \varphi_n$, all agents a, b , and all pairwise distinct agents $a_1, \dots, a_{n+1}$, the following are axioms:⁷

- $\boxplus_a \top$ (Existence of actions)
- $\boxplus_a \alpha \leftrightarrow \boxplus_b \alpha$ ⁸ (Uniform range)

⁷Recall that we assume that $\mathcal{A}$ contains at least two agents. If $\mathcal{A}$ contains a single agent a , an axiomatization of InqAL is obtained by following two axioms: (i) $\boxplus_a \top$ (existence of actions); (ii) $\boxplus_a ?\alpha$ for any declarative α (determinacy). The former captures the condition that $\Sigma_a(w) \neq \emptyset$, the latter the condition that each $s \in \Sigma_a(w)$ is a singleton. As mentioned in Footnote 2, these conditions guarantee that Σ_a is the actual effectivity function of some CGS. The completeness proof for this case follows the one below, except that the proof of the Lemma 5.12 needs to be adapted. We leave this as an exercise.

⁸Note that the restriction to declaratives in this axiom is crucial; for instance, $\boxplus_a ?p \leftrightarrow \boxplus_b ?p$ is not valid in InqAL . Alternatively, we may take as axioms all formulas of the form $\Box_a \varphi \leftrightarrow \Box_b \varphi$; in this case, no restriction is needed.

$$\bullet \Diamond_{a_1} \varphi_1 \wedge \cdots \wedge \Diamond_{a_n} \varphi_n \rightarrow \neg \Diamond_{a_{n+1}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n) \quad (\text{Independence})$$

We write $\vdash_A \psi$ if ψ is derivable in this extended system. Similarly, the notations $\Phi \vdash_A \Psi$ and $\varphi \dashv\vdash_A \psi$ are defined as for $\vdash_N$, but with reference to the extended axiom system. Our task is now to prove that this extended system is sound and (weakly) complete for InqAL.⁹

Theorem 5.2 (Soundness and completeness for InqAL). *For all $\varphi \in \mathcal{L}_{\mathcal{A}}$, $\models_{\text{InqAL}} \varphi \iff \vdash_A \varphi$.*

For soundness, we just have to check that the axioms are valid and the inference rules preserve validity. We leave this as an exercise, noting only that the three extra axioms for InqAL owe their validity to the three corresponding properties of actual effectivity functions listed in Theorem 2.2. Towards completeness, we adapt a finite canonical model construction from [12] (in turn building on ideas from [7]).

Definition 5.3. For $n \in \mathbb{N}$, an *n-bounded complete theory of declaratives* (or *nCTD* for short), is a set of declaratives $\Gamma \subseteq \mathcal{L}_{\mathcal{A}}^n$ satisfying three conditions: (i) deductive closure relative to $\mathcal{L}_{\mathcal{A}}^n$: if $\Gamma \vdash_A \alpha$ and $\alpha \in \mathcal{L}_{\mathcal{A}}^n$, then $\alpha \in \Gamma$; (ii) consistency: $\perp \notin \Gamma$; and (iii) completeness: for all $\alpha \in \mathcal{L}_{\mathcal{A}}^n$, $\alpha \in \Gamma$ or $\neg \alpha \in \Gamma$. We denote the set of all *nCTDs* by $\mathcal{K}_n$.

We now show that each $\mathcal{K}_n$ is a non-empty finite set, and that the sets $\mathcal{K}_n$ and $\mathcal{K}_m$ are disjoint for $n \neq m$.

Lemma 5.4. *If $\Delta \subseteq \mathcal{L}_{\mathcal{A}}^n$ and $\Delta \not\vdash_A \perp$, then $\Delta \subseteq \Gamma$ for some $\Gamma \in \mathcal{K}_n$. In particular, $\mathcal{K}_n \neq \emptyset$.*

Proof. A simple adaptation of the usual saturation argument. □

Lemma 5.5. *Each set $\mathcal{K}_n$ is finite.*

Proof. Since $\vdash_A$ extends $\vdash_N$, the equivalence relation $\dashv\vdash_N$ refines $\dashv\vdash_A$. By Theorem 5.1, $\dashv\vdash_N$ coincides with the semantic equivalence relation $\equiv_{\text{InqNL}_{\mathcal{A}}}$. By Prop. 3.7, the quotient of $\mathcal{L}_{\mathcal{A}}^n / \equiv_{\text{InqNL}_{\mathcal{A}}}$ is finite, and a fortiori, so is the quotient $\mathcal{L}_{\mathcal{A}}^n / \dashv\vdash_A$. Since an *nCTD* is a subset of $\mathcal{L}_{\mathcal{A}}^n$ which is deductively closed, it is fully determined by the equivalence classes of its elements modulo $\dashv\vdash_A$, i.e., it is fully determined by a subset of the quotient $\mathcal{L}_{\mathcal{A}}^n / \dashv\vdash_A$. Since these subsets are finitely many, so are the *nCTDs*. □

Lemma 5.6. *$\mathcal{K}_n \cap \mathcal{K}_m = \emptyset$ for $n \neq m$.*

Proof. Let a be an arbitrary agent and let $\boxplus_a^h \top$ abbreviate $\boxplus_a \dots \boxplus_a \top$ with h occurrences of $\boxplus_a$. If $\Gamma \in \mathcal{K}_n$, by deductive closure relative to $\mathcal{L}_{\mathcal{A}}^n$ we have $\boxplus_a^h \top \in \Gamma \iff h \leq n$, and so $n = \max\{h \mid \boxplus_a^h \top \in \Gamma\}$. As a consequence, if $\Gamma \in \mathcal{K}_n \cap \mathcal{K}_m$ then $n = \max\{h \mid \boxplus_a^h \top \in \Gamma\} = m$. □

We now define for each $n \in \mathbb{N}$ a canonical model suitable for formulas in $\mathcal{L}_{\mathcal{A}}^n$.

Definition 5.7. For $n \in \mathbb{N}$, we define the main-model $M_n = (W_n, (\Sigma_a^n)_{a \in \mathcal{A}}, V_n)$ as follows: $W_n = \bigcup_{m \leq n} \mathcal{K}_m$; $V_n(p) = \{\Gamma \in W_n \mid p \in \Gamma\}$; finally, the neighborhood map Σ_a^n is defined as follows:

- if $\Gamma \in \mathcal{K}_0$ then $\Sigma_a^n(\Gamma) = \{\mathcal{K}_0\}$
- if $\Gamma \in \mathcal{K}_{m+1}$ then $\Sigma_a^n(\Gamma) = \{S \subseteq \mathcal{K}_m, S \neq \emptyset \mid \text{for all } (\psi \Rightarrow_a \chi) \in \Gamma : \bigcap S \vdash_A \psi \text{ implies } \bigcap S \vdash_A \chi\}$

For any $n \in \mathbb{N}$, the model M_n is finite by Lemma 5.5. The following four lemmas play a key role in the completeness proof. The proofs are simple adaptations of those of the corresponding results for InqNL (Lemmas 6.10, 6.12, 6.15, and 6.16 in [12]). We provide the details in Appendix A for completeness.

⁹An interesting question that we will leave open is whether Theorem 5.2 can be extended to a strong completeness result.

Lemma 5.8 (Intersection Lemma).

For a set $\Delta \subseteq \mathcal{L}_{\mathcal{A}}^{ln}$, define $S_{\Delta}^n = \{\Gamma \in \mathcal{K}_n \mid \Delta \subseteq \Gamma\}$. For every $\varphi \in \mathcal{L}_{\mathcal{A}}^n$ we have $\Delta \vdash_A \varphi \iff \bigcap S_{\Delta}^n \vdash_A \varphi$.¹⁰ In particular, since $S_{\emptyset}^n = \mathcal{K}_n$, for every $\varphi \in \mathcal{L}_{\mathcal{A}}^n$ we have $\vdash_A \varphi \iff \bigcap \mathcal{K}_n \vdash_A \varphi$.

Lemma 5.9 (Existence Lemma).

For $m \leq n$, if $\Gamma \in \mathcal{K}_m$ and $\neg(\varphi \Rightarrow_a \psi) \in \Gamma$, then there is $S \in \Sigma_a^n(\Gamma)$ such that $\bigcap S \vdash_A \varphi$ and $\bigcap S \not\vdash_A \psi$.

Lemma 5.10 (Range Lemma).

For all $\Gamma \in \mathcal{K}_m$ with $m > 0$ we have $\bigcup \Sigma_a^n(\Gamma) = \{\Gamma' \in \mathcal{K}_{m-1} \mid \forall \alpha \in \mathcal{L}_{\mathcal{A}}^! : \boxplus_a \alpha \in \Gamma \text{ implies } \alpha \in \Gamma'\}$.

Lemma 5.11 (Support Lemma).

For all $m \leq n$, all non-empty states $S \subseteq \mathcal{K}_m$, and all formulas $\varphi \in \mathcal{L}_{\mathcal{A}}^m$: $M_n, S \models \varphi \iff \bigcap S \vdash_A \varphi$.

The part of the proof which is genuinely novel, and crucial for our purposes, lies in showing that the canonical models M_n so constructed are induced by some corresponding concurrent game models.

Lemma 5.12 (Representation Lemma). For each $n \in \mathbb{N}$ there is a CGM $\mathcal{M}_n$ such that $M_{\mathcal{M}_n} = M_n$.

Proof. It suffices to show that the maps $(\Sigma_a^n)_{a \in \mathcal{A}}$ of our canonical model satisfy the conditions (a)-(c) of Theorem 2.2. Consider a world $\Gamma \in W_n$. We have $\Gamma \in \mathcal{K}_m$ for some $m \leq n$. If $m = 0$ then $\Sigma_a^n(\Gamma) = \{\mathcal{K}_0\}$ for every agent a and conditions (a)-(c) are obviously satisfied. So, we may assume $m > 0$.

(a) *Existence of actions.* Since Γ contains the axiom $\boxplus_a \top$, which is short for $\neg(\top \Rightarrow_a \perp)$, the existence of some $S \in \Sigma_a^n(\Gamma)$ follows directly from Lemma 5.9.

(b) *Uniform range.* Take any agents $a, b \in \mathcal{A}$. For any declarative α , $\boxplus_a \alpha \leftrightarrow \boxplus_b \alpha$ is an axiom, so $\boxplus_a \alpha \vdash_A \boxplus_b \alpha$. Since the formulas $\boxplus_a \alpha$ and $\boxplus_b \alpha$ also have the same modal depth, Γ contains one iff it contains the other. Now using Lemma 5.10 for both agents a and b we have:

$$\begin{aligned} \bigcup \Sigma_a^n(\Gamma) &= \{\Gamma' \in \mathcal{K}_{m-1} \mid \forall \alpha \in \mathcal{L}_{\mathcal{A}}^! : \boxplus_a \alpha \in \Gamma \text{ implies } \alpha \in \Gamma'\} \\ &= \{\Gamma' \in \mathcal{K}_{m-1} \mid \forall \alpha \in \mathcal{L}_{\mathcal{A}}^! : \boxplus_b \alpha \in \Gamma \text{ implies } \alpha \in \Gamma'\} = \bigcup \Sigma_b^n(\Gamma) \end{aligned}$$

(c) *Independence.* Take any neighborhoods $S_1 \in \Sigma_{a_1}^n(\Gamma), \dots, S_k \in \Sigma_{a_k}^n(\Gamma)$. We must prove $S_1 \cap \dots \cap S_k \neq \emptyset$. We start by proving the following claim:

$$\bigcap S_1 \cup \dots \cup \bigcap S_k \not\vdash_A \perp \quad (*)$$

Towards a contradiction, suppose this is false. Then there are formulas $\alpha_1 \in \bigcap S_1, \dots, \alpha_k \in \bigcap S_k$ such that $\alpha_1, \dots, \alpha_k \vdash_A \perp$ (it suffices to consider a single α_i from each $\bigcap S_i$, since $\bigcap S_i$ is closed under conjunction).

By definition of $\Sigma_{a_i}^n$, each S_i is non-empty and thus $\bigcap S_i \not\vdash_A \perp$. Since $S_i \in \Sigma_{a_i}^n(\Gamma)$ and $\bigcap S_i \vdash_A \alpha_i$ but $\bigcap S_i \not\vdash_A \perp$, again by definition of $\Sigma_{a_i}^n$ it follows that $(\alpha_i \Rightarrow_{a_i} \perp) \notin \Gamma$. Also, since $\Gamma \in \mathcal{K}_m$, by definition of $\Sigma_{a_i}^n(\Gamma)$ we have $S_i \subseteq \mathcal{K}_{m-1}$, which means that $\text{md}(\alpha_i) \leq m-1$, and thus $\text{md}(\alpha_i \Rightarrow_{a_i} \perp) \leq m$. Since $\Gamma \in \mathcal{K}_m$ and $(\alpha_i \Rightarrow_{a_i} \perp) \in \mathcal{L}_{\mathcal{A}}^{!m}$, from $(\alpha_i \Rightarrow_{a_i} \perp) \notin \Gamma$ we can conclude by the completeness condition on Γ that $\neg(\alpha_i \Rightarrow_{a_i} \perp) \in \Gamma$, that is, $\boxplus_{a_i} \alpha_i \in \Gamma$. So, for each $i \leq k$ we have $\boxplus_{a_i} \alpha_i \in \Gamma$. But now, by deductive closure relative to $\mathcal{L}_{\mathcal{A}}^{!m}$, Γ contains both the axiom

$$\boxplus_1 \alpha_1 \wedge \dots \wedge \boxplus_{k-1} \alpha_{k-1} \rightarrow \neg \boxplus_k \neg(\alpha_1 \wedge \dots \wedge \alpha_{k-1})$$

and its antecedent. So, Γ must contain the consequent, $\neg \boxplus_k \neg(\alpha_1 \wedge \dots \wedge \alpha_{k-1})$.

¹⁰To be fully precise, for $S \subseteq \mathcal{K}_n$, we define $\bigcap S = \{\alpha \in \mathcal{L}_{\mathcal{A}}^{ln} \mid \alpha \in \Gamma \text{ for all } \Gamma \in S\}$. This definition implies that $\bigcap \emptyset = \mathcal{L}_{\mathcal{A}}^{ln}$, ensuring that the lemma holds also when we have $\Delta \vdash_A \perp$, and therefore $S_{\Delta}^n = \emptyset$. In this case, the sets of formulas Δ and $\bigcap S_{\Delta}^n = \bigcap \emptyset = \mathcal{L}_{\mathcal{A}}^{ln}$ indeed derive the same formulas from $\mathcal{L}_{\mathcal{A}}^n$: all of them.

On the other hand, from $\alpha_1, \dots, \alpha_k \vdash_A \perp$ we have that $\alpha_k \vdash_A \neg(\alpha_1 \wedge \dots \wedge \alpha_{k-1})$, and since $\alpha_k \in \bigcap S_k$, also $\neg(\alpha_1 \wedge \dots \wedge \alpha_{k-1}) \in \bigcap S_k$. Reasoning again as in the previous paragraph, from this and the fact that $S_k \in \Sigma_{\alpha_k}^n(\Gamma)$ we may conclude that Γ contains $\bigoplus_k \neg(\alpha_1 \wedge \dots \wedge \alpha_{k-1})$.

So, Γ contains both $\bigoplus_k \neg(\alpha_1 \wedge \dots \wedge \alpha_{k-1})$ and the negation of this formula; by deductive closure, it must contain $\perp$. But this is a contradiction, since Γ is consistent by assumption. Thus, $(*)$ is true.

Thus, $\bigcap S_1 \cup \dots \cup \bigcap S_k$ is a consistent subset of $\mathcal{L}_{\mathcal{A}}^{m-1}$. By Lemma 5.4 there is a $\Delta \in \mathcal{K}_{m-1}$ with $\bigcap S_1 \cup \dots \cup \bigcap S_k \subseteq \Delta$. To conclude, we show that $\Delta \in S_1 \cap \dots \cap S_k$, which implies that $S_1 \cap \dots \cap S_k \neq \emptyset$.

Take any $i \leq k$. We know that $\bigcap S_i \subseteq \Delta$ and we want to prove $\Delta \in S_i$. Towards a contradiction, suppose that $\Delta \notin S_i$. By Lemma 5.5, we know that S_i is a finite set, $S_i = \{\Theta_1, \dots, \Theta_\ell\}$ for some ℓ and $\Theta_1, \dots, \Theta_\ell \in \mathcal{K}_{m-1}$. For $j \leq \ell$, $\Theta_j \neq \Delta$ and therefore we can find $\theta_j \in \mathcal{L}_{\mathcal{A}}^{m-1}$ with $\theta_j \in \Theta_j$ and $\neg\theta_j \in \Delta$. Now let $\theta = \theta_1 \vee \dots \vee \theta_\ell$. We have $\theta \in \bigcap S_i$ but $\theta \notin \Delta$, which is a contradiction since $\bigcap S_i \subseteq \Delta$. $\square$

Finally, we can now put everything together and establish completeness.

Proof of Theorem 5.2, left-to-right. Suppose $\not\vdash_A \varphi$ and let $n = \text{md}(\varphi)$. By Lemma 5.8, we have that $\bigcap \mathcal{K}_n \not\vdash_A \varphi$, which by Lemma 5.11 implies $M_n, \mathcal{K}_n \not\models \varphi$. By Lemma 5.12, there is a CGM $\mathcal{M}_n$ that induces M_n , and so we have $\mathcal{M}_n, \mathcal{K}_n \not\models \varphi$. Since φ can be falsified in some CGM, $\not\models_{\text{InqAL}} \varphi$. $\square$

Note that the proof yields, for any φ which is not provable in $\vdash_A$, a *finite* countermodel. Therefore, our proof also establishes the finite model property of InqAL. In turn, the finite model property together with our (recursive) axiomatization yields the decidability of InqAL.

Corollary 5.13 (Finite model property). *If $\not\models_{\text{InqAL}} \varphi$, φ can be refuted within a finite cg-model.*

Corollary 5.14 (Decidability). *The problem of deciding if a given $\varphi \in \mathcal{L}_{\mathcal{A}}$ is valid in InqAL is decidable.*

6 Connection with Socially Friendly Coalition Logic

In this section we relate InqAL to Socially Friendly Coalition Logic (SFCL) [20], a generalization of coalition logic [26] developed by Goranko and Enqvist based on instantial neighborhood logic [4]. Modal formulas in SFCL have the form $[C](\sigma; \pi_1, \dots, \pi_n)$ where $C \subseteq \mathcal{A}$ and $\sigma, \pi_1, \dots, \pi_n$ are formulas. The primitive connectives are $\neg$ and $\vee$. To compare it with InqAL, we restrict to the individual-agent fragment of SFCL, where C is a singleton $\{a\}$, which we may identify with the agent a .¹¹ Formulas are interpreted relative to CGMs in a standard truth-conditional fashion. The clause for modal formulas is as follows:

$$\mathcal{M}, w \models [a](\sigma; \pi_1, \dots, \pi_n) \iff \exists \tau_a \in \text{act}(a, w) : O_a(w, \tau_a) \subseteq |\sigma|_{\mathcal{M}} \text{ and } O_a(w, \tau_a) \cap |\pi_i|_{\mathcal{M}} \neq \emptyset \text{ for } i \leq n$$

Based on the observations in §4, we can define a translation $(\cdot)^*$ from the individual-agent fragment of SFCL to InqAL in the following way: $p^* = p$; $(\neg\varphi)^* = \neg\varphi^*$; $(\varphi \vee \psi)^* = \varphi^* \vee \psi^*$; and, finally:

$$[a](\sigma; \pi_1, \dots, \pi_n)^* = \neg(\sigma^* \Rightarrow_a \neg\pi_1^* \vee \dots \vee \neg\pi_n^*)$$

It is easy to check that the translation σ^* of a SFCL-formula is always a declarative with the same truth conditions as σ . Translating in the opposite direction, from InqAL to SFCL, is more tricky. First, since the language of SFCL includes only statements, we can only expect to faithfully translate declaratives,

¹¹The connection discussed in this section extends straightforwardly to one between full SFCL and the natural generalization of InqAL with coalitions. The translations would work in the same way, and in particular, the same exponential blowup discussed below would result when translating a formula from the coalitional version of InqAL to SFCL.

and not arbitrary formulas. Moreover, even though $\varphi \Rightarrow_a \psi$ is a declarative, the formulas φ and ψ need not be declarative, and so, they need not have a translation. Still, we can cook up a translation via their resolutions. We define a translation from the declarative fragment of InqAL to SFCL as follows: $p^* = p$; $\perp^* = (p \wedge \neg p)$ for an arbitrary $p \in \mathcal{P}$; $(\alpha \wedge \beta)^* = \alpha^* \wedge \beta^*$; $(\alpha \rightarrow \beta)^* = \neg(\alpha^* \wedge \neg \beta^*)$; and, crucially,

$$(\varphi \Rightarrow_a \psi)^* = \bigwedge_{i=1}^n \neg[a](\alpha_i^*; \neg \beta_1^*, \dots, \neg \beta_m^*)$$

where $\{\alpha_1, \dots, \alpha_n\} = \mathcal{R}(\varphi)$ and $\{\beta_1, \dots, \beta_m\} = \mathcal{R}(\psi)$. One can prove that for any declarative $\alpha \in \mathcal{L}_{\mathcal{A}}^!$, α and its translation α^* have the same truth conditions. The proof is identical to the one given for the translation of InqNL into instantial neighborhood logic (Proposition 10.2 in [12]).

In sum, InqAL and SFCL are equi-expressive with respect to statements about individual agents. Still, the way things are expressed in these logics is rather different. Note, in particular, that the number of resolutions $\mathcal{R}(\varphi)$ of a formula φ can grow exponentially relative to the size of φ , and thus, so can the length of the translation of a modal formula containing φ . It is plausible to conjecture that this blowup is unavoidable, and so, that InqAL is in general exponentially more succinct than SFCL. It is also worth noting that there is currently no complete axiomatization of SFCL (an axiomatization is presented in [20], but by the authors' admission (p.c.) the completeness proof contains a mistake). It can be hoped that our Theorem 2.2 can be used to establish completeness at least for the individual-agent fragment of SFCL.

7 Conclusions and future work

We have motivated and investigated InqAL, a logic that allows us to reason not only about what agents can *force* through their actions, but also about what they *enable* or *determine*. An obvious goal for future work is to generalize InqAL from individual agents to coalitions. Semantically, this is straightforward. The difficulty lies in establishing a characterization theorem analogous to Theorem 2.2, which played a crucial role in our proof of completeness and decidability. A further important goal is to extend InqAL with temporal operators, yielding an inquisitive version of strategic multi-agent logics like ATL [2, 19]. In a different direction, it would be interesting to ask if InqAL, or an extension, can regiment claims to the effect that an agent can partly influence (though perhaps not fully determine) the answer to a question.

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A Appendix: Proofs of standard results

A.1 Preliminaries

We start with some basic results about derivability and resolutions. Essentially the same results appear in many completeness results for inquisitive propositional logic [10] and inquisitive modal logics [7, 8, 12]. In each case, we outline the proof and provide a reference where an analogous proof is spelled out.

Lemma A.1 (Provable normal form). *For all $\varphi \in \mathcal{L}_{\mathcal{A}}$, $\varphi \Vdash_A \bigvee \mathcal{R}(\varphi)$.*

Proof. This uses only the propositional component of the axiomatization. See Lemma 4.3.8 in [10]. $\square$

Lemma A.2. *For all $\varphi \in \mathcal{L}_{\mathcal{A}}$, if $\vdash_A \varphi$ then $\vdash_A \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$.*

Proof. It suffices to check that this property holds for axioms and is preserved by inference rules. When an axiom is a declarative α , the claim is trivially true since $\mathcal{R}(\alpha) = \{\alpha\}$. Since all our modal axioms are declaratives, the only axioms we need to check are the propositional ones. This is a simple and standard exercise (see the proof of Lemma 5.6 in [8]). As for the inference rules, if φ was obtained by Conditional Necessitation then φ is declarative and the claim is trivially true. If φ was obtained by Modus Ponens from ψ and $\psi \rightarrow \varphi$, then by induction hypothesis some resolutions $\alpha \in \mathcal{R}(\psi)$ and $\beta \in \mathcal{R}(\psi \rightarrow \varphi)$ are derivable. By definition of resolutions of an implication, β is a conjunction with one conjunct of the form $\alpha \rightarrow \gamma$ where $\gamma \in \mathcal{R}(\varphi)$. Since α and $\alpha \rightarrow \gamma$ are derivable, γ is derivable by Modus Ponens. $\square$

Lemma A.3. *If $\Gamma \subseteq \mathcal{L}_{\mathcal{A}}^!$ and $\Gamma \vdash_A \varphi$, then $\Gamma \vdash_A \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$*

Proof. If $\Gamma \vdash_A \varphi$, this means that there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\bigwedge \Gamma_0 \rightarrow \varphi$ is derivable. Let $\gamma = \bigwedge \Gamma_0$. Since Γ is a set of declaratives, γ is a declarative, so $\mathcal{R}(\gamma) = \{\gamma\}$. Then, by definition of resolutions $\mathcal{R}(\gamma \rightarrow \varphi) = \{\gamma \rightarrow \alpha \mid \alpha \in \mathcal{R}(\varphi)\}$. By the previous lemma, since $\gamma \rightarrow \varphi$ is derivable, some particular resolution $\gamma \rightarrow \alpha$ is derivable. This implies that $\Gamma \vdash_A \alpha$, and since $\alpha \in \mathcal{R}(\varphi)$ we are done. $\square$

Definition A.4 (Resolutions for sets). A *resolution function* for a set $\Phi \subseteq \mathcal{L}_{\mathcal{A}}$ is a function f assigning to each $\varphi \in \Phi$ a corresponding resolution $\alpha \in \mathcal{R}(\varphi)$. The image of Φ under a resolution function is called a *resolution* of Φ . More formally, the set of resolutions of φ is defined in the following way: $\mathcal{R}(\Phi) = \{f[\Phi] \mid f \text{ a resolution function for } \Phi\}$. Note that if $\Gamma \in \Phi$, then for each $\varphi \in \Phi$ there is some resolution $\alpha \in \mathcal{R}(\varphi)$ with $\alpha \in \Gamma$.

Lemma A.5. *If $\Phi, \Psi \subseteq \mathcal{L}_{\mathcal{A}}$ and $\Phi \not\vdash_A \Psi$, then $\Gamma \not\vdash_A \Psi$ for some $\Gamma \in \mathcal{R}(\Phi)$.*

Proof. Repeated application of Lemma A.1, by standard properties of $\mathbb{W}$. See Lemma 4.3.7 in [10]. $\square$

A.2 Intersection Lemma (Lemma 5.8)

Consider a set of declaratives $\Delta \subseteq \mathcal{L}_{\mathcal{A}}^{!n}$ and its corresponding set of n -bounded complete extensions, $S_{\Delta}^n = \{\Gamma \in \mathcal{K}_n \mid \Delta \subseteq \Gamma\}$. We want to show that for all $\varphi \in \mathcal{L}_{\mathcal{A}}^n$: $\Delta \vdash_A \varphi \iff \bigcap S_{\Delta}^n \vdash_A \varphi$.

The direction $\Rightarrow$ is obvious since $\Delta \subseteq \bigcap S_{\Delta}^n$. For the converse, suppose for a contradiction that for some φ we had $\bigcap S_{\Delta}^n \vdash_A \varphi$ but $\Delta \not\vdash_A \varphi$. Since $\bigcap S_{\Delta}^n$ is a set of declaratives, by Lemma A.3 we have $\bigcap S_{\Delta}^n \vdash_A \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$. Since $\Delta \not\vdash_A \varphi$, it follows by Lemma A.1 that $\Delta \not\vdash_A \alpha$. By the axiom $\neg\neg\alpha \rightarrow \alpha$, this implies $\Delta \not\vdash_A \neg\neg\alpha$, and therefore $\Delta \cup \{\neg\alpha\} \not\vdash_A \perp$. Since $\varphi \in \mathcal{L}_{\mathcal{A}}^n$ we have $\alpha \in \mathcal{L}_{\mathcal{A}}^{!n}$ and thus also $\Delta \cup \{\neg\alpha\} \subseteq \mathcal{L}_{\mathcal{A}}^{!n}$. So by Lemma 5.4 there is $\Gamma \in \mathcal{K}_n$ such that $\Delta \cup \{\neg\alpha\} \subseteq \Gamma$. Now since $\Gamma \in S_{\Delta}^n$ and $\bigcap S_{\Delta}^n \vdash_A \alpha$ we have $\Gamma \vdash_A \alpha$. So we have $\Gamma \vdash_A \neg\alpha$ and $\Gamma \vdash_A \alpha$, whence $\Gamma \vdash_A \perp$ and, by deductive closure relative to $\mathcal{L}_{\mathcal{A}}^{!n}$, $\perp \in \Gamma$. But this is a contradiction since $\Gamma \in \mathcal{K}_n$ is consistent by assumption. $\square$

A.3 Existence Lemma (Lemma 5.9)

We follow closely the proof of Lemma 6.12 in [12], adapting it to our multi-agent setting and to our modal depth-bounded canonical model construction. We first establish some preliminary results.

Let Γ be an n CTD, with $n > 0$. Given two sets $\Phi, \Psi \subseteq \mathcal{L}_{\mathcal{A}}^{n-1}$, we write $\Phi \Rightarrow_{\Gamma}^a \Psi$ if there are finite subsets $\Phi_0 \subseteq \Phi$ and $\Psi_0 \subseteq \Psi$ such that the formula $\bigwedge \Phi_0 \Rightarrow_a \bigvee \Psi_0$ is in Γ . Note the following fact.

Lemma A.6. *If $\Phi, \Psi \subseteq \mathcal{L}_{\mathcal{A}}^{n-1}$ and $\Phi \vdash_A \Psi$, then $\Phi \Rightarrow_{\Gamma}^a \Psi$.*

1.	$\varphi_1 \wedge \chi \Rightarrow_a \psi_1$	(premise)
2.	$\varphi_2 \Rightarrow_a \psi_2 \vee \chi$	(premise)
3.	$\varphi_1 \wedge \varphi_2 \Rightarrow_a \varphi_1$	(CN) from axiom $\varphi_1 \wedge \varphi_2 \rightarrow \varphi_1$
4.	$\varphi_1 \wedge \varphi_2 \Rightarrow_a \varphi_2$	(CN) from axiom $\varphi_1 \wedge \varphi_2 \rightarrow \varphi_2$
5.	$\varphi_1 \wedge \varphi_2 \Rightarrow_a \psi_2 \vee \chi$	(MP) from Transitivity, 4, 2
6.	$\varphi_1 \wedge \varphi_2 \Rightarrow_a \varphi_1 \wedge (\psi_2 \vee \chi)$	(MP) from Right Conjunction, 3, 5
7.	$\varphi_1 \wedge (\psi_2 \vee \chi) \Rightarrow_a \psi_2 \vee (\varphi_1 \wedge \chi)$	(CN) from prop. validity $\varphi_1 \wedge (\psi_2 \vee \chi) \rightarrow \psi_2 \vee (\varphi_1 \wedge \chi)$
8.	$\varphi_1 \wedge \varphi_2 \Rightarrow_a \psi_2 \vee (\varphi_1 \wedge \chi)$	(MP) from Transitivity, 6, 7
9.	$\psi_1 \Rightarrow_a \psi_1 \vee \psi_2$	(CN) from axiom $\psi_1 \rightarrow \psi_1 \vee \psi_2$
10.	$\psi_2 \Rightarrow_a \psi_1 \vee \psi_2$	(CN) from axiom $\psi_2 \rightarrow \psi_1 \vee \psi_2$
11.	$\varphi_1 \wedge \chi \Rightarrow_a \psi_1 \vee \psi_2$	(MP) from Transitivity, 1, 9
12.	$\psi_2 \vee (\varphi_1 \wedge \chi) \Rightarrow_a \psi_1 \vee \psi_2$	(MP) from Left Disjunction, 10, 11
13.	$\varphi_1 \wedge \varphi_2 \Rightarrow_a \psi_1 \vee \psi_2$	(MP) from Transitivity, 8, 12

Figure 2: Sketch of a proof showing $(\varphi_1 \wedge \chi \Rightarrow_a \psi_1), (\varphi_2 \Rightarrow_a \psi_2 \vee \chi) \vdash_A (\varphi_1 \wedge \varphi_2 \Rightarrow_a \psi_1 \vee \psi_2)$

Proof. If $\Phi \vdash_A \Psi$, there are finite subsets $\Phi_0 \subseteq \Phi$ and $\Psi_0 \subseteq \Psi$ such that $\vdash_A \bigwedge \Phi_0 \rightarrow \bigvee \Psi_0$. By Conditional Necessitation, also $\vdash_A \bigwedge \Phi_0 \Rightarrow_a \bigvee \Psi_0$. Since $\Phi, \Psi \subseteq \mathcal{L}_{\mathcal{A}}^{n-1}$, the modal depth of the formula $\bigwedge \Phi_0 \Rightarrow_a \bigvee \Psi_0$ is at most n , and so by deductive closure, this formula must be in Γ , witnessing $\Phi \Rightarrow_{\Gamma}^a \Psi$. $\square$

Importantly, the relation $\Rightarrow_{\Gamma}^a$ also enjoys the following cut-like property.

Lemma A.7. *For any two sets $\Phi, \Psi \subseteq \mathcal{L}_{\mathcal{A}}^{n-1}$ and formula $\chi \in \mathcal{L}_{\mathcal{A}}^{n-1}$: $\Phi \cup \{\chi\} \Rightarrow_{\Gamma}^a \Psi$ and $\Phi \Rightarrow_{\Gamma}^a \Psi \cup \{\chi\}$ implies $\Phi \Rightarrow_{\Gamma}^a \Psi$.*

Proof. Suppose $\Phi \cup \{\chi\} \Rightarrow_{\Gamma}^a \Psi$ and $\Phi \Rightarrow_{\Gamma}^a \Psi \cup \{\chi\}$. This means that there are finite sets of formulas $\Phi_0, \Phi_1 \subseteq \Phi$ and $\Psi_0, \Psi_1 \subseteq \Psi$ such that Γ contains the following formulas:

$$(\chi \wedge \bigwedge \Phi_0) \Rightarrow_a \bigvee \Psi_0 \quad \bigwedge \Phi_1 \Rightarrow_a (\chi \vee \bigvee \Psi_1)$$

We prove that Γ must contain $\bigwedge (\Phi_0 \cup \Phi_1) \Rightarrow_a \bigvee (\Psi_0 \cup \Psi_1)$, thus witnessing $\Phi \Rightarrow_{\Gamma}^a \Psi$. To ease notation, we spell out the details for the case in which the relevant sets are all singletons $\Phi_0 = \{\varphi_0\}, \Phi_1 = \{\varphi_1\}, \Psi_0 = \{\psi_0\}, \Psi_1 = \{\psi_1\}$, but the general case is analogous.

So, we know that Γ contains the formulas $(\varphi_1 \wedge \chi \Rightarrow_a \psi_1)$ and $(\varphi_2 \Rightarrow_a \psi_2 \vee \chi)$, and we want to show that it contains $(\varphi_1 \wedge \varphi_2 \Rightarrow_a \psi_1 \vee \psi_2)$. Since this formula is a declaratives with modal depth $\leq n$, and since Γ is closed under deduction relative to such formulas, it suffices to show that:

$$(\varphi_1 \wedge \chi \Rightarrow_a \psi_1), (\varphi_2 \Rightarrow_a \psi_2 \vee \chi) \vdash_A (\varphi_1 \wedge \varphi_2 \Rightarrow_a \psi_1 \vee \psi_2)$$

A derivation is given in Figure 2. In the derivation, we indicate explicitly only the modal axioms and rules involved in the reasoning, omitting reference to propositional axioms. We write (MP) to indicate *modus ponens* and (CN) for *conditional necessitation*. For simplicity, we use the formulas $(\varphi_1 \wedge \chi \Rightarrow_a \psi_1)$ and $(\varphi_2 \Rightarrow_a \psi_2 \vee \chi)$ as if they were premises; this is legitimate since we will not use the conditional necessitation rule (CN) on these formulas or anything inferred from them. Rewriting the argument with the relevant formulas used throughout as conditional antecedents is tedious but straightforward. $\square$

Lemma A.8 (Splitting lemma). *Let Γ be a nCTD with $n > 0$ and take $\Phi, \Psi \subseteq \mathcal{L}_{\mathcal{A}}^{n-1}$ with $\Phi \not\Rightarrow_{\Gamma}^a \Psi$. The set $\mathcal{L}_{\mathcal{A}}^{n-1}$ can be partitioned into sets L and R such that $\Phi \subseteq L$, $\Psi \subseteq R$, and $L \not\Rightarrow_{\Gamma}^a R$.*

Proof. Fix an enumeration $(\chi_i)_{i \in \mathbb{N}}$ of $\mathcal{L}_{\mathcal{A}}^{n-1}$. Define a sequence of sets $(L_i)_{i \in \mathbb{N}}$ and $(R_i)_{i \in \mathbb{N}}$ as follows:

- $L_0 = \Phi, R_0 = \Psi$
- if $L_i \cup \{\chi_i\} \not\equiv_{\Gamma}^a R_i$ we let $L_{i+1} := L_i \cup \{\chi_i\}$ and $R_{i+1} = R_i$
- if $L_i \cup \{\chi_i\} \equiv_{\Gamma}^a R_i$ we let $L_{i+1} := L_i$ and $R_{i+1} = R_i \cup \{\chi_i\}$

We show by induction on i that $L_i \not\equiv_{\Gamma}^a R_i$. For $n = 0$ this is true by assumption. Now suppose this is true for i and consider $i + 1$. If $L_i \cup \{\chi_i\} \not\equiv_{\Gamma}^a R_i$, the claim is obvious by definition of L_{i+1} and R_{i+1} . So, suppose $L_i \cup \{\chi_i\} \equiv_{\Gamma}^a R_i$. Since by induction hypothesis $L_i \not\equiv_{\Gamma}^a R_i$, Lemma A.7 implies $L_i \not\equiv_{\Gamma}^a R_i \cup \{\chi_i\}$, which by definition amounts to $L_{i+1} \not\equiv_{\Gamma}^a R_{i+1}$.

Now let $L = \bigcup_{i \in \mathbb{N}} L_i$ and $R = \bigcup_{i \in \mathbb{N}} R_i$. By construction, $\Phi \subseteq L$ and $\Psi \subseteq R$. We have $L \not\equiv_{\Gamma}^a R$, otherwise there would be an $i \in \mathbb{N}$ such that $L_i \equiv_{\Gamma}^a R_i$, contrary to what we just saw. Moreover, L and R form a partition of $\mathcal{L}_{\mathcal{A}}^{n-1}$. By construction, every formula of $\mathcal{L}_{\mathcal{A}}^{n-1}$ occurs in either set. Moreover, no formula cannot occur in both: to see why, suppose for a contradiction that for some $\chi \in \mathcal{L}_{\mathcal{A}}^{n-1}$ we have $\chi \in L \cap R$; since $\chi \equiv_a \chi$ is a valid declarative with modal depth $\leq n$, and since Γ is deductively closed with respect to such formulas, we would need to have $(\chi \equiv_a \chi) \in \Gamma$, contradicting $L \equiv_{\Gamma}^a R$. $\square$

With these preliminaries at hand, we are now ready to complete the proof of the existence lemma.

Proof of Lemma 5.9. Let Γ be an n CTD with $\neg(\varphi \equiv_a \psi) \in \Gamma$. This implies $n > 0$ (otherwise Γ would contain only propositional formulas) and furthermore φ and ψ must be in $\mathcal{L}_{\mathcal{A}}^{n-1}$. Since Γ is consistent, $(\varphi \equiv_a \psi) \notin \Gamma$, and so $\{\varphi\} \not\equiv_{\Gamma}^a \{\psi\}$. Now extend $\{\varphi\}$ and $\{\psi\}$ to sets L and R as in the previous lemma.

Since $L \not\equiv_{\Gamma}^a R$, by Lemma A.6 we have $L \not\vdash_A R$. By Lemma A.5 there is a set $\Delta \in \mathcal{R}(L)$ with $\Delta \not\vdash_A R$. Since $L \subseteq \mathcal{L}_{\mathcal{A}}^{n-1}$, we have $\Delta \subseteq \mathcal{L}_{\mathcal{A}}^{n-1}$. We can now take $S = S_{\Delta}^{n-1} = \{\Gamma' \in \mathcal{K}_{n-1} \mid \Delta \subseteq \Gamma'\}$. We need to verify that (i) $\bigcap S \vdash_A \varphi$, (ii) $\bigcap S \not\vdash_A \psi$ and (iii) $S \in \Sigma_a^n(\Gamma)$.

- For (i), we have $\varphi \in L$. Since $\Delta \in \mathcal{R}(L)$, for some $\alpha \in \mathcal{R}(\varphi)$ we have $\alpha \in \Delta$. By Lemma A.1, $\Delta \vdash_A \varphi$, and thus, since $\varphi \in \mathcal{L}_{\mathcal{A}}^{n-1}$, by Lemma 5.8 also $\bigcap S \vdash_A \varphi$.
- For (ii), we have $\psi \in R$. Since $\Delta \not\vdash_A R$, also $\Delta \not\vdash_A \psi$. Since $\psi \in \mathcal{L}_{\mathcal{A}}^{n-1}$, Lemma 5.8 gives $\bigcap S \not\vdash_A \psi$.
- For (iii), first note that since $\Delta \not\vdash_A R$, we have $\Delta \not\vdash_A \perp$, so by Lemma 5.4, $S \neq \emptyset$. Next, suppose $(\chi \equiv_a \xi) \in \Gamma$ and $\bigcap S \vdash_A \chi$. We need to show that $\bigcap S \vdash_A \xi$. Since $(\chi \equiv_a \xi) \in \Gamma$ and $\Gamma \in \mathcal{K}_n$ we have $\chi, \xi \in \mathcal{L}_{\mathcal{A}}^{n-1}$. Since $\bigcap S \vdash_A \chi$ and $\chi \in \mathcal{L}_{\mathcal{A}}^{n-1}$, Lemma 5.8 gives $\Delta \vdash_A \chi$. Since by construction $\Delta \not\vdash_A R$, it follows that $\chi \notin R$, and since R and L partition the set $\mathcal{L}_{\mathcal{A}}^{n-1}$ we have $\chi \in L$. Now we must have $\xi \in L$ as well, for if we had $\xi \in R$ it would follow from $(\chi \equiv_a \xi) \in \Gamma$ that $L \equiv_{\Gamma}^a R$, contrary to what we know. Since $\xi \in L$ and $\Delta \in \mathcal{R}(L)$, for some $\alpha \in \mathcal{R}(\xi)$ we have $\alpha \in \Delta$, so by Lemma A.1, $\Delta \vdash_A \xi$. Finally, since $\xi \in \mathcal{L}_{\mathcal{A}}^{n-1}$, Lemma 5.8 implies $\bigcap S \vdash_A \xi$, as desired. $\square$

A.4 Range Lemma (Lemma 5.10)

We follow the proof of Lemma 6.15 in [12], adapting it to our multi-agent and depth-bounded setting.

Consider a $\Gamma \in \mathcal{K}_m$ with $m > 0$. We want to show the identity:

$$\bigcup \Sigma_a^n(\Gamma) = \{\Gamma' \in \mathcal{K}_{m-1} \mid \forall \alpha \in \mathcal{L}_{\mathcal{A}}^1 : \boxplus_a \alpha \in \Gamma \text{ implies } \alpha \in \Gamma'\}$$

Proof. ($\subseteq$) Suppose $\Gamma' \in \bigcup \Sigma_a^n(\Gamma)$, that is, $\Gamma' \in S$ for some $S \in \Sigma_a^n(\Gamma)$. By definition of Σ_a^n , since $\Gamma \in \mathcal{K}_m$ we have $\Gamma' \in \mathcal{K}_{m-1}$. Now let α be a declarative and suppose $\boxplus_a \alpha \in \Gamma$, that is, $(\top \equiv_a \alpha) \in \Gamma$. Since $S \in \Sigma_a^n(\Gamma)$ and $\bigcap S \vdash_A \top$, it follows that $\bigcap S \vdash_A \alpha$. Since $\Gamma' \in S$, we have $\bigcap S \subseteq \Gamma'$, and so also $\Gamma' \vdash_A \alpha$.

Since $\boxplus_a \alpha \in \Gamma$ and $\Gamma \in \mathcal{K}_m$, it follows that $\alpha \in \mathcal{L}_{\mathcal{A}}^{!m-1}$. Since $\Gamma' \vdash_A \alpha$ and Γ' is closed under deduction relative to $\mathcal{L}_{\mathcal{A}}^{!m-1}$, we have $\alpha \in \Gamma'$.

($\supseteq$) Consider $\Gamma' \in \mathcal{K}_{m-1}$ and suppose for all $\alpha \in \mathcal{L}_{\mathcal{A}}^!$, $\boxplus_a \alpha \in \Gamma$ implies $\alpha \in \Gamma'$. We must show that $\Gamma' \in S$ for some $S \in \Sigma_a^n(\Gamma)$. First, we claim that $\emptyset \not\equiv_{\Gamma}^a \{\neg\alpha \mid \alpha \in \Gamma'\}$. Towards a contradiction, suppose not: then there are $\alpha_1, \dots, \alpha_n \in \Gamma'$ such that $(\top \Rightarrow_a \neg\alpha_1 \vee \dots \vee \neg\alpha_n) \in \Gamma$. Since $\neg\alpha_1 \vee \dots \vee \neg\alpha_n \vdash_A \neg(\alpha_1 \wedge \dots \wedge \alpha_n)$, by Conditional Necessitation and Transitivity we have $(\top \Rightarrow_a \neg\alpha_1 \vee \dots \vee \neg\alpha_n) \vdash_A (\top \Rightarrow_a \neg(\alpha_1 \wedge \dots \wedge \alpha_n))$. Since $(\top \Rightarrow_a \neg(\alpha_1 \wedge \dots \wedge \alpha_n)) \in \mathcal{L}_{\mathcal{A}}^m$ and Γ is deductively closed with respect to $\mathcal{L}_{\mathcal{A}}^m$, we conclude $(\top \Rightarrow_a \neg(\alpha_1 \wedge \dots \wedge \alpha_n)) \in \Gamma$, that is, $\boxplus_a \neg(\alpha_1 \wedge \dots \wedge \alpha_n) \in \Gamma$. By our assumption on Γ' , we must have $\neg(\alpha_1 \wedge \dots \wedge \alpha_n) \in \Gamma'$. But this is impossible, since each α_i is in Γ' and Γ' is consistent.

We have thus established the claim $\emptyset \not\equiv_{\Gamma}^a \{\neg\alpha \mid \alpha \in \Gamma'\}$. By Lemma A.8, we can partition the language $\mathcal{L}_{\mathcal{A}}^{m-1}$ into sets L, R with $L \not\equiv_{\Gamma}^a R$ and $\{\neg\alpha \mid \alpha \in \Gamma'\} \subseteq R$. Reasoning as in the previous lemma, we can find a $\Delta \in \mathcal{R}(L)$ with $\Delta \not\vdash R$, and we can show that the corresponding set of $m-1$ -bounded complete extensions S_{Δ}^{m-1} is in $\Sigma_a^n(\Gamma)$. We now claim that $\Gamma' \in S_{\Delta}^{m-1}$. To show this, it suffices to show that $\Delta \cup \Gamma' \not\vdash_A \perp$: if this holds, it follows by Lemma 5.4 that there is a $\Gamma'' \in \mathcal{K}_{m-1}$ with $\Delta \cup \Gamma' \subseteq \Gamma''$. Since two bounded elements of $\mathcal{K}_{m-1}$ cannot be properly included in one another, we must have $\Gamma' = \Gamma''$, and therefore $\Delta \subseteq \Gamma'$, showing that $\Gamma' \in S_{\Delta}^{m-1}$ as desired.

So, towards a contradiction, suppose $\Delta \cup \Gamma' \vdash_A \perp$. Since Γ' is closed under conjunction, this means that there is a single formula $\alpha \in \Gamma'$ such that $\Delta \cup \{\alpha\} \vdash_A \perp$, and so, $\Delta \vdash_A \neg\alpha$. But this is impossible, since by construction $\neg\alpha \in R$ and $\Delta \not\vdash R$.

To conclude, we have found a state S_{Δ}^{m-1} such that $\Gamma' \in S_{\Delta}^{m-1}$ and $S_{\Delta}^{m-1} \in \Sigma_a^n(\Gamma)$, thus showing that $\Gamma' \in \bigcup \Sigma_a^n(\Gamma)$, as required. $\square$

A.5 Support Lemma (Lemma 5.11)

We must show that for all $m \leq n$, all formulas $\varphi \in \mathcal{L}_{\mathcal{A}}^m$, and all non-empty states $S \subseteq \mathcal{K}_m$ we have

$$M_n, S \models \varphi \iff \bigcap S \vdash_A \varphi$$

The proof is by induction on φ , simultaneously for all $S \subseteq \mathcal{K}_m$. The cases for atoms and connectives are standard (see the proof of Lemma 4.3.15 in [10]). We spell out the inductive step for a modal formula $\varphi = (\psi \Rightarrow_a \chi)$. Note that since we are assuming $\varphi \in \mathcal{L}_{\mathcal{A}}^m$ we have $m > 0$ and $\psi, \chi \in \mathcal{L}_{\mathcal{A}}^{!m-1}$.

Suppose $\bigcap S \vdash_A (\psi \Rightarrow_a \chi)$. We must show $M_n, S \models (\psi \Rightarrow_a \chi)$. For this, take an arbitrary world $\Gamma \in S$ and a state $T \in \Sigma_a^n(\Gamma)$ with $M_n, T \models \psi$. We need to show that $M_n, T \models \chi$. By definition of Σ_a^n we have $T \subseteq \mathcal{K}_{m-1}$. By induction hypothesis on ψ , from $M_n, T \models \psi$ we obtain $\bigcap T \vdash_A \psi$. Since $\Gamma \in S$ we have $\bigcap S \subseteq \Gamma$, and since $\bigcap S \vdash_A (\psi \Rightarrow_a \chi)$ also $\Gamma \vdash (\psi \Rightarrow_a \chi)$. Since $(\psi \Rightarrow_a \chi) \in \mathcal{L}_{\mathcal{A}}^{!m}$ and $\Gamma \in \mathcal{K}_m$, it follows that $(\psi \Rightarrow_a \chi) \in \Gamma$. By definition of Σ_a^n , from $T \in \Sigma_a^n(\Gamma)$, $(\psi \Rightarrow_a \chi) \in \Gamma$, and $\bigcap T \vdash_A \psi$ we can conclude $\bigcap T \vdash_A \chi$. Finally, by induction hypothesis on χ , this gives $M_n, T \models \chi$, as required.

For the converse, suppose $\bigcap S \not\vdash_A (\psi \Rightarrow_a \chi)$. Then there is some $\Gamma \in S$ such that $(\psi \Rightarrow_a \chi) \notin \Gamma$. Since $(\psi \Rightarrow_a \chi) \in \mathcal{L}_{\mathcal{A}}^m$ and $\Gamma \in \mathcal{K}_m$, by completeness we have $\neg(\psi \Rightarrow_a \chi) \in \Gamma$. By the Existence Lemma (Lemma 5.9) there is a state $T \in \Sigma_a^n(\Gamma)$ such that $\bigcap T \vdash_A \psi$ and $\bigcap T \not\vdash_A \chi$. By induction hypothesis on ψ and χ , this means that $M_n, T \models \psi$ and $M_n, T \not\models \chi$. Hence, $M_n, S \not\models (\psi \Rightarrow_a \chi)$. $\square$

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Labelled Sequents for Inquisitive First-Order Modal Logic

Ivano Ciardelli

Dipartimento FISPPA
University of Padua, Italy
ivano.ciardelli@unipd.it

Simone Conti

Dipartimento FISPPA
University of Padua, Italy
simone.conti@phd.unipd.it

In recent work [5], an inquisitive first-order modal logic has been proposed to reason about relations of modal dependence, including the notion of global supervenience (functional dependence among the extensions of predicates relative to a space of possibilities). At present, no proof system exists for this logic. We provide a complete labelled sequent calculus, extending a calculus developed by Litak and Sano [22] for a weak version of inquisitive first-order logic. We prove strong completeness for the calculus and show that it enjoys desirable structural properties, including the invertibility of its rules and the admissibility of cut.

1 Introduction

Inquisitive logics [8, 4, 29, 30] extend existing versions of classical or non-classical logics with formulas representing questions. This extension is typically achieved by lifting the semantics from standard points of evaluations (e.g., valuation functions, relational models, or possible worlds) to sets of such points, called *information states* (or just *states* for short), relative to which a notion of *support* is defined.¹

The standard system of inquisitive first-order logic, InqBQ, extends classical first-order logic with two question-forming operators: inquisitive disjunction, $\vee$, and the inquisitive existential quantifier, $\exists$. In terms of $\vee$, a polar question operator $?$ is defined by letting $?\varphi := (\varphi \vee \neg\varphi)$. In spite of important advances [14, 15, 16, 7], the meta-theoretic properties of this system remain rather mysterious: in particular, it is not known whether InqBQ is recursively axiomatizable, or whether it is entailment-compact.²

Recent work [7, 10] has focused on the $\exists$ -free fragment of this logic, denoted InqWQ. This fragment is sufficiently expressive to regiment a broad variety of questions; for instance, in addition to standard sentences like $\forall xPx$, regimenting the statement “every object is P ”, InqWQ also contains sentences like $?\forall xPx$ and $\forall x?Px$, regimenting, respectively, the questions “whether or not every object is P ” and “which objects are P ”. At the same time, InqWQ enjoys a strong semantic property which is not shared by all of InqBQ: *finite coherence*.³ A formula φ is said to be n -coherent for a number $n \in \mathbb{N}$ if, in order to decide whether φ is supported by a state s , it suffices to check whether it is supported by the subsets $s' \subseteq s$ of cardinality up to n . As shown by Ciardelli and Grilletti [7], every formula φ in InqWQ is n -coherent for some number n which is computable from φ . This fact has far-reaching repercussions: one can use it to show, among other things, that InqWQ is entailment-compact and recursively axiomatizable [7].

In recent work, two complete proof systems have been provided for InqWQ: Conti [10], building on [7], gave a natural deduction system, while Litak and Sano [22] gave a labelled sequent calculus. Both

¹An analogous move is made in the *team semantics* tradition, where assignments are replaced by sets of assignments called *teams* (see, a.o., [17, 32, 11, 13]). For connections between these two lines of work see, among others, [33, 2].

²A logic is said to be entailment-compact if whenever a formula ψ follows from a set of formulas Φ , it follows from a finite subset $\Phi_0 \subseteq \Phi$. In inquisitive logic, this is not equivalent to the more common formulation of compactness in terms of satisfiability, essentially due to the fact that the double negation law does not hold in general.

³The notion of coherence goes back to the team semantics literature, where it was first investigated by Jarmo Kontinen [19].

systems rely crucially on the finite coherence property of InqWQ . In the former system, this property shows up through a specific *coherence rule* which allows one to discharge certain assumptions about the cardinality of the state of evaluations. In the latter system, coherence ensures that it is sufficient to work with labels which are finite sequences of indices, representing finite states.

In recent work, Ciardelli [5] investigated a modal logic $\text{InqQML}_{\Box}^-$ obtained by extending InqWQ with a generalization of the Kripke modality $\Box$. The motivation for this work came from the analysis of *global supervenience*, a notion of dependence that has received attention in the philosophical literature [18, 31, 23, 1, 20], where it has been invoked to give a precise formulation of certain philosophical theses, such as David Lewis's thesis of *Humean supervenience* [21]. Intuitively, global supervenience captures the idea that the overall distribution of certain properties or relations in the world is fully determined by the overall distribution of other properties or relations. An example: if we fix who is a parent of whom, we thereby also fix who is a grandparent of whom; so, the *grandparent-of* relation globally supervenes on the *parent-of* relation. Formally, we may say that a predicate Q globally supervenes on another predicate P at a possible world w if any two successors of w which assign the same extension to P also assign the same extension to Q . (The notion extends straightforwardly to the case of several predicates supervening on several other predicates.) Thus, global supervenience captures functional dependencies between the extensions of predicates across a space of possibilities. As Ciardelli [5] showed, global supervenience claims cannot be expressed in standard quantified modal logic, but they *can* be expressed in $\text{InqQML}_{\Box}^-$ in a particularly perspicuous way, namely, as strict conditionals whose antecedents and consequents are questions. Thus, e.g., the claim that Q globally supervenes on P is formalized by the strict conditional

$$\Box(\forall x?Px \rightarrow \forall x?Qx)$$

having as its antecedent the subvenient question $\forall x?Px$ ("which objects are P "), and as its consequent the supervenient question $\forall x?Qx$ ("which objects are Q "). As discussed in [5], this analysis is insightful, since it allows us to trace back logical properties of global supervenience to familiar properties of strict conditionals and questions. The inquisitive modal logic $\text{InqQML}_{\Box}^-$, then, provides an attractive system to reason about global supervenience claims, as well as modal dependence claims more generally [3].

In view of this motivation, it seems important to have a proof system for this logic: such a system would allow one to formally establish the validity of certain inferences concerning modal dependencies and, ideally, it could be used to get further insight about the logic of these dependence notions. While Ciardelli [5] proved (by means of a translation to two-sorted first-order logic) that the set of validities of $\text{InqQML}_{\Box}^-$ is recursively enumerable, he left the development of a proof system as an open problem.

In this paper, we fill this gap, providing a labelled sequent calculus for $\text{InqQML}_{\Box}^-$. Our calculus builds on Litak and Sano's calculus for InqWQ [22], where labels are finite sequences $w_1 \dots w_n$ of indices;⁴ intuitively, indices represent worlds, and so, labels represent finite states. To this calculus we add rules for the modality $\Box$. These rules are inspired by the standard rules for $\Box$ in labelled sequent calculi [26] but, at the same time, they use in a crucial way the finite coherence property of $\text{InqQML}_{\Box}^-$, inherited from InqWQ : every formula φ in $\text{InqQML}_{\Box}^-$ is n_{φ} -coherent for some number n_{φ} computable from φ [5]. This is crucial, in particular, for our right rule for $\Box$. Semantically, $\Box\varphi$ is true at a world w if φ is supported by the state consisting of all the successors of w ; however, by finite coherence, this reduces to φ being supported by all sets consisting of at most n_{φ} -many successors of w . This fact allows us to formulate a

⁴One may wonder if it is also possible to obtain a proof system for $\text{InqQML}_{\Box}^-$ by extending the natural deduction system in [10]. This seems difficult, since the completeness proof in [10] relies on the finite model property of InqWQ , which does not extend to $\text{InqQML}_{\Box}^-$. Regardless of this problem, the labelled approach of [22] seems preferable due to its better proof-theoretic properties.

right rule for $\Box$ that, in essence, says the following: in order to prove a labelled formula $w : \Box\varphi$, introduce n_φ -many fresh indices $v_1 \dots v_{n_\varphi}$ standing for successors of w , and aim to prove that $v_1 \dots v_{n_\varphi} : \varphi$.

In addition to establishing the soundness and completeness of this system, we prove the invertibility of its rules, and the admissibility of the rules of weakening, contraction, and cut.

Inquisitive modal logic is currently an active and rapidly growing field. While several proof systems have been developed for *propositional* inquisitive modal logics, including labelled sequent calculi [24, 25], our calculus represents the first example of a proof system for a *first-order* inquisitive modal logic. We think that, in addition to its direct relevance as a tool to study the logic $\text{InqQML}_\Box^-$, our contribution might serve as a model to build proof systems for other first-order inquisitive modal logics.

The paper is structured as follows: Section 2 provides the relevant background on the logic $\text{InqQML}_\Box^-$; Section 3 presents our contribution; Section 4 outlines directions for future work.

2 Background

In this section, we provide the necessary technical background on the inquisitive modal logic $\text{InqQML}_\Box^-$. For a thorough presentation of this logic and proofs of the facts mentioned in this section, we refer to [5]. For a more general introduction to inquisitive logic, see [4].

Syntax. We start with a countable first-order signature Σ . For simplicity, we focus on the case where Σ is a set of predicates (each having an associated *arity*), and doesn't contain function symbols nor the identity symbol. The language of $\text{InqQML}_\Box^-$ is given by the following definition, where $P \in \Sigma$ is an n -ary predicate and $x, x_1, \dots, x_n$ range over Var , a countably infinite set of variables:

$$\varphi ::= \perp \mid P(x_1, \dots, x_n) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \forall x \varphi \mid \Box \varphi$$

We also consider the following defined symbols: $\neg\varphi = \varphi \rightarrow \perp$, $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\exists x \varphi = \neg\forall x \neg\varphi$ and the inquisitive operator $?\varphi = \varphi \vee \neg\varphi$. We call formulas where $\vee$ does not appear *classical formulas*. These formulas can be identified with those of standard modal logic, with a particular choice of primitive operators. The operator $\vee$ is called *inquisitive disjunction*; informally, it is intended to formalize disjunctive questions. Thus, for instance, whereas the classical disjunction $P(x) \vee \neg P(x)$ regiments the (tautological) statement that *either x is P or it isn't*, the inquisitive disjunction $P(x) \vee \neg P(x)$ regiments the question *whether x is P or not* (which explains why this formula is abbreviated as $?P(x)$).

Semantics. Models for $\text{InqQML}_\Box^-$ are defined as tuples $M = \langle W, D, R, I \rangle$, where $W \neq \emptyset$ is a set of possible worlds, $D \neq \emptyset$ is a domain of individuals, $R \subseteq W \times W$ is an accessibility relation, associating each world $w \in W$ with a set of *successors* $R[w] = \{v \in W \mid wRv\}$, and I maps each $w \in W$ to an interpretation function that assigns to each n -ary predicate P a corresponding extension $I_w(P) \subseteq D^n$. An *information state* (or, simply, a *state*) is a subset of W . As usual, an assignment is a function $g : \text{Var} \rightarrow D$. Given an assignment g , a variable x , and an individual $d \in D$, $g[x \mapsto d]$ is the assignment that maps x to d and agrees with g on all other variables.

In $\text{InqQML}_\Box^-$, formulas φ are evaluated in terms of a relation of support, written as $M, s \models_g \varphi$, relative to a model M , a state s and an assignment g . We define support conditions inductively as follows:

- $M, s \models_g \perp \iff s = \emptyset$
- $M, s \models_g P(x_1, \dots, x_n) \iff \text{for all } w \in s, \langle g(x_1), \dots, g(x_n) \rangle \in I_w(P)$
- $M, s \models_g \varphi \wedge \psi \iff M, s \models_g \varphi \text{ and } M, s \models_g \psi$

- $M, s \models_g \varphi \vee \psi \iff M, s \models_g \varphi \text{ or } M, s \models_g \psi$
- $M, s \models_g \varphi \rightarrow \psi \iff \text{for all } t \subseteq s, M, t \models_g \varphi \text{ implies } M, t \models_g \psi$
- $M, s \models_g \forall x \varphi \iff \text{for all } d \in D, M, s \models_{g[x \mapsto d]} \varphi$
- $M, s \models_g \Box \varphi \iff \text{for all } w \in s : M, R[w] \models_g \varphi$

Entailment is defined as preservation of support: for a set of formulas $\Phi \cup \{\psi\}$, $\Phi \models \psi$ if for all models M , states s and assignments g , $M, s \models_g \varphi$ for all $\varphi \in \Phi$ implies $M, s \models_g \psi$. Additionally, we define a restriction of entailment to states s of cardinality at most n (in symbols, $\#s \leq n$): $\Phi \models_n \psi$ if for all models M , states s such that $\#s \leq n$ and assignments g , $M, s \models_g \varphi$ for all $\varphi \in \Phi$ implies $M, s \models_g \psi$.

The above semantics for $\text{InqQML}_{\Box}^-$ satisfies the following standard properties of inquisitive logics:

- Persistency: if $M, s \models_g \varphi$, then for all $t \subseteq s$, $M, t \models_g \varphi$
- Empty state property: $M, \emptyset \models_g \varphi$ for any formula φ

Another important desideratum of inquisitive extensions is conservativity over the original system being extended. We say that a formula φ is true at a world w (under g) if $\{w\} \models_g \varphi$. It is easy to check that, for classical formulas, the truth conditions delivered by our definition coincide with those given by standard Kripke semantics. Moreover, classical formulas are *truth-conditional*, meaning that they are supported by a state iff they are true at each of its possible worlds. Using these facts, it is easy to see that entailment among classical formulas in $\text{InqQML}_{\Box}^-$ coincides with entailment in standard constant-domain quantified modal logic. $\text{InqQML}_{\Box}^-$ can thus be seen as a conservative extension of the latter logic. Note, furthermore, that the support clause for $\Box$ implies that $\Box \varphi$ is always truth-conditional for any φ .

Coherence. A key property of formulas in $\text{InqQML}_{\Box}^-$ is *finite coherence* [19, 7], defined as follows.

Definition 2.1 (*n-coherence*). Given a state t , we denote its cardinality by $\#t$. We say that a formula φ is *n-coherent* for $n \in \mathbb{N}$ if for all M, s, g :

$$M, s \models_g \varphi \iff \text{for all } t \subseteq s \text{ such that } \#t \leq n, M, t \models_g \varphi$$

Intuitively, *n-coherence* of φ means that in order to check if φ is supported at a state, it suffices to check if it is supported by all substates of cardinality at most n . Crucially, this implies that if φ is *not* supported at a state, then there is some finite, at most n -sized substate that doesn't support φ . Notice that, in particular, 1-coherence coincides with being truth-conditional.

Proposition 2.2 ([5, Prop. 6.3]). *For every formula φ of $\text{InqQML}_{\Box}^-$ there is a number $n_{\varphi} \in \mathbb{N}$, computable from φ , such that φ is n_{φ} -coherent. In particular, n_{φ} can be computed inductively as follows: $n_{\varphi} = 1$ for any atomic formula; $n_{(\psi \wedge \xi)} = \max\{n_{\psi}, n_{\xi}\}$; $n_{(\psi \vee \xi)} = n_{\psi} + n_{\xi}$; $n_{(\psi \rightarrow \xi)} = n_{\xi}$; $n_{(\forall x_i \psi)} = n_{\psi}$ and $n_{\Box \varphi} = 1$.*

Importantly, combining this result with persistency implies that the validity of any $\text{InqQML}_{\Box}^-$ entailment can be reduced to its validity over a class of finite states with bounded cardinality.

Proposition 2.3. *For any set of $\text{InqQML}_{\Box}^-$ formulas $\Phi \cup \{\psi\}$, $\Phi \models \psi \iff \Phi \models_{n_{\psi}} \psi$.*

Using this fact, one can show that $\text{InqQML}_{\Box}^-$ is *entailment-compact*, in the following sense.

Proposition 2.4 (Entailment-compactness, [5, Prop. 6.8]). *For any set of $\text{InqQML}_{\Box}^-$ formulas $\Phi \cup \{\psi\}$, if $\Phi \models \psi$, then $\Phi_0 \models \psi$ for some finite $\Phi_0 \subseteq \Phi$.*

3 Labelled sequent calculus and completeness proof

In this section, we describe a labelled sequent calculus called IWMC (for inquisitive weak modal calculus) and prove its soundness and strong completeness for $\text{InqQML}_{\square}^-$.

Intuitively, labels represent finite states, and labelled formulas translate the support relation. Formally, labels are non-empty finite sets of indices, where each index is a natural number.⁵ We use the meta-variables w, v, u for indices, and $s, t, u, \dots$ for labels. We consider two kinds of expressions:

- Labelled formulas of the form $s : \varphi$, where φ is a formula and $s \subseteq_{\text{fin}} \mathbb{N}$
- Relational atoms of the form wRv where $w, v \in \mathbb{N}$.

As suggested by the notation, the former intuitively represent support at a state, while the latter syntactically encode instances of the accessibility relation. A sequent is an ordered pair $\Gamma \Rightarrow \Delta$ consisting of finite multisets Γ and Δ , where Γ can contain labelled formulas and relational atoms, while Δ contains only labelled formulas.

Below, we provide the rules of the sequent calculus IWMC. We write wRt for the set $\{wRv \mid v \in t\}$. The universe $W_{\Gamma, \Delta}$ of a sequent $\Gamma \Rightarrow \Delta$ is the set of indices that occur in it, within labels or in relational atoms.

The Sequent Calculus IWMC

$$\begin{array}{c}
\frac{}{s : P(\bar{x}), \Gamma \Rightarrow \Delta, t : P(\bar{x})} \text{(id) where } s \supseteq t \qquad \frac{}{s : \perp, \Gamma \Rightarrow \Delta} (\perp \Rightarrow) \\
\\
\frac{\{\Gamma \Rightarrow \Delta, \{k\} : P(\bar{x}) \mid k \in s\}}{\Gamma \Rightarrow \Delta, s : P(\bar{x})} (\Rightarrow \text{at}) \\
\\
\frac{\Gamma \Rightarrow \Delta, s : \varphi \quad \Gamma \Rightarrow \Delta, s : \psi}{\Gamma \Rightarrow \Delta, s : \varphi \wedge \psi} (\Rightarrow \wedge) \qquad \frac{s : \varphi, s : \psi, \Gamma \Rightarrow \Delta}{s : \varphi \wedge \psi, \Gamma \Rightarrow \Delta} (\wedge \Rightarrow) \\
\\
\frac{\Gamma \Rightarrow \Delta, s : \varphi, s : \psi}{\Gamma \Rightarrow \Delta, s : \varphi \vee \psi} (\Rightarrow \vee) \qquad \frac{s : \varphi, \Gamma \Rightarrow \Delta \quad s : \psi, \Gamma \Rightarrow \Delta}{s : \varphi \vee \psi, \Gamma \Rightarrow \Delta} (\vee \Rightarrow) \\
\\
\frac{\{t : \varphi, \Gamma \Rightarrow \Delta, t : \psi \mid t \subseteq s\}}{\Gamma \Rightarrow \Delta, s : \varphi \rightarrow \psi} (\Rightarrow \rightarrow) \\
\\
\frac{s : \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta, t : \varphi \quad t : \psi, s : \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}{s : \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta} (\rightarrow \Rightarrow) \text{ where } t \subseteq s \\
\\
\frac{\Gamma \Rightarrow \Delta, s : \varphi[z/x]}{\Gamma \Rightarrow \Delta, s : \forall x \varphi} (\Rightarrow \forall)^\dagger \qquad \frac{s : \varphi[y/x], s : \forall x \varphi, \Gamma \Rightarrow \Delta}{s : \forall x \varphi, \Gamma \Rightarrow \Delta} (\forall \Rightarrow) \\
\\
\frac{\{wRt, \Gamma \Rightarrow \Delta, t : \varphi \mid w \in s\}}{\Gamma \Rightarrow \Delta, s : \Box \varphi} (\Rightarrow \Box)^\ddagger \qquad \frac{t : \varphi, s : \Box \varphi, wRt, \Gamma \Rightarrow \Delta}{s : \Box \varphi, wRt, \Gamma \Rightarrow \Delta} (\Box \Rightarrow) \text{ where } w \in s
\end{array}$$

$^\dagger = \text{“}z \text{ does not occur in the conclusion”}$

$^\ddagger = \text{“}\#t = n_\varphi \text{ and } t \cap (W_{\Gamma, \Delta} \cup s) = \emptyset\text{”}$

⁵We exclude empty labels for simplicity. While the empty state is officially allowed in the semantics of $\text{InqQML}_{\square}^-$, it plays a trivial role, since it supports any formula whatsoever, and can be omitted without affecting the logic.

As mentioned in the introduction, the rules $(\Box \Rightarrow)$ and $(\Rightarrow \Box)$ are inspired by standard rules for $\Box$ in labelled sequent calculi [26]. The side condition for the right rule, however, is connected to the finite coherence property stated in Proposition 2.2. The rule can be read as follows: if for each world w in a state s , an arbitrary set t of successors of w of cardinality $\leq n_\varphi$ supports φ , then s supports $\Box\varphi$. This rule is sound (as we will see) since the support of φ on an arbitrary set t of at most n_φ -many successors of w guarantees its support on the whole set $R[w]$ of successors by Proposition 2.2.

Definition 3.1 (Derivability). A derivation in IWMC is a finite tree generated using the inference rules of IWMC whose initial sequents are instances of (id) or $(\perp \Rightarrow)$. We say that a sequent $\Gamma \Rightarrow \Delta$ is derivable in IWMC, written $\vdash \Gamma \Rightarrow \Delta$, if there is a derivation having $\Gamma \Rightarrow \Delta$ as its root.

It is clear by inspecting the rules that IWMC satisfies an appropriate notion of analyticity. Let the set of subformulas of a formula φ , $\text{SubF}(\varphi)$ be given in the usual way. Then, if we define the set of subexpressions of φ as $\text{Sub}(\varphi) = \text{SubF}(\varphi) \cup \{\psi[y/x] \mid \psi \in \text{SubF}(\varphi), y \text{ free for } x \text{ in } \psi\}$, the following holds.

Proposition 3.2 (Analyticity). *Given a derivation of a sequent $\Gamma \Rightarrow \Delta$ in IWMC, for any labelled formula $t : \psi$ appearing in the derivation there is some labelled formula $s : \varphi \in \Gamma \cup \Delta$ such that $\psi \in \text{Sub}(\varphi)$.*

3.1 Interpretation and validity for sequents

We start by defining a notion of satisfaction for labelled formulas, relational atoms and labelled sequents. For this, we make use of the notion of *mappings* into a model, i.e. functions associating each natural number to some possible world in the universe of the model.

Definition 3.3 (Satisfaction). Given a model $M = \langle W, R, D, I \rangle$, an assignment $g : \text{Var} \rightarrow D$ and a mapping $f : \mathbb{N} \rightarrow W$, we define satisfaction for labelled formulas $s : \varphi$ and relational atoms wRv as follows,

$$\begin{aligned} M, f \Vdash_g s : \varphi &\iff M, f[s] \models_g \varphi \\ M, f \Vdash_g wRv &\iff f(w)Rf(v) \end{aligned}$$

For multisets Γ of labelled formulas or relational atoms, we write $M, f \Vdash_g \Gamma$ if $M, f \Vdash_g \gamma$ for all $\gamma \in \Gamma$. Note that in particular, for a set $wRs = \{wRv \mid v \in s\}$ of relational atoms we have

$$M, f \Vdash_g wRs \iff f(w)Rf(v) \text{ for all } v \in s \iff f[s] \subseteq R[f(w)]$$

Finally, for sequents $\Gamma \Rightarrow \Delta$, we let:

$$M, f \Vdash_g \Gamma \Rightarrow \Delta \iff M, f \Vdash_g \Gamma \text{ implies } M, f \Vdash_g \delta \text{ for some } \delta \in \Delta$$

We may now define a sequent to be *valid* if it is satisfied under any interpretation.

Definition 3.4 (Validity). We say that a sequent $\Gamma \Rightarrow \Delta$ is valid, and write $\models \Gamma \Rightarrow \Delta$, if for any model M , assignment g and mapping f into M we have $M, f \Vdash_g \Gamma \Rightarrow \Delta$.

The notion of validity for sequents is connected to the notion of entailment between $\text{InqQML}_\Box^-$ -formulas via coherence. Indeed, due to Proposition 2.3, an entailment $\Phi \models \psi$ is valid iff it is valid on arbitrary states of cardinality at most n_ψ (where n_ψ is the coherence estimate for ψ given by 2.2). These states are exactly the possible interpretations, via mappings, of a label s of cardinality n_ψ . This underpins the following connection.

Proposition 3.5. *For any finite set of $\text{InqQML}_{\square}^{-}$ formulas $\Phi \cup \{\psi\}$ and an arbitrary label s with $\#s \geq n_{\psi}$, letting $s : \Phi$ denote $\{s : \varphi \mid \varphi \in \Phi\}$ we have*

$$\Phi \models \psi \quad \text{iff} \quad \text{the sequent } (s : \Phi \Rightarrow s : \psi) \text{ is valid}$$

Proof. $(\Rightarrow)$: By contraposition. Assume that $s : \Phi \Rightarrow s : \psi$ is not valid. Then, there is some model M , mapping f and assignment g such that $M, f \not\models_g s : \Phi \Rightarrow s : \psi$. Therefore, $M, f[s] \models_g \varphi$ for all $\varphi \in \Phi$ and $M, f[s] \not\models_g \psi$. It follows that $\Phi \not\models \psi$.

$(\Leftarrow)$: By contraposition. Assume that $\Phi \not\models \psi$. Then, by Proposition 2.3, $\Phi \not\models_{n_{\psi}} \psi$, i.e., there exists a model M , a state s with $\#s \leq n_{\psi}$ and an assignment g such that $M, s \models_g \varphi$ for all $\varphi \in \Phi$ and $M, s \not\models_g \psi$. Now, let f be any mapping such that $f[s] = s$. Such a mapping exists, since $\#s \leq n_{\psi} \leq \#s$. Clearly, $M, f \models_g s : \varphi$ for all $\varphi \in \Phi$ and $M, f \not\models_g s : \psi$, which implies that $M, f \not\models_g s : \Phi \Rightarrow s : \psi$. $\square$

Given this connection, we say that a derivation in our proof system is a *proof* of an entailment $\Phi \models \psi$ in case its conclusion is a sequent of the form $s : \varphi_1, \dots, s : \varphi_n \Rightarrow s : \psi$ for some formulas $\varphi_1, \dots, \varphi_n \in \Phi$ and some label s with $\#s \geq n_{\psi}$.

3.2 Illustration

To illustrate the calculus, we provide a proof of the $\text{InqQML}_{\square}^{-}$ entailment $\square \forall x ?Px \models \square ?\forall x Px$. Since the coherence estimate for the conclusion is $n_{\square ?\forall x Px} = 1$, it suffices to give a proof of a sequent of the form $\{w\} : \square \forall x ?Px \Rightarrow \{w\} : \square ?\forall x Px$, involving a label of size 1.

To ease notation, we display labels as sequences of indices rather than sets; for instance, instead of $\{1\} : \varphi$ we write simply $1 : \varphi$. For simplicity, we make use of weakening rules, whose admissibility will be proved in Section 3.4, and we occasionally apply multiple rules in one step, when no ambiguity arises. We also derive two branches, $\star_2$ and $\star_3$, separately, due to space constraints.

$$\begin{array}{c}
 \frac{23 : Py, 23 : \forall x Px, 23 : \forall x ?Px \Rightarrow 23 : Py, 23 : \perp}{23 : \forall x Px, 23 : \forall x ?Px \Rightarrow 23 : Py, 23 : \perp} \text{(id)} \quad \vdots \quad \vdots \\
 \frac{}{} (\forall \Rightarrow) \quad (\star_2) \quad (\star_3) \quad (\Rightarrow \rightarrow) \\
 \hline
 \frac{23 : \forall x ?Px \Rightarrow 23 : \neg \forall x Px, 23 : Py}{23 : \forall x ?Px \Rightarrow 23 : \forall x Px, 23 : \neg \forall x Px} (\Rightarrow \forall) \\
 \frac{}{} (\Rightarrow \forall) \\
 \frac{23 : \forall x ?Px \Rightarrow 23 : ?\forall x Px}{23 : \forall x ?Px, 1R23, 1 : \square \forall x ?Px \Rightarrow 23 : ?\forall x Px} (\Rightarrow \forall) \\
 \frac{}{} (\square \Rightarrow) \\
 \frac{1R2, 1R3, 1 : \square \forall x ?Px \Rightarrow 23 : ?\forall x Px}{1 : \square \forall x ?Px \Rightarrow 1 : \square ?\forall x Px} (\Rightarrow \square)
 \end{array}$$

The steps of this first half of the derivation are straightforward, and no choices need to be made to perform them. The only exception is the final application of $(\forall \Rightarrow)$ in the leftmost branch, where using the variable y is the only reasonable choice to obtain an (id) initial sequent. The application of rule $(\Rightarrow \rightarrow)$ generates three branches, one for each non-empty subset of the label 23. The branch corresponding to the subset 23 is displayed in the proof; the branch $\star_2$ corresponding to the subset 2 is shown below; finally, the branch $\star_3$ is analogous to $\star_2$, but with the label 3 playing the role of the label 2.

$$\begin{array}{c}
 \frac{}{} \text{(id)} \quad \frac{\dots, 2 : Py \Rightarrow \dots, 2 : Py}{2 : \perp, \dots \Rightarrow \dots} (\perp \Rightarrow) \\
 \frac{23 : Py, \dots \Rightarrow 23 : Py}{23 : \neg Py, 2 : Py \Rightarrow 23 : Py} (\Rightarrow \rightarrow) \\
 \frac{}{} (\forall \Rightarrow) \\
 \frac{2 : Py, 23 : ?Py \Rightarrow 23 : Py}{2 : Py, 23 : ?Py, 2 : \forall x Px, 23 : \forall x ?Px \Rightarrow 23 : Py, 2 : \perp} (\Rightarrow \forall), (\Rightarrow \forall) \\
 \frac{}{} (\forall \Rightarrow), (\forall \Rightarrow) \\
 \frac{2 : \forall x Px, 23 : \forall x ?Px \Rightarrow 23 : Py, 2 : \perp}{\star_2}
 \end{array}$$

Semantically, providing a derivation of branch $\star_2$ corresponds to showing that, if the truth value of Py is uniform in the state 23 (which we know from 23 : $\forall x?Px$) and the world 2 makes Py true (from 2 : $\forall xPx$), then we can show that Py is true across the whole state 23. Therefore, we proceed as follows: we apply $(\forall \Rightarrow)$ twice and instantiate it using the variable y . Then, in the right branch generated by $(\forall \Rightarrow)$, we apply $(\rightarrow \Rightarrow)$ picking 2 as our choice of subset for the label 23. This allows us to use the information that Py is true at world 2 to conclude the derivation.

3.3 Soundness.

The soundness of our system amounts to the claim that if a sequent is derivable, it is valid.

Proposition 3.6 (Soundness). *For any sequent $\Gamma \Rightarrow \Delta$ we have: $\vdash \Gamma \Rightarrow \Delta$ implies $\models \Gamma \Rightarrow \Delta$.*

Proof. We prove soundness by induction on the structure of a derivation of $\Gamma \Rightarrow \Delta$. Therefore, it suffices to show that validity is preserved by the rules. Below, we give the proof steps for the cases of $(\Box \Rightarrow)$ and $(\Rightarrow \Box)$. The proof of the remaining cases remains unchanged from [22].

$(\Box \Rightarrow)$: Assume that, for some $w \in s$, the sequent $(\Gamma, s : \Box\varphi, wRt, t : \varphi \Rightarrow \Delta)$ is satisfied by all models, mappings and assignments. Let $M = \langle W, R, D, I \rangle$ be a model, g an assignment and f a mapping over M . Now, suppose that $M, f \Vdash_g \Gamma, s : \Box\varphi, wRt$. Then, in particular, we have that $M, f[s] \models_g \Box\varphi$ and $f[t] \subseteq R[f(w)]$. By the semantics of $\Box$ and by persistency, this gives us $M, f[t] \models_g \varphi$, or, equivalently, $M, f \Vdash_g t : \varphi$. Therefore, we have $M, f \Vdash_g \Gamma, s : \Box\varphi, wRt, t : \varphi$. By our initial assumption, this implies that $M, f \Vdash_g \delta$ for some $\delta \in \Delta$. This shows that the sequent $(\Gamma, s : \Box\varphi, wRt \Rightarrow \Delta)$ is valid.

$(\Rightarrow \Box)$: Assume that there is a label t with $\#t = n_\varphi$, $t \cap (W_{\Gamma, \Delta} \cup s) = \emptyset$ and such that, for all $w \in s$, for any model, mapping and assignment, the sequent $(\Gamma, wRt \Rightarrow \Delta, t : \varphi)$ is satisfied. Then, let $M = \langle W, R, D, I \rangle$ be a model, g an assignment and f a mapping over M , and assume that $M, f \Vdash_g \Gamma$, and that for all $\delta \in \Delta$, $M, f \not\Vdash_g \delta$. We prove by contradiction that $M, f \Vdash_g s : \Box\varphi$. Suppose otherwise. We have:

$$\begin{aligned}
& M, f \not\Vdash_g s : \Box\varphi, \\
& \iff M, f[s] \not\models_g \Box\varphi, \\
& \iff \text{for some } w \in f[s], M, R[w] \not\models_g \varphi \\
& \iff \text{for some } w \in s, M, R[f(w)] \not\models_g \varphi \\
& \iff \text{for some } w \in s \text{ and some } s' \subseteq R[f(w)] \text{ s.t. } \#s' \leq n_\varphi : M, s' \not\models_g \varphi. \quad (\text{by Prop. 2.2})
\end{aligned}$$

Now, we define a new mapping f' such that $f'[t] = s'$ and f' coincides with f on all indices not in t . Note that we can define such an f' because $\#t = n_\varphi \geq \#s'$. For some $w \in s$ we have $M, f' \Vdash_g wRt$ and $M, f' \not\Vdash_g t : \varphi$, since $f'[t] = s'$, $s' \subseteq R[f(w)]$ and $M, s' \not\models_g \varphi$. Moreover, $M, f' \Vdash_g \Gamma$ and $M, f' \not\Vdash_g \delta$ for all $\delta \in \Delta$, since $t \cap (W_{\Gamma, \Delta} \cup s) = \emptyset$, so f' agrees with f on all indices occurring in Γ or Δ . Therefore, for some $w \in s$, the sequent $(\Gamma, wRt \Rightarrow \Delta, t : \varphi)$ is not satisfied by M, f', g , contradicting the assumption that this sequent is valid. $\square$

We have thus proved that our proof system IWMC is sound for $\text{InqQML}_{\Box}^-$. Before proving that it is also complete, we study the proof-theoretic properties of the calculus.

3.4 Structural properties

In this section, we show the admissibility of weakening, contraction and cut rules for IWMC, as well as the invertibility of all rules in the system. Besides their intrinsic interest, these results will play a role below in the completeness proof.

We will use the notion of *height* of a derivation, defined as the length of one of its longest branches, and we will write $\vdash_n \Gamma \Rightarrow \Delta$ when there is a derivation of $\Gamma \Rightarrow \Delta$ of height at most n . We will say that a rule is *admissible* if whenever its premisses are derivable, then its conclusion is also derivable. A rule is said to be *height-preserving admissible* if whenever all its premisses have a derivation of height at most n , then its conclusion also has a derivation of height at most n . By a rule being *invertible*, we mean that all rules having its conclusion as premiss and one of its premisses as conclusion are admissible.

Definition 3.7 (Label substitution). Given a multiset Γ of labelled formulas and relational atoms and indices $w, v \in \mathbb{N}$, we write $\Gamma[v/w]$ to indicate the multiset of labelled formulas obtained by replacing, in each element of Γ , every occurrence of w with an occurrence of v . Similarly, given a label $s = \{w_1, \dots, w_n\}$ with $w_1 < w_2 < \dots < w_n$, and a sequence $u = v_1, \dots, v_n$ of (not necessarily distinct) indices, whose length matches the size of s , we write $\Gamma[u/s]$ to mean $\Gamma[v_1/w_1, \dots, v_n/w_n]$, where all substitutions are performed simultaneously.

Lemma 3.8 (Relabeling lemma). *If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma[u/s] \Rightarrow \Delta[u/s]$ for any label s and sequence of natural numbers u s.t. $\text{length}(u) = \#s$.*

Proof. By induction on the height n of the derivation. If $n = 0$, the sequent is either an instance of (id) or $(\perp \Rightarrow)$ and the result is trivial. Assume then that the claim holds for all $k < n + 1$. Then we can proceed by cases on the last rule applied. The cases of $\text{at}, \wedge, \vee, \rightarrow, \forall$ -rules all follow immediately by induction (note that for $(\rightarrow \Rightarrow)$, the condition on the labels remains satisfied after any substitution), and so does the $(\Box \Rightarrow)$ case. This leaves out only the $(\Rightarrow \Box)$ case.

Assume $\vdash_{n+1} \Gamma \Rightarrow \Delta, s : \Box \varphi$ and that for some t s.t. $\#t = n_\varphi$ and $t \cap (W_{\Gamma, \Delta} \cup s) = \emptyset$: for all $w \in s$, $\vdash_n \Gamma, wRt \Rightarrow \Delta, t : \varphi$.

If $u \cap t = \emptyset$, we apply the inductive hypothesis to get that $\vdash_n \Gamma[u/s], wRt[u/s] \Rightarrow \Delta[u/s], t : \varphi[u/s]$ for all $w \in s$. We conclude by applying $(\Rightarrow \Box)$, which is possible because $t \cap s = \emptyset$ and, therefore, $t[u/s] = t$. As $u \cap t = \emptyset$, $t[u/s]$ still satisfies the side conditions.

If $u \cap t \neq \emptyset$, we first apply the inductive hypothesis to replace t with a label t' of equal cardinality, but disjoint from both u , t and $W_{\Gamma, \Delta} \cup s$, and we get: for all $w \in s$, $\vdash_n \Gamma, wRt' \Rightarrow \Delta, t' : \varphi$. Since t' satisfies the side conditions of $(\Rightarrow \Box)$, we then apply the inductive hypothesis to perform the label substitution $[u/s]$, and we conclude as in the previous case. $\square$

Proposition 3.9. *Let σ stand for either a labelled formula $u : \varphi$ or a relational atom wRv .*

1. *The following weakening rules are height-preserving admissible in IWMC:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, u : \varphi} (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \sigma \Rightarrow \Delta} (w \Rightarrow)$$

2. *All rules of IWMC are height-preserving invertible.*

3. *The following contraction rules are height-preserving admissible in IWMC:*

$$\frac{\Gamma \Rightarrow \Delta, u : \varphi, u : \varphi}{\Gamma \Rightarrow \Delta, u : \varphi} (\Rightarrow c) \quad \frac{\Gamma, \sigma, \sigma \Rightarrow \Delta}{\Gamma, \sigma \Rightarrow \Delta} (c \Rightarrow)$$

Proof of 1. By induction on the height of the derivation of the premiss $\Gamma \Rightarrow \Delta$. The case of $n = 0$ is standard for both weakening rules. For the inductive case, assume $\vdash_{n+1} \Gamma \Rightarrow \Delta$. The proof proceeds by cases on the last applied rule in the derivation. We only detail the cases involving the modal rules for $(\Rightarrow w)$: the modal cases for $(w \Rightarrow)$ are analogous, and all non-modal cases are standard.

- $(\Box \Rightarrow)$: If the last applied rule in the derivation is $(\Box \Rightarrow)$, then $\Gamma = \Gamma', s : \Box \psi, wRt$ for some $w \in s$ and the premiss is $\Gamma', s : \Box \psi, wRt, t : \psi \Rightarrow \Delta$. Therefore, $\vdash_n \Gamma', s : \Box \psi, wRt, t : \psi \Rightarrow \Delta$. By induction, $\vdash_n \Gamma', s : \Box \psi, wRt, t : \psi \Rightarrow \Delta, u : \phi$. By applying $(\Box \Rightarrow)$, we then find a derivation of height $n + 1$ of $\Gamma \Rightarrow \Delta, u : \phi$.
- $(\Rightarrow \Box)$: If the last applied rule in the derivation is $(\Rightarrow \Box)$, then $\Delta = \Delta', s : \Box \psi$, and, for some t s.t. $\#t = n_\psi$ and $t \cap W_{\Gamma, \Delta} = \emptyset$: for all $w \in s$, $\vdash_n \Gamma, wRt \Rightarrow \Delta', t : \psi$. Now, we distinguish two cases. If $t \cap u = \emptyset$, we proceed as above. If $t \cap u \neq \emptyset$, thanks to Lemma 3.8 and to the fact that u and $W_{\Gamma, \Delta}$ are finite, we can replace the label t with a label t' s.t. $t' \cap (t \cup W_{\Gamma, \Delta} \cup u) = \emptyset$ and we have, for all $w \in s$, $\vdash_n \Gamma, wRt' \Rightarrow \Delta', t' : \psi$. By the inductive hypothesis, for all $w \in s$, $\vdash_n \Gamma, wRt' \Rightarrow \Delta', u : \phi, t' : \psi$. Since t' still satisfies the side conditions of $(\Rightarrow \Box)$, we have $\vdash_{n+1} \Gamma \Rightarrow \Delta, u : \phi$. $\square$

Proof of 2. To show that all rules are invertible, we structure our proof following standard arguments from [28]. We proceed one rule at a time, by induction on the height of the derivation. The base step of the induction is straightforward and standard for both the first-order rules and the modal rules. The inductive step is divided into four cases, depending on (i) whether the last rule applied is the one being inverted and (ii) whether the principal formula of the last applied rule is principal in the inverted rule. Below, we only detail the steps involving the modal rules, as the rest remain unchanged from [22].

Invertibility of the non-modal rules. We need to add two clauses to the case where the last rule applied is not the one being inverted, and it is a modal rule. If the last rule is $(\Box \Rightarrow)$, the proof is straightforward. If the last rule is $(\Rightarrow \Box)$, the argument is the same, but it requires observing that the natural numbers occurring in labels in the premisses are always a subset of those found in the conclusion, so the side conditions on t remain satisfied.

Invertibility of $(\Box \Rightarrow)$. Follows immediately from the height-preserving admissibility of weakening.

Invertibility of $(\Rightarrow \Box)$. Assume $\vdash_{n+1} \Gamma \Rightarrow \Delta, s : \Box \phi$. We distinguish several cases:

- Last applied rule is $(\Rightarrow \Box)$ and $s : \Box \phi$ is principal: immediate.
- Last applied rule is $(\Rightarrow \Box)$ and $s : \Box \phi$ is not principal: then, for some u and ψ , letting $\Delta = \Delta', u : \Box \psi$ we have $\vdash_{n+1} \Gamma \Rightarrow \Delta', u : \Box \psi, s : \Box \phi$ and, for some t s.t. $\#t = n_\psi$ and $t \cap (W_{\Gamma, \Delta'} \cup s \cup u) = \emptyset$: for all $v \in u$, $\vdash_n \Gamma, vRt \Rightarrow \Delta', t : \psi, s : \Box \phi$.

By inductive hypothesis, for each $v \in u$ there is t_v with $\#t_v = n_\phi$ and $t_v \cap (W_{\Gamma, \Delta'} \cup s \cup t \cup \{v\}) = \emptyset$ and such that: for all $w \in s$, $\vdash_n \Gamma, vRt, wRt_v \Rightarrow \Delta', t : \psi, t_v : \phi$. We then apply Lemma 3.8 to substitute all the labels t_v with a single label t_s , taken to be disjoint from any labels included so far. As a result, we can reframe the set of derivable sequents as follows. For any $w \in s$, for all $v \in u$, $\vdash_n \Gamma, vRt, wRt_s \Rightarrow \Delta', t : \psi, t_s : \phi$. For each $w \in s$, we can consider the corresponding set of sequents, to which, since t still satisfies the side conditions, we can then apply rule $(\Rightarrow \Box)$ (with vRt and $t : \psi$ principal), and conclude $\vdash_{n+1} \Gamma, wRt_s \Rightarrow \Delta', u : \Box \psi, t_s : \phi$ for all $w \in s$.

- Last applied rule is not $(\Rightarrow \Box)$: we only show how to proceed for the rule $(\Rightarrow \forall)$, which has side conditions, and for the case of $(\Rightarrow at)$, which requires additional observations. The case of $(\Rightarrow \rightarrow)$ is analogous to that of $(\Rightarrow at)$.

- Last applied rule is $(\Rightarrow\forall)$: then, letting $\Delta = \Delta', u : \forall x\psi$, we have $\vdash_{n+1} \Gamma \Rightarrow \Delta', s : \Box\varphi, u : \forall x\psi$ and, for some z not occurring in the conclusion, $\vdash_n \Gamma \Rightarrow \Delta', s : \Box\varphi, u : \psi[z/x]$. By inductive hypothesis, we get, for some t s.t. $\#t = n_\varphi$ and $t \cap (W_{\Gamma, \Delta'} \cup s \cup u) = \emptyset$: for all $w \in s$, $\vdash_n \Gamma, wRt \Rightarrow \Delta', t : \varphi, u : \psi[z/x]$. We apply $(\Rightarrow\forall)$ (since z did not occur in the conclusion and, therefore, in φ), and conclude that, for some t s.t. $\#t = n_\varphi$ and $t \cap (W_{\Gamma, \Delta'} \cup s \cup u) = \emptyset$: for all $w \in s$, $\vdash_{n+1} \Gamma, wRt \Rightarrow \Delta', t : \varphi, u : \forall x\psi$.
- Last applied rule is $(\Rightarrow\text{at})$: then, letting $\Delta = \Delta', u : P(\bar{x})$, we have $\vdash_{n+1} \Gamma \Rightarrow \Delta', s : \Box\varphi, u : P(\bar{x})$ and, for all $k \in u$, $\vdash_n \Gamma \Rightarrow \Delta', s : \Box\varphi, k : P(\bar{x})$. Then, by inductive hypothesis, we get that, for all $k \in u$, there is some t_k such that $\#t_k = n_\varphi$, $t_k \cap (W_{\Gamma, \Delta'} \cup s \cup \{k\}) = \emptyset$ and such that for all $w \in s$, $\vdash_n \Gamma, wRt_k \Rightarrow \Delta', t_k : \varphi, k : P(\bar{x})$. Then, for each $k \in u$, we apply Lemma 3.8 to each sequent in the corresponding set and use it to replace t_k with t' , a label of the same cardinality which is disjoint both from $W_{\Gamma, \Delta}$, s , and u and from $\bigcup_{k \in u} t_k$ (such t' exists, since Γ and Δ are finite). Then, we have that for some t' s.t. $\#t' = n_\varphi$ and $t' \cap (W_{\Gamma, \Delta} \cup s \cup u) = \emptyset$ for all $k \in u$, for all $w \in s$, $\vdash_n \Gamma, wRt' \Rightarrow \Delta', t' : \varphi, k : P(\bar{x})$. We then apply $(\Rightarrow\text{at})$ to get that for some t' s.t. $\#t' = n_\varphi$ and $t' \cap (W_{\Gamma, \Delta} \cup s \cup u) = \emptyset$: for all $w \in s$, $\vdash_{n+1} \Gamma, wRt' \Rightarrow \Delta', t' : \varphi, u : P(\bar{x})$ $\square$

Proof of 3. By induction on the height of the derivation of the premiss. The initial step of the induction is straightforward. The inductive step is divided by cases, depending on the last rule applied in the derivation and on whether the contracted formula was principal in the last step. Those cases where the contracted formula is not principal in the last applied rule in the derivation are standard: we apply the inductive hypothesis to each premiss, then apply the last rule to the contracted sequents.

Of the cases where the formula being contracted is principal in the last applied rule, we only spell out those that were not subsumed in the first-order case:

$(\Box\Rightarrow)$ for $(c\Rightarrow)$: There are two possibilities: either we want to contract a double occurrence of $s : \Box\varphi$ or one of wRv . Both follow easily, by applying the inductive hypothesis to the premiss, where the duplicate formula is still present.

$(\Rightarrow\Box)$ for $(\Rightarrow c)$: Assuming contraction holds up to height n , suppose that $\vdash_{n+1} \Gamma \Rightarrow \Delta, s : \Box\varphi, s : \Box\varphi$. Then, for some t s.t. $\#t = n_\varphi$ and $t \cap (W_{\Gamma, \Delta} \cup s) = \emptyset$: for all $w \in s$, $\vdash_n \Gamma, wRt \Rightarrow \Delta, s : \Box\varphi, t : \varphi$. By the height-preserving invertibility of $(\Rightarrow\Box)$, we get that for the same t as above, for all $w \in s$, there is a t_w with $\#t_w = n_\varphi$ and $t_w \cap (W_{\Gamma, \Delta} \cup s \cup t) = \emptyset$ and such that: for all $v \in s$, $\vdash_n \Gamma, wRt, vRt_w \Rightarrow \Delta, t : \varphi, t_w : \varphi$. By Lemma 3.8, since both t and t_w have cardinality n_φ , we can perform the label substitution $[t/t_w]$ and get that for all $w, v \in s$, $\vdash_n \Gamma, wRt, vRt \Rightarrow \Delta, t : \varphi, t : \varphi$. In particular, this holds in the case where $v = w$, so we have that, for all $w \in s$, $\vdash_n \Gamma, wRt, wRt \Rightarrow \Delta, t : \varphi, t : \varphi$. By inductive hypothesis, we can apply left contraction $\#t$ times (recall that $wRt = \{wRv \mid v \in t\}$) and right contraction once to get $\vdash_n \Gamma, wRt \Rightarrow \Delta, t : \varphi$ for all $w \in s$. The conclusion then follows by applying $(\Rightarrow\Box)$. $\square$

Proposition 3.10. *The rule of cut is admissible in IWMC:*

$$\frac{\Gamma \Rightarrow \Delta, s : \varphi \quad s : \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (cut)}$$

Proof. The proof follows a standard structure for proofs of cut admissibility: an induction on the length of the *cut formula* φ and a subinduction on the sum of the heights of the derivations of the premisses. We start from two assumptions, $\vdash_n \Gamma \Rightarrow \Delta, s : \varphi$ and $\vdash_m s : \varphi, \Pi \Rightarrow \Sigma$, and we aim to reach the conclusion $\vdash \Gamma, \Pi \Rightarrow \Delta, \Sigma$. Let us call a formula *principal* in a premiss when it is principal in the last applied rule in the derivation of that premiss. There are two cases that we need to consider that cannot be transferred or straightforwardly adapted from the cut admissibility proof given in [22].

First, suppose that the cut formula is not principal in at least one premiss and that the last applied rule in that premiss is $(\Box \Rightarrow)$ or $(\Rightarrow \Box)$. We spell out a representative case, as the others follow similar arguments. Assume that the cut formula $s : \varphi$ is not principal in the left premiss and that the last applied rule in that premiss is $(\Rightarrow \Box)$. Then, the assumption for the left premiss has the form $\vdash_n \Gamma \Rightarrow \Delta', u : \Box \psi, s : \varphi$, and it is the case that for some t s.t. $\#t = n_\psi$ and $t \cap (W_{\Gamma, \Delta'} \cup s \cup u) = \emptyset$, for all $w \in u$, $\vdash_{n-1} \Gamma, wRt \Rightarrow \Delta', t : \psi, s : \varphi$. We can then combine the $(n-1)$ -derivability of these sequents with the hypothesis about the right premiss, $\vdash_m s : \varphi, \Pi \Rightarrow \Sigma$, to apply the cut rule (available here by subinduction hypothesis on the sum of the heights $n+m$) and obtain, for all $w \in u$, $\vdash \Gamma, wRt, \Pi \Rightarrow \Delta', t : \psi, \Sigma$. At this point, if needed, we can invoke Lemma 3.8 to substitute t with a label t' such that $t' \cap (W_{\Gamma, \Pi, \Delta', \Sigma} \cup u \cup t) = \emptyset$, and we can then conclude by an application of $(\Rightarrow \Box)$:

$$\frac{\{\Gamma, wRt', \Pi \Rightarrow \Delta', t' : \psi, \Sigma \mid w \in u\}}{\Gamma, \Pi \Rightarrow \Delta', u : \Box \psi, \Sigma} (\Rightarrow \Box)$$

Now, suppose instead that the cut formula is $s : \Box \varphi$ and that the last step of both derivations is a modal rule with $s : \Box \varphi$ principal. Then, from the right premiss, we get $\Pi = \Pi', wRt$ for some $w \in s$ and $\vdash_{m-1} \Pi', s : \Box \varphi, wRt, t : \varphi \Rightarrow \Sigma$. From the left premiss, we have that for some u s.t. $\#u = n_\varphi$ and $u \cap (W_{\Gamma, \Delta} \cup s) = \emptyset$, for all $v \in s$, $\vdash_{n-1} \Gamma, vRu \Rightarrow \Delta, u : \varphi$. From the last item, we can isolate the specific case of $v = w$ and, by Lemma 3.8, we can perform the substitution of u with t , obtaining $\vdash_{n-1} \Gamma, wRt \Rightarrow \Delta, t : \varphi$. We prove the conclusion $\vdash \Gamma, \Pi', wRt \Rightarrow \Delta, \Sigma$ by constructing the following derivation:

$$\frac{\frac{\Gamma, wRt \Rightarrow \Delta, t : \varphi}{\Gamma, wRt, \Gamma, \Pi', wRt \Rightarrow \Delta, \Delta, \Sigma} (\text{cut})}{\Gamma, \Pi', wRt \Rightarrow \Delta, \Sigma} (\text{cut})$$

$$\frac{\Gamma \Rightarrow \Delta, s : \Box \varphi \quad \Pi', s : \Box \varphi, wRt, t : \varphi \Rightarrow \Sigma}{\Gamma, \Pi', wRt, t : \varphi \Rightarrow \Delta, \Sigma} (\text{cut})$$

$$\frac{\Gamma, wRt, \Gamma, \Pi', wRt \Rightarrow \Delta, \Delta, \Sigma}{\Gamma, \Pi', wRt \Rightarrow \Delta, \Sigma} (c \Rightarrow), (\Rightarrow c)$$

The first application of the cut rule is enabled by the subinductive hypothesis about the heights n and m , while the second application relies on the inductive hypothesis about the length of φ . The third step implicitly includes multiple applications of the contraction rules. $\square$

3.5 Completeness

We prove completeness by showing that from any non-derivable sequent we can build a canonical model that refutes it. The proof generalizes the approach of [22] to the modal case. First, we define a notion of saturation and prove that non-derivable sequents can be extended to saturated ones. Then, we show how to obtain a canonical model from a saturated sequent, and prove that this model refutes the original sequent. This establishes that each valid sequent is derivable. Finally, we lift this to a strong completeness theorem for $\text{InqQML}_{\Box}^-$ by using the compactness and coherence properties of the logic.

For the proof, we need to extend our range of syntactic objects to include infinite sequents. Therefore, we start by defining a generalized class of sequents which we will call *g-sequents*. A *g-sequent* is an ordered pair $\Gamma \Rightarrow \Delta$ consisting of (possibly infinite) multisets Γ and Δ , where Γ can contain labelled formulas and relational atoms, while Δ contains only labelled formulas. For a *g-sequent* $\Gamma \Rightarrow \Delta$, we define derivability in IWMC , denoted by $\vdash \Gamma \Rightarrow \Delta$, as the existence of a finite sequent $\Gamma' \Rightarrow \Delta'$ with $\Gamma' \subseteq \Gamma$, and $\Delta' \subseteq \Delta$ which is derivable in the sense of Definition 3.1. Note that every sequent $\Gamma \Rightarrow \Delta$ is also a *g-sequent*, and that the derivability of $\Gamma \Rightarrow \Delta$ as a sequent, as given by Definition 3.1, coincides with its derivability as a *g-sequent* by the admissibility of weakening.

- Case 1: $s_i : \phi_i$ is neither in Γ_i nor in Δ_i . We simply set $\Gamma_{i+1} = \Gamma_i$, $\Delta_{i+1} = \Delta_i$.
- Case 2: $s_i : \phi_i \in \Gamma_i$. Consider the set $\{t \mid \text{for some } w \in s_i, wRt \subseteq \Gamma_i\}$. This set is finite, since Γ_i is finite. Let $t_1, \dots, t_k$ be all the elements. Now we define a sequence $(\Gamma_i^j)_{j \leq k}$ by letting $\Gamma_i^0 = \Gamma_i$ and $\Gamma_i^{j+1} = \Gamma_i^j \cup \{t_{j+1} : \psi\}$.

Inductively, each sequent $\Gamma_i^j \Rightarrow \Delta_i$ is not derivable. To see this, suppose $\Gamma_i^j \Rightarrow \Delta_i$ is not derivable but $\Gamma_i^{j+1} \Rightarrow \Delta_i$ is. The latter means that we can derive $\Gamma_i^j \cup \{t_{j+1} : \psi\} \Rightarrow \Delta_i$. However, since Γ_i^j contains $s_i : \Box\psi$ and also wRt_{j+1} for some $w \in s_i$, by the rule $(\Box\Rightarrow)$ we have that $\Gamma_i^j \Rightarrow \Delta_i$ is derivable, contrary to assumption.

Finally, we set $\Gamma_{i+1} = \Gamma_i^k = \Gamma_i \cup \{t_j : \psi \mid 1 \leq j \leq k\}$ and $\Delta_{i+1} = \Delta_i$.

- Case 3: $s_i : \phi_i \in \Delta_i$. Note that since Γ_i and Δ_i are finite, the corresponding universe W_{Γ_i, Δ_i} is finite. Therefore, there are infinitely many natural numbers that are not in this universe. Let $v_1, \dots, v_{n_\psi}$ be the first n_ψ such labels and let $t = \{v_1, \dots, v_{n_\psi}\}$. Now consider for each $w \in s_i$ the corresponding sequent:

$$\Gamma_i, wRt \Rightarrow \Delta_i, t : \psi$$

If all these sequents were derivable, then by the rule $(\Rightarrow\Box)$, $\Gamma_i \Rightarrow \Delta_i, s_i : \Box\psi$ would be derivable; and since $s_i : \Box\psi \in \Delta_i$, by contraction, $\Gamma_i \Rightarrow \Delta_i$ would be derivable, contrary to assumption. So, for at least one $w \in s_i$, the corresponding sequent is not derivable. Let w^* be the least such number and define $\Gamma_{i+1} = \Gamma_i \cup w^*Rt$ and $\Delta_{i+1} = \Delta_i \cup \{t : \psi\}$.

Finally we set $\Gamma^+ = \bigcup_i \Gamma_i$ and $\Delta^+ = \bigcup_i \Delta_i$. Clearly, $\Gamma^+ \Rightarrow \Delta^+$ is not derivable: if that was the case, there would be some finite subsets $\Gamma' \subseteq \Gamma^+$ and $\Delta' \subseteq \Delta^+$ such that $\vdash \Gamma' \Rightarrow \Delta'$. Due to their finiteness, there must be some $i \in \mathbb{N}$ such that $\Gamma' \subseteq \Gamma_i$ and $\Delta' \subseteq \Delta_i$. By weakening, this would imply $\vdash \Gamma_i \Rightarrow \Delta_i$, which we know not to be the case. The fact that $\Gamma^+ \Rightarrow \Delta^+$ is saturated is straightforwardly ensured by the inductive construction. As an illustration, we show that the $(\Box L)$ condition is satisfied.

Suppose that $s : \Box\psi \in \Gamma$ and suppose that for some $w \in s$ we have $wRt \subseteq \Gamma$ for some finite set t . We need to show $t : \psi \in \Gamma$. Let i be a number such that Γ_i contains $s : \Box\psi$ as well as all the relational atoms in the set wRt (such a number exists since t is finite and so wRt is a finite set of atoms). Let $j \geq i$ be a number that enumerates the labelled formula $s : \Box\psi$ (which exists since we assumed that all labelled formulas are enumerated infinitely many times). Then since Γ_j contains $s : \Box\psi$ as well as wRt , our procedure makes sure that Γ_{j+1} (and so also Γ^+) contains $t : \psi$, as required. $\square$

We have thus shown that any non-derivable sequent can be extended to a saturated g -sequent. From any such g -sequent, we can then construct a canonical model that refutes it. The saturation conditions will guarantee that this model acts as a countermodel for the original sequent.

Canonical model construction Let $\Gamma \Rightarrow \Delta$ be a saturated g -sequent. We define a canonical model $M_{\Gamma, \Delta}^c = \langle W_{\Gamma, \Delta}, D^c, R^c, I^c \rangle$ as follows:

- $W_{\Gamma, \Delta}$ is the universe of the g -sequent $\Gamma \Rightarrow \Delta$, i.e. the set of all indices that occur in it;
- D^c is the set of variables occurring in $\Gamma \Rightarrow \Delta$;
- $wR^c v \iff wRv \in \Gamma$;
- $\langle x_1, \dots, x_n \rangle \in I_w^c(P) \iff \{w\} : P(x_1, \dots, x_n) \in \Gamma$.

Lemma 3.12 (Support lemma). *For all labelled formulas $s : \phi$,*

- if $s : \varphi \in \Gamma$ then $M_{\Gamma, \Delta}^c, s \models_{id} \varphi$
- if $s : \varphi \in \Delta$ then $M_{\Gamma, \Delta}^c, s \not\models_{id} \varphi$

Proof. By induction on φ . We consider the cases of $\varphi = P(\bar{x})$ and $\varphi = \Box\psi$ (we omit the subscript for the assignment, for readability).

- Suppose $s : P(\bar{x}) \in \Gamma$. By saturation (atL), for all $w \in s$ we have that $w : P(\bar{x}) \in \Gamma$. By definition of $M_{\Gamma, \Delta}^c$, for all $w \in s$: $\bar{x} \in I_w^c(P)$. Therefore, $M^c, s \models P(\bar{x})$.
- Suppose $s : P(\bar{x}) \in \Delta$. By saturation (atR), for some $w \in s$, we have that $w : P(\bar{x}) \in \Delta$. Clearly, then, $w : P(\bar{x}) \notin \Gamma$, or we would have that $\Gamma \Rightarrow \Delta$ is an instance of (id), contradicting (unprov). This implies that $\bar{x} \notin I_w^c(P)$, meaning that $M^c, s \not\models P(\bar{x})$.
- Suppose $s : \Box\psi \in \Gamma$. We need to show that for any $w \in s$, $M_{\Gamma, \Delta}^c, R^c[w] \models \psi$. Since ψ is finitely coherent, it suffices to show that for any $w \in s$ and any finite $t \subseteq R^c[w]$, $M_{\Gamma, \Delta}^c, t \models \psi$. So, take a $w \in s$ and a finite $t \subseteq R^c[w]$. By definition of R^c , the latter means that for each $v \in t$, $wRv \in \Gamma$. So, we have $wRt \subseteq \Gamma$. By saturation, $s : \Box\psi \in \Gamma$ and $wRt \subseteq \Gamma$ together imply $t : \psi \in \Gamma$. By induction hypothesis, $M_{\Gamma, \Delta}^c, t \models \psi$, as required.
- Suppose $s : \Box\psi \in \Delta$. By saturation, for some $w \in s$ and some finite set t we have $wRt \subseteq \Gamma$ and $t : \psi \in \Delta$. By induction hypothesis, this gives $M_{\Gamma, \Delta}^c, t \models \psi$. Since $wRt \subseteq \Gamma$ we have $t \subseteq R^c[w]$. By persistency, $M_{\Gamma, \Delta}^c, R^c[w] \not\models \psi$. Since $w \in s$, we conclude $M_{\Gamma, \Delta}^c, s \not\models \Box\psi$. $\square$

We can now prove that IWMC is complete in the sense that it derives any valid (finite) sequent.

Proposition 3.13 (Completeness for sequents). *For any sequent $\Gamma \Rightarrow \Delta$ we have:*

$$\models \Gamma \Rightarrow \Delta \quad \text{implies} \quad \vdash \Gamma \Rightarrow \Delta$$

Proof. By contraposition. Assume that $\not\models \Gamma \Rightarrow \Delta$. Then, by Lemma 3.11, $\Gamma \Rightarrow \Delta$ has a saturated extension $\Gamma^+ \Rightarrow \Delta^+$ such that $\not\models \Gamma^+ \Rightarrow \Delta^+$. By Lemma 3.12, there is M, f , and g such that $M, f \not\models_g \Gamma^+ \Rightarrow \Delta^+$ which, in particular, implies $M, f \not\models_g \Gamma \Rightarrow \Delta$. $\square$

We have thus established that IWMC is complete with respect to sequents: it derives all and only the valid ones. Via the connection provided by Proposition 3.5, it follows that IWMC is also complete (and indeed, by compactness, strongly complete) with respect to entailment in the logic $\text{InqQML}_{\Box}^-$.

Theorem 3.14 (Strong completeness for $\text{InqQML}_{\Box}^-$). *For any set of $\text{InqQML}_{\Box}^-$ formulas $\Phi \cup \{\psi\}$, letting $s_{\psi} = \{1, \dots, n_{\psi}\}$ where n_{ψ} is the coherence estimate for ψ as given by Prop. 2.2, we have:*

$$\Phi \models \psi \quad \text{iff} \quad \vdash \{s_{\psi} : \varphi \mid \varphi \in \Phi\} \Rightarrow s_{\psi} : \psi$$

Proof. ($\Rightarrow$): If $\Phi \models \psi$, by compactness (Proposition 2.4), there is some finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi$. By the definition of $\vdash$, it suffices to prove that $\vdash \{s_{\psi} : \varphi \mid \varphi \in \Phi_0\} \Rightarrow s_{\psi} : \psi$. Assume towards a contradiction that $\not\models \{s_{\psi} : \varphi \mid \varphi \in \Phi_0\} \Rightarrow s_{\psi} : \psi$. By the completeness result for sequents, this implies that $\not\models \{s_{\psi} : \varphi \mid \varphi \in \Phi_0\} \Rightarrow s_{\psi} : \psi$. Then, since Φ_0 is finite and $\#s_{\psi} = n_{\psi}$, by Proposition 3.5 we have that $\Phi_0 \not\models \psi$, which contradicts the initial assumptions.

($\Leftarrow$): By contraposition, assume that $\Phi \not\models \psi$. Then, for all finite $\Phi_0 \subseteq \Phi$, $\Phi_0 \not\models \psi$. By Proposition 3.5, since $\#s_{\psi} = n_{\psi}$, this implies that for any such Φ_0 , $\not\models \{s_{\psi} : \varphi \mid \varphi \in \Phi_0\} \Rightarrow s_{\psi} : \psi$. By the soundness result for sequents (Proposition 3.6), we have that for all finite $\Phi_0 \subseteq \Phi$, $\not\models \{s_{\psi} : \varphi \mid \varphi \in \Phi_0\} \Rightarrow s_{\psi} : \psi$ which, by the definition of $\vdash$ combined with the admissibility of $(\Rightarrow w)$, gives us $\not\models \{s_{\psi} : \varphi \mid \varphi \in \Phi\} \Rightarrow s_{\psi} : \psi$. $\square$

3.6 Adding identity.

In this section, we show how to generalize our approach to the case of signatures including the identity predicate. We start with a brief introduction to the implementation of identity in $\text{InqQML}_{\square}^-$. For a more complete presentation of identity in first-order inquisitive logic, we refer to [4, Section 5.4].

Identity in $\text{InqQML}_{\square}^-$ In first-order inquisitive logics, identity is seen as a binary predicate whose interpretation can change across possible worlds. Syntactically, we extend the definition of the language of $\text{InqQML}_{\square}^-$ to include, amongst the predicate atoms, identity atoms of the form $x = y$, where $x, y \in \text{Var}$.

Semantically, the identity predicate $=$ is interpreted as a standard binary predicate. Therefore, in a model $M = \langle W, D, R, I \rangle$, for any world $w \in W$, $I_w(=) \subseteq D^2$. However, the interpretation of the identity predicate must satisfy two additional constraints. For a predicate P and world w , let P_w and $=_w$ denote, respectively, $I_w(P)$ and $I_w(=)$. Then, for any $w \in W$:

- Congruence: for any $d_1, d'_1, \dots, d_n, d'_n \in D$ and n -ary predicate P , if $d_1 =_w d'_1, \dots, d_n =_w d'_n$, then:

$$\langle d_1, \dots, d_n \rangle \in P_w \iff \langle d'_1, \dots, d'_n \rangle \in P_w$$

- Equivalence: $=_w$ is an equivalence relation on D

The recursive definition of the support relation remains the same as in the case without identity, and the clause for identity atoms can be obtained from the one for predicates:

- $M, s \models_g x = y \iff \text{for all } w \in s, \langle g(x), g(y) \rangle \in I_w(=)$

Extending the proof system. Since we see identity as a binary predicate, we extend the applicability of $(\Rightarrow \text{at})$ and (id) to labelled identity atoms $s : x = y$. However, we also need to extend IWMC with rules encoding the specific properties of identity. We add two rules, inspired by [28, Section 6.5], and call the extended calculus $\text{IWMC}^=$. In the rules, $x, y \in \text{Var}$, $\bar{z}$ is a sequence of variables, and $\phi[[y/x]]$ denotes the substitution of an arbitrary number of instances of x in ϕ with instances of y :

$$\frac{s : x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ (Ref)} \qquad \frac{s : x = y, s : P(\bar{z}), s : P(\bar{z})[[y/x]], \Gamma \Rightarrow \Delta}{s : x = y, s : P(\bar{z}), \Gamma \Rightarrow \Delta} \text{ (Repl)}$$

The extended system $\text{IWMC}^=$ satisfies analyticity and the structural properties that we proved for IWMC in Section 3.4. Additionally, the following rules encoding the properties of $=$ as an equivalence relation are admissible:

$$\frac{s : x = y, s : y = x, \Gamma \Rightarrow \Delta}{s : x = y, \Gamma \Rightarrow \Delta} \text{ (Sym)} \qquad \frac{s : x = y, s : y = z, s : x = z, \Gamma \Rightarrow \Delta}{s : x = y, s : y = z, \Gamma \Rightarrow \Delta} \text{ (Trans)}$$

For any formula ϕ , the replacement axiom $s : x = y, s : \phi \Rightarrow s : \phi[[y/x]]$ can be shown to be derivable by adapting the argument provided in [28, Lemma 6.5.2]. Our proof uses the admissibility of the following sub-label replacement rule:

$$\frac{s : x = y, s' : \phi, s' : P(\bar{z})[[y/x]], \Gamma \Rightarrow \Delta}{s : x = y, s' : P(\bar{z}), \Gamma \Rightarrow \Delta} \text{ (Sub-Repl) where } s' \subseteq s$$

We give an informal sketch of the proof, which is mostly routine. We prove, by induction on the length of ϕ , the stronger claim that $s : x = y, s' : \phi \Rightarrow s' : \phi[[y/x]]$ is derivable for any $s' \subseteq s$. The base case for ϕ atomic is immediate by the admissibility of (Sub-Repl). The stronger claim is needed to prove the

inductive case of implication, where subsets of s are generated by the rules for $\rightarrow$. Here, the structure of the derivation is the same as in the original proof, but the inductive hypotheses involve subsets of the initial label. The remaining cases, including the inductive case of $\Box$, follow from standard arguments.

Combining the structural properties of $\text{IWMC}^\equiv$ with the derivability of the replacement axiom and the admissibility of the above rules makes it straightforward to prove the admissibility of (G-Repl), which generalizes (Repl) to arbitrary formulas, and of ($\Rightarrow$ G-Repl), a generalized right-replacement rule:

$$\frac{s : x = y, s : \varphi, s : \varphi[[y/x]], \Gamma \Rightarrow \Delta}{s : x = y, s : \varphi, \Gamma \Rightarrow \Delta} \text{ (G-Repl)} \quad \frac{s : x = y, \Gamma \Rightarrow \Delta, s : \varphi, s : \varphi[[y/x]]}{s : x = y, \Gamma \Rightarrow \Delta, s : \varphi} (\Rightarrow\text{G-Repl})$$

Using analogous arguments, one can prove the admissibility of their sub-label versions (Sub-G-Repl) and ($\Rightarrow$ Sub-G-Repl), where $s : \varphi$ and $s : \varphi[[y/x]]$ are replaced, respectively, by $s' : \varphi$ and $s' : \varphi[[y/x]]$, with the requirement that $s' \subseteq s$ (notice that the stronger claim used in the derivability proof for the replacement axiom above is exactly what is needed in this more general case).

Completeness proof. To adapt the definition of saturated sequent, we extend (atL) and (atR) to labelled identity atoms of the form $s : x = y$ and we add the following clauses:

- (=L) If $s : x = y \in \Gamma$, then: (i) if $s : \varphi \in \Gamma$, then for any possible substitution $\varphi[[y/x]]$, $s : \varphi[[y/x]] \in \Gamma$ and (ii) if $s : \varphi \in \Delta$, then for any possible substitution $\varphi[[y/x]]$, $s : \varphi[[y/x]] \in \Delta$
- (=Ref) For all $x \in \text{Var}$ that occur in $\Gamma \cup \Delta$, for all singleton labels $w \in W_{\Gamma, \Delta}$, $w : x = x \in \Gamma$

The proof of the saturation lemma requires two changes. We fix an enumeration $(z_i)_{i \in \mathbb{N}}$ of all variables in the extended language. First, for each $i \in \mathbb{N}$, we add to Γ_{i+1} the finite set of labelled formulas $\text{ld}(i) := \{k : z_j = z_j \mid 0 \leq k, j \leq i, k \in W_{\Gamma_i, \Delta_i}, \text{ and } z_j \text{ occurs in } \Gamma_i \cup \Delta_i\}$. Underivability holds by (Ref). We also extend the case of an atom $s_i : P(\bar{x})$ being in Δ_i to include identity atoms of the form $s_i : x = y$.

Then, we include one additional inductive case for identity atoms. If φ_i is $x = y$ and $s_i : x = y \in \Gamma_i$, we let $\Gamma_i^{\text{at}} = \Gamma_i \cup \{w : x = y \mid w \in s_i\}$ and we let $\text{Sub}_i(\varphi)$ be the set of all possible substitutions $s' : \varphi[[y/x]]$ for all $s' \subseteq s_i$. We define $\Gamma_{i+1} = \Gamma_i^{\text{at}} \cup (\bigcup_{s_i : \varphi \in \Gamma_i^{\text{at}}} \text{Sub}_i(\varphi)) \cup \text{ld}(i)$. and $\Delta_{i+1} = \Delta_i \cup (\bigcup_{s_i : \varphi \in \Delta_i} \text{Sub}_i(\varphi))$. To prove that $\Gamma_i^{\text{at}} \Rightarrow \Delta_i$ is underivable, we proceed as in the proof of Lemma 3.11 for the atomic case. The underivability of $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is guaranteed by the rule (Ref) in the case of $\text{ld}(i)$ and by the rules (Sub-G-Repl) and ($\Rightarrow$ Sub-G-Repl) for the remaining additions to Γ_{i+1} and Δ_{i+1} . With this construction, verifying the saturation conditions is straightforward.

The definition of the canonical model remains unchanged from the case without identity, since we view identity as a binary predicate. In particular, the derived definition for the extension of identity is $\langle x, y \rangle \in I_w^c(=) \iff w : x = y \in \Gamma$. The verification of the congruence property of $=$ is immediate thanks to the saturation conditions. The fact that $=$ is an equivalence relation also follows easily: for symmetry, if $w : x = y \in \Gamma$, then $w : x = x \in \Gamma$ by (=Ref) and, therefore, $w : y = x \in \Gamma$ by (=L); for transitivity, if $w : x = y$, $w : y = z \in \Gamma$, then we have that $w : y = x \in \Gamma$, which can be used in combination with $w : y = z \in \Gamma$ to get that $w : x = z \in \Gamma$ by (=L). The remaining steps of the completeness proof are analogous to the identity-free case.

4 Future work

We conclude by outlining some directions for potential future extensions of this work.

A natural question is whether the approach can be extended to arbitrary signatures containing function symbols as well as the identity predicate. In inquisitive first-order logics, the interpretation of function symbols can vary between possible worlds. For this reason, there is a distinction between *rigid* function symbols and terms, whose interpretation is constant, and *non-rigid* ones. We denote the first in bold (e.g., **f**, **t**). These two classes of syntactic objects are known to satisfy different properties semantically (see [4, Chapter 5] for a detailed discussion). For signatures containing function symbols, it seems natural to assume that $\text{IWMC}^\equiv$ could be adapted by using arbitrary terms instead of variables in the rules (id) and ($\Rightarrow$ at), and by replacing ($\forall\Rightarrow$) with the following two rules, where α stands for a classical formula, t for an arbitrary fresh term and **t** for a fresh rigid term:

$$\frac{s : \varphi[t/x], s : \forall x. \varphi, \Gamma \Rightarrow \Delta}{s : \forall x. \varphi, \Gamma \Rightarrow \Delta} (\forall^{\text{gen}}\Rightarrow) \qquad \frac{s : \alpha[t/x], s : \forall x. \alpha, \Gamma \Rightarrow \Delta}{s : \forall x. \alpha, \Gamma \Rightarrow \Delta} (\forall^{\text{cl}}\Rightarrow)$$

In Section 3.4, we showed that IWMC enjoys key structural properties. It remains an open question whether the properties of our calculus can be used to obtain metatheoretical results about the logic $\text{InqQML}_\square$, such as interpolation. The structural properties we established, and in particular invertibility, may also allow us to define a systematic proof search procedure for our calculus; completeness could then be established by showing that a failed proof search always produces a countermodel, following a well-established strategy for labelled sequent calculi for modal logics (see, e.g., [12, 27]).

Other inquisitive modalities are natural candidates for extending the present approach. A particularly interesting case is that of neighborhood-based inquisitive modalities (see, for instance, the ones proposed in [6] and [9]). In neighborhood frames, the accessibility relation associates to each possible world a set of sets of possible worlds. It would be interesting to explore whether our labelled sequent calculus can be adapted to this semantic setting with similarly positive results.

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On Strong Structural Completeness of Varieties and Quasivarieties

Alex Citkin

Metropolitan Telecommunications
New York, USA
acitkin@gmail.com

We study structural completeness in the infinitary sense (strong structural completeness) in an algebraic setting. A variety is structurally complete (SCpl) if it is generated, as a quasivariety, by its free algebras, and it is strongly structurally complete (SSCpl) if it is generated, as a prevariety, by its free algebras. A quasivariety is SSCpl if it is generated, as a prevariety, by its free algebras.

We prove that every quasivariety of finite type with the CEP that is generated by finite algebras and contains an infinite irreducible algebra is not SSCpl. Moreover, every congruence meet-semidistributive variety of finite type generated by finite algebras is SSCpl if and only if it is tabular. Thus, Dummett’s and Medvedev’s logics are SCpl but not SSCpl.

A variety is primitive if it is SCpl and all its subvarieties are SCpl; it is strongly primitive if it is SSCpl and all its subvarieties are SSCpl. We prove that in primitive congruence-distributive varieties of finite type, the tabular subvarieties, and only those, are strongly primitive. This observation also yields a criterion for strong primitivity.

Keywords: structural completeness, structural completeness in the infinitary sense, varieties of algebras, quasivarieties, prevarieties, primitive varieties

1 Introduction

Propositional logic, understood as a consequence relation, can be defined in two ways: syntactically, by a deductive system—a pair consisting of a set of axioms and a set of structural inference rules—and semantically, by a class of models that, for each valuation (assignment), validates the conclusion whenever all the premises are validated. Since derivations are finite sequences of formulas, every deductive system defines a finitary structural consequence relation, whose equivalent algebraic semantics is a quasivariety (see [3]). Moreover, any finitary structural consequence relation can be axiomatized by a deductive system (see [16]). In addition, it was proved there that if the defining matrix is finite, then the consequence relation (or operator) determined by it is finitary.

However, in general, consequence relations determined by an infinite algebra or by a class of algebras need not be finitary: even the simplest infinite Heyting algebra of order type $\omega + 1$ determines a non-finitary consequence relation. Note that this consequence relation is also determined by the class of all finite linearly ordered Heyting algebras; thus, infinite classes of finite algebras may define a non-finitary consequence relation.

Not surprisingly, any finite set of similar finite algebras of finite type always determines a finitary consequence relation (see Section 3, where we give an algebraic proof of a generalization of the theorem from [16, Section 8], stating that a consequence relation defined by a finite matrix is finitary).

On the other hand, in Section 2, we prove that any infinite set of pairwise nonisomorphic finite algebras from a congruence semi-distributive variety always defines a non-finitary consequence relation. This justifies the interest in studying infinitary consequence relations.

The algebraic counterparts of finitary consequence relations are quasivarieties, that is, classes of algebras that can be defined by a set of quasiequations. A *quasiequation* is a κ -quasiequation of the form $\varepsilon_1, \dots, \varepsilon_n \Rightarrow \varepsilon$, where $\varepsilon, \varepsilon_i$ ($1 \leq i \leq n$) are equations. Thus, every quasiequation contains only finitely many (possibly none) premises and therefore only finitely many variables.

At the same time, the algebraic counterparts of non-finitary consequence relations are prevarieties (also known as implicative classes [25] or generalized quasivarieties [17]; cf. also [1]), that is, classes of algebras defined by κ -quasiequations of the form $E \Rightarrow \varepsilon$, where E is a (possibly empty or infinite) set of equations. If E is infinite, there are two important parameters associated with such a κ -quasiequation: λ —the cardinality of E —and κ —the cardinality of the set of variables occurring in the κ -quasiequation.

For any regular cardinal $\lambda > 0$ and any infinite regular cardinal κ , we define a (λ, κ) -quasiequation as a κ -quasiequation $E \Rightarrow \varepsilon$ such that $|E| < \lambda$ and E contains fewer than κ distinct variables (see [11, Section 2.3.4]). By a κ -*quasiequation* we mean a (κ, κ) -quasiequation. Thus, $(1, \kappa)$ -quasiequations are equations, and ω -quasiequations are the usual (finitary) quasiequations.

Prevarieties that can be defined by a set of κ -quasiequations will be called κ -*quasivarieties*. By $\mathbf{Q}_\kappa(\mathbf{K})$ we denote the smallest κ -quasivariety containing the class of similar algebras $\mathbf{K}$. Thus, $\mathbf{Q}_\omega(\mathbf{K}) = \mathbf{Q}(\mathbf{K})$ is the quasivariety generated by $\mathbf{K}$.

The notion of structural completeness was introduced by W. Pogorzelski in his seminal 1971 paper [18]. It was defined for consequence operators determined by a deductive system $\langle R, A \rangle$, where R is a finite set of (structural finitary) rules and A is a set of propositional formulas. A rule r is *admissible* for the operator if the operator determined by the extended deductive system $\langle R \cup r, A \rangle$ has the same set of theorems as the original operator, and r is *derivable* if the deductive system $\langle R \cup r, A \rangle$ determines the same operator. The operator is said to be *structurally complete* if every rule admissible for it is derivable. In the algebraic setting, a variety $\mathbf{V}$ is structurally complete if it is generated as a quasivariety by its free algebras, that is, if $\mathbf{V} = \mathbf{Q}(\mathbf{F}_\mathbf{V}(\omega))$.

We say that a variety $\mathbf{V}$ is κ -*structurally complete* if $\mathbf{V} = \mathbf{Q}_\kappa(\mathbf{F}_\mathbf{V}(\omega))$ (in Section 2 we justify why it is sufficient to consider free algebras of countable rank; cf. Theorem 2.12). Obviously, ω -structural completeness coincides with ordinary structural completeness.

The goal of this paper is to demonstrate that for all $\kappa > \omega$, the property of κ -structural completeness is both very strong and rare. Thus, on the one hand, for varieties generated by a finite algebra of finite type, κ -structural completeness coincides with structural completeness for all infinite κ (Theorem 3.3). On the other hand, the following holds:

Theorem. 3.2 *Suppose that $\mathbf{V}$ is a structurally complete congruence distributive variety of finite type and $\kappa > \omega$. Then $\mathbf{V}$ is κ -structurally complete if and only if it is generated by a finite algebra.*

Thus, even the variety of Heyting algebras $\mathbf{LC}$ generated by all linearly ordered Heyting algebras, which is structurally complete (see, e.g., [9]), is not κ -structurally complete for any $\kappa > \omega$. The same holds for the variety $\mathbf{ML}$ corresponding to Medvedev logic, which is structurally complete (see [20]).

In Section 4, we consider hereditarily κ -structurally complete varieties—the κ -primitive varieties. In [6], it was observed that the Heyting algebra $\mathbf{C}_{\omega+1}$ of order type $\omega + 1$ defines a non-finitary consequence relation: the prevariety defined by this algebra is not a quasivariety, because the following infinite rule is valid in $\mathbf{C}_{\omega+1}$, whereas it can be refuted in the algebra $\mathbf{C}_{\omega+2}$ of order type $\omega + 2$:

$$\tau = \frac{\neg\neg p_0, (p_j \rightarrow p_i) \rightarrow p_j, \quad 0 \leq i < j \leq \omega}{p_\omega},$$

whereas $\mathbf{Q}(\mathbf{C}_{\omega+1}) = \mathbf{Q}(\mathbf{C}_{\omega+2})$.

Note that the rule

$$\mathfrak{b} = \frac{(p_i \leftrightarrow p_j) \rightarrow p_\omega, \quad 0 \leq i < j < \omega}{p_\omega}$$

introduced independently in [22] and [23], and also playing an important role in [26, 27], separates aforementioned algebras. Moreover, for every $\kappa > \omega$, a primitive variety of Heyting algebras $\mathbf{V}$ is κ -primitive if and only if the rule $\mathfrak{r}$ (or $\mathfrak{b}$) is valid in $\mathbf{V}$.

2 Preliminaries

Throughout the paper, we use an arbitrary but fixed signature Ω (not containing the symbol $\Rightarrow$); all algebras and terms are assumed to be of this type. Terms are built in the usual way from a set of variables and symbols from Ω , and Eq denotes the set of all equations.

If $\Gamma \subseteq Eq$ and $\gamma \in Eq$, the pair $\Gamma \Rightarrow \gamma$ is called a *sequent*, or an *implication* (e.g., [25]), a *generalized quasiequation* (e.g., [17]), or a *consecution* (e.g., [15]). If Γ is empty, we omit the antecedent, and the sequent reduces to an equation. If Γ is finite, the sequent is a *quasiequation*. If λ and κ are cardinals such that $|\Gamma| < \lambda$ and the sequent $\Gamma \Rightarrow \gamma$ contains fewer than κ distinct variables, then it is called a (λ, κ) -*quasiequation*; if $\lambda = \kappa$, it is called a κ -*quasiequation*.

If $\mathbf{A}$ is an algebra, a map from the set of all variables into $\mathbf{A}$ is called a *valuation* in $\mathbf{A}$. Clearly, any valuation ν in $\mathbf{A}$ can be extended to a map from the set of all terms into $\mathbf{A}$: for a term $t(x_1, \dots, x_n)$, we define

$$\nu(t(x_1, \dots, x_n)) = t(\nu(x_1), \dots, \nu(x_n)).$$

We say that ν *validates* the equation $t \approx s$ (in symbols, $\mathbf{A} \models_\nu t \approx s$) if $\nu(t) = \nu(s)$. If Γ is a set of equations, we write $\mathbf{A} \models_\nu \Gamma$ to indicate that ν validates every equation in Γ . Throughout the paper, we assume that $\kappa \geq \omega$ is a regular cardinal.

Given an algebra $\mathbf{A}$ and a κ -quasiequation $\Gamma \Rightarrow \gamma$, we say that this κ -quasiequation is *valid* in $\mathbf{A}$ (in symbols, $\mathbf{A} \models \Gamma \Rightarrow \gamma$) if, for every valuation ν in $\mathbf{A}$,

$$\mathbf{A} \models_\nu \Gamma \text{ implies } \mathbf{A} \models_\nu \gamma.$$

A κ -quasiequation $\mathfrak{s}$ is *valid in a class of algebras* $\mathbf{K}$ if it is valid in every algebra from $\mathbf{K}$. A set of κ -quasiequations $\mathfrak{S}$ is valid in $\mathbf{K}$ if every κ -quasiequation in $\mathfrak{S}$ is valid in $\mathbf{K}$. Two sets of κ -quasiequations that are valid in exactly the same algebras from $\mathbf{K}$ are called *K-equivalent*.

Observe that, since a κ -quasiequation contains fewer than κ distinct variables, a κ -quasiequation $\mathfrak{s}$ is valid in an algebra $\mathbf{A}$ if and only if it is valid in every λ -generated subalgebra of $\mathbf{A}$ for all $\lambda < \kappa$.

Let $\mathbf{K}$ be a class of algebras. As usual, $\mathbf{I}$, $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{P}_u$ denote the class operators: $\mathbf{I}(\mathbf{K})$, $\mathbf{H}(\mathbf{K})$, $\mathbf{S}(\mathbf{K})$, $\mathbf{P}(\mathbf{K})$, and $\mathbf{P}_u(\mathbf{K})$ are, respectively, the classes of all isomorphic copies, homomorphic images, subalgebras, Cartesian products (including the product over the empty set), and ultraproducts of algebras from $\mathbf{K}$. Classes of algebras closed under $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$ are called *varieties*; classes closed under $\mathbf{I}$, $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{P}_u$ are called *quasivarieties*; and classes closed under $\mathbf{I}$, $\mathbf{S}$, and $\mathbf{P}$ are called *prevarieties*.

It is well known that every variety can be defined by a set of equations, and every set of equations defines a variety. Similarly, every quasivariety can be defined by a set of quasiequations, and every set of quasiequations defines a quasivariety. A class of algebras defined by a set of κ -quasiequations is called a κ -*quasivariety*.

Given a class of algebras K , we use $V(K)$, $Q(K)$, and $P(K)$ to denote, respectively, the smallest variety, quasivariety, and prevariety containing K ; we say that these classes are *generated* by K . It is well known that

$$V(K) = HSP(K), \quad Q(K) = ISPP_u(K), \quad P(K) = ISP(K).$$

A class of algebras K is called κ -local if, for every algebra A , whenever every λ -generated subalgebra of A (for all $\lambda < \kappa$) belongs to K , it follows that $A \in K$. Clearly, κ -quasivarieties are κ -local classes (cf. [11, Section 2.3.4]).

Let us introduce an additional class operator (similar to the one in [1]): for any cardinal $\kappa \geq \omega$ and any class of algebras K , let $U_\kappa(K)$ denote the class of all algebras whose every λ -generated subalgebra, for $\lambda < \kappa$, belongs to K .

Given a class of algebras K and an infinite cardinal κ , denote by $Q_\kappa(K)$ the smallest κ -quasivariety containing K . Then

$$Q_\kappa(K) = U_\kappa, ISP(K).$$

Note that $Q_\kappa(K)$ is always a prevariety, although $Q_\kappa(K)$ and $P(K)$ may be distinct. It is also clear that for any $\omega \leq \kappa_1 \leq \kappa_2$,

$$Q_{\kappa_2}(K) \subseteq Q_{\kappa_1}(K) \subseteq Q_\omega(K) = Q(K).$$

The following notion will play an important role in what follows. An algebra $A \in K$ is *weakly projective* in K (or weakly K -projective) if A embeds into each of its homomorphic preimages in K ; that is, whenever $A \in H(B)$ for some $B \in K$, then $A \in IS(B)$. For example, every free algebra in a variety V is weakly V -projective (indeed, projective; see, e.g., [12]).

Congruences and irreducibility.

Definition 1. Let A be a nontrivial algebra. Denote by μ_A the meet of all congruences on A distinct from the identity congruence, and call it the *monolith* of A . Algebra A is said to be *subdirectly irreducible* (SI) if μ_A is not the identity congruence.

Definition 2. Let K be a class of algebras and $A \in K$. Congruence ϑ on A is a K -congruence provided $A/\vartheta \in K$. Denote by $\mu_K(A)$ the meet of all K -congruences on A distinct from the identity congruence, and call it the K -*monolith* of A . Algebra A is said to be K -*irreducible* if $\mu_K(A)$ is not the identity congruence. We will omit the reference to the class if no confusion arises.

Clearly, if K is a variety, then K -irreducible algebras are precisely the subdirectly irreducible (SI) algebras.

As usual, if $a, b \in A$, we denote by $\vartheta_A(a, b)$ the principal congruence on A generated by the pair (a, b) , and we omit the reference to A when it is clear from the context.

Proposition 2.1. Let Q be a quasivariety. Then the following hold:

- (a) Every nontrivial algebra in Q is a subdirect product of Q -irreducible algebras;
- (b) If Q has finite type and $Q = Q(K)$, where K is a class of algebras, then every finite Q -irreducible algebra belongs to $IS(K)$;
- (c) If $Q = Q(A)$ and A is finite, then all Q -irreducible algebras belong to $IS(A)$ and are therefore finite.

For (a), see, e.g., [11, Theorem 3.1.1]; (b) follows from [11, Proposition 2.1.15]; and (c) follows from the observation that, by [7, Lemma 1.5], for any irreducible algebra $B \in Q(A)$, we have $B \in ISP_u(A)$, and $ISP_u(A) = IS(A)$ because A is finite.

Corollary 2.2. *If P is a proper subprevariety of a quasivariety Q , then $Q \setminus P$ contains a countable Q -irreducible algebra.*

Proof. Let $P \subsetneq Q$ be a proper subprevariety. Recall that every algebra in Q is a subdirect product of Q -irreducible algebras. Since the prevariety P is closed under subdirect products and $P \subsetneq Q$, the difference $Q \setminus P$ must contain a Q -irreducible algebra. $\square$

A variety for which every SI member has cardinality less than m , for some $m < \omega$, is said to have a *finite residual bound* (see, e.g., [10, Section 4.3]). Recall the following property of congruence meet-semidistributive varieties.

Theorem 2.3 ([13, Theorem 4.1]). *Suppose that V is a congruence meet-semidistributive variety of finite type. If V contains arbitrarily large finite SI algebras, then V contains an infinite SI algebra.*

We will use this in the following form.

Corollary 2.4. *If a congruence meet-semidistributive variety of finite type is not residually m -small for some finite m , then it contains an infinite SI algebra.*

Characteristic κ -quasiequations. With every K -irreducible algebra $\mathbf{A} \in K$ of finite type and cardinality at most κ , we associate a κ -quasiequation $c_{\mathbf{A}}$, which we call a *characteristic κ -quasiequation* of $\mathbf{A}$ (an infinite analogue of the Jankov characteristic formula). To each element $a \in \mathbf{A}$, we assign a variable x_a . Since $\mathbf{A}$ is K -irreducible, the monolith $\mu_K(\mathbf{A})$ is nontrivial and contains a pair (c, d) with $c \neq d$. For each operation $f \in \Omega$, let k_f denote the arity of f . We then define

$$c_{\mathbf{A}} = \{f(x_{a_1}, \dots, x_{a_{k_f}}) \approx x_{f(a_1, \dots, a_{k_f})} \mid \text{for all } f \in \Omega \text{ and all } a_1, \dots, a_{k_f} \in \mathbf{A}\} \Rightarrow x_c \approx x_d.$$

Observe that the valuation $v: x_a \mapsto a$ refutes $c_{\mathbf{A}}$ in $\mathbf{A}$; that is, $\mathbf{A} \not\models c_{\mathbf{A}}$.

The following proposition justifies the name “characteristic.”

Proposition 2.5. *Let K be a class of algebras of finite type, and let $\mathbf{A} \in K$ be a K -irreducible algebra of cardinality at most κ . Then, for every algebra $\mathbf{B} \in K$,*

$$\mathbf{B} \not\models c_{\mathbf{A}} \quad \text{if and only if} \quad \mathbf{A} \in \mathbf{IS}(\mathbf{B}).$$

Proof. The implication from right to left is immediate.

For the converse, suppose that there exists a valuation v that refutes $c_{\mathbf{A}}$ in $\mathbf{B}$. Then all premises of $c_{\mathbf{A}}$ are satisfied under v , while $v(x_c) \neq v(x_d)$. Define a map $h: \mathbf{A} \rightarrow \mathbf{B}$ by $h(a) = v(x_a)$. Since the premises are satisfied, h preserves all operations in Ω , and hence h is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$.

Moreover, since v refutes $c_{\mathbf{A}}$, we have $h(c) = v(x_c) \neq v(x_d) = h(d)$. Thus $h(c) \neq h(d)$, and since $(c, d) \in \mu_K(\mathbf{A})$, it follows from the definition of the K -monolith that h is injective. Therefore, h is an embedding, and hence $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$. $\square$

Example 1. Consider the Heyting algebras $\mathbf{C}_{\omega+1}$ and $\mathbf{C}_{\omega+2}$ whose Hasse diagrams are depicted in Fig. 1.

It is not hard to see that the algebra $\mathbf{C}_{\omega+2}$ is SI, since $(a_{\omega+1}, a_{\omega+2}) \in \mu(\mathbf{C}_{\omega+2})$. Therefore, it has a characteristic κ -quasiequation for every $\kappa > \omega$: we take variables x_{a_i} , where $i \in [0, \omega + 2]$, and define

$$c := \neg \neg x_{a_0} \approx x_{a_{\omega+2}}, \quad x_{a_j} \rightarrow x_{a_i} \approx x_{a_i}, \quad 0 \leq i < j \leq \omega + 2 \Rightarrow x_{a_{\omega+1}} \approx x_{a_{\omega+2}}.$$

Clearly, the valuation $x_{a_i} \mapsto a_i$ refutes c in $\mathbf{C}_{\omega+2}$; that is, $\mathbf{C}_{\omega+2} \not\models c$.

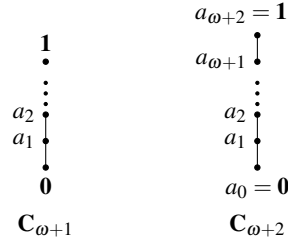


Figure 1: Chain Heyting algebras.

On the other hand, $\mathbf{C}_{\omega+2}$ cannot be embedded into $\mathbf{C}_{\omega+1}$, and by Proposition 2.5, it follows that

$$\mathbf{C}_{\omega+1} \models \mathfrak{c}.$$

Thus, for every $\kappa > \omega$,

$$\mathbf{C}_{\omega+2} \notin \mathbf{Q}_{\kappa}(\mathbf{C}_{\omega+1}). \quad (1)$$

Remark 1. In Example 1, we simplified the κ -characteristic quasiequation by omitting from the antecedent some equations that are consequences of the remaining ones. For example, if $x_{a_j} \rightarrow x_{a_i} = x_{a_i}$, then $x_{a_i} \leq x_{a_j}$ and therefore $x_{a_i} \wedge x_{a_j} = x_{a_i}$, $x_{a_i} \vee x_{a_j} = x_{a_j}$, and $x_{a_i} \rightarrow x_{a_j} = x_1$. Moreover, it is not hard to see that the equations $\neg x_{a_i} = x_0$ can be omitted for all $a_i \neq a_0$.

Congruence extension property. An algebra $\mathbf{A}$ has the *congruence extension property* (CEP) if, for every subalgebra $\mathbf{B} \in \mathcal{S}(\mathbf{A})$ and every congruence θ on $\mathbf{B}$, there exists a congruence θ' on $\mathbf{A}$ such that $\theta = \theta' \cap B^2$. A class of algebras has the CEP if every algebra in the class has the CEP.

In general, a subalgebra of an SI algebra need not be SI; indeed, consider the algebras $\mathbf{C}_{\omega+1}$ and $\mathbf{C}_{\omega+2}$ from Example 1: $\mathbf{C}_{\omega+1} \in \mathbf{IS}(\mathbf{C}_{\omega+2})$, but $\mathbf{C}_{\omega+2}$ is SI, whereas $\mathbf{C}_{\omega+1}$ is not. Nevertheless, the following holds.

Proposition 2.6. *Let $\mathbf{K}$ be a class of algebras, let $\mathbf{A} \in \mathbf{K}$ be a $\mathbf{K}$ -irreducible algebra with the CEP, and let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$ containing two distinct elements from $\mu_{\mathbf{K}}(\mathbf{A})$. Then $\mathbf{B}$ is $\mathbf{K}$ -irreducible.*

Proof. Suppose $(a, b) \in \mu_{\mathbf{K}}(\mathbf{A})$ with $a \neq b$ and $a, b \in \mathbf{B}$. Assume, for a contradiction, that $\mathbf{B}$ is not $\mathbf{K}$ -irreducible. Then $\mu_{\mathbf{K}}(\mathbf{B})$ is the identity congruence, and hence there exists a $\mathbf{K}$ -congruence θ on $\mathbf{B}$ such that $(a, b) \notin \theta$. Since $\mathbf{A}$ has the CEP, there exists a congruence θ' on $\mathbf{A}$ such that $\theta = \theta' \cap B^2$. Consequently, $(a, b) \notin \theta' \cap B^2$, which implies $(a, b) \notin \theta'$, contradicting the assumption that $\mathbf{A}$ is $\mathbf{K}$ -irreducible. $\square$

Corollary 2.7. *If $\mathbf{K}$ is a class of algebras of finite type, then every $\mathbf{K}$ -irreducible algebra with the CEP contains a countable $\mathbf{K}$ -irreducible subalgebra. Moreover, if $\mathbf{A}$ is an infinite $\mathbf{K}$ -irreducible algebra of cardinality λ , then for every infinite cardinal $\lambda' \leq \lambda$, there exists a $\mathbf{K}$ -irreducible subalgebra $\mathbf{B} \in \mathbf{IS}(\mathbf{A})$ of cardinality λ' .*

Proof. Since $\mathbf{A}$ is $\mathbf{K}$ -irreducible, there exist distinct elements $a, b \in \mathbf{A}$ such that $(a, b) \in \mu_{\mathbf{K}}(\mathbf{A})$. The subalgebra generated by a, b is countable because the similarity type is finite, and it is $\mathbf{K}$ -irreducible by Proposition 2.6.

If $\mathbf{A}$ is infinite of cardinality λ and $\lambda' \leq \lambda$, take λ' distinct elements of $\mathbf{A}$, adjoin two distinct elements from $\mu_K(\mathbf{A})$, and consider the subalgebra $\mathbf{B}$ generated by this set. Since the type is finite, $|\mathbf{B}| = \lambda'$, and by Proposition 2.6, $\mathbf{B}$ is K -irreducible. $\square$

Corollary 2.8. *Let K be a class of algebras of finite type. If a κ -quasiequation s is refuted in a K -irreducible algebra $\mathbf{A} \in K$, then there exists a K -irreducible subalgebra $\mathbf{B} \in \mathbf{S}(\mathbf{A})$ of cardinality κ that also refutes s .*

Proof. Suppose that $\mathbf{A}$ is K -irreducible and $\mathbf{A} \not\models s$, and let $a, b \in \mathbf{A}$ with $(a, b) \in \mu_K(\mathbf{A})$.

Let $x_k, k \in \kappa$, be all variables occurring in s , and let v be a valuation in $\mathbf{A}$ that refutes s . Then the subalgebra $\mathbf{B}$ of $\mathbf{A}$ generated by the elements a, b , and $v(x_k)$ (for $k \in \kappa$) has cardinality at most κ , since $\mathbf{A}$ has finite type. By Proposition 2.6, $\mathbf{B}$ is K -irreducible. Clearly, v is a valuation refuting s in $\mathbf{B}$. $\square$

Free algebras. Suppose that K is a class of algebras and $\mathbf{F} \in K$ is generated by a set $G \subseteq \mathbf{F}$. Then G *freely generates $\mathbf{F}$ relative to K* if, for every algebra $\mathbf{A} \in K$, any map $f: G \rightarrow \mathbf{A}$ extends to a homomorphism from $\mathbf{F}$ into $\mathbf{A}$. In this case, $\mathbf{F}$ is called a *K -free algebra of rank ρ* , where $\rho = |G|$. The K -free algebra of rank ρ is denoted by $\mathbf{F}_K(\rho)$, and the class of all K -free algebras is denoted by K_F . Free K -algebras of a given rank are unique up to isomorphism (cf. [4, Theorem 10.7]).

One of the most important properties of prevarieties is that every nontrivial prevariety contains free algebras of all ranks (see, e.g., [4, Theorem 10.12]). This yields the following observation, which will be important in what follows.

Proposition 2.9. *For every class of algebras K ,*

$$\mathbf{P}(K)_F = \mathbf{Q}(K)_F = \mathbf{V}(K)_F.$$

Proof. If K contains only trivial algebras, then the statement is immediate, since $\mathbf{P}(K) = \mathbf{Q}(K) = \mathbf{V}(K)$. Now assume that K contains a nontrivial algebra. Then the classes $\mathbf{V}(K)$, $\mathbf{Q}(K)$, and $\mathbf{P}(K)$ —being prevarieties—all contain free algebras of arbitrary ranks.

Moreover, by [11, Corollary 2.1.13], for any class of algebras K , every K -free algebra is free in $\mathbf{V}(K)$. Since $\mathbf{V}(K) \supseteq \mathbf{Q}(K) \supseteq \mathbf{P}(K)$, it follows that every $\mathbf{V}(K)$ -free algebra is also free in $\mathbf{Q}(K)$ and $\mathbf{P}(K)$. By uniqueness (up to isomorphism) of free algebras of a given rank, the free algebras in $\mathbf{P}(K)$, $\mathbf{Q}(K)$ and $\mathbf{V}(K)$ coincide. $\square$

Corollary 2.10. *Let V be a variety. Then, for every infinite cardinal κ , the class $\mathbf{Q}_\kappa(\mathbf{F}_V(\omega))$ is the smallest κ -quasivariety $\mathbf{Q}_\kappa$ such that $\mathbf{V}(\mathbf{Q}_\kappa) = V$, and $\mathbf{P}(\mathbf{F}_V(\omega))$ is the smallest prevariety $\mathbf{P}$ such that $\mathbf{V}(\mathbf{P}) = V$.*

Proof. First, observe that since $\mathbf{V}(\mathbf{F}_V(\omega)) = V$, we have $\mathbf{V}(\mathbf{Q}_\kappa(\mathbf{F}_V(\omega))) = V$.

Next, suppose that K is a κ -quasivariety such that $\mathbf{V}(K) = V$. Then K contains all V -free algebras, and hence $\mathbf{F}_V(\omega) \in K$. Therefore, $\mathbf{Q}_\kappa(\mathbf{F}_V(\omega)) \subseteq K$.

The statement for prevarieties follows by a similar argument. $\square$

Clearly, in contrast to varieties, prevarieties and quasivarieties need not be generated by their free algebras. Nevertheless, the following holds.

Corollary 2.11. *For any prevariety K , $\mathbf{V}(K_F) = \mathbf{V}(K)$.*

Proof. Let $V = V(K)$. Then $K_F = V_F$, therefore

$$V(K_F) = V(V_F) = V = V(K).$$

□

Moreover, the following holds.

Theorem 2.12. *For any prevariety K and any infinite cardinal κ ,*

$$V(F_K(\omega)) = V(K_F), \quad Q_\kappa(F_K(\omega)) = Q_\kappa(K_F), \quad P(F_K(\omega)) = P(K_F).$$

In particular, for any variety V ,

$$V(F_V(\omega)) = V(V_F), \quad Q_\kappa(F_V(\omega)) = Q_\kappa(V_F), \quad P(F_V(\omega)) = P(V_F).$$

Proof. Let $W = V(K)$. Since K is a prevariety, $W_F = K_F$; in particular, $F_K(\omega) \cong F_W(\omega)$. Thus,

$$V(F_K(\omega)) = V(F_W(\omega)) = V(W_F) = V(K_F).$$

Next, since $F_K(\omega) \in K_F$, we have

$$Q_\kappa(F_K(\omega)) \subseteq Q_\kappa(K_F) \quad \text{and} \quad P(F_K(\omega)) \subseteq P(K_F),$$

and it remains to prove the reverse inclusions.

Since $K_F = W_F \subseteq W$, we have $K_F \subseteq V(F_K(\omega))$. By Proposition 2.9,

$$V(F_K(\omega))F = P(F_K(\omega))F,$$

and therefore

$$K_F \subseteq V(F_K(\omega))F = P(F_K(\omega))F \subseteq Q_\kappa(F_K(\omega)).$$

Thus,

$$P(K_F) \subseteq P(F_K(\omega)) \quad \text{and} \quad Q_\kappa(K_F) \subseteq Q_\kappa(F_K(\omega)).$$

□

Varieties with EDPC.

Let us recall from [2] the definition of varieties with equationally definable principal congruences and some of their properties that will be used later.

Definition 3. A variety V is said to have *equationally definable principal congruences (EDPC)* if there exist 4-ary terms $p_i(x, y, z, w), q_i(x, y, z, w), i = 1, \dots, n$, such that for every algebra $A \in V$ and all $a, b, c, d \in A$,

$$(c, d) \in \vartheta(a, b) \text{ if and only if } A \models p_i(a, b, c, d) \approx q_i(a, b, c, d), \quad 1 \leq i \leq n. \quad (2)$$

To simplify notation, we let $E(x, y, z, w) := \{p_i(x, y, z, w) \approx q_i(x, y, z, w) \mid i = 1, \dots, n\}$, and we can rephrase (2) in the following way:

$$(c, d) \in \vartheta(a, b) \text{ if and only if } A \models E(a, b, c, d). \quad (3)$$

Recall that every variety with EDPC is congruence-distributive and has the CEP (see [2, Theorem 1.2]).

Bounding κ -quasiequation.

Let $\mathbf{V}$ be a variety with EDPC witnessed by $E(x, y, z, w)$, and let κ be an infinite cardinal.

Consider the following κ^+ -quasiequation, which is a generalization of the rule introduced in [21] and used in [22, 23, 26, 8]:

$$\mathbf{b}_{\mathbf{V}}(\kappa) := \{E(x_j, x_k, z, w) \mid 0 \leq j < k < \kappa\} \Rightarrow z \approx w. \quad (4)$$

Theorem 2.13. *Let $\mathbf{V}$ be a variety with EDPC and κ be an infinite cardinal. Then:*

- (i) $\mathbf{b}_{\mathbf{V}}(\kappa)$ is valid in every algebra from $\mathbf{V}$ of cardinality less than κ ;
- (ii) $\mathbf{b}_{\mathbf{V}}(\kappa)$ fails in every SI algebra from $\mathbf{V}$ of cardinality at least κ .

Proof. (i) Suppose that $\mathbf{A} \in \mathbf{V}$ and $|\mathbf{A}| < \kappa$. Then the number of distinct pairs $(a, b) \in \mathbf{A}^2$ is less than κ . Therefore, for any valuation ν in $\mathbf{A}$, there exist j, k with $0 \leq j < k < \kappa$ such that $\nu(x_j) = \nu(x_k)$, and by (3) we obtain $\nu(z) = \nu(w)$. Hence, $\mathbf{A} \models \mathbf{b}_{\mathbf{V}}(\kappa)$.

(ii) Suppose that $\mathbf{A} \in \mathbf{V}$ is an SI algebra with $|\mathbf{A}| \geq \kappa$. Then the monolith $\mu(\mathbf{A})$ contains a pair (c, d) with $c \neq d$. Since $|\mathbf{A}| \geq \kappa$, there exist pairwise distinct elements $a_j \in \mathbf{A}$ for $0 \leq j < \kappa$. Consider the valuation

$$\nu(x_j) = a_j \ (0 \leq j < \kappa), \quad \nu(z) = c, \quad \nu(w) = d.$$

Since $(c, d) \in \mu(\mathbf{A})$, we have $(c, d) \in \theta(a, b)$ for all $a, b \in \mathbf{A}$ with $a \neq b$. Consequently, for every equation $p_i(x, y, z, w) \approx q_i(x, y, z, w) \in E(x, y, z, w)$, we have

$$p_i(a_j, a_k, c, d) = q_i(a_j, a_k, c, d)$$

for all $0 \leq j < k < \kappa$. Thus, all premises of $\mathbf{b}_{\mathbf{V}}(\kappa)$ are satisfied under ν , while $\nu(z) \neq \nu(w)$, and hence ν refutes $\mathbf{b}_{\mathbf{V}}(\kappa)$. □

Example 2. The variety of all Heyting algebras $\mathbf{HA}$ has EDPC witnessed by $(x \leftrightarrow y) \rightarrow z \approx (x \leftrightarrow y) \rightarrow w$. Thus, the bounding κ -quasiequation for $\mathbf{HA}$ is

$$\mathbf{b}_{\mathbf{HA}}(\omega) = \{(x_j \leftrightarrow x_k) \rightarrow z \approx (x_j \leftrightarrow x_k) \rightarrow w \mid 0 \leq j < k < \omega\} \Rightarrow z \approx w.$$

Clearly, $\mathbf{b}_{\mathbf{HA}}(\omega)$ is valid in every finite Heyting algebra, and it is refuted in every infinite SI Heyting algebra $\mathbf{A}$. Indeed, let a_i , for $i = 1, 2, \dots$, be distinct elements of $\mathbf{A}$, and let b be a pre-top element, which exists because $\mathbf{A}$ is SI. Then $(a_i \leftrightarrow a_j) \rightarrow b = \mathbf{1}$ for all distinct a_i, a_j , whereas $b \neq \mathbf{1}$. Thus, the valuation ν that sends x_i to a_i , z to b , and w to $\mathbf{1}$ refutes $\mathbf{b}_{\mathbf{HA}}(\omega)$.

3 Strong structural completeness

Admissible and derivable κ -quasiequations. If $\mathbf{Q}$ is a κ -quasivariety, a κ -quasiequation $\mathfrak{s}$ is said to be *derivable* in $\mathbf{Q}$ provided that $\mathbf{Q} \models \mathfrak{s}$, and $\mathfrak{s}$ is *admissible* in $\mathbf{Q}$ if $\mathbf{V}(\mathbf{Q}) = \mathbf{V}(\mathbf{Q}^{\mathfrak{s}})$, where $\mathbf{Q}^{\mathfrak{s}}$ is the class of all algebras from $\mathbf{Q}$ in which $\mathfrak{s}$ is valid. In other words, $\mathfrak{s}$ is admissible in $\mathbf{Q}$ if the set of all equations valid in $\mathbf{Q}$ is closed under $\mathfrak{s}$. Thus, admissibility of $\mathfrak{s}$ in $\mathbf{Q}$ amounts to validity of $\mathfrak{s}$ in $\mathbf{F}_{\mathbf{V}(\mathbf{Q})}(\omega)$; hence, in view of Proposition 2.9, $\mathfrak{s}$ is admissible in $\mathbf{Q}$ if and only if $\mathbf{F}_{\mathbf{Q}}(\omega) \models \mathfrak{s}$.

In other words, if $\mathfrak{s}$ is admissible in $\mathbf{Q}$, it may happen that $\mathbf{Q}^{\mathfrak{s}} \subsetneq \mathbf{Q}$, but the equations valid in both κ -quasivarieties coincide.

Proposition 3.1. *Let K be a class of algebras and let $\mathfrak{s}$ be a κ -quasiequation. Then $\mathfrak{s}$ is admissible in $V(K)$ (respectively, for $Q_\kappa(K)$) provided that $K \models \mathfrak{s}$.*

Proof. Suppose that $K \models \mathfrak{s}$. Let $V = V(K)$ and assume, for a contradiction, that $F_V(\omega) \not\models \mathfrak{s}$.

Let $P \subseteq V$ be the subclass defined (relative to V) by $\mathfrak{s}$:

$$P = \{A \in V \mid A \models \mathfrak{s}\}.$$

Since $K \models \mathfrak{s}$, we have $K \subseteq P$, and therefore $V = V(K) \subseteq V(P) \subseteq V$, whence $V = V(P)$. By Corollary 2.10, $P(F_V(\omega)) \subseteq P$, and hence $F_V(\omega) \in P$, contradicting the assumption that $F_V(\omega) \not\models \mathfrak{s}$.

Thus, $F_V(\omega) \models \mathfrak{s}$. By Proposition 2.9, $F_V(\omega) \cong F_Q(\omega)$ (where $Q = Q_\kappa(K)$), and hence $F_Q(\omega) \models \mathfrak{s}$. Therefore, $\mathfrak{s}$ is admissible in Q . $\square$

Structural completeness

Definition 4. Let V be a variety and κ be an infinite cardinal. Then V is

structurally complete (SCpl) if every admissible in V quasiequation is derivable in V , that is, $V = Q(F_V(\omega))$;

strongly structurally complete (SSCpl) if for every infinite cardinal κ , every admissible in V κ -quasiequation is derivable in V , that is, $V = P(F_V(\omega))$;

κ -structurally complete (κ -SCpl) if every admissible in V κ -quasiequation is derivable in V , that is, $V = Q_\kappa(F_V(\omega))$.

Quasivariety Q is

strongly structurally complete (SSCpl) if for every infinite cardinal κ , every admissible in Q κ -quasiequation is derivable in Q , that is, $Q = P(F_Q(\omega))$.

κ -structurally complete (κ -SCpl) if every admissible in Q κ -quasiequation is derivable in Q , that is, $Q = Q_\kappa(F_Q(\omega))$.

Let us observe that since $Q = Q_\omega$, structural completeness and ω -structural completeness coincide.

If V is a variety, the κ -quasivariety $Q_\kappa(F_V(\omega))$ is called the κ -structural core of V and is denoted by V^κ , and the prevariety $P(F_V(\omega))$ is called the strong structural core of V and is denoted by V^p . Observe that if V is strongly structurally complete, then νV is κ -structurally complete for all infinite κ . Also, if $\kappa_1 \leq \kappa_2$ are infinite cardinals, then $V^{\kappa_2} \subseteq V^{\kappa_1}$, and hence, κ_1 -incompleteness entails κ_2 -incompleteness. Thus, if a variety V is ω^+ -incomplete, then it is κ -incomplete for all $\kappa \geq \omega^+$, and it is not strongly structurally complete.

Clearly, ω -structural completeness coincides with ordinary structural completeness, and, as we shall show, even ω^+ -structural completeness is a very strong property. In particular, the remainder of this section is devoted to proving the following theorem.

Theorem 3.2. *Suppose that V is a structurally complete congruence distributive variety of finite type generated by finite algebras, and $\kappa > \omega$. Then the variety V is κ -structurally complete if and only if it is tabular.*

This theorem follows from the following two theorems.

Theorem 3.3. *Let K be a finite set of finite algebras of finite type. Then $P(K) = Q(K)$.*

Since every finitely generated variety V is generated by a single finite V -free algebra $\mathbf{F}$, the following corollary holds, and it yields the implication from left to right in Theorem 3.2.

Corollary 3.4. *Let V be a finitely generated variety of finite type. Then V is SCpl if and only if it is SSCpl.*

The implication from left to right in Theorem 3.2 also follows from the following theorem together with the observation that, by Jónsson's Lemma, congruence distributive varieties of finite type have a finite residual bound if and only if they are tabular.

Theorem 3.5. *Let V be an SCpl congruence meet-semidistributive variety of finite type generated by finite algebras. Then V is SSCpl if and only if it has a finite residual bound.*

Proof of Theorem 3.2

Proof. Let K be a finite set of finite algebras of finite type, and let $P = \mathbf{P}(K)$ and $Q = \mathbf{Q}(K)$. Clearly, $P \subseteq Q$, and it remains to prove that $Q \subseteq P$.

Suppose that n is the maximal cardinality of algebras from K . Then, since $Q = \mathbf{Q}(K)$, every Q -irreducible algebra belongs to $\mathbf{ISP}_u(K)$, and hence every Q -irreducible algebra has cardinality at most n and is therefore finite.

By Proposition 2.1.b, every finite Q -irreducible algebra is in $\mathbf{IS}(K)$. Let K' be the class of all Q -irreducible algebras, and observe that $K' \subseteq \mathbf{P}(K)$.

By Proposition 2.1.a, every algebra in Q is a subdirect product of Q -irreducible algebras; that is,

$$Q \subseteq \mathbf{ISP}(K').$$

Therefore,

$$Q \subseteq \mathbf{ISP}(K') = \mathbf{P}(K') \subseteq \mathbf{P}(K),$$

and hence $Q = \mathbf{P}(K)$. □

Remark 2. Theorem 3.3 is an algebraic generalization of a theorem from [16], which states that every consequence relation determined by a finite matrix is finitary. This also follows from [11, Theorem 2.3.1].

Remark 3. If K is an arbitrary class of finite algebras, the prevariety $\mathbf{P}(K)$ need not be a quasivariety. For example, the prevariety of Heyting algebras generated by the class K of all finite linearly ordered Heyting algebras is not a quasivariety. The class K generates the variety LC, and it is known that LC contains linearly ordered Heyting algebras of arbitrary cardinality.

Thus, the Heyting algebra $\mathbf{C}_{\omega+2}$ of order type $\omega + 2$ belongs to LC. Note that $\mathbf{C}_{\omega+2}$ is SI, and for its characteristic κ -quasiequation $\mathfrak{c}$ we have $\mathbf{C}_{\omega+2} \not\models \mathfrak{c}$ (cf. Example 1), whereas, by Proposition 2.5, $K \models \mathfrak{c}$, since all algebras in K are finite and $\mathbf{C}_{\omega+2}$ is infinite and does not embed into any algebra in K . On the other hand, $\mathbf{C}_{\omega+2}$ is locally embeddable in K , and hence $\mathbf{C}_{\omega+2} \in \mathbf{Q}(K)$. This shows that $\mathbf{P}(K) \subsetneq \mathbf{Q}(K)$.

Proof of Theorem 3.5

We start with the following simple observation.

Proposition 3.6. *If the variety V is generated by its finite algebras, then $\mathbf{F}_V(\omega)$ has no infinite SI subalgebras.*

Proof. For a contradiction, suppose that V is generated by finite algebras and that $\mathbf{A}$ is an infinite SI subalgebra of $\mathbf{F}_V(\omega)$. Then there exist two distinct elements $a, b \in \mathbf{A} \subseteq \mathbf{F}_V(\omega)$ that belong to the monolith of $\mathbf{A}$.

Since V is generated by finite algebras, for any two distinct elements $a, b \in \mathbf{F}_V(\omega)$, there exists a homomorphism $f: \mathbf{F}_V(\omega) \rightarrow \mathbf{B}$ into a finite algebra $\mathbf{B}$ such that $f(a) \neq f(b)$. The restriction $f|_{\mathbf{A}}$ of f to $\mathbf{A}$ is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$, and $f|_{\mathbf{A}}(a) \neq f|_{\mathbf{A}}(b)$. Since $\mathbf{A}$ is infinite and $\mathbf{B}$ is finite, the homomorphism $f|_{\mathbf{A}}$ is not injective, which contradicts the fact that a and b lie in the monolith of $\mathbf{A}$. $\square$

Corollary 3.7. *If a variety V of finite type with the CEP is generated by finite algebras, then V^{ω^+} contains no infinite SI algebras.*

Proof. For a contradiction, assume that $\mathbf{A} \in V^{\omega^+} = \mathbf{Q}_{\omega^+}(\mathbf{F}_V(\omega))$ is an infinite SI algebra. Then, by Proposition 2.6, $\mathbf{A}$ contains a countable SI subalgebra $\mathbf{B}$, and $\mathbf{B} \in \mathbf{Q}_{\omega^+}(\mathbf{F}_V(\omega))$. Let c be the characteristic ω^+ -quasiequation of $\mathbf{B}$. Then $\mathbf{B} \not\models c$, and since $\mathbf{B} \in \mathbf{Q}_{\omega^+}(\mathbf{F}_V(\omega))$, it follows that $\mathbf{F}_V(\omega) \not\models c$. Thus, by Proposition 2.5, $\mathbf{B}$ is embeddable into $\mathbf{F}_V(\omega)$, contradicting the assumption. $\square$

Now we are in a position to prove Theorem 3.5.

Proof. Suppose that V is an SCpl congruence meet-semidistributive variety of finite type generated by finite algebras.

If V has a finite residual bound, then, since its type is finite, V has only finitely many finite SI algebras up to isomorphism; hence it is finitely generated. Since, by assumption, V is SCpl, it follows from Corollary 3.4 that it is SSCpl.

Conversely, suppose that V is an SCpl congruence meet-semidistributive variety of finite type that does not have a finite residual bound. Then SCpl implies that $V = V^{\omega}$, and therefore all SI algebras in V belong to V^{ω} . On the other hand, since V is congruence meet-semidistributive, of finite type, and does not have a finite residual bound, by Corollary 2.4, V contains an infinite SI algebra $\mathbf{A}$. Hence $\mathbf{A} \in V = V^{\omega}$, but by Corollary 3.7, $\mathbf{A} \notin V^{\omega^+}$. Therefore, $V^{\omega} \subseteq V^{\omega^+} \subsetneq V$, and V is not SSCpl. This completes the proof of Theorem 3.5. $\square$

Recall from [2] that varieties with EDPC are congruence distributive and have the CEP. Hence, we have the following.

Corollary 3.8. *Suppose that V is a variety of finite type with EDPC generated by finite algebras. Then $b_V(\omega^+)$ is admissible for V .*

Proof. By Corollary 3.7, V^{ω^+} contains no infinite SI algebras. Hence, $V^{\omega^+} \models b_V(\omega^+)$, and since $\mathbf{F}_V(\omega) \in V^{\omega^+}$, we have $\mathbf{F}_V(\omega) \models b_V(\omega^+)$; that is, $b_V(\omega^+)$ is admissible. $\square$

Thus, for all non-finitely generated varieties of finite type with EDPC that are generated by finite algebras, $b_V(\omega^+)$ is an admissible but not derivable ω^+ -quasiequation.

Corollary 3.9. *None of the non-finitely generated varieties of finite type with EDPC generated by finite algebras are ω^+ -structurally complete; hence, none of them are SSCpl.*

Remark 4. Theorem 3.5 generalizes [23, Corollary 1].

Remark 5. Observe that, in general, subquasivarieties of a variety generated by finite algebras need not themselves be generated by finite algebras, even if the subquasivariety is SCpl. Indeed, the variety of all Heyting algebras is generated by finite algebras, whereas its structural core, the quasivariety $\mathbf{Q}(\mathbf{F}_{\mathbf{HA}}(\omega))$, is not (cf. [24, Corollary 6.3.11]).

Some applications

Corollary 3.9 has several immediate consequences.

There is a well-known correspondence between intermediate propositional logics and varieties of Heyting algebras, and between normal extensions of the logic $S4$ and varieties of interior algebras; these correspondences preserve structural completeness and strong structural completeness. Recall that the variety HA and the variety of interior algebras have $EDPC$.

Example 3. It is known that Dummett logic LC is structurally complete, and so is the corresponding variety LC of Heyting algebras generated by linearly ordered algebras (see, e.g., [5]). The variety LC is locally finite and contains infinite SI algebras; for example, the Heyting algebra of order type $\omega + 2$. Thus, by Corollary 3.9, variety LC , as well as logic LC , is not strongly structurally complete. Moreover, LC is not even ω^+ -structurally complete.

Example 4. It was observed in [19] that Medvedev logic ML , and consequently the corresponding variety of Heyting algebras ML , is structurally complete, and ML is generated by finite algebras (see, e.g., [5]). By Corollary 3.9, variety ML , as well as logic ML , is not strongly structurally complete, and is not even ω^+ -structurally complete.

Example 5. The variety of Brouwerian algebras (implicative semilattices) is locally finite (see, e.g., [14]); hence any variety of Brouwerian algebras containing an infinite SI algebra is neither strongly structurally complete nor ω^+ -structurally complete.

Example 6. By a similar argument, logic $S4.3$ is not strongly structurally complete (infinitary structural completeness of $S4.3$ and its extensions was studied in depth in [8]).

4 Strongly primitive varieties

In this section we discuss how hereditary structural completeness and hereditary strong structural completeness are related.

Definition 5. Let V be a variety and κ be an infinite cardinal. Then,

V is *hereditarily structurally complete*, or *primitive*, if V and all its subvarieties are $SCpl$;

V is *hereditarily κ -strongly structurally complete*, or *κ -primitive*, if V and all its subvarieties are κ -structurally complete;

V is *hereditarily strongly structurally complete*, or *strongly primitive*, if V and all its subvarieties are $SSCpl$.

Theorem 3.2 yields the following.

Corollary 4.1. *If V is a congruence distributive $HSCpl$ variety of finite type, every subvariety of which is generated by finite algebras (in particular, if V is locally finite), then the finitely generated subvarieties of V , and only those, are $HSSCpl$.*

It follows from [6] that all primitive varieties of Heyting algebras are locally finite. Hence, by Corollary 4.1, only tabular primitive varieties are strongly primitive. In particular, the following holds.

Theorem 4.2 ([6, Section 4]). *A variety of Heyting algebras is strongly primitive if and only if it is primitive and omits the Heyting algebra of order type $\omega + 2$.*

The $\{\rightarrow\}$, $\{\rightarrow, \wedge\}$, and $\{\rightarrow, \wedge, \perp\}$ -fragments of intuitionistic propositional logic are $HSCpl$ and are known to be locally tabular. Thus, the corresponding varieties are primitive and locally finite. By Corollary 4.1, all tabular subvarieties of these varieties, and only those, are strongly primitive.

Remark 6. Hereditary structural completeness in the infinitary sense for the $\{\rightarrow\}$ and $\{\rightarrow, \wedge\}$ -fragments was observed and studied in [21, 22, 26, 27].

Example 7. Denote by $K4$ the variety of modal algebras corresponding to the logic $K4$. Recall (see, e.g., [24, Section 5]) that all primitive subvarieties of $K4$ are locally finite. By Corollary 4.1, the tabular subvarieties of $K4$, and only those, are strongly primitive.

Criterion of strong primitivity

Similarly to the case of primitivity for locally finite quasivarieties, we have the following.

Theorem 4.3. *A variety V of finite type with the CEP is strongly primitive if and only if all its countable SI algebras are weakly V -projective.*

Proof. Let V be a variety of finite type.

From left to right. Suppose that V is strongly primitive and that $\mathbf{A} \in V$ is a countable SI algebra. We prove the contrapositive: if $\mathbf{A}$ is not weakly V -projective, then there exists a subquasivariety $V' \subseteq V$ that is not SSCpl.

Assume that $\mathbf{B} \in V$ is such that $\mathbf{A} \in \mathbf{H}(\mathbf{B})$ whereas $\mathbf{A} \notin \mathbf{IS}(\mathbf{B})$. We show that $\mathbf{V}(\mathbf{B})$ is not SSCpl.

Since $\mathbf{A} \notin \mathbf{IS}(\mathbf{B})$, by Proposition 2.5, $\mathbf{B} \models c_{\mathbf{A}}$. Moreover, since $\mathbf{B}$ generates $\mathbf{V}(\mathbf{B})$, by Proposition 3.1, $c_{\mathbf{A}}$ is admissible for $\mathbf{V}(\mathbf{B})$. However, $\mathbf{A} \in \mathbf{H}(\mathbf{B}) \subseteq \mathbf{V}(\mathbf{B})$ and $\mathbf{A} \not\models c_{\mathbf{A}}$, which entails that $\mathbf{V}(\mathbf{B})$ is not SSCpl.

From right to left. Suppose that all countable SI algebras in V are weakly V -projective. Again, we prove the contrapositive: let $V' \subseteq V$ and suppose that V' is not SSCpl. Then $\mathbf{P}(\mathbf{F}_{V'}(\omega)) \subsetneq V'$. Observe that $V' \setminus \mathbf{P}(\mathbf{F}_{V'}(\omega))$ must contain an SI algebra; otherwise, since every algebra in V' is a subdirect product of its SI algebras, and prevarieties are closed under subdirect products, we would have $V' = \mathbf{P}(\mathbf{F}_{V'}(\omega))$, contrary to our assumption.

Suppose that $\mathbf{A} \in V' \setminus \mathbf{P}(\mathbf{F}_{V'}(\omega))$ is an SI algebra. Then there is a κ -quasiequation s such that $\mathbf{P}(\mathbf{F}_{V'}(\omega)) \models s$ whereas $\mathbf{A} \not\models s$. By Corollary 2.8, $\mathbf{A}$ contains a countable SI subalgebra $\mathbf{B}$ with $\mathbf{B} \not\models s$. Because $\mathbf{B}$ is countable, it is a homomorphic image of $\mathbf{F}_{V'}(\omega)$, but $\mathbf{B} \not\models s$ and hence $\mathbf{B} \notin \mathbf{IS}(\mathbf{F}_{V'}(\omega))$, that is, $\mathbf{B}$ is not weakly V -projective. $\square$

5 Conclusion

In conclusion, we formulate some open problems.

1. Is it true that for every variety V of finite type generated by finite algebras with EDPC, the strong structural core can be defined relative to the structural core by $b_V(\omega^+)$?
2. Is it true that, by adding the bounding rule to the Visser rules, one obtains an axiomatization of the strongly structurally complete consequence relation for IPC?
3. Is it true that the bounding rule yields an axiomatization of the strongly structurally complete consequence relation for LC and ML ?

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Intuitionistic Monotone Modal Logic: Proof Theory and Semantics

Tiziano Dalmonte*

Free University of Bozen-Bolzano
Bozano, Italy

tiziano.dalmonte@unibz.it

Jim de Groot[†]

University of Bern
Bern, Switzerland

jim.degroot@unibe.ch

We study the intuitionistic monotone modal logic IM recently introduced in [27]. We first provide a semantic characterisation for a family of natural extensions of IM in terms of constructive neighbourhood models. We then present a calculus for IM and its extensions, obtained by adapting a structured calculus for the classical monotone modal logic M. Based on the calculus, we prove some preliminary results for IM, including its decidability. Our calculus also reveals an interesting analogy between constructive and intuitionistic variants of M and the corresponding variants of K, thereby further justifying IM as a faithful intuitionistic variant of M.

1 Introduction

Constructive or intuitionistic modal logics arise from extending intuitionistic logic with modal operators, such as necessity or possibility modalities. While early work in the area is motivated by mathematical curiosity and the more philosophical “attempt to find modal logics acceptable to intuitionists” [8, 9], intuitionistic modal logics soon proved useful much more broadly. For example, Goldblatt developed an intuitionistic modal logic to reason about the Grothendieck topology [24], and various constructive systems of epistemic reasoning were studied in e.g. [60, 33, 50, 36, 4]. Moreover, intuitionistic modal logics are used to model countless computational phenomena, with applications ranging from hardware verification [18] to access control [23] to staged computation [15, 16, 43], and from type inhabitation of arrows in functional programming [35, 38], [39, Section 7.1] to program evaluation [48] to knowledge representation in AI [40, 41, 44, 42].

A peculiar feature of intuitionistic modal logics is that, similar to the intuitionistic connectives, the modal operators need not be interdefinable. This disconnection often gives rise to a wide spectrum of intuitionistic counterparts of the same classical system. On one side of this spectrum sit for example logics studied by Sotirov [53, Section 4], Wolter and Zakharyashev [61, Section 2], and Dalmonte, Grellois and Olivetti [13], in which $\Box$ and $\Diamond$ are completely unrelated. At the other extremity, we have systems studied by Bull [8] and Božić and Došen [6, Section 11], in which the diamond operator can be viewed as an abbreviation for $\neg\Box\neg$.

The study of constructive analogues of classical normal modal logics, such as K and S4, has resulted in two prominent ways to establish a more subtle interaction between the modalities. First, *constructive* counterparts are obtained by restricting a sequent calculus for the classical modal logic in question to (at most) one conclusion. This originates from Wijesekera [58], and gives rise to analogues of K such as CK [5], WK [58, 59] and CS4 [1]. Second, *intuitionistic* counterparts of normal modal logics are

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obtained by taking the set of formulas whose standard translation is derivable in *intuitionistic* first-order logic [19, 52]. This gives rise to *intuitionistic* K (IK) studied by [19, 20, 49], as well as intuitionistic counterparts of other normal modal logics including IS4 and IKD [17, 52].

Besides intuitionistic analogues of K, extensions of intuitionistic logic with monotone modal operators have been studied as well. Monotone modal operators are modalities ∇ for which $\varphi \rightarrow \psi$ implies $\nabla\varphi \rightarrow \nabla\psi$, or equivalently (in presence of the congruence rule), that satisfy $\nabla(\varphi \wedge \psi) \rightarrow \nabla\varphi$. The study of these in an intuitionistic context dates back to Bull [7, Page 5], who studied a single monotone modality that satisfies the S4 axioms. Discarding the S4 axioms yields Goldblatt's intuitionistic logic with a geometric modality [24, 26], which was further researched in e.g. [29, 30, 55]. However, compared to the wealth of research on intuitionistic analogues of K, the study of intuitionistic versions of the classical monotone modal logic M is still in its infancy. This is surprising, because classical monotone modal logic has been researched extensively [10, 31, 32, 51, 21] and encompasses modal logics such as game logic [45], concurrent propositional dynamic logic [25], coalition logic [46, 47] and alternating-time temporal logic [2].

The study of intuitionistic monotone modal logics with two modalities is even more recent. The analogy with the normal modal case gives three ways to decide on the level of interaction between the modalities. First, restricting a (suitable) sequent calculus for M to precisely one conclusion yields a logic in which the modal operators are completely independent [12]. The resulting logic, called CM in analogy with CK, may be seen as a bimodal version of Goldblatt's intuitionistic logic with a geometric modality. Second, the restriction of a sequent calculus for M to at most one conclusion (i.e. to zero or one conclusion) is investigated in [11], and gives rise to the logic WM (and extensions).¹ Finally, in [27] an *intuitionistic* approach was taken: classical monotone modal logic was translated into a suitable first-order logic, and changing this first-order logic to an intuitionistic first-order logic gave rise to the intuitionistic monotone modal logic IM. This mimics the method of obtaining IK from K using intuitionistic first-order logic [52]. Given its recent introduction, the study of IM is still lacks a proof theoretic treatment, a study of its extensions, and basic decidability results. This is the gap we aim to close in this paper.

We start by equipping IM with a semantics by means of *constructive neighbourhood frames*, derived from [11]. We use a canonical model construction to prove completeness of extensions of IM with respect to suitable classes of such frames. This differs from the *intuitionistic neighbourhood models* used in [27]. While the latter were useful for bridging the gap between the first-order and modal perspectives of the logic, the former semantics provides us with valuable intuition for the sequent calculus.

Next, we present a sound and complete calculus for IM. Considering that restrictions of Gentzen-style sequent calculi yield constructive monotone modal logics [11, 12], and by analogy with the nested sequent calculus for IK [54], we now start from a *structured* sequent calculus for classical M and adapt it to IM. We consider to this purpose the calculus HM from [14]. Besides the comma and the sequent arrow, HM employs two additional structural connectives: the hypersequent bar, which allows one to represent sets of sequents and to efficiently extract neighbourhood countermodels from failed derivations, and blocks of formulas, which intuitively represent neighbourhood sets associated with the corresponding sequent/worlds, and also allow for a modular extension of the base calculus by means of rules handling blocks. Since countermodel extraction is outside the scope of the present work, we disregard here hypersequents (which is possible to do by still preserving the syntactic completeness of the calculus) and restrict to simple sequents endowed with blocks. Our main contributions are as follows:

¹The logic WM from [11] is called IM in [13]. We reserve the name IM for the logic from [27]. This aligns with the naming convention for constructive and intuitionistic normal modal logics [28].

1. We provide a cut-free, terminating calculus C_{IM} for IM by restricting a suitable sequent calculus with blocks to single-succedent sequents.
2. We show that this is closely related to a calculus for WM: C_{IM} is based on an operation of pruning that cuts out output formulas in the application of an implication rule, and changing this to a more radical form of pruning (that we call output thinning) provides a calculus for WM.
3. We prove that IM is decidable, and re-prove [27, Theorem 4.11] that IM and WM have different $\Box$ -free fragments
4. We obtain calculi for extensions of IM by using block rules for extensions of M [14].

2 Intuitionistic monotone modal logic

We briefly recall the logics WM and IM from [11, 27]. We then give a canonical model construction for IM and some of its extensions to prove completeness with respect to constructive neighbourhood frames (as opposed to the *intuitionistic* neighbourhood frames from [27]).

2.1 The logics

Let $\mathcal{L}$ be the language generated by the grammar $\varphi ::= p \mid \perp \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Box \varphi \mid \Diamond \varphi$, where p ranges over some arbitrary but fixed set Prop of proposition letters. As usual, we abbreviate $\neg \varphi := \varphi \rightarrow \perp$ and $\top := \neg \perp$. Besides, the *complexity* of a formula is recursively defined via $c(p) = c(\perp) = 0$, $c(\varphi \wedge \psi) = c(\varphi \vee \psi) = c(\varphi \rightarrow \psi) = c(\varphi) + c(\psi) + 1$, and $c(\Box \varphi) = c(\Diamond \varphi) = c(\varphi) + 1$.

Intuitionistic analogues of monotone modal logics can be described using the following axioms:

$$\text{mon}_{\Box} \frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}, \quad \text{mon}_{\Diamond} \frac{\varphi \rightarrow \psi}{\Diamond \varphi \rightarrow \Diamond \psi}, \quad \text{mnc} (\Box \varphi \wedge \Diamond \neg \varphi) \rightarrow \perp, \quad \text{I}_{\Diamond} (\Box \top \rightarrow \Diamond \varphi) \rightarrow \Diamond \varphi.$$

Definition 2.1. Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be a set of axioms and rules. Then we write $\text{IPL} \oplus \mathcal{R}$ or $\text{IPL} \oplus R_1 \oplus \dots \oplus R_n$ for the logic over the language $\mathcal{L}$ obtained by extending an axiomatisation for intuitionistic propositional logic with the axioms and rules in $\mathcal{R}$, and closing it under modus ponens and uniform substitution. We can define *constructive*, *Wijesekera* and *intuitionistic monotone modal logic* as follows:

$$\text{CM} := \text{IPL} \oplus \text{mon}_{\Box} \oplus \text{mon}_{\Diamond}, \quad \text{WM} := \text{CM} \oplus \text{mnc}, \quad \text{IM} := \text{WM} \oplus \text{I}_{\Diamond}.$$

We are also interested in extensions of IM with the axioms N, P, T, D and K, which are well known in the classical modal logic literature. For example, in the context of coalition logic N and P embody the idea that coalitions can bring about anything that is already true, but cannot achieve the impossible (because classically P is equivalent to $\neg \Box \perp$), and in an epistemic context T states that only true things can be known. The axiom K is of interest because it yields intuitionistic normal modal logics weaker than IK. In this paper, we consider:

$\begin{array}{ll} \text{N} & \Box \top \\ \text{T}_{\Box} & \Box \varphi \rightarrow \varphi \\ \text{K}_{\Box} & \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \end{array}$	$\begin{array}{ll} \text{P} & \Diamond \top \\ \text{T}_{\Diamond} & \varphi \rightarrow \Diamond \varphi \\ \text{K}_{\Diamond} & \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \end{array}$	$\begin{array}{ll} \text{D} & \Box \varphi \rightarrow \Diamond \varphi \\ \text{T} & (\text{T}_{\Box} \wedge \text{T}_{\Diamond}) \\ \text{K} & (\text{K}_{\Box} \wedge \text{K}_{\Diamond}) \end{array}$

For any logic L, we write $\Phi \vdash_L \varphi$ if there exist $\varphi_1, \dots, \varphi_n \in \Phi$ such that $\vdash_L (\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \varphi$. Since in this paper we focus on the logic IM, if Ax is a set of axioms we abbreviate $\Phi \vdash_{\text{IM} \oplus \text{Ax}} \varphi$ to $\Phi \vdash_{\text{Ax}} \varphi$.

We will also refer to the following modal axioms, which relate to the preceding ones via Lemma 2.2.

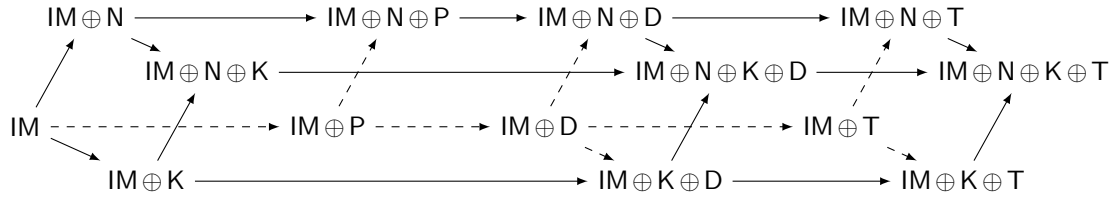
$$\begin{array}{lll}
C_{\Box} & \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi) & W \quad \Box\varphi \wedge \Diamond\psi \rightarrow \Diamond(\varphi \wedge \psi) \quad P_{\Box} \quad \Box\perp \rightarrow \perp \\
str & \frac{\varphi \wedge \psi \rightarrow \perp}{\Box\varphi \wedge \Diamond\psi \rightarrow \perp} & RD \quad \frac{\varphi \wedge \psi \rightarrow \perp}{\Box\varphi \wedge \Box\psi \rightarrow \perp}
\end{array}$$

Lemma 2.2. *Over CM, mnc is equivalent to str. Over WM, RD is derivable from D. Also, we have:*

$$\begin{array}{lllll}
\vdash_{WM \oplus K_{\Box}} C_{\Box} & \vdash_{WM \oplus C_{\Box}} K_{\Box} & \vdash_{WM \oplus P} P_{\Box} & \vdash_{WM \oplus T} D & \vdash_{WM \oplus N \oplus D} P \\
& \vdash_{WM \oplus K_{\Diamond}} W & \vdash_{WM \oplus N} I_{\Diamond} & \vdash_{WM \oplus T} P & \vdash_{WM \oplus K \oplus P} D
\end{array}$$

Furthermore, we have $\vdash_{IM \oplus D} P$. Since IM is stronger than WM, we may replace every occurrence of WM with IM.

Derivations can be found in [11, 12]. In light of Lemma 2.2, combinations of axioms in $\{N, P, T, D, K\}$ give rise to 14 different systems, depicted below, where arrows denote inclusion of logical systems. Each of the systems arises from extending the corresponding system from [11] with $I_{\Diamond}$.



2.2 Constructive neighbourhood semantics

Let $(X, \leq)$ be a preordered set. For $a \subseteq X$ we define $\uparrow a = \{y \in X \mid x \leq y \text{ for some } x \in a\}$. Then a is called an *upset* if $\uparrow a = a$, and we write $\text{up}(X, \leq)$ for the collection of upsets of $(X, \leq)$.

Definition 2.3. A *constructive neighbourhood frame* is a tuple $\mathfrak{F} = (W, \leq, N)$ consisting of a set W , a preorder $\leq$ on W , and a function $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ called a *neighbourhood function*. A *constructive neighbourhood model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ comprising an constructive neighbourhood frame $\mathfrak{F}$ and a valuation $V : \text{Prop} \rightarrow \text{up}(X, \leq)$ that assigns an upset of $(X, \leq)$ to each proposition letter p .

The interpretation of $\mathcal{L}$ -formulas at a world w of such a model $\mathfrak{M}$ is defined recursively by

$$\begin{array}{ll}
\mathfrak{M}, w \Vdash p & \text{iff } w \in V(p) \\
\mathfrak{M}, w \Vdash \perp & \text{never} \\
\mathfrak{M}, w \Vdash \varphi \wedge \psi & \text{iff } \mathfrak{M}, w \Vdash \varphi \text{ and } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \varphi \vee \psi & \text{iff } \mathfrak{M}, w \Vdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \varphi \rightarrow \psi & \text{iff for all } v \geq w, \mathfrak{M}, v \Vdash \varphi \text{ implies } \mathfrak{M}, v \Vdash \psi \\
\mathfrak{M}, w \Vdash \Box\varphi & \text{iff for all } v \geq w \text{ there exists } a \in N(v) \text{ such that for all } u \in a \text{ we have } \mathfrak{M}, u \Vdash \varphi \\
\mathfrak{M}, w \Vdash \Diamond\varphi & \text{iff for all } v \geq w \text{ and all } a \in N(v) \text{ there exists } u \in a \text{ such that } \mathfrak{M}, u \Vdash \varphi
\end{array}$$

Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}$. A constructive neighbourhood frame $\mathfrak{F}$ *validates* ψ if $(\mathfrak{F}, V), w \Vdash \psi$ for every valuation V for $\mathfrak{F}$ and every world w in $\mathfrak{F}$, and it *validates* Φ if it validates every formula in Φ , notation: $\mathfrak{F} \Vdash \psi$ and $\mathfrak{F} \Vdash \Phi$. If Ax is a set of axioms and $\varphi \in \mathcal{L}$, then we write $\Vdash_{Ax} \varphi$ if every constructive neighbourhood frame that validates the formulas in Ax also validates φ .

It was shown in [11] that constructive neighbourhood frames provide sound and complete semantics with respect to various extensions of WM. They can also be used for (extensions of) IM.

Lemma 2.4. *Let $\mathfrak{F} = (W, \leq, N)$ be a constructive neighbourhood frame.*

1. *If $N(w) \neq \emptyset$ for all $w \in W$, then $\mathfrak{F}$ validates N;*
2. *If $\emptyset \notin N(w)$ for all $w \in W$, then $\mathfrak{F}$ validates P;*
3. *If $w \in U$ for all $U \in N(w)$ and $w \in W$, then $\mathfrak{F}$ validates T;*
4. *If $U_1 \cap U_2 \neq \emptyset$ for all $U_1, U_2 \in N(w)$ and $w \in W$, then $\mathfrak{F}$ validates D;*
5. *If $N(w)$ is closed under binary intersections for all $w \in W$, then $\mathfrak{F}$ validates K;*
6. *If $(N(w) \neq \emptyset \text{ and } w \leq v) \text{ implies } N(v) \neq \emptyset$, for all $w, v \in W$, then $\mathfrak{F}$ validates $\Box\Diamond$.*

Proof. The first five items can be found in [11]. For the last one, suppose $\mathfrak{F}$ satisfies the given condition, and let $\mathfrak{M} = (\mathfrak{F}, V)$ be a model based on $\mathfrak{F}$. Let $w \in W$ and suppose $w \Vdash \Box\top \rightarrow \Diamond\phi$. We show that any neighbourhood U of any $v \geq w$ contains an element that satisfies ϕ . Let $v \geq w$. Then $v \Vdash \Box\top \rightarrow \Diamond\phi$. If $N(v) \neq \emptyset$, then by the frame condition we have $v \Vdash \Box\top$, so that $v \Vdash \Diamond\phi$, which implies that every neighbourhood of v contains an element satisfying ϕ . If, on the other hand, $N(v) = \emptyset$, then it is trivially true that every neighbourhood of v contains a world satisfying ϕ . Therefore $w \Vdash \Diamond\phi$. Since w was arbitrary, this proves that the frame validates $\Box\Diamond$. $\square$

Definition 2.5. Constructive neighbourhood frames that satisfy Item (6) above are called *continual*.

2.3 Canonical model constructions

Throughout this subsection, let $Ax \subseteq \{N, P, T, D, K\}$. We prove that $IM \oplus Ax$ is sound and complete with respect to a suitable class of continual constructive neighbourhood frames. Akin to [11, 58], we use so-called segments, rather than prime theories, to construct a canonical model. The case for IM without additional axioms was proven (indirectly) in [27, Theorem 4.8].

Definition 2.6. An *Ax-prime theory* is a subset $\Phi \subseteq \mathcal{L}$ that is *consistent* ($\Phi \not\vdash_{Ax} \perp$), *deductively closed* ($\Phi \vdash_{Ax} \phi$ implies $\phi \in \Phi$) and *prime* ($\phi \vee \psi \in \Phi$ implies $\phi \in \Phi$ or $\psi \in \Phi$).

We write PF_{Ax} for the set of Ax-prime theories. We can prove the Lindenbaum lemma as usual:

Lemma 2.7. *Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}$ be such that $\Phi \not\vdash_{Ax} \psi$. Then there exists an Ax-prime theory Φ' such that $\Phi \subseteq \Phi'$ and $\psi \notin \Phi'$.*

Our canonical model is not built from prime theories, but rather from pairs consisting of a prime theory and a family of sets of prime theories that encodes the neighbourhoods of the world.

Definition 2.8. An *Ax-segment* is a pair $(\Phi, \mathcal{N})$ consisting of an Ax-prime theory Φ and a collection $\mathcal{N}$ of sets of Ax-prime theories, such that:

- If $\Box\phi \in \Phi$ then there exists a $U \in \mathcal{N}$ such that $\Psi \in U$ implies $\phi \in \Psi$;
- If $\Diamond\phi \in \Phi$ then for all $U \in \mathcal{N}$ there exists some $\Psi \in U$ such that $\phi \in \Psi$.

Next, we consider several particular segments based on a given prime theory Φ . This also shows that we can always extend a prime theory to a segment. For a formula ϕ , define $\tilde{\phi} := \{\Psi \in PF_{Ax} \mid \phi \in \Psi\}$ and $\tilde{\phi}^c := PF_{Ax} \setminus \tilde{\phi}$. (When using the notation $\tilde{\phi}$, the set Ax will be clear from context.)

Lemma 2.9. *Let Φ be an Ax-prime theory and $\psi \in \mathcal{L}$ such that $\diamond\psi \notin \Phi$. Define*

$$\mathcal{B}_\Phi := \{\tilde{\varphi} \mid \Box\varphi \in \Phi\} \quad \text{and} \quad \mathcal{D}_{\Phi,\psi} := \begin{cases} \mathcal{B}_\Phi \cup \{\tilde{\psi}^c\} & \text{if } K \notin \text{Ax} \\ \mathcal{B}_\Phi \cup \{\tilde{\psi}^c\} \cup \{\tilde{\varphi} \cap \tilde{\psi}^c \mid \Box\varphi \in \Phi\} & \text{if } K \in \text{Ax} \end{cases}$$

Then $(\Phi, \mathcal{B}_\Phi)$ and $(\Phi, \mathcal{D}_{\Phi,\psi})$ are Ax-segments.

Proof. In each of the cases, the $\Box$ -condition is clearly satisfied, so we prove the $\diamond$ -conditions. For $(\Phi, \mathcal{B}_\Phi)$, we need to show that for all $\Box\varphi, \diamond\chi \in \Phi$ we have $\tilde{\varphi} \cap \tilde{\chi} \neq \emptyset$. Suppose towards a contradiction that this were not the case, then $\varphi, \chi \vdash_{\text{Ax}} \perp$, hence by str $\Box\varphi, \diamond\chi \vdash_{\text{Ax}} \perp$. But then $\perp \in \Phi$, a contradiction.

For $(\Phi, \mathcal{D}_{\Phi,\psi})$, if $K \notin \text{Ax}$ then we need to additionally show that $\tilde{\psi}^c \cap \tilde{\chi} \neq \emptyset$ for any $\diamond\chi \in \Phi$. If $\tilde{\psi}^c \cap \tilde{\chi} = \emptyset$, then $\tilde{\chi} \subseteq \tilde{\psi}$, which implies $\chi \vdash_{\text{Ax}} \psi$. But then mon $_\diamond$ entails $\diamond\chi \vdash_{\text{Ax}} \diamond\psi$, so that $\diamond\psi \in \Phi$ by deductive closure of Φ , a contradiction.

Finally, if $K \in \text{Ax}$ then we need to also prove that $\Box\varphi, \diamond\chi \in \Phi$ we have $(\tilde{\varphi} \cap \tilde{\psi}^c) \cap \tilde{\chi} \neq \emptyset$. Suppose towards a contradiction that this intersection is empty. Then $\tilde{\varphi} \cap \tilde{\chi} \subseteq \tilde{\psi}$, hence $\varphi, \chi \vdash_{\text{Ax}} \psi$, so by $K_\diamond$ also $\Box\varphi, \diamond\chi \vdash_{\text{Ax}} \diamond\psi$. This implies $\diamond\psi \in \Phi$, a contradiction. $\square$

Lemma 2.10. *Let $(\Phi, \mathcal{N})$ be an Ax-segment and ψ a formula with $\diamond\psi \notin \Phi$, such that $\mathcal{N} \in \{\mathcal{B}_\Phi, \mathcal{D}_{\Phi,\psi}\}$.*

1. *If $\mathcal{N} \in \text{Ax}$ then $\mathcal{N} \neq \emptyset$;*
2. *If $\mathcal{P} \in \text{Ax}$ then $\emptyset \notin \mathcal{N}$;*
3. *If $\mathcal{T} \in \text{Ax}$, then $\Phi \in U$ for all $U \in \mathcal{N}$;*
4. *$\mathcal{D} \in \text{Ax}$ then $U_1 \cap U_2 \neq \emptyset$ for all $U_1, U_2 \in \mathcal{N}$;*
5. *If $\mathcal{K} \in \text{Ax}$ then $U_1 \cap U_2 \in \mathcal{N}$ for all $U_1, U_2 \in \mathcal{N}$.*

Proof. (1) Since $\Box\top \in \Phi$ we always get $\text{PF}_{\text{Ax}} \in \mathcal{N}$.

(2) By assumption $\diamond\top \in \Phi$, hence it follows from mnc that $\Box\perp \notin \Phi$. This entails $\emptyset \notin \mathcal{B}_\Phi$. If $\emptyset \in \mathcal{D}_{\Phi,\psi}$ then we must have $\tilde{\psi}^c = \emptyset$ (if $K \notin \text{Ax}$) or $\tilde{\varphi} \cap \tilde{\psi}^c = \emptyset$ for some $\Box\varphi \in \Phi$ (if $K \in \text{Ax}$). In the former case ψ must be equivalent to $\top$, but then the $\diamond\top \in \Phi$ implies $\diamond\psi \in \Phi$, a contradiction. In the latter case, combining $\mathcal{P}$ and $K_\diamond$ yields $\mathcal{D}$, so $\Box\varphi \in \Phi$ implies $\diamond\varphi \in \Phi$. Moreover, the assumption $\tilde{\varphi} \cap \tilde{\psi}^c = \emptyset$ yields $\varphi \vdash_{\text{Ax}} \psi$, hence by monotonicity $\diamond\varphi \vdash_{\text{Ax}} \diamond\psi$ and therefore $\diamond\psi \in \Phi$, and we have reached a contradiction again. Therefore we conclude $\emptyset \notin \mathcal{D}_{\Phi,\psi}$, as desired.

(3) If $U = \tilde{\varphi}$ for some $\Box\varphi \in \Phi$, then using $\mathcal{T}_\Box$ we find $\varphi \in \Phi$, hence $\Phi \in \tilde{\varphi} = U$. If $U = \tilde{\psi}^c$ for some $\diamond\psi \notin \Phi$ then using $\mathcal{T}_\diamond$ we find $\psi \notin \Phi$, hence $\Phi \in \tilde{\psi}^c = U$. Finally, if U is of the form $\tilde{\varphi} \cap \tilde{\psi}^c$ for some $\Box\varphi \in \Phi$ and $\diamond\psi \notin \Phi$, then combining the previous two cases gives $\Phi \in U$ again.

(4) Suppose $U_1 = \tilde{\varphi}$ and $U_2 = \tilde{\chi}$ for some $\Box\varphi, \Box\chi \in \Phi$. Then $\mathcal{D}$ entails $\diamond\chi \in \Phi$, hence $\Box\varphi, \diamond\chi \vdash_{\text{Ax}} \perp$. Now str entails $\varphi, \chi \vdash_{\text{Ax}} \perp$, so that the Lindenbaum lemma tells us that $\tilde{\varphi} \cap \tilde{\chi} \neq \emptyset$, as desired.

Next, suppose $U_1 = \tilde{\varphi}$ and $U_2 = \tilde{\psi}^c$ for some $\Box\varphi \in \Phi$ and $\diamond\psi \notin \Phi$. If $\tilde{\varphi} \cap \tilde{\psi}^c = \emptyset$ then $\tilde{\varphi} \subseteq \tilde{\psi}$, so $\varphi \vdash_{\text{Ax}} \psi$. Now mon $_\Box$ yields $\Box\varphi \vdash_{\text{Ax}} \Box\psi$ and by $\mathcal{D}$ we have $\Box\psi \vdash_{\text{Ax}} \diamond\psi$, so that deductive closure of Φ entails $\diamond\psi \in \Phi$, a contradiction. If $K \notin \text{Ax}$ then this completes the proof for $(\Phi, \mathcal{D}_{\Phi,\psi})$. If $K \in \text{Ax}$ then $\mathcal{C}_\Box$ entails that we can reduce any intersection of the sets in $\mathcal{D}_{\Phi,\psi}$ to a set of the form $\tilde{\varphi} \cap \tilde{\chi}$ or $\tilde{\varphi} \cap \tilde{\psi}^c$, where $\Box\varphi, \Box\chi \in \Phi$ and $\diamond\psi \notin \Phi$, and we have already seen that such intersections are nonempty.

(5) Both cases follow immediately from the fact that derivability of $\mathcal{C}_\Box$ entails that $\mathcal{B}_\Phi$ is closed under binary intersections. $\square$

Lemma 2.11. *Let Φ be an Ax-prime theory such that $\Box\top \notin \Phi$ and $\diamond\varphi \notin \Phi$. Then there exists an Ax-prime theory Φ' extending Φ that contains $\Box\top$ but not $\diamond\varphi$.*

Proof. It suffices to prove $\Phi, \Box\top \not\vdash_{Ax} \Diamond\varphi$, because then the Lindenbaum lemma yields the desired Φ' . If this is not the case, i.e. if $\Phi, \Box\top \vdash_{Ax} \Diamond\varphi$, then there exists $\gamma \in \Phi$ such that $\gamma, \Box\top \vdash_{Ax} \Diamond\varphi$, hence $\gamma \vdash_{Ax} \Box\top \rightarrow \Diamond\varphi$. Then $\Diamond$ entails $\gamma \vdash \Diamond\varphi$, so that $\Diamond\varphi \in \Phi$ by deductive closure of Φ , a contradiction. $\square$

We base our canonical model on segments arising from Lemma 2.9. To ensure that the canonical model is continual (Definition 2.5), we impose the restriction that if $\Box\top \notin \Phi$ then we only use the segment $(\Phi, \mathcal{B}_\Phi)$.

Definition 2.12. The *canonical model* is the tuple $\mathfrak{M}_{Ax} = (W_{Ax}, \subseteq, N_{Ax}, V_{Ax})$, where:

$$\begin{aligned} W_{Ax} &:= \{(\Phi, \mathcal{B}_\Phi) \mid \Phi \in PF_{Ax}\} \cup \{(\Phi, \mathcal{D}_{\Phi, \psi}) \mid \Phi \in PF_{Ax}, \Box\top \in \Phi, \Diamond\psi \notin \Phi\} \\ (\Phi, \mathcal{N}) &\subseteq (\Psi, \mathcal{M}) \quad \text{iff} \quad \Phi \subseteq \Psi \\ N_{Ax}(\Phi, \mathcal{N}) &:= \{(\Psi, \mathcal{M}) \in W_{Ax} \mid \Psi \in \mathcal{U}\} \mid \mathcal{U} \in \mathcal{N}\} \\ V_{Ax}(p) &:= \{(\Phi, \mathcal{N}) \in W_{Ax} \mid p \in \Phi\} \end{aligned}$$

Note that the definition above defines two different canonical models, depending on whether or not $K \in Ax$ (see Lemma 2.9). Both are continual: if $(\Phi, \mathcal{N})$ is a world such that $N(\Phi, \mathcal{N}) = \emptyset$, then $\mathcal{N} = \emptyset$ and we must have $\Box\top \notin \Phi$. This implies $\mathcal{N} = \mathcal{B}_\Phi$. Then for any $(\Psi, \mathcal{M}) \subseteq (\Phi, \mathcal{N})$ we find $\mathcal{M} = \mathcal{B}_\Psi = \emptyset$.

Lemma 2.13 (Truth lemma). *For all $\varphi \in \mathcal{L}$ and $(\Phi, \mathcal{N}) \in W_{Ax}$ we have $\mathfrak{M}_{Ax}, (\Phi, \mathcal{N}) \Vdash \varphi$ iff $\varphi \in \Phi$.*

Proof. The proof proceeds by induction on the structure of φ . The propositional cases are routine.

Case $\varphi = \Box\psi$. By definition of a segment, $\Box\psi \in \Phi$ implies $(\Phi, \mathcal{N}) \Vdash \Box\psi$. Conversely, if $(\Phi, \mathcal{N}) \Vdash \Box\psi$ then since $(\Phi, \mathcal{N}) \subseteq (\Phi, \mathcal{B}_\Phi)$ we have $(\Phi, \mathcal{B}_\Phi) \Vdash \Box\psi$. This means that there exists some $\Box\chi \in \Phi$ such that every prime theory in $\tilde{\chi}$ satisfies ψ , hence by induction every such prime theory contains ψ , i.e. $\tilde{\chi} \subseteq \tilde{\psi}$. Therefore $\chi \vdash \psi$, so combining monotonicity and the fact that $\Box\chi \in \Phi$ entails $\Box\psi \in \Phi$.

Case $\varphi = \Diamond\psi$. By definition $\Diamond\psi \in \Phi$ implies $(\Phi, \mathcal{N}) \Vdash \Diamond\psi$. For the converse, suppose $\Diamond\psi \notin \Phi$. If $\Box\top \in \Phi$, then $(\Phi, \mathcal{D}_{\Phi, \psi})$ is a segment in W_{Ax} such that $\tilde{\psi}^c \in \mathcal{D}_{\Phi, \psi}$, hence $\{(\Psi, \mathcal{M}) \mid \Psi \in \tilde{\psi}^c\} \in N_{Ax}(\Phi, \mathcal{D}_{\Phi, \psi})$. Since $\psi \notin \Psi$ for all such $(\Psi, \mathcal{M})$, by the induction hypothesis no segment in $\{(\Psi, \mathcal{M}) \mid \Psi \in \tilde{\psi}^c\}$ satisfies ψ . Since $(\Phi, \mathcal{N}) \subseteq (\Phi, \mathcal{D}_{\Phi, \psi})$ this proves $(\Phi, \mathcal{N}) \not\Vdash \Diamond\psi$. If $\Box\top \notin \Phi$ then Lemma 2.11 yields a prime theory Φ' containing Φ and $\Box\top$ but not $\Diamond\psi$, so that $(\Phi, \mathcal{N}) \subseteq (\Phi', \mathcal{D}_{\Phi', \psi})$ witnesses $(\Phi, \mathcal{N}) \not\Vdash \Diamond\psi$. $\square$

In the usual way, we can now obtain soundness and completeness:

Theorem 2.14. *Let $Ax \subseteq \{N, P, T, D, K\}$. Then $\Vdash_{IM \oplus Ax} \varphi$ if and only if $\vdash_{IM \oplus Ax} \varphi$.*

3 A sequent calculus for IM

We present the sequent calculus C_{IM} , provide a syntactic proof of cut elimination, show that it is sound and complete with respect to IM, and observe that a small modification yields a calculus for WM.

3.1 The sequent calculus C_{IM}

Definition 3.1.

1. A *marked formula* is a formula $\varphi \in \mathcal{L}$ marked with the *input polarity* $\bullet$ or the *output polarity* $\circ$. Marked formulas $\varphi^\bullet$ and φ° are called *input* and *output formulas*, respectively. We write $\mathcal{L}^\bullet$ for the collection of marked formulas.

2. A *block* $\mathcal{B}$ is a structure of the form $\{\Delta\}$, where Δ is a set of formulas that contains one or zero formulas. We write $|\Delta|$ for the number of formulas in Δ . A block $\{\Delta\}$ is an *input block*, denoted $\mathcal{B}^\bullet$, if Δ contains an input formula or is empty, and is an *output block* if Δ contains an output formula. The set of blocks is denoted by $\mathcal{L}^{\{\bullet\}}$.
3. A *sequent* Γ is defined by the following grammar, where $n, k \geq 0$, and Λ is understood as a multiset:

$$\Gamma ::= \Lambda, \Pi \quad \Lambda ::= \varphi_1^\bullet, \dots, \varphi_n^\bullet, \mathcal{B}_1^\bullet, \dots, \mathcal{B}_k^\bullet \quad \Pi ::= \emptyset \mid \varphi^\circ \mid \{\varphi^\circ\}.$$

In other words, we consider (single-succedent) sequents with at most one output formula, that can occur inside or outside a block. We use Γ, Δ, Σ as variables for sequents, Λ as variable for input sequents (with only input components), and Π as variable for output sequents (with only output components).

Definition 3.2. For a sequent Γ , we define its *output pruning* $\Gamma^\downarrow$ to be the same sequent with the output formula removed (if present).

Note that if the output formula occurs inside a block, the block is not deleted but only emptied. For instance, $(\varphi^\bullet, \{\psi^\bullet\}, \chi^\circ)^\downarrow = (\varphi^\bullet, \{\psi^\bullet\})^\downarrow = \varphi^\bullet, \{\psi^\bullet\}$, and $(\varphi^\bullet, \{\psi^\bullet\}, \{\chi^\circ\})^\downarrow = \varphi^\bullet, \{\psi^\bullet\}, \{\}$, where in the latter pruning the output block $\{\chi^\circ\}$ is turned into the input block $\{\}$. The *formula interpretation* i maps sequents to formulas in $\mathcal{L}$:

Definition 3.3. Define $\iota : \mathcal{L}^\bullet \cup \mathcal{L}^{\{\bullet\}} \cup \{\emptyset\} \rightarrow \mathcal{L}$ by $\iota(\varphi^\bullet) = \iota(\varphi^\circ) = \varphi$ and

$$\iota(\{\}) = \Box \top, \quad \iota(\{\varphi^\bullet\}) = \Box \varphi, \quad \iota(\{\varphi^\circ\}) = \Diamond \varphi \quad \text{and} \quad \iota(\emptyset) = \perp.$$

Then the *formula interpretation* of a sequent $\Gamma = \Lambda, \Pi$ is given by $i(\Gamma) := (\bigwedge_{x \in \Lambda} \iota(x)) \rightarrow \iota(\Pi)$, where we take the empty meet to be $\top$.

For instance, $i(\varphi^\bullet, \{\psi^\bullet\}, \{\}) = \varphi \wedge \Box \psi \wedge \Box \top \rightarrow \perp$ and $i(\{\varphi^\bullet\}, \{\varphi^\circ\}) = \Box \varphi \rightarrow \Diamond \varphi$.

Definition 3.4. The calculus C_{IM} for IM is defined by the rules in Figure 1.

The calculus is built on a standard G3-style single-succedent calculus for IPL [56], and contains input and output rules for all connectives. Ignoring the hypersequent structure (which is not needed for completeness), C_{IM} can be seen as a single-succedent restriction of the calculus HM for the classical monotone modal classical logic M defined in [14]. Note also that contrary to nested calculi [3, 54], the blocks of C_{IM} can only contain (sets of) formulas, and therefore cannot be nested.

$\perp^\bullet \frac{}{\Gamma, \perp^\bullet}$	$\text{id} \frac{}{\Lambda, p^\bullet, p^\circ}$	$\wedge^\bullet \frac{\Gamma, \varphi^\bullet, \psi^\bullet}{\Gamma, \varphi \wedge \psi^\bullet}$	$\wedge^\circ \frac{\Lambda, \varphi^\circ \quad \Lambda, \psi^\circ}{\Lambda, \varphi \wedge \psi^\circ}$
$\vee^\bullet \frac{\Gamma, \varphi^\bullet \quad \Gamma, \psi^\bullet}{\Gamma, \varphi \vee \psi^\bullet}$	$\vee^\circ \frac{\Lambda, \varphi^\circ}{\Lambda, \varphi \vee \psi^\circ}$	$\vee^\circ \frac{\Lambda, \psi^\circ}{\Lambda, \varphi \vee \psi^\circ}$	
$\rightarrow^\bullet \frac{\Gamma^\downarrow, \varphi \rightarrow \psi^\bullet, \varphi^\circ}{\Gamma, \varphi \rightarrow \psi^\bullet}$	$\rightarrow^\circ \frac{\Lambda, \varphi^\bullet, \psi^\circ}{\Lambda, \varphi \rightarrow \psi^\circ}$		
$\Box^\bullet \frac{\Gamma, \{\varphi^\bullet\}}{\Gamma, \Box \varphi^\bullet}$	$\Box^\circ \frac{\Delta, \varphi^\circ}{\Lambda, \{\Delta\}, \Box \varphi^\circ} \quad (\Delta \leq 1)$	$\Diamond^\bullet \frac{\varphi^\bullet, \Delta}{\Gamma, \Diamond \varphi^\bullet, \{\Delta\}} \quad (\Delta \leq 1)$	$\Diamond^\circ \frac{\Lambda, \{\varphi^\circ\}}{\Lambda, \Diamond \varphi^\circ}$

Figure 1: The rules of C_{IM} .

3.2 Structural properties and cut elimination

We prove standard structural properties of the calculus C_{IM} , most importantly admissibility of cut. As a consequence, we obtain that C_{IM} is complete with respect to IM and that IM is decidable. As usual, a rule is called *admissible* in C_{IM} if derivability of the premises implies derivability of the conclusion. A one-premiss rule is *height-preserving admissible* (hp-admissible) if derivability of the premiss via a derivation of height h implies derivability of the conclusion with a derivation of height $\leq h$. Finally, a rule $R = (\Gamma_1, \dots, \Gamma_n / \Gamma)$ is *height-preserving invertible* (hp-invertible) if the rule $R_{in} = (\Gamma / \Gamma_k)$ is hp-admissible for all $1 \leq k \leq n$.

Proposition 3.9. *The sequent $\Lambda, \varphi^\bullet, \varphi^\circ$ is derivable in C_{IM} for all Λ, φ .*

Proof. By induction on the complexity c of φ . If $c = 0$, then $\Lambda, \varphi^\bullet, \varphi^\circ$ is derived via id or $\perp^\bullet$. If $\varphi = \Box\psi$, then by the induction hypothesis, $(\psi^\bullet, \psi^\circ)$ is derivable in C_{IM} . By applying $\Box^\circ$ to this sequent we can then obtain $(\Lambda, \{\psi^\bullet\}, \Box\psi^\circ)$, and by $\Box^\bullet$, $(\Lambda, \Box\psi^\bullet, \Box\psi^\circ)$, which means that $\Lambda, \Box\psi^\bullet, \Box\psi^\circ$ is derivable as well. All other cases are similar. $\square$

Proposition 3.10. *The rules $\wedge^\bullet, \wedge^\circ, \vee^\bullet, \vee^\circ, \rightarrow^\circ, \Box^\bullet, \Diamond^\circ$ are height-preserving invertible. The rule $\rightarrow^\bullet$ is height-preserving invertible with respect to the right premiss.*

Proof. For all rules, the proof is by induction on the height h of the derivation of the conclusion. We show as an example the case of $\Box^\bullet$. Suppose $(\Gamma, \Box\varphi^\bullet)$ is derivable with height h . If $h = 0$, then $(\Gamma, \{\varphi^\bullet\})$ is also derivable with height 0. If $h > 0$, we have two cases. (1) $\Box\varphi^\bullet$ is not principal in the last rule application R in the derivation of $(\Gamma, \Box\varphi^\bullet)$. If $R = \Box^\circ$ or $R = \Diamond^\bullet$, we apply R to its premiss introducing $\{\varphi^\bullet\}$ instead of $\Box\varphi^\bullet$. Otherwise we apply the induction hypothesis to the premiss(es) of R , then we apply R , obtaining $(\Gamma, \{\varphi^\bullet\})$. (2) $\Box\varphi^\bullet$ is principal in R . Then $(\Gamma, \Box\varphi^\bullet)$ is obtained via $\Box^\bullet$ from $(\Gamma, \{\varphi^\bullet\})$, which means that $(\Gamma, \{\varphi^\bullet\})$ is derivable with height $h' < h$. $\square$

Proposition 3.11. *The following rules are height-preserving admissible in C_{IM} , where x is either a formula or a block:*

$$wk^\bullet \frac{\Gamma}{\Gamma, x^\bullet} \quad wk^\circ \frac{\Lambda}{\Lambda, x^\circ} \quad ctr^\bullet \frac{\Gamma, x^\bullet, x^\bullet}{\Gamma, x^\bullet} \quad elim \frac{\Gamma, \{\}, \mathcal{B}}{\Gamma, \mathcal{B}}$$

Proof. For all rules, we use induction on the height h of the derivation of the premiss.

Cases for $wk^\bullet$ and wk° . If $h = 0$, then $(\Gamma, x^\bullet)$ (resp. (Λ, x°)) is derivable with height 0. If $h > 0$, consider the last rule application R in the derivation of Γ (Λ). If $R = \Box^\circ$ or $R = \Diamond^\bullet$, we apply R to its premiss introducing also $x^\bullet$ (x°). Otherwise we apply $wk^\bullet$ (wk°) to the premiss(es) of R (which is hp-admissible by the induction hypothesis), then we apply R .

Case for $ctr^\bullet$. If $h = 0$, then $(\Gamma, x^\bullet)$ is derivable with height 0. If $h > 0$, we consider the last rule application R in the derivation of $(\Gamma, x^\bullet, x^\bullet)$. We distinguish two cases. (1) $x^\bullet$ is not principal in R . If $R = \Box^\circ$ or $R = \Diamond^\bullet$, we apply R to its premiss introducing one occurrence of $x^\bullet$ only. Otherwise we apply $ctr^\bullet$ to the premiss(es) of R (hp-admissible by the induction hypothesis), then we apply R . (2) $x^\bullet$ is principal in R . We consider the following three examples.

(1. $x^\bullet = \varphi \rightarrow \psi^\bullet$) We have $(\Gamma, \varphi \rightarrow \psi^\bullet, \varphi \rightarrow \psi^\bullet)$ obtained via $\rightarrow^\bullet$ from $(\Gamma^\downarrow, \varphi \rightarrow \psi^\bullet, \varphi \rightarrow \psi^\bullet, \psi^\circ)$ and $(\Gamma, \psi^\bullet, \varphi \rightarrow \psi^\bullet)$. On the first premiss, we apply $ctr^\bullet$ (hp-admissible by the induction hypothesis) and obtain $\Gamma_1 = (\Gamma^\downarrow, \varphi \rightarrow \psi^\bullet, \psi^\circ)$. On the second premiss, we first consider the hp-invertibility of $\rightarrow^\bullet$ with respect to this premiss (Proposition 3.10), obtaining $(\Gamma, \psi^\bullet, \psi^\circ)$, then we apply $ctr^\bullet$ (hp-admissible

by the induction hypothesis) and obtain $\Gamma_2 = (\Gamma, \psi^\bullet)$. With an application of $\rightarrow^\bullet$ on Γ_1 and Γ_2 we then obtain $(\Gamma, \varphi \rightarrow \psi^\bullet)$.

(2. $x^\bullet = \Box \varphi^\bullet$) We have $(\Gamma, \Box \varphi^\bullet, \Box \varphi^\bullet)$ obtained via $\Box^\bullet$ from $(\Gamma, \{\varphi^\bullet\}, \Box \varphi^\bullet)$. Considering the hp-invertibility of $\Box^\bullet$ (Proposition 3.10), from the premiss we obtain $(\Gamma, \{\varphi^\bullet\}, \{\varphi^\bullet\})$, then by $\text{ctr}^\bullet$ (hp-admissible by the induction hypothesis) we get $(\Gamma, \{\varphi^\bullet\})$, finally by $\Box^\bullet$ we have $(\Gamma, \Box \varphi^\bullet)$.

(3. $x^\bullet = \{\varphi^\bullet\}$) There are two cases to consider. (3.1) We have $(\Gamma', \{\varphi^\bullet\}, \{\varphi^\bullet\}, \Box \psi^\circ)$ obtained via $\Box^\circ$ from $(\varphi^\bullet, \psi^\circ)$. We can apply $\Box^\circ$ on this premiss without introducing the second occurrence of $\{\varphi^\bullet\}$, obtaining $(\Gamma', \{\varphi^\bullet\}, \Box \psi^\circ)$. (3.2) We have $(\Gamma', \{\varphi^\bullet\}, \{\varphi^\bullet\}, \Diamond \psi^\bullet)$ obtained via $\Diamond^\bullet$ from $(\varphi^\bullet, \psi^\bullet)$. Again, we can apply $\Diamond^\bullet$ on this premiss and obtain $(\Gamma', \{\varphi^\bullet\}, \Diamond \psi^\bullet)$.

Case for elim. If $h = 0$, then $(\Gamma, \mathcal{B})$ is derivable with height 0. If $h > 0$, we consider the last rule application R in the derivation of $(\Gamma, \{\}, \mathcal{B})$. We distinguish three cases.

(1) If $\{\}$ is principal in R , then $R = \Box^\circ$ or $R = \Diamond^\bullet$. If $R = \Box^\circ$, we have $(\Gamma', \{\}, \mathcal{B}, \Box \psi^\circ)$ obtained from (ψ°) , where $(\Gamma', \Box \psi^\circ) = \Gamma$ and $\mathcal{B} = \{\varphi^\bullet\}$ or $\mathcal{B} = \{\}$. If $\mathcal{B} = \{\varphi^\bullet\}$, then from (ψ°) , by $\text{wk}^\bullet$ we obtain $(\varphi^\bullet, \psi^\circ)$, and by $\Box^\circ$, $(\Gamma', \{\varphi^\bullet\}, \Box \psi^\circ)$. If $\mathcal{B} = \{\}$, then from (ψ°) , by $\Box^\circ$ we immediately obtain $(\Gamma', \{\}, \Box \psi^\circ)$. In either case, the consequence $(\Gamma, \mathcal{B})$ of elim is derivable with height h . The case $R = \Diamond^\bullet$ is similar.

(2) If $\mathcal{B}$ is principal in R , then again $R = \Box^\circ$ or $R = \Diamond^\bullet$. We obtain $\Gamma, \mathcal{B}$ by applying the same rule R but without introducing $\{\}$.

(3) If neither $\{\}$ nor $\mathcal{B}$ is principal in R , then in case $R = \Box^\circ$ or $R = \Diamond^\bullet$, we apply R without introducing $\{\}$. Otherwise we reverse the order of the applications of R and elim , applying elim on the premiss(es) of R (with elim hp-admissible by the induction hypothesis). For instance, suppose that $(\Gamma, \{\}, \mathcal{B})$ is obtained via $\rightarrow^\bullet$. Then $\Gamma = (\Gamma', \varphi \rightarrow \psi^\bullet)$ and $\mathcal{B} = \{\chi^\circ\}$, and the premisses of $\rightarrow^\bullet$ have the form $(\Gamma', \varphi \rightarrow \psi^\bullet, \{\}, \{\}, \varphi^\circ)$ and $(\Gamma', \psi^\bullet, \{\}, \{\chi^\circ\})$. By applying elim to each premiss we then obtain $(\Gamma', \varphi \rightarrow \psi^\bullet, \{\}, \varphi^\circ)$ and $(\Gamma', \psi^\bullet, \{\chi^\circ\})$, thus by $\rightarrow^\bullet$ we get $(\Gamma', \varphi \rightarrow \psi^\bullet, \{\chi^\circ\}) = (\Gamma, \mathcal{B})$. $\square$

Theorem 3.12 (Cut admissibility). *The following rules are admissible in $\mathbf{C}_{\text{IM}}$, where $|\Lambda| \leq 1$ in $\text{sub}^\bullet$, and $|\Delta| \leq 1$ in sub° :*

$$\text{cut} \frac{\Lambda, \varphi^\circ \quad \varphi^\bullet, \Delta}{\Lambda, \Delta} \quad \text{sub}^\bullet \frac{\Lambda, \varphi^\circ \quad \{\varphi^\bullet\}, \Delta}{\{\Lambda\}, \Delta} \quad \text{sub}^\circ \frac{\Lambda, \{\varphi^\circ\} \quad \varphi^\bullet, \Delta}{\Lambda, \{\Delta\}}$$

Proof. For an application of cut , we call *cut formula* the formula φ deleted by that application, while for $\text{sub}^\bullet$, sub° we call *subs-formula* the formula φ inside the block replaced by Λ , resp. Δ . Moreover, we call *cut height* the sum of the heights of the derivations of the premisses of cut . The theorem is a consequence of the following four claims, where $\text{Cut}(c, h)$ means that all applications of cut of height h with cut formula of complexity c are admissible, and $\text{Sub}(c)$ means that all applications of $\text{sub}^\bullet$ and sub° with subs-formula of complexity c are admissible:

- (A) $\forall c. \text{Cut}(c, 0)$. (B) $\forall h. \text{Cut}(0, h)$. (C) $\forall c. (\forall h. \text{Cut}(c, h) \rightarrow \text{Sub}(c))$.
(D) $\forall c. \forall h. (\forall c' < c. (\text{Sub}(c') \wedge \forall h'. \text{Cut}(c', h')) \wedge \forall h'' < h. \text{Cut}(c, h'') \rightarrow \text{Cut}(c, h))$.

(A) If $h = 0$, then the premisses of cut are instances of id or $\perp^\bullet$. In either case the consequence of cut is an instance of id or $\perp^\bullet$.

(B) If $c = 0$, then the cut formula is $\perp$ or a propositional variable p . The proof proceeds by complete induction on h . For the base case $h = 0$, then the claim follows from (A). For the inductive step, we distinguish three cases: (i) The cut formula is not principal in the last rule R applied in the derivation of the left premiss of cut . We consider the following three examples.

$$(R = \rightarrow^\bullet) \quad \rightarrow^\bullet \frac{\frac{\Lambda', \psi \rightarrow \chi^\bullet, \psi^\circ \quad \Lambda', \chi^\bullet, p^\circ}{\text{cut} \frac{\Lambda', \psi \rightarrow \chi^\bullet, p^\circ}{\Lambda', \psi \rightarrow \chi^\bullet, \Delta}} \quad p^\bullet, \Delta \quad (\text{where } \Lambda' = (\Lambda')^\downarrow)$$

The derivation is transformed as follows. From $(\Lambda', \psi \rightarrow \chi^\bullet, \psi^\circ)$, by some applications of $\text{wk}^\bullet$ we introduce $\Delta^\downarrow$, obtaining $\Gamma_1 = (\Lambda', \Delta^\downarrow, \psi \rightarrow \chi^\bullet, \psi^\circ)$. Moreover, from $(\Lambda', \chi^\bullet, p^\circ)$ and $(p^\bullet, \Delta)$, by cut we obtain $\Gamma_2 = (\Lambda', \chi^\bullet, \Delta)$. Finally, from Γ_1 and Γ_2 , since $\Lambda', \Delta^\downarrow = (\Lambda', \Delta)^\downarrow$, by $\rightarrow^\bullet$ we obtain $(\Lambda', \psi \rightarrow \chi^\bullet, \Delta)$.

$$(R = \Box^\bullet) \quad \Box^\bullet \frac{\frac{\Lambda', \{\psi^\bullet\}, p^\circ}{\text{cut} \frac{\Lambda', \Box \psi^\bullet, p^\circ}{\Lambda', \Box \psi^\bullet, \Delta}} \quad p^\bullet, \Delta$$

The derivation is transformed as follows. From $(\Lambda', \{\psi^\bullet\}, p^\circ)$ and $(p^\bullet, \Delta)$, by cut we obtain $(\Lambda', \{\psi^\bullet\}, \Delta)$, then by $\Box^\bullet$ we obtain $(\Lambda', \Box \psi^\bullet, \Delta)$.

$$(R = \Diamond^\bullet) \quad \Diamond^\bullet \frac{\frac{\psi^\bullet, \Sigma}{\text{cut} \frac{\Lambda', \Diamond \psi^\bullet, \{\Sigma\}, p^\circ}{\Lambda', \Diamond \psi^\bullet, \{\Sigma\}, \Delta}} \quad p^\bullet, \Delta$$

The derivation is transformed as follows. From $(\psi^\bullet, \Sigma)$, by $\Diamond^\bullet$ we obtain $(\Lambda', \Diamond \psi^\bullet, \{\Sigma\}, \Delta)$.

(ii) The cut formula is not principal in the last rule R applied in the derivation of the right premiss of cut. We consider the following examples.

$$(R = \rightarrow^\bullet) \quad \frac{\Lambda, p^\circ \quad \frac{p^\bullet, (\Delta')^\downarrow, \psi \rightarrow \chi^\bullet, \psi^\circ \quad p^\bullet, \Delta', \chi^\bullet}{\text{cut} \frac{p^\bullet, \Delta', \psi \rightarrow \chi^\bullet}{\Lambda, \Delta', \psi \rightarrow \chi^\bullet}} \rightarrow^\bullet$$

The derivation is transformed as follows. From (Λ, p°) and $(p^\bullet, (\Delta')^\downarrow, \psi \rightarrow \chi^\bullet, \psi^\circ)$, by cut we obtain $\Gamma_1 = (\Lambda, (\Delta')^\downarrow, \psi \rightarrow \chi^\bullet, \psi^\circ)$. Moreover, from (Λ, p°) and $(p^\bullet, \Delta', \chi^\bullet)$, by cut we obtain $\Gamma_2 = (\Lambda, \Delta', \chi^\bullet)$. Finally, from Γ_1 and Γ_2 , by $\rightarrow^\bullet$ we obtain $(\Lambda, \Delta', \psi \rightarrow \chi^\bullet)$.

$$(R = \Diamond^\circ) \quad \frac{\Lambda, p^\circ \quad \frac{p^\bullet, \Delta', \{\psi^\circ\}}{\text{cut} \frac{p^\bullet, \Delta', \Diamond \psi^\circ}{\Lambda, \Delta', \Diamond \psi^\circ}} \Diamond^\circ$$

The derivation is transformed as follows. From (Λ, p°) and $(p^\bullet, \Delta', \{\psi^\circ\})$, by cut we obtain $(\Lambda, \Delta', \{\psi^\circ\})$, then by $\Diamond^\circ$ we obtain $(\Lambda, \Delta', \Diamond \psi^\circ)$.

(iii) The cut formula is principal on both sides. Then both premisses of cut are obtained via id, hence $h = 0$ and we are back to the case (A).

(C) Assume $\forall h. \text{Cut}(c, h)$. We need to prove that all applications of $\text{sub}^\bullet$ and sub° where φ has complexity c are admissible. The proof is by induction on the height j of the derivation $\mathcal{D}$ of $(\{\varphi^\bullet\}, \Delta)$, for $\text{sub}^\bullet$, and of $(\Lambda, \{\varphi^\circ\})$, for sub° . If $j = 0$ or $\{\varphi^\bullet\}$ (resp. $\{\varphi^\circ\}$) is not principal in the last rule application R of $\mathcal{D}$, we proceed as in (A) or (B). If $j > 0$ and $\{\varphi^\bullet\}$ (resp. $\{\varphi^\circ\}$) is principal in R , then $R = \Box^\circ$ or $R = \Diamond^\bullet$.

Suppose $R = \Box^\circ$. Then, only the premiss $(\{\varphi^\bullet\}, \Delta)$ of $\text{sub}^\bullet$ is derivable via $R = \Box^\circ$ (the premiss $(\Lambda, \{\varphi^\circ\})$ of sub° is not). In this case, $(\{\varphi^\bullet\}, \Delta)$ has the form $(\{\varphi^\bullet\}, \Delta', \Box \chi^\circ)$, derived via $\Box^\circ$ from $(\varphi^\bullet, \chi^\circ)$. We can then obtain the consequence $(\{\Lambda\}, \Delta', \Box \chi^\circ)$ of $\text{sub}^\bullet$ as follows. If $\Lambda = \emptyset$, then the other premiss of $\text{sub}^\bullet$ is (φ°) . From (φ°) and $(\varphi^\bullet, \chi^\circ)$, by cut we obtain (χ°) , and by $\Box^\circ$ we obtain $(\{\}, \Delta', \Box \chi^\circ)$. If instead $\Lambda = \psi^\bullet$, then the other premiss of $\text{sub}^\bullet$ is $(\psi^\bullet, \varphi^\circ)$. From $(\psi^\bullet, \varphi^\circ)$ and $(\varphi^\bullet, \chi^\circ)$, by cut we obtain $(\psi^\bullet, \chi^\circ)$, and by $\Box^\circ$ we obtain $(\{\psi^\bullet\}, \Delta', \Box \chi^\circ)$. In both cases, the application of cut is admissible by the induction hypothesis.

Now suppose $R = \Diamond^\bullet$. Let us consider first the premiss $(\{\varphi^\bullet\}, \Delta)$ of $\text{sub}^\bullet$, that in this case has the form $(\{\varphi^\bullet\}, \Delta', \Diamond \chi^\bullet)$, derived via $\Diamond^\bullet$ from $(\varphi^\bullet, \chi^\bullet)$. We obtain the consequence $(\{\Lambda\}, \Delta', \Diamond \chi^\bullet)$ of $\text{sub}^\bullet$ as follows. If $\Lambda = \emptyset$, then from (φ°) and $(\varphi^\bullet, \chi^\bullet)$, by cut we obtain $(\chi^\bullet)$, then by $\Diamond^\bullet$ we obtain $(\{\}, \Delta', \Diamond \chi^\bullet)$. If $\Lambda = \psi^\bullet$, then from $(\psi^\bullet, \varphi^\circ)$ and $(\varphi^\bullet, \chi^\bullet)$, by cut we obtain $(\psi^\bullet, \chi^\bullet)$, then by $\Diamond^\bullet$ we obtain $(\{\psi^\bullet\}, \Delta', \Diamond \chi^\bullet)$. In both cases, the application of cut is admissible by the induction hypothesis.

Let us now consider the premiss $(\Lambda, \{\varphi^\circ\})$ of sub° , that in this case has the form $(\Lambda', \Diamond \chi^\bullet, \{\varphi^\circ\})$, derived via $\Diamond^\bullet$ from $(\chi^\bullet, \varphi^\circ)$. We obtain the consequence $(\Lambda', \Diamond \chi^\bullet, \{\Delta\})$ of sub° as follows. If $\Delta = \emptyset$, then from $(\chi^\bullet, \varphi^\circ)$ and $(\varphi^\bullet)$, by cut we obtain $(\chi^\bullet)$, then by $\Diamond^\bullet$ we obtain $(\Lambda', \Diamond \chi^\bullet, \{\})$. If $\Delta = \psi^\circ$, then from $(\chi^\bullet, \varphi^\circ)$ and $(\varphi^\bullet, \psi^\circ)$, by cut we obtain $(\chi^\bullet, \psi^\circ)$, then by $\Diamond^\bullet$ we obtain $(\Lambda', \Diamond \chi^\bullet, \{\psi^\circ\})$. In both cases, the application of cut is admissible by the induction hypothesis.

(D) Assume $\forall c' < c. (\text{Sub}(c') \wedge \forall h'. \text{Cut}(c', h')) \wedge \forall h'' < h. \text{Cut}(c, h'')$. We show that all applications of cut of height h on a cut formula of complexity c can be replaced by different applications of cut of smaller height or on a cut formula of smaller complexity. We can assume $h, c > 0$ as the cases $h = 0$ and $c = 0$ have been already considered in **(A)** and **(B)**. We distinguish two cases: (i) The cut formula is not principal in the last rule application in the derivation of at least one of the two premisses of cut. This case is analogous to (i) or (ii) in **(B)**. (ii) The cut formula φ is principal in the last rule application in the derivation of both premisses. We show the cases $\varphi = \psi \rightarrow \chi, \Box \psi, \Diamond \psi$. For $\varphi = \psi \rightarrow \chi$, we consider two subcases, depending on whether the output component is a formula or a block.

$$(\varphi = \psi \rightarrow \chi, \text{i}) \quad \rightarrow^\circ \frac{\frac{\Lambda, \psi^\bullet, \chi^\circ}{\Lambda, \psi \rightarrow \chi^\circ} \quad \frac{\psi \rightarrow \chi^\bullet, \Delta', \psi^\circ \quad \chi^\bullet, \Delta', \theta^\circ}{\psi \rightarrow \chi^\bullet, \Delta', \theta^\circ}}{\Lambda, \Delta', \theta^\circ} \text{cut} \rightarrow^\bullet (\Delta' = (\Delta')^\downarrow)$$

The derivation is transformed as follows. First, we cut $\psi \rightarrow \chi$ from $(\Lambda, \psi \rightarrow \chi^\circ)$ and $(\psi \rightarrow \chi^\bullet, \Delta' \psi^\circ)$, obtaining $(\Lambda, \Delta', \psi^\circ)$. From this and $(\Lambda, \psi^\bullet, \chi^\circ)$, we cut ψ and obtain $(\Lambda, \Lambda, \Delta', \chi^\circ)$. From this and $(\chi^\bullet, \Delta', \theta^\circ)$ we cut χ and obtain $(\Lambda, \Lambda, \Delta', \Delta', \theta^\circ)$. Finally, by some applications of $\text{ctr}^\bullet$ we obtain $(\Lambda, \Delta', \theta^\circ)$.

$$(\varphi = \psi \rightarrow \chi, \text{ii}) \quad \rightarrow^\circ \frac{\frac{\Lambda, \psi^\bullet, \chi^\circ}{\Lambda, \psi \rightarrow \chi^\circ} \quad \frac{\psi \rightarrow \chi^\bullet, \Delta', \{\}, \psi^\circ \quad \chi^\bullet, \Delta', \{\theta^\circ\}}{\psi \rightarrow \chi^\bullet, \Delta', \{\theta^\circ\}}}{\Lambda, \Delta', \{\theta^\circ\}} \text{cut} \rightarrow^\bullet$$

The derivation is transformed as follows. First, we cut $\psi \rightarrow \chi$ from $(\Lambda, \psi \rightarrow \chi^\circ)$ and $(\psi \rightarrow \chi^\bullet, \Delta', \{\}, \psi^\circ)$, obtaining $(\Lambda, \Delta', \{\}, \psi^\circ)$. From this and $(\Lambda, \psi^\bullet, \chi^\circ)$, we cut ψ and obtain $(\Lambda, \Lambda, \Delta', \{\}, \chi^\circ)$. From this and $(\chi^\bullet, \Delta', \{\theta^\circ\})$ we cut χ and obtain $(\Lambda, \Lambda, \Delta', \Delta', \{\}, \{\theta^\circ\})$. Finally, by one application of elim and some applications of $\text{ctr}^\bullet$, we obtain $(\Lambda, \Delta', \{\theta^\circ\})$.

$$(\varphi = \Box \psi) \quad \Box^\circ \frac{\frac{\Sigma, \psi^\circ}{\Lambda', \{\Sigma\}, \Box \psi^\circ} \quad \frac{\{\psi^\bullet\}, \Delta}{\Box \psi^\bullet, \Delta}}{\Lambda', \{\Sigma\}, \Delta} \text{cut} \Box^\bullet$$

The derivation is transformed as follows. From (Σ, ψ°) and $(\{\psi^\bullet\}, \Delta)$, by $\text{sub}^\bullet$ we obtain $(\{\Sigma\}, \Delta)$, then by wk^* we obtain $(\Lambda', \{\Sigma\}, \Delta)$.

$$(\varphi = \Diamond \psi) \quad \Diamond^\circ \frac{\frac{\Lambda, \{\psi^\circ\}}{\Lambda, \Diamond \psi^\circ} \quad \frac{\psi^\bullet, \Sigma}{\Diamond \psi^\bullet, \Delta', \{\Sigma\}}}{\Lambda, \Delta', \{\Sigma\}} \text{cut} \Diamond^\bullet$$

The derivation is transformed as follows. From $(\Lambda, \{\psi^\circ\})$ and $(\psi^\bullet, \Sigma)$, by sub° we obtain $(\Lambda, \{\Sigma\})$, then by wk^* we obtain $(\Lambda, \Delta', \{\Sigma\})$. $\square$

Theorem 3.13 (Completeness). *If $\vdash_{\text{IM}} \varphi$, then $\vdash_{\text{CIM}} \varphi^\circ$.*

Proof. This follows from induction on the height of the derivation of φ in IM, using the fact that the axioms of IM are derivable in C_{IM} , and the rules $\text{mon}_{\square}$, $\text{mon}_{\diamond}$ and modus ponens are simulated in C_{IM} via cut. For instance, from the premiss $(\varphi \rightarrow \psi^{\circ})$ and the derivable sequent $(\varphi \rightarrow \psi^{\bullet}, \varphi^{\bullet}, \psi^{\circ})$, cutting on $\varphi \rightarrow \psi$ gives $(\varphi^{\bullet}, \psi^{\circ})$. Applying $\diamond^{\bullet}$ and $\diamond^{\circ}$ yields $(\diamond\varphi^{\bullet}, \diamond\psi^{\circ})$, so that $\rightarrow^{\circ}$ entails $(\diamond\varphi \rightarrow \diamond\psi^{\circ})$. $\square$

As is often the case, one can infer important properties of a logic based on its analytic cut-free calculus. As a first application of the calculus, we prove here that IM is decidable.

Theorem 3.14 (Decidability). *For any $\varphi \in \mathcal{L}$, it is decidable whether or not $IM \vdash \varphi$.*

Proof. By adapting the decidability argument for G3ip in [56], we can prove that for any sequent Γ , it is decidable whether or not Γ is derivable in C_{IM} . To this end, consider that (1) for all rules R of C_{IM} , the premiss(es) of R only contain subformulas of formulas in the conclusion; (2) with the exception of $\rightarrow^{\bullet}$, the backward application of all rules of C_{IM} reduces the sequent complexity according to the following measure $C(\Gamma) = \sum_{x \in \Gamma} c'(x)$, where for $*$ $\in \{\bullet, \circ\}$ and $\star \in \{\wedge, \vee, \rightarrow\}$, $c'(\varphi^*) = c(\varphi)$ if $\varphi = p, \perp, \psi \star \chi$; $c'(\square\varphi^*) = c'(\diamond\varphi^*) = c'(\varphi^*) + 2$; $c'(\{\}) = 1$; $c'(\{\varphi^*\}) = c'(\varphi^*) + 1$. Moreover, for a sequent Γ , let us denote $Set(\Gamma)$ the support of Γ , that is, the set of formulas and blocks in Γ disregarding multiplicities. We can then consider bottom-up proof search in C_{IM} respecting the following *loop checking* condition LC: the backward application of $\rightarrow^{\bullet}$ is not allowed if a premiss Γ of $\rightarrow^{\bullet}$ is such that $Set(\Gamma) = Set(\Delta)$ for a sequent Δ already occurring at a lower depth in the same proof-search branch. Because of the subformula property, only a finite number of applications of $\rightarrow^{\bullet}$ are possible respecting LC. Therefore, bottom-up proof search eventually terminates on every Γ . As usual, the completeness of such a proof-search procedure is ensured by the admissibility of contraction in C_{IM} . $\square$

3.3 A calculus for WM

Interestingly, C_{IM} can be turned into a calculus for WM by changing from *output pruning* $\Gamma^{\downarrow}$ in $\rightarrow^{\bullet}$ to *output thinning*² $\Gamma^{\downarrow}$, which is defined by not only removing output formulas, but also output blocks. For instance, $(\varphi^{\bullet}, \{\psi^{\bullet}\}, \chi^{\circ})^{\downarrow} = (\varphi^{\bullet}, \{\psi^{\bullet}\}, \{\chi^{\circ}\})^{\downarrow} = \varphi^{\bullet}, \{\psi^{\bullet}\}$. The calculus C_{WM} is defined by the rules in Figure 1, where context pruning in $\rightarrow^{\bullet}$ is replaced with context thinning.

C_{WM} can be shown equivalent to WM by adjusting the structural analysis of C_{IM} in all cases where pruning comes into play. Note that empty blocks never occur in derivations of formulas, and could therefore be disallowed. If empty blocks are preserved, we observe that the rule elim is no longer admissible (one can verify that the soundness of elim depends on the presence of $\text{I}_{\diamond}$) but it is not needed for cut elimination. Alternatively, completeness of C_{WM} can be proved by translating derivations in the sequent calculus S.WM [11] into derivations in C_{WM} , with applications of the single $\square$ -rule (respectively $\diamond$ -rule) of S.WM corresponding to an application of $\square^{\circ}$ ($\diamond^{\bullet}$) immediately followed by an application of $\square^{\bullet}$ ($\diamond^{\circ}$).

It was shown in [27] that IM and WM share the same $\diamond$ -free fragment, but have different $\square$ -free fragments. We can re-prove the same results using the calculi C_{IM} and C_{WM} . First, note that for every $\diamond$ -free formula φ derivable in IM, a bottom-up derivation $\mathcal{D}$ in C_{IM} of φ° never introduces output blocks, which means that context pruning and thinning are indistinguishable in $\mathcal{D}$, and $\mathcal{D}$ can be therefore seen as a derivation in C_{WM} . On the other hand, using the bottom-up proof-search strategy for C_{IM} given in the proof of Theorem 3.14, which is equally applicable to C_{WM} , one can verify that the formula $((\diamond\perp \rightarrow \perp) \rightarrow \perp) \rightarrow \diamond\perp$ is not derivable in WM, as the proof-search deadlocks on a sequent of the form $((\diamond\perp \rightarrow \perp) \rightarrow \perp^{\bullet}, \diamond\perp^{\bullet}, \diamond\perp \rightarrow \perp^{\circ})$. Example 3.5 shows that this formula is derivable in IM.

²“A more drastic form of pruning, a thinning out cut, is the removal of an entire shoot, limb, or branch at its point of origin” (<https://en.wikipedia.org/wiki/Pruning>).

4 Extensions

We now present calculi for the axiomatic extensions of IM with axioms from $\{N, P, T, D, K\}$. The design of these follows *Dosen's principle* [57]: calculi for extensions are defined by modularly adding rules dealing with structural components, while the rules for logical connectives remain untouched. As for C_{IM} , these calculi can be seen as single-succedent restrictions of corresponding classical calculi from [14].

4.1 Calculi for extensions without K

Calculi for extensions of IM are characterised by the following rules:

$$n \frac{\Gamma, \{\}}{\Gamma} \quad p \frac{\Delta}{\Gamma, \{\Delta\}} \quad d \frac{\Delta, \Sigma}{\Gamma, \{\Delta\}, \{\Sigma\}} \quad t \frac{\Gamma, \{\Delta\}^\downarrow, \Delta}{\Gamma, \{\Delta\}} \quad (|\Delta| = |\Sigma| = 1)$$

Definition 4.1. For any logic $IM \oplus Ax$, with $Ax \subseteq \{N, P, D, T\}$, the calculus $C_{IM \oplus Ax}$ for $IM \oplus Ax$ is defined as the extension of C_{IM} with the corresponding rules among n, p, d, t above. If $D \in Ax$, the corresponding calculus additionally contains the rule p .

Example 4.2. Sequents corresponding to some of the axioms can be derived in $C_{IM \oplus Ax}$ as follows:

$$\begin{array}{ccccc} \frac{\top^\circ}{\{\}, \square \top^\circ} \square^\circ & \frac{\top^\circ}{\{\top^\circ\}} p & \frac{\varphi^\bullet, \varphi^\circ}{\{\varphi^\bullet\}, \{\varphi^\circ\}} d & \frac{\{\varphi^\bullet\}, \varphi^\circ}{\{\varphi^\bullet\}, \varphi^\circ} t & \frac{\varphi^\bullet, \{\}, \varphi^\circ}{\varphi^\bullet, \{\varphi^\circ\}} t \\ \frac{\{\}, \square \top^\circ}{\square \top^\circ} n & \frac{\{\top^\circ\}}{\diamond \top^\circ} \diamond^\circ & \frac{\{\varphi^\bullet\}, \{\varphi^\circ\}}{\{\varphi^\bullet\}, \diamond \varphi^\circ} \diamond^\circ & \frac{\{\varphi^\bullet\}, \varphi^\circ}{\square \varphi^\bullet, \varphi^\circ} \square^\bullet & \frac{\varphi^\bullet, \{\varphi^\circ\}}{\varphi^\bullet, \diamond \varphi^\circ} \diamond^\circ \\ & & \frac{\square \varphi^\bullet, \diamond \varphi^\circ}{\square \varphi \rightarrow \diamond \varphi^\circ} \rightarrow^\circ & \frac{\square \varphi^\bullet, \varphi^\circ}{\square \varphi \rightarrow \varphi^\circ} \rightarrow^\circ & \frac{\varphi^\bullet, \diamond \varphi^\circ}{\varphi \rightarrow \diamond \varphi^\circ} \rightarrow^\circ \end{array}$$

Remark 4.3.

1. The rule p is added to the calculus in case $D \in Ax$ to ensure the admissibility of contraction. This addition is sound since $\vdash_{IM \oplus D} P$.
2. Adding the rule n to either C_{IM} or C_{WM} generates the same calculus. Indeed, for every output block deleted via output thinning in a backward application of $\rightarrow^\bullet$, a corresponding empty block can be reintroduced using n . This matches the axiomatic equivalence between $IM \oplus N$ and $WM \oplus N$.

We can prove their soundness and completeness with respect to the corresponding logics $IM \oplus Ax$ by extending the proofs for C_{IM} from Section 3 with the additional cases determined by the new rules.

Theorem 4.4 (Soundness). *If $\vdash_{C_{IM \oplus Ax}} \Gamma$ then $\vdash_{IM \oplus Ax} i(\Gamma)$.*

Proof. For any pair $(X, x) \in \{(N, n), (P, p), (D, d), (T, t)\}$, we show that the rule x is sound with respect to $IM \oplus X$. We let $\Gamma = \Lambda, \Pi$, and shorten $(\bigwedge_{x \in \Lambda} i(x)) = \lambda$ and $i(\Pi) = \pi$.

Case for rule n. Suppose $\vdash_{IMN} i(\Gamma, \{\}) = \lambda \wedge \square \top \rightarrow \pi$. Then by axiom N, $\vdash_{IMN} \lambda \rightarrow \pi = i(\Gamma)$.

Case for rule p. If $\Delta = \varphi^\bullet$, then from $\vdash_{IMP} i(\varphi^\bullet) = \varphi \rightarrow \perp$, by $\text{mon}_\square$ we get $\vdash_{IMP} \square \varphi \rightarrow \square \perp$, then by $P_\square$, $\vdash_{IMP} \square \varphi \rightarrow \perp$, hence $\vdash_{IMN} \lambda \wedge \square \varphi \rightarrow \pi = i(\Gamma, \{\varphi^\bullet\})$. If $\Delta = \varphi^\circ$, then $\Gamma = \Lambda$, moreover from $\vdash_{IMP} i(\varphi^\circ) = \varphi$, we get $\vdash_{IMP} i(\varphi^\circ) = \top \rightarrow \varphi$, hence by $\text{mon}_\diamond$, $\vdash_{IMP} i(\varphi^\circ) = \diamond \top \rightarrow \diamond \varphi$. Thus, being $\lambda \rightarrow \diamond \top$ derivable from $P_\diamond$, we get $\vdash_{IMP} \lambda \rightarrow \diamond \varphi = i(\Lambda, \{\varphi^\circ\})$.

Case for rule d. If $\Delta = \varphi^\bullet$ and $\Sigma = \psi^\bullet$, then from $\vdash_{IMD} i(\varphi^\bullet, \psi^\bullet) = \varphi \wedge \psi \rightarrow \perp$, by the rule RD (Lemma 2.2), we get $\vdash_{IMD} \square \varphi \wedge \square \psi \rightarrow \perp$, hence $\vdash_{IMD} \lambda \wedge \square \varphi \wedge \square \psi \rightarrow \pi = i(\Gamma, \{\varphi^\bullet\}, \{\psi^\bullet\})$. If (w.l.o.g.) $\Delta = \varphi^\bullet$ and $\Sigma = \psi^\circ$, then $\Gamma = \Lambda$, moreover from $\vdash_{IMD} i(\varphi^\bullet, \psi^\circ) = \varphi \rightarrow \psi$, by $\text{mon}_\diamond$ we get $\vdash_{IMD} \diamond \varphi \rightarrow \diamond \psi$, then by the axiom D, $\vdash_{IMD} \square \varphi \rightarrow \diamond \psi$, therefore $\vdash_{IMD} \lambda \wedge \square \varphi \rightarrow \diamond \psi = i(\Lambda, \{\varphi^\bullet\}, \{\psi^\circ\})$.

Case for rule t. If $\Delta = \varphi^\bullet$, then from $\vdash_{\text{IMT}} i(\Lambda, \{\varphi^\bullet\}, \varphi^\bullet, \Pi) = \lambda \wedge \Box \varphi \wedge \varphi \rightarrow \pi$, by axiom $T_\Box$ we get $\vdash_{\text{IMT}} \lambda \wedge \Box \varphi \rightarrow \pi = i(\Lambda, \{\varphi^\bullet\}, \Pi)$. If $\Delta = \varphi^\circ$, then $\Gamma = \Lambda$, moreover from $\vdash_{\text{IMT}} i(\Lambda, \{\ }, \varphi^\circ) = \lambda \wedge \Box \top \rightarrow \varphi$, by axiom $T_\Diamond$ we get $\vdash_{\text{IMT}} \lambda \wedge \Box \top \rightarrow \Diamond \varphi$, thus $\vdash_{\text{IMT}} \lambda \rightarrow (\Box \top \rightarrow \Diamond \varphi)$, hence by $I_\Diamond$, $\vdash_{\text{IMT}} \lambda \rightarrow \Diamond \varphi = i(\Lambda, \{\varphi^\circ\})$. $\square$

Proposition 4.5. *The rules $\text{wk}^\bullet$, wk° , $\text{ctr}^\bullet$ and elim are height-preserving admissible in $\text{C}_{\text{IM} \oplus \text{Ax}}$.*

Proof. By extending the proof of Proposition 3.11 with the additional cases determined by the new rules. We show only a few significant cases. For $\text{ctr}^\bullet$, if its premiss $(\Gamma, \{\varphi^\bullet\}, \{\varphi^\bullet\})$ is derived via d from $(\varphi^\bullet, \varphi^\bullet)$, then by a hp-application of $\text{ctr}^\bullet$ to the premiss of d we obtain $(\varphi^\bullet)$, then by p we have $(\Gamma, \{\varphi^\bullet\})$. For elim , note that $\{\ }$ is never principal in (bottom-up) applications of the rules n, p, d, t . If the premiss $(\Gamma, \{\ }, \mathcal{B})$ of elim is obtained via n from $(\Gamma, \{\ }, \{\ }, \mathcal{B})$, then by two hp-applications of elim to this sequent we obtain $(\Gamma, \mathcal{B})$. If the premiss $(\Lambda, \{\ }, \{\varphi^\circ\})$ of elim is obtained via t from $(\Lambda, \{\ }, \{\ }, \varphi^\circ)$, then by a hp-application of elim to this sequent we obtain $(\Lambda, \{\ }, \varphi^\circ)$, and by t we obtain $(\Lambda, \{\varphi^\circ\})$. $\square$

Theorem 4.6. *The rules cut , $\text{sub}^\bullet$ and sub° are admissible in $\text{C}_{\text{IM} \oplus \text{Ax}}$.*

Proof. We extend the proofs of the claims (A), (B), (C), (D) in the proof of Theorem 3.12. (A) is as before. (B, D) Suppose the last rule R applied in the derivation of a premiss of cut is among n, p, d, t . Then the cut formula is not principal in R . If $R \in \{n, t\}$, we first apply cut on the premiss of R and the other premiss of cut , then we apply R . If $R \in \{p, d\}$, we obtain the conclusion of cut by considering a different application of R . (C) Suppose the premiss $(\{\varphi^\bullet\}, \Delta)$ of $\text{sub}^\bullet$, resp. the premiss (Λ, φ°) of sub° , is obtained by $R \in \{n, p, d, t\}$. If the block $\{\varphi^\bullet\}$, resp. φ° , is not principal in R , then the analysis is essentially as before. Otherwise, we consider the following representative cases.

(i) Assume the premiss $(\{\varphi^\bullet\}, \Delta)$ of $\text{sub}^\bullet$ is derived via p from $(\varphi^\bullet)$. Then Λ is not empty in the other premiss (Λ, φ°) of $\text{sub}^\bullet$, otherwise both $(\varphi^\bullet)$ and (φ°) would be derivable, against the soundness of the calculi. Let $\Lambda = \psi^\bullet$. By cutting on $(\psi^\bullet, \varphi^\circ)$ and $(\varphi^\bullet)$, we obtain $(\psi^\bullet)$, then by p we get $(\{\psi^\bullet\}, \Delta)$.

(ii) Assume the premiss $(\Lambda', \{\psi^\bullet\}, \{\varphi^\circ\})$ of sub° is derived via d from $(\psi^\bullet, \varphi^\circ)$. Then by cutting on $(\psi^\bullet, \varphi^\circ)$ and the other premiss of sub° $(\varphi^\bullet, \Delta)$, we obtain $(\psi^\bullet, \Delta)$, then by d we get $(\Lambda', \{\psi^\bullet\}, \{\Delta\})$.

(iii) Assume the premiss $(\Lambda, \{\varphi^\circ\})$ of sub° is derived via t from $(\Lambda, \{\ }, \varphi^\circ)$. Consider the other premiss of sub° $(\varphi^\bullet, \Delta)$. If $\Delta = \psi^\circ$, then by cutting on $(\Lambda, \{\ }, \varphi^\circ)$ and $(\varphi^\bullet, \psi^\circ)$ we obtain $(\Lambda, \{\ }, \psi^\circ)$, then by t we get $(\Lambda, \{\psi^\circ\})$. If $\Delta = \psi^\bullet$, then by cutting on $(\Lambda, \{\ }, \varphi^\circ)$ and $(\varphi^\bullet, \psi^\bullet)$ we obtain $(\Lambda, \{\ }, \psi^\bullet)$, then by $\text{wk}^\bullet$, $(\Lambda, \{\ }, \{\psi^\bullet\}, \psi^\bullet)$, by elim , $(\Lambda, \{\psi^\bullet\}, \psi^\bullet)$, finally by t , $(\Lambda, \{\psi^\bullet\})$. Finally, if $\Delta = \emptyset$, by cutting on $(\Lambda, \{\ }, \varphi^\circ)$ and $(\varphi^\bullet)$ we obtain $(\Lambda, \{\ })$. $\square$

Theorem 4.7 (Completeness). *If $\vdash_{\text{IM} \oplus \text{Ax}} \varphi$, then $\vdash_{\text{C}_{\text{IM} \oplus \text{Ax}}} \varphi^\circ$.*

4.2 Calculi for extensions with K

So far, blocks contained at most one formula. While not strictly necessary, it allowed us to simplify the calculus. To formulate calculi for logics with K, we need to adopt a more general formulation of the calculi where blocks contain finite *multisets* of marked formulas. We sketch the required modifications.

We now call *input blocks* those containing only input formulas (or no formula at all), and *output blocks* those containing an output formula (possibly together with some input formulas). As before, sequents are single-succedent, meaning that their only output component is either an output formula, or an output block, or is not present. We extend the formula interpretation from Definition 3.3 via:

$$\iota(\{\ }) = \Box \top, \quad \iota(\{\varphi_1^\bullet, \dots, \varphi_k^\bullet\}) = \Box(\varphi_1 \wedge \dots \wedge \varphi_k), \quad \iota(\{\varphi_1^\bullet, \dots, \varphi_k^\bullet, \psi^\circ\}) = \Diamond(\varphi_1 \wedge \dots \wedge \varphi_k \rightarrow \psi)$$

For instance, $i(\{\varphi^\bullet, \psi^\bullet\}, \{\chi^\bullet, \theta^\circ\}) = \Box(\varphi \wedge \psi) \rightarrow \Diamond(\chi \rightarrow \theta)$.

Furthermore, in this setting the output pruning $\Gamma^\downarrow$ deletes output formulas that do not occur inside a block, and empties the *entire* block if an output formula occurs inside it. So for instance we have $(\varphi^\bullet, \{\psi^\bullet\}, \chi^\bullet, \theta^\circ)^\downarrow = \varphi^\bullet, \{\psi^\bullet\}, \chi^\bullet$ and $(\varphi^\bullet, \{\psi^\bullet\}, \{\chi^\bullet, \theta^\circ\})^\downarrow = \varphi^\bullet, \{\psi^\bullet\}, \{\}$.

Definition 4.8. The calculus $C_{IM \oplus K}$ for $IM \oplus K$ is defined by the rules in Figure 1, where the restriction $|\Delta| \leq 1$ for the rules $\Box^\circ$ and $\Diamond^\bullet$ is dropped, together with the following rule:

$$k \frac{\Gamma, \{\Delta, \Sigma\}}{\Gamma, \{\Delta\}, \{\Sigma\}}$$

Example 4.9. The axioms $K_\Box$ and $K_\Diamond$ can be derived in $C_{IM \oplus K}$ as follows:

$$\begin{array}{c} \frac{\varphi \rightarrow \psi^\bullet, \varphi^\bullet, \varphi^\circ \quad \psi^\bullet, \varphi^\bullet, \psi^\circ}{\varphi \rightarrow \psi^\bullet, \varphi^\bullet, \psi^\circ} \rightarrow^\bullet \\ \frac{\varphi \rightarrow \psi^\bullet, \varphi^\bullet, \psi^\circ}{\{\varphi \rightarrow \psi^\bullet, \varphi^\bullet\}, \Box \psi^\circ} \Box^\circ \\ \frac{\{\varphi \rightarrow \psi^\bullet, \varphi^\bullet\}, \Box \psi^\circ}{\{\varphi \rightarrow \psi^\bullet\}, \{\varphi^\bullet\}, \Box \psi^\circ} k \\ \frac{\{\varphi \rightarrow \psi^\bullet\}, \{\varphi^\bullet\}, \Box \psi^\circ}{\{\varphi \rightarrow \psi^\bullet\}, \Box \varphi^\bullet, \Box \psi^\circ} \Box^\bullet \\ \frac{\{\varphi \rightarrow \psi^\bullet\}, \Box \varphi^\bullet, \Box \psi^\circ}{\Box(\varphi \rightarrow \psi)^\bullet, \Box \varphi^\bullet, \Box \psi^\circ} \Box^\bullet \\ \frac{\Box(\varphi \rightarrow \psi)^\bullet, \Box \varphi^\bullet, \Box \psi^\circ}{\Box(\varphi \rightarrow \psi)^\bullet, \Box \varphi \rightarrow \Box \psi^\circ} \rightarrow^\circ \\ \frac{\Box(\varphi \rightarrow \psi)^\bullet, \Box \varphi \rightarrow \Box \psi^\circ}{\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)^\circ} \rightarrow^\circ \end{array} \quad \begin{array}{c} \frac{\varphi^\bullet, \varphi \rightarrow \psi^\bullet, \varphi^\circ \quad \varphi^\bullet, \psi^\bullet, \psi^\circ}{\varphi^\bullet, \varphi \rightarrow \psi^\bullet, \psi^\circ} \rightarrow^\bullet \\ \frac{\varphi^\bullet, \varphi \rightarrow \psi^\bullet, \psi^\circ}{\Diamond \varphi^\bullet, \{\varphi \rightarrow \psi^\bullet, \psi^\circ\}} \Diamond^\bullet \\ \frac{\Diamond \varphi^\bullet, \{\varphi \rightarrow \psi^\bullet, \psi^\circ\}}{\{\varphi \rightarrow \psi^\bullet\}, \Diamond \varphi^\bullet, \{\psi^\circ\}} k \\ \frac{\{\varphi \rightarrow \psi^\bullet\}, \Diamond \varphi^\bullet, \{\psi^\circ\}}{\{\varphi \rightarrow \psi^\bullet\}, \Diamond \varphi^\bullet, \Diamond \psi^\circ} \Diamond^\circ \\ \frac{\{\varphi \rightarrow \psi^\bullet\}, \Diamond \varphi^\bullet, \Diamond \psi^\circ}{\Box(\varphi \rightarrow \psi)^\bullet, \Diamond \varphi^\bullet, \Diamond \psi^\circ} \Box^\bullet \\ \frac{\Box(\varphi \rightarrow \psi)^\bullet, \Diamond \varphi^\bullet, \Diamond \psi^\circ}{\Box(\varphi \rightarrow \psi)^\bullet, \Diamond \varphi \rightarrow \Diamond \psi^\circ} \rightarrow^\circ \\ \frac{\Box(\varphi \rightarrow \psi)^\bullet, \Diamond \varphi \rightarrow \Diamond \psi^\circ}{\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi)^\circ} \rightarrow^\circ \end{array}$$

Proposition 4.10. The rule k is sound with respect to $IM \oplus K$.

Proof. If both $\{\Delta\}$ and $\{\Sigma\}$ are input blocks, then soundness is immediate using $C_\Box$ (Lemma 2.2). Suppose $\Sigma = (\Theta, \varphi^\circ)$. Then $\Gamma = \Lambda$. Let us shorten $(\bigwedge_{x \in \Lambda} \iota(x)) = \lambda$; $(\bigwedge_{y \in \Delta} \iota(y)) = \delta$; and $(\bigwedge_{z \in \Theta} \iota(z)) = \theta$. We assume $\vdash_{IM \oplus K} i(\Lambda, \{\Delta, \Sigma\}) = \lambda \rightarrow \Diamond(\delta \wedge \theta \rightarrow \varphi)$. By the formula W (Lemma 2.2), we have $\vdash_{IM \oplus K} \Box \delta \wedge \Diamond(\delta \wedge \theta \rightarrow \varphi) \rightarrow \Diamond(\delta \wedge (\delta \wedge \theta \rightarrow \varphi))$. Moreover, by $\text{mon}_\Diamond$, $\vdash_{IM \oplus K} \Diamond(\delta \wedge (\delta \wedge \theta \rightarrow \varphi)) \rightarrow \Diamond(\theta \rightarrow \varphi)$. Combining everything, we get $\vdash_{IM \oplus K} \lambda \wedge \Box \delta \rightarrow \Diamond(\theta \rightarrow \varphi) = i(\Lambda, \{\Delta\}, \{\Sigma\})$. $\square$

The generalisation of the formalism to multisets within blocks slightly complicates the structural analysis of the calculus (which motivates keeping it simple when possible). Apart from adjusting all cases from the previous proofs with blocks possibly containing more than one formula (which is in most cases simply a routine task that does not affect them significantly), the main changes to the previous analysis are the following: (1) The rule $\text{ctr}^{\{\bullet\}}$ below is hp-admissible in $C_{IM \oplus K}$; in particular, the rules $\text{ctr}^\bullet$ and $\text{ctr}^{\{\bullet\}}$ are simultaneously proved admissible by mutual induction on the height of the derivation of their premiss. (2) The rules $\text{sub}^\bullet$ and sub° now take the form below.

$$\text{ctr}^{\{\bullet\}} \frac{\Gamma, \{\Delta, \varphi^\bullet, \varphi^\bullet\}}{\Gamma, \{\Delta, \varphi^\bullet\}} \quad \text{sub}^\bullet \frac{\Lambda, \varphi^\circ \quad \{\varphi^\bullet, \Sigma\}, \Delta}{\{\Lambda, \Sigma\}, \Delta} \quad \text{sub}^\circ \frac{\Lambda, \{\Sigma, \varphi^\circ\} \quad \varphi^\bullet, \Delta}{\Lambda, \{\Sigma, \Delta\}}$$

Proposition 4.11. The rule k is height-preserving invertible in $C_{IM \oplus K}$.

Proof. By induction on the height h of the derivation of the conclusion $C = (\Gamma, \{\Delta\}, \{\Sigma\})$ of k . As an example, suppose that $h > 0$ and consider the last rule R applied in the derivation of C . If $R = k$ with $\{\Delta\}, \{\Sigma\}$ principal, then the claim trivially follows. If $R = k$ with only $\{\Delta\}$ principal, then $C = (\Gamma', \{\Delta\}, \{\Sigma\}, \{\Theta\})$ is obtained from $(\Gamma', \{\Delta, \Theta\}, \{\Sigma\})$. Applying the induction hypothesis on this sequent we get $(\Gamma', \{\Delta, \Theta, \Sigma\})$, then by k , $(\Gamma', \{\Delta, \Sigma\}, \{\Theta\})$. If $R \neq k$, we apply the induction hypothesis on the premiss of R , then we apply R . $\square$

Proposition 4.12. *The rules $wk^\bullet$, wk° , $ctr^\bullet$, $ctr^{\{\bullet\}}$, $elim$ are height-preserving admissible in $C_{IM \oplus K}$.*

Proof. Let us just consider a few most significant cases in the proof of hp-admissibility of $ctr^\bullet$ and $ctr^{\{\bullet\}}$. Suppose the premiss of $ctr^\bullet$, resp. $ctr^{\{\bullet\}}$, is derivable with height h , and that $ctr^\bullet$ and $ctr^{\{\bullet\}}$ are hp-admissible up to height $h - 1$. If the premiss $(\Gamma, \{\Delta\}, \{\Delta\})$ of $ctr^\bullet$ is obtained by k from $(\Gamma, \{\Delta, \Delta\})$, then by $|\Delta|$ hp-applications of $ctr^{\{\bullet\}}$ (induction hypothesis) we obtain $(\Gamma, \{\Delta\})$. If the premiss $(\Gamma, \{\Delta\}, \{\Delta\}, \{\Sigma\})$ of $ctr^\bullet$ is obtained by k from $(\Gamma, \{\Delta, \Sigma\}, \{\Delta\})$, then by hp-invertibility of k we have $(\Gamma, \{\Delta, \Sigma, \Delta\})$, and by $|\Delta|$ hp-applications of $ctr^{\{\bullet\}}$ (induction hypothesis), $(\Gamma, \{\Delta, \Sigma\})$, finally by k , $(\Gamma, \{\Delta\}, \{\Sigma\})$. If the premiss $(\Delta, \{\Delta, \varphi^\bullet, \varphi^\bullet\}, \Box \psi^\circ)$ of $ctr^{\{\bullet\}}$ is obtained by $\Box^\circ$ from $(\Delta, \varphi^\bullet, \varphi^\bullet, \psi^\circ)$, then by a hp-application of $ctr^\bullet$ (induction hypothesis) we obtain $(\Delta, \varphi^\bullet, \psi^\circ)$, and by $\Box^\circ$ we get $(\Delta, \{\Delta, \varphi^\bullet\}, \Box \psi^\circ)$. $\square$

Theorem 4.13. *The rules cut , $sub^\bullet$, sub° are admissible in $C_{IM \oplus K}$.*

Proof. Despite the different formulation of $sub^\bullet$ and sub° , the proof resembles that of Theorem 3.12. The only significant changes concern the proof of claim (C): Assume $\forall h. Cut(c, h)$. As before, admissibility of $sub^\bullet$ is proved by induction on the height j of the derivation of $(\{\varphi^\bullet, \Sigma\}, \Delta)$. Assume $j > 0$ and $\{\varphi^\bullet, \Sigma\}$ principal in the last rule R applied in the derivation of $(\{\varphi^\bullet, \Sigma\}, \Delta)$. If $R = \Box^\circ$, then we have $(\{\varphi^\bullet, \Sigma\}, \Delta', \Box \psi^\circ)$ obtained from $(\varphi^\bullet, \Sigma, \psi^\circ)$. By cutting $\varphi^\bullet, \varphi^\circ$ from this sequent and the other premiss (Δ, φ°) of $sub^\bullet$, we obtain $(\Delta, \Sigma, \psi^\circ)$, then by $\Box^\circ$, $(\Delta, \Sigma, \Delta', \Box \psi^\circ)$. If $R = \Diamond^\bullet$, the case is similar. If $R = k$, then we have $(\{\varphi^\bullet, \Sigma\}, \Delta', \{\Theta\})$ obtained from $(\{\varphi^\bullet, \Sigma, \Theta\}, \Delta')$. Then by applying $sub^\bullet$ to this sequent and the other premiss (Δ, φ°) of $sub^\bullet$ (which is admissible by the induction hypothesis), we get $(\Delta, \Sigma, \Theta, \Delta')$, then by k , $(\Delta, \Sigma, \Delta', \{\Theta\})$. The admissibility of sub° is proved similarly. $\square$

Finally, as a consequence of Proposition 4.10, Theorem 4.13 and derivability of $K_\Box, K_\Diamond$ in $C_{IM \oplus K}$:

Theorem 4.14 (Soundness and completeness). $\vdash_{IM \oplus K} \varphi$ if and only if $\vdash_{C_{IM \oplus K}} \varphi^\circ$.

5 Conclusion

We studied the intuitionistic monotone modal logic IM and several extensions, providing a semantics by means of constructive neighbourhood frames as well as cut-free proof systems. The proof-theoretical analysis reveals the following analogy between constructive and intuitionistic versions of M and K:

Constructive modal logics are single-succedent restrictions of Gentzen calculi;

Intuitionistic modal logics are single-succedent restrictions of structured sequent calculi.

This immediately raises the question whether this pattern restricts to constructive and intuitionistic versions of M and K, or if it applies more generally.

Furthermore, Arisaka, Das and Strassburger [3] provide a nested calculus for CK which requires a more radical form of pruning than that employed for IK. Similarly, we turned the calculus for IM into one for WM by replacing context pruning with context thinning. Again, it would be interesting to investigate if this is merely a coincidence or part of a pattern.

Other interesting questions include: Does the semantic reading from Remark 3.6 allow one to extract countermodels for failed derivations in C_{IM} ? What other variants of context thinning and pruning can be defined, and what logics do they give rise to? What kind of intuitionistic variants of M arise from the single-succedent restriction of other structured calculi, such as the nested calculus from [37]? We hope to address some of these in future research.

Finally, we would like to study the complexity of the derivability problem in IM and its extensions by either extending an optimal calculus for IPL like the one from [34], or considering reductions to classical multi-modal logics similarly to [12, 19], [22, Chapter 10.2].

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A General Theory of Propositional Modal Bundled Modalities

Yifeng Ding

Yuanzhe Yang

Department of Philosophy
Peking University*

yf.ding@pku.edu.cn

2301210999@pku.edu.cn

In studies of bundled modalities, we encode a complex conceptual notion into the semantics of a single modal operator and study its logic. Although there is already a substantial body of work on various concrete bundled operators, we still lack a general understanding of them. In this paper, we provide a general theory of the expressivity and axiomatization of bundled modalities. We offer a uniform way to define bisimulations for arbitrary bundled modalities and justify our definition by the corresponding Hennessy-Milner property. We also define a special class of bundled modalities called positive-negative-independent bundles. This class of bundles, together with their duals, cover most bundled modalities studied in the literature, and their axiomatizations can be done with the help of a more abstract notion of convex neighborhood semantics and corresponding representation results. As case studies, we axiomatize the “someone knows” bundle $\bigvee_{a \in A} \Box_a \phi$ over *S5*-models, the “disagreement within group” bundle $\bigvee_{a, b \in A} \Box_a \phi \wedge \Box_b \neg \phi$ over *KD45*-models, and the “belief without knowledge” bundle $B\phi \wedge \neg K\phi$ over *S4.2*-models.

1 Introduction

The idea of a *bundled modality* is to encode a complex conceptual notion into the semantics of a single modal operator and study the logic that takes this operator as the primitive modality. The logics of bundled modalities, which have more complicated semantics yet weaker expressive power compared to full modal logics, often have interesting behaviors on both conceptual and technical levels.

Research on “bundled modalities” started long before the name first appeared in the literature. For example, the *non-contingency modality* $\Delta\phi := \Box\phi \vee \Box\neg\phi$ received attention since 1966 [30, 24, 25, 41], and the bundle $\circ\phi := \phi \rightarrow \Box\phi$ was studied under different names since 2005 [29, 35, 34]. The terminology “bundle” was first introduced in recent studies of first-order modal logic [38, 28, 27, 37]; during the same period, not only did studies of the aforementioned modalities revive [13, 14, 6, 8, 17, 18, 19], but there also emerged studies of many other bundled modalities, e.g. false belief $\Box\phi \wedge \neg\phi$ [36, 18, 8, 39, 10], strong disagreement between two agents $(\Box_a\phi \wedge \Box_b\neg\phi) \vee (\Box_a\neg\phi \wedge \Box_b\phi)$ [2], someone knows $\bigvee_{a \in A} \Box_a\phi$ [42, 4], and secret knowledge $\Box_a\phi \wedge \bigwedge_{b \in A \setminus \{a\}} \Box_a \neg \Box_b\phi$ [40], just to name a few (more examples can be found in Example 4.3).

Given the multitudinous papers on concrete bundled modalities, one naturally wants to find a general theory subsuming them. There are indeed a few attempts towards such a theory in the literature, but none of them is completely satisfactory. For example, [1] introduced a method to automatically define bisimulations for bundled modalities (which are called *molecular connectives* there), but such a method applies only to a restricted class of modalities called *uniform connectives*; [12] introduced a uniform approach to axiomatizing logics of bundled modalities called *the method based on almost-definability schemas*, but no uniform way to find such schemas for an arbitrary bundled modality is provided, and

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it is also unclear to which kinds of bundled modalities this method applies. As a result, in studies of concrete bundled modalities, bisimulations are often left as open problems, while radically different notions of canonical models are introduced to prove completeness theorems, even though the logics in question often have a lot in common.

Our goal in this paper, then, is to provide a general theory for both expressivity and axiomatization of bundled modalities. For expressivity, we view the semantics of bundled modalities as a special kind of *neighborhood semantics* and uniformly define bisimulations for *all* bundled modalities. As for axiomatizations, we focus on a particularly well-behaved class of bundled modalities called *positive-negative-independent bundles* (*PN-independent bundles* for short). This class of bundles satisfy a natural convexity axiom and, together with their duals, cover most bundled modalities studied in the literature and yet a lot more. We show that the fairly standard two-step strategy of proving completeness theorems, namely canonical model building followed by transforming to a semantically equivalent intended model, applies with a high degree of uniformity. Specifically, canonical models can be constructed uniformly with a rich supply of canonical axioms, and model transformation does not require much action since, from our analysis of bundled semantics, we know exactly what it takes to preserve truth values.

The structure of the paper is as follows. In Section 2, we introduce the general language and semantics for bundled modalities. Then, in Section 3, we show how to uniformly define bisimulations for bundled modalities and prove a Hennessy-Milner theorem for the resulting bisimulations. In Section 4, we introduce the notion of PN-independent bundles, characterize their expressivity, and also introduce a novel notion of *convex neighborhood logic*,¹ which lies between minimal neighborhood logic and monotonic neighborhood logic, and abstracts the core feature shared by all PN-independent bundles. Then, in Section 5, to illustrate our recipe for axiomatizations, we offer complete axiomatizations for three concrete bundled modalities: the “someone knows” bundle $\bigvee_{a \in A} \Box_a \phi$ [42, 4] over *S5*-models, the “disagreement within group” bundle $\bigvee_{a, b \in A} \Box_a \phi \wedge \Box_b \neg \phi$ (which generalizes the two-agent case [2]) over *KD45*-models, and the “belief without knowledge” bundle $B\phi \wedge \neg K\phi$ over *S4.2*-models (and also *S4F*-models). To our knowledge, none of them is available in the literature. Finally, in Section 6, we summarize our results and suggest some directions for future work.

2 Language and Semantics

We now define the language and semantics for bundled modalities in general. For simplicity, we focus on cases where there is only one unary bundled modality; generalization to multiple bundled modalities with arbitrary arities is left for future work. Formally, we introduce only *one* modal language with a unary modal operator, and different bundled modalities are viewed as different *semantics* for the same modal operator.

Fix a countable set $\mathbf{P}$ of propositional letters.

Definition 2.1 (Language). Formulas in $\mathcal{L}_\circ$ are defined recursively as follows:

$$\phi ::= p \mid \top \mid \neg \phi \mid (\phi \wedge \phi) \mid \circ \phi \quad (\text{where } p \in \mathbf{P}).$$

Let $\bullet \phi$ be the abbreviation for $\circ \neg \phi$ ($\bullet \phi$ becomes useful when we offer axiomatizations).

Next, we define the *bundled semantics* for the above language. We fix a countable set $\mathbf{A}$ of indices, together with an arity function ρ from $\mathbf{A}$ to $\mathbb{N}$ s.t. for any $n \in \mathbb{N}$, $\rho^{-1}[\{n\}]$ is infinite. We also assume that for each $n \in \mathbb{N}$, there is $\iota_n \in \mathbf{A}$ with $\rho(\iota_n) = n$. We consider Kripke models of the following kind.

¹The same kind of logic is also discussed in [42].

Definition 2.2 (Kripke Models). A *Kripke model* is a triple $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$, where $W \neq \emptyset$ is the set of worlds; for each $a \in \mathbf{A}$, R_a is a $(\rho(a) + 1)$ -ary relation on W and in particular, $R_{\iota_n} = \bigcup_{w \in W} \{w\}^{n+1}$; and V is a valuation function from $\mathbf{P}$ to $\wp(W)$. For $w \in W$ and $a \in \mathbf{A}$, we use $R_a(w)$ to denote $\{(v_0, \dots, v_{n-1}) \in W^n \mid (w, v_0, \dots, v_{n-1}) \in R_a\}$ and do not distinguish (v) from v .

The structures of the bundles are expressed by the following *bundle terms*, which are essentially formulas of a polyadic modal language with only one propositional letter in *negation normal form* (where $(+)$ is the only propositional letter, and $(-)$ is its negation):

Definition 2.3 (Bundle Terms). Terms in $\mathcal{T}_{\mathbf{A}}$ are defined recursively as follows:

$$\tau ::= (+) \mid (-) \mid \nabla_a(\underbrace{\tau, \dots, \tau}_{\rho(a)\text{-many}}) \mid \Delta_a(\underbrace{\tau, \dots, \tau}_{\rho(a)\text{-many}}) \quad (\text{where } a \in \mathbf{A}).$$

As usual, when $\rho(a) = 1$, we may use $\Box_a \tau$ and $\Diamond_a \tau$ to denote $\nabla_a(\tau)$ and $\Delta_a(\tau)$, respectively.

Note that Boolean connectives are not included as primitive components in our definition of bundle terms: this is because they can be viewed as *modalities* corresponding to the ι_n 's. For example, binary disjunction and conjunction can be viewed as ∇_{ι_2} and Δ_{ι_2} , respectively.²

Then, for any bundle term τ , we define the corresponding bundled semantics for $\mathcal{L}_{\bigcirc}$ as follows.

Definition 2.4 (Bundled Semantics). Let $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$ be a Kripke model.

First, for any $w \in W$ and $X \subseteq W$, we define the following satisfaction relation $\models$ for terms in $\mathcal{T}_{\mathbf{A}}$:

$\mathcal{M}, w, X \models (+)$	$\iff$	$w \in X$
$\mathcal{M}, w, X \models (-)$	$\iff$	$w \notin X$
$\mathcal{M}, w, X \models \nabla_a(\tau_0, \dots, \tau_{n-1})$	$\iff$	$\forall (v_0, \dots, v_{n-1}) \in R_a(w) \exists i < n. \mathcal{M}, v_i, X \models \tau_i$
$\mathcal{M}, w, X \models \Delta_a(\tau_0, \dots, \tau_{n-1})$	$\iff$	$\exists (v_0, \dots, v_{n-1}) \in R_a(w) \forall i < n. \mathcal{M}, v_i, X \models \tau_i$

Then, for any $\tau \in \mathcal{T}_{\mathbf{A}}$, $w \in W$ and $\phi \in \mathcal{L}_{\bigcirc}$, the satisfaction relation $\models_{\tau}$ for τ is defined recursively as follows (Boolean cases are omitted, and $|\phi|_{\mathcal{M}}^{\tau} = \{w \in W \mid \mathcal{M}, w \models_{\tau} \phi\}$):

$\mathcal{M}, w \models_{\tau} p$	$\iff$	$w \in V(p)$
$\mathcal{M}, w \models_{\tau} \phi \circ \psi$	$\iff$	$\mathcal{M}, w, \phi _{\mathcal{M}}^{\tau} \models \psi$

Intuitively, we “unzip” the modal structure encoded by τ when we compute the truth value of ϕ . This formalizes the usual way in which semantics of bundled modalities are defined.

Note that since $R_{\iota_2} = \{(w, w, w) \mid w \in W\}$, $\mathcal{M}, w, X \models \nabla_{\iota_2}(\tau_0, \tau_1)$ iff $(\mathcal{M}, w, X \models \tau_0) \vee (\mathcal{M}, w, X \models \tau_1)$, while $\mathcal{M}, w, X \models \Delta_{\iota_2}(\tau_0, \tau_1)$ iff $(\mathcal{M}, w, X \models \tau_0) \wedge (\mathcal{M}, w, X \models \tau_1)$, so ∇_{ι_2} and Δ_{ι_2} can be safely viewed as $\vee$ and $\wedge$. For the sake of readability, we shall directly write $\vee$ and $\wedge$ rather than ∇_{ι_2} and Δ_{ι_2} from now on.

As a concrete example, if we want to give $\bigcirc \phi$ the non-contingency semantics, we may take some $a \in \mathbf{A}$ s.t. $\rho(a) = 1$, and let $\tau_{\Delta} = \Box_a(+) \vee \Box_a(-)$. Then, it is not hard to see that $\mathcal{M}, w \models_{\tau_{\Delta}} \bigcirc \phi$ iff $(\forall v \in R_a(w). \mathcal{M}, v \models_{\tau_{\Delta}} \phi) \vee (\forall v \in R_a(w). \mathcal{M}, v \not\models_{\tau_{\Delta}} \phi)$, as intended.

3 Bisimulations for Bundled Modalities

In this section, we offer a uniform way to define bisimulations for bundled modalities and show that the bisimulations defined in this way correctly characterize the expressivity of the corresponding bundled modalities. A key observation leading to these bisimulations is that the bundled semantics can be viewed as a special kind of *neighborhood semantics*. The neighborhood structure generated by a bundle term over a Kripke model is defined as follows (from which one could see why we choose to define bundle terms in negation normal forms).

²The idea of viewing Boolean operators as modalities can also be found in [20, 21], for example.

Definition 3.1 (Generated Neighborhood). Let $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$ be a Kripke model.

We define a function $\text{Nbh}_{\mathcal{M}}$ from $W \times \mathcal{T}_{\mathbf{A}}$ to $\wp(\wp(W) \times \wp(W))$ recursively as follows:³

$$\begin{aligned} \text{Nbh}_{\mathcal{M}}(w, (+)) &= \{(\{w\}, \emptyset)\} \\ \text{Nbh}_{\mathcal{M}}(w, (-)) &= \{(\emptyset, \{w\})\} \\ \text{Nbh}_{\mathcal{M}}(w, \nabla_a(\tau_0, \dots, \tau_{n-1})) &= \{\bigsqcup \text{ran}(f) \mid f \in \prod_{(v_0, \dots, v_{n-1}) \in R_a(w)} \bigcup_{i < n} \text{Nbh}_{\mathcal{M}}(v_i, \tau_i)\} \\ \text{Nbh}_{\mathcal{M}}(w, \Delta_a(\tau_0, \dots, \tau_{n-1})) &= \bigcup_{(v_0, \dots, v_{n-1}) \in R_a(w)} \{\bigsqcup \text{ran}(g) \mid g \in \prod_{i < n} \text{Nbh}_{\mathcal{M}}(v_i, \tau_i)\} \\ &\quad (\text{where } \bigsqcup \{(X_i, Y_i) \mid i \in I\} = (\bigcup_{i \in I} X_i, \bigcup_{i \in I} Y_i)) \end{aligned}$$

Intuitively, given a pointed model $\mathcal{M}, w$, each $(X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \tau)$ represents a minimal way to satisfy and falsify ϕ in the model in order for $\bigcirc \phi$ to be true at $\mathcal{M}, w$ under the τ -bundled semantics (X and Y mark where ϕ must be satisfied and falsified, respectively). Following this intuition, an element in $\text{Nbh}_{\mathcal{M}}(w, \nabla_a(\tau_0, \dots, \tau_{n-1}))$ is obtained by choosing for each successor $(v_0, \dots, v_{n-1})$ of w one $i < n$ and one element from $\text{Nbh}_{\mathcal{M}}(v_i, \tau_i)$, and then merging them together, while an element in $\text{Nbh}_{\mathcal{M}}(w, \Delta_a(\tau_0, \dots, \tau_{n-1}))$ is obtained by first choosing one successor $(v_0, \dots, v_{n-1})$ of w , then choosing for each $i < n$ one element from $\text{Nbh}_{\mathcal{M}}(v_i, \tau_i)$, and merging them together.

Our intuition above is verified by the following proposition, which shows that $\bigcirc \phi$ is true at $\mathcal{M}, w$ iff there is a pair $(X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \tau)$ s.t. $X \subseteq |\phi|_{\mathcal{M}}$ and $Y \subseteq W \setminus |\phi|_{\mathcal{M}}^{\tau}$ — in other words, X and $W \setminus |\phi|_{\mathcal{M}}^{\tau}$ constitute a pair of lower and upper bounds for $|\phi|_{\mathcal{M}}^{\tau}$. It is worth noting that the *Axiom of Choice* (AC) is indispensable for its proof.

For simplicity, we write $(X, Y) \sqsubseteq (X', Y')$ for $X \subseteq X'$ and $Y \subseteq Y'$.

Proposition 3.2. (AC) For any model $\mathcal{M}, w$, any $\tau \in \mathcal{T}_{\mathbf{A}}$ and $\phi \in \mathcal{L}_{\bigcirc}$,

$$\mathcal{M}, w \models_{\tau} \bigcirc \phi \iff \exists (X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \tau) ((X, Y) \sqsubseteq (|\phi|_{\mathcal{M}}^{\tau}, W \setminus |\phi|_{\mathcal{M}}^{\tau})).$$

Proof. Fix a model $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$. We use induction on the structure of terms to show that for any $\tau \in \mathcal{T}_{\mathbf{A}}$, $w \in W$ and $Z \subseteq W$, $\mathcal{M}, w, Z \models \tau$ iff $\exists (X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \tau) ((X, Y) \sqsubseteq (Z, W \setminus Z))$.

The base cases for $(+)$ and $(-)$ are obvious by definition, so we focus on the inductive steps.

For $\nabla_a(\tau_0, \dots, \tau_{n-1})$, we have

$$\begin{aligned} \mathcal{M}, w, Z \models \nabla_a(\tau_0, \dots, \tau_{n-1}) &\iff \forall \vec{v} \in R_a(w) \exists i < n. \mathcal{M}, v_i, Z \models \tau_i \\ &\stackrel{\text{IH}}{\iff} \forall \vec{v} \in R_a(w) \exists i < n \exists (X, Y) \in \text{Nbh}_{\mathcal{M}}(v_i, \tau_i) ((X, Y) \sqsubseteq (Z, W \setminus Z)) \\ &\stackrel{\text{AC}}{\iff} \exists f \in \prod_{\vec{v} \in R_a(w)} \bigcup_{i < n} \text{Nbh}_{\mathcal{M}}(v_i, \tau_i) \forall \vec{v} \in R_a(w) (f(\vec{v}) \sqsubseteq (Z, W \setminus Z)) \\ &\iff \exists f \in \prod_{\vec{v} \in R_a(w)} \bigcup_{i < n} \text{Nbh}_{\mathcal{M}}(v_i, \tau_i) (\bigsqcup \text{ran}(f) \sqsubseteq (Z, W \setminus Z)) \\ &\iff \exists (X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \nabla_a(\tau_0, \dots, \tau_{n-1})) ((X, Y) \sqsubseteq (Z, W \setminus Z)). \end{aligned}$$

For $\Delta_a(\tau_0, \dots, \tau_{n-1})$, similarly, we have

$$\begin{aligned} \mathcal{M}, w, Z \models \Delta_a(\tau_0, \dots, \tau_{n-1}) &\iff \exists \vec{v} \in R_a(w) \forall i < n. \mathcal{M}, v_i, Z \models \tau_i \\ &\stackrel{\text{IH}}{\iff} \exists \vec{v} \in R_a(w) \forall i < n \exists (X, Y) \in \text{Nbh}_{\mathcal{M}}(v_i, \tau_i) ((X, Y) \sqsubseteq (Z, W \setminus Z)) \\ &\iff \exists \vec{v} \in R_a(w) \exists g \in \prod_{i < n} \text{Nbh}_{\mathcal{M}}(v_i, \tau_i) \forall i < n (g(i) \sqsubseteq (Z, W \setminus Z)) \\ &\iff \exists \vec{v} \in R_a(w) \exists g \in \prod_{i < n} \text{Nbh}_{\mathcal{M}}(v_i, \tau_i) (\bigsqcup \text{ran}(g) \sqsubseteq (Z, W \setminus Z)) \\ &\iff \exists (X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \Delta_a(\tau_0, \dots, \tau_{n-1})) ((X, Y) \sqsubseteq (Z, W \setminus Z)). \end{aligned}$$

³ $\prod_{x \in X} Y_x$ denotes the set of functions from X to $\bigcup_{x \in X} Y_x$ that map each $x \in X$ to an element in Y_x .

Note that we do not need AC in the case for $\Delta_a(\tau_0, \dots, \tau_{n-1})$ since the g we need is finite. $\square$

Before moving on to define the bisimulations, as preparation, we first introduce the following notions of *domains* and *completions* for the neighborhood structures generated by terms.

Definition 3.3 (Domain and Completion). Let $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$ be a model.

We define a function $\text{Dom}_{\mathcal{M}}$ from $W \times \mathcal{T}_{\mathbf{A}}$ to $\wp(W)$ recursively as follows:

$$\begin{aligned} \text{Dom}_{\mathcal{M}}(w, (+)), \text{Dom}_{\mathcal{M}}(w, (-)) &= \{w\} \\ \text{Dom}_{\mathcal{M}}(w, \nabla_a(\tau_0, \dots, \tau_{n-1})), \text{Dom}_{\mathcal{M}}(w, \Delta_a(\tau_0, \dots, \tau_{n-1})) &= \bigcup_{(v_0, \dots, v_{n-1}) \in R_a(w)} \bigcup_{i < n} \text{Dom}_{\mathcal{M}}(v_i, \tau_i). \end{aligned}$$

Then, for any term $\tau \in \mathcal{T}_{\mathbf{A}}$, the *completion* of $\text{Nbh}_{\mathcal{M}}(w, \tau)$ is defined as

$$\text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau) = \{Z \subseteq \text{Dom}_{\mathcal{M}}(w, \tau) \mid \exists (X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \tau) ((X, Y) \sqsubseteq (Z, \text{Dom}_{\mathcal{M}}(w, \tau) \setminus Z))\}.$$

Intuitively, the truth value of $\bigcirc \phi$ at $\mathcal{M}, w$ under τ -semantics depends only on the truth values of ϕ in $\text{Dom}_{\mathcal{M}}(w, \tau)$. This is formally expressed by following proposition, which can be checked via an easy induction on the structure of terms.

Proposition 3.4. For any model $\mathcal{M}, w$ and any $\tau \in \mathcal{T}_{\mathbf{A}}$, $\bigcup_{(X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \tau)} X \cup Y \subseteq \text{Dom}_{\mathcal{M}}(w, \tau)$. Thus, by Proposition 3.2, for any $\phi \in \mathcal{L}_{\bigcirc}$, $\mathcal{M}, w \models_{\tau} \bigcirc \phi \iff |\phi|_{\mathcal{M}} \cap \text{Dom}_{\mathcal{M}}(w, \tau) \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$.

With the help of the above notions, we can define bisimulations for bundled modalities in a uniform way. Essentially, we are just applying the kind of neighborhood bisimulation introduced in [23] to the neighborhood structures generated by bundle terms, but there are also some minor adjustments.

Definition 3.5 (Bundled Bisimulation). Let $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$ be a model, and let $\tau \in \mathcal{T}_{\mathbf{A}}$. Let $\mathcal{Z}$ be a binary relation on W . For any $w, v \in W$ and $X \subseteq \text{Dom}_{\mathcal{M}}(w, \tau)$, $Y \subseteq \text{Dom}_{\mathcal{M}}(v, \tau)$, we say X and Y are $\mathcal{Z}$ -coherent, if $\mathcal{Z}[X \cup Y] \cap \text{Dom}_{\mathcal{M}}(w, \tau) \subseteq X$ and $\mathcal{Z}[X \cup Y] \cap \text{Dom}_{\mathcal{M}}(v, \tau) \subseteq Y$.

Then, we say $\mathcal{Z}$ is a τ -bisimulation on $\mathcal{M}$, if for all $(w, v) \in \mathcal{Z}$, the following are satisfied:

- (Inv) For all $p \in \mathbf{P}$, $w \in V(p)$ iff $v \in V(p)$;
- (τ -Coh) For all $X \subseteq \text{Dom}_{\mathcal{M}}(w, \tau)$ and $Y \subseteq \text{Dom}_{\mathcal{M}}(v, \tau)$ s.t. X and Y are $\mathcal{Z}$ -coherent, $X \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$ iff $Y \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$.

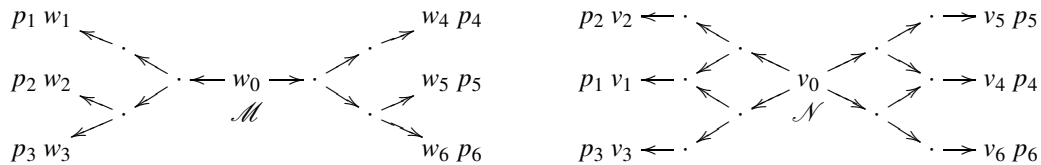
We say $\mathcal{M}, w$ and $\mathcal{N}, v$ are τ -bisimilar, denoted by $\mathcal{M}, w \stackrel{\tau}{\sim} \mathcal{N}, v$, if there is a τ -bisimulation $\mathcal{Z}$ on the disjoint union of $\mathcal{M}$ and $\mathcal{N}$ s.t. $(w, v) \in \mathcal{Z}$.

Note that the condition (τ -Coh) only quantifies over pairs of $\mathcal{Z}$ -coherent sets, from which many (but maybe not all) pairs of sets that are undefinable over $\text{Dom}_{\mathcal{M}}(w, \tau)$ and $\text{Dom}_{\mathcal{M}}(v, \tau)$ are excluded. If we quantify over all pairs of sets instead, then the resulting notion of bisimulation would become too strong, and the Hennessy-Milner theorem (Theorem 3.8) would no longer hold. One could refer to [23] for more details concerning the notion of $\mathcal{Z}$ -coherence in the more abstract context of neighborhood logic.

It is rather routine to check that $\stackrel{\tau}{\sim}$ is indeed an equivalence relation. It is also interesting to note that τ -bisimulation is *not* first-order definable when τ is complicated enough; see Appendix A for an example. Below is a simple example of a pair of τ -bisimilar models.

Example 3.6. Let $\tau = \diamond \square \diamond (+) \wedge \diamond \square \diamond (-)$, and let $\mathcal{M}, \mathcal{N}$ be the following models.

$\mathcal{M}, w_0$ and $\mathcal{N}, v_0$ are not bisimilar in the standard sense, but they are τ -bisimilar.



- $\text{Dom}_{\mathcal{M}}(w_0, \tau) = \{w_i \mid 1 \leq i \leq 6\}$ and $\text{Dom}_{\mathcal{N}}(v_0, \tau) = \{v_i \mid 1 \leq i \leq 6\}$.
- $\text{Nbh}_{\mathcal{M}}(w_0, \tau) = N \times N$, where $N = \{\{w_1, w_2\}, \{w_1, w_3\}, \{w_4, w_5\}, \{w_4, w_6\}\}$.
- $\text{Nbh}_{\mathcal{N}}(v_0, \tau) = N' \times N'$, where $N' = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_4, v_5\}, \{v_4, v_6\}\}$.
- For any $1 \leq i \leq 6$, $\text{Dom}_{\mathcal{M}}(w_i, \tau) = \text{Dom}_{\mathcal{N}}(v_i, \tau) = \text{Nbh}_{\mathcal{M}}(w_i, \tau) = \text{Nbh}_{\mathcal{N}}(v_i, \tau) = \emptyset$.
- $\mathcal{Z} = \bigcup_{0 \leq i \leq 6} (\{w_i, v_i\} \times \{w_i, v_i\})$ forms a τ -bisimulation on the disjoint union of $\mathcal{M}$ and $\mathcal{N}$, so $\mathcal{M}, w_0 \stackrel{\tau}{\sim} \mathcal{N}, v_0$.

Then, we check that the truth values of formulas under bundled semantics are indeed preserved by bundled bisimilarity. For simplicity, we write $\mathcal{M}, w \equiv_{\tau} \mathcal{N}, v$ when $\mathcal{M}, w \models_{\tau} \phi \iff \mathcal{N}, v \models_{\tau} \phi$ holds for all $\phi \in \mathcal{L}_{\circ}$.

Proposition 3.7. *For any models $\mathcal{M}, w$ and $\mathcal{N}, v$ and $\tau \in \mathcal{T}_{\mathbf{A}}$, $\mathcal{M}, w \stackrel{\tau}{\sim} \mathcal{N}, v \implies \mathcal{M}, w \equiv_{\tau} \mathcal{N}, v$.*

Proof. It suffices to show by induction that for any model $\mathcal{M}$ and τ -bisimulation $\mathcal{Z}$ on $\mathcal{M}$, for any $\phi \in \mathcal{L}_{\circ}$, if $(w, v) \in \mathcal{Z}$, then $\mathcal{M}, w \models_{\tau} \phi$ iff $\mathcal{M}, v \models_{\tau} \phi$. We only show the case for $\circ\phi$.

Assume that $(w, v) \in \mathcal{Z}$, and let $X = |\phi|_{\mathcal{M}}^{\tau} \cap \text{Dom}_{\mathcal{M}}(w, \tau)$, $Y = |\phi|_{\mathcal{M}}^{\tau} \cap \text{Dom}_{\mathcal{M}}(v, \tau)$. By IH, it is not hard to see that X and Y are $\mathcal{Z}$ -coherent, so $X \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$ iff $Y \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$. Then, by proposition 3.4, $\mathcal{M}, w \models_{\tau} \circ\phi$ iff $X \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$ iff $Y \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$ iff $\mathcal{M}, v \models_{\tau} \circ\phi$. $\square$

Finally, we show that the converse direction of the above implication holds when the models in question are *image-finite* (i.e. for any $w \in W$ and $a \in \mathbf{A}$, $R_a(w)$ is finite). This shows that the notion of τ -bisimilarity indeed characterizes the expressivity of $\mathcal{L}_{\circ}$ under τ -semantics in a natural way.

Theorem 3.8 (Hennessy-Milner). *For any image-finite models $\mathcal{M}, w, \mathcal{N}, v$ and $\tau \in \mathcal{T}_{\mathbf{A}}$,*

$$\mathcal{M}, w \stackrel{\tau}{\sim} \mathcal{N}, v \iff \mathcal{M}, w \equiv_{\tau} \mathcal{N}, v.$$

Proof. Let $\tau \in \mathcal{T}_{\mathbf{A}}$ be arbitrary. Using disjoint unions, it suffices to show that $\mathcal{Z} = \{(w, v) \mid \mathcal{M}, w \equiv_{\tau} \mathcal{M}, v\}$ forms a τ -bisimulation for any image-finite $\mathcal{M}$.

The condition (Inv) is trivially satisfied, so we only need to check (τ -Coh). Assume that $(w, v) \in \mathcal{Z}$. Since $\mathcal{M}$ is image-finite, it is easy to check that $\text{Dom}_{\mathcal{M}}(w, \tau)$ and $\text{Dom}_{\mathcal{M}}(v, \tau)$ are finite, so we can define every $\mathcal{Z}$ -equivalence class C over $\text{Dom}_{\mathcal{M}}(w, \tau) \cup \text{Dom}_{\mathcal{M}}(v, \tau)$ with a formula ψ_C . Now, assume that $X \subseteq \text{Dom}_{\mathcal{M}}(w, \tau)$ and $Y \subseteq \text{Dom}_{\mathcal{M}}(v, \tau)$ are $\mathcal{Z}$ -coherent, and consider $\phi = \bigvee_{u \in X \cup Y} \psi_{[u]_{\mathcal{Z}}}$, where $[u]_{\mathcal{Z}}$ is the $\mathcal{Z}$ -equivalence class over $\text{Dom}_{\mathcal{M}}(w, \tau) \cup \text{Dom}_{\mathcal{M}}(v, \tau)$ containing u . Since X and Y are $\mathcal{Z}$ -coherent, it is not hard to check that $X = |\phi|_{\mathcal{M}}^{\tau} \cap \text{Dom}_{\mathcal{M}}(w, \tau)$ and $Y = |\phi|_{\mathcal{M}}^{\tau} \cap \text{Dom}_{\mathcal{M}}(v, \tau)$. Then, since $\mathcal{M}, w \equiv_{\tau} \mathcal{M}, v$, $\mathcal{M}, w \models_{\tau} \circ\phi$ iff $\mathcal{M}, v \models_{\tau} \circ\phi$, so by Proposition 3.4, $X \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$ iff $Y \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$. $\square$

4 PN-Independent Bundled Modalities

Now, we turn to a special class of bundled modalities we call *positive-negative-independent* bundled modalities (*PN-independent* bundles for short). As we shall see, most bundled modalities studied in the literature are either PN-independent or duals of PN-independent bundles, and such bundles are indeed well-behaved on the technical level.

Definition 4.1 (PN-Independent Bundles). Let $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$ be a model, and let $\tau \in \mathcal{T}_{\mathbf{A}}$. We say τ is *PN-independent* over $\mathcal{M}$, if for all $w \in W$ and $(X, Y), (X', Y') \in \text{Nbh}_{\mathcal{M}}(w, \tau)$, if $X \cap Y = X' \cap Y' = X \cap Y' = \emptyset$, then $(X, Y') \in \text{Nbh}_{\mathcal{M}}(w, \tau)$.

We say a family N of sets is *convex*, if for any $X, Y \in N$ and any set Z , if $X \subseteq Z \subseteq Y$, then $Z \in N$. Note that if τ is PN-independent over $\mathcal{M}$, then for any w in $\mathcal{M}$, $\text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$ is convex. This feature of PN-independent modalities validates the axiom of convexity $\text{CONV} : \bigcirc(\phi \wedge \psi) \wedge \bigcirc(\phi \vee \chi) \rightarrow \bigcirc\phi$, which, as we shall see, plays a crucial role in axiomatizations of PN-independent bundles.

There is an easy syntactic sufficient condition for PN-independence, which covers a great amount of PN-independent bundles in the literature in their exact syntactic forms, and probably all such bundles in the literature up to logical equivalence.

Proposition 4.2. *Define bundle terms in $\mathcal{T}_{\mathbf{A}}^{\times}$ by the following grammar:*

$$\begin{aligned}\mathcal{T}_{\mathbf{A}}^+ \ni \tau^+ &::= (+) \mid \nabla_a(\tau^+, \dots, \tau^+) \mid \Delta_a(\tau^+, \dots, \tau^+) \quad (a \in \mathbf{A}) \\ \mathcal{T}_{\mathbf{A}}^- \ni \tau^- &::= (-) \mid \nabla_a(\tau^-, \dots, \tau^-) \mid \Delta_a(\tau^-, \dots, \tau^-) \quad (a \in \mathbf{A}) \\ \mathcal{T}_{\mathbf{A}}^{\times} \ni \tau^{\times} &::= \tau^+ \mid \tau^- \mid \tau^{\times} \wedge \tau^{\times} \mid \Box_a \tau^{\times} \quad (\tau^+ \in \mathcal{T}_{\mathbf{A}}^+, \tau^- \in \mathcal{T}_{\mathbf{A}}^-, \rho(a) = 1).\end{aligned}$$

Then, for any $\tau^{\times} \in \mathcal{T}_{\mathbf{A}}^{\times}$ and any model $\mathcal{M}$, $\tau^{\times}$ is PN-independent over $\mathcal{M}$.

Proof Sketch. Let $\mathcal{M} = (W, \{R_a\}_{a \in \mathbf{A}}, V)$ be arbitrary. First, note that for any $w \in W$ and $\tau^+ \in \mathcal{T}_{\mathbf{A}}^+$, there is $N_{w, \tau^+}^+ \in \wp(\wp(W))$ s.t. $\text{Nbh}_{\mathcal{M}}(w, \tau^+) = N_{w, \tau^+}^+ \times \{\emptyset\}$; similarly, for any $\tau^- \in \mathcal{T}_{\mathbf{A}}^-$, there is $N_{w, \tau^-}^- \in \wp(\wp(W))$ s.t. $\text{Nbh}_{\mathcal{M}}(w, \tau^-) = \{\emptyset\} \times N_{w, \tau^-}^-$. Then, we check by induction that for any $w \in W$ and $\tau^{\times} \in \mathcal{T}_{\mathbf{A}}^{\times}$, there are $N_{w, \tau^{\times}}^+, N_{w, \tau^{\times}}^- \in \wp(\wp(W))$ s.t. $\text{Nbh}_{\mathcal{M}}(w, \tau^{\times}) = N_{w, \tau^{\times}}^+ \times N_{w, \tau^{\times}}^-$ (which is actually stronger than the requirement in Definition 4.1). The base case for τ^+ and τ^- was just noted above. For the inductive step, if $\text{Nbh}_{\mathcal{M}}(w, \tau_0^{\times}) = N_{w, \tau_0^{\times}}^+ \times N_{w, \tau_0^{\times}}^-$ and $\text{Nbh}_{\mathcal{M}}(w, \tau_1^{\times}) = N_{w, \tau_1^{\times}}^+ \times N_{w, \tau_1^{\times}}^-$, then it is routine to compute that $\text{Nbh}_{\mathcal{M}}(w, \tau_0^{\times} \wedge \tau_1^{\times}) = \{X_0 \cup X_1 \mid X_0 \in N_{w, \tau_0^{\times}}^+, X_1 \in N_{w, \tau_1^{\times}}^+\} \times \{Y_0 \cup Y_1 \mid Y_0 \in N_{w, \tau_0^{\times}}^-, Y_1 \in N_{w, \tau_1^{\times}}^-\}$; similarly, if $\text{Nbh}_{\mathcal{M}}(v, \tau^{\times}) = N_{v, \tau^{\times}}^+ \times N_{v, \tau^{\times}}^-$ for all $v \in R_a(w)$, then $\text{Nbh}_{\mathcal{M}}(w, \Box_a \tau^{\times}) = \{\bigcup \text{ran}(g) \mid g \in \prod_{v \in R_a(w)} N_{v, \tau^{\times}}^+\} \times \{\bigcup \text{ran}(h) \mid h \in \prod_{v \in R_a(w)} N_{v, \tau^{\times}}^-\}$. $\square$

Example 4.3. By Proposition 4.2, the following bundles are PN-independent over all models: $\Diamond(+)$ $\wedge$ $\Diamond(-)$ (contingency / ignorance [30, 24, 25, 41, 13, 14, 6]); $(+) \wedge \Diamond(-)$ (unknown truth / accident [29, 35, 34, 8, 17, 18, 19]); $\Box(+)$ $\wedge$ $(-)$ (false belief [36, 18, 8, 39, 10]); $\Box(-)$ $\wedge$ $(+)$ (strong accident [32]); $((+) \vee \Diamond(+)) \wedge \Diamond(-)$ (semi-weak contingency / disjunctive ignorance [7]); $((+) \vee \Diamond(+)) \wedge ((-) \vee \Diamond(-))$ (weak contingency [5]); $BK(+)$ $\wedge$ $\tilde{K}(-)$ (Dunning-Kruger ignorance [11]); $\Box_a(+)$ $\wedge$ $\Diamond_b(-)$, $\Box_a(+)$ $\wedge$ $\Box_b(-)$, $\Diamond_a(+)$ $\wedge$ $\Box_b(-)$, $\Diamond_a(+)$ $\wedge$ $\Diamond_b(-)$ ([12]); $\bigvee_{a \in A} \Box_a(+)$ (someone knows [42, 4]); $\Box_a(+)$ $\wedge$ $\bigwedge_{b \in B} \Box_b(+)$ (secret knowledge [40]); $(\Box_a(+)$ $\wedge$ $\Diamond_b(-)) \vee (\Box_b(+)$ $\wedge$ $\Diamond_a(-))$ (moderate disagreement [3]; it is equivalent to $(\Box_a(+)$ $\vee$ $\Box_b(+)) \wedge (\Diamond_a(-)$ $\vee$ $\Diamond_b(-))$).

Moreover, assuming seriality of the models, the following bundles also become PN-independent: $(\Box(+)$ $\wedge$ $(-)) \vee (\Box(-)$ $\wedge$ $(+))$ (radical ignorance [15, 10]; equivalent to $((+) \vee \Box(+)) \wedge ((-) \vee \Box(-))$ over serial models); $(\Box_a(-)$ $\wedge$ $\Box_b(+)) \vee (\Box_a(+)$ $\wedge$ $\Box_b(-))$ (strong disagreement [2]; equivalent to $(\Box_a(+)$ $\vee$ $\Box_b(+)) \wedge (\Box_a(-)$ $\vee$ $\Box_b(-))$ over serial models).

On the other hand, one non-ad hoc example of non-PN-independent bundles is the so-called *Rumsfeld ignorance* or *unknown unknowns* $\Diamond(+)$ $\wedge$ $\Diamond(-)$ $\wedge$ $\Diamond(\Box(+)$ $\vee$ $\Box(-))$ [16, 9]: both itself and its dual are not PN-independent, even over models with rather strong properties (see Appendix B).

For PN-independent bundles, we can also define a stronger notion of bisimulation, which does not quantify over *all* subsets of the τ -domains, but only the neighborhoods generated by τ . Compared to the kind of neighborhood bisimulation defined in [23], our bisimulation for PN-independent bundles shows greater resemblance to the bisimulation for monotonic neighborhood models (see e.g. [31] for the definition) and the bisimulation for (non-)contingency modality defined in [13].

Definition 4.4 (Independent Bisimulation). Let $\mathcal{M}$ be a model, and let $\tau \in \mathcal{T}_A$. Let $\mathcal{Z}$ be a binary relation on $\mathcal{M}$. We say $\mathcal{Z}$ is an *independent τ -bisimulation* on $\mathcal{M}$, if for all $(w, v) \in \mathcal{Z}$, we have

- (Inv) For all $p \in \mathbf{P}$, $w \in V(p)$ iff $v \in V(p)$;
- (τ -Zig) For all $(X_0, X_1) \in \text{Nbh}_{\mathcal{M}}(w, \tau)$ s.t. $\mathcal{Z}[X_0] \cap X_1 = \emptyset$, there is $(Y_0, Y_1) \in \text{Nbh}_{\mathcal{M}}(v, \tau)$ s.t. $(Y_0, Y_1) \subseteq (\mathcal{Z}[X_0], \mathcal{Z}[X_1])$;
- (τ -Zag) For all $(Y_0, Y_1) \in \text{Nbh}_{\mathcal{M}}(v, \tau)$ s.t. $\mathcal{Z}[Y_0] \cap Y_1 = \emptyset$, there is $(X_0, X_1) \in \text{Nbh}_{\mathcal{M}}(w, \tau)$ s.t. $(X_0, X_1) \subseteq (\mathcal{Z}^{-1}[Y_0], \mathcal{Z}^{-1}[Y_1])$.

We say $\mathcal{M}, w$ and $\mathcal{N}, v$ are *independently τ -bisimilar*, denoted by $\mathcal{M}, w \stackrel{\times}{\sim}_{\tau} \mathcal{N}, v$, if there is an independent τ -bisimulation $\mathcal{Z}$ on the disjoint union of $\mathcal{M}$ and $\mathcal{N}$ s.t. $(w, v) \in \mathcal{Z}$.

Note that when $\tau^+ \in \mathcal{T}_A^+$ (so that for any w in $\mathcal{M}$, $\text{Nbh}_{\mathcal{M}}(w, \tau^+) = N_{w, \tau^+}^+ \times \{\emptyset\}$ for some N_{w, τ^+}^+), for any relation $\mathcal{Z}$ on $\mathcal{M}$ and any $(X, Y) \in \text{Nbh}_{\mathcal{M}}(w, \tau^+)$, we trivially have $\mathcal{Z}[X] \cap Y = \emptyset$, so (τ^+ -Zig) and (τ^+ -Zag) in Definition 4.4 can be simplified as follows:

- (τ^+ -Zig) For all $(X, \emptyset) \in \text{Nbh}_{\mathcal{M}}(w, \tau^+)$ there is $(Y, \emptyset) \in \text{Nbh}_{\mathcal{M}}(v, \tau^+)$ s.t. $Y \subseteq \mathcal{Z}[X]$;
- (τ^+ -Zag) For all $(Y, \emptyset) \in \text{Nbh}_{\mathcal{M}}(v, \tau^+)$, there is $(X, \emptyset) \in \text{Nbh}_{\mathcal{M}}(w, \tau^+)$ s.t. $X \subseteq \mathcal{Z}^{-1}[Y]$.

In other words, independent bisimulation for $\tau^+ \in \mathcal{T}_A^+$ is essentially the bisimulation for *monotonic neighborhood models* [31] applied to τ^+ -generated neighborhood structures.

Also note that for the contingency bundle $\tau_{\nabla} = \Diamond(+) \wedge \Diamond(-)$, $\text{Nbh}_{\mathcal{M}}(w, \tau_{\nabla}) = \{\{v\} \mid v \in R(w)\}^2$ for any $\mathcal{M}, w$, so the conditions (τ_{∇} -Zig) and (τ_{∇} -Zag) can be simplified as follows:

- (τ_{∇} -Zig) For all $w_0, w_1 \in R(w)$ s.t. $(w_0, w_1) \notin \mathcal{Z}$, there are $v_0, v_1 \in R(v)$ s.t. $(w_0, v_0), (w_1, v_1) \in \mathcal{Z}$;
- (τ_{∇} -Zag) For all $v_0, v_1 \in R(w)$ s.t. $(v_0, v_1) \notin \mathcal{Z}$, there are $w_0, w_1 \in R(v)$ s.t. $(w_0, v_0), (w_1, v_1) \in \mathcal{Z}$.

Then, what we obtain is essentially the bisimulation for (non-)contingency defined in [13].

Again, it is rather routine to check that $\stackrel{\times}{\sim}_{\tau}$ is an equivalence relation. The reader could also check that $\mathcal{M}, w_0$ and $\mathcal{N}, v_0$ given in Example 3.6 are independently τ -bisimilar. We then show that $\stackrel{\times}{\sim}_{\tau}$ always coincides with $\stackrel{\sim}{\sim}_{\tau}$ when τ is PN-independent (proof in Appendix C).

Proposition 4.5. For any models $\mathcal{M}, w, \mathcal{N}, v$ and $\tau \in \mathcal{T}_A$, $\mathcal{M}, w \stackrel{\times}{\sim}_{\tau} \mathcal{N}, v \implies \mathcal{M}, w \stackrel{\sim}{\sim}_{\tau} \mathcal{N}, v$; moreover, if τ is PN-independent over $\mathcal{M}$ and $\mathcal{N}$, then $\mathcal{M}, w \stackrel{\sim}{\sim}_{\tau} \mathcal{N}, v \implies \mathcal{M}, w \stackrel{\times}{\sim}_{\tau} \mathcal{N}, v$.

Corollary 4.6. For any models $\mathcal{M}, w, \mathcal{N}, v$ and $\tau \in \mathcal{T}_A$, $\mathcal{M}, w \stackrel{\times}{\sim}_{\tau} \mathcal{N}, v \implies \mathcal{M}, w \equiv_{\tau} \mathcal{N}, v$; moreover, if $\mathcal{M}, \mathcal{N}$ are also image-finite and τ is PN-independent over $\mathcal{M}$ and $\mathcal{N}$, then $\mathcal{M}, w \stackrel{\sim}{\sim}_{\tau} \mathcal{N}, v \iff \mathcal{M}, w \equiv_{\tau} \mathcal{N}, v$.

4.1 Convex Neighborhood Semantics

Next, we consider axiomatizations of PN-independent bundled modalities. Our strategy is to first consider a kind of *convex neighborhood semantics* for $\mathcal{L}_{\circ}$, which abstract the core structure of the bundled semantics for all PN-independent bundles, and then transfer completeness results under such a semantics to concrete bundled semantics on Kripke models using representation theorems.

Definition 4.7 (Convex Neighborhood Models). A *convex neighborhood model* is a 4-tuple $\mathcal{N} = (W, N^+, N^-, V)$, where $W \neq \emptyset$ is the set of worlds, N^+, N^- are both functions from W to $\wp(\wp(W))$, and $V : \mathbf{P} \rightarrow \wp(W)$ is the valuation function. A convex neighborhood frame is a model without the V part.

We say $\mathcal{N}$ is a *core model* if for all $w \in W$, we have (i) for all $X \in N^+(w)$, there is $Y \in N^-(w)$ s.t. $X \cap Y = \emptyset$; (ii) for all $Y \in N^-(w)$, there is $X \in N^+(w)$ s.t. $X \cap Y = \emptyset$, and (iii) for all $\star \in \{+, -\}$ and $Z, Z' \in N^{\star}(w)$, if $Z \subseteq Z'$, then $Z = Z'$.⁴

⁴Our core convex neighborhood models generalize core monotonic neighborhood models [22, 31].

Then, we define the following kind of *support semantics* on convex neighborhood models:

Definition 4.8 (Neighborhood Semantics). Let $\mathcal{N} = (W, N^+, N^-, V)$ be a convex neighborhood model. For any $w \in W$ and $\phi \in \mathcal{L}_\circ$, the satisfaction relation $\models$ is defined recursively as usual, with the clause for $\circ$ being $(\phi|_{\mathcal{N}} = \{w \in W \mid \mathcal{N}, w \models \phi\})$:

$$\mathcal{N}, w \models \circ\phi \iff \exists X \in N^+(w) \exists Y \in N^-(w) ((X, Y) \sqsubseteq (\phi|_{\mathcal{N}}, W \setminus \phi|_{\mathcal{N}})).$$

As usual, validity on frames is defined as truth at all worlds under all valuations.

Equivalently, given a convex neighborhood model $\mathcal{N} = (W, N^+, N^-, V)$, the above semantics can also be defined by first generating a neighborhood function N from N^+ and N^- , and then applying the exact neighborhood semantics to this N (i.e. $\mathcal{M}, w \models \circ\phi$ iff $\phi|_{\mathcal{N}} \in N(w)$), where the N in question is defined by $N(w) = \{Z \subseteq W \mid \exists X \in N^+(w) \exists Y \in N^-(w) ((X, Y) \sqsubseteq (Z, W \setminus Z))\}$ for all $w \in W$. Conversely, given an ordinary neighborhood model $\mathcal{N} = (W, N, V)$ where N maps each world to a *convex* family of neighborhoods, we can also generate a pair of positive and negative neighborhood functions N^+ and N^- by letting $N^+ = N$ and $N^-(w) = \{W \setminus X \mid X \in N(w)\}$ for all $w \in W$; then, the exact neighborhood semantics w.r.t. N also coincides with the above support semantics w.r.t. N^+ and N^- (i.e. $\phi|_{\mathcal{N}} \in N(w)$ iff $\exists X \in N^+(w) \exists Y \in N^-(w) ((X, Y) \sqsubseteq (\phi|_{\mathcal{N}}, W \setminus \phi|_{\mathcal{N}}))$). The situation here resembles monotonic neighborhood logic, where we also have an equivalence between the support semantics w.r.t. an arbitrary neighborhood function N^+ and the exact semantics w.r.t. its upward closure N (i.e. $\phi|_{\mathcal{N}} \in N(w)$ iff $\exists X \in N^+(w) X \subseteq \phi|_{\mathcal{N}}$, where $N(w) = \{Z \subseteq W \mid \exists X \in N^+(w) X \subseteq Z\}$), and the two kinds of semantics coincide if the neighborhood function in question is upward closed.

Next, by a *convex modal logic*, we mean a set of $\mathcal{L}_\circ$ -formulas that contains all instances of the axioms and is closed under the rules displayed in the following table as well as uniform substitution. The minimal convex modal logic is **CONV**.⁵

Shared axioms and rules of convex modal logics			
Axioms		Rules	
TAUT	All propositional tautologies	MP	From ϕ and $\phi \rightarrow \psi$, infer ψ
CONV	$\circ(\phi \wedge \psi) \wedge \circ(\phi \vee \chi) \rightarrow \circ\phi$	RE	From $\phi \leftrightarrow \psi$, infer $\circ\phi \leftrightarrow \circ\psi$

Proposition 4.9. **CONV** is sound w.r.t. the class of all convex neighborhood models (frames).

For the remainder of this section, we construct canonical models for convex modal logics. Note that the canonical models we define are not the usual kind of canonical models for neighborhood logics in general; they should rather be viewed as a generalization of the kind of canonical models for *monotonic* neighborhood logics defined in [4]. An important feature of our canonical models (as well as the canonical models in [4]) is that the canonical models are automatically *core* models, and correspondingly, such canonical models admit a greater amount of canonical axioms compared to the standard canonical models for neighborhood logics in general.

Definition 4.10 (Core). Let **L** be a convex modal logic. An **L**-filter is a subset F of $\mathcal{L}_\circ$ s.t. (i) $\top \in F$, (ii) $\phi, \psi \in F$ implies $\phi \wedge \psi \in F$, and (iii) $\phi \in F, \vdash_{\mathbf{L}} \phi \rightarrow \psi$ implies $\psi \in F$. Note that $\mathcal{L}_\circ$ is a filter.

Given a set Δ of $\mathcal{L}_\circ$ -formulas, we say an **L**-filter F is *+coherent* (resp. *--coherent*) with Δ if for all $\phi \in F$, there is $\psi \in F$ s.t. $\circ(\phi \wedge \psi) \in \Delta$ (resp. $\bullet(\phi \wedge \psi) \in \Delta$).⁶ Let $\text{Core}_{\mathbf{L}}^+(\Delta)$ (resp. $\text{Core}_{\mathbf{L}}^-(\Delta)$) denote the set of *maximal L-filters* among those *+coherent* (resp. *--coherent*) with Δ .

⁵It is worth noting that in the literature, the axiom CONV sometimes appears in the form of the rule that allows one to infer $\circ\psi \wedge \circ\chi \rightarrow \circ\phi$ from $\psi \rightarrow \phi$ and $\phi \rightarrow \chi$ (see [2, 3, 40] for example). This is derivable in **CONV**, since if $\vdash \psi \rightarrow \phi$, then $\vdash \psi \leftrightarrow (\phi \wedge \psi)$, and if $\vdash \phi \rightarrow \chi$, then $\vdash \chi \leftrightarrow (\phi \vee \chi)$. When we cite CONV, we occasionally mean this rule.

⁶Equivalently, an **L**-filter F is *+coherent* with Δ if $\{\phi \in F \mid \circ\phi \in \Delta\}$ is *dense* in F in the sense that for any $\phi \in F$ there is $\psi \in F$ with $\vdash_{\mathbf{L}} \psi \rightarrow \phi$ such that $\circ\psi \in \Delta$; similarly for *--coherence* with Δ .

Definition 4.11 (Canonical Models). Let $\mathbf{L}$ be a convex modal logic. The canonical model w.r.t. $\mathbf{L}$ is $\mathcal{N}_{\mathbf{L}}^c = (W_{\mathbf{L}}^c, N_{\mathbf{L}}^{\oplus}, N_{\mathbf{L}}^{\ominus}, V_{\mathbf{L}}^c)$, where $W_{\mathbf{L}}^c = \{\Delta \subseteq \mathcal{L}_{\circ} \mid \Delta \text{ is a maximally } \mathbf{L}\text{-consistent set}\}$, $V^c(p) = \{\Delta \in W_{\mathbf{L}}^c \mid p \in \Delta\}$ for all $p \in \mathbf{P}$, and for any $\Delta \in W_{\mathbf{L}}^c$, $N_{\mathbf{L}}^{\oplus}(\Delta)$ and $N_{\mathbf{L}}^{\ominus}(\Delta)$ are defined as follows:

$$\|F\| = \{\Theta \in W_{\mathbf{L}}^c \mid F \subseteq \Theta\}; N_{\mathbf{L}}^{\oplus}(\Delta) = \{\|F\| \mid F \in \text{Core}_{\mathbf{L}}^+(\Delta)\}; N_{\mathbf{L}}^{\ominus}(\Delta) = \{\|F'\| \mid F' \in \text{Core}_{\mathbf{L}}^-(\Delta)\}.$$

Intuitively, if a filter F is $+$ -coherent (resp. $-$ -coherent) w.r.t. Δ , then it is consistent with the information in Δ to have a positive (resp. negative) neighborhood supporting all formulas in F . When defining the canonical models, we want to realize the formulas in an MCS Δ with a *minimal* number of positive and negative neighborhoods, so we want each neighborhood to support as many formulas as possible; thus, the positive (resp. negative) neighborhoods of Δ in the canonical models are constructed from the *maximal* filters among the ones that are $+$ -coherent (resp. $-$ -coherent) with Δ .

Now, we fix an arbitrary convex modal logic $\mathbf{L}$. To avoid notational clutter, from now on, we omit all prefixes and subscripts “ $\mathbf{L}$ ”. The following shows that the canonical model w.r.t. $\mathbf{L}$ indeed has all the properties we need.

Lemma 4.12. *For any $\Delta \in W^c$, if F_0 is a filter $+$ -coherent (resp. $-$ -coherent) with Δ , then there is $F \in \text{Core}^+(\Delta)$ (resp. $\text{Core}^-(\Delta)$) s.t. $F_0 \subseteq F$. Thus, for any $\phi \in \Delta$, there is $F \in \text{Core}^+(\Delta)$ and $F' \in \text{Core}^-(\Delta)$ s.t. $\phi \in F$ and $\neg\phi \in F'$.*

Proof. Observe that if $\mathcal{F}$ forms a $\subseteq$ -chain of filters $+$ -coherent (resp. $-$ -coherent) with Δ , then $\bigcup \mathcal{F}$ is also a filter that is $+$ -coherent (resp. $-$ -coherent) with Δ . Thus, by Zorn’s Lemma, every $+$ -coherent (resp. $-$ -coherent) filter can be extended into a maximal one.⁷ If $\phi \in \Delta$, then the filter F_0 generated by $\{\phi\}$ consisting of all logical consequences of ϕ under $\mathbf{L}$ is $+$ -coherent with Δ , since for any $\psi \in F_0$, $\vdash \phi \leftrightarrow (\psi \wedge \phi)$ and thus $\phi \in \Delta$. Similarly, the filter generated by $\{\neg\phi\}$ is $-$ -coherent with Δ . $\square$

Lemma 4.13. *For any $\Delta \in W^c$, if $\phi \notin \Delta$, then there is no $F \in \text{Core}^+(\Delta)$ and $F' \in \text{Core}^-(\Delta)$ satisfying $\phi \in F$ and $\neg\phi \in F'$.*

Proof. Note that if there is $F \in \text{Core}^+(\Delta)$ and $F' \in \text{Core}^-(\Delta)$ s.t. $\phi \in F$ and $\neg\phi \in F'$, then since F and F' are $+$ -coherent and $-$ -coherent with Δ respectively, there is $\psi \in F$ and $\chi \in F'$ s.t. $\phi \in \Delta$. Also note that $\bullet(\neg\phi \wedge \chi) \in \Delta$ implies $\phi \in \Delta$ by RE. Thus $\phi \in \Delta$ by CONV. $\square$

It is worth noting that in our proof for the completeness of the minimal convex modal logic **CONV**, Lemma 4.13 is the only place where the axiom CONV is used.

Proposition 4.14. $\mathcal{N}^c$ is a core model.

Proof. Let $\Delta \in W^c$ be arbitrary. For requirement (i), take any $F \in \text{Core}^+(\Delta)$. Since F is $+$ -coherent, there is $\phi \in F$ such that $\phi \in \Delta$. So by Lemma 4.12, there is $F' \in \text{Core}^-(\Delta)$ s.t. $\neg\phi \in F'$, and thus $\|F\| \cap \|F'\| = \emptyset$. Requirement (ii) can be proved similarly. For (iii), take $F, G \in \text{Core}^+(\Delta)$ s.t. $\|F\| \subseteq \|G\|$. Now F must extend G , since otherwise there would have been a $\phi \in G \setminus F$, and $F \cup \{\neg\phi\}$ can be extended to a maximally consistent set $\Theta \in W^c$ such that $\Theta \in \|F\| \setminus \|G\|$, contradicting $\|F\| \subseteq \|G\|$. But then $G = F$ since F is maximal among those filters $+$ -coherent with Δ , of which G is a member. Hence, $\|F\| = \|G\|$. The case for $F', G' \in \text{Core}^-(\Delta)$ is similar. $\square$

Lemma 4.15 (Truth). *For any $\Delta \in W^c$ and $\phi \in \mathcal{L}_{\circ}$, $\mathcal{N}^c, \Delta \Vdash \phi \iff \phi \in \Delta$.*

⁷Since the language $\mathcal{L}_{\circ}$ is countable, our use of Zorn’s Lemma here is not essential. For example, we could enumerate all formulas in $\mathcal{L}_{\circ}$, add them one by one to the filter in question if admissible, and repeat the procedure for countably many times.

Proof. As usual, the lemma is proved by induction. We only consider the inductive case for $\bigcirc\phi$.

By Lemma 4.12 and 4.13, $\bigcirc\phi \in \Delta$ iff there is $F \in \text{Core}^+(\Delta)$ and $F' \in \text{Core}^-(\Delta)$ s.t. $\phi \in F$ and $\neg\phi \in F'$. Moreover, by IH, $\phi \in F$ iff for any $\Theta \in \|\mathcal{F}\|$, $\mathcal{N}^c, \Theta \Vdash \phi$, and $\neg\phi \in F'$ iff for any $\Theta \in \|\mathcal{F}'\|$, $\mathcal{N}^c, \Theta \nVdash \phi$. Thus, $\bigcirc\phi \in \Delta$ iff there is $F \in \text{Core}^+(\Delta)$ and $F' \in \text{Core}^-(\Delta)$ s.t. $(\|\mathcal{F}\|, \|\mathcal{F}'\|) \sqsubseteq (|\phi|_{\mathcal{N}^c}, W^c \setminus |\phi|_{\mathcal{N}^c})$ iff $\mathcal{N}^c, \Delta \Vdash \bigcirc\phi$. $\square$

Thus, when the logic $\mathbf{L}$ in question is just **CONV**, we obtain the following completeness theorem:

Theorem 4.16. *Under convex neighborhood semantics, **CONV** is sound and strongly complete w.r.t. the class of all models, and also the class of all core models.*

5 Case Studies

In this section, we show how the above canonical model construction could help us achieve completeness results for several concrete PN-independent bundled modalities over classes of models with rather strong properties. To our knowledge, none of the axiomatizations presented below have appeared in the literature. The following notion of *representation* is key in linking neighborhood completeness to Kripke completeness under bundled semantics.

Definition 5.1 (Representation). Let $\mathcal{N} = (W, N^+, N^-, V)$ be a convex neighborhood model, $\mathcal{M} = (W^*, \{R_a^*\}_{a \in A}, V^*)$ be a Kripke model, and let $\tau \in \mathcal{T}_A$, $v \in W$, $w \in W^*$. We say $\mathcal{N}, v$ is τ -represented by $\mathcal{M}, w$, if there is a function $\pi : W^* \rightarrow W$ s.t. $\pi(w) = v$, and the following are satisfied for all $u \in W^*$:

- (Inv) For any $p \in \mathbf{P}$, $u \in V^*(p)$ iff $\pi(u) \in V(p)$;
- (Forth) For any $(X, Y) \in \text{Nbh}_{\mathcal{M}}(u, \tau)$ s.t. $\pi[X] \cap \pi[Y] = \emptyset$, there is $X' \in N^+(\pi(u))$ and $Y' \in N^-(\pi(u))$ s.t. $(X', Y') \sqsubseteq (\pi[X], \pi[Y])$;
- (Back) For any $X' \in N^+(\pi(u))$ and $Y' \in N^-(\pi(u))$ s.t. $X' \cap Y' = \emptyset$, there is $(X, Y) \in \text{Nbh}_{\mathcal{M}}(u, \tau)$ s.t. $(\pi[X], \pi[Y]) \sqsubseteq (X', Y')$.

With the help of Proposition 3.2, it is rather routine to check the following by induction:

Proposition 5.2. *If a convex neighborhood model $\mathcal{N}, v$ is τ -represented by a Kripke model $\mathcal{M}, w$, then for any $\phi \in \mathcal{L}_{\bigcirc}$, $\mathcal{N}, v \Vdash \phi \iff \mathcal{M}, w \models_{\tau} \phi$.*

5.1 Someone Knows and Disagreement within Group

In this subsection, we fix a non-empty finite set of indices $A \subseteq \rho^{-1}[\{1\}]$, and consider the “someone knows” bundle $\tau_{\text{SK}}(A) = \bigvee_{a \in A} \Box_a(+)$ studied in [42, 4] and the bundle $\tau_{\text{DG}}(A) = \bigvee_{a, b \in A} \Box_a(+) \wedge \Box_b(-)$ for disagreement among a group of agents based on A , where the latter generalizes the bundle for disagreement between two agents studied in [2]. We focus on axiomatizations of the $\tau_{\text{SK}}(A)$ -bundle on S5-models and the $\tau_{\text{DG}}(A)$ -bundle on KD45-models.

Table 1 lists axioms and properties of convex neighborhood models relevant to our axiomatizations. It is routine to check that for any convex neighborhood frame, if it has any property listed in the table, then the corresponding axiom is valid over it. Moreover, we can show that these axioms correspond to the properties *canonically* (the proof of the following lemma is in Appendix D).

Lemma 5.3. *For any convex modal logic $\mathbf{L}$ and any axiom α in Table 1, if all instances of α is in $\mathbf{L}$, then the canonical model of $\mathbf{L}$ has the property corresponding to α .*

Axioms	Properties
EQU $\circ\phi \rightarrow \bullet\phi$	(Sym) $N_w^+ = N_w^-$
N $_{\circ}$ $\circ\top$	(Pur) $^+$ $N_w^+ \neq \emptyset \wedge \emptyset \in N_w^-$
D $_{\circ}$ $\neg\circ\perp$	(Ser) $^+$ $\emptyset \notin N_w^+$
T $_{\circ}$ $\circ\phi \rightarrow \phi$	(Refl) $^+$ $\forall X \in N_w^+ w \in X$
C $_{\circ}^n$ $\bigwedge_{0 \leq i \leq n} \circ\phi_i \rightarrow \bigvee_{0 \leq i < j \leq n} \circ(\phi_i \wedge \phi_j)$	(n-d-Bd) $^+$ $ N_w^+ \leq n$
4' $_{\circ}$ $\circ\phi \rightarrow \circ(\phi \wedge (\circ(\phi \vee \psi) \rightarrow \circ\phi))$	(PI') $^+$ $\forall X \in N_w^+ \forall v \in X ((\exists Y \in N_v^- X \cap Y = \emptyset) \Rightarrow X \in \uparrow N_v^+)$
5C $_{\circ}^n$ $\circ\phi \wedge \bigwedge_{0 \leq i \leq n} \neg\circ(\phi \wedge \psi_i) \rightarrow \circ(\phi \wedge (\bigwedge_{0 \leq i < j \leq n} \circ\psi_i \rightarrow \bigvee_{0 \leq i < j \leq n} \circ(\psi_i \wedge \psi_j)))$	(n-Coa) $^+$ $\forall X \in N_w^+ \forall v \in X N_v^+ \setminus \uparrow\{X\} \leq n$
N $_{\bullet}$, D $_{\bullet}$, T $_{\bullet}$, C $_{\bullet}^n$, 4' $_{\bullet}$, 5C $_{\bullet}^n$ are axioms obtained by replacing $\circ$ with $\bullet$ in the original axioms.	(Pur) $^-$, (Ser) $^-$, (Refl) $^-$, (n-d-Bd) $^-$, (PI') $^-$, (n-Coa) $^-$ are defined by switching $+$ and $-$ in the original definitions.

Table 1: The first set of axioms and corresponding properties. In definitions of the properties, we always omit the first universal quantifier $\forall w \in W$; we also use N_w^* to denote $N^*(w)$ for any $\star \in \{+, -\}$, and $\uparrow N$ to denote $\{Z \subseteq W \mid \exists X \in N X \subseteq Z\}$.

We then define the following two particular systems using axioms in Table 1:

$$\begin{aligned} \mathbf{SKS5}(A) &:= \mathbf{CONV} \oplus \{N_{\circ}, T_{\circ}, C_{\circ}^{|A|}, 4'_{\circ}, 5C_{\circ}^{|A|-1}\} \\ \mathbf{DGD45}(A) &:= \mathbf{CONV} \oplus \{\mathbf{EQU}, D_{\circ}, C_{\circ}^{|A|}, 4'_{\circ}, 5C_{\circ}^{|A|-1}\}. \end{aligned}$$

By the canonical model construction in the previous section and Lemma 5.3, we easily obtain the following neighborhood completeness results:

Lemma 5.4. *Under convex neighborhood semantics, $\mathbf{SKS5}(A)$ is strongly complete w.r.t. the class of core models satisfying $(\text{Pur})^+$, $(\text{Refl})^+$, $(|A|-\text{d-Bd})^+$, $(\text{PI}')^+$, $((|A|-1)-\text{Coa})^+$, and $\mathbf{DGD45}(A)$ is strongly complete w.r.t. the class of core models satisfying (Sym) , $(\text{Ser})^+$, $(|A|-\text{d-Bd})^+$, $(\text{PI}')^+$, $((|A|-1)-\text{Coa})^+$.*

Next, in order to turn the above neighborhood completeness into Kripke completeness under the corresponding bundled semantics, it suffices to show that neighborhood models with the sets of properties listed in Lemma 5.4 can be represented by the corresponding kinds of Kripke models.

Lemma 5.5. *Let $\mathcal{N}, w$ be a core model, and assume that $\mathcal{N}$ satisfies $(|A|-\text{d-Bd})^+$, $(\text{PI}')^+$ and $((|A|-1)-\text{Coa})^+$. If $\mathcal{N}$ also satisfies $(\text{Pur})^+$ and $(\text{Refl})^+$, then $\mathcal{N}$ can be $\tau_{\text{SK}}(A)$ -represented by an S5-Kripke model; on the other hand, if $\mathcal{N}$ also satisfies (Sym) and $(\text{Ser})^+$, then $\mathcal{N}$ can be $\tau_{\text{DG}}(A)$ -represented by a KD45-Kripke model.*

Proof Sketch. Let $\mathcal{N} = (W, N^+, N^-, V)$ be a core model satisfying $(|A|-\text{d-Bd})$, $(\text{PI}')^+$ and $((|A|-1)-\text{Coa})^+$, and let $w \in W$ be arbitrary. When $N^+(w) = \emptyset$, the construction is trivial, so we assume that $N^+(w) \neq \emptyset$. First, as preparation, we show that for any $u \in W$, $X \in N^+(u)$ and $v \in X$, $|N^+(v) \cup \{X\}| \leq |A|$. We consider two cases:

- Case 1: If $X \notin \uparrow N^+(v)$, then by $(\text{PI}')^+$, for any $Y \in N^-(v)$, $X \cap Y \neq \emptyset$. Then, since $\mathcal{N}$ is a core model, there cannot be any $X' \in N^+(v)$ s.t. $X \subseteq X'$ (for otherwise there should also be $Y' \in N^-(v)$ s.t. $X' \cap Y' = \emptyset$, and thus $X \cap Y' = \emptyset$). Hence, by $((|A|-1)-\text{Coa})^+$, $|N^+(v)| < |A|$, and thus $|N^+(v) \cup \{X\}| \leq |A|$.
- Case 2: If $X \in \uparrow N^+(v)$, then there is $Z \in N^+(v)$ s.t. $Z \subseteq X$. Then, since $\mathcal{N}$ is a core model, there is no $X' \in N^+(v)$ s.t. $Z \subseteq X \subsetneq X'$, so $|N^+(v) \setminus \{X\}| = |N^+(v) \setminus \uparrow\{X\}| < |A|$ by $((|A|-1)-\text{Coa})^+$, and thus $|N^+(v) \cup \{X\}| \leq |A|$.

Then, we construct the Kripke model $\mathcal{M}$ using a kind of unraveling. We first recursively define a countable sequence $(W_k, \{R_a^k\}_{a \in A})_{k \in \mathbb{N}}$ s.t. $W_k \subseteq (A \times W)^{k+1}$, R_a^k is a binary relation s.t. $\text{dom}(R_a^k) \subseteq W_k$ and $W_{k+1} \subseteq \bigcup_{a \in A} \text{ran}(R_a^k)$, and for all $\xi \in W_k$, $R_a^k(\xi) \cap W_{k+1} \subseteq \{(\xi, a)\} \times W$ and $\text{end}[R_a^k(\xi)] \in \bigcup_{u \in W} N^+(u)$, where $\text{end}((\xi, a, v)) = v$ for all $(\xi, a, v) \in \bigcup_{k \in \mathbb{N}} (A \times W)^{k+1}$.

- Let $W_0 = \{(c, w)\}$ for some random $c \in A$. By $(|A|-\text{d-Bd})^+$, $|N^+(w)| \leq |A|$, so we may index $N^+(w)$ with A as $\{X_a\}_{a \in A}$. Then, for each $a \in A$, let $R_a^0((c, w)) = \{(c, w, a, v) \mid v \in X_a\} \cup \{(c, w)\}$ if $w \in X_a$, and $R_a^0((c, w)) = \{(c, w, a, v) \mid v \in X_a\}$ otherwise. Let $W_1 = \{(c, w, a, v) \mid a \in A, v \in X_a\}$.
- Assume that $W_k, \{R_a^k\}_{a \in A}$, and W_{k+1} are constructed. For all $\xi \in W_k$, $a \in A$ and $\zeta = (\xi, a, v) \in R_a^k(\xi)$, let $X = \text{end}[R_a^k(\xi)]$; then, since $v \in X \in \bigcup_{u \in W} N^+(u)$, as we proved above, $|N^+(v) \cup \{X\}| \leq |A|$, so we may index $N^+(v) \cup \{X\}$ with A as $\{X_b\}_{b \in A}$, where $X_a = X$. Then, let $R_a^{k+1}(\zeta) = R_a^k(\xi)$, and for each $b \in A \setminus \{a\}$, let $R_b^{k+1}(\zeta) = \{(\zeta, b, u) \mid u \in X_b\} \cup \{\zeta\}$ if $v \in X_b$, and $R_b^{k+1}(\zeta) = \{(\zeta, b, u) \mid u \in X_b\}$ otherwise. Let $W_{k+2} = \{(\zeta, b, u) \mid b \in A \setminus \{a\}, u \in X_b\}$.

Then, let $\mathcal{M} = (W^*, \{R_a^*\}_{a \in A}, V^*)$, where $W^* = \bigcup_{k \in \mathbb{N}} W_k$, $R_a^* = \bigcup_{k \in \mathbb{N}} R_a^k$, and $V^*(p) = W^* \cap \text{end}^{-1}[V(p)]$ for all $p \in \mathbf{P}$. Note that $\mathcal{M}$ is by construction a $KD45$ -model, and when $\mathcal{N}$ satisfies $(\text{Ref})^+$, $\mathcal{M}$ is also an $S5$ -model. Moreover, note that $\{\text{end}[R_a^*((c, w))]\mid a \in A\} = N^+(w)$, and for any $k > 0$ and $\xi \in W_k$ $\{\text{end}[R_a^*(\xi)] \mid a \in A\} = N^+(\text{end}(\xi)) \cup \{X\}$ for some $X \in \bigcup_{u \in W} N^+(u)$ s.t. $\text{end}(\xi) \in X$. Thus, we could check that when $\mathcal{N}$ has the corresponding properties, end (restricted to W^*) satisfies the conditions listed in Definition 5.1 for $\tau_{\text{SK}}(A)$ or $\tau_{\text{DG}}(A)$, so $\mathcal{N}, w$ is $\tau_{\text{SK}}(A)$ -represented or $\tau_{\text{DG}}(A)$ -represented by $\mathcal{M}, (c, w)$ since $\text{end}((c, w)) = w$. $\square$

Then, combining the lemmas above, we obtain the following completeness theorem.

Theorem 5.6. *$\text{SKS5}(A)$ is sound and strongly complete w.r.t. the class of $S5$ -models under $\tau_{\text{SK}}(A)$ -bundled semantics, and $\text{DGD45}(A)$ is sound and strongly complete w.r.t. the class of $KD45$ -models under $\tau_{\text{DG}}(A)$ -bundled semantics.*

Using axioms in Table 1, we can also axiomatize even more complicated bundles, say the bundle $\bigvee_{a \in A} B_a(+) \wedge \bigvee_{b \in B} K_b(-)$ expressing that someone in A erroneously believes in something known to be false by someone else in B . However, we shall not discuss such cases here due to limited space.

5.2 Belief without Knowledge

In this section, we consider the bundled modality $\tau_{\text{MB}} = \Diamond \Box (+) \wedge \Diamond (-)$ over $S4.2$ -models. The class of $S4.2$ -models is noteworthy since it validates a desirable logic of knowledge and belief when we take $\Box \phi$ as the knowledge modality $K\phi$, and define belief $B\phi$ as $\Diamond \Box \phi$ (for more details, see [26, 33] for example). Then, $\Diamond \phi$ with the above bundled semantics expresses $B\phi \wedge \neg K\phi$, which is a kind of *belief without knowledge* or *mere belief*. Note that on $S4.2$ -models, $\Diamond \phi$ is also equivalent to *Dunning-Kruger ignorance* $BK\phi \wedge \neg K\phi$ [11],⁸ and $\Diamond \phi \vee \bullet \phi$ expresses the aforementioned “Rumsfeld ignorance”.

Table 2 lists relevant axioms and properties that are not covered by Table 1. Again, frames with the property corresponding to an axiom in Table 2 validate the axiom. These axioms also correspond to the properties canonically (the proof is again in Appendix D).

Lemma 5.7. *For any convex modal logic $\mathbf{L}$ and any axiom α in Table 2, if all instances of α is in $\mathbf{L}$, then the canonical model of $\mathbf{L}$ has the property corresponding to α .*

⁸In [11], a logic with both the bundle $\neg K\phi \wedge BK\phi$ and $K\phi$ is axiomatized over $S4.2$ -models (which are viewed as epistemic-doxastic models validating certain interaction principles there). But our result here is novel since $K\phi$ is not in our language.

Axioms	Properties
$\text{DIV}''_{\circ} \quad \circ \bigvee_{0 \leq i \leq n} \phi_i \rightarrow \bigvee_{0 \leq i \leq n} \circ \bigvee_{j \neq i} \phi_j$	$(n\text{-s-Bd})^+ \quad \forall X \in N_w^+ X \leq n$
$4''_{\circ} \quad \circ \phi \wedge \circ(\phi \vee \psi) \rightarrow \circ(\phi \wedge \circ(\phi \vee \psi))$	$(\text{PI}'')^+ \quad \forall X \in N_w^+ \forall v \in X (X \in \uparrow N_v^+ \wedge \forall Y \in N_w^- (X \cap Y = \emptyset \Rightarrow Y \in \uparrow N_v^-))$
$4'''_{\circ} \quad \circ \phi \wedge \circ \psi \rightarrow \circ(\phi \wedge \circ \psi)$	$(\text{PI}''')^+ \quad \forall X \in N_w^+ \forall v \in X (N_w^+ \subseteq \uparrow N_v^+ \wedge N_w^- \subseteq \uparrow N_v^-)$
$5'''_{\circ} \quad \circ \phi \wedge \neg \circ \psi \rightarrow \circ(\phi \wedge \neg \circ \psi)$	$(\text{NI}''')^+ \quad \forall X \in N_w^+ \forall v \in X (N_v^+ \subseteq \uparrow N_w^+ \wedge N_v^- \subseteq \uparrow N_w^-)$
$\text{DE}_{\circ} \quad \circ \phi \rightarrow \circ(\phi \wedge \neg \circ \psi)$	$(\text{Emp})^+ \quad \forall X \in N_w^+ \forall v \in X (N_v^+ = N_v^- = \emptyset)$
$\text{T}^{\text{op}}_{\circ} \quad \phi \wedge \circ(\phi \vee \psi) \rightarrow \circ \phi$	$(\text{Id})^+ \quad (\exists Y \in N_w^- w \notin Y) \Rightarrow \{w\} \in \uparrow N_w^+$
● and -- versions of the above axioms and properties are defined in the same way as Table 1.	

Table 2: The second set of axioms and corresponding properties. Conventions for the abbreviations we use are the same as Table 1.

We define the following two systems using the axioms presented in Table 1 and Table 2 (note that their only difference is that $4''_{\bullet}$ is strengthened to $4'''_{\bullet}$ in **MBS4F**):

$$\begin{aligned} \mathbf{MBS4.2} &:= \mathbf{CONV} \oplus \{\text{D}_{\circ}, \text{C}_{\circ}^1, \text{DE}_{\circ}, \text{D}_{\bullet}, \text{DIV}_{\bullet}^1, 4''_{\bullet}, 5'''_{\bullet}, \text{T}_{\bullet}^{\text{op}}\} \\ \mathbf{MBS4F} &:= \mathbf{CONV} \oplus \{\text{D}_{\circ}, \text{C}_{\circ}^1, \text{DE}_{\circ}, \text{D}_{\bullet}, \text{DIV}_{\bullet}^1, 4'''_{\bullet}, 5'''_{\bullet}, \text{T}_{\bullet}^{\text{op}}\}. \end{aligned}$$

Again, using the canonical construction and Lemma 5.3, Lemma 5.7, we could obtain the following neighborhood completeness:

Lemma 5.8. *MBS4.2 is strongly complete w.r.t. the class of core models satisfying $(\text{Ser})^+$, $(1\text{-d-Bd})^+$, $(\text{Emp})^+$, $(\text{Ser})^-$, $(1\text{-s-Bd})^-$, $(\text{PI}'')^-$, $(\text{NI}''')^-$ and $(\text{Id})^-$, and MBS4F is strongly complete w.r.t. the class of core models satisfying $(\text{Ser})^+$, $(1\text{-d-Bd})^+$, $(\text{Emp})^+$, $(\text{Ser})^-$, $(1\text{-s-Bd})^-$, $(\text{PI}''')^-$, $(\text{NI}''')^-$ and $(\text{Id})^-$.*

Then, the Kripke completeness of the above logics under bundled semantics follows from the representation result below.

Lemma 5.9. *Let N, w be a core model s.t. $\mathcal{N}$ satisfies $(\text{Ser})^+$, $(1\text{-d-Bd})^+$, $(\text{Emp})^+$, $(\text{Ser})^-$, $(1\text{-s-Bd})^-$, $(\text{PI}'')^-$, $(\text{NI}''')^-$ and $(\text{Id})^-$. Then, $\mathcal{N}, w$ can be τ_{MB} -represented by an S4.2-Kripke model; moreover, if $\mathcal{N}$ further satisfies $(\text{PI}''')^-$, then $\mathcal{N}, w$ can be τ_{MB} -represented by an S4F-Kripke model.⁹*

Proof Sketch. Assume that $\mathcal{N} = (W, N^+, N^-, V)$ is a core model satisfying $(\text{Ser})^+$, $(1\text{-d-Bd})^+$, $(\text{Emp})^+$, $(\text{Ser})^-$, $(1\text{-s-Bd})^-$, $(\text{PI}'')^-$, $(\text{NI}''')^-$ and $(\text{Id})^-$, and let $w \in W$ be arbitrary. When $N^+(w) = \emptyset$, the case is trivial, so we assume that $N^+(w) \neq \emptyset$.

Since $N^+(w) \neq \emptyset$, by $(\text{Ser})^+$ and $(1\text{-d-Bd})^+$, there is a non-empty $X_w \subseteq W$ s.t. $N^+(w) = \{X_w\}$. Then, Let R be the binary relation on W s.t. vRu iff $\{u\} \in N^-(v)$. Note that R is transitive: if vRu and uRs , i.e. $\{u\} \in N^-(v)$ and $\{s\} \in N^-(u)$, then by $(\text{NI}''')^-$ and $(\text{Ser})^-$, $\{s\} \in N^-(v)$, i.e. vRs . Then, consider $R(w)$. We show that $w \in R(w)$, and for any $v \in R(w)$ we have vRv and $N^+(v) = \{X_w\}$. By $(\text{Emp})^+$, for any $u \in X_w$, $N^+(u) = \emptyset$. Hence, by our assumption that $N^+(w) \neq \emptyset$, $w \notin X_w$. Thus, by $(\text{Id})^-$ and $(\text{Ser})^-$, $\{w\} \in N^-(w)$, i.e. wRw . Next, take an arbitrary $v \in R(w)$. Then, $\{v\} \in N^-(w)$, so by $(\text{PI}'')^-$ and $(\text{Ser})^-$, $\{v\} \in N^-(v)$, i.e. vRv . Moreover, since $\mathcal{N}$ is a core model and $\{v\} \in N^-(w)$, clearly $\{v\} \cap X_w = \emptyset$, so by $(\text{PI}'')^-$, $X_w \in \uparrow N^+(v)$, i.e. there is $X \in N^+(v)$ s.t. $X \subseteq X_w$. Then, by $(\text{NI}''')^-$, $X \in N^+(w)$, so $X = X_w$ since $\mathcal{N}$ is a core model, and thus $X_w \in N^+(v)$. Thus, by $(1\text{-d-Bd})^+$, $N^+(v) = \{X_w\}$. Finally, note that when $\mathcal{N}$ further satisfies $(\text{PI}''')^-$, R becomes Euclidean: if vRu and vRs , i.e. $\{u\} \in N^-(v)$ and

⁹ $\mathcal{M} = (W, R, V)$ is an S4F-model if there is $X \subseteq W$ s.t. $R = (W \setminus X)^2 \cup (W \times X)$. In [33], such models are viewed as the maximal epistemic extensions of KD45-doxastic models. An intuitive interpretation of such models can also be found in [33].

$\{s\} \in N^-(v)$, then by $(PI''')^-$ and $(Ser)^-$, $\{s\} \in N^-(u)$, i.e. uRs . Then, since $w \in R(w)$, it follows that $R \cap (R(w))^2 = (R(w))^2$.

Then, we define the Kripke model we need as $\mathcal{M} = (W^*, R^*, V^*)$, where $W^* = R(w) \cup X_w$, $R^* = (R \cap (R(w))^2) \cup (W^* \times X_w)$, and $V^*(p) = V(p) \cap W^*$ for all $p \in \mathbf{P}$. By $(Ser)^+$, $X_w \neq \emptyset$, so $\mathcal{M}$ according to what we proved above, $\mathcal{M}$ is indeed an S4.2-model; moreover, when $\mathcal{N}$ satisfies $(PI''')^-$, $R \cap (R(w))^2 = (R(w))^2 = (W^* \setminus X_w)^2$, so $\mathcal{M}$ is an S4F-model. Then, we could check that the identity function (restricted to W^*) satisfies the conditions listed in Definition 5.1 for τ_{MB} , so $\mathcal{N}, w$ is τ_{MB} -represented by $\mathcal{M}, w$. $\square$

Theorem 5.10. *Under τ_{MB} -bundled semantics, **MBS4.2** is sound and strongly complete w.r.t. the class of S4.2-models, and **MBS4F** is sound and strongly complete w.r.t. the class of S4F-models.¹⁰*

6 Conclusion

In this paper, we have studied propositional bundled modalities at a general level. We showed how to view bundled semantics as a kind of neighborhood semantics and uniformly defined bisimulations for bundled modalities from this viewpoint. We also studied the expressivity and axiomatizations of PN-independent bundles and offered axiomatizations for several concrete PN-independent bundles using our general method.

There are several directions for future work. One is to further study the bundled bisimulation we defined. For example, a *characterization theorem* for these bisimulations is highly desirable. A source of difficulty here is that notions of modal saturation for ordinary neighborhood models appear too strong for the neighborhood structures generated by bundle terms, so the usual ultrapower construction might not work here.

It also seems interesting to further study the class of PN-independent bundles. For example, it might be interesting to try to offer a purely syntactic characterization of the class of PN-independent bundles over a given class of Kripke models — though a grammar to generate PN-independent bundles is given in Proposition 4.2, it is still unclear whether the bundles it generates exhaust all PN-independent bundles (up to logical equivalence) over, say, the class of all models, or the class of serial models.

It should also be interesting to try to find more uniform ways to generate axiomatizations for bundled modalities. For example, if we could prove Sahlqvist-style correspondence and completeness theorems for convex neighborhood logics, and at the same time develop a method to automatically obtain representation results for certain bundles, then we could immediately obtain a great amount of complete axiomatizations for these bundles.

Finally, we believe the class of convex neighborhood logics deserves a more fully developed theory. On the one hand, some fine distinctions can only be made in this class of logics. For example, the axioms $4'_\circ$ and $4''_\circ$ we introduced above both collapse to $\circ\phi \rightarrow \circ(\phi \wedge \circ\phi)$ in monotonic logics with $N_\circ$, and $4'_\circ$, $4''_\circ$, $4'''_\circ$ all collapse to $\circ\phi \rightarrow \circ\circ\phi$ in normal modal logics. On the other hand, logics in this class admit adequate semantics based on core models, so we may hope for a more fruitful theory than that of all neighborhood modal logics. Hence, it should be interesting to further study the general theory of convex modal logics.

¹⁰Strictly speaking, the Kripke *soundness* of **MBS4.2** and **MBS4F** under τ_{MB} -bundled semantics does not follow directly from their neighborhood soundness w.r.t. models with the corresponding properties listed in Table 1 and Table 2, since properties of the τ_{MB} -generated neighborhoods of S4.2- and S4F-Kripke models might be weaker than the ones listed in the tables. Nevertheless, they are strong enough to validate all corresponding axioms, which is rather routine to check, so we leave the details to the reader.

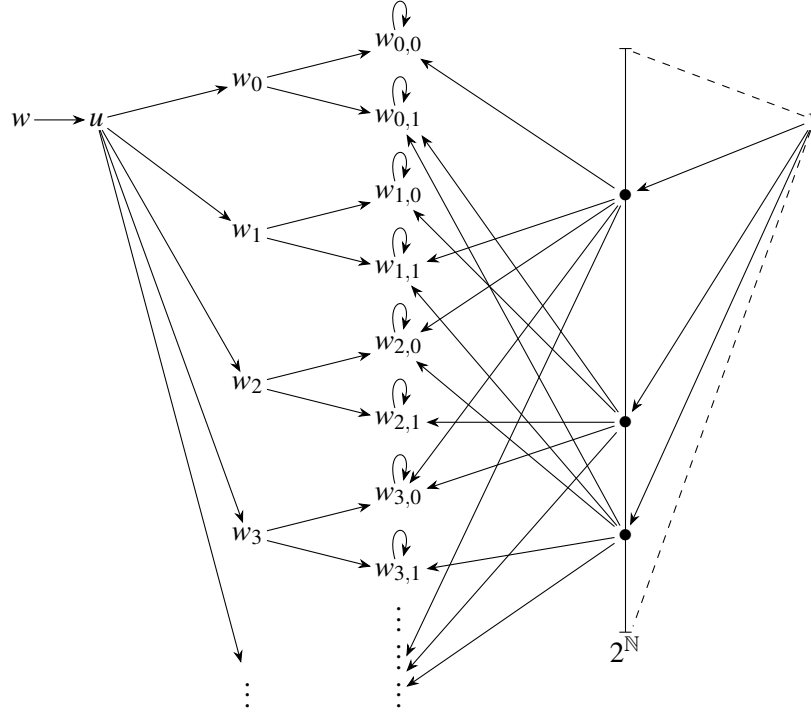


Figure 1: A schematic picture of the Kripke frame (W, R) of $\mathcal{M}$ (the valuation is empty), in which $2^{\mathbb{N}}$ is represented by a segment of the real line.

A First-Order Undefinability of Bundled Bisimulation

In this section we consider only bisimulation for $\tau = \Diamond \Box \Diamond (+)$ and show that it is not first-order definable in the sense that there is no first-order formula Φ with predicates for R and $V(p)$ ($p \in \mathbf{P}$) in Kripke models and a special predicate Z such that Φ is satisfied by a Kripke model iff the interpretation of Z is a τ -bisimulation on that model.

Let $\mathcal{M}$ be the Kripke model (W, R, V) where (as usual $2 = \{0, 1\}$)

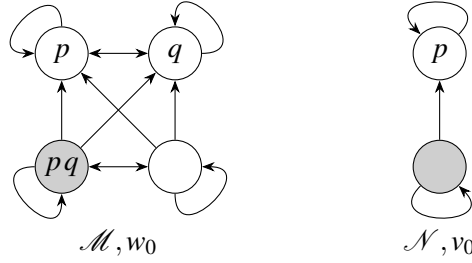
$$\begin{aligned} W &= \{w, u, v\} \cup \{w_i \mid i \in \mathbb{N}\} \cup \{w_{i,j} \mid i \in \mathbb{N}, j \in \{0, 1\}\} \cup 2^{\mathbb{N}}. \\ R &= \{(w, u)\} \cup \{(u, w_i) \mid i \in \mathbb{N}\} \cup \{(w_i, w_{i,j}), (w_{i,j}, w_{i,j}) \mid i \in \mathbb{N}, j \in \{0, 1\}\} \cup \\ &\quad (\{v\} \times 2^{\mathbb{N}}) \cup \{(f, w_{i,f(i)}) \mid f \in 2^{\mathbb{N}}, i \in \mathbb{N}\}. \\ V &= \mathbf{P} \times \{\emptyset\}. \end{aligned}$$

Note that $\text{Nbh}_{\mathcal{M}}(w, \tau) = \text{Nbh}_{\mathcal{M}}(v, \tau) = \{(\{w_{i,f(i)} \mid i \in \mathbb{N}\}, \emptyset) \mid f \in 2^{\mathbb{N}}\}$ and $\text{Dom}_{\mathcal{M}}(w, \tau) = \text{Dom}_{\mathcal{M}}(v, \tau) = \{w_{i,j} \mid i \in \mathbb{N}, j \in \{0, 1\}\}$. Then, it is not hard to check that the binary relation $\mathcal{Z}$ defined as follows is a τ -bisimulation on $\mathcal{M}$:

$$\mathcal{Z} = \{(w, v)\} \cup \{(w_{i,j}, w_{i,j}) \mid i \in \mathbb{N}, j \in \{0, 1\}\}.$$

Observe also that when the part of $\mathcal{M}$ restricted to $W \setminus 2^{\mathbb{N}}$ remains intact, removing any $f \in 2^{\mathbb{N}}$ from the model would result in $\mathcal{Z}$ being no longer a τ -bisimulation.

If τ -bisimulation were first-order definable, then we could take the Skolem hull of $\mathcal{M}$ from $W \setminus 2^{\mathbb{N}}$, and $\mathcal{Z}$ would still be a τ -bisimulation in this Skolem hull. But such a Skolem hull is a countable

Figure 2: A diagram of the countermodels $\mathcal{M}, w_0$ and $\mathcal{N}, v_0$.

submodel of $\mathcal{M}$ and thus must omit one (indeed most) $f \in 2^{\mathbb{N}}$, so $\mathcal{Z}$ in this submodel cannot be a τ -bisimulation. Hence, bisimulation for $\tau = \Diamond \Box \Diamond (+)$ is not first-order definable.

B Non-PN-Independence of Rumsfeld Ignorance

Let τ_{RI} stand for the bundle $\Diamond(+)\wedge\Diamond(-)\wedge\Diamond(\Box(+)\vee\Box(-))$ of Rumsfeld ignorance, and let $\tau_{\text{RI}}^{\text{op}}$ stand for its dual $\Box(+)\vee\Box(-)\vee\Box(\Diamond(+)\wedge\Diamond(-))$. In order to show that τ_{RI} (resp. $\tau_{\text{RI}}^{\text{op}}$) is not PN-independent over a model, it suffices to show that the axiom of convexity $\text{CONV} : \Box(\phi \wedge \psi) \wedge \Box(\phi \vee \chi) \rightarrow \Box\phi$ can be falsified in the model in question under τ_{RI} -bundled (resp. $\tau_{\text{RI}}^{\text{op}}$ -bundled) semantics.

The countermodels are the following ones: let $\mathcal{M} = (W, R, V)$ and $\mathcal{N} = (W', R', V')$, where

- $W = \{w_0, w_1, w_2, w_3\}$, $W' = \{v_0, v_1\}$;
- $R = \{w_0, w_1\}^2 \cup (W \times \{w_2, w_3\})$, $R' = \{v_0\}^2 \cup (W' \times \{v_1\})$;
- $V(p) = \{w_0, w_2\}$, $V(q) = \{w_0, w_3\}$, and $V'(p) = \{v_1\}$, $V'(q) = \emptyset$.

See Figure 2 for a diagram of the above models.

Then, $\mathcal{M}, w_0 \not\models_{\tau_{\text{RI}}} \Box(p \wedge q) \wedge \Box(p \vee q) \rightarrow \Box p$, and $\mathcal{N}, v_0 \not\models_{\tau_{\text{RI}}^{\text{op}}} \Box(p \wedge q) \wedge \Box(p \vee \neg q) \rightarrow \Box p$.

C Proof for Proposition 4.5

Proof. It suffices to show that for any model $\mathcal{M}$, any $\tau \in \mathcal{T}_{\mathbf{A}}$, any binary relation $\mathcal{Z}$ on $\mathcal{M}$ and any $(w, v) \in \mathcal{Z}$, the conditions $(\tau\text{-Zig})$ and $(\tau\text{-Zag})$ imply $(\tau\text{-Coh})$, and the converse holds when τ is PN-independent over $\mathcal{M}$. Without loss of generality, we may also assume that $\mathcal{Z}$ is an equivalence relation, since we can always use the bisimilarity relation, which is itself a bisimulation.

First, assume that $(w, v) \in \mathcal{Z}$ satisfies $(\tau\text{-Zig})$ and $(\tau\text{-Zag})$. Also assume that $X \subseteq \text{Dom}_{\mathcal{M}}(w, \tau)$ and $Y \subseteq \text{Dom}_{\mathcal{M}}(v, \tau)$ are $\mathcal{Z}$ -coherent, and $X \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$. We show that $Y \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$. By definition, there is $(X_0, X_1) \in \text{Nbh}_{\mathcal{M}}(w, \tau)$ s.t. $(X_0, X_1) \sqsubseteq (X, \text{Dom}_{\mathcal{M}}(w, \tau) \setminus X)$, so $\mathcal{Z}[X_0] \cap X_1 = \emptyset$ by the $\mathcal{Z}$ -coherence of X and Y (in particular, $\mathcal{Z}[X] \cap \text{Dom}_{\mathcal{M}}(w, \tau) \subseteq X$). Hence, by $(\tau\text{-Zig})$, there is $(Y_0, Y_1) \in \text{Nbh}_{\mathcal{M}}(v, \tau)$ s.t. $Y_0 \subseteq \mathcal{Z}[X_0]$, so $Y_0 \subseteq \mathcal{Z}[X_0] \cap \text{Dom}_{\mathcal{M}}(v, \tau) \subseteq \mathcal{Z}[X] \cap \text{Dom}_{\mathcal{M}}(v, \tau) \subseteq Y$; also, $Y_1 \subseteq \mathcal{Z}[X_1]$, and thus $Y_1 \subseteq \mathcal{Z}^{-1}[X_1]$ since $\mathcal{Z}$ is symmetric, which implies that $Y_1 \cap Y = \emptyset$, since otherwise there would be $w' \in X_1$ s.t. $w' \in \mathcal{Z}[Y] \cap \text{Dom}_{\mathcal{M}}(w, \tau) \subseteq X$, causing a contradiction. Therefore, we have $(Y_0, Y_1) \sqsubseteq (Y, \text{Dom}_{\mathcal{M}}(v, \tau) \setminus Y)$, and thus $Y \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$. The converse direction of $(\tau\text{-Coh})$ can be proved symmetrically using $(\tau\text{-Zag})$.

Next, assume that τ is PN-independent over $\mathcal{M}$ and $(w, v) \in \mathcal{Z}$ satisfies $(\tau\text{-Coh})$. We only show that (w, v) satisfies $(\tau\text{-Zig})$, since the case for $(\tau\text{-Zag})$ is symmetric. Let $(X_0, X_1) \in \text{Nbh}_{\mathcal{M}}(w, \tau)$ be arbitrary, and assume that $\mathcal{Z}[X_0] \cap X_1 = \emptyset$. First, let $X = \mathcal{Z}[X_0] \cap \text{Dom}_{\mathcal{M}}(w, \tau)$ and $Y = \mathcal{Z}[X_0] \cap \text{Dom}_{\mathcal{M}}(v, \tau)$. It is not hard to see that X and Y are $\mathcal{Z}$ -coherent and $X \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$. Thus, we have $Y \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$, so there is $(Y_0, Y_1) \in \text{Nbh}_{\mathcal{M}}(v, \tau)$ s.t. $Y_0 \subseteq Y \subseteq \mathcal{Z}[X_0]$. Next, let $X' = \text{Dom}_{\mathcal{M}}(w, \tau) \setminus \mathcal{Z}[X_1]$ and $Y' = \text{Dom}_{\mathcal{M}}(v, \tau) \setminus \mathcal{Z}[X_1]$. Again, X' and Y' are $\mathcal{Z}$ -coherent and $X' \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(w, \tau)$, so $Y' \in \text{Nbh}_{\mathcal{M}}^{\text{com}}(v, \tau)$. Hence, there is also $(Y'_0, Y'_1) \in \text{Nbh}_{\mathcal{M}}(v, \tau)$ s.t. $Y'_1 \subseteq \text{Dom}_{\mathcal{M}}(v, \tau) \setminus Y' \subseteq \mathcal{Z}[X_1]$. Thus, $(Y_0, Y'_1) \subseteq (\mathcal{Z}[X_0], \mathcal{Z}[X_1])$. Finally, note that $\mathcal{Z}[X_0] \cap \mathcal{Z}[X_1] = \emptyset$, so $Y_0 \cap Y'_1 = \emptyset$. Then, since τ is PN-independent over $\mathcal{M}$, $(Y_0, Y'_1) \in \text{Nbh}_{\mathcal{M}}(v, \tau)$. $\square$

D Proofs for Lemma 5.3 and Lemma 5.7

The following observations are useful.

Proposition D.1. *For any maximally consistent set Δ and $F \in \text{Core}^+(\Delta)$,*

- (i) *There is $\phi \in F$ s.t. $\bigcirc\phi \in \Delta$, and for any $\phi, \psi \in F$, if $\bigcirc\phi \in \Delta$, then $\bigcirc(\phi \wedge \psi) \in \Delta$.¹¹*
- (ii) *For any χ , if for all $\phi \in F$, there is $\psi \in F$ s.t. $\vdash \psi \rightarrow \phi$ and $\bigcirc(\psi \wedge \chi) \in \Delta$, then $\chi \in F$.*

The above also hold after replacing ‘+’ with ‘−’ and $\bigcirc$ with $\bullet$.

Proof. We only prove the case for $+$ -coherent filters, since the case for $-$ -coherent filters is symmetric.

First, assume that F is a filter $+$ -coherent with Δ . Since $\top \in F$, by $+$ -coherence, there is $\phi \in F$ s.t. $\bigcirc(\top \wedge \phi) \in \Delta$ and thus $\bigcirc\phi \in \Delta$ by RE. Moreover, for any $\phi, \psi \in F$, we have $\phi \wedge \psi \in F$, so by $+$ -coherence, there is $\chi \in F$ s.t. $\bigcirc(\phi \wedge \psi \wedge \chi) \in \Delta$. Thus, if $\bigcirc\phi \in \Delta$, by CONV and RE, we also have $\bigcirc(\phi \wedge \psi) \in \Delta$.

Next, assume that for any $\phi \in F$, there is $\psi \in F$ s.t. $\vdash \psi \rightarrow \phi$ and $\bigcirc(\psi \wedge \chi) \in \Delta$. Then, note that the filter G generated by $F \cup \{\chi\}$ is also $+$ -coherent with Δ , and $F \subseteq G$. Then, by the maximality of F , we must have $F = G$, so $\chi \in F$. $\square$

We now present the proofs for Lemma 5.3 and Lemma 5.7. Proofs for the canonicity of $N_{\bigcirc}, D_{\bigcirc}, T_{\bigcirc}$ are omitted since they are rather trivial. Moreover, we only prove canonicity for $\bigcirc$ -axioms, since the cases for their $\bullet$ -counterparts are completely symmetric. Also note that the order in which the following axioms are dealt is slightly different from the order they are introduced in Table 1 and Table 2.

$$(1) \quad \boxed{\text{EQU} \quad \bigcirc\phi \rightarrow \bullet\phi} \quad \boxed{(\text{Sym}) \quad N_w^+ = N_w^-}$$

Proof. Just notice that when EQU is an axiom, we have $\vdash \bigcirc\phi \leftrightarrow \bullet\phi$ for all $\phi \in \mathcal{L}_{\bigcirc}$, so for any filter F , F is $+$ -coherent with Δ iff it is $-$ -coherent with Δ . Hence, we obviously have $\text{Core}^+(\Delta) = \text{Core}^-(\Delta)$, and thus $N_{\Delta}^+ = N_{\Delta}^-$. $\square$

$$(2) \quad \boxed{\mathcal{C}_{\bigcirc}^n \quad \bigwedge_{0 \leq i \leq n} \bigcirc\phi_i \rightarrow \bigvee_{0 \leq i < j \leq n} \bigcirc(\phi_i \wedge \phi_j)} \quad \boxed{(n\text{-d-Bd})^+ \quad |N_w^+| \leq n}$$

Proof. Suppose that there are $F_0, \dots, F_n \in \text{Core}^+(\Delta)$ which are pairwise distinct. For each $i \leq n$, we list formulas in F_i as $(\phi_i^k)_{k \in \mathbb{N}}$; by (i) of Proposition D.1, we may also assume that $\bigcirc\phi_i^0 \in \Delta$. Then, let $\psi_i^m = \bigwedge_{k \leq m} \phi_i^k$; by (i) of Proposition D.1, $\bigcirc\psi_i^m \in \Delta$. Then, for any $m \in \mathbb{N}$, we have $\bigwedge_{0 \leq i \leq n} \bigcirc\psi_i^m \in \Delta$, so by $\mathcal{C}_{\bigcirc}^n$, there are $0 \leq i < j \leq n$ s.t. $\bigcirc(\psi_i^m \wedge \psi_j^m) \in \Delta$. Thus, there must be some $0 \leq i < j \leq n$ s.t.

¹¹In other words, $\{\phi \in F \mid \bigcirc\phi \in \Delta\}$ is not only dense, but also *open dense* in F , i.e. it is downward closed in F .

for infinitely many $m \in \mathbb{N}$, $\circ(\psi_i^m \wedge \psi_j^m) \in \Delta$. Then, consider the filter G generated by $F_i \cup F_j$. For any $\phi_i^k \in F_i$, $\phi_j^l \in F_j$, we can find $m > \max\{k, l\}$ s.t. $\circ(\psi_i^m \wedge \psi_j^m) \in \Delta$. Also note that $\vdash \psi_i^m \wedge \psi_j^m \rightarrow \phi_i^k \wedge \phi_j^l$ and $\psi_i^m \wedge \psi_j^m \in G$. Thus, G is $+$ -coherent with Δ , so by the maximality of F_i and F_j , we must have $F_i = F_j = G$, causing a contradiction. $\square$

$$(3) \quad \boxed{\text{DIV}_{\circ}^n \quad \circ \bigvee_{i \leq n} \phi_i \rightarrow \bigvee_{i \leq n} \circ \bigvee_{j \neq i} \phi_j \quad | \quad (n\text{-s-Bd})^+ \quad \forall X \in N_w^+ | X| \leq n}$$

Proof. Let $F \in \text{Core}^+(\Delta)$ be arbitrary, and suppose that there are pairwise distinct $\Theta_0, \dots, \Theta_n \in \|F\|$. Then, we can find $\theta_0, \dots, \theta_n$ s.t. $\vdash \bigvee_{i \leq n} \theta_i$, and for all $i \neq j \leq n$, $\theta_i \in \Theta_i$ and $\theta_j \notin \Theta_j$. Now, enumerate formulas in F as $(\phi_k)_{k \in \mathbb{N}}$; by (i) of Proposition D.1, we may also assume that $\circ\phi_0 \in \Delta$. Then, let $\psi_m = \bigwedge_{k \leq m} \phi_k$; by (i) of Proposition D.1, $\circ\psi_m \in \Delta$. Then, by RE and $\text{DIV}_{\circ}^n$, for each $m \in \mathbb{N}$, there is $i \leq n$ s.t. $\circ(\psi_m \wedge \bigvee_{j \neq i} \theta_j) \in \Delta$. Hence, there is some $i \leq n$ for which there are infinitely many $m \in \mathbb{N}$ s.t. $\circ(\psi_m \wedge \bigvee_{j \neq i} \theta_j) \in \Delta$. Then, for any $\phi \in F$, there is m s.t. $\vdash \psi_m \rightarrow \phi$ and $\circ(\psi_m \wedge \bigvee_{j \neq i} \theta_j) \in \Delta$, so by (ii) of Proposition D.1, $\bigvee_{j \neq i} \theta_j \in F$. But then, since $F \subseteq \Theta_i$, $\bigvee_{j \neq i} \theta_j \in \Theta_i$, causing a contradiction. $\square$

$$(4) \quad \boxed{\begin{array}{l} 4'_{\circ} \quad \circ\phi \rightarrow \circ(\phi \wedge (\circ(\phi \vee \psi) \rightarrow \circ\phi)) \\ (\text{PI}')^+ \quad \forall X \in N_w^+ \forall v \in X (\exists Y \in N_v^- (X \cap Y = \emptyset) \Rightarrow X \in \uparrow N_v^+) \end{array}}$$

Proof. Let $F \in \text{Core}^+(\Delta)$ and $\Theta \in \|F\|$ be arbitrary. Also assume that there is $F' \in \text{Core}^-(\Theta)$ s.t. $\|F\| \cap \|F'\| = \emptyset$. Our goal is to find $G \in \text{Core}^+(\Theta)$ extending F , so that $\|G\| \subseteq \|F\|$, and by Lemma 4.12, it is enough to show that F is $+$ -coherent with Θ . Since $\|F\| \cap \|F'\| = \emptyset$, there is a $\phi \in F$ s.t. $\neg\phi \in F'$. Since F' is $--$ -coherent with Θ , we may also assume that $\bullet\neg\phi \in \Theta$, i.e. $\circ\phi \in \Theta$. Now, to show that F is $+$ -coherent with Θ , it is enough to show that for every $\psi \in F$, $\circ\phi \rightarrow \circ(\phi \wedge \psi) \in F$ since this implies that $\circ(\phi \wedge \psi) \in \Theta$. To do this, we use (ii) of Proposition D.1. Let $\chi \in F$ be arbitrary. By $+$ -coherence, there is $\theta \in F$ s.t. $\vdash \theta \rightarrow \phi \wedge \psi \wedge \chi$ and $\circ\theta \in \Delta$ (note $\phi, \psi, \chi \in F$). Thus, by $4'_{\circ}$ with θ for ϕ and ϕ for ψ and detaching $\circ\theta \in \Delta$ and replacing $\theta \vee \phi$ by the equivalent ϕ , $\circ(\theta \wedge (\circ\phi \rightarrow \circ\theta)) \in \Delta$. Note that, by CONV, $\vdash (\circ\phi \rightarrow \circ\theta) \rightarrow (\circ\phi \rightarrow \circ(\phi \wedge \psi))$, so by CONV again (easier to see in its rule form), we have $\vdash \circ\theta \wedge \circ(\theta \wedge (\circ\phi \rightarrow \circ\theta)) \rightarrow \circ(\theta \wedge (\circ\phi \rightarrow \circ(\phi \wedge \psi)))$. Summarizing above, we have both $\vdash \theta \rightarrow \chi$ and $\circ(\theta \wedge (\circ\phi \rightarrow \circ(\phi \wedge \psi))) \in \Delta$, completing the requirement for (ii) of Proposition D.1. So $\circ\phi \rightarrow \circ(\phi \wedge \psi) \in F$. $\square$

$$(5) \quad \boxed{\begin{array}{l} 4''_{\circ} \quad \circ\phi \wedge \circ(\phi \vee \psi) \rightarrow \circ(\phi \wedge \circ(\phi \vee \psi)) \\ (\text{PI}'')^+ \quad \forall X \in N_w^+ \forall v \in X (X \in \uparrow N_v^+ \wedge \forall Y \in N_v^- (X \cap Y = \emptyset) \Rightarrow Y \in \uparrow N_v^-) \end{array}}$$

Proof. Let $F \in \text{Core}^+(\Delta)$ and $\Theta \in \|F\|$ be arbitrary. First, for any $\phi, \psi \in F$, if $\circ\phi \in \Delta$, then $\circ(\phi \wedge \psi) \in \Delta$ by (i) of Proposition D.1, and thus, by $4''_{\circ}$ ($\phi \wedge \psi$ for ϕ) and RE, $\circ(\phi \wedge \psi \wedge \circ\phi) \in \Delta$. Hence, by (ii) of Proposition D.1, $\circ\phi \in F \subseteq \Theta$. Then, it is not hard to see that F is also $+$ -coherent with Θ , so by Lemma 4.12, there is $G \in \text{Core}^+(\Theta)$ extending F , and thus $\|G\| \subseteq \|F\|$.

Next, let $F' \in \text{Core}^-(\Delta)$ be arbitrary, and assume that $\|F\| \cap \|F'\| = \emptyset$. Then, there is $\phi \in F$ s.t. $\neg\phi \in F'$, and we may assume that $\bullet\neg\phi \in \Delta$ (i.e. $\circ\phi \in \Delta$) by the $--$ -coherence of F' . Then, for any $\psi \in F'$, we have $\bullet(\neg\phi \wedge \psi) \in \Delta$ by (i) of Proposition D.1, i.e. $\circ(\phi \vee \neg\psi) \in \Delta$. Then, for any $\chi \in F$, since $\circ(\phi \wedge \chi) \in \Delta$, by $4''_{\circ}$, $\circ(\phi \wedge \chi \wedge \bullet(\neg\phi \wedge \psi)) \in \Delta$. Thus, by (ii) of Proposition D.1, $\bullet(\neg\phi \wedge \psi) \in F \subseteq \Theta$. In sum, $\psi \in F' \Rightarrow \bullet(\neg\phi \wedge \psi) \in \Theta$, so F' is $--$ -coherent with Θ , so there is $G' \in \text{Core}^-(\Theta)$ extending F' , and thus $\|G'\| \subseteq \|F'\|$. $\square$

$$(6) \quad \boxed{4'''_{\circ} \quad \circ\phi \wedge \circ\psi \rightarrow \circ(\phi \wedge \circ\psi) \quad | \quad (\text{PI}''')^+ \quad \forall X \in N_w^+ \forall v \in X (N_v^+ \subseteq \uparrow N_v^+ \wedge N_v^- \subseteq \uparrow N_v^-)}$$

Proof. Let $F \in \text{Core}^+(\Delta)$ and $\Theta \in \|F\|$ be arbitrary. By $4'''$ and (ii) of Proposition D.1, for any $\bigcirc\psi \in \Delta$, we have $\bigcirc\psi \in F \subseteq \Theta$. Thus, if a filter G is $+$ -coherent (resp. $-$ -coherent) with Δ , it would also be $+$ -coherent (resp. $-$ -coherent) with Θ , so by Lemma 4.12, there is $H \in \text{Core}^+(\Theta)$ (resp. $\text{Core}^-(\Theta)$) extending G , and thus $\|H\| \subseteq \|G\|$. $\square$

$$(7) \quad \boxed{5'''_{\bigcirc} \quad \bigcirc\phi \wedge \neg\bigcirc\psi \rightarrow \bigcirc(\phi \wedge \neg\bigcirc\psi) \quad | \quad (\text{NI}''')^+ \quad \forall X \in N_w^+ \forall v \in X (N_v^+ \subseteq \uparrow N_w^+ \wedge N_v^- \subseteq \uparrow N_w^-)}$$

Proof. Let $F \in \text{Core}^+(\Delta)$ and $\Theta \in \|F\|$ be arbitrary. By $5'''$ and (ii) of Proposition D.1, for any $\bigcirc\psi \notin \Delta$, we have $\neg\bigcirc\psi \in F \subseteq \Theta$, and thus $\bigcirc\psi \notin \Theta$. Thus, if G is $+$ -coherent (resp. $-$ -coherent) with Θ , it would also be $+$ -coherent (resp. $-$ -coherent) with Δ , so by Lemma 4.12, there is $H \in \text{Core}^+(\Delta)$ (resp. $\text{Core}^-(\Delta)$) extending G , and thus $\|H\| \subseteq \|G\|$. $\square$

$$(8) \quad \boxed{\begin{array}{l} 5C_{\bigcirc}^n \quad \bigcirc\phi \wedge \bigwedge_{0 \leq i \leq n} \neg\bigcirc(\phi \wedge \psi_i) \rightarrow \bigcirc(\phi \wedge (\bigwedge_{0 \leq i \leq n} \bigcirc\psi_i \rightarrow \bigvee_{0 \leq i < j \leq n} \bigcirc(\psi_i \wedge \psi_j))) \\ (n\text{-Coa}) \quad \forall X \in N_w^+ \forall v \in X |N_v^+ \setminus \uparrow\{X\}| \leq n \end{array}}$$

Proof. Let $F \in \text{Core}^+(\Delta)$, $\Theta \in \|F\|$ be arbitrary, and suppose that there are pairwise distinct $G_0, \dots, G_n \in \text{Core}^+(\Theta)$ s.t. for each $i \leq n$, $\|F\| \not\subseteq \|G_i\|$. Then, for each $i \leq n$, there is $\phi_i^0 \in G_i \setminus F$; since G_i is $+$ -coherent with Θ , we may also assume that $\bigcirc\phi_i^0 \in \Theta$. Now, we list formulas in G_i as $(\phi_i^k)_{k \in \mathbb{N}}$, where ϕ_i^0 is selected in the way above; then, for each $m \in \mathbb{N}$, let $\psi_i^m = \bigwedge_{k \leq m} \phi_i^k$. By (i) of Proposition D.1, $\bigcirc\psi_i^m \in \Theta$. Then, let $m \in \mathbb{N}$ be arbitrary; we show that $\bigwedge_{0 \leq i \leq n} \bigcirc\psi_i^m \rightarrow \bigvee_{0 \leq i < j \leq n} \bigcirc(\psi_i^m \wedge \psi_j^m) \in \Theta$. Since $\phi_i^0 \notin F$, clearly $\psi_i^m \notin F$, so by (ii) of Proposition D.1 in contraposition, there must be $\chi_i^m \in F$ s.t. $\bigcirc\chi_i^m \in \Delta$, but for any $\theta \in F$, $\bigcirc(\chi_i^m \wedge \theta \wedge \psi_i^m) \notin \Delta$. Then, let $\chi^m = \bigwedge_{i \leq n} \chi_i^m$. For any $\theta \in F$ and $i \leq n$, $\bigcirc(\chi^m \wedge \theta) \in \Delta$ (by (i) of Proposition D.1) and $\bigcirc(\chi^m \wedge \theta \wedge \psi_i^m) \notin \Delta$, so by $5C_{\bigcirc}^n$, $\bigcirc(\chi^m \wedge \theta \wedge (\bigwedge_{0 \leq i \leq n} \bigcirc\psi_i^m \rightarrow \bigvee_{0 \leq i < j \leq n} \bigcirc(\psi_i^m \wedge \psi_j^m))) \in \Delta$. Then, since $\theta \in F$ is arbitrary, by (ii) of Proposition D.1, $\bigwedge_{0 \leq i \leq n} \bigcirc\psi_i^m \rightarrow \bigvee_{0 \leq i < j \leq n} \bigcirc(\psi_i^m \wedge \psi_j^m) \in F \subseteq \Theta$. The rest is the same as the case for $C_{\bigcirc}^n$. $\square$

$$(9) \quad \boxed{\text{DE}_{\bigcirc} \quad \bigcirc\phi \rightarrow \bigcirc(\phi \wedge \neg\bigcirc\psi) \quad | \quad (\text{Emp})^+ \quad \forall X \in N_w^+ \forall v \in X (N_v^+ = N_v^- = \emptyset)}$$

Proof. Let $F \in \text{Core}^+(\Delta)$ and $\Theta \in \|F\|$ be arbitrary. By $\text{DE}_{\bigcirc}$, for any ψ and any $\phi \in F$ s.t. $\bigcirc\phi \in \Delta$, we have $\bigcirc(\phi \wedge \neg\bigcirc\psi) \in \Delta$. Thus, by (ii) of Proposition D.1, $\neg\bigcirc\psi \in F \subseteq \Theta$. Therefore, there is no ψ s.t. $\bigcirc\psi \in \Theta$, so $\text{Core}^+(\Theta) = \text{Core}^-(\Theta) = \emptyset$. $\square$

$$(10) \quad \boxed{\text{T}_{\bigcirc}^{\text{op}} \quad \phi \wedge \bigcirc(\phi \vee \psi) \rightarrow \bigcirc\phi \quad | \quad (\text{Id})^+ \quad (\exists Y \in N_w^- w \notin Y) \Rightarrow \{w\} \in \uparrow N_w^+}$$

Proof. Assume that there is $F' \in \text{Core}^-(\Delta)$ s.t. $\Delta \notin \|F'\|$. Then, there is $\phi \in F'$ s.t. $\neg\phi \in \Delta$; since F' is $-$ -coherent w.r.t. Δ , we may also assume that $\bigcirc\neg\phi \in \Delta$. Then, let $\psi \in \Delta$ be arbitrary. Since $\neg\phi \wedge \psi \in \Delta$ and $\bigcirc\neg\phi \in \Delta$, by $\text{T}_{\bigcirc}^{\text{op}}$, $\bigcirc(\neg\phi \wedge \psi) \in \Delta$. Thus, Δ , viewed as a filter, is $+$ -coherent with Δ itself. Then, by Lemma 4.12, there is $G \in \text{Core}^+(\Delta)$ extending Δ , and thus $\|G\| \subseteq \{\Delta\}$. $\square$

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On Interpretations of Normal Modal Logics

Lev V. Dvorkin *

Lomonosov Moscow State University, Russia

lev_135@mail.ru

We study interpretations of modal logics in one another where the Boolean connectives are interpreted identically and the modal operator diamond is interpreted by an arbitrary formula $\alpha(p)$. Clearly, such a formula $\alpha(p)$ defines an interpretation of a normal modal logic whenever $\alpha(p)$ is additive (that is, preserves disjunction) and normal (that is, preserves bottom) in the target logic. In the present paper, we provide a complete description of all additive and normal formulas in five prominent modal logics: K, GL, Grz, S4, and S5. For K, GL, and S5, we also describe all additive and normal formulas with parameters.

1 Introduction

Let $\mathcal{V} = \{p_0, p_1, \dots\}$ be a countable set of *propositional variables*, $\text{Fm}_{\mathcal{L}}$ and $\text{Fm}_{\mathcal{L}_0}$ be the sets of all formulas in propositional languages $\mathcal{L}$ and $\mathcal{L}_0$ with variables from $\mathcal{V}$. A *translation* from $\mathcal{L}$ into $\mathcal{L}_0$ is an arbitrary mapping $\tau : \text{Fm}_{\mathcal{L}} \rightarrow \text{Fm}_{\mathcal{L}_0}$. If logics $\Lambda \subseteq \text{Fm}_{\mathcal{L}}$ and $\Lambda_0 \subseteq \text{Fm}_{\mathcal{L}_0}$ are such that $\Lambda = \tau^{-1}\Lambda_0$ (i.e., $\Lambda \vdash \varphi \Leftrightarrow \Lambda_0 \vdash \tau\varphi$), then we say that τ *defines an (exact) interpretation of Λ in Λ_0* . There are several well-known examples of such interpretations:

- Glivenko’s interpretation [10] of classical logic in intuitionistic logic:

$$\text{Cl} \vdash \varphi \Leftrightarrow \text{Int} \vdash \neg\neg\varphi$$

and its modal analog [22]: $\text{S5} \vdash \varphi \Leftrightarrow \text{S4} \vdash \Box\Diamond\varphi$.

- Gödel’s interpretation [11] of intuitionistic logic in S4:

$$\text{Int} \vdash \varphi \Leftrightarrow \text{S4} \vdash \tau_G\varphi,$$

where τ_G adds $\Box$ on each subformula.

- Interpretations of KT in K [26], of S4 in K4 [23], and of Grz in GL [12, 18]:

$$\begin{aligned} \text{KT} \vdash \varphi &\Leftrightarrow \text{K} \vdash \tau_{\text{refl}}\varphi, \\ \text{S4} \vdash \varphi &\Leftrightarrow \text{K4} \vdash \tau_{\text{refl}}\varphi, \\ \text{Grz} \vdash \varphi &\Leftrightarrow \text{GL} \vdash \tau_{\text{refl}}\varphi, \end{aligned}$$

where τ_{refl} preserves variables, commutes with all Boolean connectives, and maps formulas of the form $\Diamond\psi$ to $\Diamond\tau_{\text{refl}}\psi \vee \tau_{\text{refl}}\psi$.

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- The interpretation of Solovay's logic in GL [29]:

$$S \vdash \varphi \Leftrightarrow GL \vdash \bigwedge_{\Diamond \psi \in \text{Sub}(\varphi)} (\Box \psi \rightarrow \psi) \rightarrow \varphi,$$

where $\text{Sub}(\varphi)$ is the set of all subformulas of φ .

Let us consider interpretations of modal logics in each other given by translations of the following form: for each modality $\Diamond_i$ in $\mathcal{L}$, we fix some formula $\alpha_i(p) \in \text{Fm}_{\mathcal{L}_0}$. Then $\tau_{\alpha} : \text{Fm}_{\mathcal{L}} \rightarrow \text{Fm}_{\mathcal{L}_0}$ is the translation that maps p_k to p_{2k}^1 , commutes with Boolean connectives, and sends $\Diamond_i \psi$ to $\alpha_i(\tau_{\alpha} \psi)$. Such translations are called *modal-to-modal* translations in [8]. This notion is quite restrictive. For instance, translations from S5 to S4 and from S to GL mentioned above are not of this form (and cannot be replaced by translations of this form, see [31, Theorem 5.10]). Strictly speaking, τ_{refl} is also not of this form, but it can be replaced by $\tau_{\Diamond p \vee p}$ equally well. There are other interesting examples of interpretations given by modal-to-modal translations:

- Kracht and Wolter [16] showed that every monotone monomodal logic Λ is interpretable in a normal bimodal logic via the translation $\tau_{\Box_1 \Diamond_2 p}$.
- Thomason's translation [30] provides an interpretation of normal polymodal logics in monomodal ones. It is not a modal-to-modal translation but has the form $\tau_T \varphi := \Diamond \Box \perp \rightarrow \tau_{\beta} \varphi$, where τ_{β} is a particular modal-to-modal translation.
- The logics that are interpreted in monomodal logics by the translation $\tau_{\Diamond p \wedge \Diamond \neg p}$ are known as *non-contingency logics* [24, 13, 32].
- Translation $\tau_{\Box \Diamond p}$ defines an interpretation of the minimal monotone logic EM in K [8, Theorem 6.1.11]. Some other interpretations (including interpretations of normal logics in each other) via this translation are discussed in [14, Section 2].
- The provability logic of Niebergall arithmetic w.r.t. Peano arithmetic, investigated in [6], is a normal logic, which is interpreted in the polymodal provability logic GLP by $\tau_{\Diamond_1 p \vee \Diamond_0 (\Diamond_1 \top \wedge p)}$.

In this paper, we consider only *congruential modal logics*, i.e., the sets of formulas that contain classical tautologies and are closed under the rules of modus ponens, substitution, and *equivalent replacement* $\frac{\varphi \leftrightarrow \psi}{\Diamond \varphi \leftrightarrow \Diamond \psi}$. It is easy to see that if Λ is a modal logic in this sense, then so is the set $\tau_{\alpha}^{-1} \Lambda$ for any modal-to-modal translation τ_{α} . Hence, every modal-to-modal translation defines an interpretation of some logic in a given one. In [31], it was shown that, for every logic Λ such that $K \subseteq \Lambda \subseteq GL$ or $K \subseteq \Lambda \subseteq Grz$, infinitely many monomodal logics are interpretable in Λ by modal-to-modal translations. What happens when we restrict our attention to normal logics? It is well known that a congruential logic is normal iff it contains the axioms $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ and $\Diamond \perp \leftrightarrow \perp$. Therefore, τ_{α} defines an interpretation of a normal logic in Λ iff α is *additive* and *normal* in Λ , i.e., the following equivalences hold in Λ :

$$\alpha(p \vee q) \leftrightarrow \alpha(p) \vee \alpha(q) \quad \text{and} \quad \alpha(\perp) \leftrightarrow \perp.$$

Thus, to understand which normal logics are interpretable in Λ , it is helpful to describe all formulas that are additive and normal in Λ . Such a description for K was obtained in [2, 3] using model-theoretic methods. In the present paper, we reprove this result using a different, more constructive approach and extend the analysis to four other central modal logics: GL, S5, S4, and Grz. In particular, we show that there are exactly five pairwise non-equivalent normal additive formulas in S4 and in Grz.

¹We use this technical trick to reserve odd-indexed variables for parameters (see below).

One can also consider interpretations with parameters. A classic example is the interpretation of minimal logic in intuitionistic logic [27] via the following translation with one parameter p_1 :

$$\begin{aligned}\tau \perp &:= p_1, & \tau p_k &:= p_{2k} \text{ for } k \in \omega, \\ \tau(\varphi \circ \psi) &:= \tau \varphi \circ \tau \psi \text{ for } \circ \in \{\vee, \wedge, \rightarrow\}.\end{aligned}$$

Note that, under the general definition of interpretation, there is no distinction between parameters and other variables. Differences only emerge when we consider interpretations of a certain type — for example, modal-to-modal ones. Recall that $\tau_{\vec{\alpha}}$ maps all variables to variables with even indices. Variables $p_{2k+1}, k \in \omega$ will be treated as parameters. Parametric modal-to-modal translations are defined as follows: for each modality $\Diamond_i$, we fix a formula $\alpha_i(p, \vec{r})$, where $\vec{r} = \langle r_1, \dots, r_n \rangle$ is a tuple of parameters. Then the translation $\tau_{\vec{\alpha}}$ maps p_k to p_{2k} , commutes with Boolean connectives, and sends $\Diamond_i \psi$ to $\alpha_i(\tau_{\vec{\alpha}} \psi, \vec{r})$. It is easy to verify that, for every logic Λ , $\tau_{\vec{\alpha}}^{-1} \Lambda$ is a logic, whence every modal-to-modal translation with parameters defines an interpretation of some logic in a given logic. For instance, weak Grzegorczyk logic wGrz is interpreted in GL via the parametric translation $\tau_{\Diamond p \vee (r \wedge p)}$ (see Example 7.10). Moreover, the use of parameters allows us to interpret K, K4, and GL in KT, S4, and Grz respectively (see Example 7.11). We will provide a characterization of additive formulas with parameters for the logics K, GL, and S5. The structure of additive formulas in S4 and Grz is much more complicated, so the parametric case for these logics is postponed to a subsequent paper.

The syntactic characterization of additive formulas can also be studied independently of interpretations, simply as the problem of characterizing formulas with a certain semantic property. The most well-known results of this type are the Łos-Tarski and Lyndon preservation theorems from classical model theory. Modal analogs of these results were considered in [28, 17, 3, 7]. Additive operators in Boolean algebras trace back to the classical work of Jónsson and Tarski [15]. The results of the present paper provide a syntactic characterization of the modal formulas that define such operators.

The structure of the paper is as follows. In Section 2, we recall some basic notions from modal logic and introduce the main objects for the present paper: additive and normal formulas and operators. In Section 3, we provide some basic results on additive formulas. In Section 4, we characterize all additive formulas with parameters in S5. This is the simplest, yet non-trivial, case. In Section 5, we do the same for K and GL. In Section 6, we characterize additive formulas without parameters in S4 and Grz. In Section 7, we make some remarks regarding the interpretations of normal logics, partly building on the results of the previous sections. In Section 8, we state some open questions for future research.

2 Preliminaries

2.1 Modal formulas

We fix a countable set $\mathcal{V} = \{p_0, p_1, \dots\}$ of *propositional variables*, $p := p_0$, and $q := p_2$. Variables with odd indices are called *parameters*. The letter r (possibly with indices) will be used as a metavariable ranging over parameters. *Modal formulas* are built from variables p_k , their negations $\neg p_k$, the constants $\perp$ and $\top$ using $\wedge, \vee, \Diamond$, and $\Box$. The negation of a formula is defined recursively by the duality laws. The Boolean connectives $\rightarrow$ and $\leftrightarrow$ are treated as abbreviations. The set of all modal formulas is denoted by Fm . For a tuple of parameters $\vec{r} = \langle r_j \rangle_{j < n}$, let $\text{Fm}(p, \vec{r})$ denote the set of formulas that contain no variables other than $p, r_0, \dots, r_{n-1}$. A formula φ is *p-positive* if it has no occurrence of $\neg p$. The set of all formulas from $\text{Fm}(p, \vec{r})$ that are *p-positive* is denoted by $\text{Fm}^+(p, \vec{r})$. We will also use the notation $\text{Fm}(p)$, $\text{Fm}^+(p)$, and $\text{Fm}(\vec{r})$ with the obvious meanings. Note that $\text{Fm}(\langle \rangle) \subset \text{Fm}(\vec{r}) \subset \text{Fm}^+(p, \vec{r}) \subset \text{Fm}(p, \vec{r})$, where $\text{Fm}(\langle \rangle)$ is the set of all variable-free formulas. For $\alpha = \alpha(p, \vec{r}) \in \text{Fm}(p, \vec{r})$, we put $\alpha_{\perp} := \alpha(\perp, \vec{r}) \in \text{Fm}(\vec{r})$.

2.2 Normal modal logic and Kripke semantics

Normal modal logics, Kripke frames and models, and p-morphisms are defined as usual (see, e.g., [5]). We also use standard notation for some particular logics: K, GL, S4, S5, Grz, etc. For a frame $\mathcal{F}$ (class of frames $\mathcal{C}$) we denote by $\text{Log } \mathcal{F}$ ($\text{Log } \mathcal{C}$) its logic (i.e., the set of all formulas that are valid in it). For a logic Λ , we say that formulas φ and ψ are Λ -equivalent if $\Lambda \vdash \varphi \leftrightarrow \psi$ and denote this by $\varphi \sim_\Lambda \psi$.

A *general frame* is a pair $\mathcal{G} = (\mathcal{F}, P)$, where $\mathcal{F} = (W, R)$ is a Kripke frame and $P \subseteq \mathcal{P}(W)$ is closed under Boolean operations and the *full preimage* operator $R^{-1} : U \mapsto \{w \in W \mid \exists u \in U \ w R u\}$. The *logic of $\mathcal{G}$* is the set $\text{Log } \mathcal{G}$ of all formulas that are true in all models $(\mathcal{F}, \vartheta)$, where $\vartheta(p_k) \in P$ for all $k \in \omega$. Clearly, $\text{Log } \mathcal{F} \subseteq \text{Log } \mathcal{G}$. For a class of general frames $\mathcal{D}$, $\text{Log } \mathcal{D} := \bigcap_{\mathcal{G} \in \mathcal{D}} \text{Log } \mathcal{G}$.

2.3 Monotone, additive, and normal operators and formulas

Let $\mathcal{F} = (W, R)$ be a Kripke frame. For a formula $\varphi \in \text{Fm}(p, \vec{r})$, $\vec{r} = \langle r_j \rangle_{j < n}$, consider the operator

$$\alpha_{\mathcal{F}} : \mathcal{P}(W)^{1+n} \rightarrow \mathcal{P}(W) \\ (U, \vec{V}) \mapsto \vartheta(\varphi), \text{ where } \vartheta(p) := U, \vartheta(\vec{r}) := \vec{V}.$$

Notice that $\cdot_{\mathcal{F}}$ is a homomorphism from the algebra of formulas with $1+n$ variables into the algebra of $(1+n)$ -place operators on $\mathcal{P}(W)$. It is easy to see that, for any two formulas $\varphi(p, \vec{r})$ and $\psi(p, \vec{r})$,

$$\mathcal{F} \models \varphi \leftrightarrow \psi \Leftrightarrow \varphi_{\mathcal{F}} = \psi_{\mathcal{F}}.$$

An operator $f : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ is

- *monotone* if $U \subseteq V \Rightarrow fU \subseteq fV$ for all $U, V \subseteq W$;
- *additive* if $f(U \cup V) = fU \cup fV$ for all $U, V \subseteq W$;
- *normal* if $f\emptyset = \emptyset$;
- *completely additive* if $f \bigcup_{i \in I} U_i = \bigcup_{i \in I} fU_i$ for every collection of sets $\{U_i\}_{i \in I} \subseteq \mathcal{P}(W)$.

A formula $\varphi(p, \vec{r})$ is

- *monotone (w.r.t. p) in Λ* if $\Lambda \vdash \varphi(p, \vec{r}) \rightarrow \varphi(p \vee q, \vec{r})$;
- *additive (w.r.t. p) in Λ* if $\Lambda \vdash \varphi(p \vee q, \vec{r}) \leftrightarrow \varphi(p, \vec{r}) \vee \varphi(q, \vec{r})$;
- *normal (w.r.t. p) in Λ* if $\Lambda \vdash \varphi(\perp, \vec{r}) \leftrightarrow \perp$.

Notice that additivity implies monotonicity in both cases, and all complete additive operators are additive and normal². Formulas that are additive in Λ will also be called Λ -additive. Λ -additive formulas that are normal in Λ will be called *normal Λ -additive* formulas.

For a class of frames $\mathcal{C}$, we say that a formula $\varphi(p, \vec{r})$ is *monotone (additive, normal, completely additive) in $\mathcal{C}$* if, for all $\mathcal{F} = (W, R) \in \mathcal{C}$ and $\vec{V} \in \mathcal{P}(W)^n$, the operator

$$\varphi_{\mathcal{F}}(\cdot, \vec{V}) : \mathcal{P}(W) \rightarrow \mathcal{P}(W) \\ U \mapsto \varphi_{\mathcal{F}}(U, \vec{V})$$

is monotone (additive, normal, completely additive). The following can be easily checked:

²We assume that the union of the empty collection of sets equals $\emptyset$.

Proposition 2.1. *Let $\Lambda = \text{Log } \mathcal{C}$. Then $\varphi(p, \vec{r})$ is monotone (additive, normal) in Λ iff it is monotone (additive, normal) in $\mathcal{C}$.*

Since our language is finitary, we have no syntactic counterpart to complete additivity. One can show (see [3, Proposition 2.3]) that a formula is completely additive in the class of all Kripke frames iff it is additive in this class, i.e., iff it is K-additive. It is easy to adapt this argument to the classes of frames defined by some of the usual first-order conditions such as reflexivity, symmetry, and seriality. At the same time, the situation is quite different for the classes of transitive frames:

Proposition 2.2. *The formula $\alpha = \Diamond \Box \Diamond p$ is normal S4-additive, but it is not completely additive in the class of all reflexive transitive frames.*

Proof. One can check that $\alpha(p \vee q) \leftrightarrow \alpha(p) \vee \alpha(q)$ and $\alpha(\perp) \leftrightarrow \perp$ are derivable in S4. Here we give a semantic argument which is interesting in its own right. We say that a world v of an S4-frame $\mathcal{F} = (W, R)$ is *maximal* if $v R u \Rightarrow u R v$ for all $u \in W$. Let R' be the relation on W such that $w R' v$ iff $w R v$ and v is maximal. It is easy to check that, $\alpha_{\mathcal{F}} = (R')^{-1}$ whenever $\mathcal{F}$ is finite, whence α is (completely) additive in the class of finite S4-frames. By Proposition 2.1, α is S4-additive.

Now let us consider the frame $\mathcal{F} = (\omega, \leq)$. It is easy to see that $\alpha_{\mathcal{F}} \omega = \omega$ and $\alpha_{\mathcal{F}} \{n\} = \emptyset$ for each $n \in \omega$. Therefore, $\alpha_{\mathcal{F}}$ is not completely additive, and α is not completely additive in the class of all reflexive transitive frames. $\square$

Corollary 2.3. *The formula $\beta = \tau_{\Diamond p \vee p} \alpha$ is normal K4-additive, but it is not completely additive in the class of all transitive frames.*

Proof. Notice that $\text{S4} = \tau_{\Diamond p \vee p}^{-1} \text{K4}$ and $\beta_{\mathcal{F}} = \alpha_{\mathcal{F}}$ for a reflexive frame $\mathcal{F}$. Then the claim follows from Proposition 2.2. $\square$

We now turn to monotone formulas. It is easy to see that every p -positive formula is monotone in any normal modal logic. The converse does not hold in general: in some frames, even completely additive formulas may fail to be equivalent to any positive formula.

Example 2.4. Let Λ be the logic of the frame $\mathcal{F} = (\{0, 1\}, \{0, 1\}^2)$, φ be the formula $\Box p \vee \neg p \wedge \Diamond p$. It is easy to see that $\varphi_{\mathcal{F}}$ is (completely) additive, whence φ is Λ -additive. At the same time, it was shown in [7, Proposition 4.8] that φ is not equivalent to any p -positive formula in Λ .

Notice that Λ from Example 2.4 has many good properties: it is tabular, its class of frames is first-order definable, it even has the Craig interpolation property (CIP) [20] but lacks the Lyndon interpolation property (LIP) [21]. The last fact turns out to be essential: for logics with LIP, we have the following analog of the classical Lyndon's result [19]:

Theorem 2.5 ([7, Theorem 4.4]). *Let Λ be a normal logic with LIP. Then a formula φ is monotone in Λ iff it is Λ -equivalent to some p -positive formula.*

It is well known that the logics we consider in this paper (K, GL, S4, S5, and Grz) have LIP [9]. Therefore, we will look for additive formulas only among p -positive ones.

3 General results on additive formulas

We say that $A \subseteq \text{Fm}(p, \vec{r})$ is a *complete set of (normal) Λ -additive formulas with parameters $\vec{r}$* if every $\alpha \in A$ is (normal) Λ -additive and, for each (normal) Λ -additive formula $\varphi \in \text{Fm}(p, \vec{r})$, there is $\alpha \in A$ such that $\varphi \sim_{\Lambda} \alpha$. The goal of our work is to provide syntactic descriptions of some complete sets of Λ -additive and normal Λ -additive formulas. These two tasks are in fact equivalent:

Lemma 3.1. *If A is a complete set of Λ -additive formulas, then $\{\alpha \in A \mid \alpha_{\perp} \sim_{\Lambda} \perp\}$ is a complete set of normal Λ -additive formulas. Conversely, if N is a complete set of normal Λ -additive formulas, then $\{\alpha \vee v \mid \alpha \in N, v \in \text{Fm}(\vec{r})\}$ is a complete set of Λ -additive formulas.*

Proof. The first claim is trivial. It is also easy to see that $\alpha \vee v$ is additive in Λ , whenever α is additive. Suppose that φ is additive in Λ . Since φ is monotone in Λ , $\varphi \sim_{\Lambda} (\varphi \wedge \neg\varphi_{\perp}) \vee \varphi_{\perp}$. It remains to notice that $\varphi \wedge \neg\varphi_{\perp}$ is normal Λ -additive and $\varphi_{\perp} \in \text{Fm}(\vec{r})$. $\square$

The following lemma can be easily checked:

Lemma 3.2. *Let Λ be a normal logic. Then $\perp$ and p are normal Λ -additive formulas. If α, β are normal Λ -additive formulas, $\varkappa \in \text{Fm}(\vec{r})$, then $\alpha \vee \beta$, $\varkappa \wedge \alpha$, and $\Diamond\alpha$ are also normal Λ -additive.*

For $\varkappa_0, \dots, \varkappa_n \in \text{Fm}(\vec{r})$, consider the formula

$$\delta_{\vec{\varkappa}} := \varkappa_0 \wedge \Diamond(\varkappa_1 \wedge \dots \wedge \Diamond(\varkappa_n \wedge p) \dots).$$

Such formulas are called δ -formulas. Notice that $\delta_{\vec{\varkappa}}$ is true at w in a Kripke model (W, R, ϑ) iff there is a sequence of worlds $v_0, \dots, v_n$ such that $w = v_0 R v_1 \dots R v_n$, $v_i \in \vartheta(\varkappa_i)$ for $i \leq n$, and $v_n \in \vartheta(p)$. Denote by Δ the set of all δ -formulas. We write $\Delta_0 \subset_{\text{fin}} \Delta$ if Δ_0 is a finite subset of Δ . From Lemma 3.2, we immediately obtain

Corollary 3.3. *Let Λ be a normal logic, $\Delta_0 \subset_{\text{fin}} \Delta$, $v \in \text{Fm}(\vec{r})$. Then $\bigvee \Delta_0$ is normal Λ -additive and $\bigvee \Delta_0 \vee v$ is Λ -additive.*

One of the difficulties in characterizing additive formulas is that complete sets of additive formulas are not preserved under extensions or weakenings of logics. Moreover, even if the same set A constitutes a complete set of additive formulas for two logics Λ_1 and Λ_2 with $\Lambda_1 \subset \Lambda_2$, A need not be a complete set of additive formulas for any logic between Λ_1 and Λ_2 :

Example 3.4. We will show that, in both K and GL, every additive formula is equivalent to a formula from the set $A_0 := \{\bigvee \Delta_0 \vee v \mid \Delta_0 \subset_{\text{fin}} \Delta, v \in \text{Fm}(\vec{r})\}$. At the same time, $K \subset K4 \subset GL$ and the formula β from Corollary 2.3 is K4-additive but is not K4-equivalent to a formula from A_0 , since formulas from A_0 are completely additive in all frames. β does not appear in the description of K- and GL-additive formulas, since it is not K-additive and it is GL-equivalent to a δ -formula $\Diamond(\Box\perp \wedge p)$.

Definition 3.5. Let Λ be a normal modal logic, $\mathcal{D}$ be a class of general frames. We say that $\mathcal{D}$ is *good* for Λ if $\text{Log } \mathcal{D} = \Lambda$ and, for all $(\mathcal{F}, P) \in \mathcal{D}$, $\mathcal{F}$ is a Λ -frame (i.e., $\mathcal{F} \models \Lambda$).

The following lemma trivially follows from the basic properties of p-morphisms:

Lemma 3.6. *Suppose that $\Lambda = \text{Log } \mathcal{C}$ for a class of Kripke frames $\mathcal{C}$ and, for each $\mathcal{F} \in \mathcal{C}$, we have defined a Λ -frame $\mathcal{F}^{\circ}$ and a p-morphism $\pi : \mathcal{F}^{\circ} \rightarrow \mathcal{F}$. Then the class*

$$\pi^{-1}\mathcal{C} := \{(\mathcal{F}^{\circ}, \pi^{-1}\mathcal{P}(W)) \mid \mathcal{F} = (W, R) \in \mathcal{C}\}$$

is good for Λ .

For a class of finite general frames $\mathcal{D}$, denote by $\mathcal{D}^{\dagger}$ the class of Kripke models

$$\{(\mathcal{F}, \vartheta) \mid (\mathcal{F}, P) \in \mathcal{D}, |\vartheta(p)| \leq 1, \text{ and } \vartheta(r) \in P \text{ for every parameter } r\}.$$

Notice that $\mathcal{D}^{\dagger} \models \varphi(p, \vec{r}) \leftrightarrow \psi(p, \vec{r})$ iff, for all $\mathcal{G} = (\mathcal{F}, P) \in \mathcal{D}$, $\vec{V} \in P^n$,

$$\varphi_{\mathcal{F}}(\emptyset, \vec{V}) = \psi_{\mathcal{F}}(\emptyset, \vec{V}) \quad \text{and} \quad \varphi_{\mathcal{F}}(\{u\}, \vec{V}) = \psi_{\mathcal{F}}(\{u\}, \vec{V}) \quad \text{for every world } u \text{ in } \mathcal{F}$$

Lemma 3.7. *Let Λ be a normal logic, $\mathcal{D}$ be a good class of finite general frames for Λ , α and β be Λ -additive formulas. Then $\mathcal{D}^\dagger \models \alpha \leftrightarrow \beta \Leftrightarrow \Lambda \vdash \alpha \leftrightarrow \beta$.*

Proof. Suppose that $\Lambda \not\vdash \alpha \leftrightarrow \beta$. Since $\text{Log } \mathcal{C} = \Lambda$, there is $(\mathcal{F}, P) \in \mathcal{D}$ and valuation $\vartheta : \mathcal{V} \rightarrow P$ such that $\vartheta(\alpha) \neq \vartheta(\beta)$. Let $U := \vartheta(p)$ and $\vec{V} := \vartheta(\vec{r})$. If $U = \emptyset$, then $(\mathcal{F}, \vartheta) \in \mathcal{D}^\dagger$, whence $\mathcal{D}^\dagger \models \alpha \leftrightarrow \beta$. Otherwise, since α and β are Λ -additive and $\mathcal{F}$ is finite,

$$\vartheta(\alpha) = \alpha_{\mathcal{F}}(U, \vec{V}) = \bigcup_{u \in U} \alpha_{\mathcal{F}}(\{u\}, \vec{V}) \quad \text{and} \quad \vartheta(\beta) = \beta_{\mathcal{F}}(U, \vec{V}) = \bigcup_{u \in U} \beta_{\mathcal{F}}(\{u\}, \vec{V}).$$

Therefore, $\alpha_{\mathcal{F}}(\{u\}, \vec{V}) \neq \beta_{\mathcal{F}}(\{u\}, \vec{V})$ for some $u \in U$ and $\mathcal{D}^\dagger \not\models \alpha \leftrightarrow \beta$.

The converse implication is trivial, since for every $(\mathcal{F}, P) \in \mathcal{D}$, $\mathcal{F}$ is a Λ -frame. $\square$

Suppose that Λ has LIP. Then, by Theorem 2.5, all monotone formulas, and even more so all additive ones, are Λ -equivalent to p -positive formulas. Let $\varphi \sim_{\mathcal{D}^\dagger} \psi \Leftrightarrow \mathcal{D}^\dagger \models \varphi \leftrightarrow \psi$. It is easy to check that $\sim_{\mathcal{D}^\dagger}$ is a congruence on the algebra of p -positive formulas $\mathfrak{Fm}^+(p, \vec{r}) := (\text{Fm}^+(p, \vec{r}), \perp, \top, \wedge, \vee, \Diamond, \Box)$. The quotient structure $\mathfrak{Fm}^+(p, \vec{r}) / \sim_{\mathcal{D}^\dagger}$ is a modal lattice (i.e., a distributive lattice with dual operators $\Diamond$ and $\Box$). By Lemma 3.7, each equivalence class in $\text{Fm}^+(p, \vec{r}) / \sim_{\mathcal{D}^\dagger}$ contains at most one Λ -additive formula up to Λ -equivalence. In particular, we have the following:

Corollary 3.8. *Let Λ be a normal modal logic with LIP, $\mathcal{D}$ be a good for Λ class of finite general frames, $A \subset \text{Fm}(p, \vec{r})$ be a set of Λ -additive formulas. Suppose that, for each p -positive formula $\varphi(p, \vec{r})$, there is $\alpha \in A$ such that $\mathcal{D}^\dagger \models \varphi \leftrightarrow \alpha$. Then A is a complete set of Λ -additive formulas with parameters $\vec{r}$.*

4 Additive formulas in S5

Consider the following class of frames:

$$\mathcal{C}_{S5} := \{(W, W \times W) \mid W \subseteq \omega \text{ is a finite non-empty set}\}.$$

It is well known that $\text{Log } \mathcal{C}_{S5} = \text{S5}$. Let $B := \{0, 1\}$. For a frame $\mathcal{F} = (W, W^2) \in \mathcal{C}_{S5}$, we put $\mathcal{F}^\circ := (W \times B, (W \times B)^2)$. Clearly $\mathcal{F}^\circ \models \text{S5}$ and the mapping $\pi : (a, j) \mapsto a$ is a p-morphism from $\mathcal{F}^\circ$ onto $\mathcal{F}$. By Lemma 3.6, the class of general frames $\mathcal{D}_{S5} := \pi^{-1}\mathcal{C}_{S5}$ is good for S5.

Consider the following sets of formulas

$$\begin{aligned} \Gamma_{S5} &:= \text{Fm}(\vec{r}) \cup \{\lambda \wedge p \mid \lambda \in \text{Fm}(\vec{r})\} \cup \{\mu \wedge \Diamond(v \wedge p) \mid \mu, v \in \text{Fm}(\vec{r})\}. \\ A_{S5} &:= \left\{ (\lambda \wedge p) \vee \bigvee_{i < k} (\mu_i \wedge \Diamond(v_i \wedge p)) \vee \varkappa \mid k \in \omega; \varkappa, \lambda, \mu_i, v_i \in \text{Fm}(\vec{r}) \right\}. \end{aligned}$$

Notice that $A_{S5} \subset A_0$ and A_{S5} contains all disjunctions of formulas from Γ_{S5} up to K-equivalence. We will show that A_{S5} is the complete set of S5-additive formulas.

Lemma 4.1. *For $\alpha, \beta \in \Gamma_{S5}$, there is $\gamma \in \Gamma_{S5}$ such that $\mathcal{D}_{S5}^\dagger \models \alpha \wedge \beta \leftrightarrow \gamma$.*

Proof. If $\alpha = \lambda_1 \wedge p$ and $\beta = \mu_2 \wedge \Diamond(v_2 \wedge p)$, where $\lambda_1, \mu_2, v_2 \in \text{Fm}(\vec{r})$, then

$$\mathcal{D}_{S5}^\dagger \models \alpha \wedge \beta \leftrightarrow (\lambda_1 \wedge \mu_2 \wedge v_2) \wedge p.$$

Indeed, if $\alpha \wedge \beta$ is true at w in $\mathcal{M} = (\mathcal{F}^\circ, \vartheta) \in \mathcal{D}_{S5}^\dagger$, then $w \in \vartheta(p)$ and there is a world $v \in \vartheta(v_2 \wedge p)$. Since $|\vartheta(p)| \leq 1$, $w = v$ and the right-hand side of the equivalence is true at w . The converse implication is trivial.

Similarly, if $\alpha = \mu_1 \wedge \Diamond(v_1 \wedge p)$ and $\beta = \mu_2 \wedge \Diamond(v_2 \wedge p)$, where $\mu_1, \mu_2, v_1, v_2 \in \text{Fm}(\vec{r})$, then

$$\mathcal{D}_{S5}^\dagger \models \alpha \wedge \beta \leftrightarrow (\mu_1 \wedge \mu_2) \wedge \Diamond(v_1 \wedge v_2 \wedge p).$$

All other cases are either trivial or symmetric to those already considered. $\square$

Lemma 4.2. *For $\alpha \in \Gamma_{S5}$, there is $\beta \in \Gamma_{S5}$ such that $S5 \vdash \Diamond \alpha \leftrightarrow \beta$.*

Proof. Notice that $S5 \vdash \Diamond(\mu \wedge \Diamond(v \wedge p)) \leftrightarrow \Diamond\mu \wedge \Diamond(v \wedge p)$. All other cases are trivial. $\square$

Lemma 4.3. *For $\alpha \in A_{S5}$, there is $\beta \in A_{S5}$ such that $\mathcal{D}_{S5}^\dagger \models \Box \alpha \leftrightarrow \beta$.*

Proof. Suppose that $\alpha = (\lambda \wedge p) \vee \bigvee_{i < k} \mu_i \wedge \Diamond(v_i \wedge p) \vee \varkappa$ for some $k \in \omega$ and $\varkappa, \lambda, \mu_i, v_i \in \text{Fm}(\vec{r})$. Let $\underline{k} := \{0, \dots, k-1\}$. For $I \subseteq \underline{k}$, we put $\mu^I := \bigvee_{i \in I} \mu_i$ and $v_I := \bigwedge_{i \in I} v_i$. Consider the formula

$$\beta := \bigvee_{\emptyset \neq I \subseteq \underline{k}} (\Box(\varkappa \vee \mu^I) \wedge \Diamond(v_I \wedge p)) \vee \Box \varkappa.$$

Clearly, $S5 \vdash \beta \rightarrow \Box \alpha$. For the converse, suppose that $\Box \alpha$ is true at w in $\mathcal{M} = (\mathcal{F}^\circ, \vartheta) \in \mathcal{D}_{S5}^\dagger$. If $\vartheta(p) = \emptyset$, then $\mathcal{M}, w \models \Box \varkappa$. Otherwise, there is a world v such that $\vartheta(p) = \{v\}$. Let $I := \{i < k \mid \mathcal{M}, v \models v_i\}$. Clearly, $\mathcal{M}, w \models \Diamond(v_I \wedge p)$ and $\mathcal{M} \models \neg \Diamond(v_i \wedge p)$ for $i \in \underline{k} \setminus I$.

Suppose that $\mathcal{M}, w \not\models \Box(\varkappa \vee \mu^I)$. Then there is a world u such that $\mathcal{M}, u \not\models \varkappa \vee \mu^I$. Let $u' := (a, 1-j)$, where $u = (a, j)$. Since $\pi u = \pi u' = a$, $\mathcal{M}, u' \not\models \varkappa \vee \mu^I$. At least one of the worlds u and u' is distinct from v . Without loss of generality, we can assume that it is u . Then $\mathcal{M}, u \not\models \lambda \wedge p$, whence $\mathcal{M}, u \not\models \alpha$ and $\mathcal{M}, w \not\models \Box \alpha$. Contradiction. $\square$

Theorem 4.4. *A formula φ is S5-additive iff it is S5-equivalent to a formula from A_{S5} .*

Proof. By Lemma 3.8, it is sufficient to prove that every $\varphi \in \text{Fm}^+(p, \vec{r})$ is equivalent to some formula from A_{S5} in $\mathcal{D}_{S5}^\dagger$. We proceed by induction on the construction of φ . The base case and the induction step for $\vee$ are trivial. The induction step for $\wedge$ follows, by distributivity, from Lemma 4.1. The induction step for $\Diamond$ follows, by additivity, from Lemma 4.2. The induction step for $\Box$ follows from Lemma 4.3. $\square$

Corollary 4.5. *φ is a normal additive formula in S5 iff it is equivalent to a formula from A_{S5} with $\varkappa = \perp$.*

Corollary 4.6. *There are exactly four S5-additive formulas without parameters up to S5-equivalence: $\perp$, p , $\Diamond p$, and $\top$. Three of them are normal: $\perp$, p , and $\Diamond p$.*

5 Additive formulas in K and GL

For a set Σ , denote by Σ^* the set of all finite sequences $\langle s_i \rangle_{i < n}$ with $s_i \in \Sigma$, including the empty one $\langle \rangle$, $\Sigma^+ := \Sigma^* \setminus \{\langle \rangle\}$. For two sequences $\vec{a}, \vec{b} \in \Sigma^*$, denote by $\vec{a}\vec{b}$ their concatenation. We consider the following relation on Σ^* :

$$\vec{a} < \vec{b} :\Leftrightarrow \exists c \in \Sigma (\vec{a} = \vec{b}\langle c \rangle).$$

Denote by $<$ the transitive closure of $<$. Notice that $<$ is a strict partial order.

Definition 5.1. A non-empty set $S \subseteq \Sigma^*$ is a *tree skeleton* if

$$\forall \vec{a}, \vec{b} \in \Sigma^* (\vec{a} < \vec{b} \wedge \vec{b} \in S \Rightarrow \vec{a} \in S).$$

Consider the following mappings:

$$\begin{aligned} \pi : (\Sigma \times B)^* &\rightarrow \Sigma^* & \text{and} & & \iota : (\Sigma \times B)^+ &\rightarrow (\Sigma \times B)^+ \\ \langle (a_i, j_i) \rangle_{i < n} &\mapsto \langle a_i \rangle_{i < n}, & & & \langle (a_i, j_i) \rangle_{i \leq n} &\mapsto \langle (a_i, j_i) \rangle_{i < n} \langle (a_n, 1 - j_n) \rangle. \end{aligned}$$

For a tree skeleton $S \subseteq \Sigma^*$, let $S^\circ := \pi^{-1}S$. Notice that $S^\circ \subseteq (\Sigma \times B)^*$ is also a tree skeleton and, for $v \in (\Sigma \times B)^+$, $\pi(\iota v) = \pi(v)$, whence $\iota v \in S^\circ$ iff $v \in S^\circ$.

Consider the following classes of frames:

$$\begin{aligned} \mathcal{C}_K &:= \{(S, <) \mid S \subseteq \omega^* \text{ is a finite tree skeleton}\}, \\ \mathcal{C}_{GL} &:= \{(S, <) \mid S \subseteq \omega^* \text{ is a finite tree skeleton}\}. \end{aligned}$$

Strictly speaking, we should write $(S, <|_S)$ and $(S, <|_S)$, but we drop the restriction for notational convenience. It is well known that $\text{Log } \mathcal{C}_K = K$ and $\text{Log } \mathcal{C}_{GL} = GL$. For $\mathcal{F} = (S, <) \in \mathcal{C}_K$, we put $\mathcal{F}^\circ := (S^\circ, <)$. For $\mathcal{F} = (S, <) \in \mathcal{C}_{GL}$, we put $\mathcal{F}^\circ := (S^\circ, <)$. Clearly, for $\Lambda \in \{K, GL\}$ and $\mathcal{F} \in \mathcal{C}_\Lambda$, $\mathcal{F}^\circ$ is a Λ -frame and $\pi : \mathcal{F}^\circ \rightarrow \mathcal{F}$ is a p-morphism. By Lemma 3.6, the class of general frames $\mathcal{D}_\Lambda := \pi^{-1}\mathcal{C}_\Lambda$ is good for Λ .

We are going to show that $A_0 = \{\bigvee \Delta_0 \vee \varkappa \mid \Delta_0 \subset_{\text{fin}} \Delta, \varkappa \in \text{Fm}(\vec{r})\}$ is a complete set of additive formulas both in K and in GL .

Lemma 5.2. For $\vec{\mu} = \langle \mu_i \rangle_{i \leq m} \in (\text{Fm}(\vec{r}))^{m+1}$ and $\vec{v} = \langle v_i \rangle_{i \leq n} \in (\text{Fm}(\vec{r}))^{n+1}$,

- if $m \neq n$, then $\mathcal{D}_K^\dagger \models \delta_{\vec{\mu}} \wedge \delta_{\vec{v}} \leftrightarrow \perp$;
- if $m = n$, then $\mathcal{D}_K^\dagger \models \delta_{\vec{\mu}} \wedge \delta_{\vec{v}} \leftrightarrow \delta_{\vec{\mu} \wedge \vec{v}}$, where $\vec{\mu} \wedge \vec{v} = \langle \mu_i \wedge v_i \rangle_{i \leq m}$.

Proof. Suppose that $\mathcal{M} = (\mathcal{F}, \vartheta) \in \mathcal{D}_K^\dagger$, and $\delta_{\vec{\mu}} \wedge \delta_{\vec{v}}$ is true at w . Then there are sequences of worlds $\langle u_i \rangle_{i \leq m}$ and $\langle v_i \rangle_{i \leq n}$ such that $u_0 = v_0 = w$, $u_0 < u_1 < \dots < u_m$, $v_0 < v_1 < \dots < v_n$, $u_i \in \vartheta(\mu_i)$, $v_i \in \vartheta(v_i)$, $u_m \in \vartheta(p)$, and $v_n \in \vartheta(p)$. Since $|\vartheta(p)| \leq 1$, $u_m = v_n$. Moreover, since every world in $\mathcal{F}$ has at most one predecessor, $m = n$ and $u_i = v_i$ for all $i \leq n$, whence $\mathcal{M}, w \models \delta_{\vec{\mu} \wedge \vec{v}}$. The remaining implications are trivial. $\square$

For $\mathcal{D}_{GL}^\dagger$, we need a more involved argument:

Lemma 5.3. For all $\vec{\mu} = \langle \mu_i \rangle_{i \leq m} \in (\text{Fm}(\vec{r}))^{m+1}$ and $\vec{v} = \langle v_i \rangle_{i \leq n} \in (\text{Fm}(\vec{r}))^{n+1}$,

$$\mathcal{D}_{GL}^\dagger \models \diamond \delta_{\vec{\mu}} \wedge \diamond \delta_{\vec{v}} \leftrightarrow \diamond(\delta_{\vec{\mu}} \wedge \delta_{\vec{v}}) \vee \diamond(\delta_{\vec{\mu}} \wedge \diamond \delta_{\vec{v}}) \vee \diamond(\diamond \delta_{\vec{\mu}} \wedge \delta_{\vec{v}}).$$

Proof. The right-to-left implication is derivable in K . Suppose that $\diamond \delta_{\vec{\mu}} \wedge \diamond \delta_{\vec{v}}$ is true at w in a model $\mathcal{M} = (S^\circ, <, \vartheta) \in \mathcal{D}_{GL}^\dagger$. Then there are $u, v \in S^\circ$ such that $w < u$, $w < v$, $\mathcal{M}, u \models \delta_{\vec{\mu}}$, and $\mathcal{M}, v \models \delta_{\vec{v}}$. Since $|\vartheta(p)| \leq 1$, there is $x \in S^\circ$ such that $u \leq x$, $v \leq x$, and $\vartheta(p) = \{x\}$. It is easy to see that in this case u and v are comparable, i.e., $u = v$, $u < v$, or $v < u$. Thus, at least one of the disjuncts of the right-hand side of the equivalence is true at w . $\square$

Lemma 5.4. For each $\alpha, \beta \in \Delta$, there is a set $\Delta_0 \subset_{\text{fin}} \Delta$ such that $\mathcal{D}_{GL}^\dagger \models \alpha \wedge \beta \leftrightarrow \bigvee \Delta_0$.

Proof. Let $\vec{\mu} = \langle \mu_i \rangle_{i \leq m} \in (\text{Fm}(\vec{r}))^{m+1}$ and $\vec{v} = \langle v_i \rangle_{i \leq n} \in (\text{Fm}(\vec{r}))^{n+1}$. We need to construct a set $\Delta_0 \subset_{\text{fin}} \Delta$ such that $\mathcal{D}_{\text{GL}}^+ \models \delta_{\vec{\mu}} \wedge \delta_{\vec{v}} \leftrightarrow \bigvee \Delta_0$. We proceed by induction on $m+n$. Without loss of generality, we may assume that $m \leq n$. The case $m = n = 0$ is trivial.

If $m = 0$ and $n > 0$, then $\delta_{\vec{\mu}} = \mu_0 \wedge p$ and $\delta_{\vec{v}} = v_0 \wedge \diamond \delta_{\vec{v}}$, where $\vec{v} := \langle v_1, \dots, v_n \rangle$. It is easy to see that $\text{K4} \vdash \diamond \delta_{\vec{v}} \rightarrow \diamond p$, whence $\text{K4} \vdash \delta_{\vec{\mu}} \wedge \delta_{\vec{v}} \rightarrow p \wedge \diamond p$. At the same time, since GL-frames are irreflexive and p holds in at most one world in any model from $\mathcal{D}_{\text{GL}}^+$, we have $p \wedge \diamond p \sim_{\mathcal{D}_{\text{GL}}^+} \perp$. Thus, $\delta_{\vec{\mu}} \wedge \delta_{\vec{v}} \sim_{\mathcal{D}_{\text{GL}}^+} \bigvee \emptyset$.

If $n, m > 0$, then $\delta_{\vec{\mu}} = \mu_0 \wedge \diamond \delta_{\vec{\mu}}$ and $\delta_{\vec{v}} = v_0 \wedge \diamond \delta_{\vec{v}}$, where $\vec{\mu} := \langle \mu_1, \dots, \mu_m \rangle$ and $\vec{v} := \langle v_1, \dots, v_n \rangle$. By Lemma 5.3,

$$\mathcal{D}_{\text{GL}}^+ \models \diamond \delta_{\vec{\mu}} \wedge \diamond \delta_{\vec{v}} \leftrightarrow \diamond(\delta_{\vec{\mu}} \wedge \delta_{\vec{v}}) \vee \diamond(\delta_{\vec{\mu}} \wedge \diamond \delta_{\vec{v}}) \vee \diamond(\diamond \delta_{\vec{\mu}} \wedge \delta_{\vec{v}}). \quad (1)$$

Notice that $\diamond \delta_{\vec{\mu}} \sim_{\text{K}} \delta_{\langle \top, \mu_1, \dots, \mu_m \rangle}$. Therefore, by the induction hypothesis, there are $\Delta_1, \Delta_2, \Delta_3 \subset_{\text{fin}} \Delta$ such that

$$\delta_{\vec{\mu}} \wedge \delta_{\vec{v}} \sim_{\mathcal{D}_{\text{GL}}^+} \bigvee \Delta_1, \quad \delta_{\vec{\mu}} \wedge \diamond \delta_{\vec{v}} \sim_{\mathcal{D}_{\text{GL}}^+} \bigvee \Delta_2, \quad \text{and} \quad \diamond \delta_{\vec{\mu}} \wedge \delta_{\vec{v}} \sim_{\mathcal{D}_{\text{GL}}^+} \bigvee \Delta_3. \quad (2)$$

Combining (1) and (2), we obtain

$$\mathcal{D}_{\text{GL}}^+ \models \diamond \delta_{\vec{\mu}} \wedge \diamond \delta_{\vec{v}} \leftrightarrow \diamond \bigvee \Delta_1 \vee \diamond \bigvee \Delta_2 \vee \diamond \bigvee \Delta_3.$$

Thus, for $\Delta_0 := \{ \delta_{\langle \mu_0 \wedge v_0, \mu_1, \dots, \mu_m \rangle} \mid \delta_{\vec{z}} \in \Delta_1 \cup \Delta_2 \cup \Delta_3 \}$, we have

$$\begin{aligned} \delta_{\vec{\mu}} \wedge \delta_{\vec{v}} &= (\mu_0 \wedge \diamond \delta_{\vec{\mu}}) \wedge (v_0 \wedge \diamond \delta_{\vec{v}}) \\ &\sim_{\mathcal{D}_{\text{GL}}^+} \mu_0 \wedge v_0 \wedge \left(\diamond \bigvee \Delta_1 \vee \diamond \bigvee \Delta_2 \vee \diamond \bigvee \Delta_3 \right) \\ &\sim_{\text{K}} \bigvee \Delta_0. \end{aligned}$$

□

Lemma 5.5. *Let φ be a p -positive formula. Then $\Box \varphi \leftrightarrow \Box \varphi_{\perp}$ is true in both $\mathcal{D}_{\text{K}}^+$ and $\mathcal{D}_{\text{GL}}^+$.*

Proof. Since φ is p -positive, $\text{K} \vdash \Box \varphi_{\perp} \rightarrow \Box \varphi$. Suppose that $\mathcal{M} = (S^{\circ}, R, \vartheta) \in \mathcal{D}_{\text{K}}^+ \cup \mathcal{D}_{\text{GL}}^+$, and $\Box \varphi_{\perp}$ is false at w . Then there is a world $v \in R(w)$ such that $\mathcal{M}, v \not\models \varphi_{\perp}$. It is easy to see from the definition of $\mathcal{D}_{\text{K}}$ and $\mathcal{D}_{\text{GL}}$ that $\iota v \in R(w)$, $\varphi_{\perp}$ is false at ιv , and the submodels generated by v and ιv are disjoint. Since $|\vartheta(p)| \leq 1$, $\varphi \leftrightarrow \varphi_{\perp}$ holds either at v or at ιv . Thus, φ is false at that world and $\mathcal{M}, w \not\models \Box \varphi$. □

Theorem 5.6. *For $\Lambda \in \{\text{K}, \text{GL}\}$, a formula φ is Λ -additive iff it is Λ -equivalent to a formula from A_0 .*

Proof. By Lemma 3.8, it is sufficient to prove that every p -positive formula φ is equivalent to a formula from A_0 in $\mathcal{D}_{\Lambda}^+$. We proceed by induction on the construction of φ . The base case and the induction steps for $\vee$ and $\diamond$ are trivial. The induction step for $\wedge$, by distributivity, follows from Lemmas 5.2 and 5.4. The induction step for $\Box$ follows from Lemma 5.5. □

Corollary 5.7. *A formula φ is normal additive in K or GL iff it is equivalent to a disjunction of δ -formulas.*

Remark 5.8. Our proof of Theorem 5.6, in contrast to the model-theoretic arguments of [2, 3], is constructive. In fact, we have the following: there exists an algorithm that, given a K-additive formula φ , computes a formula $\alpha \in A_0$ such that $\varphi \sim_{\text{K}} \alpha$. This algorithm proceeds in two steps:

1. Given a K -additive (and hence monotone in K) formula φ , compute a p -positive formula η such that $\varphi \sim_K \eta$.
2. Given a positive K -additive formula η , compute a formula $\alpha \in A_0$ such that $\eta \sim_K \alpha$.

The algorithm for the first step follows from the proof of [7, Theorem 4.4] together with the fact that Lyndon interpolants for given formulas can be effectively constructed in K (see, e.g., [4, Section 4]). The algorithm for the second step can be extracted from the proof of Theorem 5.6.

6 Additive formulas in S4 and Grz

Turning to the description of additive formulas in S4 and Grz, we face some difficulties. In these logics, there are additive formulas which are not equivalent to any formula from A_0 : by Proposition 2.2, the formula $\alpha = \Diamond\Box\Diamond p$ is S4-additive but not completely additive in the class of all S4-frames and formulas from A_0 are clearly completely additive. Moreover, it is easy to see that A_0 contains only four parameter-free formulas up to S4-equivalence: $\perp$, p , $\Diamond p$, and $\top$. Clearly, neither α nor $\alpha \vee p$ is equivalent to any of those four formulas even in Grz (though $\alpha \sim_{\text{Grz}} \Diamond\Box p$ and α is completely additive in all Grz-frames).

In this section, we characterize parameter-free additive formulas in S4 and Grz. As before, it is sufficient to consider only positive formulas. In fact, there are only 37 formulas in $\text{Fm}^+(p)$ up to S4-equivalence [25, Theorem 5.1], so in principle, the problem could be solved by exhaustive search. However, we present a more conceptual proof, which we hope can be extended to formulas with parameters. The overall strategy is similar to that used in the previous sections, although some of the steps differ:

1. Define a class of frames $\mathcal{D}$ that is good for Λ .
2. Describe the modal lattice $\mathfrak{Fm}^+(p)/\sim_{\mathcal{D}^\dagger}$.
3. For each equivalence class in $\text{Fm}^+(p)/\sim_{\mathcal{D}^\dagger}$, either exhibit a Λ -additive representative or prove that there is no Λ -additive formula in it.

Then, by Lemma 3.7, the Λ -additive formulas from the third step form a complete set of parameter-free Λ -additive formulas. To simplify the second step of our plan, we will use the following observation:

Lemma 6.1. *Suppose that $\Lambda \supseteq \text{S4}$, $\mathcal{D}$ is good for Λ . Then, for all elements x and y of $\mathfrak{Fm}^+(p)/\sim_{\mathcal{D}^\dagger}$,*

- *if $x \leq y \leq \Diamond x$, then $\Diamond y = \Diamond x$;*
- *if $\Box x \leq y \leq x$, then $\Box y = \Box x$.*

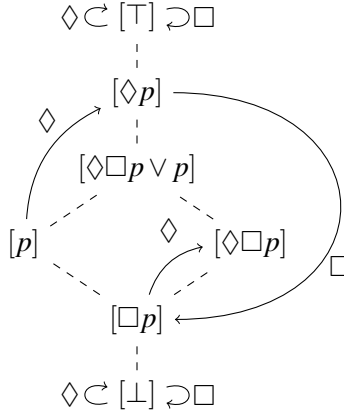
Proof. Since $\Diamond p \rightarrow \Diamond(p \vee q)$ and $\Diamond\Diamond p \leftrightarrow \Diamond p$ are derivable in S4, they are true in $\mathcal{D}^\dagger$, whence the operator $\Diamond$ in $\mathfrak{Fm}^+(p)/\sim_{\mathcal{D}^\dagger}$ is monotone and idempotent. Therefore, $\Diamond x \leq \Diamond y \leq \Diamond\Diamond x = \Diamond x$ whenever $x \leq y \leq \Diamond x$. The second claim is similar. $\square$

6.1 Additive formulas in Grz

It is well known that Grz is the logic of the following class of frames:

$$\mathcal{C}_{\text{Grz}} := \{(S, \leq) \mid S \subseteq \omega^* \text{ is a finite tree skeleton}\},$$

where $\leq$ is the reflexive closure of $<$. For $\mathcal{F} = (S, \leq) \in \mathcal{C}_{\text{Grz}}$, we put $\mathcal{F}^\circ := (S^\circ, \leq)$. Clearly, $\mathcal{F}^\circ \models \text{Grz}$, and $\pi : \mathcal{F}^\circ \rightarrow \mathcal{F}$ is a p -morphism. By Lemma 3.6, the class of general frames $\mathcal{D}_{\text{Grz}} := \pi^{-1}\mathcal{C}_{\text{Grz}}$ is good for Grz.

Figure 1: The lattice $\mathfrak{M}^+(p)/\sim_{\mathcal{D}_{\text{Grz}}^\dagger}$

Proposition 6.2. *The modal lattice $\mathfrak{M}^+(p)/\sim_{\mathcal{D}_{\text{Grz}}^\dagger}$ consists of the equivalence classes of the formulas from the set*

$$\Psi_{\text{Grz}} := \{\perp, p, \top, \diamond p, \Box p, \diamond \Box p, \diamond \Box p \vee p\},$$

where operations are shown in Figure 1 (missing arrows for $\diamond$ and $\Box$ can be restored by Lemma 6.1).

Proof. One can easily check that all formulas from Ψ_{Grz} are pairwise non-equivalent in $\mathcal{D}_{\text{Grz}}^\dagger$. Also, the ordering in Figure 1 corresponds to the ordering in Grz (i.e., if $[\alpha]$ is below $[\beta]$ in the figure, then $\text{Grz} \vdash \alpha \rightarrow \beta$). The join of $[p]$ and $[\diamond \Box p]$ is clear. We need to check that $[p] \wedge [\diamond \Box p] = [\Box p]$ or, more precisely, that $\mathcal{D}_{\text{Grz}}^\dagger \models p \wedge \diamond \Box p \rightarrow \Box p$ (the converse implication is provable in Grz). Suppose that $\mathcal{M} = (S^\circ, \leq, \vartheta) \in \mathcal{D}_{\text{Grz}}^\dagger$, $w \in \vartheta(p \wedge \diamond \Box p)$. Then there is $v \in S^\circ$ such that $\mathcal{M}, v \models \Box p$. Since $v \leq w$, $v \in \vartheta(p)$. But $|\vartheta(p)| \leq 1$, whence $w = v$ and $\mathcal{M}, w \models \Box p$.

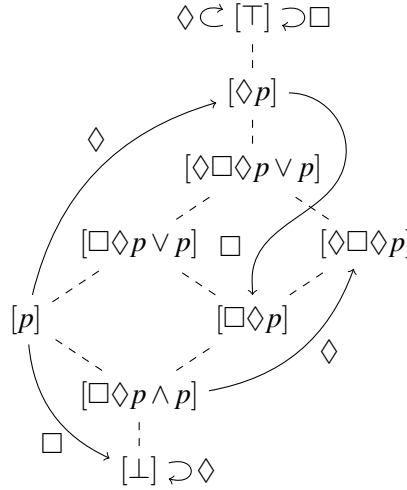
The actions of $\diamond$ and $\Box$ presented in Figure 1 are also mostly clear. We only need to check that $\Box[\diamond p] = [\Box p]$ or, more precisely, that $\mathcal{D}_{\text{Grz}}^\dagger \models \Box \diamond p \rightarrow \Box p$. Suppose that $\mathcal{M} = (S^\circ, \leq, \vartheta) \in \mathcal{D}_{\text{Grz}}^\dagger$ and $w \in \vartheta(\Box \diamond p)$. If there is $v \in S^\circ$ such that $w < v$, then $w < \iota v \in S^\circ$ and, like in the proof of Lemma 5.5, $\diamond p$ is false either in v or in ιv . This contradicts $\mathcal{M}, w \models \Box \diamond p$. Therefore, w is a maximal element of S° , i.e., $w \leq v \Leftrightarrow w = v$. Thus, since $\mathcal{M}, w \models \Box \diamond p$, $w \in \vartheta(p)$ and $\mathcal{M}, w \models \Box p$. $\square$

The first two steps of our plan are now complete. We also observed above that every formula in Ψ_{Grz} other than $\Box p$ is Grz-additive. Therefore, it remains to verify that $[\Box p]$ contains no Grz-additive representative, i.e., no Grz-additive formula is $\mathcal{D}_{\text{Grz}}^\dagger$ -equivalent to $\Box p$. Notice that this claim is stronger than the simple observation that the formula $\Box p$ itself is not Grz-additive.

Proposition 6.3. *No Grz-additive formula is $\mathcal{D}_{\text{Grz}}^\dagger$ -equivalent to $\Box p$.*

Proof. Suppose that $\varphi \in \text{Fm}(p)$ is $\mathcal{D}_{\text{Grz}}^\dagger$ -equivalent to $\Box p$. Consider the tree skeletons $S_0 := \{\langle \rangle\}$ and $S_1 := \{\langle \rangle, \langle 0 \rangle\}$. Let $\mathcal{F}_0 := (S_0^\circ, \leq)$, $\mathcal{F}_1 := (S_1^\circ, \leq)$, and $w_j := \langle \langle 0, j \rangle \rangle$ for $j = 0, 1$. Since $\varphi \sim_{\mathcal{D}_{\text{Grz}}^\dagger} \Box p$,

- $\varphi_{\mathcal{F}_1}\{\langle \rangle\} = (\Box p)_{\mathcal{F}_1}\{\langle \rangle\} = \emptyset$;
- $\varphi_{\mathcal{F}_1}\{w_j\} = (\Box p)_{\mathcal{F}_1}\{w_j\} = \{w_j\}$ for $j = 0, 1$;
- $\varphi_{\mathcal{F}_0}\{\langle \rangle\} = (\Box p)_{\mathcal{F}_0}\{\langle \rangle\} = \{\langle \rangle\}$.

Figure 2: The lattice $\mathfrak{M}^+(p)/\sim_{\mathcal{D}_{S4}^+}$

Notice that $f : x \mapsto \langle \rangle$ is a p-morphism from $\mathcal{F}_1$ onto $\mathcal{F}_0$, whence

$$\varphi_{\mathcal{F}_1} \{ \langle \rangle, w_0, w_1 \} = \varphi_{\mathcal{F}_1} f^{-1} \{ \langle \rangle \} = f^{-1} \varphi_{\mathcal{F}_0} \{ \langle \rangle \} = f^{-1} \{ \langle \rangle \} = \{ \langle \rangle, w_0, w_1 \}.$$

At the same time, $\varphi_{\mathcal{F}_1} \{ \langle \rangle \} \cup \varphi_{\mathcal{F}_1} \{ w_0 \} \cup \varphi_{\mathcal{F}_1} \{ w_1 \} = \{ w_0, w_1 \}$. Thus, φ is not Grz-additive. $\square$

Theorem 6.4. *There are exactly six Grz-additive formulas without parameters up to Grz-equivalence: $\perp$, p , $\Diamond p$, $\top$, $\Diamond \Box p$, and $\Diamond \Box p \vee p$. Five of them (all except $\top$) are normal.*

Remark 6.5. It is easy to check that there are infinitely many pairwise non-Grz-equivalent formulas in $\text{Fm}(r)$, whence there are infinitely many non-equivalent δ -formulas with one parameter in Grz and, even more so, in S4.

6.2 Additive formulas in S4

For non-empty sets T and Σ , consider the following relation on $\Sigma^* \times T$:

$$(\vec{a}, k) \preceq (\vec{b}, l) \Leftrightarrow \vec{a} \leq \vec{b}.$$

It is well known that S4 is the logic of the following class of frames:

$$\mathcal{C}_{S4} := \{ (S \times T, \preceq) \mid S \subseteq \omega^* \text{ is a finite tree skeleton, } T \subseteq \omega \text{ is a finite non-empty set} \}.$$

For $\mathcal{F} = (S \times T, \preceq) \in \mathcal{C}_{S4}$, we put $\mathcal{F}^\circ := (S^\circ \times T^\circ, \preceq)$, where $T^\circ := T \times B$. Clearly, $\mathcal{F}^\circ \models \text{S4}$ and

$$\rho : S^\circ \times T^\circ \rightarrow S \times T, \quad (v, (c, j)) \mapsto (\pi v, c)$$

is a p-morphism $\mathcal{F}^\circ \twoheadrightarrow \mathcal{F}$, whence $\mathcal{D}_{S4} := \rho^{-1} \mathcal{C}_{S4}$ is a good class of general frames for S4.

Proposition 6.6. *The modal lattice $\mathfrak{M}(p)/\sim_{\mathcal{D}_{S4}^+}$ consists of the equivalence classes of the formulas from the set*

$$\Psi_{S4} := \{ \perp, p, \top, \Diamond p, \Box \Diamond p, \Box \Diamond p \wedge p, \Box \Diamond p \vee p, \Diamond \Box \Diamond p, \Diamond \Box \Diamond p \vee p \},$$

where operations are shown in Figure 2 (missing arrows for $\Diamond$ and $\Box$ can be restored by Lemma 6.1).

Proof. Similarly to the case of Grz, it is easy to check that all formulas from Ψ_{S4} are pairwise non-equivalent in $\mathcal{D}_{S4}^\dagger$ and the ordering in Figure 2 corresponds to the ordering in S4. Most of the joins and meets are also trivial. We only need to check that $[\Box\Diamond p \vee p] \wedge [\Diamond\Box\Diamond p] = [\Box\Diamond p]$ or, more precisely, that $\mathcal{D}_{S4}^\dagger \models p \wedge \Diamond\Box\Diamond p \rightarrow \Box\Diamond p$. Suppose that $\mathcal{M} = (S^\circ \times T^\circ, \preceq, \vartheta) \in \mathcal{D}_{S4}^\dagger$, $w \in \vartheta(p)$ and $\mathcal{M}, w \models \Diamond\Box\Diamond p$. Then there is $v \in S^\circ \times T^\circ$ such that $w \preceq v$ and $\mathcal{M}, v \models \Box\Diamond p$. In particular, $\mathcal{M}, v \models \Diamond p$. Since $|\vartheta(p)| \leq 1$, $v \preceq w$. Therefore, $w \preceq u \Leftrightarrow v \preceq u$ for all u , whence $\mathcal{M}, w \models \Box\Diamond p$.

Now, let us verify that the modal operators in Figure 2 correspond to those in $\mathfrak{Fm}(p)/\sim_{\mathcal{D}_{S4}^\dagger}$. The only non-trivial implications here are $\Diamond\Box\Diamond p \rightarrow \Diamond(\Box\Diamond p \wedge p)$ and $\Box p \rightarrow \perp$. The first is derivable in S4. Let us check that $\mathcal{D}_{S4}^\dagger \models \neg\Box p$. Indeed, for each $w = (w', (c, j)) \in S^\circ \times T^\circ$, $v := (w', (c, 1-j)) \in S^\circ \times T^\circ$ and $w \preceq v$. Since the worlds w and v are distinct, p is false in at least one of them in every $\mathcal{M} \in \mathcal{D}_{S4}^\dagger$, whence $\mathcal{M}, w \not\models \Box p$. $\square$

Proposition 6.7. *No S4-additive formula is $\mathcal{D}_{S4}^\dagger$ -equivalent to any of the following formulas: $\Box\Diamond p$, $\Diamond\Box p \wedge p$, and $\Diamond\Box p \vee p$.*

Proof. Let S_0, S_1, w_0 , and w_1 be as in the proof of Proposition 6.3, $T := \{0\}$, $c_j := (0, j)$ for $j = 0, 1$. Consider the frames $\widehat{\mathcal{F}}_0 := (S_0^\circ \times T^\circ, \preceq)$ and $\widehat{\mathcal{F}}_1 := (S_1^\circ \times T^\circ, \preceq)$. Suppose that $\varphi \sim_{\mathcal{D}_{S4}^\dagger} \psi$, where $\psi \in \{\Box\Diamond p, \Diamond\Box p \wedge p, \Diamond\Box p \vee p\}$. One can check that operators $\varphi_{\widehat{\mathcal{F}}_0}$ and $\varphi_{\widehat{\mathcal{F}}_1}$ act on singletons as follows:

ψ	$\varphi_{\widehat{\mathcal{F}}_0}\{(\langle \rangle, c_1)\}$	$\varphi_{\widehat{\mathcal{F}}_1}\{(\langle \rangle, c_1)\}$	$\varphi_{\widehat{\mathcal{F}}_1}\{(w_j, c_1)\}$
$\Box\Diamond p$	$\{(\langle \rangle, c_0), (\langle \rangle, c_1)\}$	$\emptyset$	$\{(w_j, c_0), (w_j, c_1)\}$
$\Diamond\Box p \wedge p$	$\{(\langle \rangle, c_1)\}$	$\emptyset$	$\{(w_j, c_1)\}$
$\Diamond\Box p \vee p$	$\{(\langle \rangle, c_0), (\langle \rangle, c_1)\}$	$\{(\langle \rangle, c_1)\}$	$\{(w_j, c_0), (w_j, c_1)\}$

Since $\hat{f}: (x, c_j) \mapsto (\langle \rangle, c_j)$ is a p-morphism from $\widehat{\mathcal{F}}_1$ onto $\widehat{\mathcal{F}}_0$,

$$\varphi_{\widehat{\mathcal{F}}_1}\{(\langle \rangle, c_1), (w_0, c_1), (w_1, c_1)\} = \varphi_{\widehat{\mathcal{F}}_1}\hat{f}^{-1}\{(\langle \rangle, c_1)\} = \hat{f}^{-1}\varphi_{\widehat{\mathcal{F}}_0}\{(\langle \rangle, c_1)\}.$$

It is easy to check that, in all three cases,

$$\hat{f}^{-1}\varphi_{\widehat{\mathcal{F}}_0}\{(\langle \rangle, c_1)\} \neq \varphi_{\widehat{\mathcal{F}}_1}\{(\langle \rangle, c_1)\} \cup \varphi_{\widehat{\mathcal{F}}_1}\{(w_0, c_1)\} \cup \varphi_{\widehat{\mathcal{F}}_1}\{(w_1, c_1)\},$$

whence φ is not S4-additive. $\square$

Theorem 6.8. *There are exactly six S4-additive formulas without parameters up to S4-equivalence: $\perp$, p , $\Diamond p$, $\top$, $\Diamond\Box\Diamond p$, and $\Diamond\Box\Diamond p \vee p$. Five of them (all except $\top$) are normal.*

7 Some notes on interpretations

Let us make a few remarks regarding the interpretations of normal logics, partly building on the results of this paper. This task is not as straightforward as it might seem at first glance, for several reasons:

- Non-equivalent formulas can define interpretations of the same logic.

Example 7.1. In almost all standard modal logics (K, K4, GL, S4, S5, K4.3, etc.), $\tau_{\Diamond^n p}^{-1}\Lambda = \Lambda$ for all $n \geq 1$. Such logics are called *iterative* in [1]. At the same time, the formulas $\Diamond^n p$ are pairwise non-equivalent in most of them (except the ones extending S4).

- Even for fixed Λ and α , it can be challenging to provide an axiomatization of the logic interpreted in Λ by τ_α .

Example 7.2. Take $\alpha := \Diamond^2 p \vee \Diamond^3 p$ and consider the logic $\tau_\alpha^{-1}K$. This logic is strictly stronger than K : the formula $\Box^2 p \wedge \Diamond^3 \top \rightarrow \Diamond^3 p$ is derivable in it, but not in K . The complete axiomatization of $\tau_\alpha^{-1}K$ is unknown (though it follows from Proposition 7.6 below that $\tau_\alpha^{-1}K$ is the logic of the class of frames $\{(W, R \cup R^2) \mid (W, R) \in \mathcal{C}_K\}$).

Therefore, we postpone a systematic study of the interpretability of normal logics to a subsequent paper. Here we will present only the simplest observations. Firstly, one can refine [31, Theorem 4.27] by proving that infinitely many distinct *normal* logics are interpretable in every logic between K and GL :

Proposition 7.3. *Let Λ be a normal modal logic such that $K \subseteq \Lambda \subseteq GL.3$. Then infinitely many distinct normal modal logics are interpretable in Λ by modal-to-modal translations without parameters.*

Proof. Let $\alpha_n := \Diamond p \wedge \Box^{n+1} \perp$ for $n \in \omega$. Clearly, α_n are normal Λ -additive, whence the logics $\tau_{\alpha_n}^{-1}\Lambda$ are normal. It remains to show that these logics do not coincide for distinct $n \in \omega$.

By induction on $k \in \omega$, one can verify that $\tau_{\alpha_n} \Diamond^k \top \sim_K \Diamond^k \top \wedge \Box^{n+1} \perp$. Then notice that $\Diamond^k \top \wedge \Box^{n+1} \perp \sim_K \perp$ for $k > n$ and $\Diamond^k \top \wedge \Box^{n+1} \perp \sim_{GL.3} \perp$ for $k \leq n$. Thus, $\tau_{\alpha_n}^{-1}\Lambda \vdash \Box^k \perp$ iff $k > n$. $\square$

The following properties of modal-to-modal translations are straightforward:

Lemma 7.4. *For every logic Λ and formulas $\alpha, \beta \in \text{Fm}(p, \vec{r})$, $\varphi \in \text{Fm}$,*

1. $\tau_\alpha \tau_\beta \varphi = \tau_{\alpha\beta} \varphi$, whence $\tau_\beta^{-1} \tau_\alpha^{-1} \Lambda = \tau_{\alpha\beta}^{-1} \Lambda$;
2. $\tau_{\Diamond p} \varphi = \varphi$, whence $\tau_{\Diamond p}^{-1} \Lambda = \Lambda$;
3. $\tau_p^{-1} \Lambda = \text{Triv}$ and $\tau_\perp^{-1} \Lambda = \text{Ver}$.

From Corollary 4.6 and Lemma 7.4, we immediately obtain

Proposition 7.5. *The following three normal logics, and only they, are interpretable in $S5$ without parameters: Ver , Triv , and $S5$.*

For what follows, we need to describe the semantics of the interpreted logics in terms of that of the interpreting logic. It is convenient to work with modal algebras for this purpose. Basic definitions and facts can be found, for example, in [5, Section 7]. Note, however, that in the present paper we treat a modal algebra $\mathfrak{A}$ as a Boolean algebra $\mathfrak{A}^\circ$ equipped with an operator $\Diamond$ rather than $\Box$. Also, $\Diamond$ can be an arbitrary operator, possibly not normal and additive; whence $\text{Log } \mathfrak{A}$ can be an arbitrary congruential logic. For every formula $\alpha(p, \vec{r})$, consider the operator $\alpha_{\mathfrak{A}} : A^{1+n} \rightarrow A$ defined in the same way as in the case of Kripke frames (namely, $\alpha_{\mathfrak{A}}(U, \vec{V}) = \vartheta(\alpha)$ whenever $\vartheta(p) = U$ and $\vartheta(\vec{r}) = \vec{V}$). Let $\sigma_\alpha \mathfrak{A}$ denote the class of algebras $\{(\mathfrak{A}^\circ, \alpha_{\mathfrak{A}}(\cdot, \vec{V})) \mid \vec{V} \in A^n\}$. It is easy to verify that $\tau_\alpha^{-1} \text{Log } \mathfrak{A} = \text{Log } \sigma_\alpha \mathfrak{A}$. Since every logic is complete with respect to algebraic semantics, σ_α provides an equivalent view of modal-to-modal interpretations.

Recall that each Kripke frame $\mathcal{F} = (W, R)$ corresponds to the algebra $\mathfrak{A} = (\mathfrak{P}(W), R^{-1})$, where $\mathfrak{P}(W)$ is the Boolean algebra of subsets of W , in the sense that valuations (and hence all semantical notions) on $\mathcal{F}$ and $\mathfrak{A}$ coincide. Conversely, if an algebra $\mathfrak{A}$ is of the form $(\mathfrak{P}(W), \Diamond)$ with $\Diamond$ completely additive, then the frame $\mathcal{F} = (W, R_\Diamond)$, where $w R_\Diamond v : \Leftrightarrow w \in \Diamond\{v\}$, corresponds to $\mathfrak{A}$ in the same sense.

For interpretations, this correspondence yields the following: if α is completely additive in a Kripke frame $\mathcal{F} = (W, R)$, then the class of algebras $\sigma_\alpha(\mathfrak{P}(W), R^{-1})$ corresponds to the class of Kripke frames $\sigma_\alpha \mathcal{F} := \{(W, R_{\alpha_{\mathfrak{A}}(\cdot, \vec{V})}) \mid \vec{V} \in A^n\}$. Therefore, $\tau_\alpha^{-1} \text{Log } \mathcal{F} = \text{Log } \sigma_\alpha \mathcal{F}$. Similarly, if α is completely additive in a class of Kripke frames $\mathcal{C}$, then $\tau_\alpha^{-1} \text{Log } \mathcal{C} = \text{Log } \sigma_\alpha \mathcal{C}$, where $\sigma_\alpha \mathcal{C} = \bigcup_{\mathcal{F} \in \mathcal{C}} \sigma_\alpha \mathcal{F}$. This immediately gives us the following:

Proposition 7.6. *Suppose that a normal modal logic Λ is interpretable in a normal modal logic Λ_0 with the finite model property. Then Λ has the finite model property, namely $\Lambda = \text{Log } \sigma_\alpha \mathcal{C}$, where α is a formula defining interpretation and $\mathcal{C}$ is a class of finite Kripke frames such that $\text{Log } \mathcal{C} = \Lambda_0$.*

For a tree skeleton S , a non-empty set T , $v, u \in S$, and $i, j \in T$, we put $v \leq'_S u$ iff $v \leq u$ and u is maximal in $(S, \leq)$ and $(v, i) \leq'_S (u, j) \Leftrightarrow v \leq'_S u$ (compare with R' from the proof of Proposition 2.2).

Proposition 7.7. *The following five normal logics, and only they, are interpretable in Grz without parameters: Ver, Triv, Grz, $\text{Log } (S_2, \leq'_S)$, and $\text{Log } (S_2, \leq)$, where $S_2 = \{\langle \rangle\} \cup \{\langle k \rangle \mid k \in \omega\}$.*

Proof. In view of Theorem 6.4, it is sufficient to check that these five logics are interpreted in Grz by the formulas $\perp$, p , $\Diamond p$, $\Diamond \Box p$, and $\Diamond \Box p \vee p$ respectively. The first three interpretations follow from Lemma 7.4.

Let $\Lambda := \tau_\alpha^{-1} \text{Grz}$, where $\alpha := \Diamond \Box p$, $\mathcal{F}_2 := (S_2, \leq'_S)$. We need to show that $\Lambda = \text{Log } \mathcal{F}_2$. Notice that $\Diamond \Box p \sim_{\text{Grz}} \Diamond \Box \Diamond p$, whence, according to the proof of Proposition 2.2, for a Grz-frame $\mathcal{F} = (S, \leq)$, $\alpha_{\mathcal{F}}$ is completely additive and $\sigma_\alpha \mathcal{F} = \{(S, \leq'_S)\}$. In particular, $\sigma_\alpha (S_2, \leq) = \{\mathcal{F}_2\}$, whence $\mathcal{F}_2 \models \Lambda$. For the converse, we will use the following fact, which can be easily derived from the basic properties of generated frames and p-morphisms (cf. [5, Corollary 3.16]):

Lemma 7.8. *Let $\mathcal{C}$ be a class of Kripke frames and $\mathcal{F}_0 = (W_0, R_0)$ be a Kripke frame. Suppose that, for every $\mathcal{F} = (W, R) \in \mathcal{C}$ and every world $w \in W$ there is a world $w_0 \in W_0$ with a p-morphism from $\mathcal{F}_0 \uparrow w_0$ onto $\mathcal{F} \uparrow w$, where $\mathcal{F} \uparrow w$ denotes the subframe of $\mathcal{F}$ generated by w . Then $\text{Log } \mathcal{F}_0 \subseteq \text{Log } \mathcal{C}$.*

By Proposition 7.6, $\Lambda = \text{Log } \sigma_\alpha \mathcal{C}_{\text{Grz}}$. Let $\mathcal{F} = (S, \leq'_S) \in \sigma_\alpha \mathcal{C}_{\text{Grz}}$, $w \in S$. If w is maximal, then $\mathcal{F} \uparrow w$ contains only one reflexive point and is isomorphic to $\mathcal{F}_2 \uparrow \langle 0 \rangle$. Otherwise, $\mathcal{F} \uparrow w$ consists of w and some maximal points $u_0, \dots, u_{n-1}$, $n \geq 1$. Then the mapping $\langle \rangle \mapsto w$, $\langle k \rangle \mapsto u_{k \bmod n}$ is a p-morphism from $\mathcal{F}_2$ onto $\mathcal{F} \uparrow w$. Thus, by Lemma 7.8, $\text{Log } \mathcal{F}_2 \subseteq \Lambda$ and $\tau_{\Diamond \Box p}^{-1} \text{Grz} = \text{Log } (S, \leq'_S)$.

For the last interpretation, it suffices to notice that $\sigma_{\Diamond p \vee p} (S_2, \leq'_S) = (S_2, \leq)$, whence

$$\text{Log } (S_2, \leq) = \tau_{\Diamond p \vee p}^{-1} \text{Log } (S_2, \leq'_S) = \tau_{\Diamond p \vee p}^{-1} \tau_{\Diamond \Box p}^{-1} \text{Grz} = \tau_{\Diamond \Box p \vee p}^{-1} \text{Grz}$$

by the first item of Lemma 7.4. □

It is easy to adopt the above argument to the case of S4 and obtain the following characterization:

Proposition 7.9. *The following five normal logics, and only they, are interpretable in S4 without parameters: Ver, Triv, Grz, $\text{Log } (\hat{S}_2, \preceq'_S)$, and $\text{Log } (\hat{S}_2, \preceq)$, where $\hat{S}_2 := \{(\langle \rangle, 0)\} \cup \{(\langle k \rangle, i) \mid k, i \in \omega\} \subset S_2 \times \omega$.*

Now consider interpretations with parameters.

Example 7.10. Let $\alpha := \Diamond p \vee (r \wedge p)$. It is easy to see that, for a Kripke frame $\mathcal{F} = (W, R)$,

$$\sigma_\alpha \mathcal{F} = \{(W, R \cup \text{id}_V) \mid V \subseteq W\}, \quad \text{where } \text{id}_V := \{(v, v) \mid v \in V\}.$$

At the same time, it is well known that wGrz-frames can be obtained from GL-frames precisely by this procedure. Thus, by Proposition 7.6, $\text{wGrz} = \tau_\alpha^{-1} \text{GL}$.

Using the characterization of normal GL-additive formulas, one can show that wGrz is not interpretable in GL without parameters, as no disjunction of non-parametric δ -formulas can define such an interpretation. While this claim is fairly intuitive, the proof we are aware of is not entirely straightforward.

In Section 1, it was noted that $\tau_{\Diamond p \vee p}$ defines interpretations of KT, S4, and Grz in K, K4, and GL. At the same time, non-parametric interpretations in the reverse direction do not exist, since K, K4, and GL contain non-trivial variable-free formulas while KT, S4, and Grz do not (see [31, Lemma 5.1]). The situation changes if we allow parameters:

Example 7.11. Let $\alpha := (\neg r \wedge \Diamond(r \wedge p)) \vee (r \wedge \Diamond(\neg r \wedge p))$. We will show that $K = \tau_\alpha^{-1}KT$. Recall that KT is the logic of the class of frames $\mathcal{C}_{KT} = \{(S, \preceq) \mid (S, \triangleleft) \in \mathcal{C}_K\}$, where $\preceq$ is the reflexive closure of $\triangleleft$. By Proposition 7.6, $\tau_\alpha^{-1}KT = \text{Log } \sigma_\alpha \mathcal{C}_{KT}$. We claim that $\mathcal{C}_K \subseteq \sigma_\alpha \mathcal{C}_{KT}$, whence $K \supseteq \tau_\alpha^{-1}KT$ (the converse is clear). Indeed, every $\mathcal{F} = (S, \triangleleft) \in \mathcal{C}_K$ can be written as $(S, R_{\alpha_{\mathcal{F}}(\cdot, \vec{v})})$, where $\mathcal{F} := (S, \preceq) \in \mathcal{C}_{KT}$ and V consists of all sequences $v \in S$ of even length.

In a similar vein, one can show that τ_β , where $\beta := (\neg r \wedge \Diamond(r \wedge \Diamond p)) \vee (r \wedge \Diamond(\neg r \wedge \Diamond p))$, defines interpretations of $K4$ in $S4$ and of GL in Grz .

The role of parameters is also clear in the following example (cf. Proposition 7.5 and [31, Theorem 4.21]).

Proposition 7.12. *Infinitely many non-equivalent normal logics are interpretable in $S5$ with parameters.*

Proof. For $n > 0$, $\vec{r} = \langle r_l \rangle_{l < n}$, consider the formula $\alpha_n(p, \vec{r}) := \bigvee_{k \leq n} ((\gamma_n^k \vee \gamma_n^n) \wedge \Diamond(\gamma_n^k \wedge p))$, where

$$\gamma_n^k(\vec{r}) := \begin{cases} r_k \wedge \bigwedge_{l < n, l \neq k} \neg r_l & \text{for } k < n, \\ \bigwedge_{l < n} \neg \gamma_n^l & \text{for } k = n. \end{cases}$$

Let $\mathcal{F}_{3,n} := (S_{3,n}, \preceq)$, where $S_{3,n} := \{\langle \rangle, \langle 0 \rangle, \dots, \langle n-1 \rangle\} \times \omega$. It is easy to show that the logics of $\mathcal{F}_{3,n}$ are distinct: the formula $\bigwedge_{k < m} \Diamond(q_k \wedge \bigwedge_{l < m, l \neq k} \neg \Diamond q_l) \rightarrow \perp$ is valid in $\mathcal{F}_{3,n}$ iff $n < m$.

Let us show that $\tau_{\alpha_n}^{-1}S5 = \text{Log } \mathcal{F}_{3,n}$. It is well known that $S5 = \text{Log } \mathcal{F}_4$, where $\mathcal{F}_4 := (\omega, \omega \times \omega)$. By Proposition 7.6, $\tau_{\alpha_n}^{-1}S5 = \text{Log } \sigma_{\alpha_n} \mathcal{F}_4$. So, it remains to check that $\text{Log } \mathcal{F}_{3,n} = \text{Log } \sigma_{\alpha_n} \mathcal{F}_4$.

Consider the frame $\mathcal{F} = (\omega, R_{\alpha_n(\cdot, \vec{v})}) \in \sigma_{\alpha_n} \mathcal{F}_4$, where $\vec{V} := \langle V_k \rangle_{k < n}$, $V_k := \{i \in \omega \mid i \bmod (n+1) = k\}$ for $k < n$. Let $V_n := \omega \setminus \bigcup_{k < n} V_k$. Notice that, if $\vartheta(\vec{r}) = \vec{V}$, then $\vartheta(\gamma_n^k) = V_k$ for $k \leq n$, whence

$$i R_{\alpha_n(\cdot, \vec{V})} j \Leftrightarrow i \in (\alpha_n)_{\mathcal{F}_4}(\{j\}, \vec{V}) \Leftrightarrow \exists k \leq n (i \in V_k \cup V_n \wedge j \in V_k).$$

Therefore, $\mathcal{F} \cong \mathcal{F}_{3,n}$, where the isomorphism maps i to $(\langle k \rangle, s)$ if $i = s(n+1) + k$, $s \in \omega$, $k < n$ and to $(\langle \rangle, s)$ if $i = s(n+1) + n$, $s \in \omega$. Hence, $\text{Log } \mathcal{F}_{3,n} \supseteq \text{Log } \sigma_{\alpha_n} \mathcal{F}_4$.

To prove the converse inclusion, by Lemma 7.8, it suffices to show that, for each $\mathcal{F} \in \sigma_{\alpha_n} \mathcal{F}_4$ and $w \in \omega$, there is $w_0 \in S_{3,n}$ such that $\mathcal{F} \upharpoonright w$ is a p-morphic image of $\mathcal{F}_{3,n} \upharpoonright w_0$. Let $\vec{V} = \langle V_k \rangle_{k < n} \in \mathcal{P}(\omega)^n$ be such that $\mathcal{F} = (\omega, R_{\alpha_n(\cdot, \vec{V})})$, $\hat{V}_k := V_k \setminus \bigcup_{l < n, l \neq k} V_l$ for $k < n$, and $\hat{V}_n := \omega \setminus \bigcup_{k < n} \hat{V}_k$. It is easy to see that, if $\vartheta(\vec{r}) = \vec{V}$, then $\vartheta(\gamma_n^k) = \hat{V}_k$ for $k \leq n$, whence

$$i R_{\alpha_n(\cdot, \vec{V})} j \Leftrightarrow i \in (\alpha_n)_{\mathcal{F}_4}(\{j\}, \vec{V}) \Leftrightarrow \exists k \leq n (i \in \hat{V}_k \cup \hat{V}_n \wedge j \in \hat{V}_k).$$

If $w \in \hat{V}_k$ for some $k < n$ or $w \in \hat{V}_n = \omega$, then $\mathcal{F} \upharpoonright w$ consists of only one cluster (which is $\hat{V}_k \subseteq \omega$ or $\hat{V}_n = \omega$) and it is a p-morphic image of the infinite cluster $\mathcal{F}_{3,n} \upharpoonright (\langle 0 \rangle, 0)$. If $w \in \hat{V}_n \subsetneq \omega$, then $\mathcal{F} \upharpoonright w = \mathcal{F}$ and $\mathcal{F}$ consists of the root cluster $\hat{V}_n$, which sees at least one and at most n clusters (i.e., non-empty sets among $\hat{V}_k$, $k < n$). One can verify that $\mathcal{F}$ is a p-morphic image of the whole frame $\mathcal{F}_{3,n}$ in this case. $\square$

8 Problems for future research

Of course, it would be desirable to obtain a general characterization of additive formulas in normal modal logics. However, the results obtained in this paper cast doubt on the possibility of such a characterization, even for well-behaved classes of logics (for example, logics corresponding to Horn frame conditions or extensions of $S4$). Therefore, we formulate here only some of the most interesting and, at the same time, relatively accessible questions related to the characterization of additive formulas in specific logics:

1. Describe all additive formulas with parameters in S4 and Grz, all additive formulas in K4.
2. Describe all additive formulas in K4.3 and S4.3. Is it true that all additive formulas in these logics are equivalent to p -positive ones? It is known that this is not the case for monotone formulas [7, Proposition 4.7].
3. Describe all *biadditive* formulas in K, i.e., formulas $\alpha(p, q)$ such that

$$\begin{aligned}\alpha(p \vee p', q) &\sim_K \alpha(p, q) \vee \alpha(p', q), \\ \alpha(p, q \vee q') &\sim_K \alpha(p, q) \vee \alpha(p, q').\end{aligned}$$

We also pose several questions regarding interpretations of modal logics:

4. How many normal logics are interpretable with parameters in S4 and Grz? It seems plausible that there are infinitely many of them in every logic between K and Grz (cf. Remark 6.5) and possibly even in every non-tabular normal logic (cf. Proposition 7.12).
5. Is it true that every normal logic interpretable in K (GL, S4, etc.) is finitely axiomatizable? If so, what is the complexity of computing such an axiomatization from a formula defining the interpretation?

As noted after Example 7.10, the characterization of additive formulas makes it possible to prove the absence of interpretability of a certain modal logic in a given logic by checking all additive formulas. However, this approach is cumbersome. At the same time, it seems quite obvious that, for example, the logics K and K4 are not interpretable in each other, even with parameters. In this regard, it makes sense to try to develop general necessary conditions for interpretability. For example, one might suppose that transitive and non-transitive logics cannot be interpreted in each other. However, this claim is obviously wrong. To state a more accurate conjecture, we need an additional notion. A normal modal logic Λ is *pretransitive* if $\Lambda \vdash \Diamond^{n+1} p \rightarrow \bigvee_{k \leq n} \Diamond^k p$ for some $n \geq 1$.

Conjecture 8.1. *If a normal modal logic Λ is interpretable by a modal-to-modal translation (with parameters) in a pretransitive normal logic, then Λ is pretransitive.*

Notice that this conjecture, if it is true, imposes limitations on possible weaker-to-stronger translations discussed in [14].

6. Does Conjecture 8.1 hold? What can be said about interpretations of pretransitive logics in “anti-transitive” logics like K, KB, KT? What are other necessary conditions for a logic Λ to be interpretable in a logic Λ_0 with parameters? Notice that some necessary conditions for interpretations without parameters are given in [31, Section 5].

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Hyperformalism for Relevant Modal Logics

Thomas Macaulay Ferguson

Rensselaer Polytechnic Institute
Department of Cognitive Science
Troy, New York, USA
tferguson@gradcenter.cuny.edu

Shay Allen Logan

Kansas State University
Department of Philosophy
Manhattan, Kansas, USA
salogan@ksu.edu

The property of *hyperformalism* has proven to be a powerful tool in the analysis of relevant logics, revealing that increasingly weak relevant logics are closed under increasingly strong classes of non-uniform substitutions. In such substitutions, two instances of the same atom may be treated independently in virtue of syntactic features of their appearances in a complex. In this work, we extend the scope of hyperformalism to relevant modal logics by considering MPos-hyperformalism, that is, a property in which relevant modal logics are closed under substitutions in which nesting within the scope of modal operators is taken into account. We prove that the weak relevant modal logic $B^\square$ is MPos-hyperformal and investigate the classes of non-uniform substitutions under which several extensions are closed. We then consider corresponding refinements of the variable sharing property that hold of such logics. We conclude by introducing a modal logic K^{MPos} that constitutes the largest MPos-hyperformal sublogic of the classical modal logic K and provide soundness and completeness results.

1 Introduction

In typical formal logics, atomic formulas are treated as being entirely independent. This is (at least partly, and maybe wholly) captured by their being closed under uniform substitutions, since closure under uniform substitutions means that one can safely replace occurrences of distinct atoms with distinct formulas without needing to worry about in any way coordinating these substitutions.

Hyperformal logics take the notion of independence captured by uniform substitutions and run with it. More concretely (but still abstractly) a logic is hyperformal when occurrences of the same atom in sufficiently different syntactic contexts can be treated as independent. As with uniform substitutions and independence for atoms, the independence enforced by hyperformalism is made concrete in a logic L by the requirement that L -theoremhood be preserved under non-uniform substitutions treating such instances independently. *E.g.*, [10] shows that the relevant logic DR is closed under substitutions that treat atoms occurring in the scope of different numbers of conditionals—which is to say, that occur at different *depths*—as independent. For example: in the formula $p \rightarrow (q \rightarrow p)$, the two instances of the atom p appear at different depths and are thus are treated as independent, *i.e.*, as though they were distinct atoms. Consequently, the formula $p \rightarrow (q \rightarrow r)$ follows from a depth substitution. That the latter formula lacks any variable common to its antecedent and consequent ensures that it is not a theorem of DR, whence $p \rightarrow (q \rightarrow p)$ is not a theorem. Increasingly fine-grained species of hyperformalism have since been introduced (see *e.g.* [5, 21])—along with corresponding fine-grained variable sharing properties—proving hyperformalism to be a powerful tool in the analysis of relevant logics.¹

Relevant *modal* logics have nearly as long a history as relevant logics themselves.² Given how closely the analysis of modality is tied to the broader relevant logic project, it is natural to consider species of

¹For more on relevant logics see *e.g.* [11], [14], [20], or [15].

²For important early papers, see *e.g.* [16] or [8]; more recent development can be found in *e.g.* [18], [19], or [3].

hyperformalism in which occurrences of the modal $\Box$ have the same weight as $\rightarrow$ enjoys in the case of DR. The results in this paper will thus concern extending the notion of hyperformalism to modal logics whose languages include $\Box$. We will begin in Section 2 by considering modal extensions of one of the weakest standard relevant logics, B, and investigating the types of strong hyperformal properties that hold of members of this family of modal logics, including what we will call MPos-hyperformalism. We will take an alternative tack in Section 3 by describing and characterizing the largest MPos-hyperformal fragment of the modal logic K through a tableaux system. We will conclude by considering the relationship between the two approaches and the prospects for variants of hyperformalism in the modal logic setting.

2 $B^\Box$ and Its Extensions

We begin by considering hyperformalism in the context of familiar relevant modal logics, starting from the base logic $B^\Box$. To introduce the logic $B^\Box$, let $\mathbf{L}$ be a propositional language formed in the usual way from the set of atomic formulas At using the connectives $\wedge$, $\vee$, $\neg$, and $\rightarrow$. Then B is axiomatized by the following list.

BA1. $A \rightarrow A$	BA7. $\neg\neg A \rightarrow A$
BA2. $(A \wedge B) \rightarrow A; (A \wedge B) \rightarrow B$	BR1. $A, B \Rightarrow A \wedge B$
BA3. $A \rightarrow (A \vee B); B \rightarrow (A \vee B)$	BR2. $A \rightarrow B, A \Rightarrow B$
BA4. $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$	BR3. $A \rightarrow \neg B \Rightarrow B \rightarrow \neg A$
BA5. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$	BR4. $A \rightarrow B, C \rightarrow D \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow D)$
BA6. $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$	

Modal extensions of B follow the addition of one or more of the following axioms or rules; the base system with $B^\Box$ defined by adding C and RM to B:³

C $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$	K $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
T $\Box A \rightarrow A$	RM $A \rightarrow B \Rightarrow \Box A \rightarrow \Box B$
4 $\Box A \rightarrow \Box \Box A$	RN $A \Rightarrow \Box A$

2.1 Non-Uniform Substitutions in Relevant Logics

B, along with a few other relevant logics, exhibits a few forms of hyperformalism worth describing. But we will go about this in a mildly idiosyncratic way, since it will serve our aims below to do so.

To begin, let $\text{CPos} = \{c\}^*$ be the Kleene closure of the set containing just c .⁴ Say that a function $\sigma : \text{CPos} \times \text{At} \rightarrow \mathbf{L}$ is a CPos-substitution on $\mathbf{L}$. CPos-substitutions naturally extended to act on all of $\text{CPos} \times \mathbf{L}$ via the recursive clauses in Figure 1. A set of formulas X is CPos-hyperformal when for all CPos-substitutions σ , if $A \in X$, then $\sigma_e(A) \in X$. As discussed in the introduction, note that in CPos-hyperformal logics, atoms occurring ‘at’ different CPos-labels in a formula are treated as independent to

³We preserve the standard naming conventions as found in e.g. [2].

⁴Usually, instead of CPos, one uses natural numbers and refers to these numbers as ‘depths’; the numerical value of depth, clearly, can be recovered by counting the instances of c in the position. See e.g. [10, 9, 22]. Sets that play the role that CPos plays here will be called sets of *positions* below.

$$\begin{array}{ll}
\sigma_{\bar{x}}(A \wedge B) = \sigma_{\bar{x}}(A) \wedge \sigma_{\bar{x}}(B) & \sigma_{\bar{x}}(A \wedge B) = \sigma_{\bar{x}}(A) \wedge \sigma_{\bar{x}}(B) \\
\sigma_{\bar{x}}(A \vee B) = \sigma_{\bar{x}}(A) \vee \sigma_{\bar{x}}(B) & \sigma_{\bar{x}}(A \vee B) = \sigma_{\bar{x}}(A) \vee \sigma_{\bar{x}}(B) \\
\sigma_{\bar{x}}(\neg A) = \neg \sigma_{\bar{x}}(A) & \sigma_{\bar{x}}(\neg A) = \neg \sigma_{n\bar{x}}(A) \\
\sigma_{\bar{x}}(A \rightarrow B) = \sigma_{c\bar{x}}(A) \rightarrow \sigma_{c\bar{x}}(B) & \sigma_{\varepsilon}(A \rightarrow B) = \sigma_c(A) \rightarrow \sigma_c(B) \\
& \sigma_{\bar{x}}(A \rightarrow B) = \sigma_{1\bar{x}}(A) \rightarrow \sigma_{r\bar{x}}(B) \text{ for } \bar{x} \neq \varepsilon
\end{array}$$

Figure 1: CPos-substitutions on the left; LRCNPos-substitutions on the right.

the extent that they can be replaced by different formulas with no worry about coordinating between the different parts of this replacement.

It turns out, as first shown in [10], that B is CPos-hyperformal. This fact is the first of several hyperformalism results one can now find in the literature (see [21, 13, 5] for other such results). To describe another (much more powerful) such result, we first define X^{*c} to be $X^* \cup \{\bar{x}c \mid \bar{x} \in X^*\}$ and let $\text{LRCNPos} = \{1, r, n\}^{*c}$.

What we just did with CPos, we can now do with LRCNPos. Explicitly, say LRCNPos-substitutions are functions $\sigma : \text{LRCNPos} \times \text{At} \rightarrow \mathbf{L}$ for which $\sigma_{\bar{x}n\bar{y}}(p) = \sigma_{\bar{y}\bar{x}}(p)$ for all $\{\bar{x}, \bar{y}\} \subseteq \text{LRCNPos}$ and $p \in \text{At}$.⁵ We extend such a function to all of $\text{LRCNPos} \times \mathbf{L}$ using the recursive clauses in Figure 1. Say that a set of formulas X is LRCNPos-hyperformal when for all LRCNPos-substitutions σ , if $A \in X$, then $\sigma_{\varepsilon}(A) \in X$. As shown in [21], it turns out that B is *also* LRCNPos-hyperformal.

LRCNPos-hyperformalism is interesting in other ways too. To see this, let an LRCNPos-assignment be a function $\alpha : \text{LRCNPos} \times \text{At} \rightarrow \{0, 1\}$. LRCNPos-assignments extend to all of $\text{LRCNPos} \times \mathbf{L}$ by the recursive clauses given in Figure 2. Say that A is LRCNPos-valid (and we write $\models_{\text{LRCNPos}} A$) just if $\alpha_{\varepsilon}(A) = 1$ for all LRCNPos-assignments α . LRCNPos-validity can be tracked by a fairly obvious tableau system similar to the one we present below. In [21], the authors show not only that this tableau system is sound and complete for LRCNPos-validity, but that the logic it captures is the largest LRCNPos-hyperformal subset of classical logic and—more surprising yet—that it has the variable sharing property.

So not only does hyperformalism point us in the direction of interesting features of known relevant logics, it also points us in the direction of previously unknown relevant logics with simple semantic theories as well. Our goal in this paper is to extend the techniques used in these results to the case of modal logics.

Following the guide in the introduction, we begin by examining the forms of hyperformalism exhibited by known relevant modal logics.⁶ We will begin our investigation by focusing first on $B^{\square}$, and will turn to stronger relevant modal logics in the subsequent section.

To define hyperformalism for modal relevant logics, we first need to do a bit of setup. Our initial set of positions is $\text{MPos} = \{1, r, n, b\}^{*c}$. An MPos-substitution is a function $\text{MPos} \times \text{At} \rightarrow \mathbf{L}$ for which

$$\begin{array}{l}
\alpha_{\bar{x}}(A \wedge B) = \min(\alpha_{\bar{x}}(A), \alpha_{\bar{x}}(B)); \\
\alpha_{\bar{x}}(A \vee B) = \max(\alpha_{\bar{x}}(A), \alpha_{\bar{x}}(B)); \\
\alpha_{\bar{x}}(\neg A) = 1 - \alpha_{n\bar{x}}(A); \\
\alpha_{\varepsilon}(A \rightarrow B) = \max(1 - \alpha_c(A), \alpha_c(B)); \text{ and} \\
\alpha_{\bar{x}}(A \rightarrow B) = \max(1 - \alpha_{1\bar{x}}(A), \alpha_{r\bar{x}}(B)) \text{ for } \bar{x} \neq \varepsilon.
\end{array}$$

Figure 2: LRCNPos-assignments

⁵This is a feature described in [21] as “faithfulness.”

⁶For more on relevant modal logics see [6] and [8].

- $\sigma_{\bar{x}}(A \wedge B) = \sigma_{\bar{x}}(A) \wedge \sigma_{\bar{x}}(B);$
- $\sigma_{\bar{x}}(A \vee B) = \sigma_{\bar{x}}(A) \vee \sigma_{\bar{x}}(B);$
- $\sigma_{\bar{x}}(\neg A) = \neg \sigma_{\bar{x}}(A);$
- $\sigma_{\varepsilon}(A \rightarrow B) = \sigma_{\varepsilon}(A) \rightarrow \sigma_{\varepsilon}(B);$
- $\sigma_{\bar{x}}(A \rightarrow B) = \sigma_{1\bar{x}}(A) \rightarrow \sigma_{r\bar{x}}(B)$ for $\bar{x} \neq \varepsilon$; and
- $\sigma_{\bar{x}}(\Box A) = \Box \sigma_{b\bar{x}}(A).$

Figure 3: MPos-substitutions.

$\sigma_{\bar{x}nn\bar{y}}(p) = \sigma_{\bar{x}\bar{y}}(p)$ for all $\{\bar{x}, \bar{y}\} \subseteq \text{MPos}$ and $p \in \text{At}$. We extend MPos-substitutions to act on all of $\mathbf{L}$ by the recursive clauses in Figure 3.

We then say that a logic is *MPos-hyperformal* if its set of theorems is closed under MPos-substitutions.

2.2 The Base System

As promised, we begin by examining $\mathbf{B}^{\Box}$. Given the preceding definitions, we can quickly demonstrate the main result concerning $\mathbf{B}^{\Box}$:

Theorem 2.1. $\mathbf{B}^{\Box}$ is MPos-hyperformal.

Proof. By induction on derivations. The details for the axioms and rules of $\mathbf{B}$ follow from arguments virtually identical to those provided for Theorems 2 and 3 of [21], so we consider the axiom $\mathbf{C}$ and rule $\mathbf{RM}$.

For $\mathbf{C}$, we must show that every substitution applied to an instance of this axiom is itself provable. For an arbitrary substitution σ , some manipulation reveals that $\sigma_{\varepsilon}((\Box A \wedge \Box B) \rightarrow \Box(A \wedge B))$ is $(\Box \sigma_{bc}(A) \wedge \Box \sigma_{bc}(B)) \rightarrow \Box(\sigma_{bc}(A) \wedge \sigma_{bc}(B))$. But this is another instance of $\mathbf{C}$, so provable.

For $\mathbf{RM}$, we must show that for an arbitrary substitution σ that if $\Box A \rightarrow \Box B$ is a theorem due to an application of $\mathbf{RM}$, then $\sigma_{\varepsilon}(\Box A \rightarrow \Box B)$ is a theorem as well. So suppose that $\Box A \rightarrow \Box B$ is a theorem of $\mathbf{B}^{\Box}$ yielded by the application of $\mathbf{RM}$ to an earlier theorem $A \rightarrow B$. Fix an arbitrary σ and define an further substitution σ^* by the following scheme:

$$\sigma_{\bar{x}}^*(p) = \begin{cases} \sigma_{\bar{y}bc}(p) & \text{if } \bar{x} = \bar{y}c \\ \sigma_{\bar{x}}(p) & \text{otherwise} \end{cases}$$

Clearly, σ^* is faithful and is thus itself a substitution. Moreover, a trivial induction shows that $\sigma_{\bar{x}}^*(A) = \sigma_{\bar{y}bc}(A)$ when $\bar{x} = \bar{y}c$ for arbitrary (not necessarily atomic) formulas A . Consequently, by induction hypothesis, $\sigma_{\varepsilon}^*(A \rightarrow B)$ is a theorem, whence $\sigma_{\varepsilon}^*(A) \rightarrow \sigma_{\varepsilon}^*(B)$ is a theorem, whence $\sigma_{bc}(A) \rightarrow \sigma_{bc}(B)$ is a theorem. Take a derivation of this formula and apply $\mathbf{RM}$ to yield $\Box \sigma_{bc}(A) \rightarrow \Box \sigma_{bc}(B)$. With some manipulation, this establishes theoremhood of $\sigma_{\varepsilon}(\Box A \rightarrow \Box B)$; as σ was arbitrary, we infer that $\mathbf{RM}$ preserves hyperformalism. $\square$

Now, we can turn to the matter of the types of variable sharing properties that are enjoyed by $\mathbf{B}^{\Box}$. As in [10, 5, 21], non-uniform substitutions can be applied to show that $\mathbf{B}^{\Box}$ enjoys very strong versions of the traditional hallmark of relevant logics. First, consider some definitions and lemmas:

Definition 2.2. For two positions $\bar{x}$ and $\bar{y}$, say that $\bar{x} \sim_{nn} \bar{y}$ if one can produce one from the other by the addition or deletion of instances of the string nn . Say that a substitution σ is *atomic* in case its range is At and *injective up to faithfulness* in case $\sigma_{\bar{x}}(A) = \sigma_{\bar{y}}(B)$ holds if and only if $\bar{x} \sim_{nn} \bar{y}$ and $A = B$.

The “faithfulness” of the equivalence relation $\sim_{\text{nn}}$ rests in its assurance that axioms reflecting double negation introduction and elimination are preserved. In considering the results for modal logics based on weak relevant logics in which axioms like $A \rightarrow \neg\neg A$ fail—like BM—one might be interested in relaxing the assumption of faithfulness. Note also that in proving Theorem 2.1, we had assumed a condition that $\sigma_{\bar{x}\text{nn}\bar{y}}(p) = \sigma_{\bar{x}\bar{y}}(p)$ for all atoms p , considerations of elegance lead us to relax the condition and assume only injectivity up to faithfulness instead.

For a formula A and instance of a subformula B appearing in A , we define the position of B in A — $\text{pos}(A[B])$ —to be the position in (element of) MPos corresponding to the syntactic path through which one arrives at B within A . For example, $\text{pos}(\Box(p \rightarrow \Box q)[q]) = \text{brb}$. (For more precise recursive details, the reader is invited to adapt the steps described in [5] or [21].)

We will first state the strongest MPos-based variable sharing property available in the case of $B^\Box$.

Definition 2.3. A logic in the language $\mathbf{L}$ enjoys the MPos-variable sharing property in case whenever $A \rightarrow B$ is provable, there are $A[p]$ in A and $B[p]$ in B such that $\text{pos}(A[p] \rightarrow B) \sim_{\text{nn}} \text{pos}(A \rightarrow B[p])$.

To prove this, we first observe the following:

Lemma 2.4. $B^\Box$ has the variable sharing property.

Proof. This follows from Theorem 4.4 of [19], which shows that the relevant modal logic RS5 has the variable sharing property. Insofar as $B^\Box$ is a subsystem of RS5, any theorem of the former is a theorem of the latter and thus must have the variable sharing property. $\square$

Lemma 2.4 provides the necessary groundwork to begin the following theorem:

Theorem 2.5. $B^\Box$ has the MPos-variable sharing property.

Proof. Suppose that $A \rightarrow B$ is a theorem of $B^\Box$. Choose a function g from pairs $\langle \bar{x}, p \rangle \in \text{MPos} \times \text{At}$ to At such that $g(\langle \bar{x}, p \rangle) = g(\langle \bar{y}, q \rangle)$ iff $\bar{x} \sim_{\text{nn}} \bar{y}$ and $p = q$. Define the MPos-substitution σ by $\sigma_{\bar{x}}(p) = p_{g(\langle \bar{x}, p \rangle)}$. By Theorem 2.1, $\sigma_c(A) \rightarrow \sigma_c(B)$ is a theorem of $B^\Box$. By Lemma 2.4, $\sigma_c(A)$ and $\sigma_c(B)$ share some variable p_k , but consideration of the definition of g entails that an atom p_k appears in both A and B only in case $\text{pos}(A[p_k]) \sim_{\text{nn}} \text{pos}(B[p_k])$, witnessing the MPos-variable sharing property. $\square$

This also allows the description of some more simply-stated—but nevertheless strong—versions of the variable sharing property.

Corollary 2.6. If $A \rightarrow B$ is a theorem of $B^\Box$, then A and B share a propositional variable that is nested within the same number of instances of $\Box$ in both A and B .

If one, like [21], interprets relevance in terms of subject-matter, Lemma 2.4 constitutes a reflection of the discussion of [4], that is, that $\Box p$ and p may be irrelevant to one another reinforces an interpretation of the intensional $\Box$ as modifying the subject-matter of sentences to which it applies.

2.3 Extensions of $B^\Box$

Now, we proceed to consider some extensions to $B^\Box$ and consider hyperformality in such extensions. For $S \subseteq \{T, 4, K, \text{RN}\}$, let $B_S^\Box$ be the logic we get by adding the members of S as additional axioms/rules. To define hyperformalism for these logics, we define relations on MPos corresponding to T, 4, K, and RN by saying for all $\bar{x}$ and $\bar{y}$ in MPos that $\bar{x}\bar{b}\bar{y} \sim_T \bar{x}\bar{y}$, that $\bar{x}\bar{b}\bar{y} \sim_4 \bar{x}\bar{b}\bar{b}\bar{y}$, that $\bar{x}\bar{b}\bar{l}\bar{y} \sim_K \bar{x}\bar{l}\bar{b}\bar{y}$ and $\bar{x}\bar{b}\bar{r}\bar{y} \sim_K \bar{x}\bar{r}\bar{b}\bar{y}$, and that $\bar{x}\bar{l}\bar{b} \sim_{\text{RN}} \bar{x}\bar{r}\bar{b}$.

Where $S \subseteq \{T, 4, K, \text{RN}\}$, let $\sim_S$ be the smallest equivalence relation generated by $\sim_{\text{nn}} \cup \bigcup_{s \in S} \sim_s$ and write $[\bar{x}]_S$ for the $\sim_S$ -equivalence class of $\bar{x}$. Let $\text{MPos}_S = \{[\bar{x}]_S : \bar{x} \in \text{MPos}\}$ and MPos_S-substitutions

be functions $\sigma : \text{MPos}_S \times \text{At} \rightarrow \mathbf{L}$. MPos_S -substitutions extend as indicated in the previous section. Say that a set of formulas X is MPos_S -hyperformal just if for all $A \in X$ and all MPos_S -substitutions α , $\alpha_{[e]_S}(A) \in X$.

We can now state the result for extensions of $B^\square$:

Theorem 2.7. *For each $S \subseteq \{T, 4, K, RN\}$, $B_S^\square$ is MPos_S -hyperformal.*

Proof. By induction on derivations. Appealing to Theorem 2.1 leaves only the four cases to consider:

For T, suppose $T \in S$. Note that $\sigma_{[e]_S}(\Box A \rightarrow A) = \Box \sigma_{[bc]_S}(A) \rightarrow \sigma_{[c]_S}(A)$. Now observe that since $bc \sim_T c$, $\sigma_{[bc]_S}(A) = \sigma_{[c]_S}(A)$. So in fact $\Box \sigma_{[bc]_S}(A) \rightarrow \sigma_{[c]_S}(A)$ is another instance of T.

For 4, suppose that $4 \in S$. We show that $\sigma_{[e]_S}(\Box A \rightarrow \Box \Box A)$ is a theorem. Some manipulation reveals this to be equivalent to $\Box \sigma_{[bc]_S}(A) \rightarrow \Box \Box \sigma_{[bbc]_S}(A)$. Since $bc \sim_4 bbc$, $\sigma_{[bc]_S}(A) = \sigma_{[bbc]_S}(A)$, showing that this is itself an instance of 4.

For K, suppose $K \in S$. We compute that $\sigma_{[e]_S}(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))$ is

$$\Box(\sigma_{[lbc]_S}(A) \rightarrow \sigma_{[rbc]_S}(B)) \rightarrow (\Box \sigma_{[blc]_S}(A) \rightarrow \Box \sigma_{[brc]_S}(B)).$$

Now observe that since $lbc \sim_K blc$, $\sigma_{[lbc]_S}(A) = \sigma_{[blc]_S}(A)$, and since $rbc \sim_K brc$, $\sigma_{[rbc]_S}(B) = \sigma_{[brc]_S}(B)$. So the offset formula is again an instance of K.

For RN, assume that $RN \in S$ and that $\Box A$ was derived from an application of RN to an earlier theorem A. Fix a substitution σ and define a further substitution σ^{**} so that

$$\sigma_{[\bar{x}]_S}^{**}(p) = \begin{cases} \sigma_{[\bar{y}lc]_S}(p) & \text{if } \bar{x} = \bar{y}c \\ \sigma_{[\bar{x}b]_S}(p) & \text{otherwise} \end{cases}$$

Let us review three facts about σ^{**} : First, one can check that σ^{**} is faithful. Second, as before, an easy induction establishes that the two cases of the definition remain correct for any formula A . Third, σ^{**} respects each relation $\sim_S$, i.e., if $\bar{x} \sim_S \bar{y}$ then $\sigma_{[\bar{x}]_S}^{**}(A) = \sigma_{[\bar{y}]_S}^{**}(A)$. Consequently, by induction hypothesis, $\sigma_{[e]_S}^{**}(A)$ is a theorem. Now, we consider two cases based on the form of A . If A is of the form $B \rightarrow C$, then $\sigma_{[e]_S}^{**}(B \rightarrow C) = \sigma_{[c]_S}^{**}(B) \rightarrow \sigma_{[c]_S}^{**}(C)$. This by selection of σ^{**} is $\sigma_{[lb]_S}(B) \rightarrow \sigma_{[lb]_S}(C)$ and as $lb \sim_{RN} rb$, $\sigma_{[lb]_S}(B) \rightarrow \sigma_{[rb]_S}(C) = \sigma_{[b]_S}(B \rightarrow C)$ is provable as well. By an application of RN, then, $\Box(\sigma_{[b]_S}(B \rightarrow C))$ is a theorem, i.e., $\sigma_{[e]_S}(\Box(B \rightarrow C))$ is a theorem as well. In case A is not of the form $B \rightarrow C$, then by construction, $\sigma_{[e]_S}^{**}(A) = \sigma_{[b]_S}(A)$. By an application of RN, $\Box(\sigma_{[b]_S}(A))$ will be a theorem, whence $\sigma_{[e]_S}(\Box A)$ will be a theorem as well. □

As before, we can apply features of hyperformalism to demonstrate refined variable sharing properties enjoyed by relevant modal logics:

Definition 2.8. A logic in the language $\mathbf{L}$ enjoys the MPos_S -variable sharing property in case whenever $A \rightarrow B$ is provable, there are $A[p]$ in A and $B[p]$ in B such that $\text{pos}(A[p] \rightarrow B) \sim_{nn} \text{pos}(A \rightarrow B[p])$ and $\text{pos}(A[p] \rightarrow B) \sim_S \text{pos}(A \rightarrow B[p])$.

We can provide a schematic characterization of the variable sharing properties holding of these systems:

Theorem 2.9. *Each logic $B_S^\square$ has the MPos_S -variable sharing property.*

Proof. By analogous means to Theorem 2.5, modifying the definition of g to require respect for $\sim_S$. □

Many such systems will have rather elegant and novel variable sharing properties, an example of which can be illustrated by the system $B_{\{4\}}^\square$:

Corollary 2.10. *Say that an occurrence of an atom in a formula has extensional mode if it is not within the scope of a $\Box$ and intensional mode otherwise. Then if $A \rightarrow B$ is a theorem of $B_{\{4\}}^\Box$, then there is a variable occurring in both A and B with identical mode.*

The inclusion of the axiom 4 eliminates the *count* of instances of $\Box$ from playing a role in determining a variable sharing property. (If translated into terms of aboutness, this means that including axiom 4 ensures what [4] calls “K-idempotence,” i.e., that while the subject-matters of A and $\Box A$ may differ, those of $\Box A$ and $\Box\Box A$ must coincide.) Axioms K or T, themselves, have consequences for the strength of the variable sharing properties holding of systems in which they are included.

3 K^{MPos}

$B^\Box$ is a MPos-hyperformal sublogic of the classical modal logic K. We now turn to describing another MPos-hyperformal sublogic of K. We will call this logic K^{MPos} and give three different descriptions of it. The first description is in terms of MPos-varying Kripke models. The second is via a tableau system that we will show is both sound and complete for the varying Kripke models. The third description is a characterization of where K^{MPos} sits among the MPos-hyperformal subsets of K. As we will see, it is the largest such set, which gives us good reason to study it further. Unfortunately, because of space constraints, that study will have to wait for another day.

3.1 The Semantic Characterization

Given a Kripke frame $K = \langle W, \rho \rangle$, a MPos^K -assignment is a function $\alpha : \text{MPos} \times W \times \text{At} \rightarrow \{0, 1\}$. MPos^K -assignments extend via the following recursive clauses:

- $\alpha_{\bar{x}}(\omega, A \wedge B) = \min(\alpha_{\bar{x}}(\omega, A), \alpha_{\bar{x}}(\omega, B));$
- $\alpha_{\bar{x}}(\omega, A \vee B) = \max(\alpha_{\bar{x}}(\omega, A), \alpha_{\bar{x}}(\omega, B));$
- $\alpha_{\varepsilon}(\omega, A \rightarrow B) = \max(1 - \alpha_{\varepsilon}(\omega, A), \alpha_{\varepsilon}(\omega, B));$
- $\alpha_{\bar{x}}(\omega, A \rightarrow B) = \max(1 - \alpha_{\bar{x}}(\omega, A), \alpha_{\bar{x}}(\omega, B))$ for $\bar{x} \neq \varepsilon$;
- $\alpha_{\bar{x}}(\omega, \neg A) = 1 - \alpha_{\bar{x}}(\omega, A);$ and
- $\alpha_{\bar{x}}(\omega, \Box A) = \inf\{\alpha_{\bar{x}}(v, A) : \langle \omega, v \rangle \in \rho\}.$

We say that A is MPos-valid (and we write $\models_{\text{MPos}} A$) just if for all Kripke frames K , all MPos^K -assignments α , and all $\omega \in K$, $\alpha_{\varepsilon}(\omega, A) = 1$.

As a referee has helpfully pointed out, it is possible to see functions $\alpha : \text{MPos} \times W \times \text{At} \rightarrow \{0, 1\}$ as (somewhat) ordinary Kripke models with a funny set of worlds—in particular, with the set of worlds $\text{MPos} \times W$ and with assignments that, as it were, ‘happen to’ vary along one of the world-dimensions in regular ways. This sort of thing has been explored—albeit in a different setting—in [12], where some of the details were worked out. We think that this is a selling point for the logics at hand—a point that we will revisit in the conclusion.

3.2 The Tableaux Characterization

MPos-validity can be tracked by analytic tableaux that have nodes that are either quadruples $\langle \bar{x}, \omega, A, i \rangle$ or R-expressions of the form $\omega_i R \omega_j$. In quadruples, the $\bar{x}$ -index is drawn from MPos and tracks positions,

$\bar{x}, \omega, B_1 \vee B_2, 1$ $\swarrow \quad \searrow$ $\bar{x}, \omega, B_1, 1 \quad \bar{x}, \omega, B_2, 1$	$\bar{x}, \omega, B_1 \vee B_2, 0$ $\downarrow$ $\bar{x}, \omega, B_1, 0$ $\bar{x}, \omega, B_2, 0$	$\bar{x}, \omega, B_1 \wedge B_2, 1$ $\downarrow$ $\bar{x}, \omega, B_1, 1$ $\bar{x}, \omega, B_2, 1$	$\bar{x}, \omega, B_1 \wedge B_2, 0$ $\swarrow \quad \searrow$ $\bar{x}, \omega, B_1, 0 \quad \bar{x}, \omega, B_2, 0$
$\varepsilon, \omega, B_1 \rightarrow B_2$ $\swarrow \quad \searrow$ $c, \omega, B_1, 0 \quad c, \omega, B_2, 1$	$\bar{y}, \omega, B_1 \rightarrow B_2$ $\swarrow \quad \searrow$ $1\bar{y}, \omega, B_1, 0 \quad r\bar{y}, \omega, B_2, 1$	$\varepsilon, \omega, B_1 \rightarrow B_2, 0$ $\downarrow$ $c, \omega, B_1, 1$ $c, \omega, B_2, 0$	$\bar{y}, \omega, B_1 \rightarrow B_2, 0$ $\downarrow$ $1\bar{y}, \omega, B_1, 1$ $r\bar{y}, \omega, B_2, 0$
$\bar{x}, \omega, \neg B, 1$ $\downarrow$ $n\bar{x}, \omega, B, 0$	$\bar{x}, \omega, \neg B, 0$ $\downarrow$ $n\bar{x}, \omega, B, 1$	$\omega_1 R \omega_2$ $\bar{x}, \omega_1, \Box B, 1$ $\downarrow$ $b\bar{x}, \omega_2, B, 1$	$\bar{x}, \omega, \Box B, 0$ $\downarrow$ $\omega R v$ $b\bar{x}, v, B, 0$

Figure 4: Tableau rules; we assume $\bar{y} \neq \varepsilon$ and v is fresh.

the ω -index is drawn from some set $\Omega = \{\omega_i\}_{i=1}^\infty$ used to track *worlds*, A is a formula and $i \in \{0, 1\}$. R-expressions, on the other hand, are used to track accessibility facts.

Tableaux are governed by the rules in Figure 4. A *branch* on a tableau closes when it contains nodes of the form $\langle \bar{x}, \omega, A, 1 \rangle$ and $\langle \bar{y}, \omega, A, 0 \rangle$ for $\bar{x} \sim \bar{y}$. A *tableau* closes when all of its branches close. We will write $\vdash_{\text{MPos}} A$ to mean that there is a closed S -tableau that begins at $\langle \varepsilon, \omega_1, A, 0 \rangle$.⁷

3.2.1 Metatheory 1: Setup

We will say that a *branchset* is a set all of whose members are possible nodes on a tableau. So if β is a branchset, then each member of β either has the form $\langle \bar{x}, \omega, A, i \rangle$ with $\bar{x} \in \text{MPos}$, $\omega \in \Omega$, A a formula, and $i \in \{0, 1\}$ or has the form $\omega_i R \omega_j$ with $\{\omega_i, \omega_j\} \subseteq \Omega$. Given a branchset β , let Ω_β be the subset of Ω containing all ω_i that occur in some member of β . A *correspondence* is a triple consisting of a Kripke frame $K = \langle W, \rho \rangle$, a function $f : \Omega_\beta \rightarrow W$, and an MPos^K assignment α . Given a correspondence $\kappa = \langle K, f, \alpha \rangle$ with $K = \langle W_K, \rho_K \rangle$, κ is said to *conform to* β when $\alpha_{\bar{x}}(f(\omega), A) = i$ if $\langle \bar{x}, \omega, A, i \rangle \in \beta$, and $\langle f(\omega_i), f(\omega_j) \rangle \in \rho_K$ if $\omega_i R \omega_j \in \beta$.

Given a branchset β , we define the $\langle \bar{x}, \omega, A, i \rangle$ -extensions of β by cases as follows. First, β is the only $\langle \bar{x}, \omega, A, i \rangle$ -extension of β if $\langle \bar{x}, \omega, A, i \rangle \notin \beta$. Otherwise,

- if A is an atom, then β is the only $\langle \bar{x}, \omega, A, i \rangle$ -extension of β .
- Both $\beta \cup \{\langle \bar{x}, \omega, B_1, 1 \rangle\}$ and $\beta \cup \{\langle \bar{x}, \omega, B_2, 1 \rangle\}$ are $\langle \bar{x}, \omega, B_1 \vee B_2, 1 \rangle$ -extensions of β .
- $\beta \cup \{\langle \bar{x}, \omega, B_1, 0 \rangle, \langle \bar{x}, \omega, B_2, 0 \rangle\}$ is the only $\langle \bar{x}, \omega, B_1 \vee B_2, 0 \rangle$ -extension of β .
- $\beta \cup \{\langle \bar{x}, \omega, B_1, 1 \rangle, \langle \bar{x}, \omega, B_2, 1 \rangle\}$ is the only $\langle \bar{x}, \omega, B_1 \wedge B_2, 1 \rangle$ -extension of β .
- Both $\beta \cup \{\langle \bar{x}, \omega, B_1, 0 \rangle\}$ and $\beta \cup \{\langle \bar{x}, \omega, B_2, 0 \rangle\}$ are $\langle \bar{x}, \omega, B_1 \wedge B_2, 0 \rangle$ -extensions of β .
- Both $\beta \cup \{\langle c, \omega, B_1, 0 \rangle\}$ and $\beta \cup \{\langle c, \omega, B_2, 1 \rangle\}$ are $\langle \varepsilon, \omega, B_1 \rightarrow B_2, 1 \rangle$ -extensions of β .
- $\beta \cup \{\langle c, \omega, B_1, 1 \rangle, \langle c, \omega, B_2, 0 \rangle\}$ is the only $\langle \bar{x}, \omega, B_1 \rightarrow B_2, 0 \rangle$ -extension of β .

⁷One could, instead, add rules corresponding to the conditions the various logics impose on the relation R as is done in e.g. [1]. We've opted for the current approach to avoid painful, unilluminating discussions of what to do in our completeness proof when the rules governing R can be applied without bound.

- For $\bar{x} \neq \varepsilon$, both $\beta \cup \{\langle 1\bar{x}, \omega, B_1, 0 \rangle\}$ and $\beta \cup \{\langle r\bar{x}, \omega, B_2, 1 \rangle\}$ are $\langle \varepsilon, \omega, B_1 \rightarrow B_2, 1 \rangle$ -extensions of β .
- For $\bar{x} \neq \varepsilon$, $\beta \cup \{\langle 1\bar{x}, \omega, B_1, 1 \rangle, \langle r\bar{x}, \omega, B_2, 0 \rangle\}$ is the only $\langle \bar{x}, \omega, B_1 \rightarrow B_2, 0 \rangle$ -extension of β .
- $\beta \cup \{\langle n\bar{x}, \omega, B, 0 \rangle\}$ is the only $\langle \bar{x}, \omega, \neg B, 1 \rangle$ -extension of β .
- $\beta \cup \{\langle n\bar{x}, \omega, B, 1 \rangle\}$ is the only $\langle \bar{x}, \omega, \neg B, 0 \rangle$ -extension of β .
- $\beta \cup \{\langle b\bar{x}, \omega', B, 1 \rangle : \omega R \omega' \in \beta\}$ is the only $\langle \bar{x}, \omega, \Box B, 1 \rangle$ -extension of β .
- For each $\omega' \in \Omega - \Omega_\beta$, $\beta \cup \{\langle b\bar{x}, \omega', B, 0 \rangle, \omega R \omega'\}$ is an $\langle \bar{x}, \omega, \Box B, 0 \rangle$ -extension of β .

3.2.2 Metatheory 2: Soundness

Lemma 3.1. *Let β be a branchset, $\kappa = \langle K, f, \alpha \rangle$ be a correspondence, $K = \langle W, \rho \rangle$ and $\langle \bar{x}, \omega, A, i \rangle \in \beta$. Then if κ conforms to β then for some $\langle \bar{x}, \omega, A, i \rangle$ -extension β' of β , there is $g : \Omega_{\beta'} \rightarrow W$ such that $g(\omega) = f(\omega)$ for all $\omega \in \Omega_\beta$ and $\kappa_g = \langle K, g, \alpha \rangle$ conforms to β' .*

Proof. By examining cases. If A is an atom, the result is trivial. For the non-modal connectives, the arguments go as in [21]. For the modal cases, we argue as follows.

Suppose $A = \Box B$ and $i = 1$. If for no ω' is $\omega R \omega' \in \beta$, then the result is trivial. So suppose $\omega R \omega' \in \beta$. Then since κ conforms to β , $\alpha_{\bar{x}}(f(\omega), \Box B) = \inf\{\alpha_{b\bar{x}}(v, B) : \langle f(\omega), v \rangle \in \rho\} = 1$ and $\langle f(\omega), f(\omega') \rangle \in \rho$. So $\alpha_{b\bar{x}}(f(\omega'), B) = 1$, and thus κ itself conforms to $\beta \cup \{\langle b\bar{x}, \omega', B, 1 \rangle : \omega R \omega' \in \beta\}$.

Suppose $A = \Box B$ and $i = 0$. Since κ conforms to β , $\alpha_{\bar{x}}(f(\omega), \Box B) = \inf\{\alpha_{b\bar{x}}(v, B) : \langle f(\omega), v \rangle \in \rho\} = 0$. So for some $v \in W$, $\alpha_{b\bar{x}}(v, B) = 0$ and $\langle f(\omega), v \rangle \in \rho$. Choose $\omega' \in \Omega - \Omega_\beta$ and let $g(\omega) = f(\omega)$ for all $\omega \in \Omega_\beta$ and $g(\omega') = v$. Then clearly κ_g conforms to $\beta \cup \{\langle b\bar{x}, \omega', B, 0 \rangle, \omega R \omega'\}$. $\square$

Theorem 3.2. *If $\vdash_{\text{MPos}} A$, then $\models_{\text{MPos}} A$.*

Proof. Let $\vdash_{\text{MPos}} A$. Suppose $K = \langle W, \rho \rangle$ is a Kripke frame and α is an MPos^K -assignment. If $\alpha_\varepsilon(w, A) = 0$, then letting $f(\omega_1) = w$, the triple $\langle K, f, \alpha \rangle$ conforms to $\langle \varepsilon, \omega_1, A, 0 \rangle$. So by the previous lemma, it conforms to all of its extensions. But since $\vdash_{\text{MPos}} A$, all such extensions eventually contain nodes of the form $\langle \bar{x}, \omega_j, A, 1 \rangle$ and $\langle \bar{x}, \omega_j, A, 0 \rangle$ and thus cannot be conformed to. So for no K and α do we have $\alpha_\varepsilon(w, A) = 0$. So $\models_{\text{MPos}} A$. $\square$

3.2.3 Metatheory 3: Completeness

We have, to this point, been implicitly restricting our attention to finite branchsets. But (medicine and forestry aside) there's no good reason to stick to this restriction.⁸ Given a finite branchset β , say that a β -tableau is a tableau that begins with the members of β . Given *any* (finite or infinite) branchset β , say that β is *tableau-inconsistent* when for some finite $\beta' \subseteq \beta$, there is a closed β' -tableau. A branchset is *tableau-consistent* if it's not tableau inconsistent. A branchset is *maximally tableau-consistent* when it is tableau-consistent, but all proper tableau extensions of it are tableau inconsistent. A branchset is *witnessed* when it contains $\langle \bar{x}, \omega, \Box B, 0 \rangle$ only if for some ω' it contains both $\omega R \omega'$ and $\langle b\bar{x}, \omega', B, 0 \rangle$.

Lemma 3.3. *If β is tableau-consistent and Ω_β is finite, then there is a maximal, witnessed, tableau-consistent extension γ of β .*

⁸From here, what we say follows the contours of the argument in [7].

Proof. Enumerate the set of possible branchset entries as $Z_1, Z_2, \dots$. Note that some of these are quadruples and some are R-claims. Define a sequence of sets $\gamma_1, \gamma_2, \dots$ as follows.

- $\gamma_1 = \beta$
- If $\gamma_i \cup \{Z_i\}$ is tableau-inconsistent, then $\gamma_{i+1} = \gamma_i$.
- Otherwise, if Z_i is a quadruple $\langle \bar{x}, \omega_j, \Box B, 0 \rangle$, then $\gamma_{i+1} = \gamma_i \cup \{Z_i, \langle \bar{x}, \omega_k, B, 0 \rangle, \omega_j R \omega_k\}$ where k is the least n such that ω_n does not occur in γ_i .
- Otherwise $\gamma_{i+1} = \gamma_i \cup \{Z_i\}$.

Let $\gamma = \bigcup_{i=1}^{\infty} \gamma_i$. I claim that γ is a maximal, witnessed, tableau-consistent extension of β . Since it's obvious that γ is an extension of β , there are three things for us to check:

γ is maximal: Pick a possible branchset entry E . Then $E = Z_i$ for some i . By construction, either $E \in \gamma_{n+1} \subseteq \gamma$ or $\gamma_n \cup \{E\}$ is tableau-inconsistent. But in the latter case, $\gamma \cup \{E\}$ is obviously tableau-inconsistent as well. So if $E \notin \gamma$, then $\gamma_n \cup \{E\}$ is tableau-inconsistent, and thus γ is maximal.

γ is tableau-consistent: We prove that each γ_n is tableau-consistent. It follows from this that γ is tableau-consistent, but we will also want to know this below.

The proof is by contradiction. Suppose for some n that γ_n is tableau-inconsistent and let γ_k be the least such n . Since β is tableau-consistent, $k \neq 1$, so γ_{k-1} and Z_{k-1} are defined. We consider three cases:

- If $\gamma_{k-1} \cup \{Z_{k-1}\}$ is tableau-inconsistent, then $\gamma_{k-1} = \gamma_k$. But then since γ_k is tableau-inconsistent, so is γ_{k-1} , contradicting k 's minimality. So $\gamma_{k-1} \cup \{Z_{k-1}\}$ is tableau-consistent.
- If Z_{k-1} is $\langle \bar{x}, \omega_j, \Box B, 0 \rangle$, then $\gamma_k = \gamma_{k-1} \cup \{Z_{k-1}, \langle \bar{x}, \omega_l, B, 0 \rangle, \omega_j R \omega_k\}$ where l is the least m such that ω_m does not occur in γ_{k-1} .

Since γ_k is tableau-inconsistent, there is a finite subset F of γ_{k-1} and a closed tableau that starts at $F \cup \{Z_{k-1}, \langle \bar{x}, \omega_l, B, 0 \rangle, \omega_j R \omega_k\}$. But then since ω_j is fresh to $F \cup \{Z_{k-1}\}$, the very same tableau is a closed tableau that starts at $F \cup \{Z_{k-1}\}$. But this contradicts the fact that $\gamma_{k-1} \cup \{Z_{k-1}\}$ is tableau-consistent.

- Otherwise $\gamma_k = \gamma_{k-1} \cup \{Z_{k-1}\}$ and $\gamma_{k-1} \cup \{Z_{k-1}\}$ is consistent, which is a contradiction.

Thus, for no n is γ_n tableau-inconsistent. So γ is tableau-consistent.

γ is witnessed: Suppose γ contains $\langle \bar{x}, \omega, \Box B, 0 \rangle$. For some n , $\langle \bar{x}, \omega, \Box B, 0 \rangle = Z_n$. Since γ_n is consistent, there is an ω' so that γ_{n+1} —and thus γ —contains both $\omega R \omega'$ and $\langle \bar{x}, \omega', B, 0 \rangle$.

□

Given a maximal, witnessed, tableau-consistent branchset β , the canonical β -correspondence κ_β is the correspondence $\langle K, \iota, \alpha \rangle$ where $K = \langle \Omega_\beta, \{\langle \omega_i, \omega_j \rangle : \omega_i R \omega_j \in \beta\} \rangle$, ι is the identity function, and $\alpha_{\bar{x}}(\omega, p) = 1$ iff $\langle \bar{x}, \omega, p, 1 \rangle \in \beta$.

Lemma 3.4. *Let β be a maximal, witnessed, tableau-consistent branchset and let $\kappa_\beta = \langle K, \iota, \alpha \rangle$. If $\langle \bar{x}, \omega, A, 1 \rangle \in \beta$ then $\alpha_{\bar{x}}(\omega, A) = 1$ and if $\langle \bar{x}, \omega, A, 0 \rangle \in \beta$ then $\alpha_{\bar{x}}(\omega, A) = 0$*

Proof. By induction on A . If A is an atomic p , then by the definition of the canonical β -correspondence, $\langle \bar{x}, \omega, p, 1 \rangle \in \beta$ iff $\alpha_{\bar{x}}(\omega, p) = 1$ and since β is tableau-consistent, it follows that if $\langle \bar{x}, \omega, A, 0 \rangle \in \beta$ then $\alpha_{\bar{x}}(\omega, A) = 0$. Of the remaining cases, we consider the following:

Suppose $\langle \bar{x}, \omega, A \wedge B, 1 \rangle \in \beta$. If $\beta \cup \{\langle \bar{x}, \omega, A, 1 \rangle\}$ is tableau-inconsistent, then there is finite $F \subseteq \beta$ so that $F \cup \{\langle \bar{x}, \omega, A, 1 \rangle\}$ closes. But then since $\langle \bar{x}, \omega, A \wedge B, 1 \rangle \in \beta$, $F \cup \{\langle \bar{x}, \omega, A \wedge B, 1 \rangle\}$ is also a finite subset of β that must also close, contradicting the fact that β is tableau-consistent. So $\beta \cup \{\langle \bar{x}, \omega, A \wedge B, 1 \rangle\}$

is tableau-consistent. Thus by maximality, we must have that $\langle \bar{x}, \omega, A \wedge B, 1 \rangle \in \beta$. Mutatis mutandis, the same argument works for B . By the inductive hypothesis, since $\{\langle \bar{x}, \omega, A, 1 \rangle, \langle \bar{x}, \omega, B, 1 \rangle\} \subseteq \beta$, $\alpha_{\bar{x}}(\omega, A) = \alpha_{\bar{x}}(\omega, B) = 1$, and thus $\alpha_{\bar{x}}(\omega, A \wedge B) = 1$ as well.

Suppose $\langle \varepsilon, \omega, A \rightarrow B, 1 \rangle \in \beta$. By a similar argument to the one in the $\wedge$ -case, we can see that if both $\beta \cup \{\langle c, \omega, A, 0 \rangle\}$ and $\beta \cup \{\langle c, \omega, B, 1 \rangle\}$ are tableau-inconsistent, then β must be as well. So one of these sets must be tableau-consistent. Thus by the fact that β is maximal, we can see that either $\langle c, \omega, A, 0 \rangle \in \beta$ or $\langle c, \omega, B, 1 \rangle \in \beta$. Either way, by the inductive hypothesis we get that $\alpha_{\varepsilon}(\omega, A \rightarrow B) = 1$.

Suppose $\langle \bar{x}, \omega, \Box A, 0 \rangle \in \beta$. Since β is witnessed, it also contains both $\omega R \omega'$ and $\langle b\bar{x}, \omega', B, 0 \rangle$ for some ω' . By the inductive hypothesis, $\alpha_{b\bar{x}}(\omega', B) = 0$. And since $\omega R \omega' \in \beta$, it follows that $\alpha_{\bar{x}}(\omega, \Box B) = 0$. $\square$

Theorem 3.5. *If $\models_{\text{MPos}} A$, then $\vdash_{\text{MPos}} A$.*

Proof. Suppose $\not\models_{\text{MPos}} A$. Let $\beta = \{\langle \varepsilon, \omega_0, A, 0 \rangle\}$. By Lemma 3.3, there is a maximal, witnessed, tableau-consistent extension β' of β . So by Lemma 3.4, the canonical correspondence gives us a model that falsifies A at ω and ε . So $\not\models_{\text{MPos}} A$. $\square$

3.3 The Hyperformal Subset Characterization

For any set of formulas X , we define $\text{MPos}(X)$ as follows:

$$\text{MPos}(X) = \{A : \sigma_{\varepsilon}(A) \in X \text{ for all MPos-substitutions } \sigma\}$$

One can prove, by following exactly the same arguments as in Lemma 5 of [21] or Theorem 3 of [9] that $\text{MPos}(X)$ is the largest subset of X closed under MPos substitutions.

Theorem 3.6. *The following are equivalent:*

- $A \in \text{MPos}(K)$.
- For some atomic and injective-up-to-faithfulness ι , $\iota(A) \in K$.
- $A \in K^{\text{MPos}}$.

The proof of this result requires only the very mildest of changes from the proofs of similar results in [21] or [9], and is thus left to the reader. Note that this gives us a third characterization of K^{MPos} : it is the largest MPos-hyperformal subset of K .

We will end by proving that K^{MPos} has the MPos-variable sharing property. We need a bit of setup first. To begin, we identify the subsets MPos_+ and MPos_- of MPos as follows:

- $\varepsilon \in \text{MPos}_+$.
- If $\bar{x} \in \text{MPos}_+$, then $1\bar{x} \in \text{MPos}_-$, $r\bar{x} \in \text{MPos}_+$, $n\bar{x} \in \text{MPos}_-$, and $b\bar{x} \in \text{MPos}_+$.
- If $\bar{x} \in \text{MPos}_-$, then $1\bar{x} \in \text{MPos}_+$, $r\bar{x} \in \text{MPos}_-$, $n\bar{x} \in \text{MPos}_+$, and $b\bar{x} \in \text{MPos}_-$.

Let K be a one-world model with the universal accessibility relation.

Suppose A and C share no atoms. Define the MPos^K -assignments f_A^+ and f_A^- as follows:

$$f_{A,\bar{y}}^+(w, p) = \begin{cases} 1 & \text{if } \bar{y} = \bar{x}c, \bar{x} \in \text{MPos}_+, \text{ and } p \text{ occurs at } \bar{x}c \text{ in } A \\ 0 & \text{otherwise} \end{cases}$$

$$f_{A,\bar{y}}^-(w, p) = \begin{cases} 0 & \text{if } \bar{y} = \bar{x}c, \bar{x} \in \text{MPos}_+, \text{ and } p \text{ occurs at } \bar{x}c \text{ in } A \\ 1 & \text{otherwise} \end{cases}$$

Lemma 3.7. *If B is a subformula of A , then*

- $f_{A,\bar{x}c}^+(w, B) = 1$ if $\bar{x} \in \text{MPos}_+$ and B occurs in $A \rightarrow C$ at $\bar{x}c$,
- $f_{A,\bar{x}c}^+(w, B) = 0$ if $\bar{x} \in \text{MPos}_-$ and B occurs in $A \rightarrow C$ at $\bar{x}c$,
- $f_{A,\bar{x}c}^-(w, B) = 0$ if $\bar{x} \in \text{MPos}_+$ and B occurs in $C \rightarrow A$ at $\bar{x}c$,
- $f_{A,\bar{x}c}^-(w, B) = 1$ if $\bar{x} \in \text{MPos}_-$ and B occurs in $C \rightarrow A$ at $\bar{x}c$.

Proof. By induction on B . Every case except the $\Box$ -case goes exactly as in the proof of Lemmas 11 and 12 in [21]. For that case, since K has exactly one world and the universal accessibility relation, $f_{A,\bar{x}c}^+(\Box B) = f_{A,\bar{x}c}^+(B)$ and $f_{A,\bar{x}c}^-(\Box B) = f_{A,\bar{x}c}^-(B)$. So the inductive hypothesis immediately finishes the job. $\square$

Lemma 3.8. *Suppose that A and B share no atoms. Define the assignment h by*

$$h_{\bar{x}}(w, p) = \begin{cases} f_{A,\bar{x}}^+(w, p) & \text{if } p \text{ occurs in } A \text{ and occurs at } \bar{x} \text{ in } A \rightarrow B \\ f_{B,\bar{x}}^-(w, p) & \text{if } p \text{ occurs in } A \text{ and occurs at } \bar{x} \text{ in } A \rightarrow B \\ \text{whatever} & \text{otherwise.} \end{cases}$$

Then if C occurs at $\bar{x}$ in A then $h_{\bar{x}}(w, C) = f_{A,\bar{x}}^+(w, C)$ and if C occurs at $\bar{x}$ in B then $h_{\bar{x}}(w, C) = f_{B,\bar{x}}^-(w, C)$.

Proof. By induction on C , separately for each conclusion. $\square$

Theorem 3.9. K^{MPos} *enjoys the variable sharing property.*

Proof. Suppose A and B don't share atoms. Define h as above. By the preceding lemmas, $h_c(w, A) = 1$ and $h_c(w, B) = 0$, so $h_\varepsilon(w, A \rightarrow B) = 0$. $\square$

Corollary 3.10. K^{MPos} *has the MPos-variable sharing property.*

Proof. Since K^{MPos} is MPos-hyperformal, this follows immediately from the previous theorem. $\square$

4 Future Work and Concluding Remarks

We've characterized K^{MPos} semantically and via a tableaux system. To better understand it, one would like an axiom system.⁹ But to date logics formed by restricting known logics to their largest hyperformal (in some sense) fragments have resisted axiomatization. The only thing that is known on this front is the rather unsatisfying result proved in [9]: for logics like these *admit* recursive axiomatizations. Producing intelligible axiomatizations remains, in spite of this, as an outstanding project.

$B^\Box$ is MPos-hyperformal and we can use this to extract a rather shockingly strong variable sharing result for its theorems. But it's not the strongest logic with these features, nor is it the largest subset of classical logic with these features. K^{MPos} holds that honor.

Perhaps there are very good reasons for preferring $B^\Box$ to K^{MPos} . If so, they ought to be enunciated, and we now know some things they *can't* be: they can't be related to variable sharing or subclassicality. And one must admit that semantically, K^{MPos} definitely has advantages over $B^\Box$. Also note that since $B^\Box$ is subclassical and closed under MPos-substitutions, it is contained (properly as it turns out, though

⁹Thanks to a referee for pushing us on this point.

we leave it to the reader to verify this) in K^{MPos} . So anything you might want to do with a logic like $B^\square$, you can also do with K^{MPos} .

Another thing worth noting: “relevant” logics defined by pasting a simple filter on top of classical logic (a ‘syntactic sieve’, as pejoratively described by Richard Sylvan in [17]), can be easily seen to incur nontransitivity in the sense that they will, for some A , B , and C , include $A \rightarrow B$ and $B \rightarrow C$ without including $A \rightarrow C$. Were this true for K^{MPos} , it would be a fairly serious reason to set it aside. But while K^{MPos} is the largest sublogic of K with a given feature, and it is a sublogic of K with the MPos-variable sharing property, K^{MPos} is *not* the largest sublogic of K with the MPos-variable sharing property.

As an example, consider the contraction axiom: $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$. The leftmost p and the p antecedent of the terminal conditional both occur at the position lc . So this formula *does* have the MPos-variable sharing property. But it isn’t in K^{MPos} because K^{MPos} isn’t defined by its possessing a given variable sharing property but by its possessing a given closure property—namely, closure under MPos-substitutions. Since not all MPos-substitutions map contraction to a theorem of classical logic, contraction isn’t a member of K^{MPos} .

So K^{MPos} isn’t plausibly characterized as K with a syntactic sieve attached, and thus isn’t subject to any of the usual criticisms that attach to such logics. And there’s more: not only can’t it be criticized on those fronts, it also doesn’t *suffer* on those fronts. In particular, we can show the following:

Theorem 4.1. K^{MPos} is transitive.

Proof. Suppose $\sigma_e(A \rightarrow C) = 0$. Then $\sigma_c(A) = 1$ and $\sigma_c(C) = 0$. Now observe that if $\sigma_c(B) = 1$, then $\sigma_e(B \rightarrow C) = 0$ and if $\sigma_c(B) = 0$, then $\sigma_e(A \rightarrow B) = 0$. So if $A \rightarrow C \notin K^{MPos}$, then either $A \rightarrow B \notin K^{MPos}$ or $B \rightarrow C \notin K^{MPos}$. Contraposing, if $A \rightarrow B \in K^{MPos}$ and $B \rightarrow C \in K^{MPos}$, then $A \rightarrow C \in K^{MPos}$. $\square$

This observation, then, cuts against a charge of artificiality in the formulation of K^{MPos} insofar as the pathologies that follow from truly artificial approaches to relevance—like syntactic sieves—do not haunt the present system.

All told, we would like it if this paper was understood as a challenge. Those who would prefer ‘traditional’ relevant modal logics are being asked to give reasons for that preference. Absent such, we’d recommend that folks in need of relevant modal logics look to logics like K^{MPos} rather than to logics like $B^\square$.

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A Gödel Modal Logic Over Witnessed Models

Mauro Ferrari*

Dep. of Theoretical and Applied Sciences
Università degli Studi dell'Insubria
Varese, Italy.

mauro.ferrari@uninsubria.it

Camillo Fiorentini

Dep. of Computer Science
Università degli Studi di Milano
Milano, Italy

fiorentini@di.unimi.it

Paolo Giardini

Dep. of Theoretical and Applied Sciences
Università degli Studi dell'Insubria
Varese, Italy.

pcgiardini@uninsubria.it

Ricardo Oscar Rodriguez

UBA-FCEyN, Dep. De Computación
Buenos Aires, Argentina

ricardo@dc.uba.ar

We introduce **GW**, a Gödel modal logic based on Kripke models in which the value of each modal formula is witnessed by an accessible world. This witnessed semantics eliminates the limit-based phenomena that preclude the finite model property in the usual Kripke semantics for Gödel modal logics, thereby yielding a more constructive semantic framework. We present a sound and complete refutation calculus for **GW** and design a terminating backward proof-search procedure with counter-model generation. As a direct consequence of this procedure, **GW** enjoys the finite model property.

1 Introduction

Gödel modal logics, **GKs**, provide a natural framework for reasoning about graded notions of incompleteness, uncertainty, and vagueness by means of necessity and possibility operators evaluated over truth degrees. When interpreted over the real unit interval, Gödel logic captures a well-behaved notion of implication based on residuation, which makes it particularly suitable for applications in knowledge representation, as well as in multi-agent and distributed systems.

In general, the semantics of **GK** is based on Gödel Kripke models (*G-models*), a natural generalization of classical Kripke semantics in which both the valuation of propositions at each world and the accessibility relation take values in the standard Gödel algebra $[0, 1]$, i.e. the algebraic semantics of the intermediate Gödel–Dummett Logic. The minimal logics over G-models were thoroughly investigated by Caicedo and Rodríguez in [5, 6]. In this setting, the interpretation of the modal operators is defined by means of infima and suprema computed over (possibly infinite) sets of accessible worlds. However, these bounds need not be attained: the infimum may be strictly smaller than every truth value in the set, and dually for the supremum. Consequently, the truth value of a modal formula may depend on limit constructions rather than on concrete worlds of the model. This phenomenon has significant logical consequences. For instance, [5] shows that the formula $\Box \neg \neg \phi \rightarrow \neg \neg \Box \phi$ is valid in all finite Gödel Kripke models, but fails in a suitable infinite model (see Fig. 1). Hence, the underlying logic does not enjoy the finite model property, and semantic arguments may require genuinely infinitary behaviour.

Witnessed models provide a principled way to overcome these difficulties. By requiring that the relevant infima and suprema be realized by actual accessible worlds, witnessed semantics ensures that existential and universal behaviour is supported by explicit witnesses. In this way, the evaluation of modal operators more closely mirrors the behaviour of guarded quantification in first-order logic and restores a constructive reading of the semantics. Each modal statement is thus justified by concrete worlds rather than by purely asymptotic approximations.

The notion of witnessed models has appeared in several related contexts. For example, in [7] an axiomatization is given for witnessed semantics in fuzzy predicate logics. Since quantified statements can often be translated into modal ones, this connection is conceptually well grounded. Along these lines, Hájek considers in [13] the axioms $\Diamond(\varphi \rightarrow \Box\varphi)$ and $\Diamond(\Diamond\varphi \rightarrow \varphi)$, and proves that the resulting logic $S5(G)^w$ is strongly complete with respect to witnessed universal Kripke models.

Witnessed semantics has also proved fruitful in applied settings. In the context of Fuzzy Description Logics, [4] studies a system based on witnessed Gödel semantics and shows that it enjoys the finite model property and a decidable satisfiability problem with exponential-time complexity. More recently, [18] provides an axiomatization for Gödel modal logic interpreted over witnessed Kripke models.

In general, witnessed models tend to ensure compactness-like properties and often yield the finite model property and decidability results. These features are crucial for automated reasoning, since they allow infinitary semantic arguments to be replaced by finite combinatorial ones. For these reasons, studying Gödel modal logics over witnessed models not only clarifies their semantic foundations, but also strengthens their connections with well-behaved fragments of first-order logic that play a central role in computer science, thereby enhancing both their theoretical robustness and their applicability.

From a proof-theoretic perspective, witnessed semantics aligns more closely with syntactic calculi. The existence of witnesses enables canonical model constructions that are finite or finitely generated, supports filtration techniques, and allows completeness proofs to proceed by standard algebraic or canonical methods. Therefore, establishing a sound and complete calculus for Gödel modal logics over witnessed models is not only feasible but also conceptually justified, providing a proof-theoretic counterpart to their improved semantic behaviour and paving the way for effective automated reasoning procedures.

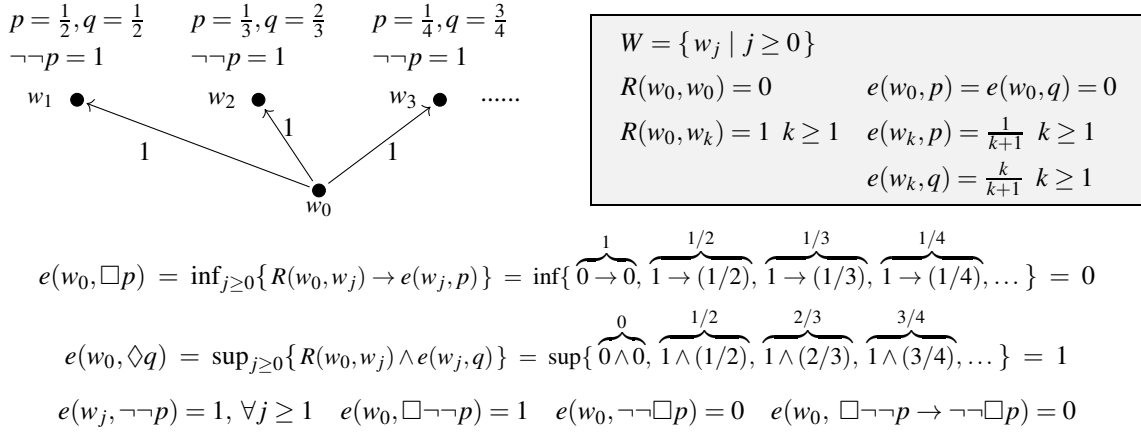
In line with the above considerations, in [11] we introduced the logic $\mathbf{GW}^c$ characterized by witnessed crisp Gödel Kripke models, where “crisp” means that the accessibility relation takes values in the set $\{0, 1\}$. For this logic, we developed a constraint-based refutation calculus inspired by the tableau system of [3], together with a terminating proof-search procedure that yields finite countermodels whenever proof search fails. As a consequence, we established that $\mathbf{GW}^c$ enjoys the finite model property. In the present work, we extend these results to the logic $\mathbf{GW}$, characterized by witnessed Gödel Kripke models without the crispness restriction. Our development follows the strategy adopted in [11]. We introduce a constraint-based refutation calculus for $\mathbf{GW}$, called $\mathcal{C}_{\mathbf{GW}}$, and prove its soundness and completeness with respect to witnessed models. Moreover, we design a terminating proof-search procedure for $\mathcal{C}_{\mathbf{GW}}$ based on a standard backward search strategy. Notably, when proof search fails, the computation can be traced to construct a finite countermodel for the given formula, and this proves the finite model property for $\mathbf{GW}$. Finally, we provide an implementation of the procedure, called *gwref* (available at [10]).

2 Basic Definitions

Formulas are built over a countable set of propositional variables $\mathcal{V}$, using the propositional connectives $\wedge, \vee, \rightarrow$, the constant $\perp$ and the modal connectives $\Box, \Diamond$. We introduce the following abbreviations: $\neg\varphi$ stands for $\varphi \rightarrow \perp$; $\varphi_1 \leftrightarrow \varphi_2$ stands for $(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$. The symbols $\wedge, \vee$ and $\rightarrow$ are also used to denote algebraic operations. The *Gödel algebra* is a structure $\langle [0, 1], \wedge, \vee, \rightarrow \rangle$ where:

$$a \wedge b = \min(a, b) \quad a \vee b = \max(a, b) \quad a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

A *G-model* (Gödel model) $\mathfrak{M}$ is a structure $\langle W, R, e \rangle$ where W is a nonempty set (the set of worlds), R is a map $W \times W \rightarrow [0, 1]$ (the accessibility relation) and $e : W \times \mathcal{V} \rightarrow [0, 1]$ (the evaluation function). If

Figure 1: A non-witnessed G-model $\mathfrak{M} = \langle W, R, e \rangle$.

$R : W \times W \rightarrow \{0, 1\}$, we say that $\mathfrak{M}$ is *crisp*; in this case, $R \subseteq W \times W$. The map e is extended to arbitrary formulas as follows:

$$\begin{aligned}
 e(w, \perp) &= 0 & e(w, \alpha \star \beta) &= e(w, \alpha) \star e(w, \beta), \star \in \{\wedge, \vee, \rightarrow\} \\
 e(w, \Box \alpha) &= \inf_{w' \in W} \{R(w, w') \rightarrow e(w', \alpha)\} & e(w, \Diamond \alpha) &= \sup_{w' \in W} \{R(w, w') \wedge e(w', \alpha)\}.
 \end{aligned}$$

Note that $e(w, \neg\alpha) = 1$ if $e(w, \alpha) = 0$ and $e(w, \neg\alpha) = 0$ if $e(w, \alpha) > 0$. A world w' is an R -successor of w iff $R(w, w') > 0$. If w has no R -successors, then $e(w, \Box \alpha) = 1$ and $e(w, \Diamond \alpha) = 0$ since, for every $w' \in W$, $R(w, w') \rightarrow e(w', \alpha) = 1$ and $R(w, w') \wedge e(w', \alpha) = 0$.

A G-model $\mathfrak{M} = \langle W, R, e \rangle$ is *witnessed*, and is called a *GW-model*, if the values of $e(w, \Box \alpha)$ and $e(w, \Diamond \alpha)$ are witnessed by some world w' in $\mathfrak{M}$, namely:

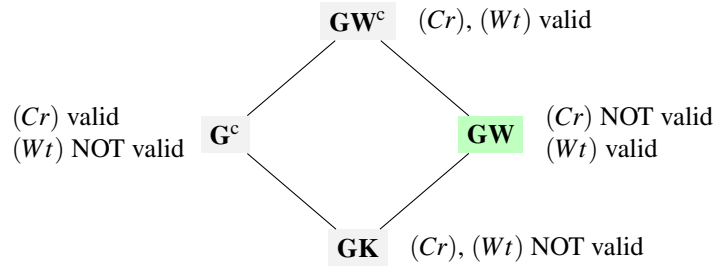
- $e(w, \Box \alpha) = r$ iff there is $w' \in W$ such that $R(w, w') \rightarrow e(w', \alpha) = r$;
- $e(w, \Diamond \alpha) = r$ iff there is $w' \in W$ such that $R(w, w') \wedge e(w', \alpha) = r$.

As a consequence, in the definitions of $e(w, \Box \alpha)$ and $e(w, \Diamond \alpha)$, the infimum and the supremum are just the minimum and the maximum. A non-witnessed model is shown in Fig. 1; whenever the value r of $R(w, w')$ or $e(w, p)$ is not displayed, it is understood that r is 0 (e.g., $R(w_i, w_j) = 0$, for every $i \geq 1$ and $j \geq 0$). It holds that $e(w_0, \Box p) = 0$, but the witnessing condition fails since there is no $w_j \in W$ such that $R(w_0, w_j) \rightarrow e(w_j, p) = 0$. Similarly $e(w_0, \Diamond q) = 1$, but there is no witness. When denoting a class of models, the superscript *c* means “crisp” (e.g., GW^c -model stands for “crisp GW-model”).

A formula φ is *valid* in $\mathfrak{M} = \langle W, R, e \rangle$ iff $e(w, \varphi) = 1$ for every world w in W . If φ is not valid in $\mathfrak{M}$, we say that $\mathfrak{M}$ is a *countermodel* for φ ; thus, $\mathfrak{M}$ contains a world w such that $e(w, \varphi) < 1$. Let $[0, 1]_{\mathbf{Q}} = [0, 1] \cap \mathbf{Q}$, where $\mathbf{Q}$ is the set of rational numbers; a model $\mathfrak{M} = \langle W, R, e \rangle$ is *discrete* if W is finite and the images of R and e are finite subsets of $[0, 1]_{\mathbf{Q}}$. We stress that every finite G-model is witnessed.

Let **G** be the Gödel-Dummett Logic, obtained by extending Intuitionistic Propositional Logic **IPL** with the linearity axiom $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$. Gödel Modal Logic **GK** is obtained by adding to **G** the following axioms and rules ($\vdash$ refers to provability in **GK**):

$$\begin{array}{ll}
 (K_{\Box}) & \Box(\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta) & (K_{\Diamond}) & \Diamond(\alpha \vee \beta) \rightarrow (\Diamond \alpha \vee \Diamond \beta) & (F_{\Diamond}) & \neg \Diamond \perp \\
 (FS_1) & \Diamond(\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Diamond \beta) & (FS_2) & (\Diamond \alpha \rightarrow \Box \beta) \rightarrow \Box(\alpha \rightarrow \beta) \\
 (N_{\Box}) & \vdash \alpha \text{ implies } \vdash \Box \alpha & (N_{\Diamond}) & \vdash \alpha \rightarrow \beta \text{ implies } \vdash \Diamond \alpha \rightarrow \Diamond \beta
 \end{array}$$



- (Cr) $\Box(\alpha \vee \beta) \rightarrow (\Box\alpha \vee \Diamond\beta)$ • $\mathbf{G}^c = \mathbf{GK} + (Cr)$ (see [17])
- (Wt) $\Box\neg\neg\alpha \rightarrow \neg\neg\Box\alpha$ • $\mathbf{GK} \subset \mathbf{G}^c \subset \mathbf{GW}^c, \mathbf{GK} \subset \mathbf{GW} \subset \mathbf{GW}^c$

Figure 2: Logics overview.

In [6] it is proved that $\mathbf{GK}$ is the set of formulas valid in every $\mathbf{G}$ -model. We introduce some logics extending $\mathbf{GK}$, by providing a semantic characterization (see Fig. 2):

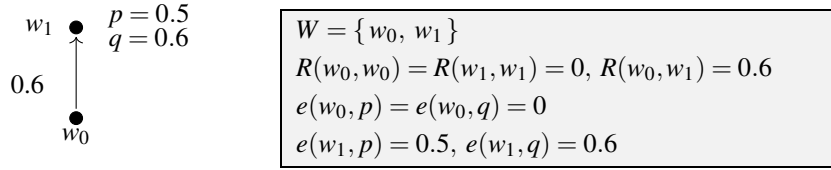
- $\mathbf{G}^c$ is the set of formulas valid in every $\mathbf{G}^c$ -model (see [17]);
- $\mathbf{GW}$ is the set of formulas valid in every $\mathbf{GW}$ -model;
- $\mathbf{GW}^c$ is the set of formulas valid in every $\mathbf{GW}^c$ -model (see [11]).

The logic $\mathbf{G}^c$ is obtained by adding the Crisp axiom $\Box(\alpha \vee \beta) \rightarrow (\Box\alpha \vee \Diamond\beta)$, which we call (Cr) , to $\mathbf{GK}$ (see [17]). The logic $\mathbf{GW}^c$ was introduced in [11] and recently an axiomatization has been proposed in [18]. By definition, $\mathbf{GK} \subseteq \mathbf{G}^c \subseteq \mathbf{GW}^c$ and $\mathbf{GK} \subseteq \mathbf{GW} \subseteq \mathbf{GW}^c$; we show that all the inclusions are strict. Let us consider the instance $\varphi = \Box(p \vee q) \rightarrow (\Box p \vee \Diamond q)$ of (Cr) and let $\mathfrak{M} = \langle W, R, e \rangle$ be the $\mathbf{GW}$ -model in Fig. 3. Since $e(w_0, \varphi) = 0.6$, $\mathfrak{M}$ is a countermodel for φ , and this certifies that $\varphi \notin \mathbf{GW}$; accordingly, axiom (Cr) is not valid in $\mathbf{GW}$. Let (Wt) be the formula schema $\Box\neg\neg\alpha \rightarrow \neg\neg\Box\alpha$; we show that (Wt) is valid in $\mathbf{GW}$. Let $\mathfrak{M} = \langle W, R, e \rangle$ be a $\mathbf{GW}$ -model and $w \in W$. If $e(w, \neg\neg\Box\alpha) = 1$, we immediately get $e(w, (Wt)) = 1$. Otherwise, $e(w, \neg\neg\Box\alpha) = 0$, hence $e(w, \Box\alpha) = 0$. Since $\mathfrak{M}$ is witnessed, there exists $w' \in W$ such that $R(w, w') \rightarrow e(w', \alpha) = 0$; it follows that $e(w', \alpha) = 0$, hence $e(w', \neg\neg\alpha) = 0$. This implies $e(w, \Box\neg\neg\alpha) = 0$, hence $e(w, (Wt)) = 1$. This proves that (Wt) is valid in $\mathbf{GW}$. Since every finite model is witnessed, a countermodel for (Wt) must be infinite. The instance $\Box\neg\neg p \rightarrow \neg\neg\Box p$ of (Wt) is not valid in the $\mathbf{G}^c$ -model $\mathfrak{M}$ in Fig. 1 (see, e.g., [11]), thus (Wt) is not valid in $\mathbf{G}^c$; this also ascertains that neither $\mathbf{G}^c$ nor $\mathbf{GK}$ enjoy the finite model property.

We show that the (classical) modal logic $\mathbf{K}$ can be embedded into each of the logics in Fig. 2, by exploiting the translation $*$ introduced in [15]. We stress that a model for the logic $\mathbf{K}$ (a $\mathbf{K}$ -model) can be viewed as a $\mathbf{GW}^c$ -model whose evaluation function takes values only in the set $\{0, 1\}$; accordingly, $\mathbf{GW}^c \subseteq \mathbf{K}$. We assume that $\Diamond$ is not a primitive connective in the language of $\mathbf{K}$ (actually, in $\mathbf{K}$ the formula $\Diamond\varphi$ can be defined as $\neg\Box\neg\varphi$). Let α be a formula of $\mathbf{K}$; the formula α^* is obtained by replacing each propositional variable p occurring in α with $\neg\neg p$. As shown in the following lemma, the evaluation of a formula α^* at any world of a $\mathbf{G}$ -model is binary (0 or 1); this is essentially due to the fact that propositional variables in α^* are double negated and $\Diamond$ does not occur in α^* .

Lemma 2.1. *Let α be a formula of $\mathbf{K}$.*

- (i) *If $\alpha \in \mathbf{K}$ then $\alpha^* \in \mathbf{GK}$.*
- (ii) *If $\alpha^* \in \mathbf{GW}^c$ then $\alpha \in \mathbf{K}$.*



$$\begin{aligned}
e(w_0, \Box(p \vee q)) &= \inf\{R(w_0, w_0) \rightarrow e(w_0, p \vee q), R(w_0, w_1) \rightarrow e(w_1, p \vee q)\} = \inf\{0 \rightarrow 0, 0.6 \rightarrow 0.6\} = 1 \\
e(w_0, \Box p) &= \inf\{R(w_0, w_0) \rightarrow e(w_0, p), R(w_0, w_1) \rightarrow e(w_1, p)\} = \inf\{0 \rightarrow 0, 0.6 \rightarrow 0.5\} = 0.5 \\
e(w_0, \Diamond q) &= \sup\{R(w_0, w_0) \wedge e(w_0, q), R(w_0, w_1) \wedge e(w_1, q)\} = \sup\{0 \wedge 0, 0.6 \wedge 0.6\} = 0.6 \\
e(w_0, \varphi) &= e(w_0, \Box(p \vee q)) \rightarrow e(w_0, \Box p \vee \Diamond q) = 1 \rightarrow 0.6 = 0.6
\end{aligned}$$

Figure 3: A countermodel for the formula $\varphi = \Box(p \vee q) \rightarrow (\Box p \vee \Diamond q)$ (instance of (Cr)).

Proof. The proof of point (ii) is immediate, since $\mathbf{GW}^c \subseteq \mathbf{K}$ and the formula $\alpha \leftrightarrow \alpha^*$ is valid in $\mathbf{K}$. We prove point (i). Let us assume that $\alpha^* \notin \mathbf{GK}$. There exists a G-model $\mathfrak{M} = \langle W, R, e \rangle$ and a world w in W such that $e(w, \alpha^*) < 1$. Let $\mathfrak{M}_1 = \langle W, R_1, e_1 \rangle$ be the K-model defined as follows:

$$R_1(w, w') = \begin{cases} 1 & \text{if } R(w, w') > 0 \\ 0 & \text{otherwise} \end{cases} \quad e_1(w, p) = \begin{cases} 1 & \text{if } e(w, p) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $w \in W$ and let φ be any formula of $\mathbf{K}$; by induction on the size of φ , we prove that:

(A) $e_1(w, \varphi) = e(w, \varphi^*)$.

The case $\varphi \in \mathcal{V} \cup \{\perp\}$ is immediate. Let $\varphi = \Box\psi$; note that $\varphi^* = \Box\psi^*$. Assume $e_1(w, \Box\psi) = 0$. There exists $w' \in W$ such that $R_1(w, w') = 1$ and $e_1(w', \psi) = 0$. Then, $R(w, w') > 0$ and, by the induction hypothesis, $e(w', \psi^*) = 0$. It follows that $R(w, w') \rightarrow e(w', \psi^*) = 0$, hence $e(w, \Box\psi^*) = 0$. Let $e_1(w, \Box\psi) = 1$ and assume, by contradiction, $e(w, \Box\psi^*) < 1$. By the induction hypothesis, for every $w' \in W$ it holds that $e(w', \psi^*) \in \{0, 1\}$, hence $R(w, w') \rightarrow e(w', \psi^*) \in \{0, 1\}$ and $e(w, \Box\psi^*) \in \{0, 1\}$. Thus, $e(w, \Box\psi^*) = 0$ and there is $w' \in W$ such that $R(w, w') \rightarrow e(w', \psi^*) = 0$. We get $R(w, w') > 0$ and $e(w', \psi^*) = 0$, hence $R_1(w, w') = 1$ and $e_1(w', \psi) = 0$, contradicting the assumption $e_1(w, \Box\psi) = 1$. Since the assumption $e(w, \Box\psi^*) < 1$ yields a contradiction, we conclude $e(w, \Box\psi^*) = 1$. The proof of (A) in the remaining cases easily follows from the induction hypothesis. Since $e(w, \alpha^*) < 1$, from (A) it follows that $e_1(w, \alpha) = 0$. Accordingly, $\alpha \notin \mathbf{K}$, and this proves (i). $\square$

As a consequence of Lemma 2.1, we get:

Proposition 2.2. *Let α be a formula of $\mathbf{K}$ and let L be any of the logics $\mathbf{GK}$, $\mathbf{G}^c$, $\mathbf{GW}$ and $\mathbf{GW}^c$. Then, $\alpha \in \mathbf{K}$ iff $\alpha^* \in L$.*

We conclude this section by discussing the non-interdefinability of $\Box$ and $\Diamond$. This was established for $\mathbf{G}^c$ in [17] using algebraic semantics. Here, we instead employ $\mathbf{GW}^c$ -models to extend the result to any modal logic contained in $\mathbf{GW}^c$. We prove that:

- (P1) there is no $\Box$ -free formula φ such that $\Box p \leftrightarrow \varphi$ is valid in $\mathbf{GW}^c$;
- (P2) there is no $\Diamond$ -free formula ψ such that $\Diamond p \leftrightarrow \psi$ is valid in $\mathbf{GW}^c$.

Lemma 2.3. *Let $\mathfrak{M}^* = \langle W, R, e \rangle$ be the $\mathbf{GW}^c$ -model defined in Fig. 4.*

Figure 4: The GW^c -model $\mathfrak{M}^* = \langle W, R, e \rangle$.

- (i) For every $\Box$ -free formula φ , $e(w_0, \varphi) \in \{0, 0.5, 1\}$.
- (ii) For every $\Diamond$ -free formula ψ , $e(w_0, \psi) \in \{0, 0.4, 1\}$.

Proof. One can easily prove that, for every formula α , the following facts hold:

- (1) $e(w_1, \alpha) \in \{0, 0.4, 1\}$.
- (2) $e(w_2, \alpha) = \Phi(e(w_1, \alpha))$, where $\Phi(0) = 0$, $\Phi(0.4) = 0.5$, $\Phi(1) = 1$.

We prove (i), by induction on the size of φ . If $\varphi \in \mathcal{V} \cup \{\perp\}$, we have $e(w_0, \varphi) = 0$. The cases $\varphi = \alpha \wedge \beta$, $\varphi = \alpha \vee \beta$ and $\varphi = \alpha \rightarrow \beta$ easily follow from the induction hypothesis. Let $\varphi = \Diamond \alpha$. By (1) and (2), one of the following properties (a)–(c) holds:

- (a) $e(w_1, \alpha) = e(w_2, \alpha) = 0$.
- (b) $e(w_1, \alpha) = 0.4$ and $e(w_2, \alpha) = 0.5$.
- (c) $e(w_1, \alpha) = e(w_2, \alpha) = 1$.

In case (a) we get $e(w_0, \Diamond \alpha) = 0$, in case (b) $e(w_0, \Diamond \alpha) = 0.5$, in case (c) $e(w_0, \Diamond \alpha) = 1$; this concludes the proof of (i). The proof of (ii) is similar. $\square$

Proposition 2.4. *Properties (P1) and (P2) hold.*

Proof. Let us assume, by contradiction, that property (P1) does not hold. Then, there exists a $\Box$ -free formula φ such that $\Box p \leftrightarrow \varphi$ is valid in GW^c . As a consequence, in the GW^c -model $\mathfrak{M}^*$ in Fig. 4 we have $e(w_0, \Box p \leftrightarrow \varphi) = 1$. Since $e(w_0, \Box p) = 0.4$, we get $e(w_0, \varphi) = 0.4$, in contradiction with Lemma 2.3(i). This proves that (P1) holds. The proof of (P2) is similar. $\square$

Proposition 2.5. $\Box$ and $\Diamond$ are not interdefinable in any modal logic L contained in GW^c .

Proof. Let $L \subseteq \text{GW}^c$ and assume that there is a $\Box$ -free formula φ such that $\Box p \leftrightarrow \varphi \in L$. Then, $\Box p \leftrightarrow \varphi \in \text{GW}^c$, in contradiction with Lemma 2.4. The proof for $\Diamond$ is analogous. $\square$

3 The Calculus $\mathcal{C}_{\text{GW}}$

Following [3, 11], we introduce the refutation calculus $\mathcal{C}_{\text{GW}}$ for GW , namely, a calculus to certify that a formula is not valid in any GW -model. The calculus exploits the constraint language $\mathcal{L}_c$ defined over a

$$\begin{array}{c}
\triangleleft \in \{<, \leq\}, \quad \triangleright \in \{>, \geq\} \qquad \overline{\Gamma}^{\text{Ax}} \quad \text{if } \text{At}^+(\Gamma) \text{ is not consistent} \\
\\
\frac{w : \alpha \triangleright t, w : \beta \triangleright t, \Gamma}{w : \alpha \wedge \beta \triangleright t, \Gamma} \wedge \triangleright \qquad \frac{w : \alpha \triangleleft t, \Gamma \quad w : \beta \triangleleft t, \Gamma}{w : \alpha \wedge \beta \triangleleft t, \Gamma} \wedge \triangleleft \\
\\
\frac{w : \alpha \triangleleft t, w : \beta \triangleleft t, \Gamma}{w : \alpha \vee \beta \triangleleft t, \Gamma} \vee \triangleleft \qquad \frac{w : \alpha \triangleright t, \Gamma \quad w : \beta \triangleright t, \Gamma}{w : \alpha \vee \beta \triangleright t, \Gamma} \vee \triangleright \\
\\
\frac{w : \alpha > w : p, w : p < t, \Gamma}{w : \alpha \rightarrow p < t, \Gamma} \rightarrow_{\text{At} <} (\dagger) \quad \frac{w : \alpha > w : q, w : \beta \leq w : q, w : q < t, \Gamma}{w : \alpha \rightarrow \beta < t, \Gamma} \rightarrow < (\dagger) \\
\\
\frac{t \geq 1, \Gamma \quad w : \alpha > w : p, w : p \leq t, \Gamma}{w : \alpha \rightarrow p \leq t, \Gamma} \rightarrow_{\text{At} \leq} (\dagger) \\
\\
\frac{t \geq 1, \Gamma \quad w : \alpha > w : q, w : \beta \leq w : q, w : q \leq t, \Gamma}{w : \alpha \rightarrow \beta \leq t, \Gamma} \rightarrow \leq (\dagger) \\
\\
\frac{w : \alpha \leq w : p, 1 \triangleright t, \Gamma \quad w : p \triangleright t, \Gamma}{w : \alpha \rightarrow p \triangleright t, \Gamma} \rightarrow_{\text{At} \triangleright} (\dagger) \\
\\
\frac{w : \alpha \leq w : q, w : \beta \geq w : q, 1 \triangleright t, \Gamma \quad w : \beta \triangleright t, \Gamma}{w : \alpha \rightarrow \beta \triangleright t, \Gamma} \rightarrow \triangleright (\dagger) \\
\\
\text{At}^+(\Gamma) = \text{At}(\Gamma) \cup \{1 > t \mid w : \Box \alpha > t \in \Gamma\} \cup \{0 < t \mid w : \Diamond \alpha < t \in \Gamma\} \\
(\dagger) \quad p \in \mathcal{V} \cup \{\perp\}, \beta \notin \mathcal{V} \cup \{\perp\}, q \text{ is a new propositional variable}
\end{array}$$

Figure 5: The calculus $\mathcal{C}_{\text{GW}}$, propositional rules (Γ is a multiset of constraints).

countable set of labels, used to represent the worlds of a GW-model. In the definitions below w, w' are labels, φ is a formula, $p \in \mathcal{V}$, R is a designated relation symbol, representing the accessibility relation.

$$\begin{array}{ll}
\text{labelled formula} & := w : \varphi \\
\text{atomic c-term } t & := 0 \mid 1 \mid w : p \mid w : \perp \mid R(w, w') \\
\text{c-term } u & := t \mid w : \varphi \mid R(w, w') \wedge w' : \varphi \mid R(w, w') \rightarrow w' : \varphi \\
\text{constraint } \chi & := u \blacktriangledown t \quad \blacktriangledown \in \{<, \leq, >, \geq\}
\end{array}$$

If $w : \varphi$ occurs in a c-term u , we say that u has label w . A constraint of the form $t \blacktriangledown t$ is called *atomic*; if $\chi = w : \varphi \blacktriangledown t$ is non-atomic and $\sharp$ is the main connective of φ , we say that χ is a $\sharp$ -*constraint*. Constraints where u is $R(w, w') \wedge w' : \varphi$ or $R(w, w') \rightarrow w' : \varphi$ are called *R-constraints*. Given a multiset of constraints Γ , by $\text{At}(\Gamma)$ we denote the set of atomic constraints in Γ . Let $\mathfrak{M} = \langle W, R, e \rangle$ be a GW-model. An $\mathfrak{M}$ -interpretation of $\mathcal{L}_c$ is a function $\mathcal{I}$ mapping labels of $\mathcal{L}_c$ to W . We extend $\mathcal{I}$ to c-terms as follows:

$$\begin{array}{lll}
\mathcal{I}(k) = k, k \in \{0, 1\} & \mathcal{I}(w : \varphi) = e(\mathcal{I}(w), \varphi) & \mathcal{I}(R(w, w')) = R(\mathcal{I}(w), \mathcal{I}(w')) \\
\mathcal{I}(R(w, w') \star w' : \varphi) = R(\mathcal{I}(w), \mathcal{I}(w')) \star e(\mathcal{I}(w'), \varphi), \star \in \{\wedge, \rightarrow\}
\end{array}$$

Note that, for every c-term u , $\mathcal{I}(u)$ belongs to $[0, 1]$. We introduce the relations $\models_{\mathcal{I}}$ and $\models$, where $\mathfrak{M}$ is a GW-model, $\mathcal{I}$ an $\mathfrak{M}$ -interpretation, Γ a multiset of constraints.

$$\begin{array}{c}
\frac{R(w, w_1) \rightarrow w_1 : \alpha \triangleleft t, \Phi^{\square, \diamond}(\Gamma, w, w_1), \Gamma}{w : \square \alpha \triangleleft t, \Gamma} \square \triangleleft \quad \frac{R(w, w_1) \wedge w_1 : \alpha \triangleright t, \Phi^{\square, \diamond}(\Gamma, w, w_1), \Gamma}{w : \diamond \alpha \triangleright t, \Gamma} \diamond \triangleright \\
\frac{R(w, w') \triangleright t, w' : \alpha \triangleright t, \Gamma}{R(w, w') \wedge w' : \alpha \triangleright t, \Gamma} R \wedge \triangleright \quad \frac{R(w, w') \triangleleft t, \Gamma \quad w' : \alpha \triangleleft t, \Gamma}{R(w, w') \wedge w' : \alpha \triangleleft t, \Gamma} R \wedge \triangleleft \\
\frac{w' : \alpha < R(w, w'), w' : \alpha < t, \Gamma}{R(w, w') \rightarrow w' : \alpha < t, \Gamma} R \rightarrow < \quad \frac{t \geq 1, \Gamma \quad w' : \alpha < R(w, w'), w' : \alpha \leq t, \Gamma}{R(w, w') \rightarrow w' : \alpha \leq t, \Gamma} R \rightarrow \leq \\
\frac{w' : \alpha \geq R(w, w'), 1 \triangleright t, \Gamma \quad w' : \alpha \triangleright t, \Gamma}{R(w, w') \rightarrow w' : \alpha \triangleright t, \Gamma} R \rightarrow \triangleright
\end{array}$$

$\Phi^{\square, \diamond}(\Gamma, w, w_1) = \{R(w, w_1) \rightarrow w_1 : \beta \triangleright' t \mid w : \square \beta \triangleright' t \in \Gamma\} \cup \{R(w, w_1) \wedge w_1 : \beta \triangleleft' t \mid w : \diamond \beta \triangleleft' t \in \Gamma\}$
 w_1 is a new label, $\triangleleft, \triangleleft' \in \{<, \leq\}$ and $\triangleright, \triangleright' \in \{>, \geq\}$.

Figure 6: The calculus $\mathcal{C}_{\text{GW}}$: modal rules and R -rules (Γ is a multiset of constraints).

- $\mathfrak{M} \models_{\mathcal{J}} u \blacktriangledown t$ iff $\mathcal{J}(u) \blacktriangledown \mathcal{J}(t)$;
- $\mathfrak{M} \models_{\mathcal{J}} \Gamma$ iff $\mathfrak{M} \models_{\mathcal{J}} \chi$, for every $\chi \in \Gamma$;
- $\mathfrak{M} \models \Gamma$ iff $\mathfrak{M} \models_{\mathcal{J}} \Gamma$ for some $\mathfrak{M}$ -interpretation $\mathcal{J}$.

A *substitution* σ is a function mapping each atomic c-term of the form $w : p$ or $R(w, w')$ to a rational number in $[0, 1]_{\mathbf{Q}}$; σ is extended to all the atomic c-terms by setting $\sigma(k) = k$, for $k \in \{0, 1\}$, and $\sigma(w : \perp) = 0$. Let Γ_{at} be a set of atomic constraints. By $\sigma(\Gamma_{\text{at}})$ we denote the set of constraints obtained by replacing every atomic c-term t occurring in Γ_{at} with $\sigma(t)$. Note that $\sigma(\Gamma_{\text{at}})$ is a set of rational constraints of the form $r_1 \blacktriangledown r_2$, with r_1 and r_2 in $[0, 1]_{\mathbf{Q}}$; if all the constraints in $\sigma(\Gamma_{\text{at}})$ hold, σ is a *solution* to Γ_{at} . The set Γ_{at} is *consistent* iff it admits at least one solution. We remark that consistency of Γ_{at} can be checked by a Constraint Solver over $\mathbf{Q}$: one has to abstract the c-terms $w : p$ and $R(w, w')$ occurring in Γ_{at} by introducing new variables ranging over $[0, 1]_{\mathbf{Q}}$, and then check the consistency of the obtained constraints by exploiting the solver.

Hereafter, in constraints $\triangleleft$ and $\triangleleft'$ are metasymbols ranging over $\{<, \leq\}$; similarly $\triangleright, \triangleright' \in \{>, \geq\}$. The rules of the calculus $\mathcal{C}_{\text{GW}}$ are displayed in Fig. 5 (propositional rules) and Fig. 6 (modal rules and R -rules). The *main constraint* of a rule application is the constraint displayed in the conclusion. Rules for implication, having a main constraint of the kind $w : \alpha \rightarrow \beta \blacktriangledown t$, are defined according to the structure of β . Let us consider the rule $\rightarrow_{\text{At} <}$, having main constraint $w : \alpha \rightarrow p < t$ ($p \in \mathcal{V} \cup \{\perp\}$). The premise contains the constraints $w : \alpha > w : p$ and $w : p < t$. This reflects the fact that, given a GW-model $\mathfrak{M}$, a world w in $\mathfrak{M}$ and an $\mathfrak{M}$ -interpretation $\mathcal{J}$, if $e(\mathcal{J}(w), \alpha \rightarrow p) < \mathcal{J}(t)$, it holds that $e(\mathcal{J}(w), \alpha) > e(\mathcal{J}(w), p)$, $e(\mathcal{J}(w), \alpha \rightarrow p) = e(\mathcal{J}(w), p)$ and $e(\mathcal{J}(w), p) < \mathcal{J}(t)$. This rule cannot be generalized to any $w : \alpha \rightarrow \beta < t$ since the constraint $w : \alpha > w : \beta$ is not allowed if $\beta \notin \mathcal{V} \cup \{\perp\}$. This case is covered by rule $\rightarrow <$, which introduces a new propositional variable q behaving in w as β , and $w : \alpha > w : \beta$ is replaced by $w : \alpha > w : q$. The correspondence between β and q is set by the constraint $w : \beta \leq w : q$, while the converse constraint $w : \beta \geq w : q$ can be omitted. There are two modal rules (see Fig. 6), the former having main constraint $w : \square \alpha \triangleleft t$ and the latter $w : \diamond \alpha \triangleright t$; both rules introduce a new label w_1 . R -rules (see Fig. 6) are similar to the corresponding propositional rules. The definitions of *tree* and *derivation* of $\mathcal{C}_{\text{GW}}$ are the usual ones (see, e.g., [19]). We remark that $\mathcal{C}_{\text{GW}}$ has

$$\begin{array}{l}
c = w_0 : q_0 \quad \Delta_0 = \{w_0 : \Box p > c, c \geq 1, c < 1\} \quad \Delta_1 = \{w_0 : \perp \leq c, c < 1\} \\
\Delta_2 = \Delta_1 \cup \{w_0 : \Box p > c, R(w_0, w_1) > w_0 : \perp\} \quad \Delta_3 = \Delta_2 \cup \{w_1 : p \leq w_1 : \perp, 1 > w_0 : \perp\} \\
\Delta_4 = \Delta_2 \cup \{R(w_0, w_1) \rightarrow w_1 : p > c, w_1 : \perp > w_0 : \perp\} \\
\\
\frac{\overline{\Delta_0} \text{ Ax} \quad \frac{w_1 : p \geq R(w_0, w_1), 1 > c, \Delta_3}{R(w_0, w_1) \rightarrow w_1 : p > c, \Delta_3} \text{ Ax} \quad \frac{w_1 : p > c, \Delta_3}{R \rightarrow \triangleright} \text{ Ax} \quad \frac{\overline{\Delta_4} \text{ Ax}}{\rightarrow_{\text{At}} \triangleright}}{\frac{R(w_0, w_1) \rightarrow w_1 : p > c, \Delta_2}{R \wedge \triangleright}} \rightarrow_{\text{At}} \triangleright \\
\frac{\overline{\Delta_0} \text{ Ax} \quad \frac{w_0 : \Box p > c, R(w_0, w_1) \rightarrow w_1 : p > c, \underline{R(w_0, w_1) \wedge w_1 : \neg p > w_0 : \perp}, \Delta_1}{w_0 : \Box p > c, w_0 : \Diamond \neg p > w_0 : \perp, \Delta_1} R \wedge \triangleright}{\frac{w_0 : \Box p > c, w_0 : \neg \Diamond \neg p \leq c, c < 1}{w_0 : \Box p \rightarrow \neg \Diamond \neg p < 1} \rightarrow_{\text{At}} \leq} \Diamond \triangleright \\
\frac{\overline{\Delta_0} \text{ Ax} \quad \frac{w_0 : \Box p > c, w_0 : \neg \Diamond \neg p \leq c, c < 1}{w_0 : \Box p \rightarrow \neg \Diamond \neg p < 1} \rightarrow_{\text{At}} \leq}{\rightarrow <}
\end{array}$$

Figure 7: A derivation of $w_0 : \Box p \rightarrow \neg \Diamond \neg p < 1$.

the *subformula property*, namely: if $\mathcal{T}$ is a tree of $\mathcal{C}_{\text{GW}}$ having Γ_0 as root, every formula occurring in $\mathcal{T}$ inside a constraint (e.g., the formula φ in the constraint $R(w, w') \rightarrow w' : \varphi$) is a subformula of a formula in Γ_0 or a new propositional variable. By $\vdash_{\text{GW}} \Gamma$ we mean that there exists a derivation of Γ . In the rest of this section we show that $\mathcal{C}_{\text{GW}}$ is a sound and complete refutation calculus for **GW**. Formally, soundness and completeness are defined as follows, where Γ is a multiset of constraints of the form $w : \varphi \blacktriangledown t$:

- (Soundness) if $\vdash_{\text{GW}} \Gamma$, then $\mathfrak{M} \not\models \Gamma$, for every GW-model $\mathfrak{M}$;
- (Completeness) if $\mathfrak{M} \not\models \Gamma$, for every GW-model $\mathfrak{M}$, then $\vdash_{\text{GW}} \Gamma$.

We stress that, if rule Ax checked the consistency of $\text{At}(\Gamma)$ instead of $\text{At}^+(\Gamma)$, the calculus $\mathcal{C}_{\text{GW}}$ would not be complete. Indeed, let $\Gamma = \{w : \Box p > w : p, w : p \leq 1, w : p \geq 1\}$. Clearly, there is no GW-model $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$ hence, by completeness, Γ must be provable in $\mathcal{C}_{\text{GW}}$. To build a derivation of Γ , one can only exploit rule Ax. Note that $\text{At}(\Gamma) = \{w : p \leq 1, w : p \geq 1\}$ is consistent (any substitution σ such that $\sigma(w : p) = 1$ is a solution); thus, if Ax checked the consistency of $\text{At}(\Gamma)$, Γ could not be proved. Instead, Γ is proved since Ax evaluates the set $\text{At}^+(\Gamma) = \text{At}(\Gamma) \cup \{1 > w : p\}$, which is not consistent.

An example of derivation is shown in Fig. 7; the main constraint of a rule application is underlined.

Let $\mathfrak{M} = \langle W, R, e \rangle$ be a GW-model and $\mathcal{I}$ an $\mathfrak{M}$ -interpretation. Let $p_0 \in \mathcal{V}$ and α a formula not containing p_0 , let $w_0 \in \mathcal{L}_c$ and $w^* \in W$. The evaluation function $e[p_0 := \alpha]$ and the $\mathfrak{M}$ -interpretation $\mathcal{I}[w_0 := w^*]$ are defined as follows ($w \in W$ and $p \in \mathcal{V}$):

$$e[p_0 := \alpha](w, p) = \begin{cases} e(w, \alpha) & \text{if } p = p_0 \\ e(w, p) & \text{otherwise} \end{cases} \quad \mathcal{I}[w_0 := w^*](w) = \begin{cases} w^* & \text{if } w = w_0 \\ \mathcal{I}(w) & \text{otherwise} \end{cases}$$

Let $e' = e[p_0 := \alpha]$ and $w \in W$; one can prove that, for every formula φ not containing p_0 , $e'(w, \varphi) = e(w, \varphi)$. Accordingly, $e'(w, p_0 \leftrightarrow \alpha) = 1$, and this implies that $\langle W, R, e' \rangle$ is a GW-model. Let $\mathcal{I}' = \mathcal{I}[w_0 := w^*]$ and let χ be a constraint not containing w_0 ; one can prove that $\mathfrak{M} \models_{\mathcal{I}'} \chi$ iff $\mathfrak{M} \models_{\mathcal{I}} \chi$.

Lemma 3.1. *Let ρ be an instance of a rule of $\mathcal{C}_{\text{GW}}$ and let Γ be the conclusion of ρ . If there is a GW-model $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$, then there exist a GW-model $\mathfrak{M}'$ and a premise Γ' of ρ such that $\mathfrak{M}' \models \Gamma'$.*

Proof. Let ρ be an instance of the rule Ax; since the rule has no premises, we have to show that there is no $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$. Let us assume, by contradiction, that there is a GW-model $\mathfrak{M} = \langle W, R, e \rangle$

and an $\mathfrak{M}$ -interpretation $\mathcal{J}$ such that $\mathfrak{M} \models_{\mathcal{J}} \Gamma$. If $w : \Box\alpha > t \in \Gamma$, then $e(\mathcal{J}(w), \Box\alpha) > \mathcal{J}(t)$, hence $1 > \mathcal{J}(t)$, which implies $\mathfrak{M} \models_{\mathcal{J}} 1 > t$. Similarly, if $w : \Diamond\alpha < t \in \Gamma$, then $\mathfrak{M} \models_{\mathcal{J}} 0 < t$. This proves that $\mathfrak{M} \models_{\mathcal{J}} \text{At}^+(\Gamma)$. By exploiting R and e , one can define a solution to $\text{At}^+(\Gamma)$; accordingly, $\text{At}^+(\Gamma)$ is consistent, a contradiction. We conclude that $\mathfrak{M}$ does not exist.

Let $\rho \neq \text{Ax}$ and let us assume $\mathfrak{M} \models_{\mathcal{J}} \Gamma$, where $\mathfrak{M} = \langle W, R, e \rangle$ is a GW-model and $\mathcal{J}$ is an $\mathfrak{M}$ -interpretation; we prove that there exist a GW-model $\mathfrak{M}'$ and a premise Γ' of ρ such that $\mathfrak{M}' \models \Gamma'$. We only detail some representative cases, by showing the corresponding rule application.

$$\frac{\overbrace{w : \alpha > w : q, w : \beta \leq w : q, w : q < t, \Delta}^{\Gamma_1}}{\underbrace{w : \alpha \rightarrow \beta < t, \Delta}_{\Gamma}} \rightarrow < \quad \begin{array}{l} \beta \notin \mathcal{V} \cup \{\perp\} \\ q \text{ does not occur in } \Gamma \end{array}$$

By the hypothesis, $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \rightarrow \beta < t$ and $\mathfrak{M} \models_{\mathcal{J}} \Delta$. This implies that $e(\mathcal{J}(w), \alpha \rightarrow \beta) < \mathcal{J}(t)$, hence $e(\mathcal{J}(w), \alpha) > e(\mathcal{J}(w), \beta)$ and $e(\mathcal{J}(w), \alpha \rightarrow \beta) = e(\mathcal{J}(w), \beta)$, thus $e(\mathcal{J}(w), \beta) < \mathcal{J}(t)$. Let $\mathfrak{M}' = \langle W, R, e[q := \beta] \rangle$; we remark that $\mathfrak{M}'$ is a GW-model. Since $\mathfrak{M} \models_{\mathcal{J}} \Delta$ and q does not occur in Δ , we get $\mathfrak{M}' \models_{\mathcal{J}} \Delta$. It is easy to check that $\mathfrak{M}' \models_{\mathcal{J}} w : \beta \leq w : q$ and $\mathfrak{M}' \models_{\mathcal{J}} w : \alpha > w : q$ and $\mathfrak{M}' \models_{\mathcal{J}} w : q < t$; we conclude $\mathfrak{M}' \models_{\mathcal{J}} \Gamma_1$.

$$\frac{\overbrace{t \geq 1, \Delta}^{\Gamma_1} \quad \overbrace{w : \alpha > w : q, w : \beta \leq w : q, w : q \leq t, \Delta}^{\Gamma_2}}{\underbrace{w : \alpha \rightarrow \beta \leq t, \Delta}_{\Gamma}} \rightarrow \leq \quad \begin{array}{l} \beta \notin \mathcal{V} \cup \{\perp\} \\ q \text{ does not occur in } \Gamma \end{array}$$

By the hypothesis, $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \rightarrow \beta \leq t$ and $\mathfrak{M} \models_{\mathcal{J}} \Delta$, hence $e(\mathcal{J}(w), \alpha \rightarrow \beta) \leq \mathcal{J}(t)$. Let us assume $e(\mathcal{J}(w), \alpha) \leq e(\mathcal{J}(w), \beta)$; then, $e(\mathcal{J}(w), \alpha \rightarrow \beta) = 1$, which implies $1 \leq \mathcal{J}(t)$, thus $\mathcal{J}(t) = 1$. We get $\mathfrak{M} \models_{\mathcal{J}} t \geq 1$, hence $\mathfrak{M} \models_{\mathcal{J}} \Gamma_1$. Assume $e(\mathcal{J}(w), \alpha) > e(\mathcal{J}(w), \beta)$; then, $e(\mathcal{J}(w), \alpha \rightarrow \beta) = e(\mathcal{J}(w), \beta)$, hence $\mathfrak{M} \models_{\mathcal{J}} w : \beta \leq t$. Let $\mathfrak{M}' = \langle W, R, e[q := \beta] \rangle$; note that $\mathfrak{M}'$ is a GW-model. Since $\mathfrak{M} \models_{\mathcal{J}} \Delta$ and q does not occur in Δ , we get $\mathfrak{M}' \models_{\mathcal{J}} \Delta$. Moreover, one can easily check that $\mathfrak{M}' \models_{\mathcal{J}} w : \beta \leq w : q$ and $\mathfrak{M}' \models_{\mathcal{J}} w : \alpha > w : q$ and $\mathfrak{M}' \models_{\mathcal{J}} w : q \leq t$; we conclude $\mathfrak{M}' \models_{\mathcal{J}} \Gamma_2$.

$$\frac{\overbrace{w : \alpha \leq w : q, w : \beta \geq w : q, 1 \triangleright t, \Delta}^{\Gamma_1} \quad \overbrace{w : \beta \triangleright t, \Delta}^{\Gamma_2}}{\underbrace{w : \alpha \rightarrow \beta \triangleright t, \Delta}_{\Gamma}} \rightarrow \triangleright \quad \begin{array}{l} \beta \notin \mathcal{V} \cup \{\perp\} \\ q \text{ does not occur in } \Gamma \end{array}$$

By the hypothesis, $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \rightarrow \beta \triangleright t$ and $\mathfrak{M} \models_{\mathcal{J}} \Delta$, hence $e(\mathcal{J}(w), \alpha \rightarrow \beta) \triangleright \mathcal{J}(t)$. Let us assume $e(\mathcal{J}(w), \alpha) > e(\mathcal{J}(w), \beta)$; then $e(\mathcal{J}(w), \alpha \rightarrow \beta) = e(\mathcal{J}(w), \beta)$, hence $\mathfrak{M} \models_{\mathcal{J}} \Gamma_2$. Let us assume $e(\mathcal{J}(w), \alpha) \leq e(\mathcal{J}(w), \beta)$; then, $e(\mathcal{J}(w), \alpha \rightarrow \beta) = 1$, which implies $1 \triangleright \mathcal{J}(t)$, thus $\mathfrak{M} \models_{\mathcal{J}} 1 \triangleright t$. Let $\mathfrak{M}' = \langle W, R, e[q := \alpha] \rangle$; note that $\mathfrak{M}'$ is a GW-model. Reasoning as in the previous cases, we can prove that $\mathfrak{M}' \models_{\mathcal{J}} \Gamma_1$.

$$\frac{\overbrace{R(w, w_1) \rightarrow w_1 : \alpha \triangleleft t, \Phi^{\Box, \Diamond}(\Gamma, w, w_1), \Delta}^{\Gamma_1}}{\underbrace{w : \Box\alpha \triangleleft t, \Delta}_{\Gamma}} \Box \triangleleft \quad w_1 \text{ does not occur in } \Gamma$$

By the hypothesis, $\mathfrak{M} \models_{\mathcal{J}} w : \Box\alpha \triangleleft t$ and $\mathfrak{M} \models_{\mathcal{J}} \Delta$, hence $e(\mathcal{J}(w), \Box\alpha) \triangleleft \mathcal{J}(t)$. Since $\mathfrak{M}$ is a GW-model, there exists a world w^* of $\mathfrak{M}$ such that $R(\mathcal{J}(w), w^*) \rightarrow e(w^*, \alpha) = e(\mathcal{J}(w), \Box\alpha)$. Let $\mathcal{J}' = \mathcal{J}[w_1 := w^*]$. We prove that:

- (i) $\mathfrak{M} \models_{\mathcal{J}'} \Delta$.
- (ii) $\mathfrak{M} \models_{\mathcal{J}'} R(w, w_1) \rightarrow w_1 : \alpha \triangleleft t$.
- (iii) $\mathfrak{M} \models_{\mathcal{J}'} \Phi^{\square, \diamond}(\Gamma, w, w_1)$.

Point (i) follows from the fact that $\mathfrak{M} \models_{\mathcal{J}} \Delta$ and w_1 does not occur in Δ . Since $e(\mathcal{J}(w), \square\alpha) \triangleleft \mathcal{J}(t)$ and $e(\mathcal{J}(w), \square\alpha) = R(\mathcal{J}(w), w^*) \rightarrow e(w^*, \alpha)$, we get $R(\mathcal{J}(w), w^*) \rightarrow e(w^*, \alpha) \triangleleft \mathcal{J}(t)$; this proves (ii). Let $\chi \in \Phi^{\square, \diamond}(\Gamma, w, w_1)$; we show that $\mathfrak{M} \models_{\mathcal{J}'} \chi$, and this proves (iii). Let $\chi = R(w, w_1) \rightarrow w_1 : \beta \triangleright t'$. Then, $w : \square\beta \triangleright t' \in \Delta$, hence $e(\mathcal{J}(w), \square\beta) \triangleright \mathcal{J}(t')$. It follows that $R(\mathcal{J}(w), w^*) \rightarrow e(w^*, \beta) \triangleright \mathcal{J}(t')$, hence $\mathfrak{M} \models_{\mathcal{J}'} \chi$. The proof of $\mathfrak{M} \models_{\mathcal{J}'} \chi$ in the case $\chi = R(w, w_1) \wedge w_1 : \beta \triangleleft' t'$ is similar, taking into account that $w : \diamond\beta \triangleleft' t' \in \Delta$. From points (i)–(iii), we conclude $\mathfrak{M} \models_{\mathcal{J}'} \Gamma_1$. $\square$

Proposition 3.2 (Soundness). (i) If $\vdash_{\text{GW}} \Gamma$, then $\mathfrak{M} \models \Gamma$, for every GW-model $\mathfrak{M}$.

(ii) $\vdash_{\text{GW}} w : \varphi < 1$ implies $\varphi \in \mathbf{GW}$.

Proof. Point (i) can be proved by induction on the depth of a derivation of Γ , by exploiting Lemma 3.1. Let us assume $\varphi \notin \mathbf{GW}$. Then, there exists a GW-model $\mathfrak{M} = \langle W, R, e \rangle$ and a world $w^* \in W$ such that $e(w^*, \varphi) < 1$. Let $\mathcal{J}$ be an $\mathfrak{M}$ -interpretation mapping w to w^* ; it holds that $\mathfrak{M} \models_{\mathcal{J}} w : \varphi < 1$. By (i), we get $\not\vdash_{\text{GW}} w : \varphi < 1$, and this proves (ii). $\square$

We show that $\mathcal{C}_{\text{GW}}$ is *strongly terminating*, namely: there exists a well-founded relation $\prec$ such that, for every application ρ of a rule of $\mathcal{C}_{\text{GW}}$, if Γ is the conclusion of ρ and Γ' any of the premises of ρ , then $\Gamma' \prec \Gamma$. Equivalently, given a finite multiset Γ and repeatedly applying the rules of $\mathcal{C}_{\text{GW}}$ upwards, proof search eventually halts, no matter which strategy is used.

To prove strong termination of $\mathcal{C}_{\text{GW}}$, we define the relation $\prec_c$ between multisets of constraints. To this aim, in Fig. 8 we introduce a weight function wg on constraints and multisets of constraints Γ , together with the multisets $\Gamma[w]$ (multiset of constraints) and $\|\Gamma\|$ (multiset of natural numbers).

Given the finite multisets of natural numbers Θ_1 and Θ_2 , we set:

$$\Theta_1 \prec_m \Theta_2 \quad \text{iff} \quad \Theta_1 \neq \Theta_2 \wedge (\forall k_1 \in \Theta_1 \setminus \Theta_2. \exists k_2 \in \Theta_2 \setminus \Theta_1. k_1 < k_2).$$

The relation $\prec_m$ is a multiset order, hence $\prec_m$ is well-founded (see [1], Th. 2.5.5 and Lemma 2.5.6). Given the multisets of constraints Γ_1 and Γ_2 , we set $\Gamma_1 \prec_c \Gamma_2$ iff $\|\Gamma_1\| \prec_m \|\Gamma_2\|$.

The relation $\prec_c$ is well-founded; in the next lemma we show that $\prec_c$ can be used to instantiate $\prec$ in the definition of strong termination.

Lemma 3.3. Let ρ be an instance of a rule of the calculus $\mathcal{C}_{\text{GW}}$, let Γ be the conclusion of ρ and Γ' any of the premises of ρ . Then, $\Gamma' \prec_c \Gamma$.

Proof. We only discuss two representative cases. Let us consider the following application of rule $\rightarrow<$:

$$\frac{\overbrace{w : \alpha > w : q, w : \beta \leq w : q, w : q < t, \Delta_w, \Theta}^{\Gamma_1}}{\underbrace{w : \alpha \rightarrow \beta < t, \Delta_w, \Theta}_{\Gamma}} \rightarrow< \quad \begin{array}{l} \text{all the constraints in } \Delta_w \text{ have label } w \\ \text{no constraint in } \Theta \text{ has label } w \end{array}$$

$$\Gamma[w] = \{w : \alpha \rightarrow \beta < t\} \cup \Delta_w$$

$$\Gamma_1[w] = \{w : \alpha > w : q, w : \beta \leq w : q, w : q < t\} \cup \Delta_w$$

$$\|\Gamma\| = \{\text{wg}(\Gamma[w])\} \cup \|\Theta\|$$

$$\|\Gamma_1\| = \{\text{wg}(\Gamma_1[w])\} \cup \|\Theta\|$$

To prove that $\|\Gamma_1\| \prec_m \|\Gamma\|$, we show that $\text{wg}(\Gamma_1[w]) < \text{wg}(\Gamma[w])$. The following holds:

$$\begin{aligned} \text{wg}(\Gamma[w]) &= \text{wg}(w : \alpha \rightarrow \beta < t) + \text{wg}(\Delta_w) = \text{wg}(\alpha) + \text{wg}(\beta) + 1 + \text{wg}(\Delta_w) \\ \text{wg}(\Gamma_1[w]) &= \text{wg}(w : \alpha > w : q) + \text{wg}(w : \beta \leq w : q) + \text{wg}(w : q < t) + \text{wg}(\Delta_w) \\ &= \text{wg}(\alpha) + \text{wg}(\beta) + 0 + \text{wg}(\Delta_w) \end{aligned}$$

Since $\text{wg}(\Gamma_1[w]) < \text{wg}(\Gamma[w])$, we get $\|\Gamma_1\| \prec_m \|\Gamma\|$, hence $\Gamma_1 \prec_c \Gamma$.

Let us consider the following application of $\Box \triangleleft$:

$$\frac{\overbrace{\Delta_{w_1}, \Delta_w, \Theta}^{\Gamma_1}}{w : \Box \alpha \triangleleft t, \Delta_w, \Theta} \Box \triangleleft \quad \begin{array}{l} \text{all the constraints in } \Delta_w \text{ have label } w \\ \text{no constraint in } \Theta \text{ has label } w \\ \Delta_{w_1} = \{R(w, w_1) \rightarrow w_1 : \alpha \triangleleft t\} \cup \Phi^{\Box, \Diamond}(\Delta_w, w, w_1) \end{array}$$

$$\begin{aligned} \Gamma[w] &= \{w : \Box \alpha \triangleleft t\} \cup \Delta_w & \Gamma_1[w] &= \Delta_w & \Gamma_1[w_1] &= \Delta_{w_1} \\ \|\Gamma\| &= \{\text{wg}(\Gamma[w])\} \cup \|\Theta\| & \|\Gamma_1\| &= \{\text{wg}(\Gamma_1[w]), \text{wg}(\Gamma_1[w_1])\} \cup \|\Theta\| \end{aligned}$$

We prove that $\|\Gamma_1\| \prec_m \|\Gamma\|$. To this aim, we show that:

- (i) $\text{wg}(\Gamma_1[w]) < \text{wg}(\Gamma[w])$.
- (ii) $\text{wg}(\Gamma_1[w_1]) < \text{wg}(\Gamma[w])$.

The proof of point (i) is immediate since $\Gamma[w] = \{w : \Box \alpha \triangleleft t\} \cup \Gamma_1[w]$. We prove (ii). We have:

$$\begin{aligned} \text{wg}(\Gamma[w]) &= \text{wg}(w : \Box \alpha \triangleleft t) + \text{wg}(\Delta_w) = \text{wg}(\alpha) + 2 + \text{wg}(\Delta_w) \\ \text{wg}(\Gamma_1[w_1]) &= \text{wg}(R(w, w_1) \rightarrow w_1 : \alpha) + \text{wg}(\Phi^{\Box, \Diamond}(\Delta_w, w, w_1)) = \text{wg}(\alpha) + 1 + \text{wg}(\Phi^{\Box, \Diamond}(\Delta_w, w, w_1)) \end{aligned}$$

To conclude the proof of (ii), we show that:

$$(iii) \text{wg}(\Phi^{\Box, \Diamond}(\Delta_w, w, w_1)) < \text{wg}(\Delta_w).$$

Let $\chi_1 \in \Phi^{\Box, \Diamond}(\Delta_w, w, w_1)$. There exists $\chi \in \Delta_w$ such that one of the two following conditions holds:

- (a) $\chi = w : \Box \beta \triangleright' t'$ and $\chi_1 = R(w, w_1) \rightarrow w_1 : \beta \triangleright' t'$;
- (b) $\chi = w : \Diamond \beta \triangleleft' t'$ and $\chi_1 = R(w, w_1) \wedge w_1 : \beta \triangleleft' t'$.

In both cases, it holds that $\text{wg}(\chi_1) = \text{wg}(\beta) + 1$ and $\text{wg}(\chi) = \text{wg}(\beta) + 2$, hence $\text{wg}(\chi_1) < \text{wg}(\chi)$; this proves (iii). From (i) and (ii) it follows that $\|\Gamma_1\| \prec_m \|\Gamma\|$, hence $\Gamma_1 \prec_c \Gamma$. $\square$

As an immediate consequence of Lemma 3.3, we get:

Proposition 3.4. *The calculus $\mathcal{C}_{\text{GW}}$ is strongly terminating.*

Consequently, any backward proof search strategy for $\mathcal{C}_{\text{GW}}$ terminates. Following [8, 12], we focus on strategies where failure is certified by countermodels, i.e.: if proof search for a multiset of constraints Γ fails, a GW-model $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$ can be built; we call $\mathfrak{M}$ a *countermodel* for Γ , since $\mathfrak{M}$ ascertains that Γ is not provable in $\mathcal{C}_{\text{GW}}$ (by its soundness). Let Γ be a multiset of constraints:

- Γ is *reduced* iff no rule of $\mathcal{C}_{\text{GW}}$ can be backward applied to Γ ; thus, every non-atomic constraint in Γ has the form $w : \Box \alpha \triangleright t$ or $w : \Diamond \alpha \triangleleft t$.

$$\begin{aligned}
\text{wg}(u) &= \begin{cases} 0 & \text{if } u \text{ is an atomic c-term} \\ |\varphi| & \text{if } u = w : \varphi \\ |\varphi| + 1 & \text{if } u = R(w, w') \star w' : \varphi \\ & \text{with } \star \in \{\wedge, \vee, \rightarrow\} \end{cases} & |\varphi| &= \begin{cases} 0 & \text{if } \varphi \in \mathcal{V} \cup \{\perp\} \\ |\alpha| + |\beta| + 1 & \text{if } \varphi = \alpha \star \beta \\ & \text{with } \star \in \{\wedge, \vee, \rightarrow\} \\ |\alpha| + 2 & \text{if } \varphi = \Box \alpha \text{ or } \varphi = \Diamond \alpha \end{cases} \\
\text{wg}(u \blacktriangledown t) &= \text{wg}(u) & \text{wg}(\Gamma) &= \sum_{\chi \in \Gamma} \text{wg}(\chi) \\
\Gamma[w] &= \{u \blacktriangledown t \in \Gamma \mid \text{label}(u) = w\} & \|\Gamma\| &= \{\text{wg}(\Gamma[w]) \mid w \text{ occurs in } \Gamma\}
\end{aligned}$$

Figure 8: Weight function wg on constraints and the multisets $\Gamma[w]$, $\|\Gamma\|$.

- Γ is *plain* iff only a modal rule can be backward applied to Γ ; thus, every non-atomic constraint in Γ has the form $w : \Box \alpha \blacktriangledown t$ or $w : \Diamond \alpha \blacktriangledown t$.

An application of a modal rule is plain iff its conclusion is plain. By $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$ we mean that Γ_{k+1} is a premise of an application of a rule ρ of $\mathcal{C}_{\text{GW}}$ having conclusion Γ_k and main constraint χ , with $\chi \in \Gamma_k$. A branch $\mathcal{B}$ is a sequence $\langle \Gamma_0, \dots, \Gamma_n \rangle$ such that, for every $0 \leq k < n$, there is χ_k such that $\Gamma_k \mapsto_{\chi_k} \Gamma_{k+1}$; thus, $\mathcal{B}$ represents a branch of a tree of $\mathcal{C}_{\text{GW}}$. The branch $\mathcal{B}$ is *saturated* iff the following holds:

- Γ_0 is a finite set of constraints of the form $w : \varphi \blacktriangledown t$ and Γ_n is reduced;
- every application of a modal rule in $\mathcal{B}$ is plain.

We show how to construct a countermodel for Γ_0 from Γ_n . Let Γ_{at} be a finite consistent set of atomic constraints and let σ be a solution to Γ_{at} . By $\text{Mod}(\Gamma_{\text{at}}, \sigma)$ we denote the G-model $\mathfrak{M} = \langle W, R, e \rangle$ such that W is the set of labels occurring in Γ_{at} and:

$$R(w, w') = \begin{cases} \sigma(R(w, w')) & \text{if } R(w, w') \text{ occurs in } \Gamma_{\text{at}} \\ 0 & \text{otherwise} \end{cases} \quad e(w, p) = \begin{cases} \sigma(w : p) & \text{if } w : p \text{ occurs in } \Gamma_{\text{at}} \\ 0 & \text{otherwise} \end{cases}$$

We remark that $\mathfrak{M}$ is a discrete GW-model. A $\text{Mod}(\Gamma_{\text{at}}, \sigma)$ -interpretation $\mathcal{I}$ is *canonical* iff $\mathcal{I}(w) = w$ for every label w occurring in Γ_{at} . In Lemma 3.6 we show that if $\mathcal{B} = \langle \Gamma_0, \dots, \Gamma_n \rangle$ is saturated and σ is a solution to $\text{At}^+(\Gamma_n)$, then $\text{Mod}(\text{At}^+(\Gamma_n), \sigma)$ is a countermodel for Γ_0 .

Example 3.5. Let $\varphi = \Box(p \vee q) \rightarrow (\Box p \vee \Diamond q)$ be an instance of (Cr) . The tree shown in Fig. 9 corresponds to a saturated branch $\mathcal{B} = \langle \Gamma_0, \dots, \Gamma_7 \rangle$, where $\Gamma_0 = \{w_0 : \varphi < 1\}$ is the root and Γ_7 is the top multiset. When the applied rule has two premises, the annotation (l) (left) or (r) (right) specifies the selected one; in every rule application, the main constraint is underlined.

The set $\text{At}^+(\Gamma_7)$ consists of the following atomic constraints:

$$w_1 : p < R(w_0, w_1), \quad w_1 : p \leq c, \quad w_1 : q \geq R(w_0, w_1), \quad 1 > c, \quad R(w_0, w_1) \leq c, \quad c < 1$$

The set $\text{At}^+(\Gamma_7)$ is consistent; a solution to $\text{At}^+(\Gamma_7)$ is any substitution σ over $\text{At}^+(\Gamma_7)$ such that

$$\sigma(w_1 : p) < \sigma(R(w_0, w_1)) \leq \sigma(c) < 1 \quad \sigma(R(w_0, w_1)) \leq \sigma(w_1 : q)$$

For every solution σ , the GW-model $\mathfrak{M} = \text{Mod}(\text{At}^+(\Gamma_7), \sigma)$ satisfies $\mathfrak{M} \models_{\mathcal{I}} w_0 : \varphi < 1$, where $\mathcal{I}$ is a canonical $\mathfrak{M}$ -interpretation; as a consequence, $\mathfrak{M}$ is a countermodel for φ , witnessing that $\varphi \notin \mathbf{GW}$. Note that the worlds of $\mathfrak{M}$ are w_0 and w_1 and, as expected, $\mathfrak{M}$ is not crisp, since $0 < \sigma(R(w_0, w_1)) < 1$. A concrete solution σ^* is obtained by setting $\sigma^*(w_1 : p) = 0.5$ and $\sigma^*(R(w_0, w_1)) = \sigma^*(c) = \sigma^*(w_1 : q) = 0.6$. The model $\text{Mod}(\text{At}^+(\Gamma_7), \sigma^*)$ is essentially the same as the one displayed in Fig. 3. $\diamond$

$$\begin{array}{c}
c = w_0 : q_0 \quad \Delta = \{w_0 : \Box(p \vee q) > c, w_0 : \Diamond q \leq c, c < 1\} \\
\\
\frac{w_1 : p < R(w_0, w_1), w_1 : p \leq c, w_1 : q \geq R(w_0, w_1), 1 > c, R(w_0, w_1) \leq c, \Delta}{w_1 : p < R(w_0, w_1), w_1 : p \leq c, w_1 : q \geq R(w_0, w_1), 1 > c, \underline{R(w_0, w_1) \wedge w_1 : q \leq c}, \Delta} R \wedge \triangleleft (l) \\
\\
\frac{\frac{w_1 : p < R(w_0, w_1), w_1 : p \leq c, w_1 : p \vee q \geq R(w_0, w_1), 1 > c, \underline{R(w_0, w_1) \wedge w_1 : q \leq c}, \Delta}{w_1 : p < R(w_0, w_1), w_1 : p \leq c, \underline{R(w_0, w_1) \rightarrow w_1 : p \vee q > c}, R(w_0, w_1) \wedge w_1 : q \leq c, \Delta} R \rightarrow \triangleright (l)}{w_1 : p < R(w_0, w_1), w_1 : p \leq c, \underline{R(w_0, w_1) \rightarrow w_1 : p \vee q > c}, R(w_0, w_1) \wedge w_1 : q \leq c, \Delta} R \rightarrow \leq (r) \\
\\
\frac{\underline{R(w_0, w_1) \rightarrow w_1 : p \leq c}, R(w_0, w_1) \rightarrow w_1 : p \vee q > c, R(w_0, w_1) \wedge w_1 : q \leq c, \Delta}{w_0 : \Box(p \vee q) > c, \underline{w_0 : \Box p \leq c}, w_0 : \Diamond q \leq c, c < 1} \Box \triangleleft \\
\\
\frac{\underline{w_0 : \Box(p \vee q) > c, \underline{w_0 : (\Box p \vee \Diamond q) \leq c}, c < 1}}{w_0 : \Box(p \vee q) \rightarrow (\Box p \vee \Diamond q) < 1} \rightarrow <
\end{array}$$

Figure 9: A saturated branch.

Lemma 3.6. Let $\mathcal{B} = \langle \Gamma_0, \dots, \Gamma_n \rangle$ be a saturated branch and σ a solution to $\text{At}^+(\Gamma_n)$. Let $\mathfrak{M} = \text{Mod}(\text{At}^+(\Gamma_n), \sigma)$, $\mathcal{J}$ a canonical $\mathfrak{M}$ -interpretation and $\chi \in \bigcup_{k \in \{0, \dots, n\}} \Gamma_k$. Then, $\mathfrak{M} \models_{\mathcal{J}} \chi$.

Proof. Since $\mathfrak{M} = \text{Mod}(\text{At}^+(\Gamma_n), \sigma)$, with σ is a solution to $\text{At}^+(\Gamma_n)$, it holds that:

(1) $\mathfrak{M} \models_{\mathcal{J}} \text{At}^+(\Gamma_n)$.

We prove $\mathfrak{M} \models_{\mathcal{J}} \chi$ by induction on $\text{wg}(\chi)$; we assume $\mathfrak{M} = \langle W, R, e \rangle$. Let $\text{wg}(\chi) = 0$; then χ is atomic, hence $\chi \in \Gamma_n$, and $\mathfrak{M} \models_{\mathcal{J}} \chi$ by (1).

Let $\text{wg}(\chi) > 0$ and let us assume that $\chi \notin \Gamma_n$. Then, there exists $k \geq 0$ such that $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$. We proceed by a case analysis on the structure of χ .

Let $\chi = w : \alpha \wedge \beta \triangleright t$. Since $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$, the applied rule is $\wedge \triangleright$, hence $w : \alpha \triangleright t \in \Gamma_{k+1}$ and $w : \beta \triangleright t \in \Gamma_{k+1}$. By the induction hypothesis $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \triangleright t$ and $\mathfrak{M} \models_{\mathcal{J}} w : \beta \triangleright t$, and this implies $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \wedge \beta \triangleright t$. The case $\chi = w : \alpha \vee \beta \triangleleft t$ is similar.

Let $\chi = w : \alpha \wedge \beta \triangleleft t$. Since $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$, the applied rule is $\wedge \triangleleft$, hence $w : \alpha \triangleleft t \in \Gamma_{k+1}$ or $w : \beta \triangleleft t \in \Gamma_{k+1}$. By the induction hypothesis $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \triangleleft t$ or $\mathfrak{M} \models_{\mathcal{J}} w : \beta \triangleleft t$, hence $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \wedge \beta \triangleleft t$. The case $\chi = w : \alpha \vee \beta \triangleright t$ is similar.

Let $\chi = w : \alpha \rightarrow \beta < t$; we only consider the case $\beta \notin \mathcal{V} \cup \{\perp\}$. Since $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$, the applied rule is $\rightarrow <$, hence:

$$(A) \{w : \alpha > w : q, w : \beta \leq w : q, w : q < t\} \subseteq \Gamma_{k+1}.$$

By the induction hypothesis, for every constraint χ' displayed in (A), $\mathfrak{M} \models_{\mathcal{J}} \chi'$, hence:

$$e(w, \alpha) > e(w, q) \quad e(w, \beta) \leq e(w, q) \quad e(w, q) < \mathcal{J}(t)$$

Since $e(w, \alpha) > e(w, \beta)$, we get $e(w, \alpha \rightarrow \beta) = e(w, \beta)$, hence $e(w, \alpha \rightarrow \beta) < \mathcal{J}(t)$, which implies $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \rightarrow \beta < t$.

Let $\chi = w : \alpha \rightarrow \beta \leq t$; we only consider the case $\beta \notin \mathcal{V} \cup \{\perp\}$. Since $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$, the applied rule is $\rightarrow \leq$, hence one of the following two subcases holds:

(B1) $t \geq 1 \in \Gamma_{k+1}$;

(B2) $\{w : \alpha > w : q, w : \beta \leq w : q, w : q \leq t\} \subseteq \Gamma_{k+1}$.

Let us assume that (B1) holds. By the induction hypothesis, we get $\mathfrak{M} \models_{\mathcal{J}} t \geq 1$, and this implies $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \rightarrow \beta \leq t$. Let us assume that (B2) holds. By the induction hypothesis, for every constraint χ' in (B2), $\mathfrak{M} \models_{\mathcal{J}} \chi'$, hence:

$$e(w, \alpha) > e(w, q) \quad e(w, \beta) \leq e(w, q) \quad e(w, q) \leq \mathcal{J}(t)$$

Since $e(w, \alpha) > e(w, \beta)$, we get $e(w, \alpha \rightarrow \beta) = e(w, \beta)$, hence $e(w, \alpha \rightarrow \beta) \leq \mathcal{J}(t)$, which implies $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \rightarrow \beta \leq t$.

Let $\chi = w : \alpha \rightarrow \beta \triangleright t$; we only consider the case $\beta \notin \mathcal{V} \cup \{\perp\}$. Since $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$, the applied rule is $\rightarrow \triangleright$, hence one of the following two subcases holds:

(C1) $\{w : \alpha \leq w : q, w : \beta \geq w : q, 1 \triangleright t\} \subseteq \Gamma_{k+1}$;

(C2) $w : \beta \triangleright t \in \Gamma_{k+1}$.

Assume (C1). By the induction hypothesis, for every constraint χ' in (C1), $\mathfrak{M} \models_{\mathcal{J}} \chi'$, hence:

$$e(w, \alpha) \leq e(w, q) \quad e(w, \beta) \geq e(w, q) \quad 1 \triangleright \mathcal{J}(t)$$

Since $e(w, \alpha) \leq e(w, \beta)$, we get $e(w, \alpha \rightarrow \beta) = 1$, hence $e(w, \alpha \rightarrow \beta) \triangleright \mathcal{J}(t)$, which implies $\mathfrak{M} \models_{\mathcal{J}} \chi$. In case (C2), by the induction hypothesis we get $\mathfrak{M} \models_{\mathcal{J}} w : \beta \triangleright t$, namely $e(w, \beta) \triangleright \mathcal{J}(t)$. Since $e(w, \alpha \rightarrow \beta) \geq e(w, \beta)$, we get $e(w, \alpha \rightarrow \beta) \triangleright \mathcal{J}(t)$, hence $\mathfrak{M} \models_{\mathcal{J}} w : \alpha \rightarrow \beta \triangleright t$. The cases concerning R -constraints can be proved as the corresponding propositional cases.

Let $\chi = w : \Box \alpha \triangleleft t$. Since $\Gamma_k \mapsto_{\chi} \Gamma_{k+1}$, the applied rule is $\Box \triangleleft$, hence:

(D) $R(w, w_1) \rightarrow w_1 : \alpha \triangleleft t \in \Gamma_{k+1}$.

Let r_1 be the value of $R(w, w_1) \rightarrow e(w_1 : \alpha)$. By the induction hypothesis $\mathfrak{M} \models_{\mathcal{J}} R(w, w_1) \rightarrow w_1 : \alpha \triangleleft t$, hence $r_1 \triangleleft \mathcal{J}(t)$. Since $e(w, \Box \alpha) \leq r_1$, we get $e(w, \Box \alpha) \triangleleft \mathcal{J}(t)$, and this implies $\mathfrak{M} \models_{\mathcal{J}} w : \Box \alpha \triangleleft t$. The case $\chi = w : \Diamond \alpha \triangleright t$ is similar.

It remains to consider the cases where $\text{wg}(\chi) > 0$ and $\chi \in \Gamma_n$, thus $\chi = w : \Box \alpha \triangleright t$ or $\chi = w : \Diamond \alpha \triangleleft t$. Let $\chi = w : \Box \alpha \triangleright t$; we show that:

(E) $\mathfrak{M} \models_{\mathcal{J}} R(w, w') \rightarrow w' : \alpha \triangleright t$, for every $w' \in W$.

Let $w' \in W$; we show that $\mathfrak{M} \models_{\mathcal{J}} R(w, w') \rightarrow w' : \alpha \triangleright t$. Let us assume that $R(w, w')$ does not occur in Γ_n . By definition of $\mathfrak{M}$, the value of $R(w, w')$ is 0, hence we have to show that $\mathfrak{M} \models_{\mathcal{J}} 1 \triangleright t$. The case where $\triangleright$ is $\geq$ is immediate. Let $\triangleright$ be $>$. Since $\chi \in \Gamma_n$, we get $1 > t \in \text{At}^+(\Gamma_n)$, thus $\mathfrak{M} \models_{\mathcal{J}} 1 > t$ by (1). Let us assume that $R(w, w')$ occurs in Γ_n . Then, there exists $k > 0$ such that $R(w, w')$ is introduced in Γ_k by an application of a modal rule ρ , namely:

- $\Gamma_{k-1} \mapsto_{\chi'} \Gamma_k$, with $\chi' = w : \Box \varphi \triangleleft t'$ or $\chi' = w : \Diamond \varphi \triangleright t'$, and the new label introduced by ρ is w' .

Since $\mathcal{B}$ is saturated, Γ_{k-1} is plain, hence the non-atomic constraints in Γ_{k-1} are modal. As a consequence, no rule applied in the sub-branch $\langle \Gamma_{k-1}, \dots, \Gamma_n \rangle$ can introduce in the premise new constraints $u \blacktriangledown t'$ such that u has label w ; this implies that $\chi \in \Gamma_{k-1}$. By definition of ρ , we get $\Phi^{\Box, \Diamond}(\Gamma_{k-1}, w, w') \subseteq \Gamma_k$, hence $R(w, w') \rightarrow w' : \alpha \triangleright t \in \Gamma_k$. By the induction hypothesis, we get $\mathfrak{M} \models_{\mathcal{J}} R(w, w') \rightarrow w' : \alpha \triangleright t$, and this concludes the proof of (E). Let $e(w, \Box \alpha) = r$. By the witnessing condition, there exists $w^* \in W$ such that the value of $R(w, w^*) \rightarrow e(w^*, \alpha)$ is r . By (E), $r \triangleright \mathcal{J}(t)$, namely $e(w, \Box \alpha) \triangleright \mathcal{J}(t)$; we conclude $\mathfrak{M} \models_{\mathcal{J}} w : \Box \alpha \triangleright t$. The case $\chi = w : \Diamond \alpha \triangleleft t$ is similar. $\square$

As a consequence, we get:

Proposition 3.7. *Let $\mathcal{B} = \langle \Gamma_0, \dots, \Gamma_n \rangle$ be a saturated branch.*

- (i) For every solution σ to $\text{At}^+(\Gamma_n)$, $\text{Mod}(\text{At}^+(\Gamma_n), \sigma)$ is a discrete countermodel for Γ_0 .
- (ii) If $\Gamma_0 = \{w : \varphi < 1\}$, then $\varphi \notin \mathbf{GW}$.

Let BS be a backward proof search strategy for $\mathcal{C}_{\text{GW}}$; we say that BS is *plain* iff all the modal rule applications performed by BS are plain.

Lemma 3.8. *Let BS be a plain proof search strategy for $\mathcal{C}_{\text{GW}}$ and let Γ_0 be a finite multiset of constraints of the form $w : \varphi \nabla t$. If BS fails to prove Γ_0 , then a discrete countermodel for Γ_0 can be built.*

Proof. Assume that BS fails. By tracing the computation, we can build an open branch $\mathcal{B} = \langle \Gamma_0, \dots, \Gamma_n \rangle$. Since BS is plain, the branch $\mathcal{B}$ is saturated. Let σ be a solution to $\text{At}^+(\Gamma_n)$; by Prop. 3.7(i), we get that $\text{Mod}(\text{At}^+(\Gamma_n), \sigma)$ is a discrete countermodel for Γ_0 . $\square$

To sum up, we get the completeness of $\mathcal{C}_{\text{GW}}$ and the finite model property for $\mathbf{GW}$.

Proposition 3.9 (Completeness). *Let Γ be a finite multiset of constraints of the form $w : \varphi \nabla t$.*

- (i) $\vdash_{\text{GW}} \Gamma$ iff, for every GW-model $\mathfrak{M}$, $\mathfrak{M} \models \Gamma$.
- (ii) $\varphi \in \mathbf{GW}$ iff $\vdash_{\text{GW}} w : \varphi < 1$; moreover, if $\varphi \notin \mathbf{GW}$, there exists a discrete countermodel for φ .

Let $\varphi \notin \mathbf{GW}$ and let $\mathfrak{M}$ be the discrete countermodel extracted from an open branch with root $w_0 : \varphi < 1$. One can easily show that the depth of $\mathfrak{M}$ is bounded by $|\varphi|$ and that every world in $\mathfrak{M}$ has at most $|\varphi|$ R -successors. Consequently, the size of $\mathfrak{M}$ is $O(|\varphi|^{|\varphi|})$; moreover, we can define formulas φ having countermodels of size $2^{|\varphi|}$ (e.g., consider the translation φ^* of a $\mathbf{K}$ -formula φ such that φ has a countermodel of exponential size). Note that $\mathfrak{M}$ cannot be constructed incrementally, one branch at a time, since the constraints generated during the expansion of a branch have global validity and must be preserved throughout the entire construction. Nevertheless, we can define a PSPACE algorithm to check if such a countermodel exists without effectively constructing it; this is discussed in the proof of the next proposition.

Proposition 3.10. *The validity problem for $\mathbf{GW}$ is PSPACE-complete.*

Proof. Since the modal logic $\mathbf{K}$ can be embedded into $\mathbf{GW}$ (Prop. 2.2) and $\mathbf{K}$ is PSPACE-complete [14], the validity problem for $\mathbf{GW}$ is PSPACE-hard. It remains to prove that $\mathbf{GW}$ -validity can be checked in PSPACE. To this aim, we sketch a PSPACE procedure to decide the membership of a formula to $\mathbf{GW}$. Henceforth, we assume that the formula to be checked is φ . As discussed above, if $\varphi \notin \mathbf{GW}$, then there exists a tree-like countermodel $\mathfrak{M} = \langle W, R, e \rangle$ for φ such that the number of worlds is $O(|\varphi|^{|\varphi|})$. We can also bound the number N of values needed to define R and e : in the worst case, the values $e(w, p)$ and $R(w, w')$ must all be pairwise distinct, for every world w, w' in $\mathfrak{M}$ and every propositional variable p occurring in φ ; accordingly, N is in $O(|\varphi|^{|\varphi|})$. We set $\mathcal{Q} = \{0\} \cup \{1/k \mid 1 \leq k \leq N\}$; note that the elements of $\mathcal{Q}$ can be generated in space $O(|\varphi|)$. By the above discussion, if $\varphi \notin \mathbf{GW}$ there exists a countermodel $\mathfrak{M}$ such that the images of R and e are contained in $\mathcal{Q}$. To define the procedure, we need a global map Φ that specifies the intended values of the subformulas of φ at each world of the model under construction. More precisely, given a world name w , $\Phi(w)$ is a set of assignments of the form $e(w, \alpha) = r$, where α is a subformula of φ and $r \in \mathcal{Q}$. We remark that $\Phi(w)$ can be stored in space $O(|\varphi|)$; moreover, to implement the search procedure, we do not need to store the whole map Φ , but at most $O(|\varphi|)$ entries of Φ . A GW-model $\mathfrak{M} = \langle W, R, e \rangle$ complies with Φ iff, for every w in W and every assignment $e(w, \alpha) = r$ stored in $\Phi(w)$, the value of $e(w, \alpha)$ (in $\mathfrak{M}$) is r . The decision procedure is implemented by the recursive procedure SEARCH having the following specification:

- Let w be a world name and let us assume that $\Phi(w) \neq \emptyset$. The procedure $\text{SEARCH}(w)$ returns SUCCESS if a GW-model complying with Φ can be built, FAIL otherwise.

The procedure, at some points, requires the selection of a value from $\mathcal{Q}$, possibly satisfying some conditions: we call this a $\mathcal{Q}$ -choice. $\mathcal{Q}$ -choices can be revoked and redone; such backtrack steps can be implemented in space $O(|\varphi|)$. The computation of $\text{SEARCH}(w)$ consists of two main steps.

(Step 1) We apply propositional saturation reductions to $\Phi(w)$ as long as possible. At each step, an assignment $e(w, \alpha \star \beta) = r$ (with $\star \in \{\wedge, \vee, \rightarrow\}$) in $\Phi(w)$ is removed and replaced by the assignments $e(w, \alpha) = r_1$ and $e(w, \beta) = r_2$ required to justify the value r , where the values r_1 and r_2 are selected with $\mathcal{Q}$ -choices. For instance, let us assume that the assignment at hand is $e(w, \alpha \rightarrow \beta) = r$.

- If $r = 1$, we must have $e(w, \alpha) \leq e(w, \beta)$; the values r_1 and r_2 are selected with $\mathcal{Q}$ -choices so that $r_1 \leq r_2$; in $\Phi(w)$, $e(w, \alpha \rightarrow \beta) = r$ is replaced with $e(w, \alpha) = r_1$ and $e(w, \beta) = r_2$.
- If $r < 1$, a value r_1 is selected with a $\mathcal{Q}$ -choice so that $r_1 > r$; in $\Phi(w)$, $e(w, \alpha \rightarrow \beta) = r$ is replaced with $e(w, \alpha) = r_1$ and $e(w, \beta) = r$ (thus, $r_2 = r$).

At the end of (Step 1), the formulas occurring in $\Phi(w)$ are either atomic or modal.

- If $\Phi(w)$ contains two evaluations $e(w, \alpha) = r_1$ and $e(w, \alpha) = r_2$ such that $r_1 \neq r_2$, a backtrack step is needed. If no new $\mathcal{Q}$ -choice is possible, the call $\text{SEARCH}(w)$ returns FAIL.
- Otherwise, if $\Phi(w)$ contains only atomic constraints, the call $\text{SEARCH}(w)$ returns SUCCESS.
- Otherwise, $\Phi(w)$ contains at least a modal formula and the computation continues with (Step 2).

(Step 2) Let $e(w, \#_1 \alpha_1) = r_1, \dots, e(w, \#_n \alpha_n) = r_n$ (with $\#_k \in \{\Box, \Diamond\}$) be all the assignments in $\Phi(w)$ involving modal formulas. We start a loop indexed by k ranging over $\{1, \dots, n\}$. Let k be the current iteration and let us assume that the assignment of index k has the form $e(w, \Box \alpha_k) = r_k$ (the case $e(w, \Diamond \alpha_k) = r_k$ is similar). We generate a new world w_k and select with a $\mathcal{Q}$ -choice a value r'_k for $R(w, w_k)$ and a value r''_k for $e(w_k, \alpha_k)$ such that $r'_k \rightarrow r''_k = r_k$. The world w_k is intended to represent the world witnessing that $e(w, \Box \alpha_k) = r_k$. We set $\Phi(w_k) = \{e(w_k, \alpha_k) = r''_k\}$. We expand $\Phi(w_k)$ by adding the assignments required to account for the assignments $e(w, \#_j \alpha_j) = r_j$ occurring in $\Phi(w)$, where $j \neq k$. More precisely:

- for every $e(w, \Box \alpha_j) = r_j$ in $\Phi(w)$ ($j \neq k$), we select with a $\mathcal{Q}$ -choice a value r'_j such that $r'_k \rightarrow r'_j \geq r_j$ and we add $e(w_k, \alpha_j) = r'_j$ to $\Phi(w_k)$;
- for every $e(w, \Diamond \alpha_j) = r_j$ in $\Phi(w)$ ($j \neq k$), we select with a $\mathcal{Q}$ -choice a value r'_j such that $r'_k \wedge r'_j \leq r_j$ and we add $e(w_k, \alpha_j) = r'_j$ to $\Phi(w_k)$.

We compute recursively $\text{SEARCH}(w_k)$; notice that the modal operators occurring in $\Phi(w_k)$ are fewer than the modal operators in $\Phi(w)$, thus the recursion is terminating. We also stress that there are at most $|\varphi|$ nested recursive calls. Let us assume that $\text{SEARCH}(w_k)$ returns SUCCESS. If $k = n$ (last iteration), the call $\text{SEARCH}(w)$ returns SUCCESS. Otherwise the computation continues with next iteration $k + 1$. We point out that the entries of the map Φ exploited in the computation of $\text{SEARCH}(w_k)$ are no longer required, so they can be removed from the memory. Let us assume that $\text{SEARCH}(w_k)$ returns FAIL. Then, a backtrack step is needed. If no new $\mathcal{Q}$ -choice is possible, the call $\text{SEARCH}(w)$ returns FAIL.

The procedure can be implemented so that $\text{SEARCH}(w)$ is computed in space $O(|\varphi|)$.

We can exploit SEARCH to decide the validity of φ in **GW**. We initialize Φ by setting $\Phi(w_0) = \{e(w_0, \varphi) = r_0\}$, where r_0 is any value in $\mathcal{Q}$ different from 1. If $\text{SEARCH}(w_0)$ returns SUCCESS, then a model $\mathcal{M}$ containing a world w_0 such that $e(w_0, \varphi) = r_0$ can be built; since $r_0 < 1$, we conclude $\varphi \notin \mathbf{GW}$. If $\text{SEARCH}(w_0)$ returns FAIL, for every possible choice of r_0 , it follows that no GW-countermodel for φ can be built; accordingly, $\varphi \in \mathbf{GW}$. $\square$

4 Conclusions and Future Work

We have implemented a proof-search procedure for the calculus $\mathcal{C}_{\mathbf{GW}}$ that supports countermodel extraction. The prover, called *gwref* [10], performs a standard backward depth-first proof search, relying on the JTabWb engine [9]. The consistency of atomic constraints is verified using the Choco-solver Java library [16]. We emphasize that the calculus $\mathcal{C}_{\mathbf{GW}}$ and the associated proof-search strategy can be smoothly adapted to treat the logic $\mathbf{GW}^c$ [11], characterized by crisp GW-models, by refining the notion of consistency for a set of atomic constraints Γ_{at} : a solution σ to Γ_{at} must satisfy $\sigma(R(w, w')) \in \{0, 1\}$ for every pair of labels w, w' . To the best of our knowledge, the only other prover available for modal fuzzy logics is mNiBLoS [20], an SMT-based solver designed for continuous t-norm-based logics, including Gödel–Dummett Logic. However, neither **GW** nor $\mathbf{GW}^c$ are supported by mNiBLoS.

The results presented in this paper are highly relevant to the study of intermediate predicate logics. It is well known that modal logics can be used to characterize relevant fragments of intermediate first-order logics, particularly certain one-variable fragments. From this perspective, the witnessed semantics, together with the consequent finite model property established here, point to deep connections with proof-theoretic properties of intermediate predicate logics, such as Herbrand’s Theorem, interpolation, and alternative forms of Skolemization investigated in [2] for broad classes of intermediate intuitionistic predicate logics with the finite model property. Motivated by these connections, and with the aim of situating **GW** within the broader landscape of intuitionistic modal logics, we plan to develop a birelational Kripke semantics for **GW**, extending to this setting the approach introduced for $\mathbf{GW}^c$ in [11].

We also plan to extend the procedure to other witnessed Gödel modal logics that impose additional constraints on the accessibility relation (e.g., reflexivity, transitivity, seriality). In contrast, treating Gödel modal logics with non-witnessed models seems to be considerably more challenging.

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Ultrafilter Extensions for Veltman Semantics

Fèlix Frigola González
University of Barcelona

Joost J. Joosten
University of Barcelona

Vicent Navarro Arroyo
Polytechnic University of València
University of Barcelona

Cosimo Perini Brogi
IMT School for Advanced Studies Lucca

In this paper, we present a first-order frame condition for interpretability logic and show that the condition is not modally definable. Yet, the frame condition holds both on ILM and on ILP frames and, hence, is of potential importance for the long-standing open problem about the interpretability logic of all reasonable arithmetical theories. In the light of the Goldblatt-Thomason Theorem, the modally inexpressible frame condition serves as motivation to develop ultrafilter extensions for interpretability logic. We develop the necessary algebraic tools to define these ultrafilter extensions and prove the main properties about both the tools and the ultrafilter extensions.

1 Introduction

Building upon early work by Montagna and Švejdar [30, 22], interpretability logics were introduced by Visser in the 1980s as a natural extension of provability logics ([32]). Since then, the field of modal interpretability logics has seen a steady process of becoming a mature discipline. A Kripke style semantics for interpretability logics goes by the name of Veltman Semantics. After some early modal completeness and decidability results in [15] it was seen that many interpretability logics satisfy various other good properties too, like the Finite Model Property, the Fixpoint Theorem, or Interpolation (see [33, 14]).

Proof calculi are mostly Hilbert style but sequent style systems and cut-elimination have been studied in [12, 26, 25]. Topological semantics for interpretability logics have been studied in [13]. Verbrugge semantics ([17]) is a generalization of Veltman semantics that is reminiscent of neighbourhood semantics. However, plain Veltman semantics is still the predominant semantics used for interpretability logics.

Unary normal modal logics are all extensions of the basic modal logic **K**. The basic modal interpretability logic is called **IL** although on occasion logics below **IL** are studied too ([20, 24]). Various extensions of **IL** by axiom schemes induce correspondences to first- or higher-order properties of frames.

To the best of our knowledge, in this paper – Section 3 – we present the first frame condition (which is first-order) for interpretability logics with a proof that the condition is not modally definable. Moreover, the frame condition is natural in light of the semantically defined interpretability logics **IL**(All) and **IL**[$\exists$ II] (see [18, 10, 23]).

Our proof of modal undefinability is rather ad-hoc and this begs the question if a Goldblatt-Thomason Theorem for interpretability logic can be proven to deal with undefinability results in a more uniform way. In the remainder of the paper we develop a first natural step toward such a Goldblatt-Thomason theorem. In Section 4 we define ultrafilter extensions and prove some elementary properties of them. More sophisticated arguments require that we develop labeling techniques in the algebraic setting which is done in Section 5. The main results on ultrafilter extensions are presented in Section 6: elementary equivalence and modal saturation. Further steps towards the Goldblatt-Thomason Theorem will be addressed in future work. We start with some preliminaries in the next section.

2 Preliminaries

Following [31, 32], throughout this paper, we will work with a multimodal language $\mathcal{L}_{\Box, \triangleright}$, whose formulas $(\alpha, \beta, \dots \in \text{Form})$ are built on top of a given denumerably infinite set of propositional atoms $(p, q, \dots \in \text{Atm})$ according to the following grammar: $\alpha, \beta \in \text{Form} ::= p \mid \perp \mid (\alpha \rightarrow \beta) \mid \Box \alpha \mid \alpha \triangleright \beta$. Working on this grammar, we introduce the following notations:

$$\begin{aligned} \top &:= \perp \rightarrow \perp & \neg \alpha &:= \alpha \rightarrow \perp & \alpha \wedge \beta &:= \neg(\alpha \rightarrow \neg \beta) \\ \alpha \vee \beta &:= (\neg \alpha \rightarrow \beta) & \alpha \leftrightarrow \beta &:= (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha) & \Diamond \alpha &:= \neg \Box \neg \alpha \end{aligned}$$

Furthermore, we adopt the convention of omitting brackets on formulas to enhance their readability, according to the following decreasing priority list: $[\neg, \Box, \Diamond; \wedge, \vee; \triangleright; \rightarrow, \leftrightarrow]$.¹

2.1 Axiomatic Calculus and Relational Semantics for IL

Definition 2.1 (Axiomatization of **IL**). The axiomatic calculus **IL** for the basic interpretability logic extends classical propositional logic by the following axiom schemas and rules:

- K: $\Box(\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta)$;
- GL: $\Box(\Box \alpha \rightarrow \alpha) \rightarrow \Box \alpha$;
- J1: $\Box(\alpha \rightarrow \beta) \rightarrow \alpha \triangleright \beta$;
- J2: $(\alpha \triangleright \beta) \wedge (\beta \triangleright \gamma) \rightarrow \alpha \triangleright \gamma$;
- J3: $(\alpha \triangleright \gamma) \wedge (\beta \triangleright \gamma) \rightarrow \alpha \vee \beta \triangleright \gamma$;
- J4: $\alpha \triangleright \beta \rightarrow (\Diamond \alpha \rightarrow \Diamond \beta)$;
- J5: $\Diamond \alpha \triangleright \alpha$;
- The modus ponens rule: $\frac{\alpha \rightarrow \beta \quad \alpha}{\beta}$;
- The necessitation rule: $\frac{\alpha}{\Box \alpha}$.

We write $\mathbf{IL} \vdash \alpha$ when the formula α is provable in the axiomatic calculus for **IL**; similarly $\Gamma \vdash_{\mathbf{IL}} \alpha$ denotes provability in **IL** of α from the *finite* set of hypotheses Γ , with the usual restriction on the application of the necessitation rule.²

The system **IL** has been extended along several directions, to cover different versions of formal interpretability between arithmetical theories: for an in-depth treatment, we refer the reader to, e.g., [29, 2, 32, 11]. On the relational semantic side, **IL** has a natural presentation in terms of Veltman frames [15].

Definition 2.2 (Veltman frames and models). A *Veltman frame* is a triple $\mathcal{F} := \langle W, R, \{S_w \mid w \in W\} \rangle$ where W is a non-empty set of *possible worlds*, R a binary relation on W which is transitive and conversely well-founded. The $\{S_w \mid w \in W\}$ is a collection of binary relations on $R[w]$ – where $R[w] := \{v \mid wRv\}$ – so that each S_w is reflexive and transitive and the restriction of R to $R[w]$ is contained in S_w , that is: $R \cap R[w] \subseteq S_w$.

A *Veltman model* consists of a Veltman frame together with a valuation function $ev : \text{Atm} \rightarrow \wp(W)$ that assigns to each propositional atom $p \in \text{Atm}$ a set of worlds $ev(p) \subseteq W$ where p is stipulated to be true.

This valuation function naturally extends to a forcing relation $\Vdash \subseteq W \times \text{Form}$ as follows:

$$\begin{aligned} w \Vdash p &\iff w \in ev(p); \text{ and } w \Vdash \perp \text{ for no } w \in W; \\ w \Vdash \alpha \rightarrow \beta &\iff w \not\Vdash \alpha \text{ or } w \Vdash \beta; \\ w \Vdash \Box \alpha &\iff \forall v. (\text{if } wRv, \text{ then } v \Vdash \alpha); \\ w \Vdash \alpha \triangleright \beta &\iff \forall u. (\text{if } wRu \text{ and } u \Vdash \alpha, \text{ then } \exists v (uS_w v \text{ and } v \Vdash \beta)). \end{aligned}$$

¹According to this notation convention, we can thus unambiguously write, e.g., $\alpha \triangleright \beta \rightarrow \alpha \wedge \Box \gamma \triangleright \beta \wedge \Box \gamma$ for $(\alpha \triangleright \beta) \rightarrow ((\alpha \wedge (\Box \gamma)) \triangleright (\beta \wedge (\Box \gamma)))$.

²Notice that $\mathbf{IL} \vdash \Box \alpha \rightarrow \Box \Box \alpha$ and $\mathbf{IL} \vdash \Box \alpha \leftrightarrow \neg \alpha \triangleright \perp$. An equivalent axiomatization of **IL** can then be given over a restricted language $\mathcal{L}_{\triangleright}$ [32].

Given a Veltman model $\mathcal{M} = \langle W, R, \{S_w \mid w \in W\}, ev \rangle$ and a formula $\alpha \in \text{Form}$, we shall write $\llbracket \alpha \rrbracket_{\mathcal{M}}$ to denote $\{x \in \mathcal{M} \mid \mathcal{M}, x \Vdash \alpha\}$ and, $\mathcal{M} \models \alpha$ as a notation for $\forall w \in W. (w \Vdash \alpha)$.

In their [15], De Jongh and Veltman proved that the calculus **IL** is sound and complete with respect to all Veltman models. Modal adequacy w.r.t. Veltman frames satisfying different conditions has been established via various techniques in subsequent work on several axiomatic extensions [15, 16, 19, 9, 3, 7].

Definition 2.3 (Frame definability). Let $\mathcal{C}$ denote a first- or higher-order formula. Let us write $\mathcal{F} \models \mathcal{C}$ for stating that the Veltman frame $\mathcal{F}$ as first- or higher-order structure satisfies the condition encoded by $\mathcal{C}$, i.e. it makes $\mathcal{C}$ true. Equivalently, we will write $\mathcal{F} \in \mathcal{C}$ for the class of frames $\mathcal{F}$ such that $\mathcal{F} \models \mathcal{C}$. The class of frames $\mathcal{C}$ is **definable** if $\exists \varphi \in \text{Form}. \forall \mathcal{F}. (\mathcal{F} \in \mathcal{C} \iff \mathcal{F} \models \varphi)$.

2.2 Labelling Techniques for Interpretability Logics

In the canonical model for interpretability logic, maximal consistent sets may need copies if they fulfill various functionalities. To keep track of the copies and their functionality in the canonical model, one typically uses labels. In the early times the labels were formulas to flag so-called critical successors (see [15]). Later the labeling techniques were generalized giving rise to so-called *assuring labels* ([8]).

Definition 2.4 (S -assuring successor). Given sets of formulas Γ, Δ, S , we say that Δ is an S -assuring successor of Γ (denoted $\Gamma \prec_S \Delta$) if, for any formula $\alpha \in \text{Form}$ and any finite subset of the label $S' \subseteq S$, the condition that $\neg\alpha$ interprets the disjunction of the negations of formulas in S' implies that α and $\Box\alpha$ are in Δ :

$$\Gamma \prec_S \Delta \iff \forall \alpha \in \text{Form}, S' \subseteq S. \left(\bigvee_{\beta \in S'} (\neg\alpha \triangleright \neg\beta) \in \Gamma \implies \alpha, \Box\alpha \in \Delta \right).$$

If that is the case, S is a **label** (for Γ and Δ).

For our aims, it suffices to recall some basic labelling lemmas for generic assuring labels. For the next lemmas, we assume that Γ and Δ are maximal consistent (with respect to some logic extending **IL**) sets of formulas. The following lemma is easy to check.

Lemma 2.5 (From [7], Lemma 4.2). *The assuring relation $\prec_S$ satisfies the following properties:*

1. *Restriction: If $S \subseteq T$ and $\Gamma \prec_T \Delta$, then $\Gamma \prec_S \Delta$.*
2. *Propagation: If $\Gamma \prec_S \Delta$ and $\Delta \prec_{\emptyset} \Theta$, then $\Gamma \prec_S \Theta$.*

The following lemma tells us that labels can be extended to theories.

Lemma 2.6 (From [7], Lemma 4.6). *The assuring relation $\prec_S$ satisfies the following closure properties:*

1. *Deducibility: For any $\varphi \in \text{Form}$, if $\Gamma \prec_S \Delta$ and $S \vdash_{\text{IL}} \varphi$, then $\Gamma \prec_{S \cup \{\varphi\}} \Delta$.*
2. *Necessitation: If $\Gamma \prec_S \Delta$, then $\Gamma \prec_{S \cup \Box S} \Delta$, where $\Box S := \{\Box\varphi \mid \varphi \in S\}$.*

Lemma 2.7 (From [7], Lemma 3.7). *For any S , the following hold:*

- *If $\Gamma \prec_S \Delta$, then $S, \Box S \subseteq \Delta$.*
- *If $\Gamma \prec_S \Delta$, then $\Diamond S \subseteq \Gamma$, where $\Diamond S := \{\Diamond\varphi \mid \varphi \in S\}$.*

2.3 Ultrafilter Extensions

(Ultra)Filters are the algebraic counterpart of (complete) theories.

Definition 2.8. Given a set W , an ultrafilter over W consists of a maximal filter over W . That is, a collection $f \subseteq \wp(W)$ is an **ultrafilter** on W , and we write $f \in \mathcal{U}(W)$, if it satisfies the following conditions: *Properness*: $\emptyset \notin f$; *Upward Closure*: If $A \in f$ and $A \subseteq B$, then $B \in f$; *Intersection Closure*: If $A, B \in f$, then $A \cap B \in f$; *Totality*: For any $A \subseteq W$, either $A \in f$ or $\bar{A} \in f$, where $\bar{A} := W \setminus A$.

Ultrafilters over a given W generated by a single fixed element $w \in W$ by considering all subsets of W containing w are called **principal ultrafilters**, denoted by Π_w .

When the starting set W is infinite, the existence of non-principal ultrafilters over W is provable modulo some choice principles over a constructive base [21, 1] as a consequence of

Lemma 2.9 (Ultrafilter Principle). *Every non-empty family s of sets with the finite intersection property (FIP) – that is, for any $t \subseteq s$, $\bigcap_{fin} t \neq \emptyset$ – generates a filter that is contained in at least one ultrafilter.*

Frames can be extended via ultrafilters as follows.

Definition 2.10. Given a relational frame $\mathcal{F} = \langle W, \{R_i\}_{i < \omega} \rangle$, its **ultrafilter extension** $\mathcal{F}^{ue}$ is given by:

- **Worlds**: The new domain is $\mathcal{U}(W)$, the set of all ultrafilters over W ;
- **Accessibility Relations** (R_i^{ue}): For any two ultrafilters $f, g \in \mathcal{U}(W)$:

$$f R_i^{ue} g \iff \forall X \subseteq W. (X \in g \implies R_i^{-1}(X) \in f),$$

$$\text{where } R_i^{-1}(X) := \{w \in W \mid \exists x \in X. (w R_i x)\}.$$

Given such an ultrafilter extension $\mathcal{F}^{ue}$, and an **evaluation** ev over $\mathcal{F}$ we define $ev^{ue}(p) := \{f \in \mathcal{U}(W) \mid ev(p) \in f\}$. Therefore, given a model $\mathcal{M}$, we denote by $\mathcal{M}^{ue}$ its ultrafilter extension.

Working with the standard forcing relation $x \Vdash \langle i \rangle \varphi :\Leftrightarrow \exists y (x R_i y \wedge y \Vdash \varphi)$, the central results on ultrafilter extensions of relational models are collected by (see, e.g. [4])

Theorem 2.11. *Given a relational model $\mathcal{M} = \langle \mathcal{F}, ev \rangle$, we have:*

- **Truth Lemma**: *For any formula φ and any ultrafilter $f \in \mathcal{U}(W)$, the following equivalence holds: $(\mathcal{M}^{ue}, f) \Vdash \varphi \iff ev(\varphi) \in f$, where $ev(\varphi) = \{w \in W \mid (\mathcal{M}, w) \Vdash \varphi\}$.*
- **Reflection of Validity**: *If a formula φ is valid in the ultrafilter extension, it is valid in the original frame, that is: if $\mathcal{F}^{ue} \models \varphi$, then $\mathcal{F} \models \varphi$.*
- **The Goldblatt-Thomason Theorem**: *An elementary (i.e. first-order definable) class of frames $\mathcal{C}$ is modally definable if and only if it is closed under: generated subframes, bounded morphisms, and disjoint unions; and reflects ultrafilter extensions, that is: if $\mathcal{F}^{ue} \in \mathcal{C}$, then $\mathcal{F} \in \mathcal{C}$.*

3 A Non-Definability Result

In [23] the class of *Pencil frames* $\mathcal{C}_{\text{Pencil}}$ is identified by the first-order frame property

$$x R_y S_x z R_u \wedge y R_v S_x u \rightarrow y R_u.$$

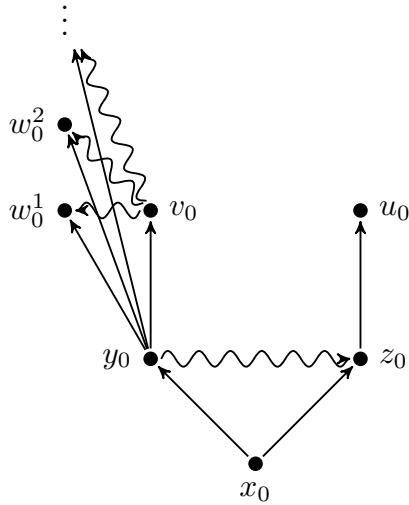
The modal logics **ILM** and **ILP** arise by adding the axiom schemes $\alpha \triangleright \beta \rightarrow \alpha \wedge \Box \gamma \triangleright \beta \wedge \Box \gamma$ and $\alpha \triangleright \beta \rightarrow \Box(\alpha \triangleright \beta)$ respectively. It is well-known [33] and easy to see that the respective frame

conditions are $yS_xzRu \rightarrow yRu$ and $xRyRzS_xu \rightarrow zS_yu$. The pencil frame condition $\mathcal{C}_{\text{Pencil}}$ can be seen to hold on all **ILP** and all **ILM** frames. Since **ILM** is the interpretability logic of all arithmetical theories with full induction [2, 29] and, since **ILP** is the interpretability logic of all finitely axiomatizable theories that prove the totality of super-exponentiation[32], the pencil frame is of interest to the long-standing open problem to determine the interpretability logic of all reasonable arithmetical theories [10, 18].

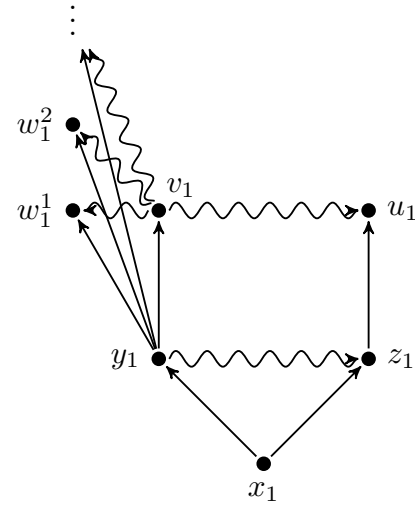
Definition 3.1. [From [33]]: A relation Z between the nodes of two models $\mathcal{M}$ and $\mathcal{N}$ (denoted $m, m_1, m_2, \dots$ and $n, n_1, n_2, \dots$ respectively) is a bisimulation if and only if it satisfies the following:

1. for all $p \in \text{Atm}$ $mZn \Rightarrow (\mathcal{M}, m \Vdash p \iff \mathcal{N}, n \Vdash p)$.
2. (Zig) If $mR^{\mathcal{M}}m_1$ and mZn , then there is some n_1 such that $nR^{\mathcal{N}}n_1$ and m_1Zn_1 and for all n_2 in $\mathcal{N}$ such that $n_1S_n^{\mathcal{N}}n_2$ there is some m_2 such that m_2Zn_2 and $m_1S_m^{\mathcal{M}}m_2$.
3. (Zag) If $nR^{\mathcal{N}}n_1$ and mZn then there is some m_1 such that $mR^{\mathcal{M}}m_1$ and m_1Zn_1 and for all m_2 such that $m_1S_m^{\mathcal{M}}m_2$ there is some n_2 such that m_2Zn_2 and $n_1S_n^{\mathcal{N}}n_2$.

As always, bisimilar worlds are modally equivalent [33] and this is the only property we use in our proof below. We prove that the class $\mathcal{C}_{\text{Pencil}}$ is not modally definable. To do so, let us consider the frames $\mathcal{F}_0$ and $\mathcal{F}_1$ from Figures 1a and 1b, respectively, where $\mathcal{F}_0$ and $\mathcal{F}_1$ satisfy the condition $x_iRy_iS_{x_i}z_iRu_i \wedge y_iRv_i \wedge \bigwedge_{j \in \omega} v_iS_{x_i}w_i^j$, but $\mathcal{F}_1$ satisfies $v_1S_{x_1}u_1$ and $\mathcal{F}_0$ does not satisfy $v_0S_{x_0}u_0$.



(a) The $\mathcal{F}_0$ frame satisfying $\mathcal{C}_{\text{Pencil}}$.



(b) The $\mathcal{F}_1$ frame not satisfying $\mathcal{C}_{\text{Pencil}}$.

Figure 1: $\mathcal{F}_0$ and $\mathcal{F}_1$ frames. Only S_{x_0} and S_{x_1} are displayed for clarity. R —transitivity and S —reflexivity, etc. are also not displayed.

More precisely, we prove the following.

Theorem 3.2. *The frame class identified by the Pencil Condition $\mathcal{C}_{\text{Pencil}}$ is not modally definable, that is: $\neg \exists \varphi \in \text{Form}. \forall \mathcal{F} (\mathcal{F} \Vdash \varphi \iff \mathcal{F} \in \mathcal{C}_{\text{Pencil}})$.*

Proof. Suppose for a contradiction that $\exists \varphi \in \text{Form}. \forall \mathcal{F} (\mathcal{F} \Vdash \varphi \iff \mathcal{F} \in \mathcal{C}_{\text{Pencil}})$. Then, pick the frames $\mathcal{F}_0$ and $\mathcal{F}_1$ depicted in Figures 1a and 1b, respectively. So, the frames are identical but for one S relation: $v_1S_{x_1}u_1$ holds whereas $v_0S_{x_0}u_0$ does not. We observe that $v_iS_{x_i}w_i^j$, and there is no S_{x_i} relation

between any w_i^j and w_i^k for different $j \neq k > 0$. Also, we observe that $\mathcal{F}_0 \in \mathcal{C}_{\text{Pencil}}$ and, $\mathcal{F}_1 \notin \mathcal{C}_{\text{Pencil}}$. Therefore, by hypothesis, $\mathcal{F}_0 \models \varphi$ and $\mathcal{F}_1 \not\models \varphi$.

Now, we prove the following claim: For every valuation ev_1 on $\mathcal{F}_1$ there is another valuation ev_0 on $\mathcal{F}_0$ such that $\mathcal{M} := \langle \mathcal{F}_1, ev_1 \rangle$ and $\mathcal{N} := \langle \mathcal{F}_0, ev_0 \rangle$ are bisimilar.

Indeed, consider an arbitrary $\mathcal{F}_1$ -valuation ev_1 and consider the model $\langle \mathcal{F}_1, ev_1 \rangle$. Given $p \in \text{Atm}$, let us define $ev_0(p)$ over $\mathcal{F}_0$ as follows:

- For $t \in \{x, y, z, u\}$:
 - $w_0^1 \in ev_0(p) : \iff u_1 \in ev_1(p)$,
 - $t_0 \in ev_0(p) : \iff t_1 \in ev_1(p)$
- $v_0 \in ev_0(p) : \iff v_1 \in ev_1(p)$,
- $w_0^{i+1} \in ev_0(p) : \iff w_1^i \in ev_1(p) \quad (i \geq 1)$.

Next, consider the relation $Z := \{(x_1, x_0), (y_1, y_0), (z_1, z_0), (v_1, v_0), (u_1, u_0), (u_1, w_0^1)\} \cup \{(w_1^i, w_0^{i+1})\}_{i \geq 1}$. It is routine to check that Z is a bisimulation between $\langle \mathcal{F}_1, ev_1 \rangle$ and $\langle \mathcal{F}_0, ev_0 \rangle$. First we check that for all $p \in \text{Atm}$, $mZn \Rightarrow (\mathcal{M}, m \Vdash p \iff \mathcal{N}, n \Vdash p)$. This is clear by the definition of ev_0 provided above. Next, the Zig condition 3.1.2. We note that $x_1 R^1 [W_1 - \{x_1\}]$, that $x_0 R^0 [W_0 - \{x_0\}]$, that $x_1 Z x_0$ and, of course, that for all $t_1 \in W_1$ there is some $t_0 \in W_0$ such that $t_1 Z t_0$. For the other points we see that $z_1 R^1 u_1$ for $z_1 Z z_0$ and $z_0 R^0 u_0$ for $u_1 Z u_0$. Also, observe that $y_1 R^1 w_1^i$ for all i , and $y_1 Z y_0$, but notice that $y_0 R^0 w_0^i$ for all i and that for all w_1^i there is some w_0^j such that $w_1^i Z w_0^j$. Besides of that, $y_1 R^1 v_1$ and $y_1 Z y_0$, but observe that $y_0 R^0 v_0$ and $v_1 Z v_0$.

S_w for $w \in W_1 - \{x_1\}$ our claim also holds following the definition. The only critical point is for the $S_{x_1}^1$ relation, for which we will only show that condition 3.1.2 holds for $v_1 S_{x_1}^1 u_1$ and $v_1 S_{x_1}^1 w_1^i$, for $i \geq 0$. We observe that $x_1 R^1 v_1$, $x_1 Z x_0$, $x_0 R^0 v_0$ and $v_1 Z v_0$. Also, notice that $v_0 S_{x_0}^0 w_0^j$, for $j \geq 0$. We need to find some $m_2 \in W_1$ such that $m_2 Z w_0^j$ and $v_1 S_{x_1}^1 m_2$. It suffices to pick $m_2 = v_1$, $m_2 = u_1$ and $m_2 = w_1^{j-1}$ for the cases $j = 0$, $j = 1$ and $j > 1$, respectively.

Condition 3.1.3 follows in a similar way.

Since $\mathcal{F}_1 \not\models \varphi$, we have that there exists a valuation ev such that $\langle \mathcal{F}_1, ev \rangle \not\models \varphi$. By the last claim, there is a valuation ev' such that $\langle \mathcal{F}_1, ev \rangle$ is bisimilar with $\langle \mathcal{F}_0, ev' \rangle$. By preservation of validity under bisimulation [5, § 3.4], we have that $\langle \mathcal{F}_0, ev' \rangle \not\models \varphi$, contra $\mathcal{F}_0 \models \varphi$. $\square$

4 Ultrafilter Extensions

To define ultrafilter extensions for interpretability logic we need to find algebraic counterparts for labels and assuring successors.

4.1 Theories, Filters and Ultrafilters

In this section we recall some basic facts of modal ultrafilter extensions and formulate the basic definitions needed to generalise all to interpretability logic.

Definition 4.1. Given a binary relation R over non-empty W , we define $\widehat{R^{-1}} : \wp(W) \longrightarrow \wp(W)$ as the dual operator for R^{-1} (from Def. 2.10) as follows: $\widehat{R^{-1}}(Y) := \{x \in W \mid \forall y (xRy \Rightarrow y \in Y)\}$. Moreover, we define $f_{\square} := \{Y \mid \widehat{R^{-1}}(Y) \in f\}$.

The following lemmas are folklore – see, e.g., [4].

Lemma 4.2. Given a binary relation R over a non-empty W , for any $Y \subseteq W$ we have $\widehat{R^{-1}}(Y) = \overline{R^{-1}(\overline{Y})}$ and $R^{-1}(g) \subseteq f \iff f_{\square} \subseteq g$.

In our context, we need to define filter-assuring successor, in analogy with Def. 2.4 and [7].

Definition 4.3. Given a Veltman frame $\mathcal{F} = \langle W, R, \{S_x \mid x \in W\} \rangle$, let us define³ $S^{-1} : \wp(W) \times \wp(W) \longrightarrow \wp(W)$ by

$$S^{-1}(X, Y) := \{w \in W \mid \forall x \in X. (wRx \Rightarrow \exists y \in Y. xS_w y)\}.$$

Let $I \subseteq \wp(W) \setminus \{\emptyset\}$ have the FIP, and let $f, g \in U(W)$. We define $f \prec_I g$ as follows:

$$f \prec_I g \iff \left[\forall A \subseteq W \forall^{\text{fin}} \{S_i \in I\} \left(S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f \Rightarrow A, \widehat{R^{-1}A} \in g \right) \right]$$

Lemma 5.1 below will prove that filter-assuring successors are indeed successors in the sense of R^{ue} .

4.2 Ultrafilter Extensions for Frames and Models

Definition 4.4. Let $\mathcal{F} = \langle W, R, \{S_w \mid w \in W\} \rangle$ be a Veltman Frame, the ultrafilter extension of the Veltman frame F^{ue} is defined recursively:

1. W^{ue} is recursively defined:
 - (a) $\langle f, \langle \rangle \rangle \in W^{\text{ue}}$, for each ultrafilter f over W .
 - (b) $\langle f, \sigma \rangle \in W^{\text{ue}} \wedge f \prec_I g \Rightarrow \langle g, \sigma \cap \langle I \rangle \rangle \in W^{\text{ue}}$.
2. R^{ue} is defined as the transitive closure of $R_{\text{one}}^{\text{ue}}$ where $R_{\text{one}}^{\text{ue}}$ is defined for $\langle f, \sigma \rangle, \langle g, \sigma \cap \langle I \rangle \rangle \in W^{\text{ue}}$ in the natural way as: $\langle f, \sigma \rangle R_{\text{one}}^{\text{ue}} \langle g, \sigma \cap \langle I \rangle \rangle \iff f \prec_I g$
3. $S_{\langle f, \sigma \rangle}^{\text{ue}}$ is defined as the smallest relation that is reflexive, transitive, so that it contains $R^{\text{ue}}[\langle f, \sigma \rangle]^2 \cap R^{\text{ue}}$ and $S_{\langle f, \sigma \rangle, \text{one}}^{\text{ue}}$ where the latter⁴ is defined as follows

$$\langle g, \tau \rangle S_{\langle f, \sigma \rangle, \text{one}}^{\text{ue}} \langle h, \tau' \rangle \iff \begin{cases} \langle f, \sigma \rangle R^{\text{ue}} \langle g, \tau \rangle \\ \langle f, \sigma \rangle R^{\text{ue}} \langle h, \tau' \rangle \\ (\tau)_{|\sigma|} = (\tau')_{|\sigma|} \end{cases}$$

Note in particular $\tau = \sigma \cap \langle I \rangle \cap \bar{\tau}$ for some I and $\bar{\tau}$ and that $\tau' = \sigma \cap \langle I \rangle \cap \bar{\tau}'$ for the very same I and some $\bar{\tau}'$. Furthermore, we have $f \prec_I g, h$ (See Lemma 5.5).

Remark 4.5. A few considerations about this type of frames:

1. Labels are a very simple way to keep track of the history of each world, which makes it possible to “localize” (in the sense of making local) the S^{ue} relation and the $\triangleright$ operator.
2. Finite frames are no longer necessarily isomorphic to their ultrafilter extensions.

Next we define a translation between Form and the language of ultrafilter extensions:

Definition 4.6. For any formula $\varphi \in \text{Form}$ we define a translation φ^t inductively by

$$\begin{aligned} \cdot \perp^t &:= \emptyset & \cdot p^t &:= A_p \text{ where } p \in \text{Atm and } A_p \text{ a set-variable} & \cdot (\varphi \rightarrow \psi)^t &:= \overline{(\varphi^t)} \cup (\psi^t) \\ \cdot (\Box \varphi)^t &:= \widehat{R^{-1}(\varphi^t)} & \cdot (\varphi \triangleright \psi)^t &:= S^{-1}((\varphi^t), (\psi^t)) \end{aligned}$$

³In [27] (corresponding to [28]) the notation $m_{\triangleright}$ is used instead of S^{-1} .

⁴We take the closure under the one-step relation here. Another approach can be to consider slightly different semantics as in [23].

We observe that $(\Box \phi)^t = \{x \in W : R[x] \subseteq \phi^t\}$ and $(\Diamond \phi)^t = \{x \in W : R[x] \cap \phi^t \neq \emptyset\}$.

For a Veltman frame $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$, once we fix a valuation e that assigns subsets of W to the set-variables A_p , this fixes in the obvious way for each formula ϕ a set that we denote by $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e$. Similarly we use p_A to denote the propositional variable such that $\llbracket p \rrbracket_{\mathcal{F}}^e = A$.

Given a set of formulas $\Gamma = \{\phi_1, \dots, \phi_n\}$ we use $\llbracket \Gamma \rrbracket_{\mathcal{F}}^e$ to denote the set $\{\llbracket \phi \rrbracket_{\mathcal{F}}^e, \dots, \llbracket \phi_n \rrbracket_{\mathcal{F}}^e\}$.

Theorem 4.7. *Let $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ be a Veltman frame. If $\mathbf{IL} \vdash \phi$, then for any valuation e we have $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$.*

Proof. Throughout the proof we fix some frame $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ and all sets mentioned below are subsets of W . Let us fix a choice of a valuation e . We first need to prove that the translation of all the $\mathbf{IL}$ axioms under e evaluate to W , that is, $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$ for every axiom ϕ of $\mathbf{IL}$. We only prove this for axioms GL and J2 as the remaining axioms follow the same lines of algebraic reasoning:

1. GL : $\Box(\Box \alpha \rightarrow \alpha) \rightarrow \Box \alpha$; the axiom translates to $\phi^t = \widehat{R^{-1}(\widehat{R^{-1}(\alpha^t)} \cup \alpha^t)} \cup \widehat{R^{-1}(\alpha^t)}$. To see that, for any choice of α , $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$, it suffices to show that, for any $A \subseteq W$, we have $\widehat{R^{-1}(\widehat{R^{-1}(A)} \cup A)} \cup \widehat{R^{-1}(A)} \subseteq \widehat{R^{-1}(A)}$. Assume $x \in \widehat{R^{-1}(\widehat{R^{-1}(A)} \cup A)}$ and assume xRy and $y \notin A$. Since R is transitive and conversely well-founded, let z be the R -maximal element in W such that $z \in \{w \in W \mid (yRw \wedge w \notin A) \vee w = y\}$. Since xRz and neither $z \in A$ nor $z \in \widehat{R^{-1}(A)}$, we have a contradiction. Therefore, $y \in A$ for our arbitrary y , and we conclude $x \in \widehat{R^{-1}(A)}$.
2. J2 : $(\alpha \triangleright \beta \wedge \beta \triangleright \gamma) \rightarrow \alpha \triangleright \gamma$; this axiom translates to $\phi^t = \widehat{S^{-1}(A, B) \cap S^{-1}(B, C)} \cup S^{-1}(A, C)$; to see that $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$, it suffices to show that $S^{-1}(A, B) \cap S^{-1}(B, C) \subseteq S^{-1}(A, C)$. Assume thus $x \in S^{-1}(A, B) \cap S^{-1}(B, C)$ and there is some $y \in W$ such that xRy and $y \in A$. Then, there is some $z \in W$ such that $yS_x z \wedge z \in B$. Since $yS_x z \rightarrow xRy \wedge xRz$, there is some $v \in W$ such that xRv and, by the definition of S^{-1} , we have that $zS_x v$ and $v \in C$. Since $yS_x z \wedge zS_x v \rightarrow yS_x v$, we conclude $x \in S^{-1}(A, C)$.

We now prove that the rules of $\mathbf{IL}$ preserve the property of derivable formulas evaluating to W under e :

1. Modus Ponens: By inductive hypothesis on $\mathbf{IL} \vdash \phi$ and $\mathbf{IL} \vdash \phi \rightarrow \psi$, we have $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$ and $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e \cup \llbracket \psi^t \rrbracket_{\mathcal{F}}^e = W$, so that $\psi^t = W$.
2. Necessitation: By inductive hypothesis on $\mathbf{IL} \vdash \phi$, we have $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$, so that, for any $x, y \in W$, $xRy \Rightarrow y \in \llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$. Hence we conclude $\llbracket \Box \phi^t \rrbracket_{\mathcal{F}}^e = \widehat{R^{-1}(\phi^t)} = W$.

□

Corollary 4.8. *Let $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ a Veltman frame. If $\mathbf{IL} \vdash \phi \rightarrow \psi$ then, for every valuation e , $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e \subseteq \llbracket \psi^t \rrbracket_{\mathcal{F}}^e$.*

Proof. By Theorem 4.7, we have $\llbracket (\phi \rightarrow \psi)^t \rrbracket_{\mathcal{F}}^e = \llbracket \phi^t \rrbracket_{\mathcal{F}}^e \cup \llbracket \psi^t \rrbracket_{\mathcal{F}}^e = W$. Then, for any $x \in \llbracket \phi^t \rrbracket_{\mathcal{F}}^e \subseteq W$, since $x \in W$, $x \in \llbracket \psi^t \rrbracket_{\mathcal{F}}^e$. Therefore, $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e \subseteq \llbracket \psi^t \rrbracket_{\mathcal{F}}^e$. □

Lemma 4.9. *Let $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ be a Veltman Frame, $\phi \in \text{Form}$, and e a valuation on $\mathcal{F}$. If, for every ultrafilter $f \in U(W)$, $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e \in f$, then $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = \llbracket \Box \phi^t \rrbracket_{\mathcal{F}}^e = W$.*

Proof. Since $\bigcap U(W) = \{W\}$, we have $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e \in \{W\}$, that is, $\llbracket \phi^t \rrbracket_{\mathcal{F}}^e = W$. But $\llbracket \Box \phi^t \rrbracket_{\mathcal{F}}^e = \widehat{R^{-1}(\llbracket \phi^t \rrbracket_{\mathcal{F}}^e)} = \widehat{R^{-1}(W)} = W$, and we are done. □

Since any **IL**-tautology evaluates to W for any e on any frame $\mathcal{F}$, we can use **IL** reasoning in the ultrafilter setting, as stated by the following

Lemma 4.10. *Given a Veltman frame $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ and a valuation e over W , if $\Gamma \vdash \varphi$ and $\llbracket \Gamma^t \rrbracket_{\mathcal{F}}^e \subseteq f$, then $\llbracket \varphi^t \rrbracket_{\mathcal{F}}^e \in f$.*

Proof. Let $\Gamma \vdash \varphi$ and e be such that $\llbracket \Gamma^t \rrbracket_{\mathcal{F}}^e \subseteq f$. By the deduction theorem, we have that **IL** $\vdash \bigwedge_i (\gamma_i) \rightarrow \varphi$ for $\gamma_i \in \Gamma$; by Theorem 4.7, we thus have $\llbracket \bigwedge_i (\gamma_i) \rightarrow \varphi \rrbracket_{\mathcal{F}}^e = W$, that is: $\llbracket \bigwedge_i (\gamma_i)^t \cup \varphi^t \rrbracket_{\mathcal{F}}^e = W$. Therefore, $\llbracket \bigwedge_i (\gamma_i)^t \rrbracket_{\mathcal{F}}^e = \bigcap (\llbracket \gamma_i^t \rrbracket_{\mathcal{F}}^e) \subseteq \llbracket \varphi^t \rrbracket_{\mathcal{F}}^e$. Since f is closed under finite intersections and, by assumption, $\llbracket \gamma_i^t \rrbracket_{\mathcal{F}}^e \in f$ for each $\gamma_i \in \Gamma$, we obtain $\bigcap (\llbracket \gamma_i^t \rrbracket_{\mathcal{F}}^e) \in f$; and since f is closed under supersets, we conclude that $\llbracket \varphi^t \rrbracket_{\mathcal{F}}^e \in f$. $\square$

Corollary 4.11. *The following hold:*

1. $S^{-1}(X, Y) \subseteq S^{-1}(X, Y \cup Z)$;
2. $\widehat{R^{-1}}(A) \subseteq \widehat{R^{-1}}(\widehat{R^{-1}}(A))$.
3. $S^{-1}(A, B) \cap S^{-1}(B, C) \subseteq S^{-1}(A, C)$.

Proof. All items follow easily from Theorem 4.7 either directly or indirectly. For example, Item 1 reflects that $(\varphi \triangleright \psi) \wedge (\psi \rightarrow \theta) \rightarrow (\varphi \triangleright \theta)$. $\square$

Finally, we extend the ultrafilter extensions of frames into ultrafilter extensions of models in a natural way:

Definition 4.12. Given a Veltman model $\mathcal{M} = \langle \mathcal{F}, ev \rangle$, we define its ultrafilter extension $\mathcal{M}^{ue}$ to be the pair $\langle \mathcal{F}^{ue}, ev^{ue} \rangle$ where $\mathcal{F}^{ue}$ is as in Def. 4.4, and ev^{ue} is such that for every $\langle f, \langle \sigma \rangle \rangle$ we have $\langle f, \langle \sigma \rangle \rangle \in ev^{ue}(p) \iff ev(p) \in f$.

5 Full Labels in Algebras

We translate here the labelling machinery from Sect. 2.2 to the setup from Sect. 4.2.

Lemma 5.1. *If $f \prec_l g$ then $R^{-1}g \subseteq f$.*

Proof. Assume that $f \prec_l g$, for some $f, g \in \mathcal{U}(W)$, and l is a proper filter over W . By Lemma 4.2, we consider some $A \subseteq W$ such that $\widehat{R^{-1}}(A) \in f$. We know that $\Box p \vdash_{\mathbf{IL}} \neg p \triangleright \perp$, for any $p \in \text{Atm}$. Now, define a valuation e such that $\llbracket p_A^t \rrbracket_{\mathcal{F}}^e = A$. Therefore, $\llbracket (\Box p_A)^t \rrbracket_{\mathcal{F}}^e = \widehat{R^{-1}}(A) \in f$ and, by Lemma 4.10, $\llbracket (\neg p_A^t \triangleright \perp)^t \rrbracket_{\mathcal{F}}^e = S^{-1}(\widehat{A}, \emptyset) \in f$. Since $f \prec_l g$, we conclude $A, \widehat{R^{-1}}(A) \in g$. $\square$

Assuring successors are successors as per Lemma 2.7:

Lemma 5.2. *If $f \prec_l g$ then $l, \widehat{R^{-1}}(l) \subseteq g$.*

Proof. Let $A \in l$. Since **IL** $\vdash \neg \alpha \triangleright \neg \alpha$, by Lemma 4.7, we have $S^{-1}(\widehat{A}, \widehat{A}) = W \in f$. Because $f \prec_l g$, we conclude that $A, \widehat{R^{-1}}(A) \in g$. $\square$

We define, from now on, the convention that whenever a function is considered on a domain, it is naturally extended to the powerset of its domain. In this case, $\widehat{R^{-1}}$ on l belongs to the powerset of the domain of R^{-1} .

Corollary 5.3. *If $f \prec_l g$, then $R^{-1}(l) \subseteq f$.*

Proof. Since f is an ultrafilter, for any $A \in l$, either $R^{-1}(A) \in f$ or $\overline{R^{-1}(A)} \in f$; that is, by Lemma 4.2, either $R^{-1}(A) \in f$ or $\widehat{R^{-1}(A)} \in f$. However, by Lemma 5.1, from the latter case we would have $\bar{A} \in g$, contra Lemma 5.2 which tells us that $A \in g$. We thus conclude $R^{-1}(A) \in f$. $\square$

Lemma 5.4. *Let $\mathcal{F} = \langle W, R, \{S_w \mid w \in W\} \rangle$ be a Veltman frame and $f, g, h \in U(W)$. If $f \prec_l g$ and $g \prec_m h$, then $f \prec_l h$.*

Proof. Assume $f \prec_l g$ and $g \prec_m h$. By Lemma 5.2, $\widehat{R^{-1}(l)} \subseteq g$. Now, let $A \in l$ and e such that $\llbracket p_A^t \rrbracket_{\mathcal{F}}^e = A$. Since $\mathbf{IL} \vdash \Box p_A \rightarrow \Box \Box p_A$, by Lemma 4.8, we have that $\widehat{R^{-1}(\widehat{R^{-1}(A)})} = \llbracket \Box \Box p_A \rrbracket_{\mathcal{F}}^e \in g$. Since $g \prec_m h$, we know that $f_{\Box} \subseteq h$, and conclude that $A, \widehat{R^{-1}(A)} \in h$. $\square$

Lemma 5.5. *Let $\mathcal{F} = \langle W, R, \{S_x \mid x \in W\} \rangle$ be a Veltman frame. Let $\mathcal{F}^{ue}$ be the ultrafilter extension of $\mathcal{F}$ and $f, g \in W^{ue}$. If $\langle f, \sigma \rangle R^{ue} \langle g, \tau \rangle$, then σ is an initial segment of τ^5 and $f \prec_l g$ for l such that $\tau = \sigma \wedge \langle l \rangle \wedge \tau'$ proper filter over W .*

Proof. By definition, $\sigma \subset \tau$. By Definition 4.4 there is a chain of worlds $\langle h_1, \sigma \wedge \langle l \rangle \rangle R_{\text{one}}^{ue} \dots R_{\text{one}}^{ue} \langle g, \tau \rangle$. Then we can apply Lemma 5.4 to obtain that $f \prec_l g$ as required. $\square$

The following reflects Lemma 3.7 in [7] about consistency formulas in labels at the ultrafilter setup:

Lemma 5.6. *Let $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ be a Veltman frame, $f \in U(W)$, and l be a proper filter over W . Let $B_f := \{A \subseteq W \mid \exists^{\text{fin}} \{S_i \in l\} (S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f)\}$. Then, if $C \in B_f$, $\widehat{R^{-1}(C)} \in B_f$.*

Proof. Assume $C \in B_f = \{A \subseteq W \mid \exists^{\text{fin}} \{S_i \in l\} (S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f)\}$. Then, for some $\{S_i\} \subseteq_{\text{fin}} l$, we have that $S^{-1}(\bar{C}, \bigcup_i \bar{S}_i) \in f$. We need to show that there exists $\{S_j\} \subseteq_{\text{fin}} l$ $S^{-1}(\widehat{R^{-1}(C)}, \bigcup_j \bar{S}_j) \in f$. Define then e such that $\llbracket p_C^t \rrbracket_{\mathcal{F}}^e = C$ and $\llbracket p_{S_k}^t \rrbracket_{\mathcal{F}}^e = S_k$ for each $S_k \in l$. By axiom J2, $\mathbf{IL} \vdash (\Diamond \neg p_C \triangleright \neg p_C \wedge \neg p_C \triangleright \bigvee_i (\neg p_{S_i})) \rightarrow \Diamond \neg p_C \triangleright \bigvee_i (\neg p_{S_i})$. Since $S^{-1}(\bar{C}, \bigcup_i \bar{S}_i) \in f$ and $\llbracket \Diamond \neg p_C \triangleright \neg p_C^t \rrbracket_{\mathcal{F}}^e = W \in f$, we have $\llbracket \neg p_C^t \triangleright \bigvee_i (\neg p_{S_i}^t) \wedge \Diamond \neg p_C \triangleright \neg p_C^t \rrbracket_{\mathcal{F}}^e \in f$. By Corollary 4.8, we conclude $\llbracket \Diamond \neg p_C^t \triangleright \bigvee_i (\neg p_{S_i}^t) \rrbracket_{\mathcal{F}}^e = S^{-1}(\widehat{R^{-1}(C)}, \bigcup_i \bar{S}_i) \in f$. $\square$

Lemma 5.7. *Let $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ be a Veltman frame, $f \in U(W)$, and l be a proper filter over W . The set $\{A \subseteq W \mid \exists^{\text{fin}} \{S_i \in l\} (S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f)\}$ is closed under intersections.*

Proof. Assume $C, D \in \{A \subseteq W \mid \exists^{\text{fin}} \{S_i \in l\} (S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f)\}$. Then, for some sets $S^C \subseteq_{\text{fin}} l$ and $S^D \subseteq_{\text{fin}} l$, we have $S^{-1}(\bar{C}, \bigcup_{i \in S^C} \bar{S}_i) \in f$ and $S^{-1}(\bar{D}, \bigcup_{j \in S^D} \bar{S}_j) \in f$. By Corollary 4.11.1, $S^{-1}(X, Y) \subseteq S^{-1}(X, Y \cup Z)$. Since f is closed under supersets, both $S^{-1}(\bar{C}, \bigcup_{i,j} \bar{S}_{i,j}^{C,D}) \in f$ and $S^{-1}(\bar{D}, \bigcup_{i,j} \bar{S}_{i,j}^{C,D}) \in f$. Define then e such that $\llbracket (p_C^t) \rrbracket_{\mathcal{F}}^e = C$, $\llbracket (p_D^t) \rrbracket_{\mathcal{F}}^e = D$, and $\llbracket p_{S_k} \rrbracket_{\mathcal{F}}^e = S_k$ for each $S_k \in l$. By this definition of e , both $\llbracket \neg p_C^t \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}}^t) \rrbracket_{\mathcal{F}}^e = S^{-1}(\bar{C}, \bigcup_{i,j} \bar{S}_{i,j}^{C,D})$ and $\llbracket \neg p_D^t \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}}^t) \rrbracket_{\mathcal{F}}^e = S^{-1}(\bar{D}, \bigcup_{i,j} \bar{S}_{i,j}^{C,D})$.

⁵Denoted by $\sigma \subset \tau$ from now on.

Since $\mathbf{IL} \vdash (\neg p_C \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}^{C,D}})) \wedge (\neg p_D \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}^{C,D}})) \rightarrow ((\neg p_C \vee \neg p_D) \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}^{C,D}}))$, by the deduction theorem $\{\neg p_C \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}^{C,D}}), p_D \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}^{C,D}})\} \vdash_{\mathbf{IL}} (\neg p_C \vee \neg p_D) \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}^{C,D}})$. Then, by Lemma 4.10, we then have $\llbracket (\neg p_C' \vee \neg p_D') \triangleright \bigvee_{i,j} (\neg p_{S_{i,j}^{C,D}}) \rrbracket_{\mathcal{F}}^e \in f$. Consequently, $\llbracket \neg(\neg p_C' \vee \neg p_D') \rrbracket_{\mathcal{F}}^e \in \{A \subseteq W \mid \exists^{\text{fin}} \{S_i \in I\} (S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f)\}$; that is: $\llbracket p_C' \wedge p_D' \rrbracket_{\mathcal{F}}^e = C \cap D \in f$, and we are done. $\square$

The above indeed suggests that each labelling lemma has its natural counterpart in the algebraic setting. Various observations from [7] are reflected in the following.

Lemma 5.8. *Let $\mathcal{F} = \langle W, R\{S_x\}_{x \in W} \rangle$ be a Veltman frame, and consider $f, g \in U(W)$ and $a, b \in \wp(W)$. The following hold:*

1. $a \subseteq b$ and $f \prec_b g$, then $f \prec_a g$;
2. $f \prec_a g$ and $R^{-1}h \subseteq g$, then $f \prec_a h$.

Proof. To prove 1, we assume that $a \subseteq b$ and $f \prec_b g$. Then, $\forall A \subseteq W \forall^{\text{fin}} \{S_i \in b\} (S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f \Rightarrow A, \widehat{R^{-1}A} \in g)$. We only need to observe that the finite choices of elements in a are finite choices of elements in b since $a \subseteq b$.

To prove 2, we assume that $f \prec_a g$ and $R^{-1}h \subseteq g$. Then, $\forall A \subseteq W \forall^{\text{fin}} \{S_i \in b\} (S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f \Rightarrow A, \widehat{R^{-1}A} \in g)$. Also, by Lemma 4.2, $g_{\square} \subseteq h$. To prove that $f \prec_a h$, pick an arbitrary $A \subseteq W$ and an arbitrary finite choice of $S_i \in a$. Now, suppose that $S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) \in f$. Since $f \prec_a g$, we have that $A, \widehat{R^{-1}(A)} \in g$. It suffices to show that $A, \widehat{R^{-1}(A)} \in h$. On the one hand, since $g_{\square} \subseteq h$ we derive that $A \in h$. On the other hand, by Lemma 4.11.2, $\widehat{R^{-1}(A)} \subseteq \widehat{R^{-1}(\widehat{R^{-1}(A)})}$ and since $\widehat{R^{-1}(A)} \in g$ and g is an ultrafilter, we know that $\widehat{R^{-1}(\widehat{R^{-1}(A)})} \in g$. Therefore, $\widehat{R^{-1}(A)} \in g_{\square}$ and, given that $g_{\square} \subseteq h$, $\widehat{R^{-1}(A)} \in h$. $\square$

Lemma 5.9. *Let $\mathcal{F} = \langle W, R, \{S_w\}_{w \in W} \rangle$ be a Veltman frame, and consider $f, g \in U(W)$ and $a, b, y \in \wp(W)$. The following hold:*

1. if $f \prec_a g$, and if $x \subseteq y$ for some $x \in a$ then $f \prec_{a \cup \{y\}} g$;
2. if $f \prec_a g$, then $f \prec_{a \cup \widehat{R^{-1}(a)}} g$, where $\widehat{R^{-1}(a)} := \{\widehat{R^{-1}(S)} \mid S \in a\}$.

Proof. To prove item 1 we pick an arbitrary $A \subseteq W$ and an arbitrary finite choice of $S_i \in a$. Now, assume that $S^{-1}(\bar{A}, \bigcup_i \bar{S}_i \cup \bar{y}) \in f$. Since $x \subseteq y$, we know that $\bar{y} \subseteq \bar{x}$. By Corollary 4.11 we have that $S^{-1}(\bar{A}, \bigcup_i \bar{S}_i \cup \bar{y}) \subseteq S^{-1}(\bar{A}, \bigcup_i \bar{S}_i \cup \bar{x})$ and, given that f is an ultrafilter, we derive that $S^{-1}(\bar{A}, \bigcup_i \bar{S}_i \cup \bar{x}) \in f$. Observe that, since $x \in a$, we have a finite choice of $S_j \in a$ consisting of the prior choice plus x and we can write $S^{-1}(\bar{A}, \bigcup_j \bar{S}_j) \in f$. Then, because $f \prec_a g$, we conclude that $A, \widehat{R^{-1}(A)} \in g$ as required.

To prove item 2, pick an arbitrary $A \subseteq W$ and an arbitrary finite choice of elements $S_i \in a$ and $\widehat{R^{-1}(S_j)} \in \widehat{R^{-1}(a)}$. Assume that $S^{-1}(\bar{A}, \bigcup_i \bar{S}_i \cup \bigcup_j \widehat{R^{-1}(S_j)}) \in f$. By Lemma 4.2,

$$S^{-1}(\bar{A}, \bigcup_i \bar{S}_i \cup \bigcup_j R^{-1}(\bar{S}_j)) \in f. \quad (1)$$

By Theorem 4.7, since $\mathbf{IL} \vdash \Diamond \alpha \triangleright \alpha$, we know that, for every S_j , $S^{-1}(R^{-1}(\bar{S}_j), \bar{S}_j) = W$ and, given that f is an ultrafilter, $S^{-1}(R^{-1}(\bar{S}_j), \bar{S}_j) \in f$, for every S_j . Also, $\mathbf{IL} \vdash \bigvee_i (\alpha_i \triangleright \beta_i) \rightarrow \bigvee_i \alpha_i \triangleright \bigvee_i \beta_i$,

so by Corollary 4.8 we deduce that $\bigcup_j S^{-1}(R^{-1}(\overline{S_j}), \overline{S_j}) \subseteq S^{-1}(\bigcup_j R^{-1}(\overline{S_j}), \bigcup_j \overline{S_j})$ which implies that $S^{-1}(\bigcup_j R^{-1}(\overline{S_j}), \bigcup_j \overline{S_j}) = W$ and, then,

$$S^{-1}(\bigcup_j R^{-1}(\overline{S_j}), \bigcup_j \overline{S_j}) \in f \quad (2)$$

Since $\mathbf{IL} \vdash \alpha \triangleright (\beta \vee \gamma) \wedge (\gamma \triangleright \eta) \rightarrow \alpha \triangleright (\beta \vee \eta)$, by Corollary 4.8 we have that

$$S^{-1}(\overline{A}, \bigcup_i \overline{S_i} \cup \bigcup_j R^{-1}(\overline{S_j})) \cap S^{-1}(\bigcup_j R^{-1}(\overline{S_j}), \bigcup_j \overline{S_j}) \subseteq S^{-1}(\overline{A}, \bigcup_i \overline{S_i} \cup \bigcup_j \overline{S_j}).$$

By 1 and 2 and the fact that f is an ultrafilter, we know that $S^{-1}(\overline{A}, \bigcup_i \overline{S_i} \cup \bigcup_j \overline{S_j}) \in f$. Finally, because $f \prec_a g$ we conclude that $A, \widehat{R^{-1}}(A) \in g$, as we wanted to show. $\square$

Definition 5.10. Given a set $a \subseteq \wp(W)$ with the finite intersection property, we define $l(a)$ to be the filter generated by a as $l(a) := \bigcap_{a \subseteq l'} \text{a filter over } W$.

The elements of the generated filter of some $a \subseteq \wp(W)$ are characterized by the elements of a .

Lemma 5.11. [From [4]] Given some set $a \subseteq \wp(W)$ we have that

$$S \in l(a) \iff (\exists S_1, \dots, S_n \in a \ S_1 \cap \dots \cap S_n \subseteq S).$$

Definition 5.10 can also be applied to arbitrary $a \subseteq \wp(W)$ and then the above lemma shows that $l(a)$ is a proper filter if and only if a has the FIP.

Lemma 5.12. Let $\mathcal{F} = \langle W, R, \{S_w \mid w \in W\} \rangle$ be a Veltman frame and consider $f, g \in U(W)$, $a \in \wp(W)$ and $f \prec_a g$. Then, $f \prec_{l(a)} g$.

Proof. By Lemma 5.11, for each $i \in \{1, \dots, n\}$, there exists a finite subset $F_i \subseteq a$ such that $\bigcap F_i \subseteq S_i$. Equivalently, for each $i \in \{1, \dots, n\}$, we have $\overline{S_i} \subseteq \bigcup_{S \in F_i} \overline{S}$. By letting $F := \bigcup_{i=1}^n F_i$, we have a finite subset $F \subseteq a$ such that $\bigcup_{i=1}^n \overline{S_i} \subseteq \bigcup_{S \in F} \overline{S}$. Now, consider some $S^{-1}(\overline{A}, \bigcup_{i=1}^n \overline{S_i}) \in f$. Because $\mathbf{IL} \vdash (\beta \rightarrow \gamma) \rightarrow (\alpha \triangleright \beta \rightarrow \alpha \triangleright \gamma)$, by Corollary 4.8 we have that $S^{-1}(\overline{A}, \bigcup_{i=1}^n \overline{S_i}) \subseteq S^{-1}(\overline{A}, \bigcup_{S \in F} \overline{S})$ ⁶. Since f is a filter, and therefore upward closed, we derive that $S^{-1}(\overline{A}, \bigcup_{S \in F} \overline{S}) \in f$. Finally, since $F \subseteq a$ is finite and $f \prec_a g$, we have that $A, \widehat{R^{-1}}(A) \in g$, and we are done. $\square$

6 Elementary Equivalence and Modal Saturation

The following two lemmas ensure that we can always find adequate ultrafilters.

Lemma 6.1. Let $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ a Veltman frame, and $f \prec_l g$, for some set l over W . If $S^{-1}(A, B) \in f$ and $A \in g$, then there is some h with $f \prec_l h$ and $B \in h$.

Proof. Assume that $S^{-1}(A, B) \in f$, that $f \prec_l g$ and $A \in g$. Seeking a contradiction, suppose that, for every h , if $f \prec_l h$, then $B \notin h$. Then, $\{B\} \cup \{C_i, \widehat{R^{-1}}(C_i) \mid \exists^{\text{fin}} S_i \in l \ S^{-1}(\overline{C_i}, \bigcup_i \overline{S_i}) \in f\}$ does not have the FIP: by Lemma 5.7, there is $C \in \{C_i, \widehat{R^{-1}}(C_i) \mid \exists^{\text{fin}} S_i \in l \ S^{-1}(\overline{C_i}, \bigcup_i \overline{S_i}) \in f\}$ such that $B \cap C \cap \widehat{R^{-1}}(C) = \emptyset$. Now, let e be a valuation such that $\llbracket p_A^t \rrbracket_e^{\mathcal{F}} = A$, $\llbracket p_B^t \rrbracket_e^{\mathcal{F}} = B$, $\llbracket p_C^t \rrbracket_e^{\mathcal{F}} = C$, and, for every S_i , $A_{p_{S_i}} = S_i$. Therefore, we have $\llbracket p_B^t \wedge p_C^t \wedge \Box p_C^t \rrbracket_e^{\mathcal{F}} = \emptyset$, that is: $\llbracket \neg(p_B^t \wedge p_C^t \wedge \Box p_C^t) \rrbracket_e^{\mathcal{F}} = W \in f'$, for every

⁶We note that given $A, B \in \wp(W)$, then $A \subseteq B \iff \overline{A} \cup B = W$

$f' \in U(W)$. Moreover, from $\neg(p_B \wedge p_C \wedge \Box p_C) \vdash_{\mathbf{IL}} p_B \rightarrow (\neg p_C \vee \Diamond \neg p_C)$ and Lemma 4.10, we have $\llbracket p'_B \rightarrow (\neg p'_C \vee \Diamond \neg p'_C) \rrbracket_{\mathcal{F}}^e \in f'$, for all ultrafilters $f' \in U(W)$. By Lemma 4.9, we have $\llbracket \Box (p'_B \rightarrow (\neg p'_C \vee \Diamond \neg p'_C)) \rrbracket_{\mathcal{F}}^e = W$, and, by $\Box (p_B \rightarrow (\neg p_C \vee \Diamond \neg p_C)) \vdash_{\mathbf{IL}} p_B \triangleright (\neg p_C \vee \Diamond \neg p_C) \vdash_{\mathbf{IL}} p_B \triangleright \neg p_C$, Lemma 4.10, and Lemma 4.9, we have that $\llbracket p'_B \triangleright \neg p'_C \rrbracket_{\mathcal{F}}^e = W \in f$.

By assumption, we know that there are some $S_i \in l$ such that $S^{-1}(\bar{C}, \bigcup_i \bar{S}_i) = \llbracket \neg p'_C \triangleright \bigvee_i \neg p'_{S_i} \rrbracket_{\mathcal{F}}^e \in f$ and that $S^{-1}(\bar{A}, B) = \llbracket \neg \neg p'_A \triangleright p'_B \rrbracket_{\mathcal{F}}^e \in f$. Given that $\neg \neg p_A \triangleright p_B$, $p_B \triangleright \neg p_C$, $\neg p_C \triangleright \bigvee_i \neg p_{S_i} \vdash_{\mathbf{IL}} \neg \neg p_A \triangleright \bigvee_i \neg p_{S_i}$, by Lemma 4.10, we derive that $S^{-1}(\bar{A}, \bigcup_i \bar{S}_i) = \llbracket \neg \neg p'_A \triangleright \bigvee_i \neg p'_{S_i} \rrbracket_{\mathcal{F}}^e \in f$ and, since $f \prec_l g$, we conclude that $\bar{A} \in g$, which contradicts our hypothesis. $\square$

Lemma 6.2. *Given $\mathcal{F} = \langle W, R, \{S_w \mid w \in W\} \rangle$ a Veltman frame, $f \in U(W)$, and $S^{-1}(A, B) \in f$, there is some ultrafilter g and some proper filter l over W with $A \in g$ and $\bar{B} \in l$ and $f \prec_l g$.*

Proof. Assume $S^{-1}(A, B) \in f$ and suppose for a contradiction that for all ultrafilters g over W and for all proper filters l , if $f \prec_l g$, then $A \notin g$. Let $l = \{S : \bar{B} \subseteq S\}^7$. Then, we have $\{A\} \cup \{C_i, \widehat{R^{-1}(C_i)} \mid \exists^{\text{fin}} \{S_i \in l \mid S^{-1}(\bar{C}_i, \bigcup_i S_i)\}\}$ does not have the finite intersection property. By Lemma 5.7, there is some $C \in \{C_i, \widehat{R^{-1}(C_i)} \mid \exists^{\text{fin}} S_i S^{-1}(\bar{C}_i, B)\}$ we have that $C \cap \widehat{R^{-1}(C)} \cap A = \emptyset$.

Let e be a valuation such that $\llbracket p_A \rrbracket_{\mathcal{F}}^e = A$, $\llbracket p_C \rrbracket_{\mathcal{F}}^e = C$ and $\llbracket p_{S_i} \rrbracket_{\mathcal{F}}^e = S_i$ for each $S_i \in l$. We observe that $\llbracket p'_A \wedge p'_C \wedge \Box p'_C \rrbracket_{\mathcal{F}}^e = \emptyset$. But then $\llbracket \neg(p'_A \wedge p'_C \wedge \Box p'_C) \rrbracket_{\mathcal{F}}^e = W \in f$. We observe that $\neg(p_A \wedge p_C \wedge \Box p_C) \vdash_{\mathbf{IL}} p_A \rightarrow \neg p_C \vee \Diamond \neg p_C$ and that by Lemma 4.10 we obtain that $\llbracket p'_A \rightarrow \neg p'_C \vee \Diamond \neg p'_C \rrbracket_{\mathcal{F}}^e \in f$. By Lemma 4.9 we obtain that $\widehat{R^{-1}(\llbracket p'_A \rightarrow \neg p'_C \vee \Diamond \neg p'_C \rrbracket_{\mathcal{F}}^e)} = W \in f$. We observe that $\Box(p_A \rightarrow \neg p_C \vee \Diamond \neg p_C) \vdash_{\mathbf{IL}} p_A \triangleright \neg p_C \vee \Diamond \neg p_C$ by J1 and, again by 4.10 $\llbracket p'_A \triangleright \neg p'_C \vee \Diamond \neg p'_C \rrbracket_{\mathcal{F}}^e \in f$. We also observe that $\mathbf{IL} \vdash \neg p_C \triangleright \neg p_C$ and $\mathbf{IL} \vdash \Diamond \neg p_C \triangleright \neg p_C$. Whence by using axioms J3 and J2 and Lemma 4.10 we resolve that $\llbracket p'_A \triangleright \neg p'_C \rrbracket_{\mathcal{F}}^e = S^{-1}(A, \bar{C}) \in f$. But then, we observe that, from the fact that for a finite choice of $S_i \in l$ we have $S^{-1}(\bar{C}, \bigcup_i \bar{S}_i) \in f$, we also have $\llbracket \neg p'_C \triangleright p'_B \rrbracket_{\mathcal{F}}^e = S^{-1}(\bar{C}, B) \in f$, by Corollary 4.11.1. It can be proven that $p_A \triangleright \neg p_C, \neg p_C \triangleright p_B \vdash_{\mathbf{IL}} p_A \triangleright p_B$. By Lemma 4.10 we have $\llbracket p'_A \triangleright p'_B \rrbracket_{\mathcal{F}}^e = S^{-1}(A, B) \in f$ which contradicts our hypothesis. $\square$

Lemma 6.3. *Given $\mathcal{M} = \langle \mathcal{F}, ev \rangle$ a Veltman model where $\mathcal{F} = \langle W, R, \{S_w \mid w \in W\} \rangle$, let $f \in U(W)$ and e be a valuation such that $\llbracket A_p \rrbracket_{\mathcal{F}}^e = \llbracket p \rrbracket_{\mathcal{M}}$ for every $p \in \text{Atm}$. Then, for any $\varphi \in \text{Form}$, we have $\llbracket \varphi' \rrbracket_{\mathcal{F}}^e \in f \iff \llbracket \varphi \rrbracket_{\mathcal{M}} \in f$.*

Proof. By a standard structural induction on the complexity of φ .

1. $\varphi = \perp$; then $\llbracket \perp' \rrbracket_{\mathcal{F}}^e = \emptyset = \llbracket \perp \rrbracket_{\mathcal{M}}$
2. $\varphi = p$ with $p \in \text{Atm}$; By definition $\llbracket p' \rrbracket_{\mathcal{F}}^e = \llbracket p \rrbracket_{\mathcal{M}}$.
3. $\varphi = \psi \rightarrow \gamma$: By definition $\llbracket \psi' \rightarrow \gamma' \rrbracket_{\mathcal{F}}^e = \llbracket \psi' \rrbracket_{\mathcal{F}}^e \cup \llbracket \gamma' \rrbracket_{\mathcal{F}}^e$. By induction hypothesis, we have that $\llbracket \psi' \rrbracket_{\mathcal{F}}^e = \llbracket \psi \rrbracket_{\mathcal{M}}$, so that $\llbracket \psi' \rrbracket_{\mathcal{F}}^e \cup \llbracket \gamma' \rrbracket_{\mathcal{F}}^e = \llbracket \psi \rrbracket_{\mathcal{M}} \cup \llbracket \gamma \rrbracket_{\mathcal{M}} = \llbracket \psi \rightarrow \gamma \rrbracket_{\mathcal{M}}$, as required.
4. $\varphi = \Box \psi$: By induction hypothesis, we have

$$\llbracket \Box \psi' \rrbracket_{\mathcal{F}}^e = \widehat{R^{-1}(\llbracket \psi' \rrbracket_{\mathcal{F}}^e)} = \widehat{R^{-1}(\llbracket \psi \rrbracket_{\mathcal{M}})} \in f \iff \llbracket \Box \psi \rrbracket_{\mathcal{M}} \in f,$$

as required.

⁷Clearly $\bar{B} \in l$ as $\bar{B} \subseteq \bar{B}$

5. $\varphi = \psi \triangleright \gamma$: By induction hypothesis, we have that

$$\llbracket \psi' \triangleright \gamma' \rrbracket_{\mathcal{F}}^e = S^{-1}(\llbracket \psi' \rrbracket_{\mathcal{F}}^e, \llbracket \gamma' \rrbracket_{\mathcal{F}}^e) = S^{-1}(\llbracket \psi \rrbracket_{\mathcal{M}}, \llbracket \gamma \rrbracket_{\mathcal{M}}) \in f \iff \llbracket \psi \triangleright \gamma \rrbracket_{\mathcal{M}} \in f,$$

as required. $\square$

Lemma 6.4. *Given $\mathcal{M} = \langle \mathcal{F}, ev \rangle$ a Veltman model where $\mathcal{F} = \langle W, R, \{S_w \mid w \in W\} \rangle$. For any formula φ and any $\langle \Pi_x, \sigma \rangle$ in the ultrafilter extension we have*

$$\mathcal{M}, x \Vdash \varphi \implies \mathcal{M}^{ue}, \langle \Pi_x, \sigma \rangle \Vdash \varphi.$$

Proof. By induction on the complexity of the formula ψ we show that $\llbracket \psi \rrbracket_{\mathcal{M}} \in f \implies \mathcal{M}^{ue}, \langle f, \sigma \rangle \Vdash \psi$.

If $\varphi = p$ for some $p \in \text{Atm}$ follows from the definition. The Boolean connectives follow directly from the inductive hypothesis, although, for clarity, we will show the case for negation: Let $\varphi = \neg\psi$. Assume $\llbracket \neg\psi \rrbracket_{\mathcal{M}} \in f$. Assume for a contradiction that for some $\langle f, \sigma \rangle \in W^{ue}, M^{ue}, \langle f, \sigma \rangle \not\Vdash \neg\psi$. Then $M^{ue}, \langle f, \sigma \rangle \Vdash \psi$. By induction hypothesis (in contraposition) this entails that $\llbracket \psi \rrbracket_{\mathcal{M}} \notin f$. But then $\llbracket \neg\psi \rrbracket_{\mathcal{M}} \notin f$ which contradicts our initial assumption.

Since the $\Box$ and $\Diamond$ modalities can be expressed in terms of $\triangleright$ we focus on the latter case: let $\psi = \alpha \triangleright \beta$.

Assume $\llbracket \alpha \triangleright \beta \rrbracket_{\mathcal{M}} \in f$ and σ is a sequence of proper filters over W in such a way that $\langle f, \sigma \rangle \in W^{ue}$. Assume there is some $\langle g, \tau \rangle \in W^{ue}$ such that $\langle f, \sigma \rangle R^{ue} \langle g, \tau \rangle$ and $\mathcal{M}^{ue}, \langle g, \tau \rangle \Vdash \alpha$. We define e so that, for all $p \in \text{Atm}$, we have $\llbracket A_p \rrbracket_{\mathcal{F}}^e = \llbracket p \rrbracket_{\mathcal{M}}$. We will denote $A = \llbracket \alpha' \rrbracket_{\mathcal{F}}^e$ and $B = \llbracket \beta \rrbracket_{\mathcal{F}}^e$. By Lemma 6.3 we have $A = \llbracket \alpha \rrbracket_{\mathcal{M}}$ and $B = \llbracket \beta \rrbracket_{\mathcal{M}}$ and that $S^{-1}(A, B) \in f$, $\langle f, \sigma \rangle R^{ue} \langle g, \tau \rangle$ and $A \in g$. We observe that $\tau = \sigma \smallfrown \langle l \rangle \smallfrown \bar{\tau}$ for some l and $\bar{\tau}$. Then, by Lemma 5.5, we have that $f \prec_l g$ and, by Lemma 6.1, we have that there is some $h \in \wp(W)$ such that $f \prec_l h$ and $B \in h$, so that that $\langle f, \sigma \rangle R^{ue} \langle h, \langle \sigma \smallfrown \langle l \rangle \rangle \rangle$ and $(\tau)_{|\sigma|} = (\sigma \smallfrown \langle l \rangle)_{|\sigma|}$. Therefore $\langle g, \tau \rangle S_{\langle f, \sigma \rangle}^{ue} \langle h, \langle \sigma \smallfrown \langle l \rangle \rangle \rangle$. Since $\langle g, \tau \rangle$ was arbitrary, we conclude that $\mathcal{M}^{ue}, f \Vdash \alpha \triangleright \beta$. $\square$

Theorem 6.5. *Given $\mathcal{M} = \langle \mathcal{F}, ev \rangle$ a Veltman model where $\mathcal{F} = \langle W, R, \{S_w \mid w \in W\} \rangle$. For any formula φ and any $\langle \Pi_x, \sigma \rangle$ in the ultrafilter extension we have*

$$\mathcal{M}, x \Vdash \varphi \iff \mathcal{M}^{ue}, \langle \Pi_x, \sigma \rangle \Vdash \varphi.$$

Proof. We will show that

$$\llbracket \psi \rrbracket_{\mathcal{M}} \in f \iff \mathcal{M}^{ue}, \langle f, \sigma \rangle \Vdash \psi$$

by induction on the complexity of the formula ψ

If $\varphi = p$ for some $p \in \text{Atm}$ it follows from the definition. For Boolean connectives it follows directly from the induction hypothesis and Lemma 6.4. As stated in the proof of the previous Lemma, $\Box$ and $\Diamond$ modalities can be expressed in terms of $\triangleright$, so we will only provide proof for the latter case. Let $\psi = \alpha \triangleright \beta$; the left-to-right direction is proven in Lemma 6.4.

For the Right-to-left direction assume that $\llbracket \alpha \triangleright \beta \rrbracket_{\mathcal{M}} \notin f$. Since f is an ultrafilter, we have that $\llbracket \alpha \triangleright \beta \rrbracket_{\mathcal{M}} \in f$ and, per definition, we obtain that $\llbracket \neg(\alpha \triangleright \beta) \rrbracket_{\mathcal{M}} \in f$. Then, by Lemma 6.4, $\mathcal{M}^{ue}, \langle f, \sigma \rangle \Vdash \neg(\alpha \triangleright \beta)$ and hence $\mathcal{M}^{ue}, \langle f, \sigma \rangle \not\Vdash (\alpha \triangleright \beta)$ as intended. $\square$

We also have that the ultrafilter extension is modally saturated.

Definition 6.6 (Modal saturation). Let $\mathcal{M}$ be a modal model and $x \in \mathcal{M}$. We call a set of Σ of modal sentences **locally possible** in x whenever for any finite $\Sigma' \subseteq \Sigma$ we have $\mathcal{M}, x \Vdash \Diamond \bigwedge \Sigma'$. The set Σ is called **possible** (outright) at x whenever there is some $y \in \mathcal{M}$ with xRy and $\mathcal{M}, y \Vdash \Sigma$.

We call a modal model $\mathcal{M}$ **modally saturated** whenever for any $x \in \mathcal{M}$ we have that any set that is locally possible at x is possible at x .

Theorem 6.7. *For any Veltman model $\mathcal{M} = \langle \mathcal{F}, ev \rangle$, the ultrafilter extension $\mathcal{M}^{ue}$ is modally saturated.*

Proof. Consider an arbitrary $\langle f, \sigma \rangle \in \mathcal{M}^{ue}$ and let Σ be locally possible at $\langle f, \sigma \rangle$ in $\mathcal{M}^{ue}$. We have to show that Σ is possible. Let $\llbracket \Sigma \rrbracket_{\mathcal{M}} := \{ \llbracket \varphi \rrbracket_{\mathcal{M}} \mid \varphi \in \Sigma \}$. By combining Lemma 4.2, Lemma 5.11, Lemma 2.9 and Theorem 6.5 it is enough to see that $f_{\square} \cup \llbracket \Sigma \rrbracket_{\mathcal{M}}$ has the finite intersection property. Since any finite part T of Σ has a R^{ue} -successor $\langle g, \sigma' \rangle$ with $\mathcal{M}^{ue}, \langle g, \sigma' \rangle \Vdash \bigwedge T$, by Theorem 6.5 we have $\bigcap_{\varphi \in T} \llbracket \varphi \rrbracket_{\mathcal{M}} \in g$. By Lemma 5.5, Lemma 5.1 and Lemma 4.2, we see that $f_{\square} \subseteq g$. Therefore, for any $A \in f_{\square}$, we have $\emptyset \neq A \cap \bigcap_{\varphi \in T} \llbracket \varphi \rrbracket_{\mathcal{M}} \in g$. Given that A and T were arbitrary, we conclude that $f_{\square} \cup \llbracket \Sigma \rrbracket_{\mathcal{M}}$ has the finite intersection property. $\square$

We also have a certain saturation at the label level.

Theorem 6.8. *Let $f \in U(W)$. If, for every finite $l' \subset l$, there exists a g such that $f \prec_{l'} g$, then there exists an h such that $f \prec_l h$.*

Proof. Suppose not. Then, for every $h \in U(W)$, there is some $A \subseteq W$ and some finite choice of $S_i \in l$ such that $S^{-1}(\overline{A}, \bigcup_i \overline{S_i}) \in f$ and $A \notin h$ or $\widehat{R^{-1}}(A) \notin h$. Observe that the finite choice of $S_i \in l$ is a finite choice of $S_i \in l'$, for some finite $l' \subseteq l$. By assumption, there is some ultrafilter $g \in U(W)$ such that $f \prec_{l'} g$. Since $S^{-1}(\overline{A}, \bigcup_i \overline{S_i}) \in f$, we have that $A, \widehat{R^{-1}}(A) \in g$, which contradicts our hypothesis. $\square$

7 Conclusions

Through our example of a modally undefinable first-order frame condition we motivated the need of a Goldblatt-Thomason theorem [6] for interpretability logic. We proceeded to give a definition for ultrafilter extensions of Veltman frames and models and proved the main properties of these. The definition that we present resembles the canonical model construction [15, 8] which came slightly unexpected since the ultrafilter extension starts from a concrete model. For this reason, and in general, it would be interesting to see how ultrafilter extensions fare under Verbrugge semantics [17].

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Topological Logics of Path-Reachability

Aleksandr Gagarin David Fernández-Duque

University of Barcelona, Spain

agagarga7@alumnes.ub.edu fernandez-duque@ub.edu

The topological semantics of modal logic has been an active area of research ever since their introduction in the 1940s, with attention shifting in recent years from standard unimodal logic to more expressive frameworks. In particular, an Until-like path-reachability modality has recently been studied in Bezhanishvili et al. (2024) in polyhedral semantics; this paper investigates its topological counterpart. Focusing on the language combining said modality with the classical Cantor derivative modality, we exhibit an axiomatic system sound and complete both for the class of T_1 topologies and for the class of all metric spaces, and establish its decidability. We also axiomatize the logic of all topological models in a weaker language obtained by substituting the closure modality for the Cantor derivative. To prove our results, we introduce an equivalent neighborhood-like semantics allowing for the finite model property.

1 Introduction

Topological semantics provide ways of interpreting modal formulas in spatial terms, by viewing propositions as regions of a topological space and modal operators as expressing some spatial transformations. Two classical variants of topological semantics, both introduced by McKinsey and Tarski [8], are the *c-semantics* and the *d-semantics*. In the *c-semantics*, the modal operators $\Diamond$ and $\Box$ are interpreted as closure and interior, respectively. The logic of all topological spaces then coincides with **S4**. Moreover, **S4** is the logic of any dense-in-itself metric space [8, 1]. In the more expressive *d-semantics*, the $\Diamond$ modality is interpreted as the Cantor derivative, which maps a region to the set of its limit points. The logic of all topologies is **wK4** [1]. The separation axiom T_D is modally definable by $\Box p \rightarrow \Box^2 p$, and **K4** = **wK4** + $\Box p \rightarrow \Box^2 p$ is the logic of metrizable Stone spaces [2].

An Until-like binary modality γ for path-reachability was introduced in [4] in the context of so-called *polyhedral models*. The formula $\gamma(\varphi, \psi)$ is defined to be true at a point x iff there exists a continuous path beginning at x , with all intermediate points validating φ , and the end point validating ψ . In [3], the logic of polyhedral models as well as the logic of Alexandroff spaces with this modality were axiomatized and shown to be decidable. In polyhedral and Alexandroff models, $\gamma(\varphi, \top)$ defines the closure of the set defined by φ , thus γ -semantics subsumes *c-semantics* [3]. This is not necessarily the case in an arbitrary topology, e.g., in any non-trivial totally path-disconnected space.

In this paper, we study logics related to γ in a general topological setting. In the language combining γ with *c-semantics*, we axiomatize the logic of all topologies and show that it coincides with the logic of metric spaces. In the language combining γ with *d-semantics*, we exhibit a formula defining the separation axiom T_1 , axiomatize the logic of all T_1 topologies, and show that it is complete for the class of metric spaces. Thus, both logics are also characterized by the class of Hausdorff spaces, by the class of regular Hausdorff spaces, etc. Further, we establish that these logics are decidable.

In proving our results, we introduce an equivalent neighborhood-like semantics (cf. [9]) for γ , where the set of intermediate points of a path is treated as a “neighborhood” of the path’s original point. Then we establish the finite model property in this semantics by constructing an appropriate filtration of the

canonical neighborhood-like model. In Sections 2–6, we obtain our result for the d-semantics, while their c-semantical counterparts are derived in Section 7.

2 Preliminaries

In this section we establish some of the notation we will use. The powerset of X is denoted $\mathcal{P}(X)$. For a transitive binary relation R on a set X , we write $R(x) := \{y \in X : x R y\}$, and call $U \subseteq X$ an R -upset if $R(x) \subseteq U$ for all $x \in U$.

Topology. We use the following standard notions from general topology; see e.g. [6] for definitions: topological space; open, closed, discrete, and connected subsets; continuous map; metric space and its induced topology. A topological space $\langle X, \tau \rangle$ is called T_1 if for all distinct $x, y \in X$ there exists an open set U with $x \in U$ and $y \notin U$. Note that all metric spaces induce T_1 topologies [6].

When a subset of $\mathbb{R}$ is mentioned (e.g., $[0, 1]$ or $\mathbb{Q}$), the usual metric given by $d(x, y) := |x - y|$ and its induced topology are implicit. A *path* in a topological space $\langle X, \tau \rangle$ is a continuous map $[0, 1] \rightarrow X$.

Miscellaneous. For a map $\pi : [0, 1] \rightarrow X$ and $a, b \in [0, 1]$, denote by $\pi \upharpoonright [a, b]$ the map $[a, b] \rightarrow X$ given by $t \mapsto \pi(bt - at + a)$. Observe that if a topology on X is given and π is a path, then $\pi \upharpoonright [a, b]$ is also a path. Note that we allow $b < a$; in particular, $\pi \upharpoonright [1, 0]$ is the “reverse” of π .

For a set X and $x, y \in X$, denote by $\iota_{x,y}$ the map $[0, 1] \rightarrow X$ given by $\iota_{x,y}(0) = x$ and $\iota_{x,y}(t) = y$ for $t \in (0, 1]$. In particular, $\iota_{x,x}$ is constant. Note that $\iota_{x,y}$ may or may not be continuous when some topology on X is given.

Language. Fix a countably infinite set of propositional variables Prop . Denote by $\mathcal{L}$ the language built from Prop and $\perp$ using the unary connective $\Box$ and binary connectives $\rightarrow$ and γ . For a formula $\varphi \in \mathcal{L}$, denote by $\text{Sub } \varphi$ the set of its subformulas.

Semantics. We interpret $\Box$ as in the usual topological d-semantics [1, 2], and γ as the reachability modality from [3, 4]. Formally, a *topological model* is a triple $\mathfrak{M} = \langle X, \tau, \llbracket \cdot \rrbracket \rangle$, where $\langle X, \tau \rangle$ is a topological space, and $\llbracket \cdot \rrbracket : \text{Prop} \rightarrow \mathcal{P}(X)$. The map $\llbracket \cdot \rrbracket$ is extended to $\mathcal{L}$ as follows:

- $\llbracket \perp \rrbracket = \emptyset$;
- $\llbracket \varphi \rightarrow \psi \rrbracket = (X \setminus \llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket$;
- $x \in \llbracket \Box \varphi \rrbracket$ iff there exists $U \in \tau$ such that $x \in U$ and $U \setminus \{x\} \subseteq \llbracket \varphi \rrbracket$;
- $x \in \llbracket \gamma(\varphi, \psi) \rrbracket$ iff there exists a path π with $\pi(0) = x$, $\pi([0, 1]) \subseteq \llbracket \varphi \rrbracket$, and $\pi(1) \in \llbracket \psi \rrbracket$.

We write $\mathfrak{M} \models \varphi$ iff $\llbracket \varphi \rrbracket = X$.

Abbreviations. We use standard abbreviations $\top$, $\neg$, $\vee$, and $\wedge$. We also abbreviate $\Box \varphi := \varphi \wedge \Box \varphi$ and $\hat{\gamma}(\varphi, \psi) := \varphi \wedge \gamma(\varphi, \varphi \wedge \psi)$. Note that in a topological model:

- $\llbracket \Box \varphi \rrbracket$ is the interior of $\llbracket \varphi \rrbracket$. In particular, all $\llbracket \Box \varphi \rrbracket$ are open sets;
- $x \in \llbracket \hat{\gamma}(\varphi, \psi) \rrbracket$ iff there exists a path π with $\pi(0) = x$, $\pi([0, 1]) \subseteq \llbracket \varphi \rrbracket$, and $\pi(1) \in \llbracket \psi \rrbracket$.

3 Axiomatization

In this section, we present an $\mathcal{L}$ -formula defining the separation axiom T_1 . We also exhibit an axiomatic system **TLR** and verify its soundness for T_1 topologies.

Proposition 3.1. *A topological space $\langle X, \tau \rangle$ is T_1 if and only if $\gamma(p \wedge \Box \neg p, \top) \rightarrow p$ is valid in all topological models based on $\langle X, \tau \rangle$.*

Proof. Suppose $\langle X, \tau \rangle$ is not T_1 , i.e., there exist distinct $x, y \in X$ such that every open neighborhood of x contains y , and thus $\iota_{x,y}$ is continuous. Take $\llbracket p \rrbracket := \{y\}$. Now $y \in \llbracket \Box \neg p \rrbracket$, whence $\iota_{x,y}$ witnesses $x \in \llbracket \gamma(p \wedge \Box \neg p, \top) \rrbracket$. However, $x \notin \llbracket p \rrbracket$.

Conversely, suppose $\langle X, \tau \rangle$ is T_1 . Consider a valuation $\llbracket \cdot \rrbracket$ and a point x with $x \in \llbracket \gamma(p \wedge \Box \neg p, \top) \rrbracket$ witnessed by some path π . The set $\llbracket p \wedge \Box \neg p \rrbracket$ is discrete, whence its connected subset $\pi[(0, 1)]$ is some singleton $\{y\}$. Now, π is continuous with $\pi(0) = x$ and $\pi[(0, 1)] = y$, thus every open neighborhood of x contains y , hence $x = y \in \llbracket p \rrbracket$. $\square$

Definition 3.2. The logic **TLR** is defined by all axioms and rules of **K4** for $\Box$ together with the following:

- (A0) $\gamma(\varphi, \psi_1 \vee \psi_2) \rightarrow \gamma(\varphi, \psi_1) \vee \gamma(\varphi, \psi_2)$;
- (A1) $\gamma(\varphi, \varphi \wedge \gamma(\varphi, \psi)) \rightarrow \gamma(\varphi, \psi)$;
- (A2) $\varphi \rightarrow \gamma(\varphi, \varphi)$;
- (A3) $\neg \gamma(\perp, \top)$;
- (A4) $\gamma(\varphi, \neg \gamma(\varphi, \psi)) \rightarrow \neg \psi$;
- (A5) $\gamma(\varphi, \psi) \rightarrow \gamma(\varphi \wedge \gamma(\varphi, \psi), \psi)$;
- (A6) $\Box \psi \wedge \gamma(\varphi, \chi) \rightarrow \gamma(\varphi \wedge \Box \psi, (\varphi \wedge \neg \Box \psi) \vee \chi)$;
- (A7) $\widehat{\gamma}(\chi \wedge (\Box \varphi_1 \vee \Box \varphi_2) \wedge (\widehat{\gamma}(\chi \wedge \Box \varphi_1, \psi) \vee \widehat{\gamma}(\chi \wedge \Box \varphi_2, \psi) \rightarrow \psi), \psi) \rightarrow \psi$;
- (A8) $\gamma(\varphi \wedge \Box \neg \varphi, \top) \rightarrow \varphi$;
- (Mon) $\frac{\varphi \rightarrow \varphi' \quad \psi \rightarrow \psi'}{\gamma(\varphi, \psi) \rightarrow \gamma(\varphi', \psi')}$.

Now we verify soundness using some natural properties of paths in a topology; e.g., (A1) expresses that a concatenation of two paths is also a path, (A2) signifies the existence of constant paths, etc.

Proposition 3.3. *The logic **TLR** is sound for the class of T_1 topologies. Moreover, (A0)–(A7) and (Mon) are valid in all topologies.*

Proof. Consider a topological model $\mathfrak{M} = \langle X, \tau, \llbracket \cdot \rrbracket \rangle$ and a point $x \in X$.

(A0): If $x \in \llbracket \gamma(\varphi, \psi_1 \vee \psi_2) \rrbracket$ is witnessed by a path π , then $\pi(1) \in \llbracket \psi_1 \vee \psi_2 \rrbracket$, hence $\pi(1) \in \llbracket \psi_i \rrbracket$ for some $i \in \{1, 2\}$, whence π witnesses $x \in \llbracket \gamma(\varphi, \psi_i) \rrbracket$.

(A1): Suppose $x \in \llbracket \gamma(\varphi, \varphi \wedge \gamma(\varphi, \psi)) \rrbracket$ is witnessed by π_1 , and $\pi_1(1) \in \llbracket \gamma(\varphi, \psi) \rrbracket$ is witnessed by π_2 . Now the path π given by $\pi \upharpoonright [0, \frac{1}{2}] = \pi_1$ and $\pi \upharpoonright [\frac{1}{2}, 1] = \pi_2$ witnesses $x \in \llbracket \gamma(\varphi, \psi) \rrbracket$.

(A2): If $x \in \llbracket \varphi \rrbracket$, then $\iota_{x,x}$ witnesses $x \in \llbracket \gamma(\varphi, \varphi) \rrbracket$.

(A3) is valid since $\pi[(0, 1)] \neq \emptyset$ for every path π .

(A4): Suppose $x \in \llbracket \gamma(\varphi, \neg \gamma(\varphi, \psi)) \rrbracket$ is witnessed by a path π , and assume $x \in \llbracket \psi \rrbracket$. Then $\pi \upharpoonright [1, 0]$ witnesses that $\pi(1) \in \llbracket \gamma(\varphi, \psi) \rrbracket$, which is a contradiction.

(A5): Suppose $x \in \llbracket \gamma(\varphi, \psi) \rrbracket$ is witnessed by a path π . Then for every $t \in (0, 1)$, the path $\pi \upharpoonright [t, 1]$ witnesses $\pi(t) \in \llbracket \gamma(\varphi, \psi) \rrbracket$. Therefore, π witnesses $x \in \llbracket \gamma(\varphi \wedge \gamma(\varphi, \psi), \psi) \rrbracket$.

(A6): Suppose $x \in \llbracket \Box \psi \rrbracket$ and $x \in \llbracket \gamma(\varphi, \chi) \rrbracket$ is witnessed by a path π . If $\pi[(0, 1)] \subseteq \llbracket \Box \psi \rrbracket$, then π witnesses $x \in \llbracket \gamma(\varphi \wedge \Box \psi, \chi) \rrbracket$. Otherwise, denote by t the least element of the closed set $\pi^{-1}[\llbracket \neg \Box \psi \rrbracket]$. As $0 < t < 1$, the path $\pi \upharpoonright [0, t]$ witnesses that $x \in \llbracket \gamma(\varphi \wedge \Box \psi, \varphi \wedge \neg \Box \psi) \rrbracket$.

(A7): Suppose $x \in \llbracket \widehat{\gamma}(\chi \wedge (\Box \varphi_1 \vee \Box \varphi_2) \wedge (\widehat{\gamma}(\chi \wedge \Box \varphi_1, \psi) \vee \widehat{\gamma}(\chi \wedge \Box \varphi_2, \psi) \rightarrow \psi), \psi) \rrbracket$ is witnessed by a path π , and assume $x \notin \llbracket \psi \rrbracket$. Denote by t the supremum of $\pi^{-1}[\llbracket \neg \psi \rrbracket]$. Fix $i \in \{1, 2\}$ such that $\pi(t) \in \llbracket \Box \varphi_i \rrbracket$ and $\varepsilon > 0$ such that $[\max\{t - \varepsilon, 0\}, \min\{t + \varepsilon, 1\}] \subseteq \pi^{-1}[\llbracket \Box \varphi_i \rrbracket]$. Now $\pi \upharpoonright [t, \min\{t + \varepsilon, 1\}]$

witnesses that $\pi(t) \in \llbracket \widehat{\gamma}(\chi \wedge \Box \varphi_i, \psi) \rrbracket$. As $\pi[[0, 1]] \subseteq \llbracket \widehat{\gamma}(\chi \wedge \Box \varphi_i, \psi) \rightarrow \psi \rrbracket$, we conclude that $\pi(t) \in \llbracket \psi \rrbracket$. By the choice of t , there exists $t' \in \pi^{-1}[\llbracket \neg \psi \rrbracket] \cap [t - \varepsilon, t]$, thus $\pi \upharpoonright [t', t]$ witnesses $\pi(t') \in \llbracket \widehat{\gamma}(\chi \wedge \Box \varphi_i, \psi) \rrbracket$, hence $\pi(t') \in \llbracket \psi \rrbracket$, which is a contradiction.

(Mon): If $\llbracket \varphi \rrbracket \subseteq \llbracket \varphi' \rrbracket$ and $\llbracket \psi \rrbracket \subseteq \llbracket \psi' \rrbracket$, then $\llbracket \gamma(\varphi, \psi) \rrbracket \subseteq \llbracket \gamma(\varphi', \psi') \rrbracket$.

If $\langle X, \tau \rangle$ is T_1 , then the validity of (A8) follows from Proposition 3.1, and the validity of **K4** is well-known [1]. $\square$

4 Neighborhood-like Semantics

In this section, we define an alternative semantics for $\mathcal{L}$ (*flanked Kripke frames*) and a certain class of finite frames in it (*suitable frames*). In the following sections, we will establish our results by proving that **TLR** is complete for suitable frames (Section 5) and that suitable frames define the same logic as T_1 topologies (Section 6).

Definition 4.1. A *flanked Kripke frame* is a triple $\langle X, R, P \rangle$, where:

- $R \subseteq X \times X$ is transitive;
- $P \subseteq X \times \mathcal{P}(X) \times X$ is *monotone*, i.e., if $\langle x, S, y \rangle \in P$ and $S \subseteq S' \subseteq X$, then $\langle x, S', y \rangle \in P$.

A *flanked Kripke model* is a flanked Kripke frame equipped with a valuation $\llbracket \cdot \rrbracket : \text{Prop} \rightarrow \mathcal{P}(X)$, which is extended to $\mathcal{L}$ as follows:

- $\llbracket \perp \rrbracket = \emptyset$;
- $\llbracket \varphi \rightarrow \psi \rrbracket = (X \setminus \llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket$;
- $x \in \llbracket \Box \varphi \rrbracket$ iff $R(x) \subseteq \llbracket \varphi \rrbracket$;
- $x \in \llbracket \gamma(\varphi, \psi) \rrbracket$ iff there exists $y \in \llbracket \psi \rrbracket$ satisfying $\langle x, \llbracket \varphi \rrbracket, y \rangle \in P$.

Remark 4.2. This semantics combines a Kripke structure R for $\Box$ and a “flanked” structure P for γ . The former is natural to consider as **K4** is the logic of T_1 topologies [1]. To motivate the latter, note that the topological interpretation of γ may be rewritten as follows: $x \in \llbracket \gamma(\varphi, \psi) \rrbracket$ iff there exists $y \in \llbracket \psi \rrbracket$ with $\langle x, \llbracket \varphi \rrbracket, y \rangle \in P$, where P is a monotone set defined as follows:

$$P := \{ \langle x, S, y \rangle \in X \times \mathcal{P}(X) \times X : \text{there exists a path } \pi \text{ with } \pi(0) = x, \pi[(0, 1)] \subseteq S, \text{ and } \pi(1) = y \}.$$

This “flanked” semantics for γ is similar to the monotone neighborhood semantics used for non-normal modal logics such as **EM** (cf. [9]).

We now list the frame properties that roughly correspond to the axioms (A1)–(A8).

Definition 4.3. A flanked Kripke frame $\langle X, R, P \rangle$ is *suitable* if it is finite and has the following 8 properties:

(F1) if $\langle x, S, z \rangle \in P$, $\langle z, S, y \rangle \in P$, and $z \in S$, then $\langle x, S, y \rangle \in P$;

(F2) $\langle x, \{x\}, x \rangle \in P$ for all $x \in X$;

and for all $\langle x, S, y \rangle \in P$:

(F3) $S \neq \emptyset$;

(F4) $\langle y, S, x \rangle \in P$;

(F5) for each $z \in S$, either $\langle x, S \setminus \{z\}, y \rangle \in P$ or $\langle z, S, y \rangle \in P$;

- (F6) for every R -upset U with $x \in U$, there exists $z \in (S \setminus U) \cup \{y\}$ with $\langle x, S \cap U, z \rangle \in P$;
- (F7) if $x, y \in S$, then, for all R -upsets U_1 and U_2 with $S \subseteq U_1 \cup U_2$, there exists an $\langle S, U_1, U_2, x, y \rangle$ -sequence. Here an $\langle S, U_1, U_2, x, y \rangle$ -sequence is a sequence $x_0, \dots, x_k$ with $k \geq 0$, $x_0 = x$, and $x_k = y$ such that for each $i < k$ there exists $j_i \in \{1, 2\}$ satisfying $x_i, x_{i+1} \in S \cap U_{j_i}$ and $\langle x_i, S \cap U_{j_i}, x_{i+1} \rangle \in P$;
- (F8) if $S = \{y\}$, then either $x = y$ or yRy .

In the remainder of this section, we briefly motivate properties (F1)–(F8) and exhibit some illustrative examples.

Remark 4.4. Note that R -upsets form an Alexandroff topology on X (cf. [1]). Given a topological space, the set P from Remark 4.2 satisfies (F1)–(F5), as well as (F6) and (F7) if we substitute “open set” for “ R -upset.” One can verify this similarly to the proof of Proposition 3.3, invoking the same topological properties.

Remark 4.5. All axioms and rules of **K4**, axiom (A0), and rule (Mon) are valid in all flanked Kripke frames. One can check that (F1)–(F4) and (F6)–(F7) are precisely the frame properties corresponding to (A1)–(A4) and (A6)–(A7). However, (F5) is defined by (A5) only over finite frames, and (F8) is defined by (A8) only over frames satisfying (F3)–(F6). We employ (F5) and (F8) merely for convenience.

Example 4.6. Take $X = \{a, b, c\}$, $R = (\{b, c\} \times X) \cup \{\langle a, a \rangle\}$, and $\langle x, S, y \rangle \in P$ iff (i) $x = y \in S$, (ii) $S = X$, or (iii) $a \in S$ and $x, y \in \{a, b\}$. A direct inspection shows that $\langle X, R, P \rangle$ is suitable. If we omit $\langle a, a \rangle$ from R , the resulting frame refutes (F8) since $\langle b, \{a\}, a \rangle \in P$, but (F1)–(F7) are unaffected. If we omit (iii) from P , the resulting frame satisfies (F1)–(F5), (F7), and (F8), but not (F6).

Example 4.7. Take $X = \{a, b, c, d, e, f\}$, $R = \{a, b\}^2 \cup \{c, d\}^2 \cup (X \times \{e, f\})$, and $\langle x, S, y \rangle \in P$ iff (i) $x = y \in S$, (ii) $x, y \in \{a, c, e\}$ and $e \in S$, (iii) $x, y \in \{b, d, f\}$ and $f \in S$, or (iv) $S = X$. This $\langle X, R, P \rangle$ refutes (F7) since $\langle a, X, f \rangle \in P$ but there are no $\langle X, \{a, b, e, f\}, \{c, d, e, f\}, a, f \rangle$ -sequences. However, one can verify (F1)–(F6) and (F8).

Example 4.8. The axiomatic system obtained from **TLR** by dropping (A8) is sound but not complete for the class of T_D spaces. Indeed, one can verify that $\varphi := \gamma(p \wedge \Box \neg p \wedge \gamma(q, \top), \top) \rightarrow \gamma(q, \top)$ is valid in all topological models. Now take $X = \{a, b, c\}$, $R = \{\langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle\}$, and $\langle x, S, y \rangle \in P$ iff (i) $x = y \in S$, (ii) $x, y \in \{a, b\}$ and $b \in S$, (iii) $x, y \in \{b, c\}$ and $c \in S$, or (iv) $x, y \in \{a, b, c\}$ and $\{b, c\} \subseteq S$. As seen by inspection, $\langle X, R, P \rangle$ satisfies (F1)–(F7) and thus validates said axiomatic system. Taking $\llbracket p \rrbracket := \{b\}$ and $\llbracket q \rrbracket := \{c\}$, we obtain $a \notin \llbracket \varphi \rrbracket$, whence φ is not derivable.

5 Finite Model Property

In this section, we verify that **TLR** has the finite model property in the flanked Kripke semantics, i.e., that **TLR** is complete for the class of suitable frames (Proposition 5.10). First, we define the canonical flanked Kripke model, similarly to canonical models in monotone neighborhood semantics (cf. [9]). Then we construct a suitable filtration.

Definition 5.1. A set $\Gamma \subseteq \mathcal{L}$ is **TLR-consistent** if there exists no finite $\Gamma' \subseteq \Gamma$ with $\mathbf{TLR} \vdash \bigwedge \Gamma' \rightarrow \perp$. The *canonical flanked Kripke model* is $\mathfrak{M}_c = \langle X_c, R_c, P_c, \llbracket \cdot \rrbracket_c \rangle$, where:

- X_c is the class of all maximal **TLR-consistent** sets;
- $\Gamma R_c \Delta$ iff for all $\Box \varphi \in \Gamma$ we have $\varphi \in \Delta$;
- $\langle \Gamma, S, \Delta \rangle \in P_c$ iff for all $\varphi \in \bigcap S$ and $\psi \in \Delta$ we have $\gamma(\varphi, \psi) \in \Gamma$;

- $\Gamma \in \llbracket p \rrbracket_c$ iff $p \in \Gamma$.

Lemma 5.2. $\mathfrak{M}_c$ is indeed a flanked Kripke model.

Proof. Since $\mathbf{TLR} \vdash \Box \varphi \rightarrow \Box^2 \varphi$, the relation R_c is transitive. It should be clear that P_c is monotone. $\square$

As for the usual neighborhood semantics of [9], we have the following:

Lemma 5.3. For all $\Gamma \in X_c$ and $\chi \in \mathcal{L}$, we have $\Gamma \in \llbracket \chi \rrbracket_c$ iff $\chi \in \Gamma$.

Proof. By induction on χ . The base case and the step for $\chi = \varphi \rightarrow \psi$ or $\chi = \Box \varphi$ are as in the standard canonical Kripke model construction for modal logic (cf. [5]); we only consider $\chi = \gamma(\varphi, \psi)$. If $\Gamma \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$, then $\langle \Gamma, \llbracket \varphi \rrbracket_c, \Delta \rangle \in P_c$ for some $\Delta \in \llbracket \psi \rrbracket_c$, whence $\varphi \in \cap \llbracket \varphi \rrbracket_c$ and $\psi \in \Delta$ by the induction hypothesis, hence $\gamma(\varphi, \psi) \in \Gamma$ by definition of P_c .

Conversely, suppose $\gamma(\varphi, \psi) \in \Gamma$. Assume $\Delta_0 := \{\psi\} \cup \{\neg \psi' : \gamma(\varphi, \psi') \notin \Gamma\}$ is not $\mathbf{TLR}$ -consistent, witnessed by $\vdash \psi \rightarrow \bigvee_{i=1}^k \psi'_i$. If $k > 0$, it follows by (Mon) and (A0) that $\vdash \gamma(\varphi, \psi) \rightarrow \bigvee_{i=1}^k \gamma(\varphi, \psi'_i)$, which is a contradiction. If $k = 0$, i.e., $\vdash \neg \psi$, then $\vdash \psi \rightarrow \neg \gamma(\varphi, \top)$ and $\vdash \gamma(\varphi, \neg \gamma(\varphi, \top)) \rightarrow \perp$ by (A4), hence $\vdash \gamma(\varphi, \psi) \rightarrow \perp$ by (Mon), which is also a contradiction. Therefore, Δ_0 is $\mathbf{TLR}$ -consistent. Fix some $\Delta \in X_c$ extending Δ_0 . Now for every $\varphi' \in \cap \llbracket \varphi \rrbracket_c$ and $\psi' \in \Delta$ we have $\vdash \varphi \rightarrow \varphi'$ and $\gamma(\varphi, \psi') \in \Gamma$, thus $\gamma(\varphi', \psi') \in \Gamma$ by (Mon). Therefore, $\langle \Gamma, \llbracket \varphi \rrbracket_c, \Delta \rangle \in P_c$ witnesses $\Gamma \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$. $\square$

For the remainder of this section, fix a formula $\varphi_0 \in \mathcal{L}$. To establish the finite model property, we take the “closure under concatenations” (i.e., under (F1)) of the “minimal φ_0 -filtration.” We first verify that it admits a “filtration lemma” (Lemma 5.7). Then we show that it is a suitable frame, by establishing that the “minimal φ_0 -filtration” itself satisfies (F2)–(F8) (Lemma 5.8) and that said “closure under concatenations” preserves (F2)–(F8) (Lemma 5.9).

Remark 5.4. This line of reasoning resembles e.g. a classical proof of the Kripke finite model property of **S4** by showing that the transitive closure of the minimal filtration is a filtration, that the minimal filtration itself is reflexive, and that transitive closure preserves reflexivity (cf. [5]).

Definition 5.5. Set $\mathfrak{M}^+ := \langle X_c / \sim, R, P^+, \llbracket \cdot \rrbracket \rangle$, where:

- $\Gamma \sim \Delta$ iff $\Gamma \in \llbracket \varphi \rrbracket_c \iff \Delta \in \llbracket \varphi \rrbracket_c$ for all $\varphi \in \text{Sub } \varphi_0$. Denote the projection $X_c \rightarrow X_c / \sim$ by g ;
- R is the transitive closure of $\{\langle g(x), g(y) \rangle : x R_c y\}$;
- $P := \{\langle g(\Gamma), g[S], g(\Delta) \rangle : \langle \Gamma, S, \Delta \rangle \in P_c\}$;
- P^+ is the smallest set extending P and satisfying (F1);
- $\llbracket p \rrbracket := g[\llbracket p \rrbracket_c]$ for all $p \in \text{Prop}$.

Remark 5.6. Observe that $\langle x, S, y \rangle \in P^+$ iff there exists a sequence $x_0, \dots, x_k$ with $k \geq 1$, $x_0 = x$, $x_k = y$, and $x_1, \dots, x_{k-1} \in S$ such that $\langle x_i, S, x_{i+1} \rangle \in P$ for each $i = 0, \dots, k-1$.

Lemma 5.7. For every $\chi \in \text{Sub } \varphi_0$ we have $\llbracket \chi \rrbracket_c = g^{-1}[\llbracket \chi \rrbracket]$.

Proof. By induction on χ . The base case and the step for $\chi = \varphi \rightarrow \psi$ or $\chi = \Box \varphi$ are as in the standard filtration construction for **K4** (cf. [5]); we only consider $\chi = \gamma(\varphi, \psi)$. If $\Gamma \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$ is witnessed by $\langle \Gamma, \llbracket \varphi \rrbracket_c, \Delta \rangle \in P_c$, then $\langle g(\Gamma), \llbracket \varphi \rrbracket, g(\Delta) \rangle \in P \subseteq P^+$ witnesses $g(\Gamma) \in \llbracket \gamma(\varphi, \psi) \rrbracket$.

Conversely, suppose $g(\Gamma) \in \llbracket \gamma(\varphi, \psi) \rrbracket$ is witnessed by $\langle g(\Gamma), \llbracket \varphi \rrbracket, y \rangle \in P^+$, and fix a corresponding sequence $x_0, \dots, x_k$ as in Remark 5.6. For each $i < k$, fix also $\Gamma_i \in g^{-1}(x_i)$ and $\Gamma'_{i+1} \in g^{-1}(x_{i+1})$ with $\langle \Gamma_i, \llbracket \varphi \rrbracket_c, \Gamma'_{i+1} \rangle \in P_c$. Let us show by descending induction on $i = k-1, \dots, 0$ that $\Gamma_i \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$. For $i = k-1$, it suffices to note that $\langle \Gamma_{k-1}, \llbracket \varphi \rrbracket_c, \Gamma'_k \rangle \in P_c$ and $\Gamma'_k \in g^{-1}(y) \subseteq g^{-1}[\llbracket \psi \rrbracket] = \llbracket \psi \rrbracket_c$. Now

consider $i < k - 1$ and suppose $\Gamma_{i+1} \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$. As $\Gamma_{i+1}, \Gamma'_{i+1} \in g^{-1}(x_{i+1})$, we have $\Gamma_{i+1} \sim \Gamma'_{i+1}$, thus $\Gamma'_{i+1} \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$, hence $\Gamma_i \in \llbracket \gamma(\varphi, \varphi \wedge \gamma(\varphi, \psi)) \rrbracket_c$. By Lemma 5.3 we have $\mathfrak{M}_c \models \mathbf{TLR}$, thus $\llbracket \gamma(\varphi, \varphi \wedge \gamma(\varphi, \psi)) \rrbracket_c \subseteq \llbracket \gamma(\varphi, \psi) \rrbracket_c$ by (A1), hence $\Gamma_i \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$. We conclude that $\Gamma_0 \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$. Since $\Gamma, \Gamma_0 \in g^{-1}(x_0)$, we have $\Gamma \sim \Gamma_0$, hence $\Gamma \in \llbracket \gamma(\varphi, \psi) \rrbracket_c$. $\square$

Lemma 5.8. *The flanked Kripke frame $\langle X_c/\sim, R, P \rangle$ satisfies (F2)–(F8).*

Proof. First note that:

- All instances of (A2)–(A8) are valid in $\mathfrak{M}_c$. Indeed, $\mathfrak{M}_c \models \mathbf{TLR}$ by Lemma 5.3.
- For every $T \subseteq X_c/\sim$ there exists a formula $\varphi \in \mathcal{L}$ with $g^{-1}[T] = \llbracket \varphi \rrbracket_c$. Moreover, if T is an R -upset, then $\llbracket \varphi \rrbracket_c = \llbracket \Box \varphi \rrbracket_c$. Indeed, an appropriate φ may be built as a Boolean combination of $\text{Sub } \varphi_0$. If T is an R -upset and $\Gamma \in \llbracket \varphi \rrbracket_c$, then for every $\Delta \in R_c(\Gamma)$ we have $g(\Gamma) R g(\Delta)$ and $g(\Gamma) \in T$, thus $g(\Delta) \in T$, whence $\Delta \in \llbracket \varphi \rrbracket_c$, hence $\Gamma \in \llbracket \Box \varphi \rrbracket_c$.

Now we check each of the requisite properties.

(F2): Consider $x \in X_c/\sim$. Fix some $\Gamma \in g^{-1}(x)$ and φ such that $\llbracket \varphi \rrbracket_c = g^{-1}(x)$. We have $\Gamma \in \llbracket \varphi \rrbracket_c \subseteq \llbracket \gamma(\varphi, \varphi) \rrbracket_c$ by (A2), whence $\langle \Gamma, \llbracket \varphi \rrbracket_c, \Gamma' \rangle \in P_c$ for some $\Gamma' \in \llbracket \varphi \rrbracket_c$, hence $\langle x, \{x\}, x \rangle \in P$.

For (F3)–(F8), consider $\langle x, S, y \rangle \in P$, and fix $\Gamma \in g^{-1}(x)$ and $\Delta \in g^{-1}(y)$ such that $\langle \Gamma, g^{-1}[S], \Delta \rangle \in P_c$.

(F3): Note that $\Gamma \notin \llbracket \gamma(\perp, \top) \rrbracket_c$ by (A3), thus $g^{-1}[S] \neq \emptyset$, hence $S \neq \emptyset$.

(F4): Fix φ and ψ such that $\llbracket \varphi \rrbracket_c = g^{-1}[S]$ and $\llbracket \psi \rrbracket_c = g^{-1}(y)$. We have $\Gamma \in \llbracket \psi \rrbracket_c$, thus $\Gamma \notin \llbracket \gamma(\varphi, \neg \gamma(\varphi, \psi)) \rrbracket_c$ by (A4), whence $\Delta \notin \llbracket \neg \gamma(\varphi, \psi) \rrbracket_c$, hence $\langle \Delta, \llbracket \varphi \rrbracket_c, \Gamma' \rangle \in P_c$ for some $\Gamma' \in \llbracket \psi \rrbracket_c$, thus $\langle y, S, x \rangle \in P$.

(F5): Consider $z \in S$ and fix φ, ψ with $\llbracket \varphi \rrbracket_c = g^{-1}[S]$ and $\llbracket \psi \rrbracket_c = g^{-1}(y)$. If there exists $\Theta \in g^{-1}(z) \cap \llbracket \gamma(\varphi, \psi) \rrbracket_c$, then $\langle \Theta, \llbracket \varphi \rrbracket_c, \Delta' \rangle \in P_c$ for some $\Delta' \in \llbracket \psi \rrbracket_c$, whence $\langle z, S, y \rangle \in P$. Now suppose $g^{-1}(z) \cap \llbracket \gamma(\varphi, \psi) \rrbracket_c = \emptyset$. We have $\Gamma \in \llbracket \gamma(\varphi, \psi) \rrbracket_c \subseteq \llbracket \gamma(\varphi \wedge \gamma(\varphi, \psi), \psi) \rrbracket_c$ by (A5), whence $\langle \Gamma, g^{-1}[S] \cap \llbracket \gamma(\varphi, \psi) \rrbracket_c, \Delta' \rangle \in P_c$ for some $\Delta' \in g^{-1}(y)$, thus $\langle \Gamma, g^{-1}[S \setminus \{z\}], \Delta' \rangle \in P_c$, hence $\langle x, S \setminus \{z\}, y \rangle \in P$.

(F6): Consider an R -upset U with $x \in U$. Fix φ, ψ , and χ such that $\llbracket \varphi \rrbracket_c = g^{-1}[S]$, $\llbracket \Box \psi \rrbracket_c = \llbracket \psi \rrbracket_c = g^{-1}[U]$, and $\llbracket \chi \rrbracket_c = g^{-1}(y)$. Now $\Gamma \in \llbracket \Box \psi \wedge \gamma(\varphi, \chi) \rrbracket_c \subseteq \llbracket \gamma(\varphi \wedge \Box \psi, (\varphi \wedge \neg \Box \psi) \vee \chi) \rrbracket_c$ by (A6), whence $\langle \Gamma, \llbracket \varphi \wedge \Box \psi \rrbracket_c, \Theta \rangle \in P_c$ for some $\Theta \in \llbracket (\varphi \wedge \neg \Box \psi) \vee \chi \rrbracket_c$, hence $\langle x, S \cap U, g(\Theta) \rangle \in P$ and $g(\Theta) \in (S \setminus U) \cup \{y\}$.

(F7): Consider R -upsets U_1 and U_2 with $x, y \in S \subseteq U_1 \cup U_2$. Denote by T the set of all $s \in S$ for which there exists an $\langle S, U_1, U_2, s, y \rangle$ -sequence. Fix $\varphi_1, \varphi_2, \psi$, and χ such that $\llbracket \Box \varphi_i \rrbracket_c = \llbracket \varphi_i \rrbracket_c = g^{-1}[U_i]$, $\llbracket \psi \rrbracket_c = g^{-1}[T]$, and $\llbracket \chi \rrbracket_c = g^{-1}[S]$. It suffices to show that $\langle \Gamma, g^{-1}[S], \Delta \rangle \in P_c$ witnesses $\Gamma \in \llbracket \hat{\gamma}(\chi \wedge (\Box \varphi_1 \vee \Box \varphi_2) \wedge (\hat{\gamma}(\chi \wedge \Box \varphi_1, \psi) \vee \hat{\gamma}(\chi \wedge \Box \varphi_2, \psi) \rightarrow \psi), \psi) \rrbracket_c$, for then $\Gamma \in \llbracket \psi \rrbracket_c$ by (A7) and thus $x \in T$. Since $y \in T$, we have $\Delta \in \llbracket \psi \rrbracket_c$. It is immediate that $\Gamma, \Delta \in g^{-1}[S] = \llbracket \chi \wedge (\Box \varphi_1 \vee \Box \varphi_2) \rrbracket_c$. Now it remains to verify that $\llbracket \hat{\gamma}(\chi \wedge \Box \varphi_i, \psi) \rightarrow \psi \rrbracket_c = X_c$ for $i = 1, 2$. Suppose $\Theta \in \hat{\gamma}(\chi \wedge \Box \varphi_i, \psi)$. Then $\Theta \in \llbracket \chi \wedge \Box \varphi_i \rrbracket_c$ and $\langle \Theta, \llbracket \chi \wedge \Box \varphi_i \rrbracket_c, \Theta' \rangle \in P_c$ for some $\Theta' \in \llbracket \chi \wedge \Box \varphi_i \wedge \psi \rrbracket_c$, thus $g(\Theta) \in S \cap U_i$ and $g(\Theta') \in S \cap U_i \cap T$. Now there exists an $\langle S, U_1, U_2, g(\Theta'), y \rangle$ -sequence, and appending $g(\Theta)$ to its beginning we obtain an $\langle S, U_1, U_2, g(\Theta), y \rangle$ -sequence, whence $g(\Theta) \in T$, hence $\Theta \in \llbracket \psi \rrbracket_c$.

(F8): Suppose $S = \{y\}$. Fix φ with $\llbracket \varphi \rrbracket_c = g^{-1}(y)$. If $\llbracket \varphi \rrbracket_c = \llbracket \varphi \wedge \Box \neg \varphi \rrbracket_c$, then $\langle \Gamma, g^{-1}(y), \Delta \rangle \in P_c$ witnesses $\Gamma \in \llbracket \gamma(\varphi \wedge \Box \neg \varphi, \top) \rrbracket_c$, whence $\Gamma \in \llbracket \varphi \rrbracket_c$ by (A8), hence $x = g(\Gamma) = y$. Otherwise, fix $\Theta \in \llbracket \varphi \wedge \neg \Box \neg \varphi \rrbracket_c$. The set $\{\varphi\} \cup \{\psi : \Box \psi \in \Theta\}$ is $\mathbf{TLR}$ -consistent, for if $\vdash \wedge \psi_i \rightarrow \neg \varphi$ then $\vdash \wedge \Box \psi_i \rightarrow \Box \neg \varphi$. Fix some $\Theta' \in \llbracket \varphi \rrbracket_c \cap \bigcap_{\Box \psi \in \Theta} \llbracket \psi \rrbracket_c$. Now $\Theta R_c \Theta'$, whence $y R y$. $\square$

Lemma 5.9. *The frame $\langle X_c/\sim, R, P^+ \rangle$ is suitable.*

Proof. The frame is clearly finite and satisfies (F1). Property (F2) holds for P^+ since it holds for P by Lemma 5.8 and $P \subseteq P^+$. By Lemma 5.8, P also satisfies (F3)–(F8), hence P^+ satisfies (F3)–(F8) for all triples $\langle x, S, y \rangle$ that are in P . Now it suffices to check that the set of triples for which P^+ satisfies (F3)–(F8) is closed under (F1), i.e., that if P^+ satisfies (F3)–(F8) for some $\langle x, S, w \rangle \in P^+$ and $\langle w, S, y \rangle \in P^+$, with $w \in S$, then P^+ also satisfies (F3)–(F8) for $\langle x, S, y \rangle$. We do this for each of these properties separately.

(F3): We have $w \in S$, thus $S \neq \emptyset$.

(F4): Since (F4) holds for $\langle x, S, w \rangle$ and $\langle w, S, y \rangle$, we have $\langle y, S, w \rangle, \langle w, S, x \rangle \in P^+$, hence $\langle y, S, x \rangle \in P^+$ by (F1).

(F5): Consider $z \in S$. If $z = w$ or $\langle z, S, y \rangle \in P^+$, there is nothing to prove. If $\langle z, S, w \rangle \in P^+$, then $\langle z, S, y \rangle \in P^+$ by (F1) since $\langle w, S, y \rangle \in P^+$. In all other cases, $\langle x, S \setminus \{z\}, w \rangle \in P^+$ by (F5) for $\langle x, S, w \rangle$, $\langle w, S \setminus \{z\}, y \rangle \in P^+$ by (F5) for $\langle w, S, y \rangle$, and $w \in S \setminus \{z\}$, hence $\langle x, S \setminus \{z\}, y \rangle \in P^+$ by (F1).

(F6): Consider an R -upset U with $x \in U$. By (F6) applied to $\langle x, S, w \rangle$ we have $\langle x, S \cap U, z \rangle \in P^+$ for some $z \in (S \setminus U) \cup \{w\}$. If $z \in S \setminus U$, there is nothing to prove. Otherwise, $z = w \in U$. Now, (F6) applied to $\langle w, S, y \rangle$ yields $\langle w, S \cap U, z' \rangle \in P^+$ for some $z' \in (S \setminus U) \cup \{y\}$. Since $\langle x, S \cap U, w \rangle \in P^+$ and $w \in U$, we obtain $\langle x, S \cap U, z' \rangle \in P$ by (F1).

(F7): Consider R -upsets U_1 and U_2 with $x, y \in S \subseteq U_1 \cup U_2$. By (F7) for $\langle x, S, w \rangle$ and $\langle w, S, y \rangle$ there exist an $\langle S, U_1, U_2, x, w \rangle$ -sequence and an $\langle S, U_1, U_2, w, y \rangle$ -sequence; concatenating them we obtain an $\langle S, U_1, U_2, x, y \rangle$ -sequence.

(F8): If $S \neq \{y\}$, there is nothing to prove. If $S = \{y\}$, then $w \in S$ implies $y = w$, hence (F8) for $\langle x, S, w \rangle$ implies (F8) for $\langle x, S, y \rangle$. $\square$

Summarizing this section, we have:

Proposition 5.10. *If $\mathbf{TLR} \not\models \varphi_0$, then φ_0 is refuted in a suitable frame of size at most $2^{|\text{Sub } \varphi_0|}$.*

Proof. The flanked Kripke frame $\langle X_c / \sim, R, P^+ \rangle$ is suitable by Lemma 5.9. Clearly, $|X_c / \sim| \leq 2^{|\text{Sub } \varphi_0|}$. If $\mathbf{TLR} \not\models \varphi_0$, then $\mathfrak{M}_c \not\models \varphi_0$ by Lemma 5.3, whence $\mathfrak{M}^+ \not\models \varphi_0$ by Lemma 5.7. $\square$

6 Tree Unraveling

In this section, we show that every formula refuted in a suitable frame is also refuted in a metric space. First, we introduce *path-morphisms* that preserve the validity of $\mathcal{L}$ -formulas (cf. [3]). Then, given a suitable frame X , it will suffice to produce a path-morphism $f : T \rightarrow X$ from some metric space T .

Definition 6.1. For a topological space $\langle T, \tau \rangle$ and a flanked Kripke frame $\langle X, R, P \rangle$, a map $f : T \rightarrow X$ is a *path-morphism* if:

($\Diamond$ -forth) every $a \in T$ has a neighborhood $U \in \tau$ with $f[U \setminus \{a\}] \subseteq R(f(a))$;

($\Diamond$ -back) if $f(a)Ry$ and $a \in U \in \tau$, then there exists $b \in U \setminus \{a\}$ with $f(b) = y$;

(γ -forth) if δ is a path in $\langle T, \tau \rangle$, then $\langle f(\delta(0)), f[\delta[(0, 1)]] \rangle \in P$; and

(γ -back) if $\langle f(a), S, y \rangle \in P$, then there exists a path δ in $\langle T, \tau \rangle$ with $\delta(0) = a$, $f[\delta[(0, 1)]] \subseteq S$, and $f(\delta(1)) = y$.

Lemma 6.2. *Let $f : \langle T, \tau \rangle \rightarrow \langle X, R, P \rangle$ be a path-morphism and $\llbracket \cdot \rrbracket : X \rightarrow \text{Prop}$. Put $\llbracket p \rrbracket_T := f^{-1}[\llbracket p \rrbracket]$ for all $p \in \text{Prop}$. Then $a \in \llbracket \chi \rrbracket_T \iff f(a) \in \llbracket \chi \rrbracket$ for all $\chi \in \mathcal{L}$ and $a \in T$.*

Proof. By induction on χ . The step for $\chi = \varphi \rightarrow \psi$ is trivial.

Consider $\chi = \Box\varphi$. If $f(a) \in \llbracket \Box\varphi \rrbracket$, then $R(f(a)) \subseteq \llbracket \varphi \rrbracket$, whence $f^{-1}[R(f(a))] \subseteq \llbracket \varphi \rrbracket_T$ by the induction hypothesis for φ , hence the set U given by $(\Diamond\text{-forth})$ witnesses $a \in \llbracket \Box\varphi \rrbracket_T$. Conversely, suppose $a \in \llbracket \Box\varphi \rrbracket_T$, i.e., there exists $U \in \tau$ with $a \in U$ and $U \setminus \{a\} \subseteq \llbracket \varphi \rrbracket_T$. By $(\Diamond\text{-back})$, for every $y \in R(f(a))$ there exists $b \in U \setminus \{a\} \subseteq \llbracket \varphi \rrbracket_T$ with $f(b) = y$, whence $y \in \llbracket \varphi \rrbracket$ by the induction hypothesis for φ . Therefore, $R(f(a)) \subseteq \llbracket \varphi \rrbracket$, i.e., $f(a) \in \llbracket \Box\varphi \rrbracket$.

Consider $\chi = \gamma(\varphi, \psi)$. If $a \in \llbracket \gamma(\varphi, \psi) \rrbracket_T$ is witnessed by a path δ , then $\langle f(a), f[\delta[(0, 1)]] \rangle \in P$ by $(\gamma\text{-forth})$, hence $f(a) \in \llbracket \gamma(\varphi, \psi) \rrbracket$. Conversely, if $f(a) \in \llbracket \gamma(\varphi, \psi) \rrbracket$ is witnessed by $\langle f(a), \llbracket \varphi \rrbracket, y \rangle \in P$, then by $(\gamma\text{-back})$ there exists a path δ in $\langle T, \tau \rangle$ witnessing $a \in \llbracket \gamma(\varphi, \psi) \rrbracket_T$. $\square$

Example 6.3. Take $\langle X, R, P \rangle$ from Example 4.6. Consider the Cantor set $\mathcal{C} \subseteq [0, 1]$ and present $[0, 1] \setminus \mathcal{C}$ as a union of pairwise-disjoint intervals $\bigcup_{i \in \mathbb{Q}} (\alpha_i, \beta_i)$. Now consider $\pi : [0, 1] \rightarrow X$ given by $\pi[[0, 1] \setminus \bigcup_{i \in \mathbb{Q}} [\alpha_i, \beta_i]] = \{c\}$ and $\pi \upharpoonright [\alpha_i, \beta_i] = \pi'$ for every i , where $\pi'(0) = \pi'(1) := b$ and $\pi'((0, 1)) := \{a\}$. A direct inspection shows that π is a path-morphism with respect to the usual topology on $[0, 1]$. By Lemma 6.2, it follows that all $\mathcal{L}$ -formulas refuted in $\langle X, R, P \rangle$ are also refuted in $[0, 1]$.

In general, given a suitable frame X , we will construct a metric tree-like structure T whose “edges” are isomorphic copies of $[0, 1]$, and a path-morphism $f : T \rightarrow X$ will be defined in terms of its restrictions to individual “edges.” For f to satisfy $(\Diamond\text{-forth})$ and $(\gamma\text{-forth})$, it is necessary that said restrictions themselves satisfy $(\Diamond\text{-forth})$ and $(\gamma\text{-forth})$, i.e., that they are maps of the following form:

Definition 6.4. A *good path* in a suitable frame $\langle X, R, P \rangle$ is a map $[0, 1] \rightarrow X$ satisfying $(\Diamond\text{-forth})$ and:

(G) for all $0 \leq t < u \leq 1$ we have $\langle \pi(t), \pi[(t, u)], \pi(u) \rangle \in P$.

Remark 6.5. It easily follows from (F1) and (F4) that (G) is equivalent to $(\gamma\text{-forth})$. A map $[0, 1] \rightarrow X$ satisfies $(\Diamond\text{-forth})$ iff preimages of R -upsets are open and preimages of irreflexive singletons are discrete. In particular, $\iota_{x,x}$ is a good path iff $x R x$.

Definition 6.6. A map $\pi : [0, 1] \rightarrow X$ *matches* a triple $\langle x, S, y \rangle \in X \times \mathcal{P}(X) \times X$ if $\pi(0) = x$, $\pi[(0, 1)] \subseteq S$, and $\pi(1) = y$.

We claim that each “non-constant” triple in P is matched either by a good path or by a constant (Lemma 6.9). This will provide us a sufficient supply of good paths to define a path-morphism $f : T \rightarrow X$ (Proposition 6.11).

Example 6.7. Each triple in P from Examples 4.6 and 6.3 is matched by one of the following good paths: $\iota_{a,a}$, $\iota_{b,b}$, $\iota_{c,c}$, $\iota_{b,a}$, $\iota_{b,a} \upharpoonright [1, 0]$, π' , the map given by $\delta(0) := c$ and $\delta(t) := \pi(t)$ for $t > 0$, the map given by $\delta'(1) := c$ and $\delta'(t) := \pi(t)$ for $t < 1$, the concatenation of $\iota_{a,b}$ with δ' , or the concatenation of δ with $\iota_{a,b} \upharpoonright [1, 0]$.

We will construct the requisite good paths from “simpler” good paths using several specific operations (among them are concatenation, reversion, the way π is constructed from π' in Example 6.3, etc.). The following lemma, informally speaking, states that all elements of P can be “constructed” from “simpler” elements of P using the corresponding set of operations on triples.

Lemma 6.8. Let $\langle X, R, P \rangle$ be a suitable frame. Then there exists a strict partial order $\triangleleft$ on P such that for every $\langle x, S, y \rangle \in P$ one of the following holds:

(C1) $x = y \in S$;

(C2) $\langle y, S, x \rangle \triangleleft \langle x, S, y \rangle$;

(C3) there is a sequence $x_0, \dots, x_k$ with $k \geq 1$, $x = x_0$, $x_k = y$, and $x_1, \dots, x_{k-1} \in S$, such that for each $i < k$ there exists $S'_i \subseteq S$ with $P \ni \langle x_i, S'_i, x_{i+1} \rangle \triangleleft \langle x, S, y \rangle$;

(C4) we have:

(C4a) $x \neq y \in S \subseteq R(x)$;

(C4b) $P \ni \langle z, S, w \rangle \triangleleft \langle x, S, y \rangle$ for all $z, w \in S$; and

(C4c) $\langle x, S, z \rangle \in P$ for all $z \in S$;

(C5) there exists $U \subseteq X$ such that:

(C5a) $x, y \in S \setminus U$;

(C5b) $(S \setminus U) \times S \subseteq R$;

(C5c) for every $z \in S \cap U$ there exists $w_z \in S \setminus U$ with $P \ni \langle z, S \cap U, w_z \rangle \triangleleft \langle x, S, y \rangle$; and

(C5d) $\langle z, S, w \rangle \in P$ for all $z, w \in S$.

Proof. For $z \in X$, put $\widehat{R}(z) := \{z\} \cup R(z)$; note that $\widehat{R}(z)$ is an R -upset. Put $\langle x, S, y \rangle \triangleleft \langle x', S', y' \rangle$ iff one of the following holds:

(L1) $S \subsetneq S'$;

(L2) $S = S'$ and $y \in S \not\supseteq y'$;

(L3) $S = S'$, $x \in S \not\supseteq x'$, and either $y, y' \in S$ or $y, y' \notin S$;

(L4) $x, y, x', y' \in S = S'$ and $\widehat{R}(y) \supsetneq \widehat{R}(y')$;

(L5) $x, y, x', y' \in S = S'$, $\widehat{R}(y) = \widehat{R}(y')$, and $\widehat{R}(x) \supsetneq \widehat{R}(x')$.

Observe that $\triangleleft$ is a strict partial order. Now consider $\langle x, S, y \rangle \in P$ and suppose that (C1), (C2), and (C3) do not hold. Then:

- (i) For all $S' \subsetneq S$ we have $\langle x, S', y \rangle \notin P$. Indeed, otherwise (C3) is true with $k = 1$.
- (ii) For every $z \in S$ we have $\langle x, S, z \rangle \in P$ and $\langle z, S, y \rangle \in P$. Indeed, $\langle x, S \setminus \{z\}, y \rangle \notin P$ by (i); it remains to apply (F5) together with (F4).
- (iii) If $y \in S$, then $\langle z, S, w \rangle \in P$ for all $z, w \in S$. Indeed, $\langle z, S, y \rangle, \langle y, S, w \rangle \in P$ by (ii) and (F4), hence $\langle z, S, w \rangle \in P$ by (F1).
- (iv) If an R -upset U satisfies $S \not\subseteq U$ and $x \in U$, then there exists $z \in S \setminus U$ with $\langle z, S, y \rangle \not\triangleleft \langle x, S, y \rangle$. Indeed, by (F6) we have $\langle x, S \cap U, z \rangle \in P$ for some $z \in (S \setminus U) \cup \{y\}$. By (i), $z \neq y$. Now we have $z \in S$, $P \ni \langle x, S \cap U, z \rangle \triangleleft \langle x, S, y \rangle$ by (L1), and $\langle z, S, y \rangle \in P$ by (ii). Therefore, $\langle z, S, y \rangle \not\triangleleft \langle x, S, y \rangle$, for otherwise (C3) holds with $k = 2$ and $x_1 = z$.

Now, to show that either (C4) or (C5) holds for $\langle x, S, y \rangle$, consider two cases.

Case 1: $x \notin S$. Let us establish (C4). By (F3) we have $S \neq \emptyset$; fix some $z' \in S$. If $y \notin S$, we have $P \ni \langle x, S, z' \rangle \triangleleft \langle x, S, y \rangle$ by (ii) and (L2), and $P \ni \langle z', S, y \rangle \triangleleft \langle x, S, y \rangle$ by (ii) and (L3), hence (C3) holds with $k = 2$. Therefore, $y \in S$.

By (L3) we have $\langle z, S, y \rangle \triangleleft \langle x, S, y \rangle$ for all $z \in S$, thus by (iv) applied to $U := \widehat{R}(x)$ we obtain $S \subseteq \widehat{R}(x)$, hence $S \subseteq R(x)$, i.e., (C4a) is true. Condition (C4b) follows from (iii) and (L3), and (C4c) holds by (ii).

Case 2: $x \in S$. If $y \notin S$, then $\langle y, S, x \rangle \triangleleft \langle x, S, y \rangle$ by (L2), whence (C2) holds. Therefore, $y \in S$.

Fix an R -upset U which is maximal with respect to $U \subsetneq \widehat{R}[S]$. (Note that U may be empty.) Since $\widehat{R}[S] \not\subseteq U$, we have $S \not\subseteq U$. Let us show that (C5) is true for U .

First we claim that every $z \in S \setminus U$ satisfies $\widehat{R}(z) = \widehat{R}[S]$. Indeed, by maximality of U we have $U \cup \widehat{R}(z) = \widehat{R}[S]$. Using (F7), fix an $\langle S, U, \widehat{R}(z), x, y \rangle$ -sequence $x_0, \dots, x_k$. Since (C1) does not hold, we

have $x \neq y$ and thus $k > 0$. Since (C3) does not hold, there exists $i < k$ with $\langle x_i, S \cap U, x_{i+1} \rangle \not\triangleleft \langle x, S, y \rangle$ or $\langle x_i, S \cap \widehat{R}(z), x_{i+1} \rangle \not\triangleleft \langle x, S, y \rangle$. As (L1) does not apply, it follows that $S \cap U = S$ or $S \cap \widehat{R}(z) = S$. Since $S \not\subseteq U$, we conclude that $S \subseteq \widehat{R}(z)$ and thus $\widehat{R}(z) = \widehat{R}[S]$.

To verify (C5a), first assume $x \in U$. Since $S \not\subseteq U$, by (iv) there exists $z \in S \setminus U$ with $\langle z, S, y \rangle \not\triangleleft \langle x, S, y \rangle$. But $x, y, z \in S$ and $\widehat{R}(x) \subseteq U \subsetneq \widehat{R}[S] = \widehat{R}(z)$, thus $\langle z, S, y \rangle \triangleleft \langle x, S, y \rangle$ by (L5), which is a contradiction. Now assume $x \in S \setminus U$ and $y \in U$. Then $\widehat{R}(y) \subsetneq \widehat{R}[S] = \widehat{R}(x)$, thus $\langle y, S, x \rangle \triangleleft \langle x, S, y \rangle$ by (L4), hence (C2) holds. Therefore, $x, y \in S \setminus U$.

To verify (C5b), recall that $S \subseteq \widehat{R}[S] = \widehat{R}(z)$ for all $z \in S \setminus U$. It remains to note that $\widehat{R}$ is total on $S \setminus U$ and $x, y \in S \setminus U$ are distinct, thus all $z \in S \setminus U$ are R -reflexive.

To check (C5c), consider $z \in S \cap U$. By (ii) we have $\langle z, S, y \rangle \in P$, thus by (F6) there exists $w_z \in S \setminus U \cup \{y\} = S \setminus U$ satisfying $\langle z, S \cap U, w_z \rangle \in P$.

Condition (C5d) follows from (iii). $\square$

Lemma 6.9. *Let $\langle X, R, P \rangle$ be a suitable frame. Then, for every triple $\langle x, S, y \rangle \in P$, either $x = y \in S$ or there exists a good path matching $\langle x, S, y \rangle$.*

Proof. Fix a strict partial order $\triangleleft$ as in Lemma 6.8. Since X and P are finite, this $\triangleleft$ is well-founded. The proof proceeds by $\triangleleft$ -induction on P .

Case (C1). Since $x = y \in S$, there is nothing to prove.

Case (C2). Since $\langle y, S, x \rangle \triangleleft \langle x, S, y \rangle$, we have $x \neq y$. By the induction hypothesis, some good path π matches $\langle y, S, x \rangle$. Now, $\pi \upharpoonright [1, 0]$ matches $\langle x, S, y \rangle$; it is a good path by (F4).

Case (C3). Consider a requisite sequence $x_0, \dots, x_k$. Without loss of generality, either $x_0 \neq x_1 \neq \dots \neq x_k$ or $k = 1$, for we can omit duplicate adjacent x_i 's. If $k = 1$ and $x = y \in S'_0$, then $x = y \in S$. Otherwise, for each $i < k$ the induction hypothesis yields a good path π_i matching $\langle x_i, S'_i, x_{i+1} \rangle$. Now the map $\pi : [0, 1] \rightarrow X$ given by $\pi \upharpoonright [\frac{i}{n}, \frac{i+1}{n}] = \pi_i$ matches $\langle x, S, y \rangle$. Since each π_i satisfies $(\diamond\text{-forth})$, π also satisfies $(\diamond\text{-forth})$. Since each π_i satisfies (G), π also satisfies (G) by (F1).

Case (C4) with $S = \{y\}$. The map $\iota_{x,y}$ matches $\langle x, S, y \rangle$. We have $x R y$ by (C4a) and $y R y$ by (F8), hence $\iota_{x,y}$ satisfies $(\diamond\text{-forth})$. By (F2) we have $\langle y, \{y\}, y \rangle \in P$, thus $\iota_{x,y}$ satisfies (G).

Case (C4) with $\{y\} \subsetneq S$. Denote by $z_0, \dots, z_{k-1}$ an enumeration of S such that $z_0 = y$, and put $z_k = z_0$. For each $i < k$ we have $P \ni \langle z_{i+1}, S, z_i \rangle \triangleleft \langle x, S, y \rangle$ by (C4b) and $z_{i+1} \neq z_i$, thus the induction hypothesis yields a good path π_i matching $\langle z_{i+1}, S, z_i \rangle$. Consider $\pi : [0, 1] \rightarrow X$ given by $\pi(0) = x$ and $\pi \upharpoonright [\frac{1}{i+2}, \frac{1}{i+1}] = \pi_{i \bmod k}$ for all $i \geq 0$. Clearly, π matches $\langle x, S, y \rangle$. This π satisfies $(\diamond\text{-forth})$ at every $t \in (0, 1]$ since π_i 's satisfy $(\diamond\text{-forth})$, and at $t = 0$ since $\pi[(0, 1)] = S \subseteq R(x)$ by (C4a). For all $u \in (0, 1]$ we have $\langle \pi(0), \pi[(0, u)], \pi(u) \rangle = \langle x, S, \pi(u) \rangle \in P$ by (C4c). Condition (G) for $0 < t < u \leq 1$ follows easily by (F1) from the fact that π_i 's satisfy (G).

Case (C5). For every $z \in S \cap U$, we claim that there exists a good path π_z with $z \in \pi_z[(0, 1)] \subseteq S$ and $\pi_z(0) = \pi_z(1) \in S \setminus U$. Indeed, by (C5c) and the induction hypothesis, there exists a good path π'_z matching $\langle z, S \cap U, w_z \rangle$ for some $w_z \in S \setminus U$. Take π_z with $\pi_z \upharpoonright [0, \frac{1}{2}] = \pi'_z \upharpoonright [1, 0]$ and $\pi_z \upharpoonright [\frac{1}{2}, 1] = \pi'_z$.

As in Example 6.3, denote by $\mathcal{C} \subseteq [0, 1]$ the standard Cantor set, and present $[0, 1] \setminus \mathcal{C}$ as a union of pairwise-disjoint intervals $\bigcup_{i \in \mathbb{Q}} (\alpha_i, \beta_i)$, with $i < j \iff \alpha_i < \alpha_j$. Fix $f : \mathbb{Q} \rightarrow S$ such that $f^{-1}(z)$ is dense

in $\mathbb{Q}$ for every $z \in S$. Define $\pi : [0, 1] \rightarrow X$ as follows:

- $\pi \upharpoonright [\alpha_i, \beta_i] = \pi_{f(i)}$ if $f(i) \in S \cap U$ and $\pi \upharpoonright [\alpha_i, \beta_i] = \iota_{f(i), f(i)}$ if $f(i) \in S \setminus U$;
- $\pi(t) = x$ for all $t \in [0, 1) \setminus \bigcup_{i \in \mathbb{Q}} [\alpha_i, \beta_i]$, and $\pi(1) = y$.

Clearly, π matches $\langle x, S, y \rangle$. Consider $t \in [0, 1]$. If $\pi(t) \in S \setminus U$, then $R(\pi(t)) \supseteq S = \pi[[0, 1]]$ by (C5a) and (C5b). Otherwise, some $i \in \mathbb{Q}$ satisfies $t \in (\alpha_i, \beta_i)$ and $\pi \upharpoonright [\alpha_i, \beta_i]$ satisfies $(\diamond\text{-forth})$. Therefore, π satisfies $(\diamond\text{-forth})$. Towards (G), consider $0 \leq t < u \leq 1$. If $[t, u] \subseteq [\alpha_i, \beta_i]$ for some i with $f(i) \in S \cap U$, it suffices to apply (G) for $\pi_{f(i)}$. If $[t, u] \subseteq [\alpha_i, \beta_i]$ with $f(i) \in S \setminus U$, apply (F2). Finally, suppose $[t, u]$ is not contained in any of the intervals $[\alpha_i, \beta_i]$. Then $[t, u]$ intersects at least two of those intervals, say $[\alpha_i, \beta_i]$ and $[\alpha_j, \beta_j]$ with $i < j$. But now for every $z \in S$ there exists $k \in f^{-1}(z) \cap (i, j)$, thus $z \in \pi[[\alpha_k, \beta_k]] \subseteq \pi[[t, u]]$. Therefore, $\pi[[t, u]] = S$, hence $\langle \pi(t), \pi[[t, u]], \pi(u) \rangle = \langle \pi(t), S, \pi(u) \rangle \in P$ by (C5d). $\square$

Now, given a suitable frame $\langle X, R, P \rangle$, we employ Lemma 6.9 to produce a metric “tree” T and a path-morphism $f : T \rightarrow X$. Let us first outline our construction. To assure $(\diamond\text{-back})$ and $(\gamma\text{-back})$, it suffices to define T as the closure of $\{a_0\}$ with $f(a_0) := x_0$ under the following two operations:

- $(\diamond\text{-lift})$ if $a \in T$ and $f(a) R y$, isometrically glue $\{0\} \cup \{\frac{1}{n} : n \geq 1\}$ to T with $a \sim 0$, taking $f(\frac{1}{n}) := y$ for all $n \geq 1$;
- $(\gamma\text{-lift})$ if $a \in T$ and $\langle f(a), S, y \rangle \in P$, isometrically glue a real interval $[0, 1]$ to T with $a \sim 0$, taking $f \upharpoonright [0, 1] := \pi$, where $\pi : [0, 1] \rightarrow X$ is a good path matching $\langle f(a), S, y \rangle$. We need not do this operation if $f(a) = y \in S$ (because in this case $(\gamma\text{-forth})$ is already witnessed by $\delta := \iota_{a,a}$); otherwise a requisite good path exists by Lemma 6.9.

For this construction, we will show that properties $(\diamond\text{-forth})$ and $(\gamma\text{-forth})$ transfer from individual good paths to the whole map f .

Remark 6.10. The metric space T we construct is similar to a *real tree* (as defined e.g. in [7]), except that in addition to “edges” isomorphic to $[0, 1]$ we also have those isomorphic to $\{0\} \cup \{\frac{1}{n} : n \geq 1\}$. We define the metric in terms of a height function h and a meet-semilattice $\wedge$ of “least common ancestors” in our “tree”; this is reminiscent of [7, Example 10.9]. In a real tree, every path passes through all the points on the unique arc between its ends (cf. e.g. [7]); we essentially make a similar observation on our metric to verify $(\gamma\text{-forth})$.

Proposition 6.11. *If $\langle X, R, P \rangle$ is a suitable frame and $x_0 \in X$, then there exists a metrizable topological space $\langle T, \tau \rangle$ and a path-morphism $f : T \rightarrow X$ with $x_0 \in f[T]$.*

Proof. In this proof, $a \frown b$ denotes the concatenation of tuples a and b .

Using Lemma 6.9, fix a set Γ containing one good path matching each triple $\langle x, S, y \rangle \in P$, except those with $x = y \in S$. Denote by T the set consisting of the empty tuple $\emptyset$ and all tuples $\langle \pi_1, u_1, \dots, \pi_k, u_k \rangle$ such that:

- for each $i = 1, \dots, k$, either:
 - π_i is an element of Γ and $u_i \in (0, 1]$; or
 - $\pi_i = \iota_{x,y}$ for some $x, y \in X$ with $x R y$, and $u_i = \frac{1}{n}$ for some integer $n > 0$; and
- $\pi_1(0) = x_0$ and $\pi_i(0) = \pi_{i-1}(u_{i-1})$ for each $i = 2, \dots, k$.

(Note that $\iota_{x,y}$ may or may not be in Γ . If $\pi_i = \iota_{x,y} \in \Gamma$, we allow any $u_i \in (0, 1]$.)

For $a, b \in T$, put $a \prec b$ iff b has a prefix $c \frown \langle \pi, u \rangle$ such that either $a = c$ or $a = c \frown \langle \pi, u' \rangle$ for some $u' < u$. Observe that $\prec$ is a partial order on T .

We claim that $\langle T, \prec \rangle$ is a meet-semilattice (i.e., that any two elements of the “tree” $\langle T, \prec \rangle$ have a least common ancestor). Indeed, consider $\prec$ -incomparable $a, b \in T$. There are unique prefixes $c^\wedge \langle \pi_a, u_a \rangle$ and $c^\wedge \langle \pi_b, u_b \rangle$ of a and b , respectively, with $\langle \pi_a, u_a \rangle \neq \langle \pi_b, u_b \rangle$. If $\pi_a = \pi_b$, then $c^\wedge \langle \pi_a, \min\{u_a, u_b\} \rangle$ is the meet of a and b . Otherwise, c is the meet of a and b . We denote $\prec$ -meets by $\wedge$.

Put $h(\langle \pi_1, u_1, \dots, \pi_k, u_k \rangle) := u_1 + \dots + u_k$, $h(\emptyset) := 0$, and $d(a, b) := h(a) + h(b) - 2h(a \wedge b)$ for $a, b \in T$. We claim that d is a metric on T . Indeed, d is clearly symmetric. Since h is $\prec$ -monotone, d is positive. For all $a, b, c \in T$ we have $h(b) \geq \max\{h(a \wedge b), h(b \wedge c)\}$ and $h(a \wedge c) \geq \min\{h(a \wedge b), h(b \wedge c)\}$, thus $h(b) + h(a \wedge c) \geq h(a \wedge b) + h(b \wedge c)$, hence $d(a, b) + d(b, c) \geq d(a, c)$.

Denote by τ the topology on T induced by d . Put $f(a^\wedge \langle \pi, u \rangle) := \pi(u)$ and $f(\emptyset) := x_0$. Now we verify that f is a path-morphism from $\langle T, \tau \rangle$ to $\langle X, R, P \rangle$.

Condition ($\diamond$ -back). Consider $a \in T$, $y \in R(f(a))$, and $U \in \tau$ with $a \in U$. For every $n > 0$, we have $b_n := a^\wedge \langle \iota_{f(a), y}, \frac{1}{n} \rangle \in T$ and $d(a, b_n) = \frac{1}{n}$. Therefore, $b_n \in U \setminus \{a\}$ for a sufficiently big n .

Condition (γ -back). Consider $a \in T$ and $\langle f(a), S, y \rangle \in P$. If $f(a) = y \in S$, it suffices to take $\delta := \iota_{a, a}$. Otherwise, fix $\pi \in \Gamma$ that matches $\langle f(a), S, y \rangle$, put $\delta(0) := a$ and $\delta(u) := a^\wedge \langle \pi, u \rangle$ for $u \in (0, 1]$, and observe that δ is isometric and thus continuous.

Condition ($\diamond$ -forth). Consider $a \in T$. If $a \neq \emptyset$, present a as $a = a'^\wedge \langle \pi_a, u_a \rangle$. Since Γ is finite and all its elements satisfy ($\diamond$ -forth), we can fix a sufficiently small $\varepsilon > 0$ such that:

- for every $\pi \in \Gamma$, we have $\pi[(0, \varepsilon)] \subseteq R(f(\pi(0)))$;
- if $a \neq \emptyset$ and $\pi_a \in \Gamma$, then $\varepsilon < u_a$ and $\pi_a[(u_a - \varepsilon, \min\{u_a + \varepsilon, 1\}) \setminus \{u_a\}] \subseteq R(f(a))$;
- if $a \neq \emptyset$ and $\pi_a \notin \Gamma$, then $\varepsilon < u_a$ and $(u_a - \varepsilon, u_a) \cup (u_a, u_a + \varepsilon)$ is disjoint from $\{\frac{1}{n} : n \geq 1\}$.

Consider $b \in T$ with $d(a, b) < \varepsilon$; it suffices to show that $b = a$ or $f(a) R f(b)$. As $d(a, a \wedge b) \leq d(a, b) < \varepsilon$, we have either $a = \emptyset$ or $a \wedge b = a'^\wedge \langle \pi_a, u \rangle$ for some $u \in (u_a - \varepsilon, u_a]$. Present b as $c^\wedge \langle \pi_1, u_1, \dots, \pi_k, u_k \rangle$ with $k \geq 0$, where $c = \emptyset$ if $a = \emptyset$ and $c = a'^\wedge \langle \pi_a, u' \rangle$ for some $u' \geq u$ otherwise. In the latter case, $0 \leq u' - u \leq d(a \wedge b, b) < \varepsilon$, whence $u' \in (u_a - \varepsilon, u_a + \varepsilon)$, hence either $c = a$ (if $u' = u_a$) or $f(a) R f(c)$ (otherwise). If $k = 0$, there is now nothing to prove. Otherwise, for each $i < k$ with $\pi_i \in \Gamma$ we have $u_i \leq d(a \wedge b, b) < \varepsilon$, whence $\pi_i(0) R \pi_i(u_i)$ by definition of ε . For each $i < k$ with $\pi_i \notin \Gamma$, we have $\pi_i = \iota_{x, y}$ for some $x R y$, thus also $\pi_i(0) R \pi_i(u_i)$. By transitivity of R , it follows that $f(c) R f(b)$. Since $c = a$ or $f(a) R f(c)$, we conclude that $f(a) R f(b)$.

Condition (γ -forth). Consider a path δ in $\langle T, \tau \rangle$. First observe the following for every $a^* \in T$:

- (i) the set $T \setminus \{a^*\}$ is the disjoint union of the following open sets:
 - $\{b \in T : a^* \not\leq b\}$;
 - $\{b \in T : a^*^\wedge \langle \pi \rangle$ is a prefix of $b\}$, for some π ;
 - $\{b \in T : a^* = a'^\wedge \langle \pi, u' \rangle$ and $a'^\wedge \langle \pi, u \rangle$ is a prefix of b for some $u > u'\}$;

thus, by connectedness of $[0, 1]$, either $a^* \in \delta[[0, 1]]$ or $\delta[[0, 1]]$ is a subset of one of these sets.

- (ii) if $a^* = a'^\wedge \langle \pi, u' \rangle$ with $\pi \notin \Gamma$, then $\{b \in T : a^* \leq b\}$ is open. Hence $\delta[[0, 1]]$ is a subset either of said set or of its complement, by connectedness of $[0, 1]$.

Now, to establish (γ -forth) for δ , consider 5 cases:

- *Case 1A:* $\delta(1) = a \wedge \langle \pi, u \rangle$ and either $\delta(0) = a \wedge \langle \pi, u' \rangle$ with $u' < u$ or $\delta(0) = a$. If $\delta(0) = a$, put $u' := 0$. Since $\delta(0) \prec \delta(1)$, by (ii) applied to $a^* := \delta(1)$ we have $\pi \in \Gamma$, thus by (G) $\langle \pi(u'), \pi[(u', u)], \pi(u) \rangle \in P$. For every $u'' \in (u', u)$, we have $\delta(0) \prec a \wedge \langle \pi, u'' \rangle \prec \delta(1)$, hence $a \wedge \langle \pi, u'' \rangle \in \delta[(0, 1)]$ by (i), whence $\pi(u'') = f(a \wedge \langle \pi, u'' \rangle) \in f[\delta[(0, 1)]]$. Therefore, $\pi[(u', u)] \subseteq f[\delta[(0, 1)]]$. As P is monotone, we obtain $\langle f(\delta(0)), f[\delta[(0, 1)]], f(\delta(1)) \rangle \in P$.
- *Case 1:* $\delta(0) \prec \delta(1)$. By induction on the length of $\delta(1)$. Present $\delta(1)$ as $a \wedge \langle \pi, u \rangle$. If Case 1A does not apply, we have $\delta(0) \prec a$. By (i) for $a^* := a$, we obtain $a = \delta(t)$ for some $t \in (0, 1)$. It remains to apply the induction hypothesis to $\delta \upharpoonright [0, t]$ and Case 1A to $\delta \upharpoonright [t, 1]$, and use (F1).
- *Case 2:* $\delta(1) \prec \delta(0)$. Apply Case 1 to $\delta \upharpoonright [1, 0]$ and use (F4).
- *Case 3:* $\delta(0) \not\prec \delta(1)$, $\delta(1) \not\prec \delta(0)$, and $\delta(0) \neq \delta(1)$. Applying (i) to $a^* := \delta(0) \wedge \delta(1)$, we obtain $\delta(t) = \delta(0) \wedge \delta(1)$ for some $t \in (0, 1)$. Now apply Case 2 to $\delta \upharpoonright [0, t]$ and Case 1 to $\delta \upharpoonright [t, 1]$, and use (F1).
- *Case 4:* $\delta(0) = \delta(1)$. If δ is constant, it suffices to use (F2). Otherwise, fix some $t \in (0, 1)$ with $\delta(t) \neq \delta(0)$, apply previous cases to $\delta \upharpoonright [0, t]$ and $\delta \upharpoonright [t, 1]$, and use (F1).

□

Summarizing this and the preceding sections, we have the following:

Theorem 6.12. *For a formula $\varphi \in \mathcal{L}$, the following are equivalent:*

- (1) φ is valid over the class of T_1 topological spaces;
- (2) φ is valid over the class of metric spaces;
- (3) φ is valid over the class of suitable frames of size at most $2^{|\text{Sub } \varphi|}$;
- (4) $\mathbf{TLR} \vdash \varphi$.

The set of formulas with these properties is decidable.

Proof. (1) implies (2) since all metric spaces are T_1 ; (2) implies (3) by Lemma 6.2 and Proposition 6.11; (3) implies (4) by Proposition 5.10; (4) implies (1) by Proposition 3.3. Decidability follows from (3). □

7 C-semantics

In this section, we axiomatize the c-semantical fragment of **TLR** and observe that it is sound for the class of all topologies. Denote by $\mathcal{L}_c \subseteq \mathcal{L}$ the set of formulas that may be written in terms of $\perp$, $\rightarrow$, γ , and $\Box$, without explicit occurrences of $\square$.

Definition 7.1. Denote by $\mathbf{TLR}_c$ the $\mathcal{L}_c$ -calculus consisting of:

- all axioms and rules of **S4** for $\Box$;
- axioms (A0)–(A7) and rule (Mon).

Lemma 7.2. *For $\xi \in \mathcal{L}$, denote by $\xi^\# \in \mathcal{L}_c$ the result of substituting each $\square$ with $\Box$.*

- 1) *If $\xi \in \mathcal{L}_c$, then $\mathbf{TLR}_c \vdash \xi \leftrightarrow \xi^\#$.*
- 2) *If $\mathbf{TLR} \vdash \xi$, then $\mathbf{TLR}_c \vdash \xi^\#$.*

Proof. 1) Since $\xi^\#$ is obtained from ξ by substituting each occurrence of $\Box\varphi = \varphi \wedge \Box\varphi$ with $\varphi \wedge \Box\varphi$, it suffices to note that $\mathbf{S4} \vdash \varphi \wedge \Box\varphi \leftrightarrow \Box\varphi$.

2) By induction on the derivation of $\mathbf{TLR} \vdash \xi$. If ξ is an axiom of $\mathbf{K4}$ or is obtained by a rule of $\mathbf{K4}$ or is obtained by (Mon), it suffices to note that those axioms and rules are also among the axioms and rules of $\mathbf{TLR}_c$ with $\Box$ substituted for $\square$. If $\xi = \gamma(\varphi \wedge \Box\neg\varphi, \top) \rightarrow \varphi$ is an instance of (A8), then $\mathbf{S4} \vdash \varphi^\# \wedge \Box\neg\varphi^\# \rightarrow \perp$ and $\mathbf{TLR}_c \vdash \gamma(\perp, \top) \rightarrow \varphi^\#$ by (A3), hence $\mathbf{TLR}_c \vdash \gamma(\varphi^\# \wedge \Box\neg\varphi^\#, \top) \rightarrow \varphi^\#$ by (Mon). Now suppose ξ is an instance of (A0)–(A7). Since $\Box$ does not occur in those axioms (except as $\Box$), we can present ξ as $\chi[p_1/\psi_1, \dots, p_k/\psi_k]$, where $\chi \in \mathcal{L}_c$ is itself an occurrence of (A0)–(A7). Since $\mathbf{TLR}_c \vdash \chi$, we have $\mathbf{TLR}_c \vdash \chi^\#$ by (1), hence $\mathbf{TLR}_c \vdash \chi^\#[p_1/\psi_1^\#, \dots, p_k/\psi_k^\#] = \xi^\#$. $\square$

Theorem 7.3. *The logic $\mathbf{TLR}_c$ is decidable, sound for the class of all topological spaces, and complete for the class of metric spaces.*

Proof. Soundness of $\mathbf{S4}$ is well-known [1]. Soundness of (A0)–(A7) and (Mon) follows from Proposition 3.3. If $\xi \in \mathcal{L}_c$ is valid over the class of metric spaces, then $\mathbf{TLR} \vdash \xi$ by Theorem 6.12, whence $\mathbf{TLR}_c \vdash \xi$ by Lemma 7.2. Decidability follows by Theorem 6.12. $\square$

8 Final Remarks

In this paper we axiomatized the logic of all topological spaces in the language with path-reachability γ and the interior modality, but in the language enriched by the Cantor derivative we only addressed T_1 spaces. As illustrated in Example 4.8, the interaction of Cantor derivative with γ in non- T_1 spaces has extra phenomena not caught by the axioms of this paper. To establish completeness for all spaces, it might be useful to enrich the tree-like construction of Proposition 6.11 with “jump” edges isomorphic to the 2-point Sierpiński space. Another problem, already mentioned in [3], is to axiomatize the γ -logic of the Euclidean plane.

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Taming Complexity in Intuitionistic Modal Logic: The Case of **FIK** and Its Shallow Calculus

Han Gao

Institute of Computer Science
Czech Academy of Sciences
gao@cs.cas.cz

Nicola Olivetti

LIS
Aix-Marseille University, CNRS
nicola.olivetti@lis-lab.fr

Intuitionistic modal logics (IML) comprise many systems: from constructive modal logics such as **CK** and Wijesekera’s **WK** to Fischer Servi/Simpson’s **IK**, as well as some recently introduced variants. All of them are characterized by bi-relational semantics and have complete axiomatisations. However, from the perspective of proof theory and complexity, there are strong differences: while for constructive modal logics simple Gentzen calculi suffice, for **IK** more complex calculi, based on nested or labeled sequents, are needed. As a consequence, the decision problem for constructive modal logics has a PSPACE upper bound, whereas for **IK** is not known and it is even conjectured to be non-elementary. We study here the proof theory and complexity of **FIK**, a natural intuitionistic modal logic recently introduced. **FIK** is strictly in between **WK** and **IK**, yet it has the same forcing conditions as **IK**. We define a “shallow” sequent calculus for **FIK** which is a nested sequent calculus where sequents have at most one level of nesting. We prove its syntactic completeness by showing the admissibility of cut. By means of this calculus we show that decision problem for **FIK** is in EXPSpace, whence significantly lower than the complexity conjectured for **IK**.

1 Introduction

Intuitionistic modal logics (IML for short) have been studied since the late 40’s [12] and comprise a variety of systems. It is usual to divide them into two main subfamilies. In the first family, sometimes referred as *intuitionistic modal logic* in *strictu sensu*, the basic system is **IK**, this logic was introduced by Fischer Servi [9, 10, 11] and later systematized by Simpson [26]; **IK** is specifically motivated by its meta-theoretical properties, as it has a standard translation into first-order intuitionistic logic (FOIL). The other family is called *constructive modal logic*. It was pioneered by Prawitz [25] and then developed by Wijesekera who proposed the first order system *constructive concurrent dynamic logic* [28], whose propositional fragment is now usually named **WK**. The most known representative of this family is **CK** as ‘constructive K’ proposed in [6], namely a slight weaker system than **WK**. The main motivation for constructive modal logics lies in their applications in computer science and proof theory, first of all their computational interpretation in terms of types and the Curry-Howard correspondence, but also for representing contextual reasoning [22, 5]. All the mentioned logics are nonetheless meant to be the intuitionistic counterpart of the classical K.

However, this overview is not exhaustive; the landscape of IML is richer than one might expect. In recent years other systems have been introduced with their own motivations. We consider in this paper one of the recent variants, *forward-confluent* intuitionistic modal logic **FIK**, proposed in [1]. This logic enjoys a natural bi-relational semantics and it is the weakest logic defined by Fischer Servi (or Simpson’s) forcing conditions for *both* modalities. Moreover, **FIK** satisfies all the criteria stated by Simpson in [26], except for its interpretability in FOIL; for this reason it seems a meaningful system in the realm of IML on its own. Notice that the logic **FIK** is strictly stronger than constructive logics **CK** and **WK** and strictly

weaker than **IK**, these (strict) inclusions hold notably already for the $\Diamond$ -free fragment of the respective logics. Other logics have been recently investigated, as we cannot be exhaustive here we just mention the logic **LIK** of *local* modalities [2] which is stronger than **FIK** and incomparable with **IK**; and the ‘minimal’ logic in [3] with a ‘dual’ forcing condition for $\Diamond$ which that makes it incomparable with **WK**. Furthermore, a systematic investigation of other variants between **CK** and **IK** is carried on recently in [19] where, among others, an exact correspondence between each axiom of **IK** and independent frame conditions is established.

From a semantic perspective, *all* the mentioned systems are characterized uniformly by the bi-relational semantics with suitable forcing and frame conditions; in contrast, their proof theory, whenever it exists, is far from uniformity. The constructive modal logics **CK** and **WK** enjoy cut-free simple Gentzen calculi, analogous to basic calculi for classical modal logic known at least since Fitting’s seminal work [13]. On the other hand, for **IK** no such simple calculi are available and it seems unlikely that they indeed exist. To the best of our knowledge, existing sequent calculi for **IK** all employ sequent with enriched structure, such as nested calculus [27, 15, 20], fully labelled calculus [21, 17] and a more recent bi-nested calculus [16]. From a computational viewpoint the difference is even more striking: by virtue of their simple calculi, it is folklore that constructive modal logic **CK** and **WK** are decidable in polynomial space; In contrast, according to e.g. [23], the decision problem for **IK** might not have an *elementary* upper bound. It is then natural to ask where **FIK** stands from a complexity perspective, since it has the same forcing conditions of **IK**: is it a neighbor of constructive modal logics or on the contrary, a neighbor of **IK**? In [1] a proof system for **FIK** is proposed in the form of a ‘bi-nested’ calculus; although this calculus provides a decision procedure for **FIK** as well as a finite counter-model construction, it is not obvious how to obtain an upper bound of decidability by means of the calculus. In [4], alternative proofs of decidability of **FIK** are provided, but the complexity of the decision problem remains open.

In this paper we tackle the problem of the complexity of **FIK**. By means of a new calculus, we show that the decision problem of the logic is in EXPSpace. This upper bound, although higher than that of constructive modal logics, is significantly lower than what is conjectured for **IK**, as it is elementary. In our opinion this result provides a further argument in favour of **FIK** as an interesting member of the IML family. The question whether this bound is strict, that is of the lower bound for decidability of **FIK**, is open at present, as it is for **IK**. Our result is obtained through proof-theoretic methods. We introduce a new calculus for **FIK**, which makes use of structured sequents that are only slightly more complex than standard Gentzen sequents, yet notably simpler than (bi-)nested sequents. A structured sequent is a Gentzen sequent possibly coupled with a tuple of ‘blocks’ containing Gentzen sequents. As a difference with (bi-)nested calculi, structured sequents have therefore at most two-levels, no further nesting needed. The rules operate either on the ‘top’ level or within a block. For this reason, we qualify this calculus as ‘shallow’, following the terminology of [18]. Semantically, a ‘shallow’ sequent can be viewed as a partial view of a (bi-)nested sequent, representing only the current world and its direct accessible successors. We establish the completeness of the shallow calculus syntactically via a proof of cut-admissibility. Beyond the current results, we believe it is worthwhile to study structured calculi of this type, whose structure lies halfway between simple Gentzen calculi and full (bi-)nested calculi, for other intuitionistic modal logics with the aim of establishing complexity bounds and other properties.

2 Syntax and semantics

In this section we introduce basics of **FIK**, defining the syntax, the bi-relational semantics as well as its Hilbert axiom system [1].

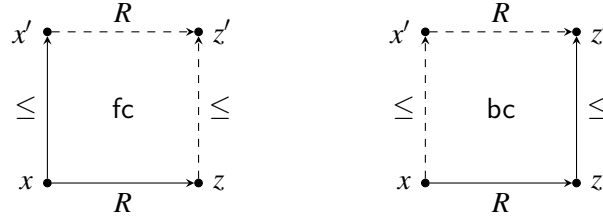


Figure 1: Forward confluence (left) and backward confluence (right)

Definition 2.1 (Formulas). Let At be a countable set of propositional variables denoted as p, q , etc. The set of formulas $\mathcal{L}$ whose elements are denoted A, B , etc., is generated by the grammar

$$A ::= p \mid \perp \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$$

where p ranges over At . Negation $\neg A$ is defined as $A \supset \perp$.

For a formula A , the *size* of it denoted $|A|$ is the number of symbols in A . The *modal degree* of A denoted $\text{md}(A)$ is defined as usual.

Semantics of the logic **FIK** is defined in terms of bi-relational Kripke models[1].

Definition 2.2 (Bi-relational frames and models). A *bi-relational frame* is a triple $(W, \leq, R)$ where W is a nonempty set of *worlds*, $\leq$ is a pre-order on W and R is a binary relation on W . A *bi-relational model* is a quadruple $(W, \leq, R, V)$, where $(W, \leq, R)$ is a bi-relational frame and $V : \text{At} \rightarrow \mathcal{P}(W)$ is a valuation function such that for all $p \in \text{At}$, $V(p)$ is upper closed with respect to $\leq$.

We postulate the following frame condition called forward confluence (fc) (cf. Figure 1 (left)):

$$(fc) \quad \forall x, x', z \in W, \text{ if } x \leq x' \text{ and } Rxz, \text{ there is } z' \in W \text{ s.t. } Rx'z' \text{ and } z \leq z'.$$

A **FIK**-frame (abbreviated as frame) is a bi-relational frame satisfying (fc) and a **FIK**-model (abbreviated as model) is a bi-relational model based on a **FIK**-frame.

Next we define the forcing conditions for the formulas in a **FIK**-model.

Definition 2.3 (Forcing conditions for **FIK**). Let $\mathcal{M} = (W, \leq, R, V)$ be a **FIK**-model. The forcing conditions of a formula at a world $w \in W$ of $\mathcal{M}$ are defined as follows:

$$\begin{aligned} \mathcal{M}, w &\not\models \perp \\ \mathcal{M}, w &\models p & \text{iff } w \in V(p) \\ \mathcal{M}, w &\models B \wedge C & \text{iff } \mathcal{M}, w \models B \text{ and } \mathcal{M}, w \models C \\ \mathcal{M}, w &\models B \vee C & \text{iff } \mathcal{M}, w \models B \text{ or } \mathcal{M}, w \models C \\ \mathcal{M}, w &\models B \supset C & \text{iff for all } w' \in W \text{ with } w \leq w', \text{ if } \mathcal{M}, w' \models B, \text{ then } \mathcal{M}, w' \models C \\ \mathcal{M}, w &\models \Box B & \text{iff for all } w', v \in W \text{ with } w \leq w' \text{ and } Rw'v, \text{ it holds } \mathcal{M}, v \models B \\ \mathcal{M}, w &\models \Diamond B & \text{iff there is } v \in W \text{ s.t. } R w v \text{ and } \mathcal{M}, v \models B \end{aligned}$$

We shall abbreviate $\mathcal{M}, w \models A$ as $w \models A$ if the model is clear from the context. A formula A in $\mathcal{L}$ is *valid*, denoted $\models A$, if for any bi-relational model $\mathcal{M}$ and any world w in it, it holds that $\mathcal{M}, w \models A$.

We observe that the forward confluence condition is needed to ensure the hereditary property for $\Diamond$ -formulas.¹ To put things in a context, the logic **FIK** is related to other well-known systems in the IML family. The logic **IK** is characterized by models based on frames satisfying *both* forward confluence and the following condition *backward confluence* (bc) (cf. Figure 1 (right)):

¹By hereditary property, we mean the following: for every formula $A \in \mathcal{L}$ if $w \models A$ and $w \leq w'$ then $w' \models A$. This property is required to extend the validity of intuitionistic axioms to the whole modal language $\mathcal{L}$.

k1 $\Box(A \supset B) \supset (\Box A \supset \Box B)$
 k2 $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
 k3 $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$
 k5 $\neg \Diamond \perp$
 wCD $\Box(A \vee B) \supset ((\Diamond A \supset \Box B) \supset \Box B)$

$$\frac{A \supset B \quad A}{B} \text{ mp} \quad \frac{A}{\Box A} \text{ nec}$$

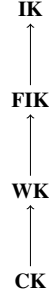
Figure 2: Axiom and rules of $\mathcal{C}_0$ 

Figure 3: Hierarchy of some IML systems

(bc) $\forall x, z, z' \in W$, if Rxz and $z \leq z'$, there is $x' \in W$ s.t. $Rx'z'$ and $x \leq x'$.

The forcing conditions of **IK** are defined in the same way as in **FIK**, which go back to Fischer Servi [11]. On the other hand, in bi-relational models of **CK** (as well as those of **WK**) no additional property of frames is assumed. The forcing condition of $\Box$ is the same as in **FIK**, whereas for $\Diamond$:

$x \Vdash_{\mathbf{CK}} \Diamond A$ iff $\forall x' \in W$ with $x \leq x'$, there is $y \in W$ s.t. $Rx'y$ & $y \Vdash_{\mathbf{CK}} A$.

This condition is equivalent to the *local* one of **FIK** with respect to **FIK**-models.

We now recall the Hilbert-style axiom system for **FIK** introduced in [1].

Definition 2.4 (Axiom systems). The Hilbert-style axiom system for **FIK** called $\mathcal{C}_{\mathbf{FIK}}$ is defined as $\mathcal{C}_{\mathbf{FIK}} = \text{IPC} \oplus \mathcal{C}_0$ where IPC denotes an axiomatization of intuitionistic propositional logic (IPL) and $\mathcal{C}_0$ contains axioms and rules in Figure 2.

The notions of proof and derivability in the axiom system are defined as usual. The Hilbert system $\mathcal{C}_{\mathbf{FIK}}$ is shown sound and complete with respect to the bi-relational semantics.

Theorem 2.5 ([1]). *A formula $A \in \mathcal{L}$ is **FIK**-valid if and only if it is provable in $\mathcal{C}_{\mathbf{FIK}}$.*

It is instructive to briefly recall the axiomatization of a few logics mentioned in the introduction. The logic **WK** ([28]) is obtained by removing k3 and (wCD)² from $\mathcal{C}_{\mathbf{FIK}}$. The logic **CK** [6] is obtained by further removing k5.³ In turn, the axiomatization of **IK** is obtained from $\mathcal{C}_{\mathbf{FIK}}$ by replacing (wCD) with the following axiom denoted k4 or (FS) (for Fischer Servi):

$$(\text{FS/k4}) (\Diamond A \supset \Box B) \supset \Box(A \supset B)$$

Notice that (wCD) is derivable in the axiom system of **IK** [1]. The diagram in Figure 3 illustrates the strict inclusion relations among the mentioned systems.

We end this section with some discussions on the legitimacy of **FIK** in the IML family. Simpson [26] proposed some criteria that a reasonable IML should satisfy: (i) it is a conservative extension of IPL, (ii) it contains all instances of IPL axioms and is closed with respect to the rule (mp), (iii) $\Box$ and $\Diamond$ are not inter-definable, (iv) it satisfies disjunction property, (v) adding classical *excluded middle law*, it collapses into its classical counterpart (say K for the basic case). Lastly the (vi)th criterion requires that axioms of the logic should be validated by the standard translation of modal logic into first-order intuitionistic logic (FOIL). As shown in [1], **FIK** satisfies all criteria (i)-(v), but not the sixth one. However in our opinion

²The name (wCD) can be spelled as *weak constant domain*, see [1] for a justification.

³There are of course other variants not included in this graph, e.g. a calculus for **FIK** excluding (wCD) is given in [8] and its semantic characterization as well as other systems like $\mathbf{CK} \oplus \mathbf{k5}$ and their relations are studied in [19].

the last criterion is controversial as the standard translation in FOIL might not be considered in itself as a main justification of a logic (alternative semantics rather than possible world relational semantics are also possible); in addition, alternative translations might be possible for IML's which do not satisfy this criterion. Thus in this sense the absence of it should not be seen as a lack of justification.

3 A shallow calculus for FIK

In this section, we define a new calculus for **FIK** called a *shallow* sequent calculus. The calculus makes use of nested sequents, generated by structural operators called 'blocks', but as a difference with nested calculi like [7, 27, 15, 20, 14], there is only one level of nesting. This means that the content of a block is just an ordinary Gentzen sequent, and no further nesting is allowed. Consequently, most of the rules of the calculus operate on both levels: either on the 'surface' or within the blocks. As in [27, 20], we adopt a 'polarised' annotation of formulas distinguishing the 'input' formulas and the unique 'output' formula. Intuitively the former play the role of the antecedent of a sequent and the latter its unique consequent.

Definition 3.1 (Sequents). Sequents are built out of annotated formulas in the form of $A^\bullet$ or A° , where a formula annotated by $\bullet$ is called an *input formulas* and a formula annotated by $\circ$ is called an *output formula*. A *simple sequent* S is of the form Φ, B° where Φ is a multi-set of $\bullet$ -formulas $A_1^\bullet, \dots, A_n^\bullet$ and B is a formula. A *shallow sequent* (or a *sequent* for short) is of the form

$$\Phi, [\Psi_1], \dots, [\Psi_l]$$

where $l \geq 0$ and exactly one of $\Phi, \Psi_1, \dots, \Psi_l$ is a simple sequent whereas the others are multi-set of $\bullet$ -formulas. When $l = 0$ a sequent is a simple sequent. We call Φ the top-level component and each nested component $[\Psi_i]$ a block. The part of sequent without the output formula is called an *input sequent*. We shall generically use $\Omega, \Phi, \Psi, \dots$ to denote sequents and input sequents.

We can view shallow sequents as a simplification of nested sequents. The idea of retaining only the necessary structure from nested sequents is also present in other formalisms, such as linear nested sequents, but the simplification is orthogonal: linear nested sequents keep only a single path of worlds, whereas shallow sequents keep only the immediate successors of the current world.

In order to define the calculus we need several operators that remove or add an output formula, transforming a sequent into an input sequent and vice versa.

Definition 3.2. The $*$ -operator is defined as follows: for a sequent Ω , we denote Ω^* as the result of removing the output formula of Ω wherever it occurs. The $+$ -operator is defined as follows: let Φ be a simple sequent, if Φ does not contain the output formula, then $\Phi^+ := \Phi, \perp^\circ$; otherwise $\Phi^+ := \Phi$. Finally we denote the formula part of Ω by $\text{Fm}(\Omega)$, obtained by removing the blocks of Ω if any.

For example, let $\Omega = A^\bullet, A \supset B^\bullet, [C \vee D^\bullet, E^\circ]$, then $\Omega^* = A^\bullet, A \supset B^\bullet, [C \vee D^\bullet]$. It is easy to see that Ω^* is an input sequent; moreover, if Ω itself is an input sequent, then $\Omega = \Omega^*$. In the same example, we have $\text{Fm}(\Omega) = \{A^\bullet, A \supset B^\bullet\}$.

We will need to argue about the size of sequents when discussing complexity bound. As we have already defined in the previous section the size $|A|$ of a formula A and its modal degree $\text{md}(A)$, let us now extend these notions to multisets: the size $|\Phi|$ of a multiset of formulas Φ is defined (in the obvious way) as the sum of the sizes of the formulas in the multi-set (counting multiplicity); the modal degree $\text{md}(\Phi) := \max\{\text{md}(A) \mid A \in \Phi\}$.

$\frac{}{\Omega, \perp^\bullet} (\perp_1^\bullet)$	$\frac{}{\Omega, [\Phi, \perp^\bullet]} (\perp_2^\bullet)$	$\frac{}{\Omega, p^\bullet, p^\circ} (\text{id}_1)$	$\frac{}{\Omega, [\Phi, p^\bullet, p^\circ]} (\text{id}_2)$
$\frac{\Omega, A^\bullet, B^\bullet}{\Omega, A \wedge B^\bullet} (\wedge_1^\bullet)$	$\frac{\Omega, [\Phi, A^\bullet, B^\bullet]}{\Omega, [\Phi, A \wedge B^\bullet]} (\wedge_2^\bullet)$	$\frac{\Omega, A_i^\circ}{\Omega, A_1 \vee A_2^\circ} (\vee_1^\circ)$	$\frac{\Omega, [\Phi, A_i^\circ]}{\Omega, [\Phi, A_1 \vee A_2^\circ]} (\vee_2^\circ)$
$\frac{\Omega, A^\circ \quad \Omega, B^\circ}{\Omega, A \wedge B^\circ} (\wedge_1^\circ)$	$\frac{\Omega, [\Phi, A^\circ] \quad \Omega, [\Phi, B^\circ]}{\Omega, [\Phi, A \wedge B^\circ]} (\wedge_2^\circ)$	$\frac{\Omega, A^\bullet \quad \Omega, B^\bullet}{\Omega, A \vee B^\bullet} (\vee_1^\bullet)$	$\frac{\Omega, [\Phi, A^\bullet] \quad \Omega, [\Phi, B^\bullet]}{\Omega, [\Phi, A \vee B^\bullet]} (\vee_2^\bullet)$
$\frac{\Omega^*, A \supset B^\bullet, A^\circ \quad \Omega, B^\bullet}{\Omega, A \supset B^\bullet} (\supset_1^\bullet)$	$\frac{\Omega^*, [\Phi^*, A \supset B^\bullet, A^\circ] \quad \Omega, [\Phi, B^\bullet]}{\Omega, [\Phi, A \supset B^\bullet]} (\supset_2^\bullet)$	$\frac{\Omega, A^\bullet, B^\circ}{\Omega, A \supset B^\circ} (\supset_1^\circ)$	$\frac{\Phi, A^\bullet, B^\circ}{\Omega, [\Phi, A \supset B^\circ]} (\supset_2^\circ)$
$\frac{\Omega, \Box A^\bullet, [\Phi, A^\bullet]}{\Omega, \Box A^\bullet, [\Phi]} (\Box^\bullet)$	$\frac{\Omega, [A^\circ]}{\Omega, \Box A^\circ} (\Box_1^\circ)$	$\frac{\Phi, [A^\circ]}{\Omega, [\Phi, \Box A^\circ]} (\Box_2^\circ)$	
$\frac{\Omega, [A^\bullet]}{\Omega, \Diamond A^\bullet} (\Diamond_1^\bullet)$	$\frac{\Phi^+, [A^\bullet]}{\Omega, [\Phi, \Diamond A^\bullet]} (\Diamond_2^\bullet)$	$\frac{\Omega, [\Phi, A^\circ]}{\Omega, \Diamond A^\circ, [\Phi]} (\Diamond^\circ)$	

Figure 4: The calculus SC_{FIK}

Definition 3.3 (Size and modal degree of a sequent). Given a sequent $\Omega = \Phi, [\Psi_1], \dots, [\Psi_l]$, its size is defined as $|\Omega| = |\Phi| + |\Psi_1| + \dots + |\Psi_l|$. The size of its output, denoted $|\text{Output}(\Omega)|$, is the size of the unique output formula of Ω ; the size of its input, denoted $|\text{Input}(\Omega)|$ is defined as $|\Omega^*|$. Moreover we define $\text{md}(\Omega) = \max\{\text{md}(\Phi), \text{md}(\Psi_1) + 1, \dots, \text{md}(\Psi_l) + 1\}$.

The calculus SC_{FIK} for **FIK** operates on sequents and contains all the rules in Figure 4. Initial sequents are of four types, namely $\perp_1^\bullet, \perp_2^\bullet, \text{id}_1$ and id_2 . Rules except $\Box^\bullet$ and $\Diamond^\circ$ all appear in pairs, denoted r_1° and r_2° (resp. $r_1^\bullet$ and $r_2^\bullet$), where the former applies to the formula part of a sequent, while the latter operates inside a block. We uniformly refer to both members of the pair as r° (resp. $r^\bullet$), e.g. when we say $\supset^\circ$, it means both $\supset_1^\circ$ and $\supset_2^\circ$.

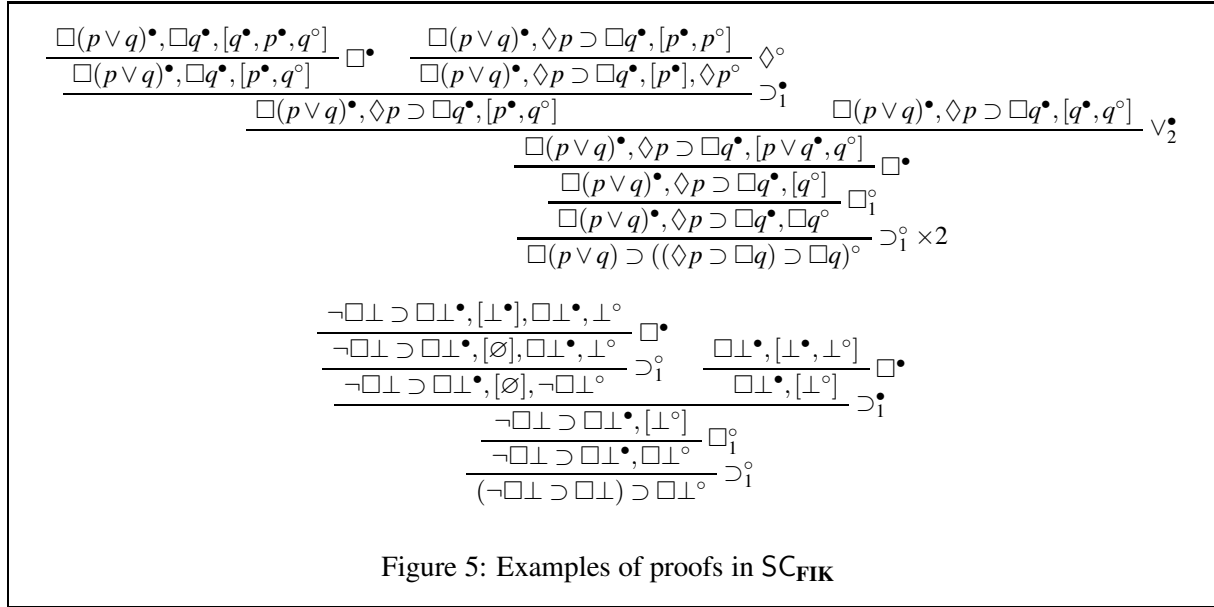
Definition 3.4 (Formula occurrences). Given an application (r) of a rule with the conclusion S , we say that an occurrence of a formula A in S is *principal* if it is explicitly treated in that application of the rule (that is reading backward the application (r), A is decomposed or propagated in at least one premise). We say that an occurrence of A is *side* (or contextual) if it is not principal and it occurs in at least one premise, and it is weak if does not occur in any premises.

For example, in $(\vee_2^\circ)$, the explicit $A \vee B^\circ$ is principal and formulas in Ω and Φ are all side; in $(\Diamond_2^\bullet)$, the explicit $\Diamond A^\bullet$ is principal, formulas in Φ are side and formulas in Ω are weak.

The notions of derivations, proofs (e.g. a derivation whose leaves are axioms namely initial sequents) and height of a derivation are defined as usual in sequent calculi. We write $\vdash_n \Omega$ to denote the proposition that Ω has a proof of height n in SC_{FIK}. Moreover, we say a formula A is provable in SC_{FIK} if so is the sequent A° .

Example 3.5. Axiom (**wCD**): $\Box(p \vee q) \supset ((\Diamond p \supset \Box q) \supset \Box q)$ and the formula $(\neg \Box \perp \supset \Box \perp) \supset \Box \perp$ are provable in SC_{FIK}.⁴ The proofs are found in Figure 5.

⁴Note that this $\Diamond$ -free formula is not valid in **CK**, which makes the $\Diamond$ -free fragments of these two logics different [8].



Sequents are interpreted according to the location of the output formula and only when the output formula occurs on the top-level the sequents have a formula interpretation.

Definition 3.6 (Semantic interpretation). Let $\mathcal{M} = (W, \leq, R, V)$ be a bi-relational model and $x, y \in W$. For a sequent Ω , we define $\mathcal{M}, x \Vdash \Omega$ according to the location of its output formula as follows

- a. if $\Omega = \Phi, [\overline{\Psi_i}]_{i \in I}, F^\circ$ where Φ is a multi-set of $\bullet$ -formulas,

$$x \Vdash \Omega \text{ iff } x \Vdash \bigwedge \Phi \wedge \bigwedge \{\Diamond \wedge \Psi_i\}_{i \in I} \supset F$$

- b. if $\Omega = \Phi, [\overline{\Psi_i}]_{i \in I}, [\Xi, F^\circ]$ where Φ is a multi-set of $\bullet$ -formulas,

$$x \Vdash \Omega \text{ iff } \forall x' \geq x \text{ if } x' \Vdash \bigwedge \Phi \wedge \bigwedge \{\Diamond \wedge \Psi_i\}_{i \in I} \text{ then } \forall y (Rx'y \ \& \ y \Vdash \bigwedge \Xi \text{ implies } y \Vdash F)$$

Ω is called valid if $\mathcal{M}, x \Vdash \Omega$ holds for any $\mathcal{M}$ and any x .

Based on the semantic interpretation, we show the soundness of $SC_{\mathbf{FIK}}$.

Theorem 3.7. *For any sequent Ω , if it is derivable in $SC_{\mathbf{FIK}}$, then it is valid in $\mathbf{FIK}$.*

Proof. We prove this by induction on the structure of a derivation. It suffices to verify that, the initial sequents are valid and the inference rules preserve validity. We only show some non-trivial cases.

- ($\supset_2^\bullet$) We only show the case when $F^\circ \in \Phi$ as the other cases are similar. Assume w.l.o.g. that Ω is block-free and the output formula of the conclusion is in Ω . Then the rule is with conclusion $\Omega^*, [\Phi, A \supset B^\bullet], F^\circ$ and two premises $\Omega^*, [\Phi^*, A \supset B^\bullet, A^\circ]$ and $\Omega^*, [\Phi, A \supset B^\bullet], F^\circ$. Assume by absurdity that this rule application does not preserve validity, then there exists a bi-relational model $\mathcal{M} = (W, \leq, R, V)$ with $x \in W$ such that $x \not\Vdash \Omega^*, [\Phi, A \supset B^\bullet], F^\circ$. By definition, there is $x' \in W$ such that $x \leq x'$ and $x' \Vdash \bigwedge \Omega \wedge \Diamond (\bigwedge \Phi \wedge (A \supset B))$ and $x' \not\Vdash F$. It follows that there is $y \in W$ such that $y \Vdash (A \supset B) \wedge \bigwedge \Phi$. Since the left premise is valid, we have $x' \Vdash \Omega^*, [\Phi^*, A \supset B^\bullet, A^\circ]$. Thus $y \Vdash A$. Recall $y \Vdash A \supset B$, so $y \Vdash B$. Meanwhile, the right premise is also valid, so $x' \Vdash \Omega^*, [\Phi, B^\bullet], F^\circ$, which implies $x' \not\Vdash \Diamond (\bigwedge \Phi \wedge B)$. Since $Rx'y$ and $y \Vdash \bigwedge \Phi$, we have $y \not\Vdash B$, a contradiction.

($\Diamond^\bullet$) The rule is of one of the two forms: conclusion $\Omega, [\Phi, \Diamond A^\bullet, F^\circ]$ and premise $\Phi, [A^\bullet], F^\circ$; conclusion $\Phi, [A^\bullet], \perp^\circ$ and premise $\Omega, [\Phi, \Diamond A^\bullet]$. For the first case, assume this rule does not preserve validity, then there exists a bi-relational model $\mathcal{M} = (W, \leq, R, V)$ with $x \in W$ such that $x \not\models \Omega, [\Phi, \Diamond A^\bullet, F^\circ]$. By definition, this means there is $x', y \in W$ such that $x' \geq x, Rx'y$ and $y \models \bigwedge \Phi \wedge \Diamond A$ and $y \not\models F$. Meanwhile, since the premise is valid, we have $y \models \Phi, [A^\bullet], F^\circ$, which means $y \models \bigwedge \Phi \wedge \Diamond A \supset F$. Since $y \leq y$, by $y \models \bigwedge \Phi \wedge \Diamond A$ it follows $y \models F$, a contradiction. For the second case, the output formula occurs in Ω . Assume this rule application does not preserve validity, then there exists a bi-relational model $\mathcal{M} = (W, \leq, R, V)$ with $x \in W$ such that $x \not\models \Omega, [\Phi, \Diamond A^\bullet]$. By definition, regardless of the exact location of the output formula, there exists $x' \in W$ such that $x' \geq x$ such that $x' \models \Diamond(\bigwedge \Phi \wedge \Diamond A)$. This further implies there is $y \in W$ such that $Rx'y$ and $y \models \bigwedge \Phi \wedge \Diamond A$. Meanwhile, since the premise is valid, we have $y \models \Phi, [A^\bullet], \perp^\circ$, which means $y \not\models \bigwedge \Phi \wedge \Diamond A$, a contradiction.

This completes the proof. $\square$

4 Admissibility of cut and completeness

In this section we prove SC_{FIK} is syntactically complete, meaning that all theorems of $\mathcal{C}_{\text{FIK}}$ can be derived by SC_{FIK} . As usual we first prove that the calculus SC_{FIK} augmented with suitable cut rules is complete. Then we show that the cut rules are admissible in SC_{FIK} . In accordance with the form of the sequents, we need to consider two cut rules that operate respectively on the top level and within a block.

Definition 4.1 (Cut-rules). The cut rules are the following:

$$\frac{\Omega^*, A^\circ \quad A^\bullet, \Omega}{\Omega} (\text{cut}) \quad \frac{\Omega^*, [\Phi^*, A^\circ] \quad \Omega, [A^\bullet, \Phi]}{\Omega, [\Phi]} (\text{cut}^{[\cdot]})$$

In both rules, the formula A which occurs explicitly in both premisses in both polarized forms $A^\bullet, A^\circ$ is called the **cut formula**.

We can easily prove the syntactic completeness of SC_{FIK} augmented by the two cut rules.

Theorem 4.2 (Completeness of $\text{SC}_{\text{FIK}} + (\text{cut}) + (\text{cut}^{[\cdot]})$). *If a formula A is provable in $\mathcal{C}_{\text{FIK}}$, then it is provable in $\text{SC}_{\text{FIK}} + (\text{cut}) + (\text{cut}^{[\cdot]})$.*

Proof. We show that any instance of axioms of $\mathcal{C}_{\text{FIK}}$ can be proved in SC_{FIK} and that the rule of (mp) is simulated by (cut) as usual. The proof of axiom (wCD) is given in Figure 5 as an example. For rule (nec), we prove by induction on a derivation that for any formula A if A° is provable in $\text{SC}_{\text{FIK}} + (\text{cut}) + (\text{cut}^{[\cdot]})$, $[A^\circ]$ is also provable in the same system, and then we conclude by an application of $(\Box_1^\circ)$. $\square$

The rest of the section is devoted to a syntactic proof of the admissibility of both cut rules. We introduce some preliminary definitions and prove some preliminary facts.

Definition 4.3 (Admissibility and invertibility). Let $\mathcal{R}$ be an inference rule of the form $\frac{S_1 (S_2)}{S}$.⁵ We say that $\mathcal{R}$ is *admissible* in SC_{FIK} if for every application of the rule $(r) = \frac{S_1 (S_2)}{S}$: if $\vdash_{n_i} S_i$ for some n_i , then $\vdash_n S$ for some n . Rule $\mathcal{R}$ is further called *height-preserving admissible* if for every application it holds $n \leq n_i$ for $i \in \{1, 2\}$. Conversely, $\mathcal{R}$ is called *invertible* in SC_{FIK} if for every application $\frac{S_1 (S_2)}{S}$ whenever $\vdash_n S$ it holds $\vdash_{n_i} S_i$ for $i \in \{1, 2\}$. and further *height-preserving invertible* if $n_i \leq n$ for $i \in \{1, 2\}$.

As usual, weakening is hp-admissible by induction on the height of a derivation:

⁵By $\frac{S_1 (S_2)}{S}$ we mean two possible rules either with one premise S_1 or two premises S_1 and S_2 .

invertible rules	$(\wedge^\bullet)(\wedge^\circ)(\vee^\bullet)(\Box^\bullet)(\supset_1^\circ)(\Box_1^\circ)(\Diamond_1^\bullet)$
non-invertible rules	$(\vee^\circ)(\supset_2^\circ)(\Box_2^\circ)(\Diamond^\circ)(\Diamond_2^\bullet)$
rules with one invertible premise	$(\supset^\bullet)$

Table 1: Invertible and non-invertible rules in SC_{FIK}

Proposition 4.4. *The following structural rules are height-preserving admissible in SC_{FIK} .*

$$\frac{\Omega}{A^\bullet, \Omega} (w_1^\bullet) \quad \frac{\Omega, [\Phi]}{\Omega, [A^\bullet, \Phi]} (w_2^\bullet) \quad \frac{\Omega}{\Omega, [\Phi]} (w_{[\cdot]}^+)$$

Lemma 4.5 (Inversion lemma). $(\wedge^\bullet)(\wedge^\circ)(\vee^\bullet)(\Box^\bullet)(\supset_1^\circ)(\Box_1^\circ)$ and $(\Diamond_1^\bullet)$ are height-preserving invertible in SC_{FIK} . Moreover, $(\supset^\bullet)$ is height-preserving invertible only for the left premise, which means if $\vdash_n \Omega, A \supset B^\bullet$ then $\vdash_n \Omega, B^\bullet$ and if $\vdash_n \Omega, [\Phi, A \supset B^\bullet]$ then $\vdash_n \Omega, [\Phi, B^\bullet]$.

According to the lemma above, the rules in SC_{FIK} are then divided into the invertible and non-invertible groups, see Table 1. Moreover, with the lemma and definition above, we show the following,

Proposition 4.6. *The following structural rules are height-preserving admissible in SC_{FIK} :*

$$\frac{A^\bullet, A^\bullet, \Omega}{A^\bullet, \Omega} (c_1^\bullet) \quad \frac{\Omega, [A^\bullet, A^\bullet, \Phi]}{\Omega, [A^\bullet, \Phi]} (c_2^\bullet) \quad \frac{\Omega, [\Phi], [\Phi^*]}{\Omega, [\Phi]} (c_{[\cdot]}^-) \quad \frac{\Phi}{\Omega, [\Phi]} ([\cdot])$$

Corollary 4.7. *If Ω, Ω^* is provable, then Ω is provable.*

Next, we introduce the following measure in order to show cut admissibility.

Definition 4.8 (rank of cut-application). To each application (r) of (cut) (resp. $(\text{cut}^{[\cdot]})$) in $\mathcal{D}$ whose premises are provable, we associate $\mathbf{r}_{\text{cut}}(d, h)$ (resp. $\mathbf{r}_{\text{cut}^{[\cdot]}}(d, h)$) as the **rank** of it, where d denotes size of the cut formula and h denotes sum of the heights of the proofs of two premises of (r) .⁶ We take the lexicographic order on the pair (d, h) as the order on the rank $\mathbf{r}_{\text{cut}}(d, h)$ (resp. $\mathbf{r}_{\text{cut}^{[\cdot]}}(d, h)$).

Theorem 4.9 (Cut-admissibility). *Both (cut) and $(\text{cut}^{[\cdot]})$ are admissible in SC_{FIK} , meaning that if a sequent Ω is derivable in $\text{SC}_{\text{FIK}} + \text{cut} + \text{cut}^{[\cdot]}$ then it is also derivable in SC_{FIK} .*

Proof. We carry on the proof by mutual induction on $\text{cut}(d_1, h_1)$ and $\text{cut}^{[\cdot]}(d_2, h_2)$, where

- $\text{cut}(d_1, h_1) :=$ every application (r) of cut in a derivation has a rank strictly smaller than (d_1, h_1)
- $\text{cut}^{[\cdot]}(d_2, h_2) :=$ every application (r) of $\text{cut}^{[\cdot]}$ in a derivation has a rank strictly smaller than (d_2, h_2)

Therefore, the induction hypotheses we use are the following:

$$\text{IH1 } \forall (d_1, h_1). \left[(\forall (d_2, h_2) < (d_1, h_1). \text{cut}^{[\cdot]}(d_2, h_2)) \ \& \ (\forall (d'_1, h'_1) < (d_1, h_1). \text{cut}(d'_1, h'_1)) \right] \rightarrow \text{cut}(d_1, h_1)$$

$$\text{IH2 } \forall (d_2, h_2). \left[(\forall (d_1, h_1) < (d_2, h_2). \text{cut}(d_1, h_1)) \ \& \ (\forall (d'_2, h'_2) < (d_2, h_2). \text{cut}^{[\cdot]}(d'_2, h'_2)) \right] \rightarrow \text{cut}^{[\cdot]}(d_2, h_2)$$

The two inductive hypotheses will take care of the two derivations of the following form, denoted as $\mathcal{D}$ and $\mathcal{E}$ respectively,

⁶Usually d denotes formula complexity but it also works here by using size, since the size of a formula strictly decreases in accordance with proper subformulas.

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D}_1 & & \mathcal{D}_2 \\
\hline
\Omega^*, A^\circ & & A^\bullet, \Omega \\
\hline
\Omega & & (\text{cut})
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccc}
\mathcal{E}_1 & & \mathcal{E}_2 \\
\hline
\Omega^*, [\Phi^*, A^\circ] & & \Omega, [A^\bullet, \Phi] \\
\hline
\Omega, [\Phi] & & (\text{cut}^{[\cdot]})
\end{array}
\end{array}$$

The structure of the proof is as follows. We consider different cases according to the role of the cut formula in the premises of (cut) for the last step in $\mathcal{D}_1$ and $\mathcal{D}_2$, then similarly for the roles of the cut formula in the premises of (cut^[·]) for the last step in $\mathcal{E}_1$ and $\mathcal{E}_2$. Our goal is to push up each application of (cut) and (cut^[·]) to some upper applications with strictly smaller ranks.

First, for $\mathcal{D}$, let us denote the last rule applied in $\mathcal{D}_1$ and $\mathcal{D}_2$ as $(r_1), (r_2)$ respectively. If either A° is weak in (r_1) or $A^\bullet$ is weak in (r_2) , we only need to apply the same rule with another context without that occurrence of A . Otherwise, neither A° nor $A^\bullet$ is weak, then one of the following holds: (D1) one of the premises is an axiom namely the inductive base (cut applied at height 0); and for inductive step, (D2) A is principal in **both** premises; (D3) A° is principal in (r_1) and $A^\bullet$ is side in (r_2) ; (D4) A° is side in (r_1) .

Next, for $\mathcal{E}$, similar to the first half of the proof, we discuss possible different roles of the cut formula in the premises of (cut^[·]) for the last step in $\mathcal{E}_1$ and $\mathcal{E}_2$. We denote the last rule applied in $\mathcal{E}_1$ and $\mathcal{E}_2$ as $(e_1), (e_2)$ respectively. In addition to the weak cases, we have: (E1) one of the premises is an axiom; (E2) cut formula A is principal in **both** premises; (E3) A° is principal in (e_1) and $A^\bullet$ is side in (e_2) ; (E4) A° is side in (e_1) .

Let us consider all the cases of (D1)-(D4) and (E1)-(E4) in turn.

(D1) one of the premises is an axiom, which implies one of the following: (1) Ω^* or Ω is an axiom, so is the sequent on the conclusion; (2) $A = p$ and $p^\bullet \in \Omega$, then we obtain Ω just by contraction from the right premise $A^\bullet, \Omega$; (3) $A = p$ and $p^\circ \in \Omega$, dual to the second case.

(D2) cut formula A is principal in both premises, we consider the following non-trivial sub-cases:

- If $A = B \supset C$, we have

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D}'_1 & & \mathcal{D}'_2 & & \mathcal{D}''_2 \\
\hline
\Omega^*, B^\bullet, C^\circ & & B \supset C^\bullet, \Omega^*, B^\circ & & C^\bullet, \Omega \\
\hline
\Omega^*, B \supset C^\circ & & B \supset C^\bullet, \Omega & & (\text{cut}) \\
\hline
\Omega & & & &
\end{array}
\end{array}
\rightsquigarrow
\begin{array}{c}
\frac{\Omega^*, B \supset C^\circ \quad B \supset C^\bullet, \Omega^*, B^\circ}{\Omega^*, B^\bullet} (\text{cut}) \quad \frac{\Omega^*, B^\bullet, C^\circ}{\Omega^*, C^\circ} (\text{cut}) \quad \frac{C^\bullet, \Omega}{\Omega} (\text{cut})
\end{array}$$

- If $A = \Box B$, further assume $\Omega = \Omega', [\Phi]$, then we have

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D}'_1 & & \mathcal{D}'_2 \\
\hline
\Omega^*, [B^\circ] & & \Omega', \Box B^\bullet, [B^\bullet, \Phi] \\
\hline
\Omega^*, \Box B^\circ & & \Omega', \Box B^\bullet, [\Phi] \\
\hline
\Omega', [\Phi] & & (\text{cut})
\end{array}
\end{array}
\rightsquigarrow
\begin{array}{c}
\frac{\Omega^*, [B^\circ] \quad \frac{\Omega', \Box B^\bullet, [B^\bullet, \Phi] \quad \frac{\Omega^*, [\Phi^*], \Box B^\circ}{\Omega^*, [B^\bullet, \Phi^*], \Box B^\circ} (w_{[\cdot]}^\bullet)}{\Omega', [\Phi^*], \Box B^\circ} (w_{[\cdot]}^\bullet) \quad \frac{\Omega', \Box B^\bullet, [B^\bullet, \Phi]}{\Omega', [B^\bullet, \Phi]} (\text{cut})}{\Omega, [\Phi]} (\text{cut}^{[\cdot]})
\end{array}$$

- If $A = \Diamond B$, assume $\Omega = \Omega', [\Phi]$ and the output formula of the conclusion is in Φ , then we have

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D}'_1 & & \mathcal{D}'_2 \\
\hline
\Omega^*, [\Phi^*, B^\circ] & & \Omega', [\Phi], [B^\bullet] \\
\hline
\Omega^*, [\Phi^*], \Diamond B^\circ & & \Diamond B^\bullet, \Omega', [\Phi] \\
\hline
\Omega', [\Phi] & & (\text{cut})
\end{array}
\end{array}
\rightsquigarrow
\begin{array}{c}
\frac{\Omega^*, [\Phi^*, B^\circ]}{\Omega^*, [\Phi^*, B^\circ], [\Phi^*]} (w_{[\cdot]}^+) \quad \frac{\Omega', [\Phi], [B^\bullet]}{\Omega', [\Phi], [\Phi^*, B^\bullet]} (w_{[\cdot]}^\bullet)}{\Omega', [\Phi^*], [\Phi]} (w_{[\cdot]}^-) \quad \frac{\Omega', [\Phi^*], [\Phi]}{\Omega', [\Phi]} (\text{cut}^{[\cdot]})
\end{array}$$

The other case when the output formula is in Ω' is similar.

(D3) A° is principal in (r_1) while $A^\bullet$ is side in (r_2) . We have the following sub-cases:

- (i) r_2 is one of the invertible rules except $\Box_1^\circ$. As an example, we show the case when $r_2 = \supset_1^\bullet$. Suppose $\Omega = B \supset C^\bullet, \Omega'$, in this case, we have the following derivation unfolding $\mathcal{D}$,

$$\frac{\frac{\mathcal{D}'_1}{B \supset C^\bullet, \Omega'^*, A^\circ} \quad \frac{\frac{\mathcal{D}'_{21}}{A^\bullet, B \supset C^\bullet, \Omega'^*, B^\circ} \quad \frac{\mathcal{D}'_{22}}{A^\bullet, C^\bullet, \Omega'} (\supset_1^\bullet)}{A^\bullet, B \supset C^\bullet, \Omega'} (\supset_1^\bullet)}{B \supset C^\bullet, \Omega'^*} (\text{cut})$$

Then we transform the derivation into the following

$$\frac{\frac{B \supset C^\bullet, \Omega'^*, A^\circ}{B \supset C^\bullet, B \supset C^\bullet, \Omega'^*, A^\circ} (w^\bullet) \quad \frac{A^\bullet, B \supset C^\bullet, \Omega'^*, B^\circ}{A^\bullet, B \supset C^\bullet, B \supset C^\bullet, \Omega'^*, B^\circ} (w^\bullet)}{B \supset C^\bullet, \Omega'^*, B^\circ} (\text{cut}) \quad \frac{\frac{B \supset C^\bullet, \Omega'^*, A^\circ}{C^\bullet, B \supset C^\bullet, \Omega'^*, A^\circ} (w^\bullet) \quad \frac{A^\bullet, C^\bullet, \Omega'}{A^\bullet, C^\bullet, B \supset C^\bullet, \Omega'} (w^\bullet)}{C^\bullet, B \supset C^\bullet, \Omega'} (\supset_1^\bullet)}{B \supset C^\bullet, B \supset C^\bullet, \Omega'} (w^\bullet)}{B \supset C^\bullet, \Omega'} (w^\bullet)$$

- (ii) suppose $r_2 = \Box_1^\circ$ and $\Omega = \Omega', \Box B^\circ$. Hence $\Omega^* = \Omega'$ and the right premise of cut is derived from $A^\bullet, \Omega', [B^\circ]$. We have the following sub-cases according to the shape of A ,

- $A = C \vee D$ and $r_1 = \vee^\circ$, in this case, the left premise of (cut) is $\Omega_1, C \vee D^\circ$ and is derived from Ω_1, C° , the right premise of (cut) is $C \vee D^\bullet, \Omega'_2, \Box B^\circ$. By Lemma 4.5, both $C^\bullet, \Omega'_2, \Box B^\circ$ and $D^\bullet, \Omega'_2, \Box B^\circ$ are derivable at the same height. Then the proof proceeds in the same way as in (D2). $A = C \wedge D$ is similar.
- $A = C \supset D$ and $r_1 = \supset_1^\circ$, we apply (cut) to the premise of $(\Box_1^\circ)$ and then apply $\Box_1^\circ$.
- $A = \Box C$ and $r_1 = \Box_1^\circ$, we have

$$\frac{\frac{\mathcal{D}'_1}{\Omega', [C^\circ]} \quad \frac{\mathcal{D}'_2}{\Box C^\bullet, \Omega', [B^\circ]} (\Box_1^\circ)}{\frac{\Omega', [C^\circ]}{\Omega', \Box C^\circ} \quad \frac{\Box C^\bullet, \Omega', [B^\circ]}{\Box C^\bullet, \Omega', \Box B^\circ} (\Box_1^\circ)} (\text{cut}) \quad \rightsquigarrow \quad \frac{\Omega', [C^\circ]}{\Omega', [C^\circ]} \quad \frac{\frac{\Omega', \Box C^\circ}{\Omega', [C^\bullet, B^\circ]} (\text{cut})}{\frac{\Omega', [B^\circ]}{\Omega', \Box B^\circ} (\Box_1^\circ)} (\text{cut}^{[1]})$$

- $A = \Diamond C$, $r_1 = \Diamond_1^\circ$ and $\Omega' = \Omega'', [\Phi]$, since $\Omega = \Omega', \Box B^\circ$, we have $\Omega^* = \Omega'', [\Phi]$.

$$\frac{\frac{\mathcal{D}'_1}{\Omega'', [\Phi, C^\circ]} \quad \frac{\mathcal{D}'_2}{\Diamond C^\bullet, \Omega'', [\Phi], [B^\circ]} (\Diamond_1^\circ)}{\frac{\Omega'', [\Phi, C^\circ]}{\Omega'', \Diamond C^\circ, [\Phi]} \quad \frac{\Diamond C^\bullet, \Omega'', [\Phi], [B^\circ]}{\Diamond C^\bullet, \Omega'', [\Phi], \Box B^\circ} (\Diamond_1^\circ)} (\text{cut}) \quad \rightsquigarrow \quad \frac{\frac{\Omega'', [\Phi, C^\circ]}{\Omega'', [\Phi], [\Phi, C^\circ]} (w_{[\Phi]}^+) \quad \frac{\Omega'', [\Phi], \Box B^\circ, [C^\bullet]}{\Omega'', [\Phi], \Box B^\circ, [C^\bullet, \Phi]} (w^\bullet)}{\frac{\Omega'', [\Phi], \Box B^\circ, [\Phi]}{\Omega'', [\Phi], \Box B^\circ} (c_{[\Phi]}^-)} (\text{cut}^{[1]})$$

- (iii) r_2 is one of the non-invertible rules, according to the form of rules, we see $r_2 \in \{\Diamond^\circ, \vee_1^\circ, \vee_2^\circ\}$. Among all these three cases can be done by switching the order of (e_2) and (cut), we only show the case when $r_2 = \vee_2^\circ$ as an example. Assume $\Omega = \Omega', [\Phi, B \vee C^\circ]$. We have

$$\frac{\frac{\mathcal{E}'_1}{\vdots} \quad \frac{\mathcal{E}'_2}{A^\bullet, \Omega', [\Phi, B^\circ]} (\vee_2^\circ)}{\frac{\Omega', [\Phi], A^\circ}{\Omega', [\Phi], A^\circ} \quad \frac{A^\bullet, \Omega', [\Phi, B \vee C^\circ]}{A^\bullet, \Omega', [\Phi, B \vee C^\circ]} (\vee_2^\circ)} (\text{cut}) \quad \rightsquigarrow \quad \frac{\Omega', [\Phi], A^\circ}{\Omega', [\Phi], A^\circ} \quad \frac{A^\bullet, \Omega', [\Phi, B^\circ]}{\Omega', [\Phi, B^\circ]} (\text{cut})}{\frac{\Omega', [\Phi, B \vee C^\circ]}{\Omega', [\Phi, B \vee C^\circ]} (\vee_2^\circ)} (\vee_2^\circ)$$

(D4) cut formula A° is *side* in the left premise. This further implies the last rule (r_1) applied in $\mathcal{D}_1$ is one of the invertible $\bullet$ -rules since other rules do not have side $\circ$ -formulas (outside a block) on the conclusion. The proof is the same as the *side* case (i) covered in **(D3)**.

(E1) one of the premises is an axiom, similar to the corresponding cases in (D1);

(E2) cut formula A is principal in both premises. Note that Φ is a simple sequent, namely block-free, so A cannot be of the form $\Box B$ or $\Diamond B$ under the assumption of (E2). $A = B \vee C$ or $B \wedge C$ is trivial, we only consider $A = B \supset C$, we have

$$\begin{array}{c}
 \begin{array}{c} \triangleleft_{\mathcal{E}_1'} \\ \Phi^*, B^\bullet, C^\circ \end{array} \quad \begin{array}{c} \triangleleft_{\mathcal{E}_2'} \\ \Omega^*, [B \supset C^\bullet, \Phi^*, B^\circ] \end{array} \quad \begin{array}{c} \triangleleft_{\mathcal{E}_2''} \\ \Omega, [C^\bullet, \Phi] \end{array} \quad \rightsquigarrow \\
 \frac{\frac{\Phi^*, B^\bullet, C^\circ}{\Omega^*, [\Phi^*, B \supset C^\circ]} (\supset_2^\circ) \quad \frac{\Omega^*, [B \supset C^\bullet, \Phi^*, B^\circ] \quad \Omega, [C^\bullet, \Phi]}{\Omega, [B \supset C^\bullet, \Phi]} (\supset_2^\bullet)}{\Omega, [\Phi]} (\text{cut}^{[\cdot]}) \\
 \\
 \frac{\frac{\Omega^*, [\Phi^*, B \supset C^\circ] \quad \Omega^*, [B \supset C^\bullet, \Phi^*, B^\circ]}{\Omega^*, [\Phi^*, B^\circ]} (\text{cut}^{[\cdot]}) \quad \frac{\Phi^*, B^\bullet, C^\circ}{\Omega^*, [\Phi^*, B^\bullet, C^\circ]} ([\cdot])}{\Omega^*, [\Phi^*, C^\circ]} (\text{cut}^{[\cdot]}) \quad \frac{\Omega, [C^\bullet, \Phi]}{\Omega, [\Phi]} (\text{cut}^{[\cdot]})
 \end{array}$$

(E3) A° is principal in (e_1) while $A^\bullet$ is *side* in (e_2) . Similar to (D3).

(E4) A° is *side* in (e_1) . This implies the last rule (e_1) applied in $\mathcal{E}_1$ is one of the $\bullet$ -invertible rules or is $(\Diamond_2^\bullet)$. For the invertible rules, the proof is similar to the *side* case (i) covered in (D3). Here we only show the case when $e_1 = \Diamond_2^\bullet$. In this case, $\Phi = \Diamond B^\bullet, \Phi'$ and derivation $\mathcal{E}$ is unfolded as

$$\begin{array}{c}
 \vdots \\
 e_1' \frac{\Phi'^*, [B^\bullet], A^\circ}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, A^\circ]} \quad e_2 \frac{\vdots}{\Omega, [A^\bullet, \Phi', \Diamond B^\bullet]} \\
 \hline
 \Omega, [\Phi', \Diamond B^\bullet]
 \end{array}$$

We discuss the role of $A^\bullet$ in (e_2) and the role of A° in (e_1') . We have the following cases where neither $A^\bullet$ nor A° is weak:

(i) both $A^\bullet$ and A° are *principal*, then according to the form of rules, A cannot be a $\Box$ -formula. Thus we have the following sub-cases:

• $A = C \wedge D$ or $C \vee D$. We only show the first one. In this case, we have

$$\begin{array}{c}
 \begin{array}{c} \triangleleft_{\mathcal{E}_{11}'} \\ \Phi'^*, [B^\bullet], C^\circ \end{array} \quad \begin{array}{c} \triangleleft_{\mathcal{E}_{12}'} \\ \Phi'^*, [B^\bullet], D^\circ \end{array} \quad \begin{array}{c} \triangleleft_{\mathcal{E}_2'} \\ \Omega, [C^\bullet, D^\bullet, \Phi', \Diamond B^\bullet] \end{array} \quad \rightsquigarrow \\
 \frac{\frac{\Phi'^*, [B^\bullet], C^\circ}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C \wedge D^\circ]} (\wedge_2^\circ) \quad \frac{\Omega, [C^\bullet, D^\bullet, \Phi', \Diamond B^\bullet]}{\Omega, [C \wedge D^\bullet, \Phi', \Diamond B^\bullet]} (\wedge_2^\bullet)}{\Omega, [\Phi', \Diamond B^\bullet]} (\text{cut}^{[\cdot]}) \\
 \\
 \frac{\frac{\Phi'^*, [B^\bullet], D^\circ}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, D^\circ]} (\Diamond_2^\bullet) \quad \frac{\frac{\Phi'^*, [B^\bullet], C^\circ}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C^\circ]} (\Diamond_2^\bullet) \quad \Omega, [C^\bullet, D^\bullet, \Phi', \Diamond B^\bullet]}{\Omega, [D^\bullet, \Phi', \Diamond B^\bullet]} (w^\bullet)}{\Omega, [\Phi', \Diamond B^\bullet]} (\text{cut}^{[\cdot]})
 \end{array}$$

• $A = C \supset D$. In this case, derivation $\mathcal{E}$ is unfolded as

$$\begin{array}{c}
\begin{array}{c} \triangleleft_{\mathcal{E}'_1} \\ \hline \frac{\Phi'^*, [B^\bullet], C^\bullet, D^\circ}{\Phi'^*, [B^\bullet], C \supset D^\circ} (e'_1) \\ \hline \frac{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C \supset D^\circ]}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C \supset D^\circ]} (e_1) \end{array} \quad \frac{\frac{\Omega^*, [C \supset D^\bullet, \Phi'^*, \Diamond B^\bullet, C^\circ]}{\Omega^*, [C \supset D^\bullet, \Phi', \Diamond B^\bullet]} \quad \frac{\Omega, [D^\bullet, \Phi', \Diamond B^\bullet]}{\Omega, [C \supset D^\bullet, \Phi', \Diamond B^\bullet]} (e_2)}{\Omega, [\Phi', \Diamond B^\bullet]} (\text{cut}^{[\cdot]})
\end{array}$$

and we transform it into

$$\frac{\frac{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C \supset D^\circ]}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C^\circ]} \quad \frac{\Omega^*, [C \supset D^\bullet, \Phi'^*, \Diamond B^\bullet, C^\circ]}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C^\circ]} (\text{cut}^{[\cdot]})}{\frac{\frac{\Phi'^*, [B^\bullet], C^\bullet, D^\circ}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, C^\bullet, D^\circ]} (\Diamond_2^\bullet)}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, D^\circ]} (\text{cut}^{[\cdot]})} \quad \frac{\Omega, [D^\bullet, \Phi', \Diamond B^\bullet]}{\Omega, [\Phi', \Diamond B^\bullet]} (\text{cut}^{[\cdot]})$$

- $A = \Diamond C$. In this case, we have

$$\begin{array}{c}
\begin{array}{c} \triangleleft_{\mathcal{E}'_1} \\ \hline \frac{\Phi'^*, [B^\bullet, C^\circ]}{\Phi'^*, [B^\bullet], \Diamond C^\circ} (e'_1) \\ \hline \frac{\Omega^*, [\Phi'^*, \Diamond B^\bullet, \Diamond C^\circ]}{\Omega^*, [\Phi'^*, \Diamond B^\bullet, \Diamond C^\circ]} (e_1) \end{array} \quad \frac{\frac{[C^\bullet], \Phi'^+, \Diamond B^\bullet}{\Omega, [\Diamond C^\bullet, \Phi', \Diamond B^\bullet]} (e_2)}{\Omega, [\Phi', \Diamond B^\bullet]} (\text{cut}^{[\cdot]}) \quad \rightsquigarrow \quad \frac{\frac{\Phi'^*, [B^\bullet, C^\circ]}{\Phi'^*, [B^\bullet, C^\circ], \Diamond B^\bullet} (w^\bullet) \quad \frac{[C^\bullet], \Phi'^+, \Diamond B^\bullet}{[C^\bullet, B^\bullet], \Phi'^+, \Diamond B^\bullet} (w^\bullet)}{\frac{[B^\bullet], \Phi'^+, \Diamond B^\bullet}{\Omega, [\Diamond B^\bullet, \Phi', \Diamond B^\bullet]} (\Diamond_2^\bullet)} (\text{cut}^{[\cdot]}) \\
\frac{\Omega, [\Diamond B^\bullet, \Phi', \Diamond B^\bullet]}{\Omega, [\Phi', \Diamond B^\bullet]} (c^\bullet)
\end{array}$$

(ii) $A^\bullet$ is *side* in (e_2) , then according to the form of rules, e_2 can be any rule of the calculus.

- if e_2 is an invertible rule or $\Diamond^\circ, \vee^\circ$, we switch the order of (e_2) and $(\text{cut}^{[\cdot]})$, that is, apply $(\text{cut}^{[\cdot]})$ to the premise of (e_2) and the conclusion of (e'_1) first, then apply (e_2) .
- if $e_2 \in \{\supset_2^\circ, \Box_2^\circ, \Diamond_2^\bullet\}$, the three cases are similar and we only show one here. Assume $\Phi' = \Psi, C \supset D^\circ$, then $\Omega^* = \Omega, \Phi'^* = \Psi$, and we have

$$\begin{array}{c}
\begin{array}{c} \triangleleft_{\mathcal{E}'_1} \\ \vdots \\ \hline \frac{\Psi, [B^\bullet], A^\circ}{\Psi, [B^\bullet], A^\circ} (e'_1) \\ \hline \frac{\Omega, [\Psi, \Diamond B^\bullet, A^\circ]}{\Omega, [\Psi, \Diamond B^\bullet, A^\circ]} (e_1) \end{array} \quad \frac{\frac{A^\bullet, \Psi, \Diamond B^\bullet, C^\bullet, D^\circ}{\Omega, [A^\bullet, \Psi, \Diamond B^\bullet, C \supset D^\circ]} (e_2)}{\Omega, [\Psi, \Diamond B^\bullet, C \supset D^\circ]} (\text{cut}^{[\cdot]}) \quad \rightsquigarrow \quad \frac{\frac{\Psi, [B^\bullet], A^\circ}{\Psi, [B^\bullet], \Diamond B^\bullet, C^\bullet, A^\circ} (w^\bullet) \quad \frac{A^\bullet, \Psi, \Diamond B^\bullet, C^\bullet, D^\circ}{A^\bullet, \Psi, [B^\bullet], \Diamond B^\bullet, C^\bullet, D^\circ} (w_1^\bullet)}{\frac{\Psi, [B^\bullet], \Diamond B^\bullet, C^\bullet, D^\circ}{\Psi, \Diamond B^\bullet, \Diamond B^\bullet, C^\bullet, D^\circ} (\Diamond_1^\bullet)} (\text{cut}) \\
\frac{\Psi, \Diamond B^\bullet, \Diamond B^\bullet, C^\bullet, D^\circ}{\Psi, \Diamond B^\bullet, C^\bullet, D^\circ} (c^\bullet) \\
\frac{\Omega, [\Psi, \Diamond B^\bullet, C \supset D^\circ]}{\Omega, [\Psi, \Diamond B^\bullet, C \supset D^\circ]} (\supset_2^\circ)
\end{array}$$

(iii) A° is *side* in (e'_1) and $A^\bullet$ is *principal* in (e_2) . Similar to (iii), A cannot be a $\Box$ -formula. For cases when $A = C \wedge D$ or $C \supset D$, the proof is the same to (iii), since the corresponding $\circ$ -rules with respect to A° are invertible. Thus we only have two cases remaining: $A = \Diamond D$ or $A = C \vee D$. We show the former case as the latter is similar.

In this case, the derivation $\mathcal{E}$ is unfolded as follows, where $\Psi = \Phi'$,

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \hline \frac{e'_1 \quad \Psi^*, [B^\bullet], \Diamond D^\circ}{\Omega^*, [\Psi^*, \Diamond B^\bullet, \Diamond D^\circ]} \\ \hline \frac{\Omega^*, [\Psi^*, \Diamond B^\bullet, \Diamond D^\circ]}{\Omega^*, [\Psi^*, \Diamond B^\bullet, \Diamond D^\circ]} (e_1) \end{array} \quad \frac{\frac{[D^\bullet], \Psi^+, \Diamond B^\bullet}{\Omega, [\Diamond D^\bullet, \Psi, \Diamond B^\bullet]} (e_2)}{\Omega, [\Psi, \Diamond B^\bullet]} \text{cut}^{[\cdot]}
\end{array}$$

- (a) If $e'_1 = \Diamond_2^\bullet$, the conclusion of cut follows from the premise of (e'_1) by $(\Diamond_2^\bullet)$ twice.
- (b) If e'_1 is one of the $\bullet_1$ -rules except $\Diamond_1^\bullet$, then from the premise of (e'_1) , we apply (e_1) first, then $(\text{cut}^{[\cdot]})$ to the conclusion of (e_1) with the conclusion of (e_2) , and the $\bullet_2$ -counterpart of (e'_1) in the end.
- (c) If e'_1 is $\Diamond^\circ$, then $\Psi, [B^\bullet, D^\circ]$ is derivable, we transform the whole derivation into

$$\begin{array}{c}
\frac{\Psi, [B^\bullet, D^\circ]}{\Psi, \Diamond B^\bullet, [B^\bullet, D^\circ]} (w^\bullet) \quad \frac{[D^\bullet], \Psi^+, \Diamond B^\bullet}{[D^\bullet, B^\bullet], \Psi^+, \Diamond B^\bullet} (w^\bullet) \\
\hline
\frac{}{[B^\bullet], \Psi^+, \Diamond B^\bullet} (\text{cut}^{[\cdot]}) \\
\hline
\frac{}{\Omega, [\Psi, \Diamond B^\bullet, \Diamond B^\bullet]} (\Diamond_2^\bullet) \\
\hline
\frac{}{\Omega, [\Psi, \Diamond B^\bullet]} (c^\bullet)
\end{array}$$

(d) If e'_1 is one of the $\bullet_2$ -rules or $\Diamond_1^\bullet$, let us consider the rules applied above e'_1 . Assume that before any $\circ$ -rule is applied, there are no $\bullet_1$ -rules applied above e'_1 , otherwise we return to case (a), switching the rule order and simulating the derivation with the $\bullet_2$ -counterpart as above. Let us enumerate the branches of $\mathcal{E}'_1$ as $\mathcal{B}_1, \dots, \mathcal{B}_k$ and for each $\mathcal{B}_i$ further enumerate sequents contained in it as $\Phi_{i_0}, \dots, \Phi_{i_j}, \dots, \Phi_{i_m}$ such that $\Phi_{i_0} = \Phi^*, [B^\bullet], \Diamond D^\circ$ and Φ_{i_m} is an axiom. Then for each $\mathcal{B}_i$, it implies one of the following situations regarding the output formula:

- (h1) $\Diamond D^\circ$ is in each Φ_{i_j} .
- (h2) there is some $i_k \geq 1$ such that $\Phi_{i_k} = \Theta, [\Lambda], \Diamond D^\circ$ for some Θ, Λ and $\Phi_{i_{k+1}} = \Theta, [\Lambda, D^\circ]$; meanwhile, for any $i_j \leq i_k$, the output formula of Φ_{i_j} is $\Diamond D^\circ$.
- (h3) there is some $i_k \geq 1$ such that $\Phi_{i_k} = \Theta, [\Psi, P \supset Q^\bullet], \Diamond D^\circ$ for some Θ, Ψ, P, Q and $\Phi_{i_{k+1}} = \Theta, [\Psi, P \supset Q^\bullet, P^\circ]$; meanwhile, for any $i_j \leq i_k$, the output formula of Φ_{i_j} is $\Diamond D^\circ$.

We process each $\mathcal{B}_i$ respectively and construct the corresponding $\mathcal{B}'_i$ for it. If (h1) holds, this means $\Diamond D^\circ$ is unused and hence side in the whole branch, so we can just replace each occurrence of D° in $\mathcal{B}_i$ with the output formula F° of Ψ^+ and then get $\mathcal{B}'_i$. If (h2) holds, we reconstruct $\mathcal{B}_i$ into $\mathcal{B}'_i$ as follows. First we keep all the components Φ_{i_j} 's for $j > k + 1$. Next, we remove Φ_{i_k} ; and as in (b), cut $\Phi_{i_{k+1}}$ with the premise of (e_2) and obtain Φ'_{i_k} as follows

$$\frac{\frac{\Theta, [\Lambda, D^\circ]}{\Psi, \Diamond B^\bullet, \Theta, [\Lambda, D^\circ]} (w^\bullet) \quad \frac{[D^\bullet], \Psi^+, \Diamond B^\bullet}{[\Lambda, D^\bullet], \Psi^+, \Diamond B^\bullet, \Theta} (w^\bullet)}{\Psi^+, \Diamond B^\bullet, \Theta, [\Lambda]} (\text{cut}^{[\cdot]})$$

Then for all the Φ_{i_j} 's with $j < k$, let $\Phi'_{i_j} = \Psi, \Diamond B^\bullet, \Phi^*, F^\circ$.

Lastly, if (h3) holds, to construct $\mathcal{B}'_i$ from the current $\mathcal{B}$, we keep all the components Φ_{i_j} 's for $j > k$ and for $j \leq k$, we replace $\Diamond D^\circ$ with F° , which is the output formula of the final conclusion of $(\text{cut}^{[\cdot]})$. Thus, for instance, $\Phi'_{i_k} = \Theta, [\Psi, P \supset Q^\bullet], F^\circ$. Note that the way we process each branch $\mathcal{B}$ in fact eliminates the cut formula $\Diamond D^\circ$ in the whole branch by substituting with the output formula at certain points, but this does not influence other rule applications in the derivation, thus we can connect each $\mathcal{B}'_i$ and reconstruct the derivation with contraction ending with $\Psi^+, \Diamond B^\bullet, [B^\bullet]$. Then by $(\Diamond_2^\bullet)$ and contraction, we conclude $\Omega, [\Psi, \Diamond B^\bullet]$.

This completes the whole proof of cut-admissibility. $\square$

By the previous theorem we finally obtain the completeness of SC_{FIK} .

Theorem 4.10 (Completeness of SC_{FIK}). *If a formula A is provable in $\mathcal{C}_{\text{FIK}}$, then it is provable in SC_{FIK} .*

5 Proof search and complexity

In this section we show that the decision problem of **FIK** is in EXPSPACE by means of proof search in the calculus SC_{FIK} with some refinement. Before presenting the main result, we can easily show the calculus SC_{FIK} provides a decision problem with some rough upper bound by just minimal means.

To obtain decidability and a loose upper bound, we proceed as follows. We know by proposition 4.6 that contraction and weakening are admissible, hence we can restrict our attention to ‘set-sequents’ in the following sense:

Definition 5.1 ((quasi-)set-sequents). Let Ω be a sequent of the form $\Phi, [\Psi_1], \dots, [\Psi_1]$. Ω is called a *quasi-set-sequent* if each Φ, Ψ_i is a set (rather than a multiset) of formulas; Ω is further called a *set-sequent* if it is a quasi-set-sequent and for $i \neq j$, $\Psi_i \neq \Psi_j$.

Set-sequents in this sense do not contain duplicated formulas in the same ‘context’ (block/top-level sequent), nor duplicated blocks. It is then easy to prove that given a formula A° (as a root of a derivation), there are only finitely many *distinct* set-sequents that can occur in any derivation of A , no matter what are the rules of the calculus:

Proposition 5.2. *Let A be a formula and $|A| = \mathcal{O}(n)$. Then the size of any set-based sequent that may occur in any possible derivation of A is $2^{\mathcal{O}(n)}$, whence the set of all possible sequents that can occur in any possible derivation has cardinality $2^{2^{\mathcal{O}(n)}}$.*

Proof. Each set-sequent Ω is a member of the set $\mathcal{P}(\text{Sub}(A)) \times \mathcal{P}(\mathcal{P}(\text{Sub}(A)))$. Let $|A| = \mathcal{O}(n)$, we have $|\text{Sub}(A)| = \mathcal{O}(n)$, whence the size of Ω is $2^{\mathcal{O}(n)}$ and therefore the cardinality of the sets of such sequents is the cardinality of the set $\mathcal{P}(\text{Sub}(A)) \times \mathcal{P}(\mathcal{P}(\text{Sub}(A)))$ which is $2^{2^{\mathcal{O}(n)}}$. $\square$

From this result we can obtain directly a decision procedure and an upper bound for it: given a formula A° as root, we carry on backward proof search in SC_{FIK} storing one branch of a derivation at a time with the following provisos: (i) we do not apply rules to an axiom, (ii) we remove duplicated formulas or blocks in order to keep the set-based structure, (iii) we stop proof-search if the same sequent already appears in the branch. By the previous result it follows that the size of each branch is bounded by $2^{2^{\mathcal{O}(n)}}$. Whence the decision procedure runs in 2-EXPSpace. Notice that this bound, although elevated is elementary whence already much lower than what is conjectured for IK .

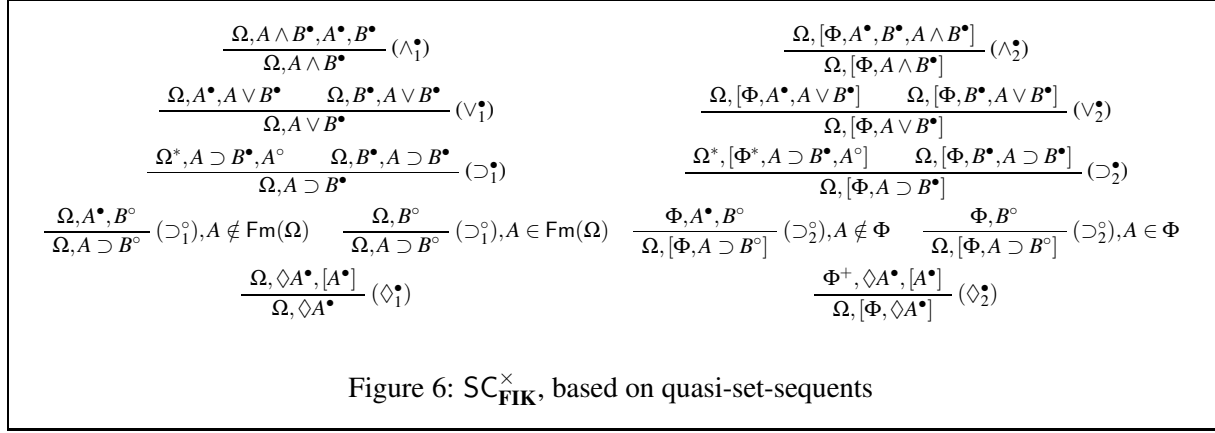
The rest of the section is devoted to refine this upper bound to EXPSpace. Although as many as $2^{2^{\mathcal{O}(n)}}$ set sequents can theoretically occur in a proof branch (by Proposition 5.2), we aim to show that only $2^{\mathcal{O}(n)}$ of them can actually occur. Our plan is as follows: We first define a variant of SC_{FIK} called $\text{SC}_{\text{FIK}}^\times$ based on quasi-set-sequents that preserves the admissibility of structural rules. Then we show that if a quasi-set-sequent is provable in SC_{FIK} then it is provable in $\text{SC}_{\text{FIK}}^\times$. Next we refine the notion of set-sequents to *tight sequents* which are set-sequents with an additional restriction. We show if a formula is provable in $\text{SC}_{\text{FIK}}^\times$ then it has a *clean* proof where any sequent is a tight sequent. Finally we define a suitable proof search strategy for finding clean proofs for formulas and show that the proof search procedure runs in EXPSpace.

We define the calculus $\text{SC}_{\text{FIK}}^\times$ for FIK based on quasi-set-sequents by modifying some of the rules in SC_{FIK} . The modified rules are found in Figure 6 and for other rules such as most $\circ$ -rules except $\supset^\circ$ as well as $\Box^\bullet$, they keep the same form as in SC_{FIK} . Comparing with SC_{FIK} there are a few changes. First, the input part of each rule now becomes *locally* cumulative by repeating the principal formula (if it is a $\bullet$ -formula) with respect to the corresponding block/top-level sequent. Second, as part of the machinery to avoid redundant rule applications, we divide both $\supset_1^\circ$ and $\supset_2^\circ$ into two sub-rules according to whether the premise A of the principal formula $A \supset B^\circ$ already occurs on formula part of the conclusion.

The following proposition is obtained directly by contraction in SC_{FIK} :

Proposition 5.3. *If a quasi-set-sequent Ω is provable in SC_{FIK} then it is provable in $\text{SC}_{\text{FIK}}^\times$.*

Note that $\text{SC}_{\text{FIK}}^\times$ operates on quasi-set-sequents for which we only assume internal contraction (contraction of formulas within a block) but allow duplicated blocks, hence weakening and external contraction (contraction of blocks) are hp-admissible as in the original SC_{FIK} . Therefore, we have



Proposition 5.4. Let $\Phi_1 \subseteq \Phi_2$ be two sets of input formulas. The following hold for quasi-set-sequents if $\vdash_{SC_{\mathbf{FIK}}^x}^n \Omega, [\Phi_1], F^°$ then $\vdash_{SC_{\mathbf{FIK}}^x}^n \Omega, [\Phi_1], [\Phi_2], F^°$ if $\vdash_{SC_{\mathbf{FIK}}^x}^n \Omega, [\Phi_1], F^°$ then $\vdash_{SC_{\mathbf{FIK}}^x}^n \Omega, [\Phi_1], [\Phi_2], F^°$

Next, we introduce a more restricted notion of set-sequents called ‘tight sequent’, which is required to generate derivations in which each sequent is set-based and at the same time its input parts are non-decreasing, meaning that the $\bullet$ -formulas are the same or cumulating along derivations, with respect to the same block. This in turn is essential to get the refined complexity bound.

Definition 5.5 (tight sequent and clean derivation). A *tight* sequent is a set-sequent satisfying the additional condition: if there is a block containing the output formula, then the input part of that block is either empty or is not identical to any other block. A derivation (resp. proof) containing only tight sequents is called *clean* derivation (resp. proof).

Example 5.6. $p \vee q^°, [p^°, q^°], [q^°], p^°$ is a tight sequent while neither (1) $p \vee q^°, [p^°, q^°], [q^°, p^°, p^°]$ nor (2) $p \vee q^°, [p^°, q^°], [p^°, q^°], p^°$ is, since in (1) the input of the block containing output formula is identical to another block; in (2) there are duplicated blocks.

Proposition 5.7. If a tight sequent $\Omega, [\Phi_1], \dots, [\Phi_k]$ (possibly $k = 0$) is provable in $SC_{\mathbf{FIK}}^x$ then there exist $\Psi_1, \dots, \Psi_l$ such that $\bigcup_{i=1}^l \Psi_i \subseteq \bigcup_{j=1}^k \Phi_j$, $l \leq k$ and the tight sequent $\Omega, [\Psi_1], \dots, [\Psi_l]$ has a clean proof.

Proof. Let $\Sigma = \Omega, [\Phi_1], \dots, [\Phi_k]$ and call sequents $\Omega, [\Psi_1], \dots, [\Psi_l]$ of the form in the proposition its sub-sequents. Assume Σ is provable in $SC_{\mathbf{FIK}}^x$ with a proof $\mathcal{D}$, we show that $\mathcal{D}$ can be transformed into a clean proof for some subsequent Σ_1 . The idea is, we find the first application which produces non-tight sequents in the premise and then reconstruct the derivation. We do this iteratively until we get a clean proof of some Σ_1 . First if a tight sequent is an initial sequent in $SC_{\mathbf{FIK}}^x$, then itself is a clean proof. Otherwise, let us consider the first rule application (r) that might produce non-tight sequents. We show the following cases and others are similar.

(1) $r = \wedge_2^*$ or $r = \vee_2^*$, we only show the case of conjunction. Consider an application with premise $\Omega', [\Phi, A \wedge B^*, A^*, B^*], [\Phi, A \wedge B^*, A^*, B^*]$ and conclusion $\Omega', [\Phi, A \wedge B^*], [\Phi, A \wedge B^*, A^*, B^*]$. From the premise by contraction in $SC_{\mathbf{FIK}}^x$, we obtain $\Omega', [\Phi, A \wedge B^*, A^*, B^*]$ which is a tight sequent. We remove this rule application in $\mathcal{D}$ and reconstruct $\mathcal{D}$ as follows: For rule applications above the original premise $\Omega', [\Phi, A \wedge B^*, A^*, B^*], [\Phi, A \wedge B^*, A^*, B^*]$, if it happens only in Ω' , we keep as it is; if it involves any of the two blocks $[\Phi, A \wedge B^*, A^*, B^*], [\Phi, A \wedge B^*, A^*, B^*]$, we apply them only in the single one. For rule applications below the original conclusion $\Omega', [\Phi, A \wedge B^*], [\Phi, A \wedge B^*, A^*, B^*]$, we remove them if it involves the block $[\Phi, A \wedge B^*]$ and keep the rest.

r	F	condition of redundancy
$\vee_1^\bullet$	$A \vee B^\bullet$	$A \vee B^\bullet \in \text{Fm}(\Omega)$, and $A^\bullet \in \text{Fm}(\Omega)$ or $B^\bullet \in \text{Fm}(\Omega)$
$\vee_2^\bullet$	$A \vee B^\bullet$	$A \vee B^\bullet \in \Phi$, and $A^\bullet \in \Phi$ or $B^\bullet \in \Phi$
$\supset_1^\bullet$	$A \supset B^\bullet$	$A \supset B^\bullet \in \text{Fm}(\Omega)$ and $B^\bullet \in \text{Fm}(\Omega)$
$\supset_2^\bullet$	$A \supset B^\bullet$	$A \supset B^\bullet \in \Phi$ and $B^\bullet \in \Phi$
$\diamond_1^\bullet$	$\diamond A^\bullet$	$\diamond A^\bullet \in \text{Fm}(\Omega)$ and $A^\bullet \in \Phi$ for some $[\Phi] \in \Omega$

Table 2: Conditions of redundancy for (r) on F

(2) $r = \diamond^\circ$. Consider an application with premise $\Omega', [\Phi, A^\circ], [\Phi, \Theta]$ and conclusion $\Omega', [\Phi], [\Phi, \Theta], \diamond A^\circ$. In this case, the premise has a block with output formula but its input part is included in some other block, so it is not tight. Assume w.l.o.g there are no other blocks such that Φ, Θ is a subset of it. By Proposition 5.4, we have $\Omega', [\Phi], [\Phi, \Theta, A^\circ]$. By contraction, we have $\Omega', [\Phi, \Theta, A^\circ]$. We replace the application by another one with premise $\Omega', [\Phi, \Theta, A^\circ]$ and conclusion $\Omega', [\Phi, \Theta], \diamond A^\circ$. Similar to the previous case, we reconstruct $\mathcal{D}$ as follows: For rule applications above the original premise $\Omega', [\Phi, A^\circ], [\Phi, \Theta]$, if it happens only in Ω' , we keep as it is; if it involves any of the two blocks $[\Phi, A^\circ], [\Phi, \Theta]$, we apply them only in the single one $[\Phi, \Theta, A^\circ]$. For rule applications below the conclusion $\Omega', [\Phi], [\Phi, \Theta], \diamond A^\circ$, if it involves the block $[\Phi]$ we remove it as well as all the occurrences of $[\Phi]$ and keep the rest.

(3) $r = \square^\bullet$. Consider the following three cases:

$$\frac{\Omega', \square A^\bullet, [\Phi, A^\bullet], [\Phi, A^\bullet, F^\circ]}{\Omega', \square A^\bullet, [\Phi], [\Phi, A^\bullet, F^\circ]} \square^\bullet \quad \frac{\Omega', \square A^\bullet, [\Phi, A^\bullet], [\Phi, A^\bullet]}{\Omega', \square A^\bullet, [\Phi], [\Phi, A^\bullet]} \square^\bullet \quad \frac{\Omega', \square A^\bullet, [\Phi, A^\bullet], [A^\bullet, \Phi, F^\circ]}{\Omega', \square A^\bullet, [A^\bullet, \Phi], [\Phi, F^\circ]} \square^\bullet$$

For the first two cases we have the reconstruction as in (2) and (1) respectively. Lastly for (3), first by weakening and contraction we have $\Omega', \square A^\bullet, [\Phi, A^\bullet, F^\circ]$. We replace the application above with another one with premise $\Omega', \square A^\bullet, [A^\bullet, \Phi, F^\circ]$ and conclusion $\Omega', \square A^\bullet, [\Phi, F^\circ]$ and reconstruct the derivation as: For rule applications above the original premise $\Omega', \square A^\bullet, [\Phi, A^\bullet], [A^\bullet, F^\circ]$, if it happens only in Ω' , we keep as it is; if it involves any of the two blocks $[\Phi, A^\bullet], [A^\bullet, \Phi, F^\circ]$, we apply them only in the single one $[\Phi, A^\bullet, F^\circ]$. For rule applications below the conclusion $\Omega', \square A^\bullet, [A^\bullet, \Phi], [\Phi, F^\circ]$, we remove these applications as well as all the occurrences of $[A^\bullet, \Phi]$ if they involves the block $[A^\bullet, \Phi]$ and keep the rest.

Each time we encounter the situations in (1)-(3), we reconstruct the proof until there are no any ‘bad’ rule applications as above. In the end, we obtain a clean proof of some sub-sequent of Σ . $\square$

Now we consider proof search in order to find a *clean proof* for any provable formula. To this end, we define the notion of redundancy in backward proof search.

Definition 5.8 (Redundant rule application). Let (r) be a $\bullet$ -rule in $\text{SC}_{\text{FIK}}^\times$. An application (r) to a sequent Ω on a formula F (and, whenever is involved, on a block $[\Phi] \in \Omega$) is called *redundant* if it satisfies one of the following conditions in Table 2.

We adopt a (backward) proof search strategy with the following constraints:

- (i) no rule is applied to an axiom; (ii) no rule is applied redundantly;
- (iii) no identical sequents occurring are allowed in the same branch;
- (iv) at each step, if the processed sequent is of the form $\Omega, [\Phi_1], [\Phi_2]$ and $\Phi_1 \subset \Phi_2$ then rule applications to Φ_1 is forbidden. As a result, for a sequent of the form $\Omega, [\Phi_1], \dots, [\Phi_k]$ with $\Phi_1 \subset \dots \subset \Phi_k$ then any rule acting on blocks ⁷ is only allowed to be applied to the maximal Φ_k or to the block containing the output formula.

⁷Namely all the $\bullet_2$ -rules (except $\diamond_2^\bullet$), together with $\diamond^\circ$ and $\square^\bullet$.

If the application of a rule produce at least one non-tight premise, the rule cannot be applied.

By adopting these restriction, we obtain all sequents that can be generated in backward proof search are tight. This is the argument: in backward proof search, duplicated blocks can be produced only in the following cases: (a) by an application of $\Diamond_1^\bullet$ to $\Omega, \Diamond A^\bullet, [A^\bullet]$; (b) by an application of $\supset^\bullet$ to $\Omega, [\Phi], [\Phi, F^\circ]$ which removes F° from the block; (c) by an application of $\bullet$ -rules to $[\Phi]$ in $\Omega, [\Phi], [\Phi, \Theta]$ which expands Φ to Φ, Θ . Note that (a) is prevented by non-redundancy condition of $\Diamond_1^\bullet$ of constraint (ii) and (b) cannot occur by definition of tight sequent itself; and (c) is prevented indeed by the constraint (iv) above. The completeness of this constrained proof search is ensured by Proposition 5.7.

Now we turn to the final step, we determine the space requirement of proof search. Preliminarily, we have the following proposition whose proof is by an easy check of each rule.

Proposition 5.9. *Let (r) be a rule in $SC_{\mathbf{FIK}}^\times$ which is of the form $\frac{S_1 S_2}{S}$ or $\frac{S_1}{S}$. Then we have*

$$\text{md}(S_1) < \text{md}(S) \quad \text{if } r \in \{\Diamond_2^\bullet, \supset_2^\circ, \Box_2^\circ\}; \quad \text{md}(S_i) \leq \text{md}(S) \text{ for } i \in \{1, 2\} \quad \text{otherwise.}$$

Lemma 5.10. Backward proof search in $SC_{\mathbf{FIK}}^\times$ for a formula A terminates with a finite clean derivation/proof where each branch is of exponential length in terms of $|A|$.

Proof. Let A be a formula of size n , $\mathcal{D}$ be a clean derivation of the tight sequent A° and $\mathcal{B}$ be an arbitrary branch in $\mathcal{D}$. We divide $\mathcal{B}$ into phases $\mathcal{B}_1, \dots, \mathcal{B}_k, \dots$ where each $\mathcal{B}_i = S_i^1, \dots, S_i^{m_i}$ such that $S_1^1 = A^\circ$ and each S_{i+1}^1 is obtained from $S_i^{m_i}$ by one of the rules $(\Diamond_2^\bullet), (\supset_2^\circ)$ and $(\Box_2^\circ)$. This means the maximal index k equals to the maximal number of applications of $(\Diamond_2^\bullet), (\supset_2^\circ)$ and $(\Box_2^\circ)$ that are applied within $\mathcal{B}$. By Proposition 5.9, $k \leq \text{md}(A)$, hence we can write $\mathcal{B} = \mathcal{B}_1, \dots, \mathcal{B}_k$ for some $k \leq \text{md}(A)$. Moreover, we claim for any $i \in \{1, \dots, k\}$, m_i as the length of $\mathcal{B}_i$ is bounded by $2^{O(n)}$.

Proof of claim: m_i as the length of $\mathcal{B}_i$ is identical to the number of sequents that are produced by backward proof search. Since $\mathcal{B}_i$ does not contain applications of $(\Diamond_2^\bullet), (\supset_2^\circ)$ and $(\Box_2^\circ)$, by definition, any backward rule application of other rules in $SC_{\mathbf{FIK}}^\times$ does not decrease the size of the input part. Since each tight sequent is also a set-sequent, by Proposition 5.2, there are at most $\mathcal{O}(2^n)$ blocks contained in a tight sequent. Also, each block has size $\mathcal{O}(n^2)$, as the input part is increasing within $\mathcal{B}_i$, hence the number of different input parts for sequents in $\mathcal{B}_i$ is $\mathcal{O}(n^2 \cdot 2^n)$. On the other hand, while keeping/increasing the set of input, the output formula might be replaced by $\supset^\bullet$ or decomposed by other $\circ$ -rules. The unique output formula can occur in any block and since there are $\mathcal{O}(n)$ output formulas and $2^{\mathcal{O}(n)}$ blocks, there are $2^{\mathcal{O}(n)} \cdot \mathcal{O}(n)$ possible occurrences of the output formula which give $\mathcal{O}(2^n \cdot n^2) \cdot \mathcal{O}(n \cdot 2^n) = 2^{\mathcal{O}(n)}$ different sequents, which is the upper bound of m_i . $\dashv$

To estimate the length of each branch in $\mathcal{D}$, since $\mathcal{B} = \mathcal{B}_1, \dots, \mathcal{B}_k$ and $k \leq \text{md}(A) = \mathcal{O}(n)$, by the claim above, the length of $\mathcal{B}$ is bounded by $2^{\mathcal{O}(n)} \cdot \mathcal{O}(n) = 2^{\mathcal{O}(n)}$. Therefore, we conclude $\mathcal{B}$ is of exponential length in terms of n . $\square$

By Lemma 5.10 and Proposition 5.2, each branch in a derivation of A has an exponential size of $|A|$.

Theorem 5.11. *The decision problem of $\mathbf{FIK}$ is in EXPSpace.*

6 Conclusion

We have presented a ‘shallow’ sequent calculus for intuitionistic modal logic $\mathbf{FIK}$, which is a close variant or weakening of Fischer Servi/Simpson’s $\mathbf{IK}$, regarded a natural and meaningful IML. By means of the shallow calculus, we have shown that the decision problem of $\mathbf{FIK}$ is in EXPSpace, much lower

than the upper bound conjectured for **IK**. However, whether this bound is tight, that is to say what is the lower bound for the decision problem of **FIK**, is an open problem at present.

Form a proof-theoretic viewpoint, our ‘shallow’ sequent calculi can be seen either as a minimal extension of standard Gentzen calculi, or as a drastic simplification of nested calculi. We believe that calculi of this type might be useful for studying certain meta-logical properties such as different kinds of interpolation. Moreover, we aim to develop similar calculi for other logics of the IML family in order to get new or better complexity bounds.

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Modal CEGAR-tableaux with RECAR and resolution-based SAT-shortcuts

Rajeev Goré

Faculty of Information Technology, Monash University, Australia
rajeev.gore@monash.edu

Cormac Kikkert

Cormac Kikkert Research

We investigate two approaches for extending CEGAR-tableaux with SAT-shortcuts using a previously known approach called RECAR but also a totally new approach using the modal resolution theorem prover K_{SP} as an oracle. Our experiments using our C++ implementation CEGARBox++ of CEGAR-tableaux show that: (1) CEGARBox++ with RECAR SAT-shortcuts is not competitive (2) CEGARBox++ using K_{SP} to provide SAT-shortcuts is superior to both CEGARBox++ and K_{SP} , particularly on large satisfiable problems. As far as we know, this is the first effective integration of SAT, tableaux and resolution methods for modal satisfiability which performs better than its parts.

1 Introduction

Propositional modal, temporal and description logics are of fundamental importance in logic-based artificial intelligence, hardware and software verification [22] and knowledge representation and reasoning [1]. Thus efficient decision procedures for modal logic are an important area of research.

Such research has not matched the advances in SAT-solving, until recently, when various authors independently used SAT-solvers for modal satisfiability [19, 8, 13, 14, 9] using a technique originally due to Claessen and Rosén [5] for propositional intuitionistic logic, which was, itself, “inspired” by the work of Goré et al. on “modal clause learning” using binary decision diagrams [11].

We previously [9] presented a new type of tableaux calculus which uses a SAT-solver and the standard counter-example guided abstraction refinement (CEGAR) methodology [6] to search for a rooted Kripke model for a given formula in modal clausal normal form (mcnf). We showed how to modify the initial mcnf to “compile in” the axioms T and 4 via a preprocessing stage, adding a standard loop-check for termination when required. Our Haskell implementation CEGARBox was, overall, the best over the standard benchmarks for the modal logics K, KT and S4, sometimes by orders of magnitude. Indeed, the only benchmark where CEGARBox did not win were the K-MQBF benchmarks, where the modal resolution theorem prover K_{SP} solved approximately 50 (out of 1000) more problems in 32 seconds each [9]: see Figure 3.

When using SAT-solvers as oracles, we usually first create a formula of classical propositional logic (cpl) which is an approximation of the given (modal) formula φ_0 . The approximation is an under-approximation $\check{\varphi}_0$ if its cpl-UNSATisfiability implies the modal unsatisfiability of φ_0 , giving us an $\check{\text{UNSAT}}$ -shortcut. The approximation is an over-approximation $\hat{\varphi}_0$ if its cpl-SATisfiability implies the modal satisfiability of φ_0 , giving us a $\hat{\text{SAT}}$ -shortcut.

Here we report on two ways to add $\hat{\text{SAT}}$ -shortcuts to CEGAR-tableaux: the RECAR approach and a totally new approach using the resolution-based solver K_{SP} as an oracle. Using our new C++ implementation CEGARBox++, our experiments show that: (1) CEGARBox++ with RECAR $\hat{\text{SAT}}$ -shortcuts is not competitive; and (2) CEGARBox++ using K_{SP} to provide $\hat{\text{SAT}}$ -shortcuts is superior to each, particularly on large satisfiable problems.

$$\begin{array}{ll}
M, w \Vdash \langle \alpha \rangle \varphi & \text{iff} \quad \exists v, R_\alpha(w, v) \ \& \ M, v \Vdash \varphi \\
M, w \Vdash [\alpha] \varphi & \text{iff} \quad \forall v, R_\alpha(w, v) \Rightarrow M, v \Vdash \varphi
\end{array}$$

Figure 1: Semantics of multi-modal logic K_n (aka ALC)

2 Preliminaries: syntax, semantics and modal clausal normal form

Let Ag be a non-empty finite set of agent names. Let Atm be a set of atomic formulae with $Atm \cap Ag = \emptyset$. Consider the language of formulae defined from atoms $p \in Atm$ and $\alpha \in Ag$ by the BNF grammar:

$$\varphi ::= \perp \mid \top \mid p \mid \neg \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid [\alpha] \varphi \mid \langle \alpha \rangle \varphi$$

Define $(\varphi_1 \rightarrow \varphi_2) := (\neg \varphi_1 \vee \varphi_2)$ and $(\varphi_1 \leftrightarrow \varphi_2) := ((\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1))$. Let $[\alpha]^0 \varphi := \varphi$ and $[\alpha]^{n+1} \varphi := [\alpha][\alpha]^n \varphi$. The modal-depth of a formula is the maximum number of nested modalities in it. For a finite set $\Gamma = \{\varphi_1, \dots, \varphi_k\}$ of formulae, let $\hat{\Gamma} = (\varphi_1 \wedge \dots \wedge \varphi_k)$ and let $[\alpha]\Gamma = \{[\alpha]\varphi_1, \dots, [\alpha]\varphi_k\}$.

The Kripke semantics for these multimodal (description) logics use Kripke models consisting of triples $M := \langle W, \{R_\alpha\}_{\alpha \in Ag}, \vartheta \rangle$ over some non-empty set W (of possible worlds), and binary relations R_α over W for every agent $\alpha \in Ag$, and a valuation $\vartheta(w, p) \subseteq W \times Atm$ telling us the truth value of each atomic formula p at each world $w \in W$. Truthhood at world w in model M , written as $M, w \Vdash \varphi$, extends the usual truth-tables for classical propositional logic (cpl): see Figure 1.

A formula φ is K_n -satisfiable if there is a Kripke model M containing a world w such that $M, w \Vdash \varphi$. A formula φ is K_n -valid if $\neg \varphi$ is not K_n -satisfiable. Formulae φ and ψ are logically equivalent if $\varphi \leftrightarrow \psi$ is K_n -valid, and are equi-satisfiable if φ is K_n -satisfiable iff ψ is K_n -satisfiable.

2.1 Separated modal clausal normal form for K_n

A *positive literal* is an atomic formula p . A *negative literal* is a negated atomic formula $\neg p$. A *literal* is a positive literal p or else a negative literal $\neg p$. We use a to f and l and r for literals. We use A, B, C and D for a set of literals. Let $\bar{l} := \neg p$ if $l = p$ and $\bar{l} := p$ if $l = \neg p$ so that $\bar{\bar{l}} = l$. A formula is in *negation normal form* (NNF) if it is implication-free and negations appear only in front of atomic formulae. An NNF can be created with a linear descent of the formula.

Proposition 2.1. *A formula φ_0 of modal depth κ can be converted into a logically equivalent formula $nnf(\varphi_0)$ in NNF which is at most polynomially longer.*

A modal clause is any formula of one of the following forms [21, 15, 10]:

cpl-clause: a formula $(l_1 \wedge \dots \wedge l_i) \rightarrow (r_1 \vee \dots \vee r_j)$ of literals;

box-clause: a formula $a \rightarrow [\alpha]b$ with literals a and b and agent name $\alpha \in Ag$;

dia-clause: a formula $c \rightarrow \langle \alpha \rangle d$ with literals c and d and agent name $\alpha \in Ag$.

Using semicolon to indicate a set-union, an arbitrary set $\mathcal{C} = (\mathcal{C}^{cpl} ; \mathcal{C}^{box} ; \mathcal{C}^{dia})$, of modal-clauses can be partitioned into the cpl-clauses $\mathcal{C}^{cpl}$ and box-clauses $\mathcal{C}^{box}$ and dia-clauses $\mathcal{C}^{dia}$ as defined above.

Our modal clausal normal form requires notation to succinctly express finite sequences of modalities such as $[\alpha_1][\alpha_2] \dots [\alpha_k]$ which naturally correspond to all finite paths $w_1 R_{\alpha_1} w_2 R_{\alpha_2} \dots R_{\alpha_k} w_k$ in a Kripke model. We therefore abuse notation to extend the language of formulae using constructs from regular expressions even though they are not part of the official syntax.

We use the composition operator “;” from regular expressions with ε for the empty regular expression obeying $(\varepsilon; \alpha) = (\alpha; \varepsilon) = \alpha$. We let $\alpha^0 = \varepsilon$ and $\alpha^{k>0} = (\alpha; \alpha^{k-1})$. We write $[\alpha_1; \alpha_2; \dots; \alpha_k]$ instead of

$[\alpha_1][\alpha_2] \cdots [\alpha_k]$, write $[\alpha^n; \beta^m]$ for $[\alpha]^n[\beta]^m$ and write $w_1 R_{\alpha_1; \dots; \alpha_k} w_k$ for $w_1 R_{\alpha_1} w_2 R_{\alpha_2} \cdots R_{\alpha_k} w_k$. Let Ag^* be the set of all finite regular expressions over Ag that use only “;” and let Ag^k be the set of all finite regular expressions of length k over Ag that use only “;”.

Proposition 2.2. *A formula φ_0 of modal depth κ can be put into an equi-satisfiable modal clausal normal form $r \wedge SNF_{mc}(\varphi_0)$ by naming subformulae using new atomic propositions, using r as the name for φ_0 , and where each $\mathcal{C}$ below is a finite set of modal clauses (essentially the SNF_{ml} of Nalon et al. [17] who prove this proposition in detail):*

$$SNF_{mc}(\varphi_0) := \mathcal{C}_\varepsilon ; \bigcup_{\sigma \in Ag^1} [\sigma] \mathcal{C}_\sigma ; \cdots ; \bigcup_{\sigma \in Ag^\kappa} [\sigma] \mathcal{C}_\sigma$$

Example 2.3. Consider the negation $\varphi_0 := \neg([\alpha](p \rightarrow q) \rightarrow ([\alpha]p \rightarrow [\alpha]q))$ of the K axiom with modal depth $\kappa = 1$. Its NNF is $[\alpha](\neg p \vee q) \wedge [\alpha]p \wedge \langle \alpha \rangle \neg q$. Putting r as the name for φ_0 gives us: $\mathcal{C}_\varepsilon := r \rightarrow [\alpha]b ; r \rightarrow [\alpha]p ; r \rightarrow \langle \alpha \rangle \neg q$ and $\mathcal{C}_\alpha := b \rightarrow \neg p \vee q$ with $SNF_{mc}(\varphi_0) = \mathcal{C}_\varepsilon ; [\alpha] \mathcal{C}_\alpha$.

Note, the set $\mathcal{C}_\varepsilon$ excludes r so we must add it explicitly to form $(r ; SNF_{mc}(\varphi_0))$.

3 SAT-solvers with internal state as CPL-oracles

CEGAR-tableaux [9] assume we have access to a SAT-solver s , with internal state, to which we can add a cpl-clause φ via $\text{addClause}(s, \varphi)$, and which accepts a set $\mathcal{A}$ of literals, called unit assumptions, such as MiniSAT [7]. Intuitively, we want all literals in $\mathcal{A}$ to be assigned to true.

When pre-loaded with a set $\mathcal{C}^{cpl}$ of cpl-clauses and called with $\text{solve}(s, \mathcal{A})$, such a SAT-solver returns one of two results:

(sat, ϑ) : if $(\mathcal{A} ; \mathcal{C}^{cpl})$ is true under some cpl-valuation $\vartheta \supseteq \mathcal{A}$ or-else

$(unsat, UC)$: if $UC \subseteq \mathcal{A}$ is a, not necessarily unique, minimal “unsatisfiable core” of $\mathcal{A}$ such that $(UC ; \mathcal{C}^{cpl})$ is cpl-unsatisfiable, and hence so is $(\mathcal{A} ; \mathcal{C}^{cpl})$.

UC itself may be cpl-satisfiable but the term “unsatisfiable core” is standard.

Since SAT-solvers handle classical propositional logic, they should return a cpl-valuation $\vartheta \subseteq Atm$. That is, strictly speaking, a cpl-valuation ϑ is just a set of atomic formulae. But in concrete applications, it may be more useful for a user to assert “I know that atomic formula c is false”, which can be achieved with $\neg c \in \mathcal{A}$. Thus, the valuations returned by a SAT-solver are actually a subset of the set of all literals that appear in $(\mathcal{A} ; \mathcal{C}^{cpl})$.

We can then easily extend such a set of literals to a valuation over all atoms that appear in an mcnf by putting all missing atoms to false.

Example 3.1. From our previous example, φ_0 is equi-satisfiable with $(r ; \mathcal{C}_\varepsilon ; [\alpha] \mathcal{C}_\alpha)$. Create a SAT-solver s_0 for modal depth 0, pre-load s_0 with $\mathcal{C}_\varepsilon^{cpl} = \emptyset$, the cpl-part of $\mathcal{C}_\varepsilon$, put $\mathcal{A}_\varepsilon = \{r\}$ because r must be true at modal depth 0, and call $\text{solve}(s_0, \mathcal{A}_\varepsilon)$: it must return $(sat, \vartheta_\varepsilon)$ where $\vartheta_\varepsilon = \{r\} \supseteq \mathcal{A}_\varepsilon$ since $(\mathcal{A}_\varepsilon ; \mathcal{C}_\varepsilon^{cpl}) = (r ; \emptyset) = \{r\}$ is true under $\vartheta_\varepsilon = \{r\}$.

Create a SAT-solver s_1 for modal depth 1, pre-load s_1 with $\mathcal{C}_\alpha^{cpl} = \{b \rightarrow \neg p \vee q\}$, the cpl-part of $\mathcal{C}_\alpha$, put $\mathcal{A}_\alpha = \{b, p, \neg q\}$, and call $\text{solve}(s_1, \mathcal{A}_\alpha)$: it must return $(unsat, UC_\alpha)$ where $UC_\alpha = \{b, p, \neg q\} \subseteq \mathcal{A}_\alpha$ since $(UC_\alpha ; \mathcal{C}_\alpha^{cpl}) = \{b, p, \neg q, b \rightarrow \neg p \vee q\}$ is cpl-unsatisfiable but every proper subset is cpl-satisfiable.

Finally, add $\neg r$ to s_0 via $\text{addClause}(s_0, \neg r)$, and restart it via $\text{solve}(s_0, \mathcal{A}_\varepsilon)$: it must return $(unsat, UC_\varepsilon)$ where $UC_\varepsilon = \{r\} \subseteq \mathcal{A}_\varepsilon$ because $(UC_\varepsilon ; (\mathcal{C}_\varepsilon^{cpl} ; \neg r)) = (r ; \neg r)$ is cpl-unsatisfiable but no proper subset is so.

Algorithm 1 CEGARTAB($A, \text{Trie}[\sigma]$)

```

1: {Inputs:  $A$  is a set of unit assumptions,
    $\text{Trie}[\sigma]$  is a node in our Trie containing modal clauses and a SAT-solver.}
2: Let  $t_\sigma := \text{solve}(\text{Trie}[\sigma].\text{sat}, A)$  {is  $\mathcal{C}_\sigma^{\text{cpl}}$  cpl-satisfiable}
3: if  $t_\sigma = (\text{unsat}, UC_\sigma)$  then
4:   return Unsatisfiable( $UC_\sigma$ ) {because  $\mathcal{C}_\sigma$  is  $K_n$ -unsatisfiable}
5: else if  $t_\sigma = (\text{sat}, \vartheta_\sigma)$  then
6:   {Check box and diamond clauses that fire under classical valuation  $\vartheta_\sigma$ }
7:   for  $(c \rightarrow \langle \alpha \rangle d) \in \text{Trie}[\sigma].\text{DiaCl}$  with  $c \in \vartheta_\sigma$  do
8:     Let  $B = \{b \mid (a \rightarrow [\alpha]b) \in \text{Trie}[\sigma].\text{BoxCl} \text{ and } a \in \vartheta_\sigma\}$ 
9:     {evaluate the next  $\alpha$ -successor using the jump rule}
10:    if CEGARTAB( $(d; B), \text{Trie}[\sigma].\text{child}(\alpha)$ ) = Unsatisfiable( $UC_{\sigma;\alpha}$ ) then
11:       $CS := \{c\} \cup \{a \mid (a \rightarrow [\alpha]b) \in \text{Trie}[\sigma].\text{BoxCl} \text{ and } a \in \vartheta_\sigma \text{ and } b \in UC_{\sigma;\alpha}\}$ 
12:      Let  $\varphi := \bigvee_{l \in CS} \neg l$ 
13:      addClause( $\text{Trie}[\sigma].\text{sat}, \varphi$ ) {Learn new clause  $\varphi := \neg \bigwedge CS$ }
14:      return CEGARTAB( $A, \text{Trie}[\sigma]$ ) {apply (restart)}
15:    end if
16:  end for
17:  return Satisfiable {because every fired diamond is fulfilled}
18: end if

```

Figure 2: Algorithm of CEGARTAB for multi-modal (description) logic K_n (ALC)

4 CEGAR-tableaux for multi-modal logic K_n (aka ALC)

We now describe the CEGAR-tableaux [9] procedure extended to multi-modal logic K_n (aka ALC) without local (ABox) and global (TBox) assumptions.

Conceptually, the extension from mono-modal logic K to multi-modal logic K_n is easy because there are no interactions between the different modalities, so we just have n “copies” of K , one for each modality $[\alpha]$ for every agent $\alpha \in Ag$.

We give the extension of the main CEGARTAB algorithm from Goré and Kikkert [9] as Algorithm 1 in Figure 2, and make the connection to the general form of $\text{SNF}_{mc}(\varphi_0)$ but with a non-singleton Ag :

$$\text{SNF}_{mc}(\varphi_0) := \mathcal{C}_\varepsilon ; \bigcup_{\sigma \in Ag^1} [\sigma] \mathcal{C}_\sigma ; \dots ; \bigcup_{\sigma \in Ag^\kappa} [\sigma] \mathcal{C}_\sigma$$

Because CEGARTAB handled only mono-modal logics, the mcnf of a given formula φ_0 of modal depth κ was as below for a singleton $Ag = \{\alpha\}$ (say) :

$$\text{SNF}_{mc}(\varphi_0) := \mathcal{C}_\varepsilon ; [\alpha] \mathcal{C}_\alpha ; [\alpha^2] \mathcal{C}_{\alpha^2} ; \dots ; [\alpha^\kappa] \mathcal{C}_{\alpha^\kappa}$$

Conceptually, these “modal contexts” were stored in a linear Trie (a.k.a. prefix tree) data structure with Node[0] initialised to contain $\mathcal{C}_\varepsilon$ and node Node[i], for $i > 0$, initialised to contain $\mathcal{C}_{\alpha^i}$, eliding the box-prefix $[\alpha^i]$ without loss of generality, but saving quadratic space.

In the multi-modal case, we use a multi-dimensional trie and all trie-edges are labelled by an agent name and the trie-root is labelled with ε . For a regular expression σ , we write $\text{Trie}[\sigma]$ for the corresponding TrieNode. For example, from the trie-root node $\text{Trie}[\varepsilon] = \mathcal{C}_\varepsilon$ there is a trie-path labelled by an α -edge followed by a β -edge to the trie-node $\text{Trie}[\alpha;\beta] = \mathcal{C}_{\alpha;\beta}$.

In Algorithm 1 in Figure 2, and in the implementation, each $\mathcal{C}_\sigma$ is stored at trie-position $\text{Trie}[\sigma]$ in four fields by partitioning it into its three disjoint components $\mathcal{C}_\sigma = (\mathcal{C}_\sigma^{cpl} ; \mathcal{C}_\sigma^{dia} ; \mathcal{C}_\sigma^{box})$ and adding “next” pointers as follows:

1. $\text{Trie}[\sigma].\text{sat}$ is a dedicated SAT-solver for this trie-node pre-loaded with $\mathcal{C}_\sigma^{cpl}$
2. $\text{Trie}[\sigma].\text{DiaCl}$ contains $\mathcal{C}_\sigma^{dia} := \bigcup_{\alpha \in Ag} \{c \rightarrow \langle \alpha \rangle d \mid (c \rightarrow \langle \alpha \rangle d) \in \mathcal{C}_\sigma\}$
3. $\text{Trie}[\sigma].\text{BoxCl}$ contains $\mathcal{C}_\sigma^{box} := \bigcup_{\alpha \in Ag} \{a \rightarrow [\alpha] b \mid (a \rightarrow [\alpha] b) \in \mathcal{C}_\sigma\}$
4. $\text{Trie}[\sigma].\text{Child}(\alpha)$ is (a pointer to) the node $\text{Trie}[\sigma; \alpha]$, for each $\alpha \in Ag$.

The second and third fields are structured further so that we can select the dia-clauses and box-clauses for a particular agent $\alpha \in Ag$ via $\text{Trie}[\sigma].\text{DiaCl}(\alpha)$ and $\text{Trie}[\sigma].\text{BoxCl}(\alpha)$, respectively. We often write “formula in a TrieNode” or even “ $\varphi \in \text{Trie}[\sigma]$ ” to refer to formulae that are stored in one of these fields.

Recall that r names the initial formula but it is not in the root $\text{Trie}[\varepsilon]$. Now we simply start by calling $\text{CEGARTAB}(\{r\}, \text{Trie}[\varepsilon])$ as shown in Figure 2.

Example 4.1. Let us continue with our example. We know that φ_0 is equi-satisfiable with $r ; \mathcal{C}_\varepsilon ; [\alpha] \mathcal{C}_\alpha$, where $\mathcal{C}_\varepsilon^{cpl} = \emptyset$ and $\mathcal{C}_\varepsilon^{dia} = (r \rightarrow \langle \alpha \rangle \neg q)$ and $\mathcal{C}_\varepsilon^{box} = (r \rightarrow [\alpha] b ; r \rightarrow [\alpha] p)$, meaning $\text{Trie}[\varepsilon].\text{sat}$ is s_0 . We also have $\mathcal{C}_\alpha^{cpl} = (b \rightarrow \neg p \vee q)$ and $\mathcal{C}_\alpha^{dia} = \emptyset$ and $\mathcal{C}_\alpha^{box} = \emptyset$, meaning $\text{Trie}[\alpha].\text{sat}$ is s_1 .

Recursion 0. The initial call to $\text{CEGARTAB}(\{r\}, \text{Trie}[\varepsilon])$ sets $\sigma := \varepsilon$ and $A := \{r\} = \mathcal{A}_\varepsilon$ so line 2 is $\text{solve}(s_0, \mathcal{A}_\varepsilon)$: it returns $(\text{sat}, \vartheta_\varepsilon)$ where $\vartheta_\varepsilon = \{r\}$ so we enter the “then” part of line 5. At line 7, there is only one “fired” dia-clause $(r \rightarrow \langle \alpha \rangle \neg q) \in \text{Trie}[\varepsilon].\text{DiaCl} = \mathcal{C}_\varepsilon^{dia}$ with $r \in \vartheta_\varepsilon$, so $c := r$ and $d := (\neg q)$ and we know that $\langle \alpha \rangle \neg q$ must be (modally) true at modal depth 0. Under $\vartheta_\varepsilon = \{r\}$, every box-clause in $\mathcal{C}_\varepsilon^{box}$ “fires”, thus the set $\{[\alpha] b, [\alpha] p\}$ is (modally) true at modal depth 0, and hence the set $B = \{b, p\}$ formed at line 8 must be classically true at some R_α -successor for $\neg q$ at modal depth 1. To evaluate this R_α -successor, we compute $(d; B) = \{b, p, \neg q\}$ at line 10 and recurse via $\text{CEGARTAB}(\{b, p, \neg q\}, \text{Trie}[\alpha].\text{Child}(\alpha))$.

Recursion 1. So $\sigma := \alpha$ and $A := \{b, p, \neg q\} = \mathcal{A}_\alpha$ and the relevant SAT-solver is the “next” one so $\text{Trie}[\alpha].\text{sat} = s_1$. Line 2 calls this SAT-solver, so this is the call $\text{solve}(s_1, \mathcal{A}_\alpha)$: it must return $(\text{unsat}, UC_\alpha)$ where $UC_\alpha = \{b, p, \neg q\}$ so we return $\text{Unsatisfiable}(\{b, p, \neg q\})$ at line 4 and pop the recursion stack.

Recursion 0. We return to line 11 where $\sigma := \varepsilon$ and so $(\sigma; \alpha) = \alpha$, so $UC_{\sigma; \alpha} = UC_\alpha = \{b, p, \neg q\}$. We know the chosen ϑ_ε at modal depth 0 caused a clash at modal depth 1 involving $\{b, p, \neg q\}$, and hence that $\{[\alpha] b, [\alpha] p, \langle \alpha \rangle \neg q\}$ cannot be jointly true at modal depth 0. At line 11 we trace the relevant box- and dia-clauses to find the conflict set $CS = \{r\}$ at modal depth 0. To avoid this “mistake”, we compute the “refinement” $\varphi = \neg r$ at line 12 of recursion level 0. Line 13 is then just $\text{addClause}(s_0, \neg r)$ so $\mathcal{C}_\varepsilon^{cpl} = \{\neg r\}$. The refined $\text{SNF}_{mc}(\varphi_0)$ demands r is false at the root. At line 14, we call $\text{CEGARTAB}(\{r\}, \text{Trie}[\varepsilon])$ but we stay at the current SAT-solver at modal depth 0 thereby restarting s_0 .

Recursion 1. Line 2 calls $\text{solve}(s_0, \mathcal{A}_\varepsilon)$: it returns $(\text{unsat}, UC_\varepsilon)$ where $UC_\varepsilon = r$ which returns $\text{Unsatisfiable}(\{r\})$ to line 14 of Recursion 0: our final answer.

Theorem 4.2. If Trie contains $\text{SNF}_{mc}(\varphi_0)$ and r names φ_0 then $\text{CEGARTAB}(\{r\}, \text{Trie}[\varepsilon])$ terminates, and returns Satisfiable iff φ_0 is K_n -satisfiable.

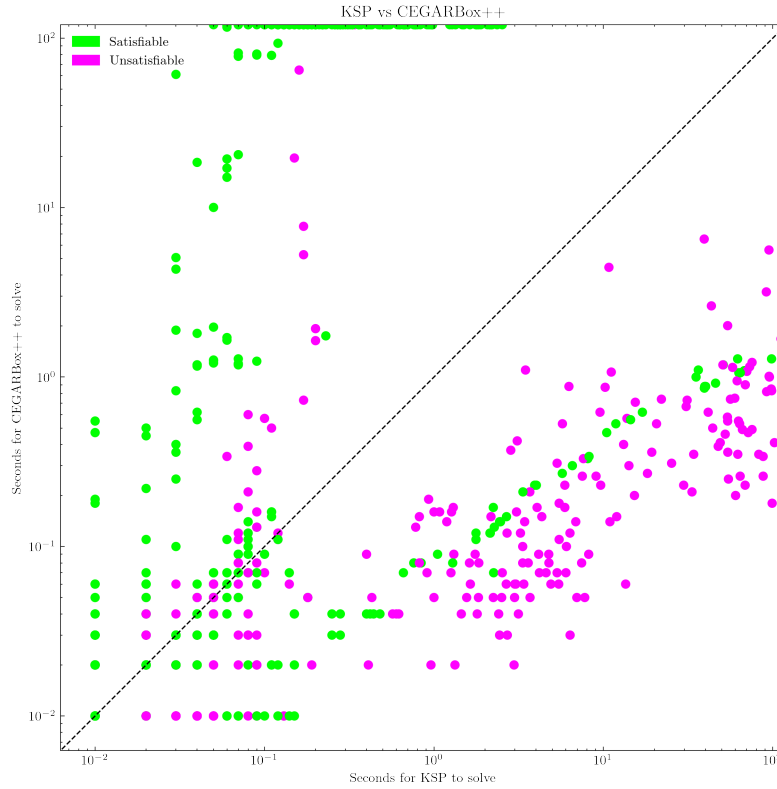


Figure 3: Performance of CEGARBox++ and K_{SP} on the MQBF benchmarks. Problems on the top/right edge indicate a timeout for CEGARBox++/K_{SP} respectively.

Proof. A simple modification of the proofs for mono-modal K from Goré and Kikkert [9] which proceed by induction on the number of restarts. $\square$

5 CEGAR-tableaux with different ($\neg$ SAT) shortcuts

We previously showed [9] that CEGARBox outperformed all solvers we tested on the extended LWB K-benchmarks, and 3CNF K-benchmarks, mostly by orders of magnitude. However, on the MQBF K-benchmarks, CEGARBox loses to K_{SP} by about 50 problems. We first determine why and then explore two solutions.

5.1 Why does CEGARBox++ not win on the MQBF benchmarks?

Figure 3 shows that CEGARBox++ solves all unsatisfiable benchmarks but struggles on the satisfiable ones, while K_{SP} solves all satisfiable benchmarks but struggles on the unsatisfiable ones. K_{SP} performs better overall as there are 617 satisfiable and 399 unsatisfiable problems in the MQBF benchmarks.

By inspecting the individual (satisfiable) benchmarks that CEGARBox++ struggles on, we saw that CEGARBox++ learns few clauses, and instead spends most of its time building a large model. That is, because CEGARBox++ explicitly builds a satisfying model, a huge model can cause it to timeout. This is particularly noticeable in benchmarks with lots of branching, as it can result in satisfying models having

an exponential number of worlds with respect to the size of the original benchmark. K_gP on the other hand doesn't need to explicitly build a model to deem a formula as satisfiable, it simply applies rules until *saturation*.

This is a fundamental weakness in not just CEGARTAB, but the CEGAR procedure itself. For a formula φ , we first test whether an under-approximation $\check{\varphi}$ consisting of the cpl-clauses of its mcnf is cpl-satisfiable. If $\check{\varphi}$ is cpl-unsatisfiable, we can perform an ($\check{\varphi}$ UNSAT) shortcut by immediately returning UNSAT as φ must also be modally unsatisfiable. However, there is no corresponding ($\check{\varphi}$ SAT) shortcut.

Our observation is not new. Brummayer [3] devised an under-approximation for SMT that allows for early termination in the UNSAT case. Wang et al. [23] used induction to show the existence of large counter-models in model-checking. These frameworks are domain specific, so it is unlikely their insights would apply to CEGARTAB. However, for SAT-solving, Lagniez et al. [12] presented a *general* variant of CEGAR, which allows for both ($\check{\varphi}$ SAT) and ($\check{\varphi}$ UNSAT) shortcuts.

We first analyse the RECAR approach of Lagniez et al. [12]. We have previously shown [9] their implementation was unsound and their benchmarks were incorrect, but the theory of RECAR is sound so we incorporate their ideas [12] to create our own variant of RECAR-Tableaux.

We also create our own novel approach by detecting fixpoints in our under-approximation. Although it is specific to recursive CEGAR algorithms, we show that it works exceptionally well, allowing CEGARBox++ to solve all problems in the MQBF benchmark set bar one. Our approach combines K_gP with CEGARBox++, giving a solver that elegantly combines SAT, tableaux and resolution methods, which previously were the three competing methods for K-satisfiability.

5.2 Background on modal resolution and RECAR

We first give some background on modal resolution and the RECAR framework.

5.2.1 Modal clausal resolution (K_gP).

Recalling the definition of SNF_{mc} , we briefly outline the modal calculus rules of K_gP [16] for multi-modal K_n , which is a resolution prover that operates on SNF_{ml} , a close variant of SNF_{mc} .

Proposition 5.1. *A formula φ_0 of modal depth κ can be put into the equi-satisfiable separated normal form with modal layers $\text{SNF}_{ml}(\varphi_0)$ of Nalon et al [17], who prove this proposition in detail, and where each $\mathcal{C}_i$ below is a set of modal clauses:*

$$\text{SNF}_{ml}(\varphi_0) := \mathcal{C}_0 ; []^1 \mathcal{C}_1 ; \dots ; []^\kappa \mathcal{C}_\kappa$$

Informally, but not exactly because K_gP requires extra clauses that are not required by CEGARBox++, we have $\mathcal{C}_l \approx \bigcup_{\sigma \in \text{Ag}^l} \mathcal{C}_\sigma$, where $\mathcal{C}_\sigma$ is from our $\text{SNF}_{mc}(\varphi_0)$. That is, each $\mathcal{C}_l$ is the “union” formed by taking a “horizontal slice” across layer $l = |\sigma|$ of our “vertical tree-like” $\text{SNF}_{mc}(\varphi_0)$, where $|\sigma|$ is the length of the Trie-path σ . But now, $\text{SNF}_{ml}(\varphi_0)$ can be implemented as a linear trie, rather than a branching trie as for $\text{SNF}_{mc}(\varphi_0)$. Let us write $n : \psi$ to indicate the clause ψ is stored at the n -th node in the (linear) trie, where the root is element 0. Using this notation, the resolution rules used by K_gP are presented in Figure 4.

They differ from normal resolution as clauses are now labelled by their modal layer. Applying these rules will either derive an empty clause, meaning φ_0 is K_n -unsatisfiable, or terminate after saturation, meaning φ_0 is K_n -satisfiable. Observe the following close connection between modal resolution and CEGARTAB: all resolvents in K_gP are classical clauses as are all learnt clauses in CEGARTAB. We will use this insight to combine both these approaches.

$$\begin{array}{ll}
\text{[LRES]} \quad \frac{n : D \vee l \quad n : D' \vee \neg l}{n : D \vee D'} & \text{[MRES]} \quad \frac{n : l_1 \Rightarrow [\alpha] l \quad n : l_2 \Rightarrow \langle \alpha \rangle \neg l}{n : \neg l_1 \vee \neg l_2} \\
\text{[GEN1]} \quad \frac{n : l'_1 \Rightarrow [\alpha] \neg l_1, \dots, n : l'_m \Rightarrow [\alpha] \neg l_m, n : l' \Rightarrow \langle \alpha \rangle \neg t, n+1 : l_1 \vee \dots \vee l_m \vee t}{n : \neg l'_1 \vee \dots \vee \neg l'_m \vee \neg l'} & \\
\text{[GEN2]} \quad \frac{n : l'_1 \Rightarrow [\alpha] l_1 \quad n : l'_2 \Rightarrow [\alpha] \neg l_1 \quad n : l'_3 \Rightarrow \langle \alpha \rangle l_2}{n : \neg l'_1 \vee \neg l'_2 \vee \neg l'_3} & \\
\text{[GEN3]} \quad \frac{n : l'_1 \Rightarrow [\alpha] \neg l_1, \dots, n : l'_m \Rightarrow [\alpha] \neg l_m, n : l' \Rightarrow \langle \alpha \rangle l, n+1 : l_1 \vee \dots \vee l_m}{n : \neg l'_1 \vee \dots \vee \neg l'_m \vee \neg l'} &
\end{array}$$

Figure 4: The resolution rules used by KsP [16].

5.2.2 The RECAR framework (MOSAIC).

RECAR (Recursive Explore and Check Abstraction Refinement) is an extension of the CEGAR approach which introduces a recursive step to allow for an additional shortcut [12]. There are two variants, RECAR-under, where the main CEGAR loop contains an UNSAT shortcut ($\not\text{UNSAT}$), whilst the recursive step allows for a SAT shortcut ($\not\text{SAT}$), and RECAR-over where the main CEGAR loop contains a SAT shortcut ($\not\text{SAT}$), whilst the recursive step allows for an UNSAT shortcut.

We present just the RECAR-under algorithm here, as the main “CEGAR” loop of the RECAR-under framework has ($\not\text{UNSAT}$) shortcuts, like CEGAR-tableaux. This will help motivate our own application of RECAR later but note that the RECAR-over algorithm can be defined analogously (and is what is used in MOSAIC [12]). We require the following assumptions, see Figure 5, where RC is a Boolean function that determines whether or not a recursive call occurs:

Definition 5.2. Assumptions for RECAR-under [12]

1. Function ‘check’ is a sound, complete and terminating implementation that decides if an input formula is K-unsatisfiable.
2. Function $\text{refine}(\varphi)$ is a function that constrains φ with more clauses so that:
 - (a) If the under-approximation $\check{\varphi}$ is K-unsatisfiable, then so is $\text{refine}(\check{\varphi})$.
 - (b) There exists an $n \in \mathbb{N}$ such that $\text{refine}^n(\check{\varphi})$ is K-satisfiable iff φ is so.
3. If the over-approximation $\hat{\varphi}$ is K-satisfiable, then so is φ .
4. Let $\text{over}(\varphi) = \hat{\varphi}$. There exists $n \in \mathbb{N}$ such that $RC(\text{over}^n(\varphi), \text{over}^{n+1}(\varphi))$ evaluates to false (guaranteeing termination).

Proposition 5.3. RECAR-under is sound, complete and terminating [12].

The RECAR framework was implemented in the solver MOSAIC [12]. Briefly, the over-approximation involves checking if some formula φ is K-satisfiable with $\leq n$ worlds via a naive SAT translation, with $\mathcal{O}(\text{Atom}(\varphi) \times n + n^2)$ variables, where $\text{Atom}(\varphi)$ denotes the number of atoms in φ . The translation has variables for each world, as well as extra variables indicating which worlds are accessible from each

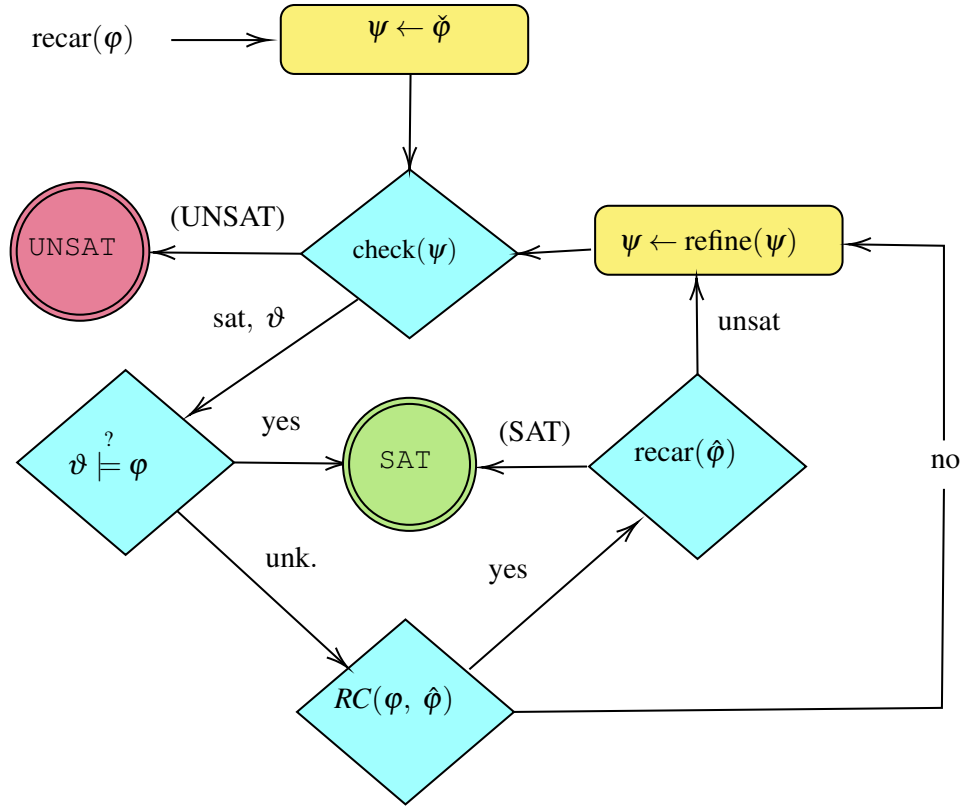


Figure 5: The RECAR-under framework [12].

other. Over the course of the algorithm, n will increase until it reaches the bound $n = Atom(\varphi)^{\text{depth}(\varphi)}$ where K-satisfiability can be determined as φ is K-satisfiable if and only if it has a model with n worlds [18]. The under-approximation involves “removing” conjuncts in φ . Thus both the under-approximation and over-approximation involve reasoning about the formula φ in its entirety via a SAT-solver. This is exactly what CEGAR-tableaux aim to avoid by using a SAT-solver to only determine worlds (and not models), thus requiring only $\mathcal{O}(n)$ variables in a SAT-solver.

5.3 RECAR_{TAB}: Extending CEGAR-tableaux with RECAR

We now extend CEGAR-tableaux with RECAR ($\not\text{SAT}$)-shortcuts.

First, we must take some liberties when applying RECAR to CEGAR-tableaux. Note our under-approximation $\check{\varphi} := \mathcal{C}^{cpl}(\varphi)$ is a classical logic formula, whilst φ is a modal logic formula. As the algorithm progresses, $\check{\varphi}$ will be refined with classical clauses only, and so whilst there exists an n where $\text{refine}^n(\check{\varphi})$ is K-satisfiable iff φ is so, we are unable to detect when we have reached this n . This differs from MOSAIC, where the upper bound $n = UB(\varphi)$ can be pre-calculated by recursively counting the number of diamond sub-formulae in φ . Second, CEGAR-tableaux already have a recursive step because they are based on recursive CEGAR loops, unlike RECAR which is based on just one CEGAR loop.

These differences arise because CEGAR_{TAB} considers only one world of the Kripke model at a time, as opposed to naive SAT-translation approaches that translate the whole input modal formula. We believe that keeping worlds separate is one of the main factors of CEGAR_{TAB}’s success, and so we keep the

under-approximation as is. Thus, instead of implementing a faithful RECAR algorithm, we shall instead perform RECAR on each world. Our under-approximation is unchanged, but our over-approximation is best motivated by an example.

Algorithm 3 Algorithm of RECAR_{TAB} for incorporating ($\not$ SAT) shortcuts

```

1: {Inputs:  $A$  is a set of unit assumptions,
    $\text{Trie}[\sigma]$  is a node in our Trie containing modal clauses and a SAT-solver.}
2: Let  $t_\sigma := \text{solve}(\text{Trie}[\sigma].\text{sat}, A)$  {is  $\mathcal{C}_\sigma^{\text{cpl}}$  cpl-satisfiable}
3: if  $t_\sigma = (\text{unsat}, UC_\sigma)$  then
4:   return Unsatisfiable( $UC_\sigma$ ) {because  $\mathcal{C}_\sigma$  is  $K_n$ -unsatisfiable}
5: else if  $t_\sigma = (\text{sat}, \vartheta_\sigma)$  then
6:   Let  $L = \{l \mid (c \rightarrow_l \langle \alpha \rangle d) \in \text{Trie}[\sigma].\text{DiaCl} \text{ and } c \in \vartheta_\sigma\}$ 
7:   {Check box and diamond clauses that fire under classical valuation  $\vartheta_\sigma$ }
8:   for  $i \in L$  do
9:     Let  $D = \{d \mid (c \rightarrow_l \langle \alpha \rangle d) \in \text{Trie}[\sigma].\text{DiaCl} \text{ and } c \in \vartheta_\sigma \text{ and } l = i\}$ 
10:    Let  $B = \{b \mid (a \rightarrow [\alpha]b) \in \text{Trie}[\sigma].\text{BoxCl} \text{ and } a \in \vartheta_\sigma\}$ 
11:    {evaluate the next  $\alpha$ -successor using the jump rule on all fired diamonds}
12:    if RECARTAB( $D \cup B, \text{Trie}[\sigma].\text{child}(\alpha)$ ) = Unsatisfiable( $UC_{\sigma;\alpha}$ ) then
13:      Let  $D_{UC} := UC_{\sigma;\alpha} \cap D$  {How many diamonds in conflict set?}
14:      if  $|D_{UC}| = 1$  then
15:        {Normal clause learning}
16:        Find  $c \rightarrow_l \langle \alpha \rangle d$  where  $d \in D_{UC}$ 
17:         $CS := \{c\} \cup \{a \mid (a \rightarrow [\alpha]b) \in \text{Trie}[\sigma].\text{BoxCl}, a \in \vartheta_\sigma, b \in UC_{\sigma;\alpha}\}$ 
18:        Let  $\varphi := \bigvee_{l \in CS} \neg l$ 
19:        addClause( $\text{Trie}[\sigma].\text{sat}, \varphi$ ) {Learn new clause  $\varphi := \neg \bigwedge CS$ }
20:      else if  $|D_{UC}| < 1$  then
21:        {No diamond in conflict — pick random  $d \in D$  and simulate normal clause learning}
22:        Pick arbitrary  $d \in D$ 
23:        Find  $c \rightarrow_l \langle \alpha \rangle d$ 
24:         $CS := \{c\} \cup \{a \mid (a \rightarrow [\alpha]b) \in \text{Trie}[\sigma].\text{BoxCl}, a \in \vartheta_\sigma, b \in UC_{\sigma;\alpha}\}$ 
25:        Let  $\varphi := \bigvee_{l \in CS} \neg l$ 
26:        addClause( $\text{Trie}[\sigma].\text{sat}, \varphi$ )
27:      else
28:        {( $\not$ SAT) shortcut failed: multiple conflicting diamonds}
29:        for  $(c \rightarrow_l \langle \alpha \rangle d) \in \text{Trie}[\sigma].\text{DiaCl}$  with  $c \in \vartheta_\sigma, l = i, d \in UC_{\sigma;\alpha}$  do
30:          Replace  $l$  with a fresh label  $l'$  in  $c \rightarrow_l \langle \alpha \rangle d$ 
31:        end for
32:      end if
33:      return RECARTAB( $A, \text{Trie}[\sigma]$ ) {apply (restart)}
34:    end if
35:  end for
36:  return Satisfiable {because every fired diamond is fulfilled}
37: end if

```

Suppose we are creating a model with depth n , and each world is ‘firing’ two diamond clauses. This results in creating $\mathcal{O}(2^n)$ worlds, and as CEGARTAB must iterate over each of these worlds, this will clearly timeout for large n . Instead, let us create an over-approximation that will aim to reduce the number of successors for each node. Instead of creating two worlds for $\langle \alpha \rangle p \wedge \langle \alpha \rangle q$, one for p and one for q , we will attempt to create one α -successor with $(p ; q)$. If this fails we will get a conflict A' , and if $\{p, q\} \subseteq A'$, we know we must separate these diamond clauses. Otherwise, we have just managed to create a model for w_0 that has one successor, not two. If every world succeeds in grouping diamond clauses we can create a model with $\mathcal{O}(n)$ worlds instead of $\mathcal{O}(2^n)$ worlds, as formalised next.

5.4 RECAR TAB: the RECAR-tableaux over-approximation algorithm

We begin by annotating each dia-clause with a ‘label’, initialised to 1: i.e. we replace $a \rightarrow \langle \alpha \rangle b$ with $a \rightarrow_1 \langle \alpha \rangle b$, for all dia-clauses. Now, we create a sole successor for each fired *label*, instead of each fired *dia-clause*. The number of labels will increase as the algorithm progresses so let the set of labels be L .

Theorem 5.4. *Algorithm 3 for RECAR TAB is a sound and complete decision procedure for K_n .*

Proof.

Termination: Separating diamond clauses can happen only finitely many times. Thus for RECAR-tableaux to not terminate, it must either jump or restart infinitely, but both are impossible by the proofs for CEGAR-tableaux [9].

Soundness: Any clause learned by RECAR TAB is of the form $\neg c \vee \neg a_1 \vee \dots \vee \neg a_m$ as in line 17 and 24 and is a logical consequence of the discovery that for some label l , the set $\{c, a_1, \dots, a_m, c \rightarrow_l \langle \alpha \rangle d, a_1 \rightarrow_l [\alpha] b_1, \dots, a_m \rightarrow_l [\alpha] b_m\}$ is K-unsatisfiable just as they are in CEGARTAB, so it is sound to learn such a clause. But when the conflict set contains multiple diamonds, as in line 28, we separate these conflicting diamonds using different labels, and restart, which will reduce the cardinality of L at line 6. This corresponds to finding that $(\langle \alpha \rangle D; [\alpha] B)$ is K-unsatisfiable and then attempting to K-satisfy $(\langle \alpha \rangle D'; [\alpha] B)$ for some $D' \subset D$ instead. The move is sound because we maintain the requirement for ($\sharp$ SAT)-shortcuts that if $(\langle \alpha \rangle D'; [\alpha] B)$ is K-satisfiable then so is each individual set $(\langle \alpha \rangle d'; [\alpha] B)$ for all $d' \in D'$. In the worst case, this will lead to every dia-clause having its own label, meaning that RECAR TAB will just simulate CEGARTAB, which we know to be sound.

Completeness Suppose RECAR TAB returns Satisfiable and proceed by induction on the number of jumps on the maximal sequence of jumps. If there are no jumps the Kripke model is just a dead-end w_ε with ϑ_ε . Otherwise, consider the first jump from this root world on this sequence for any $i \in L$. The fired diamonds consist of some set $\langle \alpha \rangle D = \{\langle \alpha \rangle d_1, \dots, \langle \alpha \rangle d_{n>0}\}$ while the fired boxes are $[\alpha] B = \{[\alpha] b_1, \dots, [\alpha] b_{m \geq 0}\}$. We jump to a child containing the assumptions $(D; B) = \{d_1, \dots, d_n, b_1, \dots, b_m\}$ and the recursive call at line 12 must have returned Satisfiable so this child w_α^i must be K-satisfiable by the induction hypothesis. Thus $(\langle \alpha \rangle D; [\alpha] B)$ is K-satisfiable at w_ε in the K-model obtained by putting $w_\varepsilon R_\alpha w_\alpha^i$. But then so must each set $(\langle \alpha \rangle d; [\alpha] B)$ for every $d \in D$ since $\langle \alpha \rangle (p \wedge q) \rightarrow \langle \alpha \rangle p \wedge \langle \alpha \rangle q$ is K-valid. Thus each $i \in L$ generates such an R_α -child for its corresponding diamond jump and the for-loop over these $i \in L$ at line 8 succeeds only when all such children return Satisfiable, meaning K-satisfiable. Thus every $(\langle \alpha \rangle d; [\alpha] B)$ fired by ϑ_ε has a K-satisfiable witness, giving a K-model for $\text{SNF}_{mc}(\varphi_0)$.

□

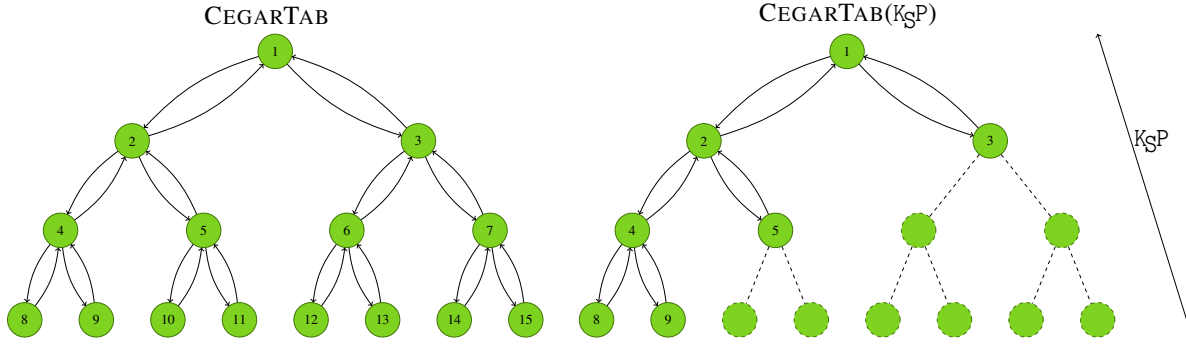


Figure 6: A comparison of the search process of CEGARTAB and CEGARTAB(KSP). CEGARTAB evaluates worlds with a pre-order traversal, and in the meantime, KSP saturates layers bottom-up. So CEGARTAB(KSP) generates no successors for “fixpoint” worlds 5 and 3 giving scenarios where CEGARTAB(KSP) evaluates a linear number of worlds, whilst CEGARTAB evaluates an exponential number.

We report on experiments on RECAR-Tableaux at the end of this section, but let us first propose our own approach to achieving ($\sharp$ SAT) shortcuts.

5.5 ($\sharp$ SAT) shortcuts via fixpoint detection using KSP

One of the main issues with RECAR-TAB is that it explicitly generates the model, meaning it suffers from the same shortcomings as regular CEGAR-tableaux. Sure, it uses heuristics to find smaller models, but there is no guarantee these smaller models exist, and this “optimism” might lead to wasted work.

We propose a different method of finding ($\sharp$ SAT) shortcuts. Instead of explicitly creating a model, we simply check if one exists by searching for fixpoints in the CEGARTAB under-approximation where, intuitively, a fixpoint is a set of clauses which will never be refined with any new clauses by CEGARTAB.

Lemma 5.5 (Fixpoints are Satisfiable). *If $\mathcal{A}_\sigma$ is a set of unit assumptions, and CEGARTAB will not refine $\mathcal{C}_\sigma^{cpl}$ with any clauses in the future, then $\mathcal{A}_\sigma ; \mathcal{C}_\sigma^{cpl}$ is classically satisfiable iff $\mathcal{A}_\sigma ; \mathcal{C}_\sigma^{cpl} ; \mathcal{C}_\sigma^{box} ; \mathcal{C}_\sigma^{dia}$ is K_n -satisfiable.*

Proof. If $\mathcal{A}_\sigma ; \mathcal{C}_\sigma^{cpl} ; \mathcal{C}_\sigma^{box} ; \mathcal{C}_\sigma^{dia}$ is K_n -satisfiable then its subset $\mathcal{A}_\sigma ; \mathcal{C}_\sigma^{cpl}$ is clearly classically satisfiable. If $\mathcal{A}_\sigma ; \mathcal{C}_\sigma^{cpl}$ is classically satisfiable then CEGARTAB($\mathcal{A}, Trie[\sigma]$) will find a classical valuation, then iterate over diamonds. For a contradiction, suppose a diamond jump returns Unsatisfiable. Then we learn a clause to add to $\mathcal{C}_\sigma^{cpl}$, contradicting that $\mathcal{C}_\sigma^{cpl}$ is a fixpoint. Thus all diamonds must be fulfilled, and so CEGARTAB($\mathcal{A}, Trie[\sigma]$) returns Satisfiable. By the correctness of CEGARTAB, the set $\mathcal{A}_\sigma ; \mathcal{C}_\sigma^{cpl} ; \mathcal{C}_\sigma^{box} ; \mathcal{C}_\sigma^{dia}$ is K_n -satisfiable. $\square$

Lemma 5.5 allows CEGARTAB to skip the potentially expensive process of generating a model if we know there are no clauses left to learn. This helps when a satisfying model is large, but “easy” to produce. But how to determine whether $\mathcal{C}_\sigma^{cpl}$ is a fixpoint? To this end we use KSP, as explained next.

5.6 CEGAR-tableaux with a resolution-based oracle

Recall that KSP uses SNF_{ml} rather than SNF_{mc} and that SNF_{ml} leads to a linear Trie data structure, as explained previously. Nevertheless, Lemma 5.5 still applies.

Algorithm 4 CEGARTAB(KSP)(A, Trie[l])

```

1: {Inputs: A is a set of unit assumptions,
   Trie[l] contains the set  $\mathcal{C}_l$  of modal clauses from  $\text{SNF}_{ml}(\varphi_0)$  and a SAT-solver.}
2: {Merge Step: Check if external KSP process has saturated layer  $l$  wrt layer  $l+1$  (bottom-up)}
3: if exists KSP-output-file for layer  $l$  then
4:   Add new KSP clauses to Trie[l].sat
5:   Fixpoint := true
6: end if
7: Let  $t_l := \text{solve}(\text{Trie}[l].\text{sat}, A)$  {is  $\mathcal{C}_l^{cpl}$  cpl-satisfiable}
8: if  $t_l = (\text{unsat}, UC_l)$  then
9:   return Unsatisfiable( $UC_l$ ) {because  $\mathcal{C}_l$  is  $K_n$ -unsatisfiable}
10: else if  $t_l = (\text{sat}, \vartheta_l)$  then
11:   if Fixpoint = true then
12:     return Satisfiable {Fixpoint reached by KSP: skip child generation}
13:   end if
14:   {Check box and diamond clauses that fire under classical valuation  $\vartheta_l$ }
15:   for every  $(c \rightarrow \langle \alpha \rangle d) \in \text{Trie}[l].\text{DiaCl}$  with  $c \in \vartheta_l$  do
16:     Let  $B = \{b \mid (a \rightarrow [\alpha]b) \in \text{Trie}[l].\text{BoxCl} \text{ and } a \in \vartheta_l\}$ 
17:     {evaluate the next layer using the jump rule on  $\alpha$ }
18:     if CEGARTAB( $(d; B)$ , Trie[l+1]) = Unsatisfiable( $UC_{l+1}$ ) then
19:        $CS := \{c\} \cup \{a \mid (a \rightarrow [\alpha]b) \in \text{Trie}[l].\text{BoxCl} \text{ and } a \in \vartheta_l \text{ and } b \in UC_{l+1}\}$ 
20:       Let  $\varphi := \bigvee_{l \in CS} \neg l$ 
21:       addClause(Trie[l].sat,  $\varphi$ ) {Learn new clause  $\varphi := \neg \bigwedge CS$ }
22:       return CEGARTAB(A, Trie[l]) {apply (restart)}
23:     end if
24:   end for
25:   return Satisfiable {because every fired diamond is fulfilled}
26: end if

```

Figure 7: Algorithm of CEGARTAB(KSP) on SNF_{ml} showing the integration of fixpoint detection.

Theorem 5.6 (KSP produces CEGARTAB fixpoints). *Suppose that a **saturated** derivation from $\text{SNF}_{ml}(\varphi_0)$ by KSP results in a new set $\mathcal{C}'_l$ of modal clauses (at layer l). Then $\mathcal{C}'_l$ is a fixpoint of CEGARTAB.*

Proof. If $\mathcal{C}'_l$ is not a CEGARTAB fixpoint, the restart rule is applicable. Then the learnt clause corresponds exactly to one of the [GEN] resolution rule applications from Figure 4 by KSP, so the set of clauses is not saturated: contradiction. $\square$

Our algorithm for “fixpoint detecting” CEGAR-Tableaux is CEGARTAB(KSP) but our implementation is CEGARBox++(KSP). For simplicity, CEGARBox++(KSP) is **multithreaded**, running CEGARBox++ and KSP independently, with naive communication via file writing/reading. We thank Cláudia Nalon for adjusting KSP to make this possible. The approach is as follows where both CEGARBox++ and KSP use the linear Trie for SNF_{ml} , not the tree-like Trie for SNF_{mc} , see Figure 6 and Figure 7:

1. Run KSP on an input formula φ_0 , and have it print out $\text{SNF}_{ml}(\varphi_0)$;

2. K_gP then performs (layered) resolution, from deepest (κ) to shallowest layers (0), and after fully saturating a layer, prints out all classical clauses it has learned;
3. CEGARTAB runs independently on $\text{SNF}_{ml}(\varphi_0)$, from shallowest (0) to deepest (κ) layers, but when it receives clauses from K_gP at layer l , it adds them to $\mathcal{C}_l^{\text{cpl}}$ (at modal layer l), marks the layer as a fixpoint and ignores the jump rule at that layer (sound by Lemma 5.5).

There are some subtle points in this integration which deserve noting. At any particular time, K_gP is saturating some layer l with respect to layer $l + 1$ using the resolution rules and adding cpl-clauses to layer l as required to avoid the jumps that would cause a clash at layer $l + 1$. If CEGARBox++ arrives at layer l from layer $i - 1$ *before* K_gP has saturated layer l wrt layer $l + 1$, then the flag *Fixpoint* will be false and line 12 will be skipped, so CEGARBox++(K_gP) mimics CEGARBox++ at layer l . Else, if K_gP has already saturated layer l wrt layer $l + 1$, and printed out new cpl-clauses, then *Fixpoint* is put to true at line 5. CEGARBox++ therefore imports the new cpl-clauses at line 4 and calls the SAT-solver at layer l at line 7 with the assumptions A . That is, CEGARBox++ checks whether the assumptions A demanded by the jump from layer $l - 1$ to layer l are jointly cpl-satisfiable wrt the old cpl-clauses of layer l (inside the SAT-solver at layer l) plus these new cpl-clauses. But surely, K_gP has already told us that layer l is satisfiable, so why do we need this check? Not quite: it is perfectly possible that the new cpl-clauses make layer l cpl-unsatisfiable under assumptions A as demanded by layer $l - 1$, since K_gP has not checked this possibility (yet). Thus the call to the SAT-solver at line 7 is essential in both cases.

At worst, this algorithm *should* be as strong as CEGARTAB. As we shall see, the combination works surprisingly well, despite the rather naive integration.

5.7 Experimental evaluation of the two new approaches

Our experiments used an Intel i7-11700F@2.50 GHz CPU with 16GB of RAM. We used the same K-benchmarks as in our original paper [9]. Our code can be found here: <https://github.com/cormackikkert/CEGARBoxCPP>. We used MiniSAT [7] as our SAT-solver, preliminary results showed that using other SAT-solvers leads to negligible performance differences.

Using CEGARBox++ as our control, we tested our two variants that implement ($\not\text{SAT}$) shortcuts: CEGARBox++(K_gP) and RECARBox. Finally, as CEGARBox++(K_gP) runs an instance of K_gP “under the hood”, we compare against K_gP too. We use K_gP(0.1.6), whereas our previous work [9] used version 0.1.3 (the latest at that time).

Since CEGARBox++(K_gP) uses multi-threading, CPU-time is not an appropriate measure of performance, so all reported times are “wall time”, not CPU-time.

Figure 8 shows that in all cases RECARBox is worse than CEGARBox++, likely meaning that the extra work required to look for smaller models is wasted. On the other hand we see that CEGARBox++(K_gP) outperforms CEGARBox++ by a large margin on the MQBF benchmarks and is equal on the 3CNF benchmarks. However, CEGARBox++ outperforms CEGARBox++(K_gP) on the LWB-K benchmarks. This results in both provers being about equal overall. RECARBox performs better than CEGARBox++(K_gP) on the LWB-K benchmarks, which we believe is due to the presence of massive formulae in these benchmarks: see our conclusion.

Synergy vs. Parallelism Finally, as the integration involves multithreading, one might wonder whether the performance gains are due to the parallelism itself (effectively “cheating” by doubling resources). Specifically for the MQBF benchmarks, is CEGARBox++ *helping* to solve the satisfiable problems, or is it simply just K_gP solving them all (since it is best on satisfiable formulae)?

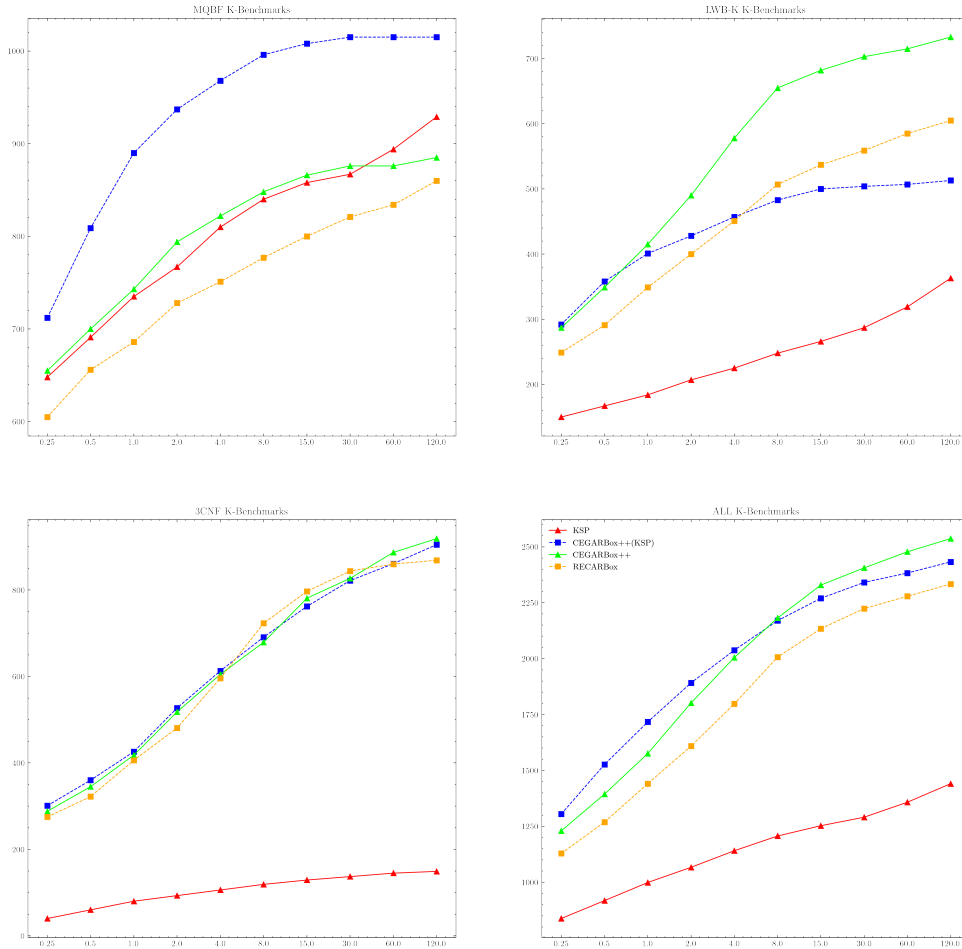


Figure 8: Performance of solvers on the standard K benchmarks

To this end, we compare CEGARBox++(KSP) with KSP in Figure 9. We can see that on hard satisfiable benchmarks, CEGARBox++(KSP) outperforms KSP, meaning that CEGARBox++ is contributing. We can see no satisfiable timeouts for CEGARBox(KSP), and even see a 100x improvement in some satisfiable problems, refuting the possibility that this improvement comes just from the doubling of resources due to parallelisation.

This highlights a distinction between a simple portfolio approach and our integration. While KSP and CEGARBox++ generally excel at different problem types (satisfiable vs. unsatisfiable), the integration appears to be better than the sum of their parts. Communicating fixpoints has allowed us to achieve results that a non-communicating portfolio could not.

6 Conclusions

We can see that RECAR-tableaux doesn't perform that well. We believe this is because it operates on a much stricter form of ($\neq$ SAT) shortcuts, where it tries to create smaller models, as opposed to CEGARTAB(KSP), which doesn't create models, but just shows they exist. Creating smallest models is intractable [4, 24] in general, and if they don't exist, RECAR-tableaux simply wastes its time.

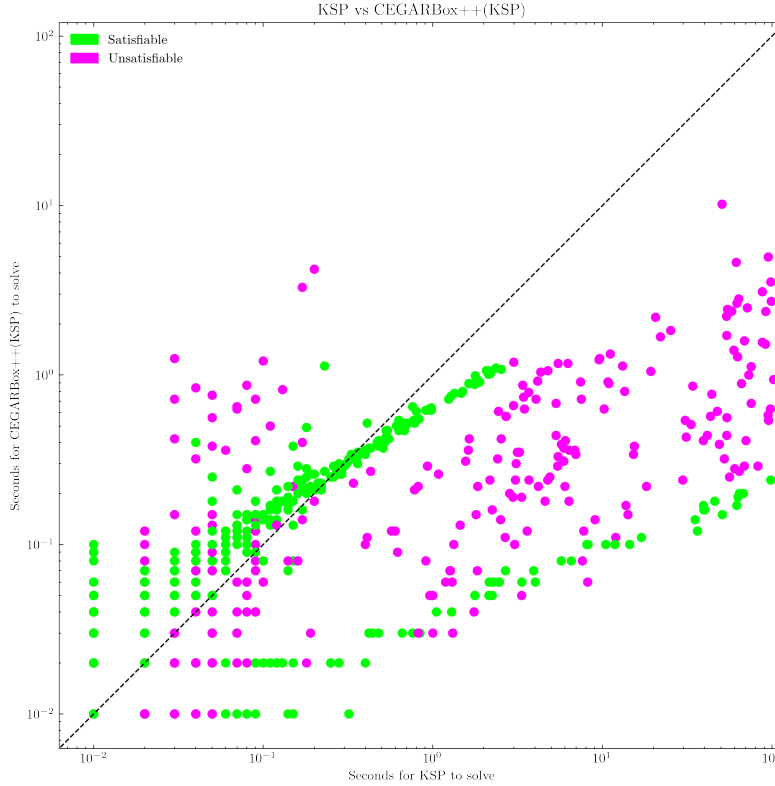


Figure 9: Performance of CEGARBox++(KSP) and KSP on the MQBF benchmarks. Problems on the top edge and the right edge indicate a timeout for CEGARBox++(KSP) and KSP respectively. Problems in the bottom right triangle/top left triangle are solved by CEGARBox++(KSP) / KSP quickest.

Our approach, CEGARTAB(KSP), for dealing with ($\mathcal{L}$ SAT) shortcuts shows a lot of promise, outperforming all solvers on the MQBF benchmarks. We believe a better implementation would result in CEGARTAB(KSP) outperforming CEGARTAB across the board. In particular, the issue with the current approach is that the KSP normal forming procedure produces approximately two times as many clauses as CEGARTAB, resulting in slowdowns on **massive** formulae, which are present in the LWB-K benchmarks. These are why RECARBox performs better than CEGARBox++(KSP) on the LWB-K benchmarks. KSP can simplify this normal form (resulting in a smaller number of clauses than CEGARTAB), but on these massive formulae the simplification times out. A more polished integration would use CEGARTAB's normal forming procedure, and have better and more real-time communication protocols. We imagine this would be capable of outperforming CEGARTAB on the 3CNF and LWB-K benchmarks, whilst also full-solving the MQBF benchmarks. An implementation of such is easier said than done as KSP has certain requirements on its normal form and must produce extra clauses that are not needed for CEGARTAB. An alternative would be to instead optimise the simplification process of KSP, so it can produce reasonable normal forms quickly.

In general, this idea of searching for fixpoints is simply a method of detecting if φ and φ are equisatisfiable, a normal idea in classical CEGAR, but more difficult in complicated CEGAR algorithms, where the under-approximation is a different type to the input (in our case a classical formula vs a modal formula). This fixpoint approach will be generally useful for other recursive CEGAR algorithms, such as

CAQE [20], or PDR [2], though of course it is most effective in applications such as CEGARTAB where branching is involved, meaning any ($\neg$ SAT) shortcut can possibly result in an exponential improvement in speed.

Whilst any approach could be used for fix-point detection, it is interesting to consider the implications of using resolution here. Previously, SAT-based, tableaux-based and resolution-based solvers were the state-of-the-art solvers used for reasoning in modal K. Now, CEGARTAB(K_SP), is an elegant combination of all these approaches, that is greater than the sum of its parts. In particular, this highlights a key similarity of CEGARTAB and resolution methods. In CEGARTAB we learn clauses when we need to (i.e. *lazily*), whereas K_SP learns clauses iteratively (i.e. *eagerly*).

We used K_SP as an oracle for CEGARTAB. From discussion with Cláudia Nalon, Clare Dixon, Ullrich Hustadt, and Fabio Papacchini, creating an efficient implementation by reversing the roles is not obvious.

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Relational Semantics for Flat Heyting-Lewis Logic

Jim de Groot*

University of Bern
Bern, Switzerland

jim.degroot@unibe.ch

Tadeusz Litak†

University of Naples Federico II
Naples, Italy

tadeusz.litak@fau.de

tadeusz.litak@unina.it

We introduce relational semantics for “flat Heyting-Lewis logic” HLC^b . This logic arises as the extension of intuitionistic logic with a Lewis-style strict implication modality that, contrary to its “sharp” counterpart $\text{HLC}^\sharp$, does not turn meets into joins in its first argument. We prove completeness and the finite model property for HLC^b and for several extensions with additional axioms.

1 Introduction

Recent years have seen a revival of the interest in intuitionistic modal logics [45, 39, 53, 3, 40, 5, 47, 34, 16, 52, 20, 1, 6, 15, 22, 11, 44, 7], including extensions with a Lewisian strict implication $\rightarrow$ [41, 42] and various types of conditional implications [59, 14]. Recall that in the intuitionistic setting, $\rightarrow$ is not definable in terms of unary $\Box$. Instead, it can be viewed as sitting between $\Box(p \rightarrow q)$ and $\Box p \rightarrow \Box q$. Indeed, the basic “flat” system¹ HLC^b proves [41]:

$$(b1) \quad \Box(p \rightarrow q) \rightarrow (p \rightarrow q)$$

$$(lb) \quad (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q).$$

The original motivation of the Utrecht school to study such a connective came from research on schematic logics of theories over intuitionistic arithmetic HA. More specifically, it arose from the study of Σ_1^0 -preservativity [55, 56, 58, 27, 28], which over PA can be seen as the contraposed variant of both Π_1^0 -conservativity and arithmetical interpretability [8, 51, 30, 57, 2]. Subsequently, many other application and interpretations were put forward, see e.g. [41, 21]. In particular, in the presence of an additional axiom *str* (see Section 5) the resulting calculus turns out to be the Curry-Howard counterpart (i.e. the inhabitation logic) of (Hughes) arrows in functional programming, in particular in Haskell [24, 25, 4, 29, 38, 37, 36]. Somewhat underdeveloped philosophical applications include a generalisation of intuitionistic epistemic logic IEL [3] to the intuitionistic logic of entailments IELE [21, Section 2.4] or a fine-grained analysis of the collapse of Lewis’ original 1918 system of strict implication caused by involutive negation [41, Appendix D].²

The flat calculus HLC^b arises from extending intuitionistic logic with a binary operator $\rightarrow$ that is normal in its second argument, transitive, and satisfies *implication necessitation*, i.e. derivability of $\varphi \rightarrow \psi$ implies derivability of $\varphi \rightarrow \psi$. From this, we can obtain the sharp calculus $\text{HLC}^\sharp$ by adding the axiom:

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¹Naming underwent several evolutions. Early references in the Utrecht school [26, 27, 64, 28] denoted the base “flat” system as iP^- and the base sharp system as iP , Litak and Visser [41] replaced iP with iA (with P standing for *preservativity* and A standing for *arrows*), and in a subsequent paper the same authors [42] finally settled for the present notation.

²It is worth noting here that in later years, having become aware of nascent study of non-classical calculi, Lewis not only followed closely the development of early multi-valued logics, but also on at least one occasion spoke favourably of Brouwer’s rejection of excluded middle. More information and detailed discussion can be found in Litak and Visser [41].

$$(di) \quad (p \multimap r) \wedge (q \multimap r) \rightarrow ((p \vee q) \multimap r)$$

This sharp version of the logic can conveniently be interpreted in Kripke-style relational semantics. This perspective has resulted in numerous correspondence, completeness and finite model property results for this “sharp” semantics [26, 27, 64, 28, 41], with recent work showing how to use the natural Gödel-McKinsey-Tarski translation to transfer metatheory of bimodal classical logics [21].

Nevertheless, from the point of view of applications and interpretations discussed above, there are few reasons to insist on the “sharp” calculus $HLC^\sharp$ as the minimal one, excluding interpretations not validating di. In more philosophical contexts such as IELE [21, Section 2.4], the present state of the research does not determine the status of di, whereas in mathematical and computer science applications enforcing it as a base axiom would simply be too restrictive. Insisting on the validity of di in the inhabitation logic of arrows in functional programming languages [24, 25, 4, 29, 38, 37, 36] would limit the Curry-Howard interpretation to so-called *arrows with choice* [25]. While this would still cover *arrows with delay* and *higher-order arrows* (corresponding, respectively, to *applicative functors* and *monads*), and some other instances such as list processors and Kleisli arrows, important examples of arrows *without* choice can be obtained using automata or functions on infinite streams [21, § 2.3].

The situation is similar when it comes to schematic logics of arithmetical theories, i.e. the original motivation of the Utrecht school to study variants and extensions of HLC: Sharpness obtains in some contexts, but not in others. For example, it fails classically, i.e. when HA is replaced by Peano Arithmetic (PA). In fact, one can argue that the main reason why classical interpretability logic is a separate subfield with its own methods and semantics, irreducible even to multi-modal and multi-dimensional normal modal logics, is precisely the failure of the validity of the (contraposed form of) di in the logic of Π_1^0 -conservativity/interpretability of PA. Some readers might find this failure surprising given that di holds in the schematic theory of HA, which is a subtheory of PA. However, non-monotonicity of schematic logics is a phenomenon observable even in the signature of provability with a single unary $\Box$ ([41, § 5.3]). A sufficient condition for sharpness of the schematic logic of a given arithmetical theory T is that T is able to prove that the set of T -theorems is *closed under q-realizability* [41, Theorem 5.6]. Note that if $T' \supseteq T$, the T -provable statements about T -theorems remain T' -provable, but there might be new T' -theorems for which q-realizability is not T' -provable (and, *a fortiori*, not T -provable).

Arguably, the main temptation to accept the sharp system as the minimal one has been that of nice completeness results and natural countermodels. In the flat setting, so far one has had to turn to algebraic semantics or a suitable adaptation of Chellas-Weiss semantics for ICK [59, 13, 17], Routley-Meyer semantics for substructural logics [48, 49, 50, 46, 10], or (generalised) Veltman semantics [31, 54, 32, 33], because a simple Kripke-style semantics for HLC^b appeared elusive.

In this paper we fill this gap by providing a Kripkean interpretation for HLC^b . This semantics is inspired by recent work on semantics of CK [22], and crucially relies on using a *preorder* $\preceq$ instead of a partial order to interpret the intuitionistic implication. Since the semantic clause for $\multimap$ directly enforces upward persistence (Definition 3.1), the most general version of the new semantics (Definition 3.1) does not impose any interaction conditions between R and $\preceq$. However, similarly to the case of intuitionistic $\Box$ and unlike the sharp interpretation, our language is oblivious to closing R under *post-composing* with $\preceq$ (Proposition 3.10), and the resulting *upward-flat* frames (Section 3.2) prove convenient for computing correspondents and obtaining completeness results.

Using a canonical model construction, we prove completeness and the finite model property for HLC^b and several of its extensions (Sections 4 and 5.1). Guided by the canonical model construction for CK, we use *segments* rather than prime theories to have a more fine-grained handle on the modal accessibility relation. Still mirroring CK, we sometimes need to restrict our choice of segments, for example when proving completeness for natural variants of K4 and S4 in our setting (Section 5.2).

When $\preceq$ is collapsed to equality, turning our frames into standard Kripke frames, our frames turn lb into a bi-implication, rather than bl. This does not mean that our semantics trivialises classically: in the preorder setting, validating excluded middle simply requires $\preceq$ to be symmetric, and such a classical variant of our semantics does not collapse $\neg$ (Example 3.3). This creates the opportunity to use our semantics for completeness results for subsystems of standard interpretability logics such as ILM and ILP.

In the CK setting, the segment approach can be used to obtain duality results [23]. While our paper does not discuss duality in depth, we include comments for the interested reader, such as Remarks 3.8 and 4.10, illustrating difficulties with more standard approaches. However, we discuss a promising application in Section 6 in the context of syntactically motivated notion of *extension stability*. We note the relationship of this notion to what one might call the *open subframe* construction, and use our semantics to show that $\text{HLC}^\#$ is not extension stable, unlike the flat base calculus.

2 Intuitionistic strict implication, sharply and flatly

This section provides preliminaries and recapitulates known material. Section 2.1 presents the base flat system HLC^b . Section 2.2 discusses the sharp variant $\text{HLC}^\#$ together with its known Kripke semantics. Section 2.3 recapitulates the algebraic semantics of both systems. Throughout the paper, we denote by $\mathcal{L}_\neg$ the language generated by the grammar

$$\varphi ::= p \mid \top \mid \perp \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \neg \varphi,$$

where p ranges over some arbitrary but fixed set Prop of proposition letters. We abbreviate $\Box\varphi := \top \neg \varphi$.

2.1 Syntax and axioms of the base flat system

A *consecution* is an expressions of the form $\Gamma \Rightarrow \varphi$, where $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\neg$.

Definition 2.1. Let Ax^b be an axiomatisation of intuitionistic logic together with the axioms

$$(\text{ka}) \quad ((p \neg q) \wedge (p \neg r)) \rightarrow (p \neg (q \wedge r)) \qquad (\text{tr}) \quad ((p \neg q) \wedge (q \neg r)) \rightarrow (p \neg r)$$

If $\text{Ax} \subseteq \mathcal{L}_\neg$, then we denote by $\mathcal{I}(\text{Ax})$ the collection of substitution instances of formulas in Ax , and define the axiomatic system $\text{HLC}^b \oplus \text{Ax}$ by:

$$(\text{Ax}) \quad \frac{\varphi \in \mathcal{I}(\text{Ax}^b) \cup \mathcal{I}(\text{Ax})}{\Gamma \Rightarrow \varphi} \qquad (\text{EI}) \quad \frac{\varphi \in \Gamma}{\Gamma \Rightarrow \varphi} \qquad (\text{MP}) \quad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \varphi \rightarrow \psi}{\Gamma \Rightarrow \psi} \qquad (\text{Na}) \quad \frac{\emptyset \Rightarrow \varphi \rightarrow \psi}{\Gamma \Rightarrow \varphi \neg \psi}$$

We say that $\Gamma \Rightarrow \varphi$ is *provable in* $\text{HLC}^b \oplus \text{Ax}$, and write $\Gamma \vdash_{\text{Ax}} \varphi$, if there exists a tree of consecutions built using the rules above with $\Gamma \Rightarrow \varphi$ as root and adequate applications of rules (EI) and (Ax) as leaves. If $\text{Ax} = \emptyset$ then we abbreviate $\vdash_{\text{Ax}}$ to $\vdash$, and if $\text{Ax} = \{\varphi_1, \dots, \varphi_n\}$ then we write $\text{HLC}^b \oplus \varphi_1 \oplus \dots \oplus \varphi_n$ for $\text{HLC}^b \oplus \text{Ax}$. Finally, we write $\Gamma \vdash_{\text{Ax}} \Delta$ if there exist $\psi_1, \dots, \psi_m \in \Delta$ such that $\Gamma \vdash_{\text{Ax}} \psi_1 \vee \dots \vee \psi_m$.

Proposition 2.2. For $\text{Ax} \subseteq \mathcal{L}_\neg$ and uniform substitution σ , the following rules are admissible in $\text{HLC}^b \oplus \text{Ax}$:

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \Gamma' \Rightarrow \varphi} \qquad \frac{\{\Gamma \Rightarrow \delta \mid \delta \in \Delta\} \quad \Delta \Rightarrow \varphi}{\Gamma \Rightarrow \varphi} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma^\sigma \Rightarrow \varphi^\sigma} \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi}$$

Proof. By induction on the height of a derivation for the premiss(es). □

The first three rules show that $\text{HLC}^b \oplus \text{Ax}$ is a monotone, compositional and structural relation, respectively. The double line in the last rule indicates that it holds both ways. Furthermore, since proof trees are finite we have $\Gamma \vdash_{\text{Ax}} \varphi$ if and only if there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\text{Ax}} \varphi$. Therefore $\text{HLC}^b \oplus \text{Ax}$ is a finitary logic [35, Definition 1.4.1]. Wherever possible, we blur the distinction between a logic and the set of its theorems (identified with derivable consecutions with an empty premise).

2.2 The sharpening

Definition 2.3. Let di be the axiom $(p \rightarrow r) \wedge (q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r)$, and define the *sharp Heyting-Lewis calculus* by $\text{HLC}^\sharp := \text{HLC}^b \oplus (\text{di})$.

The sharp systems is known to allow a simple Kripke-style semantics, with soundness, completeness and the finite model property results for many of its extensions [26, 27, 64, 28, 41, 21]:

Definition 2.4. A *sharp frame* is a tuple $\mathfrak{F} = (W, \leq, R)$ consisting of a set W , a partial order $\leq$ on W , and a relation R on W such that for all $w, v, u \in W$:

$$\text{if } w \leq v \text{ and } vRu \text{ then } wRu.$$

A *sharp model* is formed by adding a valuation that interprets proposition letters as upsets. The interpretation of $\mathcal{L}_\rightarrow$ -formulas at a world w in a sharp model $\mathfrak{M}$ is defined recursively via:

$$\begin{aligned} \mathfrak{M}, w \Vdash^s p & \quad \text{iff } w \in V(p) \\ \mathfrak{M}, w \Vdash^s \perp & \quad \text{never} \\ \mathfrak{M}, w \Vdash^s \varphi \wedge \psi & \quad \text{iff } \mathfrak{M}, w \Vdash^s \varphi \text{ and } \mathfrak{M}, w \Vdash^s \psi \\ \mathfrak{M}, w \Vdash^s \varphi \vee \psi & \quad \text{iff } \mathfrak{M}, w \Vdash^s \varphi \text{ or } \mathfrak{M}, w \Vdash^s \psi \\ \mathfrak{M}, w \Vdash^s \varphi \rightarrow \psi & \quad \text{iff for all } w' \succeq w, \text{ if } \mathfrak{M}, w' \Vdash^s \varphi \text{ then } \mathfrak{M}, w' \Vdash^s \psi \\ \mathfrak{M}, w \Vdash^s \varphi \rightarrow \psi & \quad \text{iff for all } v \text{ (if } wRv \text{ and } \mathfrak{M}, v \Vdash^s \varphi \text{ then } \mathfrak{M}, v \Vdash^s \psi) \end{aligned}$$

2.3 Algebraic semantics

While Kripke completeness has so far only been available for the sharp calculus and its reasonably well-behaved extensions, algebra provides an obvious route towards a generic completeness result.

Definition 2.5. A *flat Lewisian Heyting Algebra Expansion*, or L-HAE^3 , is a tuple $\mathcal{A} := \langle A, \wedge, \vee, \rightarrow, \rightarrow, \perp, \top \rangle$ such that $\langle A, \wedge, \vee, \rightarrow, \top, \perp \rangle$ is a Heyting algebra and the following laws are satisfied:

$$(\text{CK}) \quad (a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c),$$

$$(\text{CT}) \quad (a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c,$$

$$(\text{CI}) \quad a \rightarrow a = \top.$$

If $\mathcal{A}$ additionally satisfies

$$(\text{CD}) \quad (a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c,$$

then it is a *sharp Lewisian Heyting Algebra* (L-HAO).

³We write L-HAE s for the class of all Lewisian Heyting Algebra Expansions, following Litak and Visser [42]. The same authors call the class of sharp algebras *Lewisian Heyting Algebras with Operators* and discuss the reasons behind this terminology, whereas De Groot et al. [21] call the sharp algebras simply *Heyting-Lewis algebras*, a name which would prove rather confusing in this context.

A L-HAE is a L-HAO if and only if its *strict reduct*, i.e. the reduct without $\rightarrow$, is a weak Heyting algebra [12]. We note that CK, CD, CT, and CI are referred to as C1–C4 in [12]. A valuation v in $\mathcal{A}$, as usual, maps propositional atoms to elements of A and is inductively extended to $\hat{v}$ defined on all formulas in the obvious way. We write $\mathcal{A}, v \Vdash \varphi$ if $\hat{v}(\varphi) = \top$ and $\mathcal{A} \Vdash \varphi$ if $\mathcal{A}, v \Vdash \varphi$ for every valuation v . For $Ax \subseteq \mathcal{L}_3$, we write $\text{L-HAE}(Ax)$ for the class of L-HAE-algebras $\mathcal{A}$ such that $\mathcal{A} \Vdash \varphi$ for all $\varphi \in Ax$. Furthermore, we write $\text{L-HAE}(Ax) \Vdash \Gamma \Rightarrow \varphi$ if there exists a finite $\Gamma' \subseteq \Gamma$ such that all algebras in $\text{L-HAE}(Ax)$ validate $(\bigwedge \Gamma') \rightarrow \varphi$. Then the usual Lindenbaum-Tarski construction gives:

Theorem 2.6. *Let $Ax \subseteq \mathcal{L}_3$ be a set of axioms and $\Gamma \Rightarrow \varphi$ a consecution. Then $\Gamma \vdash_{Ax} \varphi$ if and only if $\text{L-HAE}(Ax) \Vdash \Gamma \Rightarrow \varphi$.*

3 Relational semantics for flat Heyting-Lewis logic

We introduce relational semantics for HLC^b , first in the most general version (Section 3.1), then in the “upward-flat” variant (Section 3.2) simplifying calculations of correspondents and completeness proofs. In Section 3.3 we compare the flat semantics to the sharp semantics from Definition 2.4.

3.1 Flat frames

Definition 3.1. A *flat frame* is a tuple $(W, \preceq, R)$ consisting of a nonempty set W , a preorder $\preceq$ on W and a relation R on W . A *flat model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ consisting of a flat frame $\mathfrak{F} = (W, \preceq, R)$ and a valuation $V : \text{Prop} \rightarrow \text{up}(W, \preceq)$ that assigns to each proposition letter p an upset $V(p)$ of $(W, \preceq)$. The interpretation of $\mathcal{L}_3$ -formulas at a world $w \in W$ extends the usual intuitionistic semantics with

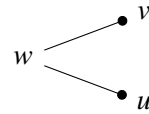
$$\mathfrak{M}, w \Vdash \varphi \rightarrow \psi \quad \text{iff} \quad \text{for all } w' \succeq w, \text{ if } \mathfrak{M}, v \Vdash \varphi \text{ for all } v \in W \text{ such that } w' R v \\ \text{then } \mathfrak{M}, v \Vdash \psi \text{ for all } v \in W \text{ such that } w' R v$$

The *truth set* of φ is given by $\llbracket \varphi \rrbracket^{\mathfrak{M}} = \{w \in W \mid \mathfrak{M}, w \Vdash \varphi\}$.

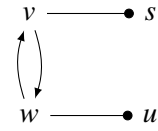
Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_3$ and let $\mathfrak{M}$ be a flat model. We write $\mathfrak{M}, w \models \Gamma$ if w satisfies all $\psi \in \Gamma$, and we say that $\mathfrak{M}$ *validates* $\Gamma \Rightarrow \varphi$ if $\mathfrak{M}, w \models \Gamma$ implies $\mathfrak{M}, w \models \varphi$ for all worlds w in $\mathfrak{M}$. A flat frame $\mathfrak{F}$ *validates* $\Gamma \Rightarrow \varphi$ if every model of the form $(\mathfrak{F}, V)$ validates the consecution, and it validates a formula φ if it validates the consecution $\emptyset \Rightarrow \varphi$. If $Ax \subseteq \mathcal{L}_3$ is a set of axioms, then we write $\Gamma \Vdash_{Ax} \varphi$, and say that Γ *semantically entails* φ on the class of flat frames for $\text{HLC}^b \oplus Ax$, if every flat frame that validates all formulas in Ax also validates the consecution $\Gamma \Rightarrow \varphi$.

Using truth-set notation, we have $\mathfrak{M}, w \Vdash \varphi \rightarrow \psi$ iff $R[w'] \subseteq \llbracket \varphi \rrbracket^{\mathfrak{M}}$ implies $R[w'] \subseteq \llbracket \psi \rrbracket^{\mathfrak{M}}$, for all $w' \succeq w$. To illustrate the subtleties of this semantics (even in the classical setting), we give two examples showing that (di) and the reverse of (lb) are not valid.

Example 3.2. Consider the flat model $(W, \preceq, R)$ where $W = \{w, v, u\}$, the intuitionistic accessibility relation $\preceq$ is the reflexive closure of the three worlds, and R is given by wRv and wRu . Let V be a valuation such that $V(p) = \{v\}$, $V(q) = \{u\}$ and $V(r) = \emptyset$. Then $w \Vdash p \rightarrow r$ because w has no intuitionistic successor x such that $R[x] \subseteq V(p)$, so the truth condition for $p \rightarrow r$ is vacuously true. Similarly, $w \Vdash q \rightarrow r$. But $w \not\Vdash (p \vee q) \rightarrow r$, because every R -successor of w satisfies $p \vee q$, but not every successor satisfies r . This shows that di is not valid on flat frames.



Example 3.3. Consider the flat model depicted on the right with $V(p) = \{s\}$ and $V(q) = \emptyset$. Then $w, v \not\models \Box p$ because $u \not\models p$, and hence $w \models \Box p \rightarrow \Box q$. On the other hand, the fact that $R[v]$ is contained in $V(p)$ but not in $V(q)$ shows that $w \not\models p \rightarrow q$. Therefore $w \not\models (\Box p \rightarrow \Box q) \rightarrow (p \rightarrow q)$, i.e. the reverse direction of (lb) is false.



A routine induction on the structure of φ allows us to prove:

Lemma 3.4 (Intuitionistic heredity). *Let $\mathfrak{M} = (W, \preceq, R, V)$ be a flat model. Then for all $\varphi \in \mathcal{L}_{\rightarrow}$ and all $w, v \in W$, if $\mathfrak{M}, w \models \varphi$ and $w \preceq v$ then $\mathfrak{M}, v \models \varphi$.*

Every flat frame gives rise to a L-HAE via its complex algebra.

Definition 3.5. The *complex algebra* of a flat frame $\mathfrak{F} = (W, \preceq, R)$ is $\mathfrak{F}^{+b} := \langle \text{up}(W, \preceq), \cap, \cup, \supseteq, \supseteq^b, W, \emptyset \rangle$, where

$$\begin{aligned} a \supseteq b &:= \{w \in W \mid \text{for all } v \succeq w, v \in a \text{ implies } v \in b\} \\ a \supseteq^b b &:= \{w \in W \mid \text{for all } v \succeq w, R[v] \subseteq a \text{ implies } R[v] \subseteq b\}. \end{aligned}$$

Lemma 3.6. *If $\mathfrak{F} = (W, \preceq, R)$ is a flat frame then $\mathfrak{F}^{+b}$ is a L-HAE, and $\mathfrak{F}$ and $\mathfrak{F}^{+b}$ validate precisely the same consecutions.*

Proof. We know that the upsets with the given operations form a Heyting algebra. A routine verification shows that $\supseteq^b$ satisfies (CK), (CT) and (CI). The second part of the lemma follows from the fact that valuations for $\mathfrak{F}$ correspond bijectively with assignments in $\mathfrak{F}^{+b}$, and that the interpretation of connectives in $\mathfrak{F}$ corresponds to that in $\mathfrak{F}^{+b}$. $\square$

Combining Lemma 3.6 and algebraic soundness proves:

Proposition 3.7. *For any $\text{Ax} \subseteq \mathcal{L}_{\rightarrow}$ and any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\rightarrow}$, we have that $\Gamma \vdash_{\text{Ax}} \varphi$ implies $\Gamma \models_{\text{Ax}} \varphi$.*

Remark 3.8. One might expect that, at least in the finite setting, turning such a complex algebra back into a flat frame should be straightforward, with join-prime elements providing the carrier set of the frame. Example 3.3 illustrates that this is not the case: the Heyting reduct of the dual algebra is the Boolean algebra with three atoms (join-primes). Collapsing the $\{v, w\}$ cluster would change the equational theory. The right approach to duality, similar to the one pursued in [60, 22], uses a suitable algebraic translation of the notion of the notion a segment introduced in Section 4, potentially blowing up the number of states (cf. estimates in the proof of Lemma 4.7). While many segments can often be eliminated (cf. Remark 4.9 and Section 5.2), care is needed.

3.2 Upward-flat frames

When only using the modal relation to interpret $\rightarrow$, we can make a simplification to our frames and assume that $R[w]$ is an upset for each w .

Definition 3.9. An *upward-flat frame* is a flat frame $(W, \preceq, R)$ such that for all $w, v, u \in W$, if $wRv \preceq u$ then wRu . A *upward-flat model* is a flat model whose underlying frame is upward-flat.

The coherence condition on the relation can be read as $(R \circ \preceq) = R$. While not strictly required, it simplifies the correspondence results for some of the additional axioms we consider in Section 5.

Proposition 3.10. *Let $\mathfrak{F} = (W, \preceq, R)$ be a flat frame. Define $R_{\preceq} := R \circ \preceq$, i.e. $wR_{\preceq}u$ if there exists a v such that $wRv \preceq u$, and let $\mathfrak{F}_{\preceq} = (W, \preceq, R_{\preceq})$. Then $\mathfrak{F}$ and $\mathfrak{F}_{\preceq}$ have the same complex algebra.*

Proof. Let a be an upset of $(W, \preceq)$ and $w \in W$. Then $R[w] \subseteq a$ if and only if $R_{\preceq}[w] \subseteq a$. This entails that the change from $\mathfrak{F}$ to $\mathfrak{F}_{\preceq}$ leaves the definition of $\neg^b$ unchanged, so that $\mathfrak{F}^{+b} = (\mathfrak{F}_{\preceq})^{+b}$. $\square$

It is often easier to find and depict frame correspondence results for upward-flat frames than for arbitrary ones. The definition and proposition above show that these can always be transformed into arbitrary frame conditions: simply replace every occurrence of R with $(R \circ \preceq)$. To illustrate the difference, consider the axiom $4_a : \varphi \rightarrow (\top \rightarrow \varphi)$.

Proposition 3.11.

1. A flat frame validates 4_a if and only if for all x, y, z, w such that $xRy \preceq zRw$, there exists v such that $xRv \preceq w$.
2. An upward-flat frame validates 4_a if and only if R is transitive.

Proof. (1) Suppose $\mathfrak{W}$ validates 4_a , and $xRy \preceq zRw$ for some worlds x, y, z, w . Let V be a valuation of p with $V(p) = \uparrow R[x]$. Then all worlds in $R[x]$ satisfy p , so by assumption they also all satisfy $\top \rightarrow p$. In particular, $y \models \top \rightarrow p$, and since $y \preceq z$ and (trivially) all worlds in $R[z]$ satisfy $\top$, we must have $w \models p$. By definition, this means that w lies above some R -successor v of x , as desired.

Conversely, suppose $\mathfrak{W}$ satisfies the frame condition. Let x be any world. To show that it satisfies 4_a , let $x \preceq x'$ and suppose all worlds in $R[x']$ satisfy p . Then we need that all worlds in $R[x]$ satisfy $\top \rightarrow p$. Let y be such a world. Since $\top$ is always true, we need to prove that $y \preceq zRw$ implies that $w \models p$. This follows from the frame condition. So 4_a is valid.

(2) This is a straightforward simplification of the above condition. For readers' convenience, we provide a direct proof. Suppose R is transitive and let w be a world such that $R[w] \subseteq V(p)$. Then by assumption $R[R[w]] \subseteq \llbracket p \rrbracket$, and since $wRv \preceq uRs$ implies $wRuRs$ we have $R[u] \subseteq \llbracket p \rrbracket$ for every $u \in R[w]$, so that $R[w] \subseteq \llbracket \top \rightarrow p \rrbracket$. Conversely, suppose $wRvRu$. Let V be a valuation such that $V(p) = R[w]$. Then $R[w] \subseteq V(p)$, so we must have $R[w] \subseteq \llbracket \top \rightarrow p \rrbracket$. This forces $R[R[w]] \subseteq V(p) = R[w]$. In particular, we have $u \in R[R[w]] \subseteq R[w]$ so wRu . Therefore R is transitive. $\square$

3.3 Relation to sharp semantics

The sharp semantics for $\text{HLC}^\sharp$ can be embedded into flat semantics in a truth preserving way. This gives rise to a completeness result for $\text{HLC}^\sharp$ with respect to flat semantics. We start with a simple sufficient condition for a flat frame to validate di. Let us call a flat frame $\mathfrak{F} = (W, \preceq, R)$ *pointwise downward directed* if $R[w]$ is a downward directed subset of $(W, \preceq)$ for every $w \in W$.

Lemma 3.12. *If a flat frame $\mathfrak{F} = (W, \preceq, R)$ is pointwise downward directed, then it validates di*

Proof. Suppose $\mathfrak{F}$ is a flat frame that is pointwise downward directed, $\mathfrak{M} = (\mathfrak{F}, V)$ is a flat model based on $\mathfrak{F}$ and $w \in W$ satisfies $p \rightarrow r$ and $q \rightarrow r$. Suppose $w' \succeq w$ and $R[w'] \subseteq \llbracket p \vee q \rrbracket = V(p) \cup V(q)$. We claim that either $R[w'] \subseteq V(p)$ or $R[w'] \subseteq V(q)$. If this is not the case, then we can find $v, u \in R[w']$ such that $v \notin V(p)$ and $u \notin V(q)$. By assumption there exists some $s \in R[w']$ such that $s \preceq v$ and $s \preceq u$. But then $s \notin V(p) \cup V(q)$, a contradiction. So we must have $R[w'] \subseteq V(p)$ or $R[w'] \subseteq V(q)$. In either case, using the assumption yields $R[w'] \subseteq V(r)$. This proves $w \models (p \vee q) \rightarrow r$, and hence di is valid on $\mathfrak{F}$. $\square$

Next, we turn a sharp model into a flat one. Intuitively, for each w we create a cluster such that each element of the cluster can modally access precisely one of the worlds in $R[w]$.

Definition 3.13. Let $\mathfrak{F} = (W, \leq, R, V)$ be a sharp frame and $\bullet$ a symbol such that $\bullet \notin W$. Define $\mathfrak{F}^b = (W^b, \preceq, \mathcal{R}, V^b)$, where

$$\begin{aligned} W^b &:= \{(w, v) \mid w \in W \text{ and } wRv\} \cup \{(w, \bullet) \mid w \in W \text{ and } R[w] = \emptyset\} & (w, v) \preceq (w', v') &\text{ iff } w \leq w' \\ V^b(p) &:= \{(w, v) \in W^b \mid w \in V(p)\} & (w, v) \mathcal{R} (w', v') &\text{ iff } v \leq v' \end{aligned}$$

Note that the definition of $\mathcal{R}$ ensures that $\mathfrak{M}^b$ is an upward-flat model. Moreover, for each $(w, v) \in W^b$ we have that $\mathcal{R}[(w, v)]$ is the upward closure (under $\preceq$) of (v, u) , for any $u \in W \cup \{\bullet\}$ such that $(v, u) \in W^b$. This implies that $\mathfrak{M}$ is pointwise downward directed.

Proposition 3.14. Let $\mathfrak{M} = (W, \leq, R, V)$ be a sharp model, and $\mathfrak{M}^b = (W^b, \preceq, \mathcal{R}, V^b)$ the corresponding flat model. Then for all $(w, v) \in W^b$ and all formulas φ , we have $\mathfrak{M}, w \Vdash^s \varphi$ if and only if $\mathfrak{M}^b, (w, v) \Vdash \varphi$.

Proof. We use induction on the φ , showcasing only the induction step for $\varphi = \psi \rightarrow \chi$. Suppose $\mathfrak{M}, w \Vdash^s \psi \rightarrow \chi$. Suppose $(w, v) \preceq (w', v')$ and $R[(w', v')] \subseteq \llbracket \psi \rrbracket^{\mathfrak{M}^b}$. Then $w \leq w'$ and $w'Rv'$, so wRv because $\mathfrak{M}$ is a sharp model. Also $R[(w', v')] = \{(u, s) \in W^b \mid v' \leq u\}$, so by the induction hypothesis we have $\mathfrak{M}, v' \Vdash^s \psi$. The assumption that $\mathfrak{M}, w \Vdash^s \psi \rightarrow \chi$ then gives $\mathfrak{M}, v' \Vdash^s \chi$, and intuitionistic heredity entails $\mathfrak{M}, u \Vdash^s \chi$ for all $u \geq v'$. Using induction again, this implies $R[(w', v')] \subseteq \llbracket \chi \rrbracket^{\mathfrak{M}^b}$, and hence $\mathfrak{M}^b, (w, v) \Vdash \psi \rightarrow \chi$. Conversely, suppose $\mathfrak{M}^b, (w, v) \Vdash \psi \rightarrow \chi$ and suppose wRu and $\mathfrak{M}, u \Vdash^s \psi$. Then $(w, v) \preceq (w, u)$ and by induction $\mathcal{R}[(w, u)] \subseteq \llbracket \psi \rrbracket^{\mathfrak{M}^b}$. This implies $\mathcal{R}[(w, u)] \subseteq \llbracket \chi \rrbracket^{\mathfrak{M}^b}$, and hence $\mathfrak{M}, u \Vdash^s \chi$. Therefore $\mathfrak{M}, w \Vdash^s \psi \rightarrow \chi$. $\square$

Combining the known completeness result for $\text{HLC}^\sharp$ with respect to sharp frames, the lemma and proposition above, and the fact that $\mathfrak{M}^b$ is a pointwise downward directed upward-flat model, gives:

Proposition 3.15. The logic $\text{HLC}^\sharp$ is sound and complete with respect to the class of pointwise downward directed flat frames.

4 Canonical models and completeness

We provide a canonical model construction relative to some set Σ that is closed under subformulas. This will give us, at once, the finite model property and strong completeness of the logic. We use a modification of the canonical model construction for CK, using so-called segments. The idea behind a segment is that it encodes both a world of the frame (a prime theory) as well as its successors. We start by defining prime Σ -theories. Throughout this subsection, we let Ax be a consistent set of formulas, and Σ denote a set of formulas that contains $\top$ and is closed under subformulas.

Definition 4.1. A *prime* (Ax, Σ) -theory is a subset $\Gamma \subseteq \Sigma$ that is *deductively closed* (i.e. if $\varphi \in \Sigma$ and $\Gamma \vdash_{\text{Ax}} \varphi$ then $\varphi \in \Gamma$), *consistent* (i.e. $\Gamma \not\vdash_{\text{Ax}} \perp$), and Σ -*prime* (i.e. if $\varphi_1, \dots, \varphi_n \in \Sigma$ and $\Gamma \vdash_{\text{Ax}} \varphi_1 \vee \dots \vee \varphi_n$ then $\varphi_i \in \Gamma$ for some $i \in \{1, \dots, n\}$). Write $\text{Th}_{\text{Ax}, \Sigma}$ for the set of prime (Ax, Σ) -theories. If $\Sigma = \mathcal{L}_{\rightarrow}$ then we omit reference to Σ and simply write *prime Ax-theory* instead of prime (Ax, Σ) -theory.

The Lindenbaum lemma can be proved as usual. We can use it to obtain prime (Ax, Σ) -theories by taking the intersection of the resulting prime Ax-theory with Σ .

Lemma 4.2 (Lindenbaum lemma). Let $\Gamma \cup \Delta \subseteq \mathcal{L}_{\rightarrow}$ and suppose $\Gamma \not\vdash_{\text{Ax}} \Delta$. Then there exists a prime theory Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \cap \Delta = \emptyset$.

Lemma 4.3. If Γ is a prime Ax-theory, then $\Gamma \cap \Sigma$ is a prime (Ax, Σ) -theory.

Segments comprise of a prime (Ax, Σ) -theory together with a suitable set of such theories that encodes the successors of the segment.

Definition 4.4. An (Ax, Σ) -segment is a pair (Γ, U) where $\{\Gamma\} \cup U \subseteq Th_{Ax, \Sigma}$ such that

(S1) if $\Delta \in U$ and $\Delta \subseteq \Delta' \in Th_{Ax, \Sigma}$ then $\Delta' \in U$;

(S2) for all $\varphi, \psi \in \Sigma$, if $\Gamma \vdash_{Ax} \varphi \rightarrow \psi$ and $\varphi \in \Delta$ for all $\Delta \in U$, then $\psi \in \Delta$ for all $\Delta \in U$.

Let $SEG_{Ax, \Sigma}$ be the set of (Ax, Σ) -segments and define relations by setting $(\Gamma, U) \subseteq (\Gamma', U')$ iff $\Gamma \subseteq \Gamma'$, and $(\Gamma, U) \mathcal{R} (\Gamma', U')$ iff $\Gamma' \in U$. Define the (canonical) valuation by $V_{Ax, \Sigma}(p) = \{(\Gamma, U) \in SEG_{Ax, \Sigma} \mid p \in \Gamma\}$. Then

$$\mathfrak{F}_{Ax, \Sigma} = (SEG_{Ax, \Sigma}, \subseteq, \mathcal{R}) \quad \text{and} \quad \mathfrak{M}_{Ax, \Sigma} = (SEG_{Ax, \Sigma}, \subseteq, \mathcal{R}, V_{Ax, \Sigma})$$

are an upward-flat frame and model, called the *full canonical frame and model (with respect to Ax and Σ)*. If $\Sigma = \mathcal{L}_{\rightarrow}$ then we abbreviate $SEG_{Ax} := SEG_{Ax, \mathcal{L}_{\rightarrow}}$ and $\mathfrak{F}_{Ax} := \mathfrak{F}_{Ax, \mathcal{L}_{\rightarrow}}$.

Lemmas 4.2 and 4.3 provide a way to construct prime (Ax, Σ) -theories, given suitable sets of formulas. The following lemma allows us to extend this to a segment:

Lemma 4.5. Let Γ be a prime (Ax, Σ) -theory and $\varphi \in \Sigma$, and define

$$U_{\Gamma, \varphi} := \{\Delta \in Th_{Ax, \Sigma} \mid \text{if } \psi \in \Sigma \text{ and } \Gamma \vdash_{Ax} \varphi \rightarrow \psi \text{ then } \psi \in \Delta\}.$$

1. $(\Gamma, U_{\Gamma, \varphi})$ is an (Ax, Σ) -segment.
2. $\varphi \in \Delta$ for all $\Delta \in U_{\Gamma, \varphi}$.
3. If $\theta \in \Sigma$ is such that $\Gamma \not\vdash_{Ax} \varphi \rightarrow \theta$, then there exists $\Delta \in U_{\Gamma, \varphi}$ such that $\theta \notin \Delta$.

Proof. (1) It follows immediately from the definition that $(\Gamma, U_{\Gamma, \varphi})$ satisfies (S1), so we focus on proving (S2). Suppose $\chi, \xi \in \Sigma$ and $\Gamma \vdash_{Ax} \chi \rightarrow \xi$ and $\chi \in \Delta$ for all $\Delta \in U_{\Gamma, \varphi}$. Then we must have

$$\{\psi \in \Sigma \mid \Gamma \vdash_{Ax} \varphi \rightarrow \psi\} \vdash \chi,$$

because otherwise we could use the Lindenbaum lemma to find some prime Σ -theory in $U_{\Gamma, \varphi}$ that does not contain χ . (We can first use the usual Lindenbaum lemma to find a prime theory containing the LHS but not φ , and then take its intersection with Σ .) By compactness, we can find $\psi_1, \dots, \psi_n \in \Sigma$ such that $\Gamma \vdash_{Ax} \varphi \rightarrow \psi_i$ for all $i \in \{1, \dots, n\}$ and $\psi_1, \dots, \psi_n \vdash_{Ax} \chi$. This implies

$$\varphi \rightarrow \psi_1, \dots, \varphi \rightarrow \psi_n \vdash_{Ax} \varphi \rightarrow \chi,$$

hence using transitivity

$$\varphi \rightarrow \psi_1, \dots, \varphi \rightarrow \psi_n, \chi \rightarrow \xi \vdash_{Ax} \varphi \rightarrow \xi.$$

Since Γ derives everything on the LHS, we also get $\Gamma \vdash_{Ax} \varphi \rightarrow \xi$, hence by definition of $U_{\Gamma, \varphi}$ we have $\xi \in \Delta$ for all $\Delta \in U_{\Gamma, \varphi}$.

(2) This follows from the fact (N_a) entails $\vdash \varphi \rightarrow \varphi$ for any $\varphi \in \mathcal{L}_{\rightarrow}$. Therefore $\Gamma \vdash_{Ax} \varphi \rightarrow \varphi$ and hence $\varphi \in \Delta$ for all $\Delta \in U_{\Gamma, \varphi}$ by definition.

(3) We claim that $\{\psi \in \Sigma \mid \Gamma \vdash_{Ax} \varphi \rightarrow \psi\} \not\vdash_{Ax} \theta$. Suppose towards a contradiction that this is not the case. Then by compactness we can find $\psi_1, \dots, \psi_n \in \Sigma$ such that

$$\psi_1, \dots, \psi_n \vdash_{Ax} \theta \tag{1}$$

and $\Gamma \vdash_{Ax} \varphi \rightarrow \psi_i$ for each $i \in \{1, \dots, n\}$. This implies $\Gamma \vdash_{Ax} \varphi \rightarrow (\psi_1 \wedge \dots \wedge \psi_n)$. Furthermore, (1) entails $\vdash_{Ax} (\psi_1 \wedge \dots \wedge \psi_n) \rightarrow \theta$, so by (N_a) we get $\vdash_{Ax} (\psi_1 \wedge \dots \wedge \psi_n) \rightarrow \theta$. In particular, this gives $\Gamma \vdash_{Ax} (\psi_1 \wedge \dots \wedge \psi_n) \rightarrow \theta$, so that tr entails $\Gamma \vdash_{Ax} \varphi \rightarrow \theta$, a contradiction. So we have $\{\psi \in \Sigma \mid \Gamma \vdash_{Ax} \varphi \rightarrow \psi\} \not\vdash_{Ax} \theta$. Then Lemma 4.2 gives a prime Ax -theory Δ containing ψ for every $\psi \in \Sigma$ such that $\Gamma \vdash_{Ax} \varphi \rightarrow \psi$, but not θ . By definition $\Delta \cap \Sigma \in U_{\Gamma, \varphi}$, so it is the desired witness. $\square$

Lemma 4.6. *For all $\varphi \in \Sigma$ and $(\Gamma, U) \in \text{SEG}_{\text{Ax}, \Sigma}$ we have $\mathfrak{M}_{\text{Ax}, \Sigma}, (\Gamma, U) \models \varphi$ iff $\varphi \in \Gamma$.*

Proof. We use induction on the structure of φ . If φ is $\top, \perp$ or a proposition letter, the statement is immediate. The cases for $\wedge$ and $\vee$ follow using induction and the fact that Γ is prime.

Case $\varphi = \psi \rightarrow \chi$. Suppose $\psi \rightarrow \chi \in \Gamma$. Let $(\Gamma, U) \subsetneq (\Gamma', U')$ and suppose $(\Gamma', U') \models \psi$. Then by definition of $\subsetneq$, deductive closure of prime (Ax, Σ) -theories, and the induction hypothesis we find $\psi \in \Gamma'$ and $\psi \rightarrow \chi \in \Gamma'$. This implies $\chi \in \Gamma'$, hence by induction $(\Gamma', U') \models \chi$. This proves $(\Gamma, U) \models \psi \rightarrow \chi$.

Conversely, suppose $\psi \rightarrow \chi \notin \Gamma$. Then $\Gamma \not\models_{\text{Ax}} \psi \rightarrow \chi$, so $\Gamma, \psi \not\models_{\text{Ax}} \chi$ and we can find a prime theory Γ' containing Γ, ψ but not χ . Then $\Gamma' \cap \Sigma$ is a prime Σ -theory and we can extend it to an (Ax, Σ) -segment (for example by using Lemma 4.5 with $\varphi = \top$) which (using induction) satisfies ψ but not χ . Therefore $(\Gamma, U) \not\models \psi \rightarrow \chi$.

Case $\varphi = \psi \rightarrow \neg \chi$. If $\psi \rightarrow \neg \chi \in \Gamma$ then we get $(\Gamma, U) \models \psi \rightarrow \neg \chi$ immediately from the definition of a segment. Now suppose $\psi \rightarrow \neg \chi \notin \Gamma$. By Lemma 4.5 $(\Gamma, U_{\Gamma, \psi})$ is a (Ax, Σ) -segment such that $\psi \in \Delta$ for all $\Delta \in U_{\Gamma, \psi}$ while $\chi \notin \Delta$ for some $\Delta \in U_{\Gamma, \psi}$. Since each Δ can be extended to a segment, this proves $(\Gamma, U) \not\models \psi \rightarrow \neg \chi$. $\square$

Depending on our choice of Ax and Σ , the canonical model construction gives rise to a finite model property and a strong completeness result.

Lemma 4.7. *Let Ax be a set of axioms.*

1. *Suppose that for every finite consecution $\Gamma \Rightarrow \varphi$ there exists a finite subformula-closed set Σ that contains Γ, φ and $\top$ such that $\mathfrak{F}_{\text{Ax}, \Sigma}$ validates Ax . Then $\text{HLC}^b \oplus \text{Ax}$ has the finite model property.*
2. *Suppose $\mathfrak{F}_{\text{Ax}}$ validates Ax . Then $\text{HLC}^b \oplus \text{Ax}$ is strongly complete with respect to the class of (upward-)flat frames validating Ax .*

Proof. (1) Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas and suppose $\Gamma \not\models_{\text{Ax}} \varphi$. Let Σ be as described. Then we can use Lemmas 4.2 and 4.3 to construct a prime Σ -theory Γ' extending Γ that does not contain φ . Let (Γ', U) be a segment in $\text{SEG}_{\text{Ax}, \Sigma}$. It follows from Lemma 4.6 that $\mathfrak{M}_{\Sigma}, (\Gamma', U) \models \psi$ for all $\psi \in \Gamma$ and $\mathfrak{M}_{\Sigma}, (\Gamma', U) \not\models \varphi$. So $\mathfrak{M}_{\Sigma} \not\models \Gamma \Rightarrow \varphi$, hence $\mathfrak{F}_{\text{Ax}, \Sigma}$ is a flat frame that does not validate $\Gamma \Rightarrow \varphi$.

By assumption Σ is finite. A prime Σ -theory is a subset of Σ , so we have at most $2^{|\Sigma|}$ many prime theories, and hence at most $2^{|\Sigma|} \times 2^{2^{|\Sigma|}}$ Σ -segments, where $|\Sigma|$ denotes the size of Σ . Hence $\mathfrak{F}_{\Sigma}$ is finite.

(2) The proof is identical to the first paragraph of item (1) with $\Sigma = \mathcal{L}_{\neg}$. $\square$

Taking $\text{Ax} = \emptyset$ and Σ the closure of $\Gamma \cup \{\varphi, \top\}$ under subformulas yields:

Theorem 4.8. *The logic HLC^b has the finite model property and is strongly complete with respect to the class of (upward-)flat frames.*

Remark 4.9. While taking the collection of all (Ax, Σ) -segments in Definition 4.4 provides a canonical choice of segments, it is not strictly necessary. Analogous to [22], we can restrict the shape of segments we use while maintaining the truth lemma and completeness result. This can help create a canonical model that satisfies additional constraints. We will see an example of a restriction in Section 5.2, where we use this strategy to ensure that the modal accessibility relation is transitive when having 4_a as an axiom.

Remark 4.10. A different method for obtaining completeness results, employed for instance for $\text{HLC}^\#$ [21] and intuitionistic modal logic with a $\Box$ [62, 63], is via a Gödel-McKinsey-Tarski translation into classical bimodal logic with an S4 -box $\Box_i$ and a normal box $\Box_m$. Our case seems amenable to this treatment: flat frames corresponds precisely to the semantics of $\text{S4} \oplus \text{K}$, and the interpretation of $\varphi \rightarrow \psi$ is given by

$\Box_i(\Box_m\varphi \rightarrow \Box_m\psi)$. However, there is a mismatch between the descriptive frames of both logics: a duality for HLC^b would resemble that for CK [23] and use segments. As a consequence it does not seem to be the case that the two types of descriptive frames line up. This frustrates the transfer of e.g. completeness.

5 Completeness and the fmp for axiomatic extensions

We investigate the extension of HLC^b with the axioms listed in Table 1 and the given correspondence conditions proven in Lemma 5.2 and Proposition 3.11. We start by using Lemma 4.7 to obtain completeness and the finite model property for certain extensions of HLC^b with the listed axioms. In Section 5.2 we modify this canonical model construction to obtain completeness for extensions that include 4_a , and to obtain the finite model property for $\text{HLC}^b \oplus t_\Box \oplus 4_a$.

Axiom	Formula	Upward-flat correspondent
em	$p \vee \neg p$	$\preceq$ is symmetric
$t_\Box$	$(\top \rightarrow p) \rightarrow p$	$(\preceq \circ R)$ is reflexive
4_a	$p \rightarrow (\top \rightarrow p)$	R is transitive
str	$(p \rightarrow q) \rightarrow (p \rightarrow q)$	wRv implies $w \preceq v$
p_a	$(p \rightarrow q) \rightarrow (\top \rightarrow (p \rightarrow q))$	if $wRvRs$ then there exists $u \succeq w$ such that uRs and $R[u] \subseteq R[v]$

Table 1: Five axioms and their correspondents for upward-flat frames $\mathfrak{F} = (W, \preceq, R)$.

Remark 5.1. Both 4_a and p_a often occur in arithmetical contexts. It is worth noting that while 4_a is the “flat” correspondent of transitivity, p_a is the “sharp” one [42]. While str is a rather degenerate axiom classically (cf. Remark 5.6), intuitionistically it plays an important role, occurring in the logics of Haskell arrows [25], guarded (co)recursion, and entailments [21, Section 2.4], and even allows a non-trivial arithmetical interpretation as *completeness principle*.

Lemma 5.2. *Let $\mathfrak{F} = (W, \preceq, R)$ be an upward-flat frame. Then*

1. $\mathfrak{F}$ validates em if and only if $\preceq$ is symmetric;
2. $\mathfrak{F}$ validates $t_\Box$ if and only if for all w there exists v such that $w \preceq vRw$;
3. $\mathfrak{F}$ validates str if and only if wRv implies $w \preceq v$;
4. $\mathfrak{F}$ validates p_a if and only if for all w, v, s satisfying $wRvRs$ there exists $u \succeq w$ such that $s \in R[u]$ and $R[u] \subseteq R[v]$.

Proof. (1) Suppose $\preceq$ is symmetric. Let V be any valuation and suppose $w \in W$ does not satisfy p . Then for all $v \succeq w$ we have $v \preceq w$ by symmetry, so $v \not\models p$. This proves $w \models \neg p$. Therefore em is valid. For the converse, suppose the frame condition does not hold, so there exist v, w such that $w \preceq v$ and $v \not\preceq w$. Let V be a valuation such that $V(p) = \uparrow v$. Then $w \not\models p$ because $w \notin V(p)$ and $w \not\models \neg p$ because $w \preceq v \models p$, so em fails.

(2) Suppose the frame condition holds and let V be any valuation. If $w \models \top \rightarrow p$ then for all $v \succeq w$ we have $R[v] \subseteq p$. By assumption there exists such a v such that $w \in R[v]$, hence $w \models p$. Therefore $t_\Box$ is valid. Conversely, suppose $t_\Box$ is valid. Let w be any world. Let V be a valuation such that $V(p) = \bigcup \{R[v] \mid v \succeq w\}$. Then $w \models \top \rightarrow p$, hence $w \models p$, so we must have $w \in R[v]$ for some $v \succeq w$, as desired.

(3) Suppose $R \subseteq \preceq$, let V be any valuation, and $w \Vdash p \rightarrow q$. If $w \preceq v$ and $R[v] \subseteq V(p)$ then by assumption $w \preceq u$ for all $u \in R[v]$, hence $u \Vdash q$ for all such u , so that $R[v] \subseteq V(q)$. This proves $w \Vdash p \rightarrow q$, so **str** is valid. For the converse, suppose the frame condition does not hold. Then we can find $w, v \in W$ such that wRv while $w \not\preceq v$. Let V be a valuation such that $V(p) = R[w]$ and $V(q) = \uparrow w$. (Recall that $R[w]$ is upwards closed in upward-flat frames.) Then w trivially satisfies $p \rightarrow q$, but $w \not\models p \rightarrow q$ because all modal successors of w satisfy p , but not all of them satisfy q (namely v does not satisfy q).

(4) Suppose the frame condition holds, and let V be any valuation. Suppose $w \Vdash p \rightarrow q$. To show that $w \Vdash \top \rightarrow (p \rightarrow q)$, we need to prove that $w \preceq w'Rv$ implies $v \Vdash p \rightarrow q$. To this end, let $v' \succeq v$ and assume $R[v'] \subseteq V(p)$. Then because the frame is upward-flat we have $w'Rv'$. Now let $s \in R[v']$. Then by assumption there exists some $u \succeq w'$ such that $s \in R[u] \subseteq R[v']$. Since $w \Vdash p \rightarrow q$ and $R[u] \subseteq V(p)$ we find $s \Vdash q$. This entails that $R[v'] \subseteq V(q)$, so $v \Vdash p \rightarrow q$, as desired.

Conversely, if the frame condition does not hold then we can find w, v, s such that $wRvRs$ and for all $u \succeq w$ either $R[u] \not\subseteq R[v]$ or $s \notin R[u]$. Taking $V(p) = R[v]$ and $V(q) = R[v] \setminus \downarrow s$ then gives $w \Vdash p \rightarrow q$, because $R[u] \not\subseteq V(p)$ for all $u \succeq w$, while $v \not\models p \rightarrow q$, so $w \not\models \top \rightarrow (p \rightarrow q)$. $\square$

5.1 Reusing the full canonical model

We begin by focussing on **em**, **t_□**, **str** and **p_a**. Towards proving completeness and the finite model property for some extensions of HLC^b with these axioms, we give conditions on Σ that guarantee that the canonical frame $\mathfrak{F}_{\text{Ax}, \Sigma}$ satisfies the correspondence conditions derived in Lemma 5.2. To this end, we use the following definition of single negations: if φ is a formula then its *single negation* $\sim\varphi$ is defined as $\sim\varphi = \psi$ if $\varphi = \neg\psi$ for some $\psi \in \mathcal{L}_{\rightarrow}$, and $\sim\varphi = \neg\varphi$ otherwise. We say that a set Σ is closed under single negations if $\varphi \in \Sigma$ implies $\sim\varphi \in \Sigma$.

Lemma 5.3. *Let Ax be a set of axioms, $\Sigma \subseteq \mathcal{L}_{\rightarrow}$ a set of formulas that is closed under subformulas, and $\mathfrak{F}_{\text{Ax}, \Sigma} = (\text{SEG}_{\text{Ax}, \Sigma}, \preceq, \mathcal{R})$ the canonical frame generated by Ax and Σ .*

1. *If Σ is closed under single negations and $\text{em} \in \text{Ax}$, then $\preceq$ is symmetric.*
2. *If $\text{t}_{\square} \in \text{Ax}$ then for all $(\Gamma, U) \in \text{SEG}_{\text{Ax}, \Sigma}$ there exists (Δ, D) such that $(\Gamma, U) \preceq (\Delta, D)\mathcal{R}(\Gamma, U)$.*
3. *If $\text{str} \in \text{Ax}$ then $(\Gamma, U)\mathcal{R}(\Gamma', U')$ implies $(\Gamma, U) \preceq (\Gamma', U')$*
4. *If $\Sigma = \mathcal{L}_{\rightarrow}$ and $\text{p}_a \in \text{Ax}$ then $\mathfrak{F}_{\text{Ax}}$ satisfies the correspondence condition for p_a .*

Proof. (1) Suppose $(\Gamma, U) \preceq (\Gamma', U')$. Then $\varphi \in \Gamma'$ implies $\sim\varphi \notin \Gamma'$. Since $\Gamma \subseteq \Gamma'$ this gives $\sim\varphi \notin \Gamma$, hence $\varphi \in \Gamma$.

(2) We can take $(\Delta, D) = (\Gamma, U_{\Gamma, \top})$. Then $(\Gamma, U) \preceq (\Gamma, U_{\Gamma, \top})$, and it follows from **t_□** that $\Gamma \in U_{\Gamma, \top}$.

(3) Suppose $(\Gamma, U)\mathcal{R}(\Gamma', U')$. Then $\varphi \in \Gamma$ implies $\Gamma \vdash_{\text{Ax}} \top \rightarrow \varphi$, hence using **str** we get $\Gamma \vdash_{\text{Ax}} \top \rightarrow \varphi$. By definition of an (Ax, Σ) -segment, this implies that $\varphi \in \Delta$ for all $\Delta \in U$. It follows that $\Gamma \subseteq \Delta$ for all $\Delta \in U$. In particular, this implies $\Gamma \subseteq \Gamma'$, hence $(\Gamma, U) \preceq (\Gamma', U')$.

(4) Suppose $(\Gamma, U)\mathcal{R}(\Delta, D)$. Then $\varphi \rightarrow \psi \in \Gamma$ implies $\top \rightarrow (\varphi \rightarrow \psi) \in \Gamma$, so that $\varphi \rightarrow \psi \in \Delta$. It follows that (Γ, D) is a segment. This implies the correspondence condition, because for any s is the correspondence condition we can take $u = (\Gamma, D)$. $\square$

Theorem 5.4. *Let $\text{Ax} \subseteq \{\text{em}, \text{t}_{\square}, \text{str}, \text{p}_a\}$. Then $\text{HLC}^b \oplus \text{Ax}$ is sound and strongly complete with respect to the class of (upward-)flat frames on which they are valid.*

Proof. Combine Lemma 4.7(2) and Lemma 5.3. $\square$

Theorem 5.5. *Let $\text{Ax} \subseteq \{\text{em}, \text{t}_{\square}, \text{str}\}$. Then $\text{HLC}^b \oplus \text{Ax}$ has the finite model property.*

Proof. Use Lemma 4.7(1), taking Σ to be the closure under subformulas and under single negations of $\Gamma \cup \{\varphi\}$. This is finite when Γ is finite. Lemma 5.3 shows that $\mathfrak{F}_{Ax, \Sigma}$ validates the required axiom(s). $\square$

Remark 5.1 indicates that each of the axioms taken in separation and even several surprising combinations thereof (for example $\text{em} \oplus \text{p}_a$) are of independent interest. In the presence of str , however, certain careless combinations may degenerate. Still, such proofs of degeneracy may also illustrate convenience of our semantics.

Remark 5.6. We note that $\text{HLC}^b \oplus \text{em} \oplus \text{str}$ is rather degenerate, reducing not only to its own $\square$ -fragment, but in fact further still to the classical propositional calculus enriched with a single constant: One can show that $p \rightarrow q$ is equivalent to $(p \rightarrow q) \vee \square \perp$. While the algebraic proof is very simple, our semantics allows an even more perspicuous argument: In upward-flat frames for this system, $\preceq$ is an equivalence relation and $R \subseteq \preceq$. In those clusters where R is non-empty, $p \rightarrow q$ is the same as $p \rightarrow q$, and otherwise it reduces to $\top \rightarrow \perp$, which in such degenerate clusters is equivalent to $\top$ (and elsewhere to $\perp$).

Remark 5.7. In the logic $\text{HLC}^b \oplus \text{t}_\square \oplus \text{str}$ strict implication collapses to $\rightarrow$. To see this, note that str already gives $(p \rightarrow q) \rightarrow (p \rightarrow q)$. Combining the correspondence conditions for str and $\text{t}_\square$ gives: $R \subseteq \preceq$ and for every $w \in W$ there exists some w' in the same $\preceq$ -cluster (i.e. $w \preceq w' \preceq w$) such that $R[w'] = \uparrow_{\preceq} w$. Let us verify that this entails $(p \rightarrow q) \rightarrow (p \rightarrow q)$.

Let w be a world in an upward-flat model such that $w \Vdash p \rightarrow q$ and let $v \succeq w$ be a world that satisfies p . Then we can find some v' in the same cluster as v such that $R[v'] = \uparrow_{\preceq} v$. By assumption and intuitionistic heredity we then get $R[v'] \subseteq V(p)$, and since $w \preceq v'$ and $w \Vdash p \rightarrow q$ this implies $R[v'] \subseteq V(q)$. In particular, this gives $v \Vdash q$, so it follows that $w \Vdash p \rightarrow q$.

5.2 Modifying the full canonical model

We turn our attention to extensions of HLC^b with sets of axioms that include 4_a . Recall that on upward-flat frames, 4_a corresponds to transitivity of the modal accessibility relation. The following example illustrates that we cannot use the full canonical model construction from Section 4.

Example 5.8. Let $Ax = \{4_a\}$ and consider $\Sigma = \{\top, q\}$. Then we have two prime Σ -theories, $\{\top\}$ and $\{\top, q\}$. Let $\{\Gamma\} \cup U \subseteq \{\{\top\}, \{\top, q\}\}$ and suppose U is upwards closed under inclusion. In order for (Γ, U) to be an (Ax, Σ) -segment, we need to show that for all $\varphi, \psi \in \{\top, q\}$, if $\Gamma \vdash_{Ax} \varphi \rightarrow \psi$ and $\varphi \in \Delta$ for all $\Delta \in U$, then $\psi \in \Delta$ for all $\Delta \in U$. This gives four cases, $\top \rightarrow q$, $q \rightarrow \top$, $q \rightarrow q$ and $\top \rightarrow \top$. The desired condition is clearly satisfied for the latter three, and a simple countermodel shows that $\Gamma \not\vdash_{Ax} \top \rightarrow q$ for either choice of Γ . Therefore (Γ, U) is an (Ax, Σ) -segment for any choice of Γ and U .

In particular, this shows that for $\Gamma := \{\top\}$ and $\Delta := \{\top, q\}$ we have $(\Gamma, \{\Delta\}) \mathcal{R}(\Delta, \{\Gamma, \Delta\}) \mathcal{R}(\Gamma, \emptyset)$ while $(\Gamma, \emptyset)$ is not modally accessible from $(\Gamma, \{\Delta\})$. So the modal accessibility relation $\mathcal{R}$ of the full canonical frame $\mathfrak{F}_{Ax, \Sigma}$ is not transitive, hence $\mathfrak{F}_{Ax, \Sigma}$ does not validate 4_a .

In order to prove completeness for extensions of HLC^b with 4_a , we used a *trimmed* version of the canonical model construction from Section 4. This is obtained by restricting the set $\text{SEG}_{Ax, \Sigma}$.

Definition 5.9. Let Ax be a consistent set of axioms and Σ a set of formulas that is closed under subformulas and contains $\top$. We call an (Ax, Σ) -segment (Γ, U) *pointed* if there exists a formula $\gamma \in \Sigma$ such that

$$U = U_{\Gamma, \gamma} := \{\Delta \in \text{Th}_{Ax, \Sigma} \mid \text{if } \psi \in \Sigma \text{ and } \Gamma \vdash_{Ax} \gamma \rightarrow \psi \text{ then } \psi \in \Delta\}.$$

By Lemma 4.5, every prime (Ax, Σ) -theory can be extended to a pointed (Ax, Σ) -segment.

Write $\text{SEG}_{Ax, \Sigma}^P$ for the set of pointed (Ax, Σ) -segments, and $\mathfrak{F}_{Ax, \Sigma}^P := (\text{SEG}_{Ax, \Sigma}^P, \subseteq, \mathcal{R})$ and $\mathfrak{M}_{Ax, \Sigma}^P := (\mathfrak{F}_{Ax, \Sigma}^P, V_{Ax, \Sigma})$ for the *pointed canonical frame* and *model*. If $\Sigma = \mathcal{L}_{\rightarrow}$ we abbreviate $\mathfrak{F}_{Ax}^P := \mathfrak{F}_{Ax, \mathcal{L}_{\rightarrow}}^P$.

Using precisely the same proof as Lemma 4.6, we get

Lemma 5.10. *Let $Ax \subseteq \mathcal{L}_{\neg}$ be a set of axioms and $\Sigma \subseteq \mathcal{L}_{\neg}$ a set of formulas that contains $\top$ and is closed under subformulas. Then for all $\varphi \in \Sigma$ and $(\Gamma, U) \in \text{SEG}_{Ax, \Sigma}$ we have $\mathfrak{M}_{Ax, \Sigma}, (\Gamma, U) \Vdash \varphi$ iff $\varphi \in \Gamma$.*

Theorem 5.11. *Let $Ax \subseteq \{\text{em}, t_{\Box}, \text{str}, 4_a\}$. Then $\text{HLC}^b \oplus Ax$ is sound and strongly complete with respect to the class of (upward-)flat frames on which Ax is valid.*

Proof. It suffices to show that $\mathfrak{F}_{Ax}^P$ validates each of the axioms in Ax . Using the same proof as in Lemma 5.3 shows that if $\text{em}, t_{\Box}$ or str is in Ax , then $\mathfrak{F}_{Ax}^P$ validates it, so we are left to consider 4_a . Suppose $4_a \in Ax$. We need to show that $\mathcal{R}$ is transitive. To this end, let $(\Gamma, U_{\Gamma, \gamma})\mathcal{R}(\Delta, U_{\Delta, \delta})\mathcal{R}(\Pi, U_{\Pi, \pi})$ in $\mathfrak{F}_{Ax}^P$. Suppose $\gamma \rightarrow \psi \in \Gamma$. By 4_a we also have $\psi \rightarrow (\top \rightarrow \psi) \in \Gamma$, so tr gives $\gamma \rightarrow (\top \rightarrow \psi) \in \Gamma$. This entails $\top \rightarrow \psi \in \Delta$, which by the definition of a segment gives $\psi \in \Pi$. This proves that $\Pi \in U_{\Gamma, \gamma}$, so that $(\Gamma, U_{\Gamma, \gamma})\mathcal{R}(\Pi, U_{\Pi, \pi})$, as desired. $\square$

Finally, using the same kind of canonical model we derive the finite model property for the logic $\text{HLC}^b \oplus t_{\Box} \oplus 4_a$. The key insight towards this is that $\top \rightarrow \varphi$ is equivalent to $\top \rightarrow (\top \rightarrow \varphi)$ in this logic, so that it suffices to close Σ under “single boxes.”

Lemma 5.12. *We have $t_{\Box, 4_a}(\top \rightarrow \varphi) \leftrightarrow (\top \rightarrow (\top \rightarrow \varphi))$.*

Proof. As a substitution instance of $t_{\Box}$ we get $t_{\Box, 4_a}(\top \rightarrow (\top \rightarrow \varphi)) \rightarrow (\top \rightarrow \varphi)$. Conversely, combining $\top \rightarrow \varphi$ with 4_a and tr yields $\top \rightarrow (\top \rightarrow \varphi)$. $\square$

Definition 5.13. For $\varphi \in \mathcal{L}_{\neg}$ we define

$$\boxdot \varphi := \begin{cases} \varphi & \text{if } \varphi = \top \rightarrow \psi \text{ for some } \psi \in \mathcal{L}_{\neg} \\ \top \rightarrow \varphi & \text{otherwise} \end{cases}$$

A set $\Sigma \subseteq \mathcal{L}_{\neg}$ is said to be *closed under single boxes* if $\varphi \in \Sigma$ implies $\boxdot \varphi \in \Sigma$.

Closing a finite set Σ under single boxes at most doubles its size, hence it stays finite. This allows us to construct a finite model with a transitive modal relation.

Theorem 5.14. *The logic $\text{HLC}^b \oplus t_{\Box} \oplus 4_a$ has the finite model property.*

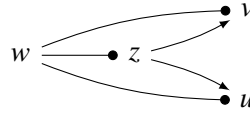
Proof. Let $\Gamma \Rightarrow \varphi$ be a finite consecution such that $\Gamma \not\vdash_{t_{\Box}, 4_a} \varphi$. Let Σ be the set of subformulas of $\Gamma \cup \{\top, \varphi\}$ closed under single boxes. Then Σ is finite, and we can use Lemmas 4.2 and 4.3 to extend Γ to a prime (Ax, Σ) -theory Γ' containing Γ but not φ . Lemma 4.5 then yields an (Ax, Σ) -segment $(\Gamma', U_{\Gamma', \top})$ which by Lemma 5.10, under the canonical valuation, invalidates $\Gamma \Rightarrow \varphi$. Therefore $\mathfrak{F}_{Ax, \Sigma}^P = (\text{SEG}_{Ax, \Sigma}^P, \subseteq, \mathcal{R})$ invalidates $\Gamma \Rightarrow \varphi$. To establish the finite model property, we now argue that $\mathfrak{F}_{Ax, \Sigma}$ validates $t_{\Box}$ and 4_a .

Using the same proof as Lemma 5.3(2) shows that $\mathfrak{F}$ validates $t_{\Box}$. For 4_a , let $(\Gamma, U_{\Gamma, \gamma}), (\Delta, U_{\Delta, \delta})$ and $(\Pi, U_{\Pi, \pi})$ be three (Ax, Σ) -segments and suppose $(\Gamma, U_{\Gamma, \gamma})\mathcal{R}(\Delta, U_{\Delta, \delta})\mathcal{R}(\Pi, U_{\Pi, \pi})$. Let $\gamma, \psi \in \Sigma$ and suppose $\Gamma \vdash_{Ax} \gamma \rightarrow \psi$. By assumption we have $\Gamma \vdash_{Ax} \psi \rightarrow (\top \rightarrow \psi)$, hence by tr we find $\Gamma \vdash_{Ax} \gamma \rightarrow (\top \rightarrow \psi)$. This entails $\Gamma \vdash_{Ax} \gamma \rightarrow \boxdot \psi$, and since $\psi \in \Sigma$ we have $\boxdot \psi \in \Sigma$. Therefore we must have $\boxdot \psi \in \Delta$, hence $\Delta \vdash_{Ax} \top \rightarrow \psi$. Finally, the definition of a segment and the fact that $(\Delta, U_{\Delta, \delta})\mathcal{R}(\Pi, U_{\Pi, \pi})$ entails $\psi \in \Pi$. Thus, we have shown that for any $\psi \in \Sigma$, $\Gamma \vdash_{Ax} \gamma \rightarrow \psi$ implies $\psi \in \Pi$, so that $\Pi \in U_{\Gamma, \gamma}$ hence $(\Gamma, U_{\Gamma, \gamma})\mathcal{R}(\Pi, U_{\Pi, \pi})$. Therefore $\mathcal{R}$ is transitive, so $\mathfrak{F}_{Ax, \Sigma}^P \Vdash 4_a$. $\square$

6 Open subframes and extension stability

Litak and Visser [42] note a direct connection between the syntactic notion of *extension stability*, motivated by arithmetical interpretations of $\neg$, and a special type of nuclei on flat algebras, more specifically *open nuclei* [19, 43]. Recall that nuclei provide an algebraic perspective on *subframes* in modal logic [18, 61, 9]. In particular, quotienting an algebra by an open nucleus generated by a chosen element a produces an algebra (isomorphic to one) whose Heyting reduct is (isomorphic to) the ideal of elements below a , with suitably restricted $\neg$. In the classical setting with a unary box, applying this construction to dual algebras of Kripke frames produces the dual algebra of the (not necessarily modally generated!) subframe induced by a ; that is, a Kripke frame whose carrier and modal accessibility relation are restricted to a . In the Heyting setting, the fact that a is an element of the upset algebra means that the carrier set of the corresponding subframe is $\preceq$ -generated, i.e. an upset. When it comes to R , Proposition 3.10 indicates a certain subtlety: unlike the classical case, the dual algebras of our frames might fail to notice the presence/absence of certain R -edges. Let us reconsider the example of 4_a from Proposition 3.11: the corresponding class of arbitrary flat frames does not appear closed with respect to the open subframe construction. However, over upward-flat frames, the situation changes: transitivity is well-known to be persistent with respect to subframes. Together with difficulties in presenting duality for flat subframes noted above (Remarks 3.8 and 4.10), this means that some care is needed. Given the space constraints of the present paper, we do not attempt a full discussion here. Nevertheless, it is illustrative to provide a semantic discussion of the failure of extension stability for $\text{HLC}^\#$.

Example 6.1. Consider the flat model $(W, \preceq, R)$ where $W = \{w, v, u, z\}$, the intuitionistic accessibility relation $\preceq$ is the reflexive closure of the four worlds together with $z \preceq v$ and $z \preceq u$, and R is given by wRv , wRu and wRz :



This frame is clearly upward-flat. Moreover, it satisfies the sufficient condition of Lemma 3.12 to validate di . However, the open subframe obtained by removing z is precisely the one used in Example 3.2 to illustrate the failure of di .

In order to turn this counterexample into a formal proof, let us recall the syntactic characterisation of extension stability [42]. Given a formula φ and a fresh propositional variable p , define the translation $\varphi^{[p]}$ inductively as commuting with the propositional variables and the connectives of IPC, with the $\neg$ clause being

$$\bullet (\psi \neg \chi)^{[p]} := ((p \rightarrow \psi^{[p]}) \neg (p \rightarrow \chi^{[p]})).$$

As $\Box \varphi$ is $\top \neg \varphi$, we get $\text{HLC}^b \vdash (\Box \varphi)^{[p]}$ iff $\text{HLC}^b \vdash \Box(p \rightarrow \varphi^{[p]})$. Note that for any logic Λ and any φ , if $\Lambda \vdash \varphi^{[p]}$, then $\Lambda \vdash \varphi$. A logic Λ is *extension stable* if, whenever $\Lambda \vdash \varphi$ and p does not appear in φ , we have $\Lambda \vdash p \rightarrow \varphi^{[p]}$.

Theorem 6.2. *The frame from Example 6.1 refutes $s \rightarrow \text{di}^{[s]}$, i.e.*

$$s \rightarrow (((s \rightarrow p) \neg (s \rightarrow r)) \wedge ((s \rightarrow q) \neg (s \rightarrow r)) \rightarrow ((s \rightarrow (p \vee q)) \neg (s \rightarrow r))).$$

Thus, $\text{HLC}^\#$ is not extension stable, and neither is any of its extensions validated by this frame.

Proof. Define $V(s)$ to be the complement of z and follow Example 3.2 for other atoms, i.e., $V(p) = \{v\}$, $V(q) = \{u\}$ and $V(r) = \emptyset$. One can then follow the reasoning from Example 3.2, with s in the antecedent used to relativize reasoning to the three-state open subframe. $\square$

For contrast, consider 4_a . One can easily see that $4_a^{[s]}$ is equivalent to a substitution instance of 4_a itself, and hence $s \rightarrow 4_a^{[s]}$ is a theorem of $HLC^b \oplus 4_a$. This shows that the closure of the corresponding upward-flat frames under open subframes is more important than the apparent failure of such closure in the broader class. In other words, narrowing down the class of frames might be essential for giving an appropriate duality account.

7 Conclusions and future work

We believe we have demonstrated the potential of the flat semantics for HLC^b . Future work needs to include general completeness and finite model property results (potentially also in the context of classical subsystems of various interpretability logics), a more systematic treatment of duality, and the open subframe construction, possibly generalising the subframe completeness result of Fine [18]. A tantalising perspective is to use the present semantics to study combinations of intuitionistic $\neg 3$ with $\Diamond$, especially on frames failing upward-flatness.

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Intuitionistic K is a Bisimulation-Invariant Fragment of Intuitionistic First-Order Logic

Jim de Groot*

University of Bern
Bern, Switzerland

jim.degroot@unibe.ch

João Marcos†

Universidade Federal de Santa Catarina
Florianópolis, Brazil

botocudo@gmail.com

Rodrigo Stefanés‡

Universidade Federal de Santa Catarina
Florianópolis, Brazil

rodrigoamstefanes@gmail.com

We define the notion of IK-bisimulation between the relational semantics for the intuitionistic modal logic IK, and prove that IK arises as the IK-bisimulation-invariant fragment of intuitionistic first-order logic. En route, we provide an intrinsic characterisation result of this logic by way of a Hennessy–Milner-style theorem and develop some intuitionistic first-order model theory, including intuitionistic analogues of Łoś’s Theorem, elementary embeddings and countable saturation.

1 Introduction

Bisimulations are an important tool in the study of modal logic and computer science. In computer science, they can be used as an equivalence relation between process graphs [23, 25]. In modal logic they provide a structural notion of equivalence: worlds linked through a bisimulation satisfy the same formulas. The converse result, also known as Hennessy–Milner property [18], implies that the language is powerful enough to track down structural differences between the linked worlds. Moreover, Van Benthem’s theorem, originally proved in [29], provides a relative characterisation theorem that states that normal modal logic is precisely the bisimulation-invariant fragment of classical first-order logic.

Similar results have been attained for several other logics, each having an appropriate notion of bisimulation encapsulating the underlying structure of the logic. These include modal logics without negation [20], logics with negative and restorative modalities [14], monotone modal logic [16], neighbourhood logics [17], (bi-)intuitionistic logic [2, 15, 24, 26], modal μ -calculi (within monadic second order logics) [8, 19] and fragments of XPath [1, 10, 5].

In this paper we prove a characterisation theorem for the intuitionistic modal logic IK. This logic was introduced by Fischer Servi [11], and was also studied by Plotkin and Stirling [27], Ewald [9] and Simpson [28]. It is one of the myriad intuitionistic counterparts of the classical normal modal logic K (see [28] for an overview), and can be viewed as the collection of modal formulas whose standard translations are provable in intuitionistic first-order logic. While existing analogues of the Van Benthem characterisation theorem for non-classical logics, such as bi-intuitionistic logic and modal extensions of positive logic, use a *classical* first-order logic [20, 2, 14], our aim is to establish IK as a bisimulation-invariant fragment of *intuitionistic* first-order logic.

We follow a standard path to obtain the characterisation theorem. This requires an extension of the intuitionistic ultraproduct construction and of Łoś’s Theorem from [13, 21] to allow for constants. Moreover, we introduce intuitionistic analogues of elementary embeddings and ω -saturation, and accompanying theorems. On the modal side, we use the birelational models of IK to provide an intuitive

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notion of IK-bisimulations and prove a Hennessy–Milner-style theorem. The embedding of intuitionistic first-order structures into birelational models then allows us to transfer this to the intuitionistic first-order setting.

Structure of the paper Sections 2 and 3 review basic notions and semantics for intuitionistic first-order logic and for IK. Section 4 defines IK-bisimulations and modal saturation, and establishes a Hennessy–Milner-style theorem. In Section 5 we define (ultra)filter products of intuitionistic first-order structures and give an intuitionistic analogue of Łoś’s Theorem, and in Section 6 we provide an intuitionistic analogue of ω -saturation. Finally, Section 7 proves that intuitionistic modal logic IK is the IK-bisimulation-invariant fragment of intuitionistic first-order logic.

2 (Intuitionistic) first-order logic

We recall some first-order logic-related material. We fix throughout the text, unless otherwise stated, an arbitrary first-order signature $(\text{Cnst}, \text{Pred})$, without function symbols, where Cnst and Pred are disjoint sets containing, respectively, constant symbols and predicate symbols, the latter with respective arities. We also fix a denumerable set Var , disjoint from $\text{Cnst} \cup \text{Pred}$, of variables.

Definition 2.1. The *language* $\mathcal{L}$ (for the signature $(\text{Cnst}, \text{Pred})$) is defined by the grammar

$$\varphi ::= P(t_1, \dots, t_n) \mid \perp \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid \forall x \varphi \mid \exists x \varphi$$

where P , with arity n , ranges over Pred , and where $t_1, \dots, t_n$ are *terms*, i.e. elements of $\text{Var} \cup \text{Cnst}$, and $x \in \text{Var}$.

Definition 2.2. A $\mathcal{CL}$ -*structure* (for the signature $(\text{Cnst}, \text{Pred})$) is a pair $\mathfrak{C} = (D, \mathcal{I})$ consisting of a nonempty set D and an interpretation $\mathcal{I}$ that assigns to each $c \in \text{Cnst}$ an individual $\mathcal{I}(c) \in D$, and to each n -ary $P \in \text{Pred}$ a relation $\mathcal{I}(P) \subseteq D^n$.

Definition 2.3. An $\mathcal{JL}$ -*structure* (for the signature $(\text{Cnst}, \text{Pred})$) is a tuple $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ consisting of a poset $(W, \leq)$ of *worlds* and for each $w \in W$ a $\mathcal{CL}$ -structure $\mathfrak{C}_w = (D_w, \mathcal{I}_w)$, for some $\text{Cnst}_w \subseteq \text{Cnst}$, such that

1. if $w \leq w'$, then $D_w \subseteq D_{w'}$ and $\mathcal{I}_w(P) \subseteq \mathcal{I}_{w'}(P)$ for any predicate symbol P ;
2. for each constant symbol $c \in \text{Cnst}$ there is an individual $c^{\mathfrak{M}} \in \bigcup_{w \in W} D_w$ such that, for each $w \in W$:
 - if $c^{\mathfrak{M}} \in D_w$, then $c \in \text{Cnst}_w$ and $\mathcal{I}_w(c) = c^{\mathfrak{M}}$;
 - if $c^{\mathfrak{M}} \notin D_w$, then $c \notin \text{Cnst}_w$ and $\mathcal{I}_w(c)$ is not defined.

We write $D_{\mathfrak{M}} := \bigcup \{D_w \mid w \in W\}$ for the *domain* of $\mathfrak{M}$.

An *assignment* for an $\mathcal{JL}$ -structure $\mathfrak{M}$ is a mapping $\rho : \text{Var} \rightarrow D_{\mathfrak{M}}$. Given $d \in D_{\mathfrak{M}}$ and $x \in \text{Var}$, we write $\rho[x := d]$ for the assignment that gives the value d to x and agrees with ρ on all other variables. An assignment $\rho : \text{Var} \rightarrow D_{\mathfrak{M}}$ can naturally be extended to a mapping $\rho^{\mathfrak{M}} : \text{Var} \cup \text{Cnst} \rightarrow D_{\mathfrak{M}}$ such that $\rho^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$ for every $c \in \text{Cnst}$.

We interpret formulas from the set $\mathcal{L}$ in $\mathcal{IL}$ -structures under an assignment ρ via:

$$\begin{aligned}
\mathfrak{M}, w \Vdash^\rho P(t_1, \dots, t_n) & \text{ iff } (\rho^{\mathfrak{M}}(t_1), \dots, \rho^{\mathfrak{M}}(t_n)) \in \mathcal{I}_w(P) \\
\mathfrak{M}, w \Vdash^\rho \perp & \text{ never} \\
\mathfrak{M}, w \Vdash^\rho \varphi \wedge \psi & \text{ iff } \mathfrak{M}, w \Vdash^\rho \varphi \text{ and } \mathfrak{M}, w \Vdash^\rho \psi \\
\mathfrak{M}, w \Vdash^\rho \varphi \vee \psi & \text{ iff } \mathfrak{M}, w \Vdash^\rho \varphi \text{ or } \mathfrak{M}, w \Vdash^\rho \psi \\
\mathfrak{M}, w \Vdash^\rho \varphi \rightarrow \psi & \text{ iff for all } w' \in W, \text{ if } w \leq w' \text{ and } \mathfrak{M}, w' \Vdash^\rho \varphi, \text{ then } \mathfrak{M}, w' \Vdash^\rho \psi \\
\mathfrak{M}, w \Vdash^\rho \forall x \varphi & \text{ iff for all } w' \in W, \text{ if } w \leq w' \text{ and } d \in D_{w'}, \text{ then } \mathfrak{M}, w' \Vdash^{\rho[x:=d]} \varphi \\
\mathfrak{M}, w \Vdash^\rho \exists x \varphi & \text{ iff there exists a } d \in D_w \text{ such that } \mathfrak{M}, w \Vdash^{\rho[x:=d]} \varphi
\end{aligned}$$

We say that a world w *satisfies* a formula $\varphi \in \mathcal{L}$ under the assignment ρ if $\mathfrak{M}, w \Vdash^\rho \varphi$. In case $\mathfrak{M}, w \not\Vdash^\rho \varphi$, we say that w *refutes* the formula φ under the assignment ρ .

If φ has no free variables then its interpretation does not depend on ρ , and we sometimes omit reference to it. If φ has one free variable x , then satisfaction of φ depends only on the action of ρ on x , and we write $\mathfrak{M}, w \Vdash^{[x:=d]} \varphi$ to mean that $\mathfrak{M}, w \Vdash^\rho \varphi$ for some arbitrary assignment ρ that maps x to d .

Note that if $\rho^{\mathfrak{M}}(t_j) \notin D_w$, then $P(t_1, \dots, t_n)$ is not satisfied at w . This agrees, as a matter of fact, with the treatment of atomic formulas provided in [12, Chapter 8]. As usual, we have persistence:

Lemma 2.4. *If $w \leq w'$ and $\mathfrak{M}, w \Vdash^\rho \varphi$, then $\mathfrak{M}, w' \Vdash^\rho \varphi$.*

Definition 2.5. A *consecution* is a pair $\langle \Gamma : \Delta \rangle$, where $\Gamma, \Delta \subseteq \mathcal{L}$. A (finite) *subconsecution* of $\langle \Gamma : \Delta \rangle$ is a consecution $\langle \Gamma' : \Delta' \rangle$ such that Γ' and Δ' are (finite) subsets of Γ and Δ , respectively. We write $\langle \Gamma : \Delta \rangle(x_1, \dots, x_n)$ to express that the set of free variables occurring in $\Gamma \cup \Delta$ is a subset of $x_1, \dots, x_n$.

Definition 2.6. Let $\mathfrak{M} = (W, \leq, \{\mathcal{C}_w\}_{w \in W})$ be an $\mathcal{IL}$ -structure. A world $w \in W$ and an assignment ρ for $\mathfrak{M}$ *separate* the consecution $\langle \Gamma : \Delta \rangle$, denoted by $\mathfrak{M}, w \Vdash^\rho \langle \Gamma : \Delta \rangle$, if, under the assignment ρ , w satisfies all formulas of Γ and refutes all formulas of Δ . We say that the consecution $\langle \Gamma : \Delta \rangle$ is *separable* in a set $U \subseteq W$ if there exist $w \in U$ and an assignment ρ such that $\mathfrak{M}, w \Vdash^\rho \langle \Gamma : \Delta \rangle$. We call $\langle \Gamma : \Delta \rangle$ *finitely separable* in a set $U \subseteq W$ if every finite subconsecution of $\langle \Gamma : \Delta \rangle$ is separable in U . A set $\Omega \subseteq \mathcal{L}$ is *satisfied* at w under the assignment ρ (notation: $\mathfrak{M}, w \Vdash^\rho \Omega$), if $\mathfrak{M}, w \Vdash^\rho \langle \Omega : \emptyset \rangle$, and Ω is called (finitely) *satisfiable* in a set U if $\langle \Omega : \emptyset \rangle$ is (finitely) separable in U . Finally, we say that Ω is *refuted* at w under ρ if $\mathfrak{M}, w \Vdash^\rho \langle \emptyset : \Omega \rangle$.

3 Intuitionistic modal logic

We recall the intuitionistic modal logic IK and its connection to intuitionistic first-order logic via the standard translation. Fix a denumerable set Prop of propositional letters. Throughout this section, we take $\text{Cnst} = \emptyset$ and $\text{Pred} = \text{Prop} \cup \{R\}$, where each symbol in Prop is taken as unary and R is a binary predicate symbol. The language $\mathcal{L}$ and the $\mathcal{IL}$ -structures in this section have $(\text{Cnst}, \text{Pred})$ as their signature.

Definition 3.1. The *modal language* $\mathcal{L}_{\Box\Diamond}$ is generated by the grammar

$$\varphi ::= P \mid \perp \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid \Box\varphi \mid \Diamond\varphi \quad (\text{where } P \in \text{Prop})$$

Definition 3.2. The *standard translation* $\text{Var} \times \mathcal{L}_{\Box\Diamond} \rightarrow \mathcal{L}$ is recursively defined by

$$\begin{aligned}
\text{st}(x, P) &= P(x) & \text{st}(x, \perp) &= \perp & \text{st}(x, \varphi \star \psi) &= \text{st}(x, \varphi) \star \text{st}(x, \psi) & (\star \in \{\wedge, \vee, \rightarrow\}) \\
\text{st}(x, \Box\varphi) &= \forall y (R(x, y) \rightarrow \text{st}(y, \varphi)) & \text{st}(x, \Diamond\varphi) &= \exists y (R(x, y) \wedge \text{st}(y, \varphi))
\end{aligned}$$

where y is a fresh variable. For $\Phi \subseteq \mathcal{L}_{\Box\Diamond}$ we define $\text{st}(x, \Phi) := \{\text{st}(x, \varphi) \mid \varphi \in \Phi\}$.

The logic IK can then be defined as a set of theorems by

$$\text{IK} = \{\varphi \in \mathcal{L}_{\Box\Diamond} \mid \mathfrak{M}, w \Vdash^\rho \text{st}(x, \varphi) \text{ for every } \mathcal{IL}\text{-structure } \mathfrak{M}, \text{ world } w \text{ and assignment } \rho\}.$$

Then we can use $\mathcal{IL}$ -structures $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ to interpret formulas $\varphi \in \mathcal{L}_{\Box\Diamond}$ at a world $w \in W$ and an individual $d \in D_w$ via

$$\mathfrak{M}, w, d \Vdash \varphi \quad \text{iff} \quad \mathfrak{M}, w \Vdash^{[x:=d]} \text{st}(x, \varphi).$$

An alternative adequate semantics for IK is given by birelational models [28, Section 8.1]:

Definition 3.3. A *birelational model* is a tuple $\mathfrak{B} = (W, \leq, R, V)$ consisting of a poset $(W, \leq)$, a valuation V that maps propositional letters to upsets of $(W, \leq)$, and a binary relation R on W such that:

- if $v \geq wRu$, then there exists a $t \in W$ such that $vRt \geq u$, for all $u, v, w \in W$; and
- if $wRu \leq v$, then there exists a $t \in W$ such that $w \leq tRv$, for all $u, v, w \in W$.

The interpretation of formulas in $\mathcal{L}_{\Box\Diamond}$ is recursively defined by

$$\begin{aligned} \mathfrak{B}, w \Vdash P & \quad \text{iff} \quad w \in V(P) \\ \mathfrak{B}, w \Vdash \perp & \quad \text{never} \\ \mathfrak{B}, w \Vdash \varphi \wedge \psi & \quad \text{iff} \quad \mathfrak{B}, w \Vdash \varphi \text{ and } \mathfrak{B}, w \Vdash \psi \\ \mathfrak{B}, w \Vdash \varphi \vee \psi & \quad \text{iff} \quad \mathfrak{B}, w \Vdash \varphi \text{ or } \mathfrak{B}, w \Vdash \psi \\ \mathfrak{B}, w \Vdash \varphi \rightarrow \psi & \quad \text{iff} \quad \text{for every } v \in W, \text{ if } w \leq v \text{ and } \mathfrak{B}, v \Vdash \varphi, \text{ then } \mathfrak{B}, v \Vdash \psi \\ \mathfrak{B}, w \Vdash \Box\varphi & \quad \text{iff} \quad \text{for every } v, u \in W, \text{ if } w \leq v \text{ and } vRu, \text{ then } \mathfrak{B}, u \Vdash \varphi \\ \mathfrak{B}, w \Vdash \Diamond\varphi & \quad \text{iff} \quad \text{there exists a } v \in W \text{ such that } wRv \text{ and } \mathfrak{B}, v \Vdash \varphi \end{aligned}$$

Worlds w_1 and w_2 are called *modally equivalent* (notation: $w_1 \leftrightarrow w_2$) if they satisfy the same formulas from $\mathcal{L}_{\Box\Diamond}$. We define separability, satisfaction and refutation as in Definition 2.6.

Birelational semantics enjoy the expected persistence property [28, Lemma 3.3.1]:

Lemma 3.4. Let $\mathfrak{B} = (W, \leq, R, V)$ be a birelational model. If $w, v \in W$ and $w \leq v$, then $\mathfrak{B}, w \Vdash \varphi$ implies $\mathfrak{B}, v \Vdash \varphi$, for all $\varphi \in \mathcal{L}_{\Box\Diamond}$.

Every $\mathcal{IL}$ -structure gives rise to a birelational model in a truth-preserving way as follows (following Section 8.1.1 of [28]).

Definition 3.5. Let $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ be an $\mathcal{IL}$ -structure. The *induced birelational model* $\mathfrak{B}_{\mathfrak{M}}$ is given by $(W^\bullet, \leq^\bullet, R^\bullet, V^\bullet)$, where:

$$\begin{aligned} W^\bullet &= \{(w, d) \mid w \in W \text{ and } d \in D_w\} & (w, d) \leq^\bullet (w', d') & \text{iff} \quad w \leq w' \text{ and } d = d' \\ V^\bullet(P) &= \{(w, d) \mid d \in \mathcal{I}_w(P)\} & (w, d) R^\bullet (w', d') & \text{iff} \quad w = w' \text{ and } (d, d') \in \mathcal{I}_w(R) \end{aligned}$$

Lemma 3.6. For any $\mathcal{IL}$ -structure $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$, $w \in W$, $d \in D_w$ and $\varphi \in \mathcal{L}_{\Box\Diamond}$ we have

$$\mathfrak{M}, w, d \Vdash \varphi \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{M}}, (w, d) \Vdash \varphi.$$

4 IK-bisimulations

We study IK-bisimulations between birelational frames. These are slightly weaker than the usual definition of a (Kripke) bisimulation from modal logic [4, Section 2.2] because, as we shall see in Example 4.3, the usual definition is too strong to prove that bisimilarity and modal equivalence coincide even on finite models. Throughout this section we use the same first-order signature as in Section 3. Given binary relations S and R , we write $S ; R := \{(x, y) \mid xSz \text{ and } zRy \text{ for some } z\}$ for their sequential composition.

Definition 4.1. Let $\mathfrak{B} = (W, \leq, R, V)$ and $\mathfrak{B}' = (W', \leq', R', V')$ be two birelational models. An *IK-bisimulation* between $\mathfrak{B}$ and $\mathfrak{B}'$ is a relation $Z \subseteq W \times W'$ such that for all $(w, w') \in Z$:

(Z_p) $w \in V(p)$ if and only if $w' \in V'(p)$, for all $p \in \text{Prop}$;

$(Z_{\leq}^1)$ if $w \leq v$, then there exists a $v' \in W'$ such that $(v, v') \in Z$ and $w' \leq' v'$;

$(Z_{\leq}^2)$ if $w' \leq' v'$, then there exists a $v \in W$ such that $(v, v') \in Z$ and $w \leq v$;

$(Z_{\Box}^1)$ if wRv , then there exists a $v' \in W'$ such that $(v, v') \in Z$ and $w'(\leq' ; R')v'$;

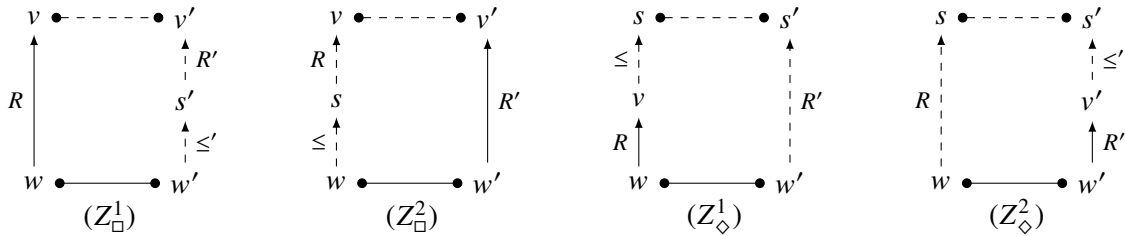
$(Z_{\Box}^2)$ if $w'R'v'$, then there exists a $v \in W$ such that $(v, v') \in Z$ and $w(\leq ; R)v$;

$(Z_{\Diamond}^1)$ if wRv , then there exist $s \in W$ and $s' \in W'$ such that $v \leq s$ and $w'R's'$ and $(s, s') \in Z$;

$(Z_{\Diamond}^2)$ if $w'R'v'$, then there exist $s' \in W'$ and $s \in W$ such that $v' \leq' s'$ and wRs and $(s, s') \in Z$.

We write $w \rightleftharpoons w'$, and call w and w' *bisimilar*, if there exists an IK-bisimulation that links w and w' .

The last four conditions can be depicted as follows, where solid and dashed arrows indicate universal and existential quantification, respectively:



Note that bisimilarity implies model equivalence, that is:

Proposition 4.2. Let $\mathfrak{B} = (W, \leq, R, V)$ and $\mathfrak{B}' = (W', \leq', R', V')$ be two birelational models. Then $w \rightleftharpoons w'$ implies $w \leftrightarrow w'$, for all $w \in W$ and $w' \in W'$.

Proof. We prove something stronger, namely that for every formula φ and for any pair of worlds w, w' such that $w \rightleftharpoons w'$ we have $\mathfrak{B}, w \Vdash \varphi$ iff $\mathfrak{B}', w' \Vdash \varphi$. This may be checked by structural induction on φ . We showcase the inductive steps for $\Diamond$ and $\Box$. Assume, then, for a certain formula ψ , the following induction hypothesis, $P[\psi]$: $\mathfrak{B}, w \Vdash \psi$ iff $\mathfrak{B}', w' \Vdash \psi$, for any pair of worlds w, w' such that $w \rightleftharpoons w'$. Accordingly, in what follows, take arbitrary worlds w, w' such that $w \rightleftharpoons w'$.

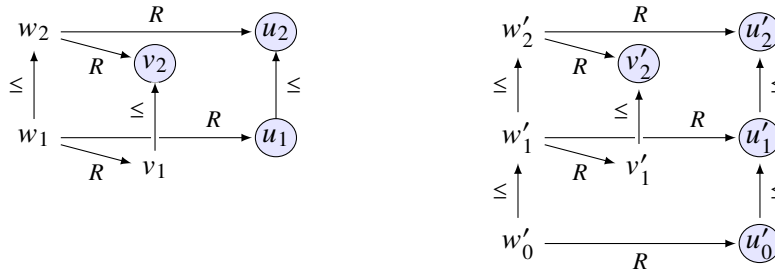
► *Case for $\varphi = \Diamond\psi$.* Suppose $\mathfrak{B}, w \Vdash \varphi$. In view of the assumption that $w \rightleftharpoons w'$, let $Z \subseteq W \times W'$ be an IK bisimulation between $\mathfrak{B}$ and $\mathfrak{B}'$ such that $(w, w') \in Z$. From $\mathfrak{B}, w \Vdash \Diamond\psi$ we may obtain a world $v \in W$ such that wRv and $\mathfrak{B}, v \Vdash \psi$. Using $(Z_{\Diamond}^1)$ we obtain $s \in W$ and $s' \in W'$ such that $v \leq s$, $w'R's'$ and $(s, s') \in Z$. The induction hypothesis and Lemma 3.4 then imply $\mathfrak{B}', s' \Vdash \psi$, hence $\mathfrak{B}', w' \Vdash \Diamond\psi$. The converse (namely, if $\mathfrak{B}', w' \Vdash \Diamond\psi$ then $\mathfrak{B}, w \Vdash \Diamond\psi$) can be proven similarly.

► *Case for $\varphi = \Box\psi$.* Suppose $\mathfrak{B}, w \Vdash \varphi$. Let $Z \subseteq W \times W'$ be an IK bisimulation between $\mathfrak{B}$ and $\mathfrak{B}'$ such that $(w, w') \in Z$. Let $t', v' \in W'$ be such that $w' \leq' t' R' v'$. Then we can use $(Z_{\leq}^2)$ and $(Z_{\Box}^2)$ to find $t, s, v \in W$ such that $w \leq t \leq s R v$ and $(v, v') \in Z$. This implies $w \leq s$, so by Lemma 3.4 we have $\mathfrak{B}, s \Vdash \Box\psi$. Hence $\mathfrak{B}, v \Vdash \psi$, and by the induction hypothesis $\mathfrak{B}', v' \Vdash \psi$. This entails that $\mathfrak{B}', w' \Vdash \Box\psi$. The converse is proven similarly.

This concludes the inductive proof of $P[\varphi]$, for every φ .

Take now worlds w, w' such that $w \rightleftharpoons w'$. Given an arbitrary formula φ , we may use $P[\varphi]$ to conclude that $w \rightleftharpoons w'$. $\square$

Example 4.3. Consider the two birelational models drawn below, where the circled worlds indicate the valuation of a proposition letter P , and the intuitionistic accessibility relation is the reflexive closure of the depicted $\leq$ -arrows:



The following relation is an IK-bisimulation:

$$Z = \{(w_1, w'_0), (w_1, w'_1), (w_2, w'_2), (v_1, v'_1)\} \cup \{(x, y) \mid x \in \{v_2, u_1, u_2\} \text{ and } y \in \{v'_2, u'_0, u'_1, u'_2\}\}.$$

In particular, w_1 and w'_0 are modally equivalent. However, there is no Kripke bisimulation between w_1 and w'_0 , since $w_1 R v_1$ cannot be mirrored at w'_0 .

Definition 4.4. We define an IK-bisimulation between two $\mathcal{IL}$ -structures $\mathfrak{M}$ and $\mathfrak{M}'$ as an IK-bisimulation between the induced birelational models $\mathfrak{B}_{\mathfrak{M}}$ and $\mathfrak{B}_{\mathfrak{M}'}$.

Next, we give a notion of saturation that encompasses image-finite birelational models, and prove that on the class of saturated birelational models bisimilarity coincides with logical equivalence.

Definition 4.5. Let $\mathfrak{B} = (W, \leq, R, V)$ be a birelational model. A subset $X \subseteq W$ is called

1. *positively saturated* if every set $\Phi \subseteq \mathcal{L}_{\Box\Diamond}$ that is finitely satisfiable in X is also satisfiable in X ;
2. *saturated* if every consecution $\langle \Phi : \Psi \rangle$ that is finitely separable in X is also separable in X .

We call $\mathfrak{B}$ *modally saturated* if for each $w \in W$ the set $R[w] := \{v \in W \mid w R v\}$ is positively saturated, and the sets $\uparrow w := \{v \in W \mid w \leq v\}$ and $R_{\uparrow}[w] := \{v \in W \mid w(\leq ; R)v\}$ are both saturated.

Theorem 4.6. Let $\mathfrak{B} = (W, \leq, R, V)$ and $\mathfrak{B}' = (W', \leq', R', V')$ be two modally saturated birelational models. Then bisimilarity and modal equivalence coincide between these models.

Proof. We prove that the relation Z of modal equivalence is an IK-bisimulation. It clearly satisfies (Z_P) , and $(Z_{\leq}^1)$ and $(Z_{\leq}^2)$ are similar to [26, Theorem 21]. We show that Z satisfies $(Z_{\Box}^1)$ and $(Z_{\Box}^2)$, the remaining conditions being similar. For $(Z_{\Box}^1)$, let $(w, w') \in Z$ and $w R v$ and suppose towards a contradiction that there exists no $v' \in R'_{\uparrow}[w'] := \{x' \in W' \mid w'(\leq' ; R')x'\}$ such that $(v, v') \in Z$. Then for each such v' either

- there exists a formula $\varphi_{v'}$ that is true at v but not at v' ; or

- there exists a formula $\psi_{v'}$ that is true at v' but not at v .

For each $v' \in R'_\uparrow[w']$, pick such $\varphi_{v'}$ or $\psi_{v'}$, and collect them in sets Φ and Ψ , respectively. Then there is no world in $R'_\uparrow[w']$ that separates $\langle \Phi : \Psi \rangle$. Since $R'_\uparrow[w']$ is saturated, this implies that there exist finite subsets $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ such that no world in $R'_\uparrow[w']$ separates $\langle \Phi' : \Psi' \rangle$. Set $\varphi := \bigwedge \Phi'$ and $\psi := \bigvee \Psi'$. It follows from frame condition (2) on $\mathfrak{B}'$ that $w'(\le' ; R' ; \le')s'$ implies $w'(\le' ; R')s'$, that is, $R'_\uparrow[w']$ is upward-closed under $\le'$. Therefore, each v' such that $w'(\le' ; R')v'$ satisfies $\varphi \rightarrow \psi$. Furthermore, we have $v \not\models \varphi \rightarrow \psi$, so that ultimately $w \not\models \Box(\varphi \rightarrow \psi)$ while $w' \models \Box(\varphi \rightarrow \psi)$, a contradiction.

Next, to see that Z satisfies $(Z^1_\Diamond)$, let $(w, w') \in Z$ and wRv and suppose towards a contradiction that there exist no $s \in W$ and $s' \in W'$ such that $v \leq s$ and $w'R's'$ and $(s, s') \in Z$. Note that, by assumption, in particular, $\uparrow v$ is saturated. So, similarly to the previous case, for each $s' \in R'[w']$ we can find formulas $\varphi_{s'}$ and $\psi_{s'}$ such that s' satisfies $\varphi_{s'}$ and refutes $\psi_{s'}$, and each $s \geq v$ either refutes $\varphi_{s'}$ or satisfies $\psi_{s'}$. Let $\xi_{s'} := \varphi_{s'} \rightarrow \psi_{s'}$. Then $v \models \xi_{s'}$ and $s' \not\models \xi_{s'}$. Now let $\Xi := \{\xi_{s'} \mid s' \in R'[w']\}$. Then no world in $R'[w']$ satisfies Ξ . Since $R'[w']$ is positively saturated by assumption, it follows that there exists a finite $\Xi' \subseteq \Xi$ such that each $s' \in R'[w']$ refutes some formula in Ξ' . Let $\xi := \bigwedge \Xi'$. Then $v \models \xi$, so $w \models \Diamond \xi$ while $w' \not\models \Diamond \xi$, a contradiction. $\square$

5 Filter products of $\mathcal{IL}$ -structures

We define an intuitionistic analogue of the ultrafilter product of classical first-order structures, and use this to prove compactness of the logic. We will also use it in Section 6 to prove that every $\mathcal{IL}$ -structure can be elementarily embedded in an ω -saturated one. The notion of (ultra)filter product of $\mathcal{IL}$ -structures—without constant symbols—may be found in [13] and [21, Chapter III]. To improve overall legibility, details of some of our constructions and proofs of some of our results were shifted to Appendix A. The remaining work in this section mirrors the development of classical model theory (see e.g. [6]).

Definition 5.1. Let I be a set, and for each $i \in I$ let $\mathfrak{M}_i = (W_i, \leq_i, \{\mathfrak{C}_{i,w}\}_{w \in W_i})$ be an $\mathcal{IL}$ -structure, where $\mathfrak{C}_{i,w} = (D_{i,w}, \mathcal{J}_{i,w})$. Let F be a filter on I . Then:

1. Take W to be the reduced product $\prod_{i \in I}^F W_i$, namely the elements in $\prod_{i \in I} W_i$ modulo the equivalence relation given by $\alpha \sim \beta$ iff $\{i \in I \mid \alpha(i) = \beta(i)\} \in F$. The equivalence class of α is denoted by α_F .
2. Define the relation $\leq$ on W by $\alpha_F \leq \beta_F$ if and only if $\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \in F$.
3. For each $i \in I$, let $D_i := D_{\mathfrak{M}_i} = \bigcup_{w \in W_i} D_{i,w}$. Let $D = \prod_{i \in I}^F D_i$ be the reduced product of the D_i , namely the elements in $\prod_{i \in I} D_i$ modulo the equivalence relation $\xi \approx \eta$ iff $\{i \in I \mid \xi(i) = \eta(i)\} \in F$, and denote the equivalence classes by ξ_F . Define the domain D_{α_F} at $\alpha_F \in W$ by

$$D_{\alpha_F} := \{\xi_F \in D \mid \{i \in I \mid \xi(i) \in D_{i,\alpha(i)}\} \in F\}.$$

4. For $c \in \text{Cnst}$, define $\tilde{c} : I \rightarrow \bigcup_{i \in I} D_i$ by $\tilde{c}(i) = c^{\mathfrak{M}_i}$, and let $\mathcal{J}_{\alpha_F}(c) := \tilde{c}_F$ if $\tilde{c}_F \in D_{\alpha_F}$ and leave the interpretation of c undefined otherwise.
5. Finally, for each n -ary predicate $P \in \text{Pred}$, define its interpretation by

$$(\xi_F^1, \dots, \xi_F^n) \in \mathcal{J}_{\alpha_F}(P) \quad \text{iff} \quad \{i \in I \mid (\xi^1(i), \dots, \xi^n(i)) \in \mathcal{J}_{i,\alpha(i)}(P)\} \in F.$$

For each $\alpha_F \in W$ we get a $\mathcal{CL}$ -structure $\mathfrak{C}_{\alpha_F} = (D_{\alpha_F}, \mathcal{J}_{\alpha_F})$. The *filter product* of the $\mathfrak{M}_i$ is given by

$$\prod_{i \in I}^F \mathfrak{M}_i = (W, \leq, \{\mathfrak{C}_{\alpha_F}\}_{\alpha_F \in W}).$$

Proposition 5.2. *Let I be a set, and for each $i \in I$ let $\mathfrak{M}_i$ be an $\mathcal{JL}$ -structure. Let F be a filter on I . Then the filter product $\prod_{i \in I}^F \mathfrak{M}_i$ is an $\mathcal{JL}$ -structure.*

Definition 5.3. Let I be a set, let $\mathfrak{M}_i$ be an $\mathcal{JL}$ -structure, for each $i \in I$, and let F be a filter over I . If F is an ultrafilter, then the filter product $\prod_{i \in I}^F \mathfrak{M}_i$ is called an *ultraproduct*. Finally, if $\mathfrak{M}_i = \mathfrak{M}$ for all $i \in I$, then the ultraproduct is called an *ultrapower* of $\mathfrak{M}$.

The next goal is the following analogue of Łoś's Theorem. This requires the following notion of a product of assignments.

Definition 5.4. Suppose ρ_i is an assignment for $\mathfrak{M}_i$, for each $i \in I$. For each $x \in \text{Var}$, define $\rho(x) : I \rightarrow \bigcup_{i \in I} D_i : i \mapsto \rho_i(x)$. Then the *product assignment* ρ_F maps x to the equivalence class of $\rho(x)$, i.e. $\rho_F(x) := \rho(x)_F$.

We obtain the following analogue of Łoś's Theorem, a proof of which can be found in Appendix A.

Theorem 5.5. *Let I be a set, and for each $i \in I$ let $\mathfrak{M}_i$ be an $\mathcal{JL}$ -structure and ρ_i an assignment for $\mathfrak{M}_i$. Let F be an ultrafilter on I , $\mathfrak{M} := \prod_{i \in I}^F \mathfrak{M}_i$ be the filter product, and ρ_F be the product assignment for $\mathfrak{M}$. Then for any world $\alpha_F \in W$ and any formula $\varphi(x_1, \dots, x_n)$ such that $\rho_F(x_1), \dots, \rho_F(x_n) \in D_{\alpha_F}$,*

$$\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \varphi \quad \text{iff} \quad \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \varphi\} \in F.$$

As an application of Łoś's Theorem, we prove that every $\mathcal{JL}$ -structure can be elementarily embedded in any of its ultrapowers. We begin by defining an intuitionistic analogue of elementary embeddings.

Definition 5.6. Let $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ and $\mathfrak{M}' = (W', \leq', \{\mathfrak{C}_{w'}\}_{w' \in W'})$ be two $\mathcal{JL}$ -structures. An *elementary embedding* of $\mathfrak{M}$ into $\mathfrak{M}'$ is a pair (ϵ, η) of injective functions $\epsilon : W \rightarrow W'$ and $\eta : D_{\mathfrak{M}} \rightarrow D_{\mathfrak{M}'}$ such that for every $\varphi \in \mathcal{L}$, $w \in W$ and assignment ρ :

$$\mathfrak{M}, w \Vdash^{\rho} \varphi \quad \text{iff} \quad \mathfrak{M}', \epsilon(w) \Vdash^{\eta \circ \rho} \varphi.$$

Definition 5.7. Let $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ be an $\mathcal{JL}$ -structure, I be a set, F be an ultrafilter over I , and write $\mathfrak{M}^* = (W^*, \leq^*, \{\mathfrak{C}_{\alpha_F}\}_{\alpha_F \in W^*})$ for the ultrapower of $\mathfrak{M}$ modulo F . As in Definition 2.3, let $D_{\mathfrak{M}^*} := \bigcup_{\alpha_F \in W^*} D_{\alpha_F}$. For each $w \in W$, let $w^* \in W^* := \prod_{i \in I}^F W$ be the equivalence class of the constant function $I \rightarrow W : i \mapsto w$ and for each $d \in D_{\mathfrak{M}}$ let $d^* \in D_{\mathfrak{M}^*} = \prod_{i \in I}^F D_{\mathfrak{M}}$ be the equivalence class of the constant map $I \rightarrow D_{\mathfrak{M}} : i \mapsto d$. The *natural embedding* of $\mathfrak{M}$ into $\mathfrak{M}^*$ is given by

$$\epsilon : W \rightarrow W^* : w \mapsto w^* \quad \text{and} \quad \eta : D_{\mathfrak{M}} \rightarrow D_{\mathfrak{M}^*} : d \mapsto d^*.$$

Proposition 5.8. *Let $\mathfrak{M}$ be an $\mathcal{JL}$ -structure, I be a set and F be an ultrafilter over I . The natural embedding of $\mathfrak{M}$ into $\mathfrak{M}^*$ is an elementary embedding.*

Proof. The maps $w \mapsto w^*$ and $d \mapsto d^*$ are clearly injective. Let ρ be any assignment for $\mathfrak{M}$, and let ρ^* be the product assignment on $\mathfrak{M}^*$ induced by ρ (as in Definition 5.4). Then $\rho^* = \eta \circ \rho$. Furthermore, for any $w \in W$ we have $w^*(i) = w$, so we get

$$\begin{aligned} \mathfrak{M}^*, w^* \Vdash^{\eta \circ \rho} \varphi & \quad \text{iff} \quad \mathfrak{M}^*, w^* \Vdash^{\rho^*} \varphi & \quad (\text{because } \rho^* = \eta \circ \rho) \\ & \quad \text{iff} \quad \{i \in I \mid \mathfrak{M}, w^*(i) \Vdash^{\rho} \varphi\} \in F & \quad (\text{Theorem 5.5}) \\ & \quad \text{iff} \quad \{i \in I \mid \mathfrak{M}, w \Vdash^{\rho} \varphi\} \in F & \quad (\text{because } w^*(i) = w) \\ & \quad \text{iff} \quad \mathfrak{M}, w \Vdash^{\rho} \varphi \end{aligned}$$

The last “iff” follows from the fact that $\{i \in I \mid \mathfrak{M}, w \Vdash^{\rho} \varphi\}$ is either empty (if $\mathfrak{M}, w \not\Vdash^{\rho} \varphi$) or equal to I (if $\mathfrak{M}, w \Vdash^{\rho} \varphi$), so it is in F if and only if the latter is the case. $\square$

Finally, still mirroring the classical case, we use Łoś's Theorem 5.5 to prove compactness of the logic.

Definition 5.9. Given $\Gamma, \Delta \subseteq \mathcal{L}$, we say that Δ is a consequence of Γ , and write $\Gamma \models \Delta$, if for every $\mathcal{IL}$ -structure $\mathfrak{M}$, there is no $w \in W$ and no assignment ρ that separate $\langle \Gamma : \Delta \rangle$ in $\mathfrak{M}$. We say that two formulas φ and ψ are *logically equivalent* if $\{\varphi\} \models \{\psi\}$ and $\{\psi\} \models \{\varphi\}$.

Theorem 5.10. Let $\langle \Gamma : \Delta \rangle(x_1, \dots, x_n)$ be a consecution in $\mathcal{L}$ and let I be the set of all of its finite subconsecutions, i.e. $I = \{\langle \Gamma_{\text{fin}} : \Delta_{\text{fin}} \rangle \mid \Gamma_{\text{fin}} \subseteq \Gamma \text{ and } \Delta_{\text{fin}} \subseteq \Delta \text{ are finite}\}$. For each $i \in I$, let $\mathfrak{M}_i = (W_i, \leq_i, \{\mathfrak{C}_{i,w}\}_{w \in W_i})$ be an $\mathcal{IL}$ -structure, $w_i \in W_i$ and ρ_i an assignment such that $\mathfrak{M}_i, w_i \Vdash^{\rho_i} i$. Then there exists an ultrafilter F over I such that $\prod_{i \in I}^F \mathfrak{M}_i, \{(i, w_i) \mid i \in I\}_F \Vdash^{\rho_F} \langle \Gamma : \Delta \rangle$, where ρ_F is the product assignment of the ρ_i .

Proof. For each pair (γ, δ) such that $\gamma \in \Gamma$ and $\delta \in \Delta$, define

$$\overline{(\gamma, \delta)} = \{\langle \Gamma_{\text{fin}} : \Delta_{\text{fin}} \rangle \in I \mid \gamma \in \Gamma_{\text{fin}} \text{ and } \delta \in \Delta_{\text{fin}}\} \quad \text{and} \quad E = \{\overline{(\gamma, \delta)} \mid \gamma \in \Gamma \text{ and } \delta \in \Delta\}.$$

Then E has the finite intersection property because for any given $\overline{(\gamma_1, \delta_1)}, \dots, \overline{(\gamma_n, \delta_n)} \in E$ we have

$$\langle \{\gamma_1, \dots, \gamma_n\} : \{\delta_1, \dots, \delta_n\} \rangle \in \overline{(\gamma_1, \delta_1)} \cap \dots \cap \overline{(\gamma_n, \delta_n)}.$$

By the ultrafilter theorem [6, Theorem 4.1.3] there exists an ultrafilter F on I such that $E \subseteq F$. Now consider an arbitrary pair (γ, δ) such that $\gamma \in \Gamma$ and $\delta \in \Delta$, and $i = \langle \Gamma_{\text{fin}} : \Delta_{\text{fin}} \rangle \in I$. If $i \in \overline{(\gamma, \delta)}$, then $\gamma \in \Gamma_{\text{fin}}, \delta \in \Delta_{\text{fin}}$ and hence $\mathfrak{M}_i, w_i \Vdash^{\rho_i} \gamma$ and $\mathfrak{M}_i, w_i \not\Vdash^{\rho_i} \delta$. Therefore, $\overline{(\gamma, \delta)} \subseteq \{i \in I \mid \mathfrak{M}_i, w_i \Vdash^{\rho_i} \gamma\}$ and $\overline{(\gamma, \delta)} \subseteq \{i \in I \mid \mathfrak{M}_i, w_i \not\Vdash^{\rho_i} \delta\}$, and since $\overline{(\gamma, \delta)} \in F$ by construction we get

$$\{i \in I \mid \mathfrak{M}_i, w_i \Vdash^{\rho_i} \gamma\} \in F \quad \text{and} \quad \{i \in I \mid \mathfrak{M}_i, w_i \not\Vdash^{\rho_i} \delta\} \in F.$$

Because F is an ultrafilter we get $\{i \in I \mid \mathfrak{M}_i, w_i \not\Vdash^{\rho_i} \delta\} \notin F$, so Theorem 5.5 entails

$$\prod_{i \in I}^F \mathfrak{M}_i, \{(i, w_i) \mid i \in I\}_F \Vdash^{\rho_F} \gamma \quad \text{and} \quad \prod_{i \in I}^F \mathfrak{M}_i, \{(i, w_i) \mid i \in I\}_F \not\Vdash^{\rho_F} \delta,$$

where $\{(i, w_i) \mid i \in I\}_F$ denotes the equivalence class of the function $I \rightarrow \bigcup_i W_i : i \mapsto w_i$. Note that ρ_i does not depend on γ or δ . Therefore $\{(i, w_i) \mid i \in I\}_F$ and ρ_F separate the consecution $\langle \Gamma : \Delta \rangle$ in $\prod_{i \in I}^F \mathfrak{M}_i$. $\square$

Corollary 5.11. If $\Gamma \models \Delta$, then there exist finite subsets $\Gamma_{\text{fin}} \subseteq \Gamma$ and $\Delta_{\text{fin}} \subseteq \Delta$ such that $\Gamma_{\text{fin}} \models \Delta_{\text{fin}}$.

Proof. Let $\Gamma \models \Delta$ and suppose towards a contradiction that for all finite $\Gamma_{\text{fin}} \subseteq \Gamma$ and $\Delta_{\text{fin}} \subseteq \Delta$, we have $\Gamma_{\text{fin}} \not\models \Delta_{\text{fin}}$. Let I be the set of finite subconsecutions of $\langle \Gamma : \Delta \rangle$. Then for every $i \in I$ there exists an $\mathcal{IL}$ -structure $\mathfrak{M}_i = (W_i, \leq_i, \{\mathfrak{C}_{i,w}\}_{w \in W_i})$, a world $w_i \in W_i$ and an assignment ρ_i such that $\mathfrak{M}_i, w_i \Vdash^{\rho_i} i$. Therefore Theorem 5.10 implies that there exists an ultrafilter F over I such that $\prod_{i \in I}^F \mathfrak{M}_i, \{(i, w_i) \mid i \in I\}_F \Vdash^{\rho_F} \langle \Gamma : \Delta \rangle$, where ρ_F is the product assignment. As a consequence $\Gamma \not\models \Delta$, a contradiction. $\square$

6 Countable saturation in the intuitionistic setting

A final ingredient we need in the proof of the characterisation theorem for IK is a way to turn an $\mathcal{IL}$ -structure into a modally saturated one. More precisely, we wish to elementarily embed any $\mathcal{IL}$ -structure $\mathfrak{M}$ into an $\mathcal{IL}$ -structure $\mathfrak{M}^*$ whose induced birelational model $\mathfrak{B}_{\mathfrak{M}^*}$ is modally saturated in the sense of Definition 4.5. Keeping in sync with the classical case, we do so using an intuitionistic variation of the notion of ω -saturation which is tailored towards our needs as an intermediary between $\mathcal{IL}$ -structures and modally saturated birelational models.

Definition 6.1. Given an $\mathcal{JL}$ -structure $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ and $A \subseteq \bigcup_{w \in W} D_w$, we write $\mathcal{L}_A$ for the extension of $\mathcal{L}$ with constants $\{a \mid a \in A\}$. We define $(\mathfrak{M})_A$ as the $\mathcal{JL}_A$ -structure that extends the $\mathcal{JL}$ -structure $\mathfrak{M}$ by interpreting each of the new constants a as a .

Definition 6.2. Let $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ be an $\mathcal{JL}$ -structure.

1. A set $U \subseteq W$ is called *positively ω -saturated* if for all finite $A \subseteq \bigcap_{w \in U} D_w$ and every set $\Gamma(x) \subseteq \mathcal{L}_A$ with one free variable x , if $\Gamma(x)$ is finitely satisfiable in U , then it is satisfiable in U .
2. The set $U \subseteq W$ is called *ω -saturated* if for all finite $A \subseteq \bigcap_{w \in U} D_w$ and all sets $\Gamma(x), \Delta(x) \subseteq \mathcal{L}_A$ with one free variable x , if $\langle \Gamma(x) : \Delta(x) \rangle$ is finitely separable in U , then it is separable in U .

The $\mathcal{JL}$ -structure $\mathfrak{M}$ is called *ω -saturated* if, for every $w \in W$, the set $\{w\}$ is positively ω -saturated and $\uparrow w$ is ω -saturated.

We verify that every ω -saturated $\mathcal{JL}$ -structure gives rise to a modally saturated birelational model.

Proposition 6.3. Let $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ be an ω -saturated $\mathcal{JL}$ -structure. Then the induced birelational model $\mathfrak{B}_{\mathfrak{M}} = (W^\bullet, \leq^\bullet, R^\bullet, V^\bullet)$ is modally saturated.

Proof. Let $(w, d) \in W^\bullet$ be an arbitrary world. One may check, similarly to [4, Theorem 2.65], that $R^\bullet[(w, d)]$ is positively saturated. We verify the two remaining saturation conditions from Definition 4.5.

$\uparrow(w, d)$ is saturated. Let $\Phi, \Psi \subseteq \mathcal{L}_{\Box\Diamond}$ be sets of formulas that are finitely separable in $\uparrow(w, d)$. Consider the consecution $\langle \text{st}(x, \Phi) : \text{st}(x, \Psi) \rangle$. Then each of its finite subconsecutions is of the form $\langle \text{st}(x, \Phi') : \text{st}(x, \Psi') \rangle$ for some finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$. By assumption, we can find some $(v, d) \in \uparrow(w, d)$ that satisfies Φ' and refutes Ψ' , which implies $\mathfrak{M}, v \models^{[x:=d]} \bigwedge \text{st}(x, \Phi')$ and $\mathfrak{M}, v \not\models^{[x:=d]} \bigvee \text{st}(x, \Psi')$. Therefore $\langle \text{st}(x, \Phi') : \text{st}(x, \Psi') \rangle$ is finitely separable in $\uparrow w$, and since $\mathfrak{M}$ is assumed to be ω -saturated it must also be separable in some world $u \in \uparrow w$. This entails that $(u, d) \in \uparrow(w, d)$ separates $\langle \Phi : \Psi \rangle$, as desired.

$R^\bullet_\uparrow[(w, d)]$ is saturated. Let $\Phi, \Psi \subseteq \mathcal{L}_{\Box\Diamond}$ and suppose that for any finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ there exists a world $(v, e) \in R^\bullet_\uparrow[(w, d)]$ separating $\langle \Phi' : \Psi' \rangle$. Note that $(v, e) \in R^\bullet_\uparrow[(w, d)]$ iff $(w, d) \leq^\bullet (v, d) R^\bullet(v, e)$, iff $w \leq v$ and $(d, e) \in \mathcal{J}_v(R)$. Define

$$\Gamma := \{Rdx\} \cup \text{st}(x, \Phi) \quad \text{and} \quad \Delta := \text{st}(x, \Psi).$$

We claim that $\langle \Gamma : \Delta \rangle$ is finitely separable in $\uparrow w$ on the $\mathcal{JL}$ -structure $\mathfrak{M}$. To see this, let $\langle \Gamma' : \Delta' \rangle$ be a finite subconsecution of $\langle \Gamma : \Delta \rangle$, and let $\Phi' := \{\varphi \in \Phi \mid \text{st}(x, \varphi) \in \Gamma'\}$ and $\Psi' := \{\psi \in \Psi \mid \text{st}(x, \psi) \in \Delta'\}$. Then Φ' and Ψ' are finite, so by assumption we can find some $(v, e) \in R^\bullet_\uparrow[(w, d)]$ that separates $\langle \Phi' : \Psi' \rangle$. This gives $\mathfrak{M}, v \models^{[x:=e]} \langle \Gamma' : \Delta' \rangle$, since $(v, e) \in R^\bullet_\uparrow[(w, d)]$ implies $\mathfrak{M}, v \models^{[x:=e]} Rdx$. As we have seen, $(v, e) \in R^\bullet_\uparrow[(w, d)]$ implies $w \leq v$, so $\langle \Gamma : \Delta \rangle$ is finitely separable in $\uparrow w$. Since $\mathfrak{M}$ is ω -saturated, we can find some $u \in \uparrow w$ and $b \in D_u$ such that $\mathfrak{M}, u \models^{[x:=b]} \langle \Gamma : \Delta \rangle$. From this it follows that $(u, b) \in W^\bullet$ is an element of $R^\bullet_\uparrow[(w, d)]$ that separates $\langle \Phi : \Psi \rangle$. We conclude that $R^\bullet_\uparrow[(w, d)]$ is saturated, as desired. $\square$

Recall that a countably incomplete ultrafilter on a set I is an ultrafilter that is not closed under countably infinite intersections [6, Propositions 4.3.4 and 4.3.5]. Modifying [6, Theorem 6.1.1] we have:

Proposition 6.4. Let $\mathcal{L}$ be a language over a countable signature, let I be a set and $\mathfrak{M}_i$ an $\mathcal{JL}$ -structure for each $i \in I$. If F is a countably incomplete ultrafilter on I , then $\prod_{i \in I}^F \mathfrak{M}_i$ is ω -saturated.

Proof. Part 1: for any world α_F in $\prod_{i \in I}^F \mathfrak{M}_i$, the set $\{\alpha_F\}$ is positively ω -saturated. Let α_F be an element of the ultraproduct $\prod_{i \in I}^F \mathfrak{M}_i$. Let $A \subseteq D_{\alpha_F}$ be a finite set of individuals of α_F and let $\mathcal{L}_A$ be the extension of the language with constants $\underline{a}$ for each $a_F \in A$. Let $A_i = \{a(i) \mid a_F \in A\}$. Then we have

$$\left(\prod_{i \in I}^F \mathfrak{M}_i \right)_A = \prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}).$$

Let $\Gamma(x) = \{\gamma_1(x), \gamma_2(x), \dots\}$ be a subset of $\mathcal{L}_A$ that is finitely satisfiable in $\{\alpha_F\}$. Since $\Gamma(x)$ is finitely satisfiable in $\{\alpha_F\}$, for each $n \in \mathbb{N}$ we can find some $d \in D_{\alpha_F}$ such that $\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}, \alpha_F \Vdash^{[x:=d]} \gamma_1(x) \wedge \dots \wedge \gamma_n(x)$. Hence for each $n \in \mathbb{N}$,

$$\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}, \alpha_F \Vdash \exists x (\gamma_1(x) \wedge \dots \wedge \gamma_n(x))$$

Using the fact that F is countably incomplete, we can find a descending chain $I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ of sets in F such that $\bigcap_{n \in \mathbb{N}} I_n = \emptyset$. Let $X_0 = I$, and for each $n \in \mathbb{N}_{>0}$ define

$$X_n = I_n \cap \{i \in I \mid (\mathfrak{M}_i)_{A_i}, \alpha(i) \Vdash \exists x (\gamma_1(x) \wedge \dots \wedge \gamma_n(x))\}$$

By Łoś's Theorem, the right part of the intersection belongs to F and hence $X_n \in F$. Moreover, we have $X_n \supseteq X_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcap_{n \in \mathbb{N}} X_n = \emptyset$. As a consequence, for each $i \in I$ there exists an $n(i) \in \mathbb{N}$ such that $n(i)$ is the greatest natural number with $i \in X_{n(i)}$, so there exists an assignment ρ_i for $\mathfrak{M}_i$ such that

$$(\mathfrak{M}_i)_{A_i}, \alpha(i) \Vdash^{\rho_i} \gamma_1(x) \wedge \dots \wedge \gamma_{n(i)}(x)$$

Now let $n \in \mathbb{N}_{>0}$ and $i \in X_n$. Then $n \leq n(i)$ and hence $(\mathfrak{M}_i)_{A_i}, \alpha(i) \Vdash^{\rho_i} \gamma_n(x)$. Since $X_n \in F$, we have

$$X_n \subseteq \{i \in I \mid (\mathfrak{M}_i)_{A_i}, \alpha(i) \Vdash^{\rho_i} \gamma_n(x)\} \in F$$

so, by Łoś's Theorem 5.5, we have $\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}, \alpha_F \Vdash^{\rho_F} \gamma_n(x)$, where ρ is the product assignment. Since this holds for any $n \in \mathbb{N}_{>0}$, it follows that $\Gamma(x)$ is satisfiable in $\{\alpha_F\}$.

Part 2: for any world α_F in $\prod_{i \in I}^F \mathfrak{M}_i$, the set $\uparrow \alpha_F$ is ω -saturated. Let α_F be an element of the ultraproduct $\prod_{i \in I}^F \mathfrak{M}_i$. Let $A \subseteq \bigcap_{\beta_F \in \uparrow \alpha_F} D_{\beta_F} = D_{\alpha_F}$ be a finite set of individuals of α_F and let $\mathcal{L}_A$ be the extension of the language with constants $\underline{a}$ for each $a_F \in A$. Let $A_i = \{a(i) \mid a_F \in A\}$. As before, we have $(\prod_{i \in I}^F \mathfrak{M}_i)_A = \prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i})$.

Let $\Gamma(x), \Delta(x) \subseteq \mathcal{L}_A$ be countable sets such that $\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i})$ finitely separates $\langle \Gamma(x) : \Delta(x) \rangle$ in the set $\uparrow \alpha_F$. Then we can write $\Gamma(x) = \{\gamma_1(x), \gamma_2(x), \dots\}$ and $\Delta(x) = \{\delta_1(x), \delta_2(x), \dots\}$. By hypothesis, for each $n \in \mathbb{N}$ there exists an $\alpha_F \leq \beta_F^n$ and an assignment ρ_F^n such that

$$\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}, \beta_F^n \Vdash^{\rho_F^n} \langle \{\gamma_1(x), \dots, \gamma_n(x)\} : \{\delta_1(x), \dots, \delta_n(x)\} \rangle$$

and hence, since $\alpha_F \leq \beta_F^n$,

$$\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}, \alpha_F \not\Vdash \forall x ((\gamma_1(x) \wedge \dots \wedge \gamma_n(x)) \rightarrow (\delta_1(x) \vee \dots \vee \delta_n(x)))$$

Again, the fact that F is countably incomplete yields an infinite descending chain $I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ of elements in F such that $\bigcap_{n \in \mathbb{N}} I_n = \emptyset$. Let $X_0 = I$, and for each $n \in \mathbb{N}_{>0}$ define

$$X_n = I_n \cap \{i \in I \mid (\mathfrak{M}_i)_{A_i}, \alpha(i) \not\models \forall x ((\gamma_1(x) \wedge \dots \wedge \gamma_n(x)) \rightarrow (\delta_1(x) \vee \dots \vee \delta_n(x)))\}$$

By Łoś's Theorem 5.5, the right part of the intersection belongs to F and hence $X_n \in F$. Moreover, $X_n \supseteq X_{n+1}$ and $\bigcap_{n \in \mathbb{N}} X_n = \emptyset$. Hence for each $i \in I$ there exists an $n(i) \in \mathbb{N}$ such that $n(i)$ is the greatest natural number such that $i \in X_{n(i)}$, so that there exists an assignment ρ_i for $\mathfrak{M}_i$ and some $a_i \geq \alpha(i)$ in W_i such that

$$(\mathfrak{M}_i)_{A_i}, a_i \not\models^{\rho_i} (\gamma_1(x) \wedge \dots \wedge \gamma_{n(i)}(x)) \rightarrow (\delta_1(x) \vee \dots \vee \delta_{n(i)}(x))$$

Therefore, for every $i \in I$, there exists some $b_i \geq_i a_i$ such that

$$(\mathfrak{M}_i)_{A_i}, b_i \models^{\rho_i} \langle \{\gamma_1(x), \dots, \gamma_{n(i)}(x)\} : \{\delta_1(x), \dots, \delta_{n(i)}(x)\} \rangle.$$

For each $n \in \mathbb{N}_{>0}$ and $i \in X_n$ we have $n \leq n(i)$, hence $(\mathfrak{M}_i)_{A_i}, b_i \models^{\rho_i} \langle \{\gamma_n(x)\} : \{\delta_n(x)\} \rangle$. Since $X_n \in F$,

$$X_n \subseteq \{i \in I \mid (\mathfrak{M}_i)_{A_i}, b_i \models^{\rho_i} \langle \{\gamma_n(x)\} : \{\delta_n(x)\} \rangle\} \in F$$

so using Łoś's Theorem 5.5 we obtain $\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}), \beta_F \models^{\rho_F} \langle \{\gamma_n(x)\} : \{\delta_n(x)\} \rangle$, where ρ is the product assignment and $\beta(i) = b_i$ for every $i \in I$. Since this holds for every $n \in \mathbb{N}_{>0}$, we may conclude that $\prod_{i \in I}^F ((\mathfrak{M}_i)_{A_i}), \beta_F \models^{\rho_F} \langle \Gamma(x) : \Delta(x) \rangle$ so that $\langle \Gamma(x) : \Delta(x) \rangle$ is separable in $\uparrow \alpha_F$. $\square$

Theorem 6.5. *Every $\mathcal{JL}$ -structure for a countable signature can be elementarily embedded in an ω -saturated $\mathcal{JL}$ -structure.*

Proof. Combine the fact that countably incomplete ultrafilters exist (see Proposition 4.3.5 of [6]) with Propositions 6.4 and 5.8. $\square$

7 Van Benthem-style characterisation theorem

By now we have developed all ingredients needed to prove the characterisation theorem for IK. Throughout this section we use the same first-order signature as in Section 3.

Definition 7.1. A formula $\varphi(x) \in \mathcal{L}$ with one free variable x is said to be *invariant under IK-bisimulations* if for every pair of $\mathcal{JL}$ -structures $\mathfrak{M} = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$ and $\mathfrak{M}' = (W', \leq', \{\mathfrak{C}_{w'}\}_{w' \in W'})$, worlds $w \in W$ and $w' \in W'$, and individuals $d \in D_w$ and $d' \in D_{w'}$, if there exists an IK-bisimulation B between $\mathfrak{B}_{\mathfrak{M}}$ and $\mathfrak{B}_{\mathfrak{M}'}$ linking (w, d) and (w', d') , then

$$\mathfrak{M}, w \models^{[x:=d]} \varphi(x) \quad \text{iff} \quad \mathfrak{M}', w' \models^{[x:=d']} \varphi(x).$$

Theorem 7.2. *A formula $\alpha(x) \in \mathcal{L}$ with one free variable x is equivalent to the translation of a formula in $\mathcal{L}_{\square\Diamond}$ if and only if it is invariant under IK-bisimulations.*

Proof. Suppose $\alpha(x)$ is equivalent to $\text{st}(x, \psi)$ for some $\psi \in \mathcal{L}_{\square\Diamond}$. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two $\mathcal{JL}$ -structures and Z be an IK-bisimulation between $\mathfrak{B}_{\mathfrak{M}}$ and $\mathfrak{B}_{\mathfrak{M}'}$ linking (w, d) to (w', d') . Then

$$\begin{aligned} \mathfrak{M}, w \models^{[x:=d]} \alpha(x) & \quad \text{iff} \quad \mathfrak{M}, w \models^{[x:=d]} \text{st}(x, \psi) \\ & \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{M}}, (w, d) \models \psi & \quad (\text{by definition and Lemma 3.6}) \\ & \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{M}'}, (w', d') \models \psi & \quad (\text{Proposition 4.2}) \\ & \quad \text{iff} \quad \mathfrak{M}', w' \models^{[x:=d']} \text{st}(x, \psi) & \quad (\text{by definition and Lemma 3.6}) \\ & \quad \text{iff} \quad \mathfrak{M}', w' \models^{[x:=d']} \alpha(x) \end{aligned}$$

So, α is invariant under IK-bisimulations.

For the converse, assume that $\alpha(x)$ is invariant under IK-bisimulations. Consider the set

$$\text{MOC}(\alpha) = \{\text{st}(x, \varphi) \mid \varphi \in \mathcal{L}_{\Box\Diamond} \text{ and } \alpha(x) \models \text{st}(x, \varphi)\}.$$

It suffices to show $\text{MOC}(\alpha) \models \alpha(x)$, because then compactness (Corollary 5.11) yields a finite subset $\{\text{st}(x, \psi_1), \dots, \text{st}(x, \psi_n)\} \subseteq \text{MOC}(\alpha)$ that has $\alpha(x)$ as a consequence, from which it follows that $\alpha(x)$ is equivalent to $\text{st}(x, \psi_1 \wedge \dots \wedge \psi_n)$.

Assume $\mathfrak{M}, w \Vdash^{[x:=d]} \text{MOC}(\alpha)$. We need to show that $\mathfrak{M}, w \Vdash^{[x:=d]} \alpha(x)$. Let

$$\Gamma(x) := \{\text{st}(x, \varphi) \mid \mathfrak{M}, w \Vdash^{[x:=d]} \text{st}(x, \varphi)\} \quad \text{and} \quad \Delta(x) := \{\text{st}(x, \psi) \mid \mathfrak{M}, w \nVdash^{[x:=d]} \text{st}(x, \psi)\}.$$

Suppose towards a contradiction that $\Gamma(x) \cup \{\alpha(x)\} \models \Delta(x)$. Then Corollary 5.11 yields finite $\Gamma_{\text{fin}}(x) \subseteq \Gamma(x)$ and $\Delta_{\text{fin}}(x) \subseteq \Delta(x)$ such that $\Gamma_{\text{fin}}(x) \cup \{\alpha(x)\} \models \Delta_{\text{fin}}(x)$. This implies $\alpha(x) \models \bigwedge \Gamma_{\text{fin}}(x) \rightarrow \bigvee \Delta_{\text{fin}}(x)$, hence $\bigwedge \Gamma_{\text{fin}}(x) \rightarrow \bigvee \Delta_{\text{fin}}(x) \in \text{MOC}(\alpha)$. By assumption, $\mathfrak{M}, w \Vdash^{[x:=d]} \text{MOC}(\alpha)$, so $\mathfrak{M}, w \Vdash^{[x:=d]} \bigwedge \Gamma_{\text{fin}}(x) \rightarrow \bigvee \Delta_{\text{fin}}(x)$. But then the fact that $\mathfrak{M}, w \Vdash^{[x:=d]} \text{st}(x, \psi)$ for all $\text{st}(x, \psi) \in \Gamma_{\text{fin}}(x)$ contradicts the fact that $\mathfrak{M}, w \nVdash^{[x:=d]} \text{st}(x, \chi)$ for all $\text{st}(x, \chi) \in \Delta_{\text{fin}}(x)$.

So we have $\Gamma(x) \cup \{\alpha(x)\} \not\models \Delta(x)$, hence there must exist an $\mathcal{SL}$ -structure $\mathfrak{N} = (V, \leq, \{\mathfrak{C}_v\}_{v \in V})$, a world $v \in V$ and an individual $e \in D_v$ such that $\mathfrak{N}, v$ separates $\langle \Gamma(x) \cup \{\alpha(x)\} : \Delta(x) \rangle$ under $[x := e]$. Then by construction we have $\mathfrak{M}, w \Vdash^{[x:=d]} \text{st}(x, \varphi)$ if and only if $\mathfrak{N}, v \Vdash^{[x:=e]} \text{st}(x, \varphi)$ for all $\varphi \in \mathcal{L}_{\Box\Diamond}$. Proposition 5.8 gives elementary embeddings of $\mathfrak{M}$ and $\mathfrak{N}$ into ω -saturated $\mathcal{SL}$ -structures $\mathfrak{M}^*$ and $\mathfrak{N}^*$, respectively. Hence, for every $\varphi \in \mathcal{L}_{\Box\Diamond}$

$$\begin{aligned} \mathfrak{M}^*, w^* \Vdash^{[x:=d^*]} \text{st}(x, \varphi) & \quad \text{iff} \quad \mathfrak{M}, w \Vdash^{[x:=d]} \text{st}(x, \varphi) \\ & \quad \text{iff} \quad \mathfrak{N}, v \Vdash^{[x:=e]} \text{st}(x, \varphi) \quad \text{iff} \quad \mathfrak{N}^*, v^* \Vdash^{[x:=e^*]} \text{st}(x, \varphi). \end{aligned}$$

By Proposition 6.3, $\mathfrak{B}_{\mathfrak{M}^*}$ and $\mathfrak{B}_{\mathfrak{N}^*}$ are modally saturated birelational models, so that Lemma 3.6 entails

$$\mathfrak{B}_{\mathfrak{M}^*}, (w^*, d^*) \Vdash \varphi \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{N}^*}, (v^*, e^*) \Vdash \varphi$$

for all $\varphi \in \mathcal{L}_{\Box\Diamond}$. Hence, by Theorem 4.6 there exists an IK-bisimulation Z between $\mathfrak{B}_{\mathfrak{M}^*}$ and $\mathfrak{B}_{\mathfrak{N}^*}$ linking (w^*, d^*) and (v^*, e^*) . Since α is invariant under IK-bisimulations, we conclude that

$$\mathfrak{M}, w \Vdash^{[x:=d]} \alpha(x) \quad \text{iff} \quad \mathfrak{M}^*, w^* \Vdash^{[x:=d^*]} \alpha(x) \quad \text{iff} \quad \mathfrak{N}^*, v^* \Vdash^{[x:=e^*]} \alpha(x) \quad \text{iff} \quad \mathfrak{N}, v \Vdash^{[x:=e]} \alpha(x).$$

By construction the right-hand side holds, so that ultimately $\mathfrak{M}, w \Vdash^{[x:=d]} \alpha(x)$, as desired. $\square$

8 Conclusions

Motivated by Van Benthem's celebrated characterisation theorem, we introduced a precise notion of IK-bisimulation between birelational models and proved that the modal logic IK corresponds exactly to the IK-bisimulation-invariant fragment of the intuitionistic first-order logic with one binary predicate and a unary predicate for each propositional letter. En route to this result, we developed intuitionistic counterparts of classical model theory machinery, including an intuitionistic version of Łoś's Theorem, a compactness theorem and notions of elementary embeddings and countable saturation.

Within the model-theoretic framework, a natural next step is to generalise our results and constructions so as to allow for function symbols in the signature, as well as to weaken the restrictions made on the

interpretation of terms, and to expand the theory of saturated models beyond the countable case. Regarding the modal logical framework, obtaining a characterisation theorem for IK relative to classical first-order logic may also be of interest, as well as to further explore the relation between Kripke bisimulations and IK-bisimulations to check whether they yield the same invariance notion.

Finally, it would be worth investigating whether the notion of IK-bisimulation developed here is robust to changes in the chosen intuitionistic modal logic (e.g. CK [3, 7, 22], WK [30, 31], FS [32], or other constructive variants), or whether each non-classical variant of K needs its own tailored bisimulation.

A The filter product construction

We elaborate on the definition of the filter product. Proposition 5.2 follows from Lemmas A.1 to A.3.

The setup Let I be a set, and for each $i \in I$ let $\mathfrak{M}_i = (W_i, \leq_i, \{\mathfrak{C}_{i,w}\}_{w \in W_i})$ be an $\mathcal{SL}$ -structure, where $\mathfrak{C}_{i,w} = (D_{i,w}, \mathcal{J}_{i,w})$. Let F be a filter on I . In what follows, we aim to define the *filter product* $\prod_{i \in I}^F \mathfrak{M}_i = (W, \leq, \{\mathfrak{C}_w\}_{w \in W})$.

Step 1: defining W . Let W be the reduced product $\prod_{i \in I}^F W_i$. In other words, W consists of elements in $\prod_{i \in I} W_i$ modulo the equivalence relation $\sim$ given by $\alpha \sim \beta$ iff $\{i \in I \mid \alpha(i) = \beta(i)\} \in F$. Elements in W are denoted by α_F .

Step 2: defining $\leq$. Define the relation $\leq$ on W by $\alpha_F \leq \beta_F$ iff $\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \in F$.

Lemma A.1.

1. The definition of $\leq$ does not depend on the choice of representative of α_F and β_F .
2. The relation $\leq$ defines a partial order on W .

Proof. (1) Suppose $\alpha \sim \alpha'$ and $\beta \sim \beta'$. Then $\{i \in I \mid \alpha(i) = \alpha'(i)\} \in F$ and $\{i \in I \mid \beta(i) = \beta'(i)\} \in F$. Note that

$$\{i \in I \mid \alpha(i) = \alpha'(i)\} \cap \{i \in I \mid \beta(i) = \beta'(i)\} \cap \{i \in I \mid \alpha(i) \leq_i \beta(i)\} \subseteq \{i \in I \mid \alpha'(i) \leq_i \beta'(i)\}.$$

Therefore, if $\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \in F$, then the fact that F is a filter, hence closed under intersections and upwards closed, entails $\{i \in I \mid \alpha'(i) \leq_i \beta'(i)\} \in F$. This proves that $\leq$ is well defined.

(2) We need to verify that $\leq$ is reflexive, antisymmetric and transitive. Reflexivity follows from the fact that $\{i \in I \mid \alpha(i) \leq_i \alpha(i)\} = I \in F$. For antisymmetry, suppose $\alpha_F \leq \beta_F$ and $\beta_F \leq \alpha_F$. Then

$$\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \cap \{i \in I \mid \beta(i) \leq_i \alpha(i)\} = \{i \in I \mid \alpha(i) = \beta(i)\} \in F,$$

so $\alpha \sim \beta$, hence $\alpha_F = \beta_F$. Lastly, suppose $\alpha_F \leq \beta_F$ and $\beta_F \leq \gamma_F$. Then

$$\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \cap \{i \in I \mid \beta(i) \leq_i \gamma(i)\} \subseteq \{i \in I \mid \alpha(i) \leq_i \gamma(i)\} \in F,$$

so $\alpha_F \leq \gamma_F$. □

Step 3: defining the domains D_{α_F} . For each $i \in I$, let $D_i := D_{\mathfrak{M}_i} = \bigcup_{w \in W_i} D_{i,w}$ be the union of the domains $D_{i,w}$ of the $\mathcal{SL}$ -structures $\mathfrak{C}_{i,w}$. Define $D := \prod_{i \in I}^F D_i$ to be the reduced product of the D_i , that is, D is equal to the product $\prod_{i \in I} D_i$ modulo the equivalence relation $\approx$ given by $\xi \approx \eta$ iff $\{i \in I \mid \xi(i) = \eta(i)\} \in F$. Now we define the domain at $\alpha_F \in W$ by

$$D_{\alpha_F} := \{\xi_F \in D \mid \{i \in I \mid \xi(i) \in D_{i,\alpha(i)}\} \in F\}.$$

Lemma A.2.

1. The definition of D_{α_F} does not depend on the choice of representative of α_F or ξ_F .
2. If $\alpha_F \leq \beta_F$, then $D_{\alpha_F} \subseteq D_{\beta_F}$.
3. $D = \bigcup_{\alpha_F \in W} D_{\alpha_F}$

Proof. Suppose $\alpha \sim \alpha'$ and $\xi \approx \xi'$, so that $\{i \in I \mid \alpha(i) = \alpha'(i)\} \in F$ and $\{i \in I \mid \xi(i) = \xi'(i)\} \in F$. We need to show that $\{i \in I \mid \xi(i) \in D_{i, \alpha(i)}\} \in F$ if and only if $\{i \in I \mid \xi'(i) \in D_{i, \alpha'(i)}\} \in F$. Suppose the former is in F . Then for any j in the intersection

$$\{i \in I \mid \alpha(i) = \alpha'(i)\} \cap \{i \in I \mid \xi(i) \in D_{i, \alpha(i)}\} \cap \{i \in I \mid \xi(i) = \xi'(i)\} \quad (1)$$

we have $\alpha(j) = \alpha'(j)$ and $\xi(j) = \xi'(j)$ and $\xi(j) \in D_{j, \alpha(j)}$, which clearly entails $\xi'(j) \in D_{j, \alpha'(j)}$. Therefore the intersection of (1) is contained in $\{i \in I \mid \xi'(i) \in D_{i, \alpha'(i)}\}$. Since filters are upwards closed and closed under finite intersections, we find $\{i \in I \mid \xi'(i) \in D_{i, \alpha'(i)}\} \in F$, as desired. The other direction of the “iff” is analogous.

For the second item, suppose $\alpha_F \leq \beta_F$ and $\xi_F \in D_{\alpha_F}$. Then $\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \in F$ and $\{i \in I \mid \xi(i) \in D_{i, \alpha(i)}\} \in F$. Now we note that

$$\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \cap \{i \in I \mid \xi(i) \in D_{i, \alpha(i)}\} \subseteq \{i \in I \mid \xi(i) \in D_{i, \beta(i)}\},$$

because $\alpha(i) \leq_i \beta(i)$ implies $D_{i, \alpha(i)} \subseteq D_{i, \beta(i)}$. Since F is a filter we find $\{i \in I \mid \xi(i) \in D_{i, \beta(i)}\} \in F$, hence $\xi_F \in D_{\beta_F}$, so that $D_{\alpha_F} \subseteq D_{\beta_F}$.

For the third item, $\bigcup_{\alpha_F \in W} D_{\alpha_F} \subseteq D$ follows from the corresponding definition. For the converse, take $\xi_F \in D$. Thus, $\xi : I \rightarrow \bigcup_{i \in I} D_i$ is such that $\xi(i) \in D_i = \bigcup_{w \in W_i} D_{i, w}$ for each $i \in I$ and hence, for each $i \in I$, there exists a $w_i \in W_i$ such that $\xi(i) \in D_{i, w_i}$. Define $\alpha_\xi : I \rightarrow \bigcup_{i \in I} W_i$ by $\alpha_\xi(i) = w_i \in W_i$. Thus,

$$\{i \in I \mid \xi(i) \in D_{i, \alpha_\xi(i)}\} = I$$

and hence $\xi_F \in D_{\alpha_{\xi_F}}$. □

Step 4: defining the constants. Let $c \in \text{Cnst}$ be a constant symbol. For each $i \in I$, let c_i be the interpretation of c in D_i , and define $\tilde{c} \in \prod_{i \in I} D_i$ by

$$\tilde{c} : I \rightarrow \bigcup_{i \in I} D_i : i \mapsto c_i.$$

Then, if $\tilde{c}_F \in D_{\alpha_F}$, we let $c \in \text{Cnst}_{\alpha_F}$ and define $\mathcal{J}_{\alpha_F}(c) := \tilde{c}_F$; otherwise, if $\tilde{c}_F \notin D_{\alpha_F}$, we let $c \notin \text{Cnst}_{\alpha_F}$ and we leave $\mathcal{J}_{\alpha_F}(c)$ undefined. The fact that D_{α_F} does not depend on the representative of α_F entails that this is well defined, and by definition it satisfies the requirements from Definition 2.3.

Step 5: defining the predicates. Finally, we define the interpretations of the predicates on the domains D_{α_F} . For an n -ary predicate $P \in \text{Pred}$, set

$$(\xi_F^1, \dots, \xi_F^n) \in \mathcal{J}_{\alpha_F}(P) \quad \text{iff} \quad \{i \in I \mid (\xi^1(i), \dots, \xi^n(i)) \in \mathcal{J}_{i, \alpha(i)}(P)\} \in F.$$

Then for each $\alpha_F \in W$ we get a $\mathcal{CL}$ -structure $\mathfrak{C}_{\alpha_F} = (D_{\alpha_F}, \mathcal{J}_{\alpha_F})$.

Lemma A.3.

1. $\mathcal{J}_{\alpha_F}(P) \subseteq D_{\alpha_F}^n$.

2. The definition of $\mathcal{J}_{\alpha_F}(P)$ does not depend on the choice of α_F or any of the ξ_F^j .
3. If $\alpha_F \leq \beta_F$, then $\mathcal{J}_{\alpha_F}(P) \subseteq \mathcal{J}_{\beta_F}(P)$.

Proof. (1) Suppose $(\xi_F^1, \dots, \xi_F^n) \in \mathcal{J}_{\alpha_F}(P)$. We aim to show $\xi_F^j \in D_{\alpha_F}$ for each $j \in \{1, \dots, n\}$. By assumption, $\{i \in I \mid (\xi^1(i), \dots, \xi^n(i)) \in \mathcal{J}_{i, \alpha(i)}(P)\} \in F$. Since $\mathcal{J}_{i, \alpha(i)}(P) \subseteq D_{i, \alpha(i)}^n$ for each $i \in I$, this implies

$$\{i \in I \mid (\xi^1(i), \dots, \xi^n(i)) \in \mathcal{J}_{i, \alpha(i)}(P)\} \subseteq \{i \in I \mid \xi^j(i) \in D_{i, \alpha(i)}\},$$

so that $\{i \in I \mid \xi^j(i) \in D_{i, \alpha(i)}\} \in F$ for each $j \in \{1, \dots, n\}$. By definition, this implies $\xi_F^j \in D_{\alpha_F}$.

(2) Suppose $\alpha \sim \beta$ and $\xi^j \approx \eta^j$ for all $j \in \{1, \dots, n\}$. Then $\{i \in I \mid \alpha(i) = \beta(i)\} \in F$ and $\{i \in I \mid \xi^j(i) = \eta^j(i)\} \in F$ for all $j \in \{1, \dots, n\}$. We need to show that

$$\{i \in I \mid (\xi^1(i), \dots, \xi^n(i)) \in \mathcal{J}_{i, \alpha(i)}(P)\} \in F \quad \text{iff} \quad \{i \in I \mid (\eta^1(i), \dots, \eta^n(i)) \in \mathcal{J}_{i, \beta(i)}(P)\} \in F.$$

The left-to-right direction follows from the fact that

$$\begin{aligned} & \{i \in I \mid (\xi^1(i), \dots, \xi^n(i)) \in \mathcal{J}_{i, \alpha(i)}(P)\} \cap \{i \in I \mid \alpha(i) = \beta(i)\} \\ & \cap \{i \in I \mid \xi^1(i) = \eta^1(i)\} \cap \dots \cap \{i \in I \mid \xi^n(i) = \eta^n(i)\} \subseteq \{i \in I \mid (\eta^1(i), \dots, \eta^n(i)) \in \mathcal{J}_{i, \beta(i)}(P)\}. \end{aligned}$$

The other direction can be proven analogously. (3) is similar to the proof of Lemma A.2. $\square$

Suppose ρ_i is an assignment for $\mathfrak{M}_i$, for each $i \in I$. For each $x \in \text{Var}$, define $\rho(x) : I \rightarrow \bigcup_{i \in I} D_i : i \mapsto \rho_i(x)$. Then the *product assignment* ρ_F is given by letting $\rho_F(x)$ be the equivalence class of $\rho(x)$, i.e. $\rho_F(x) := \rho(x)_F$. Conversely, every assignment ρ_F for $\prod_{i \in I}^F \mathfrak{M}_i$ can be obtained in this way from assignments ρ_i given by $\rho_i(x) = \rho_F(x)(i)$.

Theorem A.4. Let I be a set, and for each $i \in I$ let $\mathfrak{M}_i$ be a first-order structure and ρ_i an assignment for $\mathfrak{M}_i$. Let F be an ultrafilter on I , $\mathfrak{M} := \prod_{i \in I}^F \mathfrak{M}_i$ the filter product, and ρ_F the product assignment for $\mathfrak{M}$. Then for any world $\alpha_F \in W$ and any formula $\varphi(x_1, \dots, x_n)$ such that $\rho_F(x_1), \dots, \rho_F(x_n) \in D_{\alpha_F}$,

$$\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \varphi \quad \text{iff} \quad \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \varphi\} \in F.$$

Proof of Theorem 5.5. We use induction on the structure of φ .

► *Case for $\varphi = P(t_1, \dots, t_m)$.* Compute

$$\begin{aligned} \mathfrak{M}, \alpha_F \Vdash^{\rho_F} P(t_1, \dots, t_m) & \quad \text{iff} \quad (\rho_F^{\mathfrak{M}}(t_1), \dots, \rho_F^{\mathfrak{M}}(t_m)) \in \mathcal{J}_{\alpha_F}(P) & \quad (\text{definition of } \Vdash^{\rho_F}) \\ & \quad \text{iff} \quad \{i \in I \mid (\rho^{\mathfrak{M}}(t_1)(i), \dots, \rho^{\mathfrak{M}}(t_m)(i)) \in \mathcal{J}_{i, \alpha(i)}\} \in F & \quad (\text{definition of } \mathcal{J}_{i, \alpha(i)}) \\ & \quad \text{iff} \quad \{i \in I \mid (\rho_i^{\mathfrak{M}_i}(t_1), \dots, \rho_i^{\mathfrak{M}_i}(t_m)) \in \mathcal{J}_{i, \alpha(i)}\} \in F & \quad (\text{definition of } \rho_F) \\ & \quad \text{iff} \quad \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} P(t_1, \dots, t_m)\} \in F & \quad (\text{definition of } \Vdash^{\rho_i}) \end{aligned}$$

► *Case for $\varphi = (\psi \wedge \chi)$.* If $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \psi \wedge \chi$, then $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \psi$ and $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \chi$. By induction, $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi\} \in F$ and $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \chi\} \in F$, and using the fact that F is a filter we find

$$\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi\} \cap \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \chi\} \subseteq \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi \wedge \chi\} \in F.$$

Conversely, if $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi \wedge \chi\} \in F$, we may use the fact that F is a filter to conclude that

$$\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi \wedge \chi\} \subseteq \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi\} \in F,$$

and, similarly, we conclude that $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \chi\} \in F$. The induction hypothesis then entails $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \psi$ and $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \chi$, from which it follows that $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \psi \wedge \chi$.

► *Case for $\varphi = (\psi \vee \chi)$.* This is similar to the previous case, but using the fact that F is prime.

► *Case for $\varphi = (\psi \rightarrow \chi)$.* Suppose $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi \rightarrow \chi\} \in F$. We show that $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \psi \rightarrow \chi$. To this end, let $\alpha_F \leq \beta_F$ and suppose $\mathfrak{M}, \beta_F \Vdash^{\rho_F} \psi$. Then, using the induction hypothesis, we obtain

$$\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \in F \quad \text{and} \quad \{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i} \psi\} \in F.$$

Combining this with the assumption gives

$$\begin{aligned} & \{i \in I \mid \alpha(i) \leq_i \beta(i)\} \cap \{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i} \psi\} \\ & \cap \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi \rightarrow \chi\} \subseteq \{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i} \chi\} \in F, \end{aligned}$$

so by the induction hypothesis again we obtain $\mathfrak{M}, \beta_F \Vdash^{\rho_F} \chi$. Therefore $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \psi \rightarrow \chi$.

For the converse, suppose

$$\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \psi \rightarrow \chi\} \notin F$$

Since F is an ultrafilter, its complement is an element of F :

$$S := \{i \in I \mid \mathfrak{M}_i, \alpha(i) \nVdash^{\rho_i} \psi \rightarrow \chi\} \in F$$

Hence, for every $i \in S$, there exists a $b_i \in W_i$ such that $\alpha(i) \leq_i b_i$ and $\mathfrak{M}_i, b_i \Vdash^{\rho_i} \psi$ and $\mathfrak{M}_i, b_i \nVdash^{\rho_i} \chi$. Define $\beta \in \prod_{i \in I} W_i$ by

$$\beta(i) = \begin{cases} b_i & \text{if } i \in S \\ \alpha(i) & \text{if } i \notin S \end{cases}$$

Then $\beta_F \in W$ and $\alpha_F \leq_F \beta_F$ because $S \subseteq \{i \in I \mid \alpha(i) \leq_i \beta(i)\}$ and $S \in F$. Furthermore, $S \subseteq \{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i} \psi\}$ and $S \subseteq \{i \in I \mid \mathfrak{M}_i, \beta(i) \nVdash^{\rho_i} \chi\}$, so

$$\{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i} \psi\} \in F \quad \text{and} \quad \{i \in I \mid \mathfrak{M}_i, \beta(i) \nVdash^{\rho_i} \chi\} \in F,$$

hence $\{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i} \chi\} \notin F$. The induction hypothesis then yields $\mathfrak{M}, \beta_F \Vdash^{\rho_F} \psi$ and $\mathfrak{M}, \beta_F \nVdash^{\rho_F} \chi$. Since $\alpha_F \leq_F \beta_F$, this proves $\mathfrak{M}, \alpha_F \nVdash^{\rho_F} \psi \rightarrow \chi$.

► *Case for $\varphi = \forall x \psi$.* Suppose that $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \forall x \psi\} \notin F$. Since F is an ultrafilter,

$$S := \{i \in I \mid \mathfrak{M}_i, \alpha(i) \nVdash^{\rho_i} \forall x \psi\} \in F$$

and hence for every $i \in S$ there exist a world $b_i \in W_i$ and a domain element $d_i \in D_{i, b_i}$ such that $\alpha(i) \leq_i b_i$ and $\mathfrak{M}_i, b_i \nVdash^{\rho_i[x:=d_i]} \psi$. Define $\beta \in \prod_{i \in I} W_i$ by

$$\beta(i) = \begin{cases} b_i & \text{if } i \in S \\ \alpha(i) & \text{if } i \notin S \end{cases}$$

and $\xi^d \in \prod_{i \in I} D_i$ by

$$\xi^d(i) = \begin{cases} d_i & \text{if } i \in S \\ d_i^* & \text{if } i \notin S, \text{ where } d_i^* \text{ is any element of } D_{i, \beta(i)} \end{cases}$$

Then $\beta_F \in W$ and $\xi_F^d \in D_{\beta_F}$, because $S \subseteq \{i \in I \mid \xi^d(i) \in D_{i, \alpha(i)}\} \in F$. Since $S \subseteq \{i \in I \mid \alpha(i) \leq_i \beta(i)\} \in F$ we have $\alpha_F \leq_F \beta_F$. Furthermore, by construction we have

$$S \subseteq \{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i[x := \xi^d(i)]} \psi\} \in F$$

so $\{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i[x := \xi^d(i)]} \psi\} \notin F$. Using the fact that $\rho_F[x := \xi_F^d]$ is the product of the assignments $\rho_i[x := \xi^d(i)]$ and the induction hypothesis we obtain $\mathfrak{M}, \beta_F \Vdash^{\rho_F[x := \xi_F^d]} \psi$. Therefore $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \forall x \psi$.

For the converse, suppose that $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \forall x \psi$. Then there exist a world $\beta_F \in W$ and an individual $\xi_F \in D_{\beta_F}$ such that $\alpha_F \leq_F \beta_F$ and $\mathfrak{M}, \beta_F \Vdash^{\rho_F[x := \xi_F]} \psi$. This implies $\{i \in I \mid \alpha(i) \leq_i \beta(i)\} \in F$ and $\{i \in I \mid \xi(i) \in D_{i, \beta(i)}\} \in F$ and (by the induction hypothesis) $\{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i[x := \xi(i)]} \psi\} \notin F$. Since F is an ultrafilter, we get $\{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i[x := \xi(i)]} \psi\} \in F$. It follows that

$$\begin{aligned} & \{i \in I \mid \alpha(i) \leq_i \beta(i)\} \cap \{i \in I \mid \xi(i) \in D_{i, \beta(i)}\} \\ & \cap \{i \in I \mid \mathfrak{M}_i, \beta(i) \Vdash^{\rho_i[x := \xi(i)]} \psi\} \subseteq \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \forall x \psi\} \in F \end{aligned}$$

and hence $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \forall x \psi\} \notin F$.

► *Case for $\varphi = \exists x \psi$.* Suppose $\mathfrak{M}, \alpha_F \Vdash^{\rho_F} \exists x \psi$. Then there exists some individual $\xi_F \in D_{\alpha_F}$ such that $\mathfrak{M}, \alpha_F \Vdash^{\rho_F[x := \xi_F]} \psi$. By the induction hypothesis, and using the fact that $\rho_F[x := \xi_F]$ is the product of the assignments $\rho_i[x := \xi(i)]$, we obtain $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i[x := \xi(i)]} \psi\} \in F$. Clearly

$$\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i[x := \xi(i)]} \psi\} \subseteq \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \exists x \psi\},$$

so $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \exists x \psi\} \in F$.

Now suppose $S := \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i} \exists x \psi\} \in F$. Then, for every $i \in S$ there exists a $d_i \in D_{i, \alpha(i)}$ such that $\mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i[x := d_i]} \psi$. Define $\xi^d \in \prod_{i \in I} D_i$ by

$$\xi^d(i) = \begin{cases} d_i & \text{if } i \in S \\ d_i^* & \text{if } i \notin S, \text{ where } d_i^* \text{ is any element of } D_{i, \alpha(i)} \end{cases}$$

Then $\xi_F^d \in D_{\alpha_F}$ and

$$S \subseteq \{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i[x := \xi^d(i)]} \psi\},$$

so $\{i \in I \mid \mathfrak{M}_i, \alpha(i) \Vdash^{\rho_i[x := \xi^d(i)]} \psi\} \in F$ and by the induction hypothesis $\mathfrak{M}, \alpha_F \Vdash^{\rho_F[x := \xi_F^d]} \psi$. $\square$

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On Modal Logics of Connectedness in Metric Spaces

John Harding

Ilya Shapirovsky

New Mexico State University
NM, USA

For a positive number a , each metric space carries the relation D_a consisting of those pairs that are of distance less than a apart. A space X is said to be *a-connected*, if the graph (X, D_a) is connected (that is, there is a D_a -path between every pair of points in X). We give a complete axiomatization of *a-connected* metric spaces in the language with a family of distance modalities and the universal modality. Then we give a complete axiomatization of the logic of connected (in the classical topological sense) metric spaces in the language with the topological modality, universal modality, and a single distance modality. We also show that these logics have the finite model property.

1 Introduction

Let (X, d) be a metric space, which we usually refer to as simply X . There is a natural way to associate a relational structure, or Kripke frame, with X . For each real number $r > 0$ define a relation D_r on X by setting

$$xD_r y \Leftrightarrow d(x, y) < r.$$

This allows us to interpret in X modal formulas with the corresponding distance modalities $\Diamond_r$. We can also consider the topological closure operation of X , which allows us to interpret in X modal formulas with the closure modality $\Diamond$.

The situation described is the general setting of [8, 9, 15, 6, 7]. In this series of papers, numerous results were established related to the axiomatizability and decidability of modal logics of metric spaces with different families of operators. For instance, the axiomatization of the logic of all metric spaces with distance modalities $\Diamond_r$ for each $r > 0$ is given by the family of axioms for all $r, s > 0$ [15]:

- (1) $p \rightarrow \Diamond_r p$
- (2) $p \rightarrow \Box_r \Diamond_r p$
- (3) $\Diamond_r \Diamond_s p \rightarrow \Diamond_{r+s} p$

The first axiom says that for a subset P of a metric space, we have P is contained in the set of points of distance less than r from P ; the second expresses the symmetry of the metric; and the third comes from the triangle inequality.

In this paper, we are interested in modal logics of connected metric spaces. More precisely, we consider two forms of connectedness: in the usual topological sense, and in a weaker sense of connectedness of the graph (X, D_a) for some fixed positive a . We address the latter property as *a-connectedness*, which means that there is a D_a -path between every pair of points in X .

It is known that connectedness of a topological space is expressed in the language with the closure and universal modalities by the formula

$$\exists p \wedge \exists \neg p \rightarrow \exists (\Diamond p \wedge \Diamond \neg p) \quad (1)$$

and moreover, a complete axiomatization of the class of all connected spaces is the logic S4UC, the extension of S4 with the axioms of universal modality and the above formula [13]. The axiomatization of connected metric spaces in the language of the topological closure $\Diamond$, universal modality, and distance modalities is an open problem [15, 6]. We make partial progress in this direction, providing the following two completeness results.

- For a given positive a , we identify an axiomatization of a -connectedness in the language without topological closure, with the universal modality, and with any set of distance modalities $\Diamond_r$ containing $\Diamond_a$.
- We give an axiomatization of topological connectedness in the language of closure $\Diamond$, universal modality, and a single distance modality $\Diamond_r$.

We also show that these logics have the finite model property and, in the case of a finite language, are decidable.

The paper is organized as follows. Syntactic and semantic conventions are given in Section 2. In Section 3, we prove the finite model property theorem, one of the ingredients of the completeness proofs. In Section 4, we give the axiomatization of a -connected spaces, and the result for topological connectedness is given in Section 5.

2 Logics and models

2.1 Language

Let A be a set of positive real numbers (*parameters*), which will also be considered as indices of modalities. An A -*logic* is a normal modal logic with modalities $\{\Diamond_a \mid a \in A\} \cup \{\exists\}$, and an $(A, \Diamond)$ -*logic* is a logic in this language endowed with $\Diamond$, where $\exists$ and $\Diamond$ are two fresh symbols. We write $\forall$ for the abbreviation $\neg\exists\neg$. In relational or topological spaces, $\exists$ will be interpreted as “somewhere”, that is, formally, via the universal relation. In topological spaces, $\Diamond$ will be interpreted as the operation of closure.

For a logic L and a formula φ , the smallest logic containing $L \cup \{\varphi\}$ is denoted as $L + \varphi$.

2.2 Axioms

Let $\vec{r} = (r_1, \dots, r_m)$ be a tuple of parameters. We write $\Diamond_{\vec{r}}$ for the compound modality $\Diamond_{r_1} \dots \Diamond_{r_m}$, and $\oplus \vec{r}$ for $r_1 + \dots + r_m$. We put $\oplus \vec{r} = 0$ for the empty $\vec{r}$.

Definition 2.1. We define $\text{Metr}(A)$ as the smallest normal A -logic that, for each $r \in A$, contains the formulas

- (i) $p \rightarrow \forall \exists p, p \rightarrow \exists p, \exists \exists p \rightarrow \exists p$, and $\Diamond_r p \rightarrow \exists p$;
- (ii) $p \rightarrow \Box_r \Diamond_r p$;
- (iii) $\Diamond_{\vec{r}} p \rightarrow \Diamond_r p$ for each tuple $\vec{r}$ over A with $\oplus \vec{r} \leq r$.

Then set $\text{Metr}(A, \Diamond)$ to be the smallest normal $(A, \Diamond)$ -logic that contains $\text{Metr}(A)$ and the formulas

- (i) $p \rightarrow \Diamond p, \Diamond \Diamond p \rightarrow \Diamond p, \Diamond p \rightarrow \exists p$, and
- (ii) $\Diamond_r \Diamond p \rightarrow \Diamond_r p$ for each $r \in A$.

Remark 2.2. Note that $\Diamond p \rightarrow \exists p$ follows from other axioms, provided that A is non-empty. For the empty A , this logic is known as S4U, the logic of all (finite) preorders endowed with the universal relation [3, Theorem 5.9].

2.3 Semantics

In the following, we recall that our language includes a fixed set A of positive real numbers as parameters.

Definition 2.3. Let $S = (X, d)$ be a metric space. For $r \in A$, let D_r be the binary relation on X given by $(x, y) \in D_r$ iff $d(x, y) < r$. The *metric A-frame* of (X, d) is the relational structure $(X, (D_r)_{r \in A}, X \times X)$. Let $\text{Log } F_A(S)$ denote the set of all A -formulas that are valid in the metric A -frame of S . The A -*logic* of a class $\mathcal{K}$ of metric spaces is defined as $\bigcap \{\text{Log } F_A(S) \mid S \in \mathcal{K}\}$. To define the $(A, \Diamond)$ -*logic* of $\mathcal{K}$, we additionally interpret $\Diamond$ as the topological closure.

In [15], it was shown that $\text{Metr}(A, \Diamond)$ is the $(A, \Diamond)$ -logic of the class of all metric spaces.

Remark 2.4. In [15], the axiom system was formally different, since extra conditions were assumed for the set of parameters. The difference is minor.

2.4 Expressing connectedness

A unimodal frame $F = (X, R)$ is *connected*, if for any points x, y in X , there is a non-oriented R -path from x to y , that is: there are $n < \omega$ and points $x_0 = x, x_1, \dots, x_n = y$ such that for each $k < n$, $x_k R x_{k+1}$ or $x_{k+1} R x_k$. A frame $F = (X, (R_i)_I)$ is *a-connected* for $a \in I$, if (X, R_a) is connected.

The property of a -connectedness is expressed by the formula

$$\exists p \wedge \exists \neg p \rightarrow \exists((p \wedge \Diamond_a \neg p) \vee (\neg p \wedge \Diamond_a p)) \quad (2)$$

Here $\exists$ is interpreted as the modality of the universal relation $X \times X$. That is, we have:

$$(X, (R_i)_I) \text{ is } a\text{-connected iff } \text{Con}_a \text{ is valid in } (X, (R_i)_I, X \times X).$$

The proof is straightforward.

For a symmetric R_a , (2) simplifies to

$$\exists p \wedge \exists \neg p \rightarrow \exists(p \wedge \Diamond_a \neg p).$$

This formula will be denoted by Con_a .

For the case when R_a is reflexive, (2) simplifies to

$$\exists p \wedge \exists \neg p \rightarrow \exists(\Diamond_a p \wedge \Diamond_a \neg p). \quad (3)$$

If the modality is interpreted as the closure in a topological space, (3) expresses topological connectedness [13]; in this case, we denote this formula as $\text{Con}_\Diamond$.¹

For the empty A , the logic $\text{Metr}(A, \Diamond) + \text{Con}_\Diamond$ is known as S4UC. By [13, Theorem 10], this logic has the finite model property: it is characterized by the class of finite connected preorders endowed with the universal relation. Moreover, S4UC is the logic of all connected topological spaces, and in fact is characterized by any connected dense-in-itself separable metric space (i.e., any connected metric space with more than one point) [13, Theorem 18].

3 The finite model property of logics of connectedness

In this section, we consider logics in an abstract setting, and their models are not assumed to be related to any geometric spaces – they are just relational structures.

¹More ways to represent topological connectedness in propositional languages are discussed in [4, 5].

3.1 Filtration lemma

In this subsection, we consider models, formulas, and logics in a fixed modal alphabet.

For a set of formulas Γ , let $\text{Sub}\Gamma$ be the set of subformulas of formulas occurring in Γ . If $\Gamma = \text{Sub}\Gamma$, then Γ is said to be *Sub-closed*. For a model M and a set Γ of formulas, let $\sim_\Gamma$ be the equivalence on M induced by Γ :

$$x \sim_\Gamma y \quad \text{iff} \quad \forall \psi \in \Gamma (M, x \models \psi \text{ iff } M, y \models \psi).$$

We next recall the definition of the minimal and maximal filtrations of a relation.

Definition 3.1. Suppose Γ is a Sub-closed set of formulas, $M = (X, (R_a)_A, v)$ is a model, and $\sim$ is an equivalence relation that refines $\sim_\Gamma$. Set $\hat{X} = X/\sim$. For a relation R corresponding to a modality $\Diamond$, we define the *minimal filtered relation* $R_\sim$ and the *maximal filtered relation* $R_\sim^\Gamma$ on $\hat{X}$ by setting

$$\begin{aligned} [x]R_\sim[y] & \quad \text{iff} \quad \text{there exist } x' \sim x \text{ and } y' \sim y \text{ with } x'Ry' \\ [x]R_\sim^\Gamma[y] & \quad \text{iff} \quad \text{for all } \psi \text{ with } \Diamond\psi \in \Gamma, \text{ if } M, y \models \psi \text{ then } M, x \models \Diamond\psi. \end{aligned}$$

It is easy to check that $R_\sim$ is contained in $R_\sim^\Gamma$.

Definition 3.2. A *filtration of M through Γ* is a model $\hat{M} = (\hat{X}, (\hat{R}_a)_A, \hat{v})$ such that

- (i) $\hat{X} = X/\sim$ for an equivalence relation $\sim$, which refines $\sim_\Gamma$;
- (ii) $\hat{v}(p) = \{[x] \in \hat{X} \mid x \in v(p)\}$ for all variables $p \in \Gamma$;
- (iii) For each $a \in A$, $R_{a,\sim} \subseteq \hat{R}_a \subseteq R_{a,\sim}^\Gamma$.

The following fact is standard.

Lemma 3.3 (Filtration lemma). *Suppose that Γ is a Sub-closed set of formulas and $\hat{M}$ is a Γ -filtration of a model M . Then, for all points x in M and all formulas $\psi \in \Gamma$, we have:*

$$M, x \models \psi \quad \text{iff} \quad \hat{M}, [x] \models \psi.$$

Proof. Induction on ψ . □

For a relation R on X , let R^+ be its transitive closure $\bigcup_{0 < i < \omega} R^i$. The following is well known.

Lemma 3.4. ([11, Lemma 2]) *Let $M \models \Diamond_a \Diamond_a \psi \rightarrow \Diamond_a \psi$ for each formula ψ , and let $\sim$ be the equivalence induced by a Sub-closed Γ on M . Then $(R_{a,\sim})^+$ is contained in the maximal filtered relation $R_{a,\sim}^\Gamma$.*

Proof. By induction on n , we show $(R_{a,\sim})^n \subseteq R_{a,\sim}^\Gamma$ for all $n > 0$. Basis is trivial, since the a -th minimal filtered relation is contained in the a -th maximal. For the step, suppose $\Diamond_a \psi \in \Gamma$, $M, y \models \psi$, and $[x](R_{a,\sim})^{(n+1)}[y]$. Then $[x]R_{a,\sim}[z](R_{a,\sim})^n[y]$ for some z . By the hypothesis, $M, z \models \Diamond_a \psi$. We have $x'R_a z'$ for some $x' \sim x$ and $z' \sim z$, and since $\Diamond_a \psi \in \Gamma$, $M, z' \models \Diamond_a \psi$. Thus, $M, x' \models \Diamond_a \Diamond_a \psi$. The formula $\Diamond_a \Diamond_a \psi \rightarrow \Diamond_a \psi$ is true in M , hence $M, x' \models \Diamond_a \psi$. So $M, x \models \Diamond_a \psi$, as required. □

3.2 Metric closure operations on frames

The formula $\Diamond_c \Diamond p \rightarrow \Diamond_c p$ expresses the property $R_c \circ R \subseteq R_c$. The following construction will be used to build filtrations with this property. Let R, D be relations on X . We define recursively a relation S called *the R-closure of D* as follows:

$$S^{(0)} = D \cup D^{-1}, \quad (4)$$

$$S^{(n+1)} = (S^{(n)} \circ R) \cup (S^{(n)} \circ R)^{-1}, \quad (5)$$

$$S = \bigcup_{\omega} S^{(n)}. \quad (6)$$

The relation $S^{(n)}$ is called the *n-th grade* of the closure S .

Lemma 3.5. *S is symmetric and $S \circ R \subseteq S$.*

Proof. Each $S^{(n)}$ is symmetric, so is S . Assume $(x, y) \in S \circ R$. Then $(x, y) \in S^{(n)} \circ R$ for some n , and so $(x, y) \in S^{(n+1)} \subseteq S$. $\square$

For a non-empty tuple $\vec{r} = (r_1, \dots, r_m)$ of parameters and a frame $(X, (S_r)_A)$, put $S_{\vec{r}} = S_{r_1} \circ \dots \circ S_{r_m}$. For the empty $\vec{r}$, define $S_{\vec{r}}$ as the diagonal $\{(x, x) \mid x \in X\}$.

Definition 3.6. The *A-closure of a frame $(X, (S_r)_A)$* is the frame $(X, (H_r)_A)$ such that

$$H_r = \bigcup \{S_{\vec{r}} \mid \vec{r} \text{ is a tuple of parameters with } \oplus \vec{r} \leq r\}.$$

Lemma 3.7. *The A-closure $(X, (H_r)_A)$ of any frame $(X, (S_r)_A)$ validates the formulas $\Diamond_{\vec{r}} p \rightarrow \Diamond_r p$ for each tuple $\vec{r}$ over A and $r \in A$ with $\oplus \vec{r} \leq r$.*

Proof. Let $(x, y) \in H_r$ for $\vec{r} = (r_1, \dots, r_m)$. Then we have $(x, y) \in S_{\vec{s}_1} \circ \dots \circ S_{\vec{s}_m}$ for some $\vec{s}_i$ with $\oplus \vec{s}_i \leq r_i$. Then $(x, y) \in S_{\vec{s}}$, where $\vec{s}$ is the concatenation $\vec{s}_1 \vec{s}_2 \dots \vec{s}_m$. Hence $\oplus \vec{s} \leq r$. By the definition of H_r , we have $(x, y) \in H_r$, as desired. $\square$

Definition 3.8. The *(A, $\Diamond$)-closure of a frame $F = (X, R, (D_r)_A)$* is the frame $G = (X, \preceq, (H_r)_A, X \times X)$ defined as follows:

- (i) $\preceq$ is the transitive closure of R ;
- (ii) $(X, (H_r)_A)$ is the A-closure of $(X, (S_r)_A)$, where S_r is the $\preceq$ -closure of D_r .

Lemma 3.9. *For a frame $F = (X, R, (D_r)_A)$ with all relations reflexive, its (A, $\Diamond$)-closure validates the logic $\text{Metr}(A, \Diamond)$.*

Proof. Let $G = (X, \preceq, (H_r)_A, X \times X)$ be this closure. The axioms for the modality $\exists$ hold trivially. Since R is reflexive, $\preceq$ is a preorder. That the $\Diamond_{\vec{r}} p \rightarrow \Diamond_r p$ are valid follows from Lemma 3.7. All S_r are symmetric by Lemma 3.5, hence each H_r is the union of compositions of symmetric relations, and so is symmetric as well.

Let us check the validity of $\Diamond_r \Diamond p \rightarrow \Diamond_r p$. For this, we show that $H_r \circ \preceq \subseteq H_r$. In turn, it is enough to show that for a tuple of parameters $\vec{r}$ with $\oplus \vec{r} \leq r$ we have $S_{\vec{r}} \circ \preceq \subseteq H_r$. For any $t \in A$ we have $S_t \circ \preceq$ included in S_t by Lemma 3.5, and S_t included in $S_t \circ \preceq$ due to reflexivity of $\preceq$. So we have for any non-empty $\vec{r} = \vec{s}t$:

$$S_{\vec{r}} \circ \preceq = (S_{\vec{s}} \circ S_t) \circ \preceq = S_{\vec{s}} \circ (S_t \circ \preceq) = S_{\vec{s}} \circ S_t = S_{\vec{r}}.$$

Thus when $\vec{r}$ is non-empty, $S_{\vec{r}} \circ \preceq \subseteq H_r$. When $\vec{r}$ is empty, $S_{\vec{r}}$ is by definition the identity relation, so we must show $\preceq \subseteq H_r$. Since D_r is reflexive, so is S_r , hence $\preceq \subseteq S_r \circ \preceq$. Thus by Lemma 3.5 used again, $\preceq \subseteq S_r$, and clearly $S_r \subseteq H_r$. $\square$

3.3 The filtration

The following is a corollary of [13, Theorem 10] and [12, Lemma 2].

Lemma 3.10. *Consider a model $M = (X, (R_i)_I, X \times X, v)$ and let $\widehat{X} = X / \sim_\Delta$ for a finite Sub-closed set of formulas Δ . Let $a \in I$, and assume that a relation $\widehat{R}$ on $\widehat{X}$ includes the minimal filtered relation $R_{a, \sim_\Delta}$. Assume $M \models L$ for an A-logic L , and L contains the formula $\exists p \wedge \exists \neg p \rightarrow \exists(\Diamond_a p \wedge \Diamond_a \neg p)$ or the formula $\exists p \wedge \exists \neg p \rightarrow \exists(p \wedge \Diamond_a \neg p)$. Then the frame $(\widehat{X}, \widehat{R})$ is connected.*

Proof. Assume that $\widehat{X} = \widehat{Y} \cup \widehat{Z}$ for disjoint non-empty $\widehat{Y}$ and $\widehat{Z}$, $Y = \bigcup \widehat{Y}$, $Z = \bigcup \widehat{Z}$. We aim to show that

$$\exists y \in Y \exists z \in Z ([y] \widehat{R} [z] \text{ or } [z] \widehat{R} [y]). \quad (7)$$

Since $\widehat{Y}$ consists of $\sim_\Delta$ -classes, for some Boolean combination γ of formulas in Δ we have for all $y \in X$:

$$M, y \models \gamma \quad \text{iff} \quad y \in Y.$$

Assume that L contains the formula $\exists p \wedge \exists \neg p \rightarrow \exists(\Diamond_a p \wedge \Diamond_a \neg p)$. Then it also contains its substitution instance $\exists \gamma \wedge \exists \neg \gamma \rightarrow \exists(\Diamond_a \gamma \wedge \Diamond_a \neg \gamma)$. Since both Y and Z are non-empty, there are points x_1, x_2 , and x_3 such that $x_1 R x_2$, $x_1 R x_3$, $M, x_2 \models \gamma$, and $M, x_3 \models \neg \gamma$. Hence, $x_2 \in Y$ and $x_3 \in Z$. If $x_1 \in Y$, put $y = x_1$ and $z = x_3$; otherwise, put $y = x_2$ and $z = x_1$. In either case, we have (7) according to the definition of minimal filtration.

Now assume that L contains $\exists p \wedge \exists \neg p \rightarrow \exists(p \wedge \Diamond_a \neg p)$. Similar reasoning shows that there are points y, z such that $y R z$, $M, y \models \gamma$, and $M, z \models \neg \gamma$, and so $y \in Y$ and $z \in Z$.

Hence, we have (7), and so $(\widehat{X}, \widehat{R})$ is connected. $\square$

Lemma 3.11. *Assume A is finite. Let $L = \text{Metr}(A) + \text{Con}_a$ for some $a \in A$, or $L = \text{Metr}(A, \Diamond) + \text{Con}_a$ for some $a \in A \cup \{\Diamond\}$. Then L has the finite model property.*

Proof. The proof for the case when $\Diamond$ is in the language is more general, so we consider this case.

In the case $A = \emptyset$, L is the logic S4UC, whose fmp is known [13]. Assume $A \neq \emptyset$.

The proof is technical, and we first outline the strategy. Assume that φ is L -consistent. To establish the result, we find a finite model of L in which φ is satisfiable. (A) Begin with a certain model M of L in which φ is satisfiable. (B) Take $\Gamma = \text{Sub } \varphi$ and Δ to be a certain finite set of formulas constructed from Γ . Then form the minimal filtration of M over $\sim_\Delta$. (C) Take the $(A, \Diamond)$ -closure of this filtration $\widehat{F}$. By Lemma 3.9, $\widehat{F}$ validates $\text{Metr}(A, \Diamond)$. The details of our setup allow us to use Lemma 3.10 to obtain that $\widehat{F}$ validates Con_a . Since Δ is finite, the minimal filtration is finite, hence $\widehat{F}$ is finite. (D) Define a valuation to construct a model $\widehat{M}$ from $\widehat{F}$. Then show $\widehat{M}$ is a filtration of M through Γ , hence by the filtration lemma φ is satisfiable in $\widehat{F}$.

Step (A) Assume φ is L -consistent. Then $\varphi \in x_0$ for some x_0 in the canonical model of L . Let $M = (X, R, (D_r)_A, R_\exists, v)$ be the submodel of the canonical model of L generated by x_0 ; here R interprets $\Diamond$, $R_\exists$ interprets the modality $\exists$. The logic $\text{Metr}(A, \Diamond)$ is canonical since its axioms are Sahlquist formulas. Hence, we have: R is a preorder, all D_r are symmetric and reflexive, and for all $r \in A$

$$D_r \circ R \subseteq D_r. \quad (8)$$

Since M is point-generated, we have $R_\exists = X \times X$.

Step (B) Let $\Gamma = \text{Sub } \varphi$ and let V be the set of tuples $\vec{r}$ of parameters with $\oplus \vec{r} \leq r$ for some $r \in A$. Define

$$\Delta = \{ \Diamond_{\vec{r}} \psi, \Diamond \Diamond_{\vec{r}} \psi \mid \vec{r} \in V, \psi \in \Gamma \}.$$

Note that the empty string $\vec{r}$ has $\oplus \vec{r} = 0$, so belongs to V , and in this case $\Diamond_{\vec{r}} \psi$ is ψ for any formula. Thus $\Gamma \subseteq \Delta$. Put $\hat{X} = X/\sim$ for the equivalence $\sim = \sim_\Delta$ induced by Δ . Then $\sim$ refines $\sim_\Gamma$. Let $R_\sim$ be the minimal filtered relation induced by R on $\hat{X}$, and let E_r denote the minimal filtered relation $D_{r,\sim}$.

Step (C) Let $\hat{F} = (\hat{X}, \preceq, (H_r)_A, \hat{X} \times \hat{X})$ be the $(A, \Diamond)$ -closure of $(\hat{X}, R_\sim, (E_r)_A)$. Since R and the D_r are reflexive, so are their minimal filtrations $R_\sim$ and the E_r , and so by Lemma 3.9, $\hat{F}$ validates $\text{Metr}(A, \Diamond)$. Since L contains Con_a , $\hat{F}$ validates Con_a by Lemma 3.10. Hence, $\hat{F}$ validates L .

Step (D) Define $\hat{M} = (\hat{F}, \hat{v})$, where $\hat{v}(p) = \{[x] \in \hat{X} \mid x \in v(p)\}$ for variables $p \in \Gamma$. Our aim is to show the following:

$$\hat{M} = (\hat{X}, \preceq, (H_r)_A, \hat{X} \times \hat{X}, \hat{v}) \text{ is a filtration of } M = (X, R, (D_r)_A, R_\exists, v) \text{ through } \Gamma.$$

By definition, $\hat{X} = X/\sim$, and as noted in Step (B), $\sim$ refines $\sim_\Gamma$. The valuation $\hat{v}$ is defined to match the criterion set in Definition 3.2. It remains to show that each relation of $\hat{M}$ lies between the minimal and maximal filtered relations. Trivially, $\hat{X} \times \hat{X}$ is the minimal filtered relation of $R_\exists = X \times X$. By definition of the $(A, \Diamond)$ -closure, $\preceq$ is the transitive closure of $R_\sim$. So clearly $\preceq$ contains the minimal filtered relation $R_\sim$. Since L contains $\Diamond\Diamond p \rightarrow \Diamond p$ and $M \models L$, we have $M \models \Diamond\Diamond \psi \rightarrow \Diamond \psi$ for any formula ψ , so by Lemma 3.4,

$$\preceq \text{ is contained in } R_\sim^{(\Delta)}.$$

But $\Gamma \subseteq \Delta$ gives $R_\sim^\Delta \subseteq R_\sim^\Gamma$.

It remains to show that for each $r \in A$, the relation H_r lies between the minimal and maximal $\sim$ -filtrations of D_r . Recall that $D_{r,\sim}$ is written E_r and H_r is constructed by first taking the $\preceq$ -closure S_r of E_r as given before Lemma 3.5, then obtaining H_r from the family $(S_r)_A$ as in Definition 3.6. Note that S_r contains $D_{r,\sim}$ and $S_r \subseteq H_r$ is immediate from the definition of H_r . Thus H_r contains the minimal $\sim$ -filtration of D_r . To show that it is contained in the maximal filtration $D_{r,\sim}^\Gamma$ we must show

$$\text{If } \Diamond_r \psi \in \Gamma, [x] H_r [y], \text{ and } M, y \models \psi, \text{ then } M, x \models \Diamond_r \psi. \quad (9)$$

We establish this through a series of results.

We first claim that for $[x] \preceq [y]$:

$$\text{if } \Diamond \psi \in \Delta \text{ and } M, y \models \Diamond \psi, \text{ then } M, x \models \Diamond \psi; \quad (10)$$

$$\text{if } \Diamond_r \psi \in \Delta \text{ and } M, x \models \Diamond_r \psi, \text{ then } M, y \models \Diamond_r \psi. \quad (11)$$

We have (10) as a corollary of the fact that $\preceq \subseteq R_\sim^\Delta$. Indeed, if $M, y \models \Diamond \psi$, then $M, z \models \psi$ for some z with $y R z$. Now $[x] \preceq [z]$, so $[x] R_\sim^\Delta [z]$, giving $M, x \models \Diamond \psi$.

Let us check (11). By induction on n , we show that (11) holds for $[x](R_\sim)^n[y]$ for all $n \geq 0$. The basis $n = 0$ is trivial, since $x \sim y$ in this case, meaning that $M, x \models \psi$ iff $M, y \models \psi$ for every formula ψ in Δ and we assumed $\Diamond_r \psi \in \Delta$. Let $n > 0$. We have $[x] R_\sim [z](R_\sim)^{n-1}[y]$ for some z , and $x' R z'$ for some $x' \sim x$ and $z' \sim z$. Then $M, x' \models \Diamond_r \psi$, and we have $M, u \models \psi$ for some u with $x' D_r u$. Since D_r is symmetric, we have $u D_r x'$, and so $u D_r \circ R z'$. So $u D_r z'$ by (8). Hence, $z' D_r u$, and so $M, z' \models \Diamond_r \psi$. We have $[z'](R_\sim)^{n-1}[y]$, and $M, y \models \Diamond_r \psi$ now follows from the induction hypothesis. This completes the proof of (11).

Now by induction on n , we show that for the n -grade $S_r^{(n)}$ of S_r , we have

$$\text{if } \Diamond_r \psi \in \Delta, [x] S_r^{(n)} [y], \text{ and } M, y \models \Diamond \psi, \text{ then } M, x \models \Diamond_r \psi. \quad (12)$$

First, observe that $\Diamond_r \psi \in \Delta$ gives $\Diamond \psi \in \Delta$. Indeed, if $\Diamond_r \psi \in \Delta$, then $\Diamond_r \psi$ has the form $\Diamond_r \Diamond_{\vec{s}} \chi$ for some $\chi \in \Gamma$ and $\vec{s} \in V$. Then $\Diamond \Diamond_{\vec{s}} \chi \in \Delta$.

Let $[x]S_r^{(0)}[y]$. We have $S_r^{(0)} = E_r \cup E_r^{-1}$. Since E_r is symmetric, $S^{(0)} = E_r = D_{r, \sim}$. The assumptions of (12) give some $x' \sim x$, $y' \sim y$ with $x'D_r y'$, and since $\Diamond\psi \in \Delta$, we have $M, y' \models \Diamond\psi$. Hence, $M, x' \models \Diamond_r \Diamond\psi$, and so $M, x' \models \Diamond_r \psi$, since $\Diamond_r \Diamond\psi \rightarrow \Diamond_r \psi$ is true in M . Thus, $M, x \models \Diamond_r \psi$.

Now let $[x]S_r^{(n+1)}[y]$ and $M, y \models \Diamond\psi$. Consider two cases. First, assume that $[x]S_r^{(n)} \circ \preceq [y]$. Then we have that $[x]S_r^{(n)}[z] \preceq [y]$ for some $[z]$. So by (10) we have $M, z \models \Diamond\psi$. By induction hypothesis, we have $M, x \models \Diamond_r \psi$. Now assume that $[x](S_r^{(n)} \circ \preceq)^{-1}[y]$, that is $[y]S_r^{(n)}[z] \preceq [x]$ for some z . Since $S_r^{(n)}$ is symmetric, we have $[z]S_r^{(n)}[y]$, and so $M, z \models \Diamond_r \psi$ by the induction hypothesis. Now (11) implies that $M, x \models \Diamond_r \psi$. This completes the proof of (12).

Using (12), now we show:

$$\text{If } \Diamond_r \psi \in \Delta, [x]S_r[y], \text{ and } M, y \models \psi, \text{ then } M, x \models \Diamond_r \psi. \quad (13)$$

Since $M, y \models \psi$ and R is reflexive, $M, y \models \Diamond\psi$. We have $[x]S_r^{(n)}[y]$ for some n , and by (12), $M, x \models \Diamond_r \psi$, as desired.

Now we claim that for $\vec{r} \in V$, we have:

$$\text{If } \psi \in \Gamma, [x]S_{\vec{r}}[y], \text{ and } M, y \models \psi, \text{ then } M, x \models \Diamond_{\vec{r}} \psi. \quad (14)$$

By induction on the length of $\vec{r}$. If $\vec{r}$ is empty, $S_{\vec{r}}$ is the diagonal, so $[x] = [y]$; also, $\Diamond_{\vec{r}} \psi$ is just ψ ; now (14) holds because $\psi \in \Gamma \subseteq \Delta$. For the inductive step, suppose that $\vec{r}$ is the concatenation $r\vec{s}$ for a parameter r and a tuple of parameters $\vec{s}$. We have $[x]S_r[z]S_{\vec{s}}[y]$ for some z . Notice that $\vec{s} \in V$, so by the induction hypothesis, we have $M, z \models \Diamond_{\vec{s}} \psi$. We have $\Diamond_r \Diamond_{\vec{s}} \psi \in \Delta$, and by (13), we get $M, x \models \Diamond_{\vec{r}} \psi$. This completes the proof of (14).

To establish (9), assume that $\Diamond_r \psi \in \Gamma$, $[x]H_r[y]$, and $M, y \models \psi$. We have $[x]S_{\vec{r}}[y]$ for some tuple of parameters $\vec{r}$ with $\oplus \vec{r} \leq r$. Also, since Γ is Sub-closed, $\psi \in \Gamma$. By (14), $M, x \models \Diamond_{\vec{r}} \psi$. We have $\oplus \vec{r} \leq r$, so the formula $\Diamond_{\vec{r}} p \rightarrow \Diamond_r p$ is in L . So $\Diamond_{\vec{r}} \psi \rightarrow \Diamond_r \psi$ is true in M . Thus, $M, x \models \Diamond_r \psi$, and establishing (9).

We have established that $\widehat{M}$ is a filtration of M through Γ . It then follows by the filtration lemma that φ is satisfiable in $\widehat{F}$, completing the proof. $\square$

4 The logic of graph-connected spaces

Theorem 4.1. *Let A be a set of positive real numbers, $a \in A$. The A -logic of the class of all a -connected metric spaces, and also of all finite a -connected metric spaces, is $\text{Metr}(A) + \text{Con}_a$.*

The proof of this theorem is based on two ingredients. The first one is Kripke completeness of the logic, which follows from the finite model property established earlier. The second ingredient is the following lemma, which in fact follows from [9, 7].

Lemma 4.2 (Corollary of [9, 7]). *Assume that A is finite. Let $F = (X, (R_l)_{l \in A}, X \times X)$ be a $\text{Metr}(A)$ -frame, and assume that the frame $(X, (R_l)_{l \in A})$ is point-generated. Then there exists a metric d on X such that F is the A -metric frame of (X, d) , and $\inf\{d(a, b) \mid a \neq b \text{ and } a, b \in X\} > 0$.*

This statement is only a slight modification of known facts; for the sake of rigor, we provide its proof in the Appendix.

Proof of theorem 4.1. Soundness is straightforward.

To prove the other inclusion, assume that a formula φ is consistent with the logic $\text{Metr}(A) + \text{Con}_a$. Let B be the set consisting of a and the parameters occurring in φ , and let $L = \text{Metr}(B) + \text{Con}_a$. Then

φ is consistent with the logic L . Due to Lemma 3.11, L has the finite model property and so is Kripke complete. Hence, φ is satisfiable in a (finite) Kripke frame $F = (X, (R_l)_{l \in B}, X \times X)$, which validates L . Due to a -connectedness, the frame $(X, (R_l)_{l \in B})$ is point-generated. By Lemma 4.2, F is the B-metric frame of a metric space (X, d) . Clearly, this space is a -connected. Now consider the expansion $F_A(X, d)$ of F . The formula φ is satisfiable in $F_A(X, d)$, which completes the proof. $\square$

Remark 4.3. Theorem 4.1 generalizes the finite model property of the logic of a -connectedness (Lemma 3.11) for the case of infinite alphabets: indeed, finite metric spaces correspond to finite Kripke frames.

Remark 4.4. Theorem 4.1 does not apply (at least, directly) to the case of topological connectedness due to the following observation. If the closure modality is interpreted via a preorder in a Kripke frame, the corresponding topological space need not even be T_1 . And hence, this topological closure is not induced by any metric. This motivates our next section.

5 The logic of connected spaces

In this section, we consider the language with the topological modality, universal modality, and a single distance modality. Our aim is to axiomatize the logic of all connected metric spaces in this language.

In the next subsection, we list the axioms of the logic, which we denote $S4UC_{<1}$. We then discuss three preliminary constructions – relational and topological (introduced earlier in [13]), as well as metric; they are given in subsections 5.2, 5.3, and 5.4. Then, for a given $S4UC_{<1}$ -consistent formula φ , we construct a connected space where φ is satisfiable. This part of the proof is given in the two final subsections: we first construct a certain connected subspace X of $\mathbb{R}^3$; then, preserving its connectedness, we alter its metric to ensure satisfiability of φ .

5.1 The logic

We assume that the alphabet A of distance modalities is a singleton. Without loss of generality, we assume that $A = \{1\}$. Let

$$S4UC_{<1} = \text{Metr}(\{1\}, \Diamond) + \text{Con}_\Diamond.$$

According to Definition 2.1, the explicit set of axioms defining $S4UC_{<1}$ is the following:

$$p \rightarrow \forall \exists p, p \rightarrow \exists p, \exists \exists p \rightarrow \exists p, \text{ and } \Diamond_1 p \rightarrow \exists p; \quad (15)$$

$$p \rightarrow \Box_1 \Diamond_1 p; \quad (16)$$

$$p \rightarrow \Diamond_1 p; \quad (17)$$

$$p \rightarrow \Diamond p, \Diamond \Diamond p \rightarrow \Diamond p; \quad (18)$$

$$\Diamond_1 \Diamond p \rightarrow \Diamond_1 p; \quad (19)$$

$$\exists p \wedge \exists \neg p \rightarrow \exists (\Diamond p \wedge \Diamond \neg p), \text{ that is } \text{Con}_\Diamond. \quad (20)$$

As we mentioned earlier, the counterpart of $S4UC_{<1}$ for the empty A is the logic $S4UC$.

5.2 Quasiparks and suitable frames

As usual, a *preorder* $(W, \preceq)$ is a set with a transitive reflexive relation. We use the following notation: for $a \in W$ set $\uparrow a = \{b \mid a \preceq b\}$ and for $U \subseteq W$, set $\uparrow U = \bigcup_{a \in U} \uparrow a$. Likewise for downsets $\downarrow a$ and $\downarrow U$. A rooted poset $(P, \leq)$ is a (*transitive*) *tree*, if $\downarrow a$ is a finite chain for all a in P . A preorder $(W, \preceq)$ is a *quasitree*, if its skeleton (the associated partial order) is a tree.

Proposition 5.1. *Let $(W, \preceq, S, W \times W)$ be an $S4UC_{<1}$ -frame. We have:*

1. $\preceq$ is included in S ;
2. For $a \in W$, $\uparrow a$ is an S -clique, that is $\uparrow a \times \uparrow a \subseteq S$;
3. If $a \preceq a'$, $b \preceq b'$, and aSb , then $a'Sb'$.

Proof. Axioms (16) – (19) give that $\preceq$ is a quasi-order, S is reflexive and symmetric, and $S \circ \preceq \subseteq S$. So clearly we have (1) $\preceq \subseteq S$. For (2) suppose $a \preceq b, c$. Then $bSa \preceq c$, so $(b, c) \in S \circ \preceq \subseteq S$. For (3) assume $a \preceq a'$, $b \preceq b'$ and aSb . Then $bSa \preceq a'$, so bSa' , and then $a'Sb$. Then $a'Sb \preceq b'$ gives $a'Sb'$. $\square$

Definition 5.2. [13] In a poset $(P, \leq)$, let $\prec$ denote the immediate successor relation: $x \prec y$ iff $x < y$ but $x < z < y$ for no z . A *non-trivial loop* in $(P, \leq)$ is a non-oriented $\prec$ -path $(x_1, \dots, x_n, x_1)$ (in other words, with $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$ for each $i \leq n$) with $n \geq 3$ and all $x_1, \dots, x_n$ distinct.

A finite preorder $(W, \preceq)$ is called a *quasipark*, if:

- (i) the skeleton of $(W, \preceq)$ has no non-trivial loops, and
- (ii) one can order the minimal clusters of $(W, \preceq)$ as $C_1, \dots, C_n$ so that

$$\uparrow C_i \cap \uparrow C_j \neq \emptyset \text{ iff } |i - j| \leq 1. \quad (21)$$

The following is straightforward from the definition.

Proposition 5.3. *In a quasipark $Q = (W, \preceq)$, for each $a, b \in W$ we have:*

1. The restriction of Q to $\uparrow a$ is a quasitree;
2. If $\uparrow a \cap \uparrow b$ is non-empty, then the restriction of Q to this set is a quasitree.

Definition 5.4. An $S4UC_{<1}$ -frame $(W, \preceq, S, W \times W)$ is said to be *suitable*, if $(W, \preceq)$ is a quasipark.

It is known that $S4UC$ is characterized by quasiparks (endowed with the universal relation) [13, Theorem 14]. Together with the finite model property of $S4UC_{<1}$, this results in the following fact.

Corollary 5.5. $S4UC_{<1}$ is the logic of suitable frames.

Proof. According to the finite model property of $S4UC_{<1}$ (Lemma 3.11), it is enough to show that any finite $S4UC_{<1}$ -frame $F = (X, \sqsubseteq, T, X \times X)$ is a p-morphic image of a suitable G .

In view of [13, Lemma 13], there is a quasipark $(W, \preceq)$ and a p-morphism $f : (W, \preceq) \twoheadrightarrow (X, \sqsubseteq)$. Set $S = \{(a, b) \in W \times W \mid f(a)Tf(b)\}$, $G = (W, \preceq, S, W \times W)$. It is immediate that $f : G \twoheadrightarrow F$. Also, it is straightforward that G is an $S4UC_{<1}$ -frame. In particular, the formula $\Diamond_1 \Diamond p \rightarrow \Diamond_1 p$ holds there. For this, assume that we have $aSb \preceq c$. Then $f(a)Tf(b) \sqsubseteq f(c)$, and hence $f(a)Tf(c)$. By the definition of S , aSc . $\square$

5.3 Morphisms from topological spaces to preorders

In a topological space (X, τ) , for $Y \subseteq X$, let cY denote the closure of Y .

Definition 5.6. [13] Let (X, τ) be a topological space, $(W, \preceq)$ a preorder. A surjective $f : X \rightarrow W$ is called a *cp-morphism*, if for each $a \in W$ we have

$$cf^{-1}(a) = f^{-1}[\downarrow a]$$

(equivalently, f is an interior map from (X, τ) onto $(W, \tau_{\preceq})$, where $\tau_{\preceq}$ is the topology of upsets in $(W, \preceq)$.)

The next statement is a corollary of [10]; in explicit form, it is given in [13, Lemma 16].

Lemma 5.7. *If X is a connected dense-in-itself separable metric space and F is a finite quasitree, then $f : X \twoheadrightarrow F$ for a cp-morphism f .*

5.4 Jumps

Here we introduce a notion of jumps – a geometric tool to produce from a metric space (X, d) a new metric space (X, d_J) that is topologically equivalent, but reduces distances by making “wormholes” in space.

Definition 5.8. Let (X, d) be a metric space, $E \subseteq X \times X$, and $J : E \rightarrow \mathbb{R}_{>0}$ a map that is symmetric in that if $(x, y) \in E$, then $(y, x) \in E$ and $J(x, y) = J(y, x)$. Call $J(x, y)$ the *jump between x and y* . For $x, y \in X$, define the *weight of (x, y)* : if $(x, y) \in E$ and $J(x, y) < d(x, y)$, put $w(x, y) = J(x, y)$; otherwise, put $w(x, y) = d(x, y)$. A *path in X from x to y* is a finite non-empty tuple $\tau = (x_0, \dots, x_m)$ of elements of X with $x_0 = x$ and $x_m = y$. Put $w(\tau) = \sum_{i < m} w(x_i, x_{i+1})$. For $x, y \in X$, define

$$d_J(x, y) = \inf\{w(\tau) \mid \tau \text{ is a path from } x \text{ to } y\}. \quad (22)$$

Lemma 5.9. *If $\inf\{J(x, y) \mid (x, y) \in E\} > 0$, then d_J is a metric on X and the topologies on X induced by d and d_J are equal.*

Proof. It is easy to see that d_J is a metric on X ; in particular, the triangle inequality is straightforward from (22). Also, open balls of radii less than this infimum form bases of both these topologies. $\square$

5.5 Constructing a connected space for φ

Assume that a formula φ is $S4UC_{<1}$ -satisfiable. By Corollary 5.5, φ is satisfiable in a suitable frame

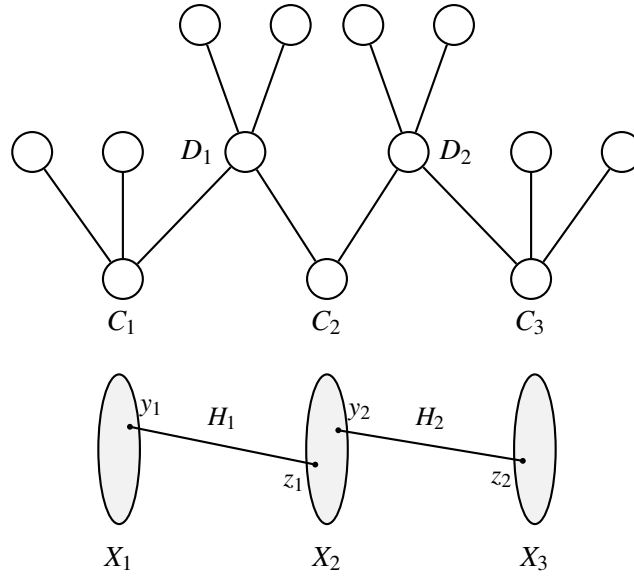
$$F = (W, \preceq, S, W \times W).$$

Our goal is to show that φ is satisfiable in a connected metric space.

First, we name parts of F . Let C_i , $1 \leq i \leq n$ be the minimal clusters of W in the order satisfying (21). We denote $\uparrow C_i$ as W_i , and the restriction of $(W, \preceq)$ to W_i as F_i . Put $V_i = W_i \cap W_{i+1}$ (here $1 \leq i < n$). By Proposition 5.3, each V_i has a least cluster D_i . Let d_i be some point in D_i . Finally, let Q_i be the restriction of $(W, \preceq)$ to $\uparrow D_i$.

Now we construct a subspace X of $\mathbb{R}^3$.

Fix a number $L > 2$. Let $1 \leq i \leq n$. In the plane $x = Li$, consider a closed disk X_i centered at $(Li, 0, 0)$ of radius $\frac{1}{L}$, and let τ_i be the standard topology on X_i . Fix a cp-morphism $f_i : (X_i, \tau_i) \twoheadrightarrow F_i$; the existence of f_i follows from Lemma 5.7.

Figure 1: A quasipark $(W, \preceq)$ and its preimage X

Choose y_i in the f_i -preimage of d_i , and z_i in the f_{i+1} -preimage of d_i . Define the i -th handle H_i as the open interval between y_i and z_i on the line through these two points; the standard topology on H_i is denoted as ρ_i . Fix a cp-morphism $g_i : (H_i, \rho_i) \rightarrow Q_i$ (we use [13, Lemma 16] again).

Let X be the (disjoint, in fact) union of these disks and handles, τ the standard topology on X . Observe that (X, τ) is connected. Now let $f = f_1 \cup g_1 \cup f_2 \cup \dots \cup g_{n-1} \cup f_n$.

Proposition 5.10. $f : (X, \tau) \rightarrow (W, \preceq)$ is a cp-morphism.

Proof. Clearly f is onto. Note that any set closed in X_i is closed in X and any set closed in H_i and containing its endpoints is closed in X . So for any $a \in W$ we have that $f^{-1}[\downarrow a]$ is a finite union of closed sets of X , so is closed in X . Thus $\text{cf}^{-1}(a) \subseteq f^{-1}[\downarrow a]$. Conversely, suppose $x \in f^{-1}[\downarrow a]$ and consider two cases. Case (i) $x \in X_i$. Then $f(x) \in W_i$, and since W_i is an upset, $a \in W_i$. Then x is in the closure of $f_i^{-1}(a)$ in X_i , hence it is in the closure of $f^{-1}(a)$ in X . Case (ii) $x \in H_i$. Then $a \in Q_i$, and x is in the closure of $g_i^{-1}(a)$ in H_i , hence in the closure of $f^{-1}(a)$ in X . $\square$

We say that two points x, y in X are *neighbours*, if $d(x, y) < L$.

Proposition 5.11. Let x, y be neighbours. Then:

1. $\downarrow f(x) \cap \downarrow f(y) \neq \emptyset$;
2. $f(x) S f(y)$;
3. $\text{cf}^{-1}(f(x)) \cap \text{cf}^{-1}(f(y)) \neq \emptyset$.

Proof. Due to the construction, a pair of adjacent handles, as well as a pair of adjacent handle and a disk, is mapped to a quasitree. Hence, if $d(x, y) < L$, then $f(x)$ and $f(y)$ belong to the same quasitree F_i .

That $f(x) S f(y)$ follows from the fact that F_i is an S -clique by Proposition 5.1.

The third statement follows from the first, since f is a cp-morphism. $\square$

5.6 Altering the metric

Let d denote the Euclidean distance in $\mathbb{R}^3$. Our goal is to define jumps J in X in such a way that φ is satisfiable in (X, d_J) .

Let $B(x, \delta)$ denote the open ball in X centered at x of radius δ (w.r.t. the Euclidean distance).

Let $x \in X$. We put

$$\delta(x) = \sup\{\delta < \frac{1}{L} \mid B(x, \delta) \subseteq f^{-1}[\uparrow f(x)]\}. \quad (23)$$

Since $f^{-1}[\uparrow f(x)]$ is open, $\delta(x) > 0$.

For $b \in W$, we define y_b and $\delta(b)$. Let i be the least such that $b \in W_i$. Take y_b in $f_i^{-1}(b)$, which is not a limit of a handle (notice that if $y \in f_i^{-1}(b)$ is a limit of a handle, then $b = d_i$ or $b = d_{i-1}$; in both cases, $\downarrow b \cap W_i$ contains the minimal points in W_i , which are distinct from b ; we have $cf_i^{-1}(b) = f_i^{-1}[\downarrow b \cap W_i] \neq f_i^{-1}(b)$, so $f_i^{-1}(b)$ is infinite.)

Let $\delta(b)$ be the supremum of δ such that $\delta < \delta(y_b)$ and $B(y_b, \delta)$ does not contain limits of handles. Let $\delta_F = \frac{1}{2} \min\{\delta(b) \mid b \in W\}$. So we have

$$f[B(y_b, \delta_F)] \subseteq \uparrow b \quad (24)$$

Also, notice that $B(y_b, \delta_F)$ and $B(y_c, \delta_F)$ are disjoint for distinct $b, c \in W$.

Assume that for $x \in X$ we have $f(x)Sb$. If $d(x, y_b) \geq 1$, then we add the jump

$$J(x, y_b) = 1 - \frac{1}{2} \min\{\delta(x), \delta_F\}.$$

Notice that $J(x, y_b) > \frac{1}{2}$.

Now consider the metric space (X, d_J) .

Proposition 5.12. (X, d_J) is connected.

Proof. Follows from Lemma 5.9 immediately. □

Proposition 5.13. $f : (X, d_J) \rightarrow (W, \preceq)$ is a cp-morphism.

Proof. Follows from Lemma 5.9 and Proposition 5.10 immediately. □

Now consider the distance relation on (X, d_J) :

$$xS^{(J)}y \Leftrightarrow d_J(x, y) < 1.$$

Proposition 5.14. $f : (X, S^{(J)}) \rightarrow (W, S)$ is a p-morphism.

Proof. That f satisfies the back property of p-morphism is immediate from the definition of J : if $f(x)Sb$ and $d(x, y_b) \geq 1$, we added the jump $J(x, y_b)$.

To check the relational homomorphism property, assume $d_J(x, y) < 1$. We aim to show that $f(x)Sf(y)$.

Let w be the weighting function induced by J on (X, d) (Definition 5.8). Then for some path $\tau = (x_0, \dots, x_m)$ from x to y we have $\sum_{i < m} w(x_i, x_{i+1}) < 1$.

Clearly, this path has at most one jump, since any jump value is greater than $\frac{1}{2}$.

If there are no jumps, then $d(x, y) < 1$, and hence x and y are neighbours, and the claim follows from Proposition 5.11.

Assume that τ has one jump (x_k, x_{k+1}) . Then x_{k+1} (or x_k) is y_b for some b . It follows that both $\sum_{i < k} w(x_i, x_{i+1})$ and $\sum_{k+1 \leq i < m} w(x_i, x_{i+1})$ are less than $\min\{\delta(x_k), \delta_F\}$. At the same time, these initial and final parts of τ have no jumps, and we have $d(x, x_k) \leq \sum_{i < k} w(x_i, x_{i+1})$ and $d(x_{k+1}, y) \leq \sum_{k+1 \leq i < m} w(x_i, x_{i+1})$.

Now it follows from (23) and (24) that $f(x_k) \preceq f(x)$ and $f(x_{k+1}) \preceq f(y)$. We also have $f(x_k)Sf(x_{k+1})$ due to the construction of jumps. By Proposition 5.1(3), it follows that $f(x)Sf(y)$. $\square$

From the p-morphism lemma [2, Section 3.3] and cp-morphism lemma [13, Lemma 15], it follows that φ is satisfiable in (X, d_J) . Hence, we have

Theorem 5.15. *In the language with the topological modality, universal modality, and a single distance modality, the logic of all connected, and also of all connected and compact metric spaces is $S4UC_{<1}$.*

6 Concluding remarks

The construction given in the previous section does not seem to be applicable to the case of more than one metric modality. The main (and the only) problem is the relational homomorphism condition in the proof of Proposition 5.14: our argument that any path was augmented with at most one jump does not extend to the case of several distance relations. Hence, the problem of axiomatization of connected metric spaces ([15, 6]) in the language of the topological closure $\diamond$, universal modality, and multiple metric modalities remains open.

Another natural question is the axiomatization of metric spaces that are a -connected for all positive a . Such spaces are said to be *well-chained* [14]. We conjecture that for any set A of parameters, the A -logic of the class of all well-chained metric spaces is $\text{Metr}(A) + \{\text{Con}_a \mid a \in A\}$.

These are only two of many open problems related to modal axiomatization of metric spaces. Of particular interest are distance logics of Euclidean spaces. Even the unimodal language with a single distance modality turns out to be surprisingly expressive: for example, it distinguishes between different dimensions and between the rational and real lines [1]. However, no complete axiomatizations are known for the corresponding logics.

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Appendix

Proof of Lemma 4.2. Very close to the argument in the proof of [7, Theorem 4.4].

The case $A = \emptyset$ is trivial. Assume $A \neq \emptyset$. Let $R = \bigcup_{l \in A} R_l$. Since $(X, (R_l)_{l \in A})$ is point-generated, (X, R) is connected.

Let

$$k = \frac{\max A + 1}{\min A},$$

and let $A^{<2k}$ be the set of tuples over A whose length is less than $2k$. Consider the set

$$S = \{\oplus \vec{r} - l \mid \vec{r} \in A^{<2k}, l \in A, \text{ and } \oplus \vec{r} > l\}.$$

Clearly, S is a finite set of positive numbers. Let

$$\varepsilon = \frac{\min(S \cup A)}{2k}.$$

Since $k > 1$ and $\min(S \cup A) \leq \min A$, we have

$$\varepsilon < \frac{\min A}{2}. \quad (25)$$

Let $e = (a, b) \in R$, $a \neq b$. Define the *index* $\text{ind}(e)$ of e as $\min\{l \in A \mid e \in R_l\}$, and its *weight* $w(e)$ as $\text{ind}(e) - \varepsilon$. By the definition, we put $w(a, a) = 0$.

For an R -path $\tau = (a_0, \dots, a_m)$, put $w(\tau) = \sum_{i < m} w(a_i, a_{i+1})$.

Finally, for $a, b \in X$, we define

$$d(a, b) = \inf\{w(\tau) \mid \tau \text{ is an } R\text{-path from } a \text{ to } b\}. \quad (26)$$

It is easy to see that d is a metric on X . In particular, since (X, R) is connected, $d(a, b)$ is defined for all $a, b \in X$. That $d(a, b) = d(b, a)$ is immediate from the symmetry of relations. The triangle inequality is straightforward from (26). It is also clear from the definition of w that $d(a, b) \geq \min A - \varepsilon$ for distinct a, b .

It remains to show that for $l \in A$,

$$aR_l b \text{ iff } d(a, b) < l. \quad (27)$$

The ‘only if’ follows from (26) and the definition of weight.

For the ‘if’ direction, assume that $d(a, b) < l \in A$. Then a and b are connected by an R -path $\tau = (a_0, \dots, a_m)$ with $w(\tau) < l$. The case $a = b$ is trivial, so let us assume that $m > 0$. We can also assume that τ is simple, that is all a_i are distinct. For $i < m$, put $l_i = \text{ind}(a_i, a_{i+1})$, and consider $\vec{r} = (l_0, \dots, l_{m-1})$. We have

$$w(\tau) = \sum_{i < m} (l_i - \varepsilon) = \oplus \vec{r} - m\varepsilon < l. \quad (28)$$

We claim that $m < 2k$. Indeed, in view of (25), we have

$$m \frac{\min A}{2} < m(\min A - \varepsilon) \leq \oplus \vec{r} - m\varepsilon = w(\tau) < l \leq \max A,$$

and hence we have $m < 2 \frac{\max A}{\min A} < 2k$.

Now we claim that

$$\oplus \vec{r} \leq l. \quad (29)$$

For the sake of contradiction, assume $\oplus \vec{r} > l$. We have $\oplus \vec{r} - l \in S$, and hence

$$\oplus \vec{r} - l \geq \min S \geq \min(S \cup A) = 2k\varepsilon > m\varepsilon,$$

so $\oplus \vec{r} - m\varepsilon > l$. This contradicts (28), which proves (29).

We have $(a, b) \in R_{\vec{r}}$. Since F is a $\text{Metr}(A)$ -frame and in view of (29), $(a, b) \in R_l$. This completes the proof of (27). $\square$

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Stratified Counterpossible Logic

Chen Huang

Institute of Logic and Cognition
Department of Philosophy
Sun Yat-sen University
Guangzhou, China
huangch58@mail2.sysu.edu.cn

Xuefeng Wen

Institute of Logic and Cognition
Department of Philosophy
Sun Yat-sen University
Guangzhou, China
wxfllogic@gmail.com

This paper presents two logic systems, **SCP** and its extension **SCP1**, to distinguish between different types of impossibility. The semantics use a stratified structure that partitions worlds into logic-normal worlds (N) and anti-logic worlds (I). By defining a metaphysical accessibility relation within N , the systems separate metaphysical impossibility from logical contradiction. This allows for logical reasoning to be maintained even when dealing with metaphysical impossibilities. To address the challenge of vacuism, **SCP1** implements a non-empty constraint on the selection function, ensuring that counterpossibles with impossible antecedents are not trivially true but depend on the connection between the antecedent and the consequent. We provide proofs for the soundness, completeness, and decidability of both systems. Finally, we indicate the possibility of applying this stratified approach to other modal domains, such as deontic or epistemic logic.

1 Introduction

Counterfactual semantics typically aim to balance formal simplicity with intuitive plausibility. While the Stalnaker-Lewis systems succeed for standard counterfactuals [10, 7, 8], they yield counterintuitive results for counterpossibles—conditionals with impossible antecedents.

1. If Hobbes had squared the circle, sick children in Afghanistan would have been cured.
2. If Hobbes had squared the circle, we would have been surprised.

For Stalnaker and Lewis, both conditionals are true because squaring the circle is impossible. Intuitively, however, the first seems false and the second true. This conflict between theory and intuition motivates the debate between vacuism and non-vacuism.

Vacuism asserts that all counterpossibles are vacuously true. While Williamson [13] defends this view for its logical strength, it has difficulty accommodating mathematical proofs by contradiction or metaphysical thought experiments. If every impossible antecedent yields a trivial truth, we cannot distinguish substantive hypotheses from absurd contradictions.

Non-vacuism, in contrast, argues that some counterpossibles are false [3]. To evaluate impossible antecedents within possible-world semantics, Nolan [9] introduced impossible worlds and the Strangeness of Impossibility Condition (SIC): any possible world is more similar to the actual world than any impossible world. This framework allows for the evaluation of counterpossibles while preserving classical logic for possible antecedents.¹ Weiss [11] evaluated the relative merits of sphere semantics and ceteris paribus logic, highlighting the potential of weak systems to provide a uniform semantic framework. French [4] proposed a conciliatory approach. By employing the translation $t(\varphi > \psi) := \Diamond t(\varphi) \rightarrow (t(\varphi) > t(\psi))$, French proved that all theorems of vacuist logic are theorems of non-vacuist logic under translation.

¹For further research on formal logical systems, see [1, 12].

This addresses the concern that non-vacuism necessarily weakens the logic by abandoning valid inference rules.

While these works progress in demarcating the possible from the impossible, they do not sufficiently model the internal structure of the impossible domain. Existing frameworks often conflate metaphysical impossibilities (e.g., “water is not H_2O ”) with logical contradictions (e.g., $P \wedge \neg P$). This leads to a trade-off: to accommodate contradictions, systems often sacrifice inference rules that are still needed for reasoning about metaphysical hypotheses.

This paper argues for both a distinction between the possible and impossible and a stratification of the impossible realm.² We propose Stratified Counterpossible Logic (SCP), which partitions worlds into logic-normal worlds (N) and anti-logical worlds (I). By nesting a metaphysical accessibility relation (R_M) within N , we create a three-layered structure that preserves logical reasoning even when metaphysical necessities are violated.

2 Philosophical Motivation

Existing literature on non-vacuism (e.g., [9, 12]) tends to adopt a binary perspective: on one side are possible worlds that obey all rules, and on the other are impossible worlds that break them. However, this approach implicitly assumes that all impossibilities have the same logical status. Treating all impossibilities identically forces logical systems into a dilemma when dealing with counterpossible reasoning.

To characterize these differences, this paper adopts classical logic as the base logic to define logically possible and logically impossible worlds.³ The logical space is partitioned into two regions:

Logic-normal worlds (N): This region contains all worlds that are consistent under the base logic (classical logic). It includes not only the actual world but also “anti-essential” worlds that violate metaphysical laws yet still follow classical inference rules, such as the Law of Excluded Middle and the Law of Non-Contradiction.

Anti-logical worlds (I): This region contains situations that violate the norms of the base logic.

In this framework, worlds closed under non-classical logics, such as paraconsistent worlds where $\phi \wedge \neg \phi$ holds, are categorized as “logically impossible” relative to the base N . While I can accommodate diverse non-classical structures, this paper treats it primarily as a domain where base logical norms fail.

Within the logically normal domain N , we distinguish between metaphysical possibility and impossibility via the relation R_M . The actual world and its R_M -accessible worlds constitute the metaphysically possible region. Conversely, R_M -inaccessible worlds within N represent “anti-essential” scenarios, such as “water is XYZ” or “gold has atomic number 50.”

This stratified $N/R_M/I$ structure echoes philosophical practice in scientific and metaphysical thought experiments. When assuming antecedents that violate natural laws or essential properties, we typically continue to reason using classical logic. This suggests that such anti-metaphysical scenarios should remain within the logic-normal domain N . Consider the following example:

Example 2.1. If gold had atomic number 50, gold would still be a metal.

Metaphysically, gold necessarily and essentially has atomic number 79; thus, “the atomic number of gold is 50” is a metaphysical impossibility. In unstratified counterpossible logics, because the antecedent is impossible, it must point to an “impossible world.” Since impossible worlds are generally

²Similar stratified approaches exist; for example, Gillies [6] used scorekeeping to establish stratified strict conditional semantics for context-sensitivity.

³While many non-classical logics, such as paraconsistent or intuitionistic logic, offer deep philosophical insights, we leave them for future research to avoid overcomplicating the semantic model.

unconstrained by logic, one might lose the inference that “gold is a metal,” which is based on chemical classification. Intuitively, even if we imagine a change in the atomic number of gold, we are still reasoning with classical logic. Therefore, such “anti-essential” scenarios should be retained within the logic-normal worlds (N).

3 Stratified Counterpossible Logic (SCP)

This section presents Stratified Counterpossible Logic (henceforth, **SCP**). The core intuition of **SCP** is to partition the logical space into three levels: metaphysically possible worlds, logically possible (but metaphysically impossible) worlds, and anti-logical worlds.

SCP employs equivalence relations to characterize metaphysical and logical possibilities, presupposing that metaphysical possibility implies logical possibility.

Definition 3.1 (Formal Language). Given a set of atomic propositions P , the formulas of the language $\mathcal{L}$ are generated by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \rightarrow \varphi) \mid \Box_M\varphi \mid \Box_L\varphi \mid (\varphi > \varphi),$$

where $p \in P$. In this paper, contradictions and tautologies refer to those in classical logic, denoted by $\perp$ and $\top$ respectively.

3.1 Semantics

We first provide the model-theoretic definition of **SCP**. The model is based on selection semantics for conditionals and introduces stratified modal accessibility relations to characterize the inclusion relationship between metaphysical and logical possibilities.

Definition 3.2. An **SCP** model is a tuple $M = (W, N, R_M, R_L, f, V)$, where:

- W is a non-empty set of worlds (including impossible worlds);
- N is a subset of W , consisting of normal worlds that follow the laws of classical logic;
- $I := W - N$ is thus a set of anti-logical worlds that are not constrained by classical logic;
- $R_M, R_L \subseteq N \times N$ are binary equivalence relations defined on N , representing the metaphysical accessibility relation and the logical accessibility relation, respectively;
- $f : N \times \mathcal{L} \rightarrow \mathcal{P}(W)$ is a selection function;
- $V : \mathcal{L} \rightarrow \mathcal{P}(W)$ is a valuation function.

Note that for brevity V is not merely defined on atoms, but on all formulas.

Definition 3.3 (Truth Conditions for **SCP**). The truth relation $M, w \Vdash \varphi$ for a formula φ at world w in model M is defined recursively as follows:

- In normal worlds ($w \in N$):
 - For an atomic proposition p , $M, w \Vdash p \iff w \in V(p)$.
 - $M, w \Vdash \neg\varphi \iff M, w \not\Vdash \varphi$.
 - $M, w \Vdash \varphi \rightarrow \psi \iff M, w \not\Vdash \varphi$ or $M, w \Vdash \psi$.
 - $M, w \Vdash \Box_M\varphi \iff R_M(w) \subseteq [\varphi]$, where $[\varphi] := \{w \in W \mid M, w \Vdash \varphi\}$.

- $M, w \Vdash \Box_L \varphi \iff R_L(w) \subseteq [\varphi]$.
- $M, w \Vdash \varphi > \psi \iff f(w, \varphi) \subseteq [\psi]$.
- In anti-logical worlds ($w \in I$):
 - For any formula $\varphi \in \mathcal{L}$: $M, w \Vdash \varphi \iff w \in V(\varphi)$.

Remark 3.4. In I , the truth values of complex formulas (e.g., $\varphi \wedge \psi$) are no longer defined recursively. This allows for logical impossibilities, such as φ being true while $\varphi \vee \psi$ is false, or $\varphi \wedge \neg\varphi$ being true.

Definition 3.5 (Semantic Constraints). For any $w \in N$, the model M must satisfy the following constraints:

(modal) $R_M(w) \subseteq R_L(w)$	(soc-m) If $w \in [\Diamond_M \varphi]$, then $f(w, \varphi) \subseteq R_M(w)$	
(id) $f(w, \varphi) \subseteq [\varphi]$	(sic) If $w \in [\Diamond_L \varphi]$, then $f(w, \varphi) \subseteq N$	

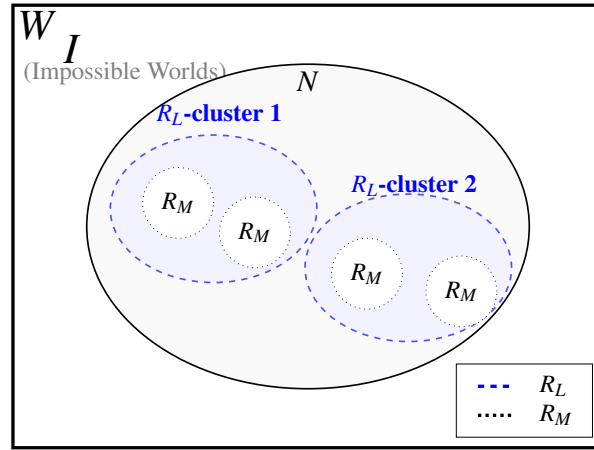


Figure 1: The Stratified World Model of the **SCP** System

Figure 1 illustrates the hierarchical structure of the **SCP** system's world model. Its semantic framework consists of the following four core components:

The Universe W and Impossible Worlds I : The outermost rectangle represents the total set W of all logically possible and impossible worlds. Points within I represent anti-logical worlds where logical laws may fail, providing a semantic buffer for handling counterpossibles.

Normal World Region N : The central oval region N represents the set of normal worlds that satisfy all logical axioms of the **SCP** system.

Logical Equivalence Classes R_L : Within N , blue dashed lines partition the space into R_L -clusters. Disjoint equivalence classes represent different boundaries of logical possibility.

Metaphysical Equivalence Classes R_M : Within each R_L equivalence class, R_M classes are further partitioned. This reflects the nested property $R_M \subseteq R_L$, indicating that metaphysical possibility is a finer-grained division restricted by logical possibility.

Definition 3.6 (Validity). Given a class S of **SCP** models, an inference from a set of formulas Γ to φ is valid in S , denoted by $\Gamma \models_S \varphi$, if and only if for any $M = (W, N, R_M, R_L, f, V) \in S$ and any $w \in N$, if $M, w \Vdash \Gamma$, then $M, w \Vdash \varphi$.

Validity is strictly restricted to the set of normal worlds N . This aligns with the standard approach in counterpossible logic: we evaluate hypotheses about the “impossible” from the standpoint of worlds where logical laws remain in effect. Let **SCP** denote the class of models satisfying all semantic constraints. In this section, $\models_{\text{SCP}}$ is often abbreviated as $\models$.

3.2 Validities of SCP

We examine the key logical properties of **SCP**. In standard counterfactual semantics (such as Lewis-Stalnaker semantics), conditionals with impossible antecedents are treated as “vacuously true.” Non-vacuism rejects this, arguing that even if an antecedent is unsatisfiable, the conditional should have non-trivial truth conditions. In other words, a contradiction or an impossibility should not imply everything. Consequently, a counterpossible logic should not validate the following two formulas:⁴

$$(EPQ) \quad \perp > \varphi \qquad (VAC) \quad \neg \Diamond \varphi \rightarrow (\varphi > \psi)$$

Proposition 3.7. *SCP is a counterpossible logic.*

$$1. \not\models \perp > \varphi \qquad 2. \not\models \neg \Diamond_L \varphi \rightarrow (\varphi > \psi) \qquad 3. \Diamond_L \varphi \models \varphi > \top$$

Regarding the first point, in **SCP**, $f(w, \perp) \subseteq I$. Since truth values are assigned arbitrarily in anti-logical worlds, $\perp > \varphi$ is not universally valid. The same reasoning applies to the second point. The third point indicates that **SCP** does not weaken logical strength in all cases. As long as the antecedent is logically possible ($\Diamond_L \varphi$), the conditional operator reverts to classical semantic rules, thereby ensuring the necessity of the tautology $\top$.

Proposition 3.8. *The following hold.*

$$1. \not\models \Box_L(\Box_M \varphi \rightarrow \varphi) \qquad 2. \Box_M \psi \not\models \Diamond_L \varphi \rightarrow (\varphi > \psi) \qquad 3. \models \Box_L \varphi \rightarrow \Box_M \Box_L \varphi$$

Proof. We only prove the Item 3. Suppose $M, w \Vdash \Box_L \varphi$. Since R_L is an equivalence relation, for any $v \in R_L(w)$, we have $M, v \Vdash \Box_L \varphi$. Given $R_M(w) \subseteq R_L(w)$, it follows that $M, w \Vdash \Box_M \Box_L \varphi$. $\square$

Item 1 shows that it is logically possible for a metaphysically necessary proposition to be false. There exists a logic-normal world $w \in N$ (logically possible relative to the actual world but metaphysically impossible) that is logically accessible, yet metaphysical laws fail in some worlds accessible from it. This ensures that we can discuss scenarios like “water is not H_2O ” without causing logical collapse.

Item 2 further indicates that metaphysically necessary truths are not necessarily preserved under anti-essential hypotheses. For instance, let ψ be “water is H_2O ” (metaphysically necessary) and φ be “water is XYZ ” (anti-essential but logically possible). This is a distinct advantage of the **SCP** system: it allows for a meaningful discussion of the consequences of “water not being H_2O ” without the derivation becoming vacuous due to the necessity of ψ .

Item 3 reflects the “overlap” of the logical hierarchy over the metaphysical one. While metaphysical laws can be false at the logical level (see the first point), logical laws must be true at the metaphysical level. Once a proposition φ is judged logically necessary (i.e., true in all worlds in R_L), it remains logically necessary regardless of how we shift our metaphysical perspective via R_M .

3.3 Axiomatic system

The axiomatic system for **SCP** is built upon classical propositional logic, supplemented by **S5** modal operators $\Box_L$ and $\Box_M$. The system is defined as follows:

⁴For a more detailed classification of counterpossible logics, see [5].

Basic Propositional and Modal Axioms

(PC)	All tautologies of propositional logic	
(KL)	$\Box_L(\varphi \rightarrow \psi) \rightarrow (\Box_L\varphi \rightarrow \Box_L\psi)$	(KM) $\Box_M(\varphi \rightarrow \psi) \rightarrow (\Box_M\varphi \rightarrow \Box_M\psi)$
(TL)	$\Box_L\varphi \rightarrow \varphi$	(TM) $\Box_M\varphi \rightarrow \varphi$
(4L)	$\Box_L\varphi \rightarrow \Box_L\Box_L\varphi$	(4M) $\Box_M\varphi \rightarrow \Box_M\Box_M\varphi$
(5L)	$\Diamond_L\varphi \rightarrow \Box_L\Diamond_L\varphi$	(5M) $\Diamond_M\varphi \rightarrow \Box_M\Diamond_M\varphi$

Modal and Conditional Axioms

(MODAL)	$\Box_L\varphi \rightarrow \Box_M\varphi$
(ID)	$\varphi > \varphi$
(SOC-M)	$\Diamond_M\varphi \wedge \Box_M\psi \rightarrow (\varphi > \psi)$

Inference Rules

$$\begin{array}{c}
\frac{\varphi, \quad \varphi \rightarrow \psi}{\psi} \text{(MP)} \quad \quad \frac{\varphi}{\Box_L\varphi} \text{(RN}_L\text{)} \quad \quad \frac{\varphi}{\Box_M\varphi} \text{(RN}_M\text{)} \\
\\
\frac{(\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \varphi}{\Diamond_L\psi \rightarrow (((\psi > \varphi_1) \wedge \dots \wedge (\psi > \varphi_n)) \rightarrow (\psi > \varphi))} \text{(SIC)}
\end{array}$$

3.4 Soundness and completeness

Theorem 3.9 (Soundness of **SCP**). *For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, if $\Gamma \vdash_{SCP} \varphi$, then $\Gamma \models_{SCP} \varphi$.*

Proof. The validity of MODAL, ID, PC, and S5-related axioms and rules is straightforward.

(SIC): Suppose $\models (\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \varphi$. Let $M, w \Vdash \Diamond_L\psi$ and $M, w \Vdash \psi > \varphi_1, \dots, M, w \Vdash \psi > \varphi_n$. According to the truth conditions and the (sic) constraint, $f(w, \psi) \subseteq [\varphi_1], \dots, f(w, \psi) \subseteq [\varphi_n]$ and $f(w, \psi) \subseteq N$. Thus, $f(w, \psi) \subseteq [\varphi_1 \wedge \dots \wedge \varphi_n]$. Since $\models (\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \varphi$, for any world $x \in N$, $x \in [\varphi_1 \wedge \dots \wedge \varphi_n] \implies x \in [\varphi]$, meaning $[\varphi_1 \wedge \dots \wedge \varphi_n] \subseteq [\varphi]$. Therefore, $M, w \Vdash \psi > \varphi$.

(SOC-M): Suppose $w \in N$ and $M, w \Vdash \Diamond_M\varphi \wedge \Box_M\psi$. Since $M, w \Vdash \Box_M\psi$, it follows that $R_M(w) \subseteq [\psi]$. Given $M, w \Vdash \Diamond_M\varphi$, the constraint (soc-m) implies $f(w, \varphi) \subseteq R_M(w)$. Therefore, $M, w \Vdash \varphi > \psi$. $\square$

To prove the completeness of the **SCP** system, we use the canonical model construction. The core objective is to show that every consistent set of formulas is satisfiable in some model. The key challenge in the construction is handling the opposition between logic-normal worlds and anti-logical worlds. We must construct maximal consistent sets (MCS) to represent normal worlds, and all subsets of the language to serve as anti-logical worlds that accommodate logical impossibilities.

Definition 3.10 (Canonical Models for **SCP**). The canonical model $M^c = (W^c, N^c, f^c, R_M^c, R_L^c, V^c)$ is defined as follows:

- $W^c = \mathcal{P}(\mathcal{L})$
- $N^c = \{w \mid w \text{ is an MCS of } \mathbf{SCP}\}$ with $I^c := W^c - N^c$.
- $\forall x, y \in N^c, xR_M^cy$ iff $\{\varphi \mid \Box_M\varphi \in x\} \subseteq y$.
- $\forall x, y \in N^c, xR_L^cy$ iff $\{\varphi \mid \Box_L\varphi \in x\} \subseteq y$.

- $f^c(w, \varphi) = \begin{cases} \{x \in N^c \mid \{\psi \mid \varphi > \psi \in w\} \subseteq x\} & \text{if } \Diamond_L \varphi \in w \\ \{x \in W^c \mid \{\psi \mid \varphi > \psi \in w\} \subseteq x\} & \text{if } \Diamond_L \varphi \notin w \end{cases}$
- $V^c(\varphi) = \{\Delta \mid \varphi \in \Delta, \Delta \in \mathcal{P}(\mathcal{L})\}$.

Let $[[\varphi]] := \{w \in W^c \mid M^c, w \Vdash \varphi\}$.

Lemma 3.11 (Truth Lemma). *For any $w \in W^c$, $M^c, w \Vdash \alpha$ iff $\alpha \in w$.*

Proof. For any $w \in I^c$ and $\alpha \in \mathcal{L}$, the truth condition gives $M^c, w \Vdash \alpha \iff w \in V^c(\alpha) \iff \alpha \in w$.

For any $w \in N^c$, we proceed by induction on the complexity of formula α . The cases for atomic propositions and Boolean connectives are trivial.

Case $\alpha = \Box_M \varphi$:

($\Leftarrow$) If $\Box_M \varphi \in w$, then for all $x \in R_M^c(w)$, $\varphi \in x$. By IH, $M^c, x \Vdash \varphi$ for all $x \in R_M^c(w)$, hence $M^c, w \Vdash \Box_M \varphi$.

($\Rightarrow$) If $\Box_M \varphi \notin w$, we construct a set $\Gamma = \{\psi \mid \Box_M \psi \in w\} \cup \{\neg \varphi\}$. If Γ were inconsistent, then $\vdash (\psi_1 \wedge \dots \wedge \psi_n) \rightarrow \varphi$ for some $\Box_M \psi_i \in w$. By normality (K and RN), $\vdash (\Box_M \psi_1 \wedge \dots \wedge \Box_M \psi_n) \rightarrow \Box_M \varphi$, implying $\Box_M \varphi \in w$, a contradiction.

Case $\alpha = \Box_L \varphi$: Similar to the $\Box_M \varphi$ case.

Case $\alpha = \varphi > \psi$:

($\Leftarrow$) If $\varphi > \psi \in w$, then by the definition of f^c , for any $x \in f^c(w, \varphi)$, we have $\psi \in x$. By IH, $M^c, x \Vdash \psi$, so $M^c, w \Vdash \varphi > \psi$.

($\Rightarrow$) Suppose $\varphi > \psi \notin w$.

If $\Diamond_L \varphi \notin w$: Let $u = \{\chi \mid (\varphi > \chi) \in w\}$. Since $\varphi > \psi \notin w$, $\psi \notin u$. If $u \in N^c$, by IH, $M^c, u \not\Vdash \psi$. Since $u \in f^c(w, \varphi)$, $M^c, w \not\Vdash \varphi > \psi$. If $u \notin N^c$, by the truth condition, we also have $M^c, w \not\Vdash \varphi > \psi$.

If $\Diamond_L \varphi \in w$: We show that $\{\chi \mid (\varphi > \chi) \in w\} \cup \{\neg \psi\}$ is consistent. If not, there exist χ_i such that $\vdash (\chi_1 \wedge \dots \wedge \chi_n) \rightarrow \psi$. By the SIC rule, $\vdash \Diamond_L \varphi \rightarrow ((\varphi > \chi_1) \wedge \dots \wedge (\varphi > \chi_n) \rightarrow (\varphi > \psi))$. This implies $\varphi > \psi \in w$, a contradiction. $\square$

We further prove that M^c satisfies the semantic constraints of the defined class of models.

Lemma 3.12. *The canonical model M^c satisfies the semantic constraints: for any $w \in N^c$,*

(modal)	$R_M^c(w) \subseteq R_L^c(w)$	(id)	$f^c(w, \varphi) \subseteq [[\varphi]]$
(soc-m)	If $w \in [[\Diamond_M \varphi]]$, then $f^c(w, \varphi) \subseteq R_M^c(w)$	(sic)	If $w \in [[\Diamond_L \varphi]]$, then $f^c(w, \varphi) \subseteq N^c$

Proof. (modal): Derived directly from the axiom $\Box_L \varphi \rightarrow \Box_M \varphi$.

(id): Derived directly from the axiom $\varphi > \varphi$.

(soc-m): Suppose $w \in [[\Diamond_M \varphi]]$. By the Truth Lemma, $\Diamond_M \varphi \in w$. Assume there exists $u \in f^c(w, \varphi)$ such that $u \notin R_M^c(w)$. Then there exists ψ such that $\Box_M \psi \in w$ but $\psi \notin u$. By SOC-M, $\varphi > \psi \in w$. Thus $\psi \in u$ by the definition of f^c , a contradiction.

(sic): Suppose $w \in [[\Diamond_L \varphi]]$. By the Truth Lemma, $\Diamond_L \varphi \in w$. From the definition of f^c in the canonical model, $f^c(w, \varphi) \subseteq N^c$. $\square$

Theorem 3.13 (Completeness). *If $\Gamma \models_{SCP} \varphi$, then $\Gamma \vdash_{SCP} \varphi$.*

3.5 Decidability

To construct a finite model, we first define a finite set of formulas Σ_φ that contains the target formula φ and all its relevant inferential components.

Definition 3.14 (Subformula Closure). For a formula φ , the closure Σ_φ is defined as the smallest set of formulas satisfying the following conditions:

- $\varphi \in \Sigma_\varphi$;
- If $\psi \in \Sigma_\varphi$ and χ is a subformula of ψ , then $\chi \in \Sigma_\varphi$;
- If $\psi \in \Sigma_\varphi$ and ψ does not start with $\neg$, then $\neg\psi \in \Sigma_\varphi$;
- If $\Box_M\psi \in \Sigma_\varphi$ or $\Diamond_M\psi \in \Sigma_\varphi$, then $\psi \in \Sigma_\varphi$;
- If $\Box_L\psi \in \Sigma_\varphi$ or $\Diamond_L\psi \in \Sigma_\varphi$, then $\psi, \Box_M\Box_L\psi \in \Sigma_\varphi$;
- If $\psi > \chi \in \Sigma_\varphi$, then $\psi, \chi, \Diamond_L\psi, \Diamond_M\psi, \Box_L\chi, \Box_M\chi, \psi > \psi \in \Sigma_\varphi$.

The inclusion of $\Box_M\Box_L\psi$ captures the nested interaction between R_M^* and R_L^* . While not a direct subformula of $\Box_L\psi$, this single-layer expansion avoids infinite recursion. Similarly, adding modal derivatives such as $\Diamond_L\psi$ and $\Box_M\chi$ provides the necessary information for the selection function to satisfy the SOC-M and SIC constraints. Crucially, since each subformula in Σ_φ generates only a fixed, finite number of such derivatives, the closure set remains finite.

Definition 3.15 (Equivalence Relation). Define an equivalence relation $\sim$ between worlds in the canonical model M^c :

$$w \sim v \iff \forall \psi \in \Sigma_\varphi, (\psi \in w \iff \psi \in v)$$

Let $\bar{w} = \{v \in W^c \mid v \sim w\}$ denote the equivalence class of w . Since Σ_φ is finite, the quotient set $W^* = W^c / \sim$ is also finite.

Definition 3.16 (Filtration Model). Given the canonical model $M^c = (W^c, N^c, R_M^c, R_L^c, f^c, V^c)$, the filtration model for Σ_φ is defined as $M^* = (W^*, N^*, R_M^*, R_L^*, f^*, V^*)$, where:

- $W^* = \{\bar{w} \mid w \in W^c\}$;
- $N^* = \{\bar{w} \mid w \in N^c\}$;
- For any $\bar{w}, \bar{v} \in N^*$, $\bar{w}R_M^*\bar{v} \iff \forall \Box_M\psi \in \Sigma_\varphi, (\Box_M\psi \in w \iff \Box_M\psi \in v)$;
- For any $\bar{w}, \bar{v} \in N^*$, $\bar{w}R_L^*\bar{v} \iff \forall \Box_L\psi \in \Sigma_\varphi, (\Box_L\psi \in w \iff \Box_L\psi \in v)$;
- For any $\psi > \chi \in \Sigma_\varphi$, let $U_{\psi,w} = \{\bar{v} \mid \forall (\psi > \chi) \in \Sigma_\varphi, (\psi > \chi \in w \Rightarrow \chi \in v)\}$. Let

$$f^*(\bar{w}, \psi) = \begin{cases} U_{\psi,w} \cap R_M^*(\bar{w}) & \text{if } \Diamond_M\psi \in w \\ U_{\psi,w} \cap N^* & \text{if } \Diamond_M\psi \notin w \text{ and } \Diamond_L\psi \in w \\ U_{\psi,w} & \text{if } \Diamond_L\psi \notin w \end{cases}$$

- For other ψ , let $f^*(\bar{w}, \psi) = \emptyset$.
- $V^*(\psi) = \{\bar{w} \mid w \in V^c(\psi)\}$.

Let $[[\varphi]]^* := \{w \in W^* \mid M^*, w \models \varphi\}$.

Lemma 3.17. *The model is well-defined; that is, for any $\bar{w}, \bar{v} \in N^*$ and any $\psi > \chi \in \Sigma_\phi$:*

1. If $w \sim v$, then $R_M^*(\bar{w}) = R_M^*(\bar{v})$;
2. R_M^* is an equivalence relation on N^* ;
3. If $w \sim v$, then $R_L^*(\bar{w}) = R_L^*(\bar{v})$;
4. R_L^* is an equivalence relation on N^* ;
5. If $w \sim v$, then $f^*(\bar{w}, \psi) = f^*(\bar{v}, \psi)$.

Proof. Suppose $w \sim v$. For any $\bar{u} \in R_M^*(\bar{w})$, we have $\forall \Box_M \psi \in \Sigma_\phi, (\Box_M \psi \in w \Rightarrow \Box_M \psi \in u)$. Since $w \sim v$, we have $\forall \psi \in \Sigma_\phi, (\psi \in w \iff \psi \in v)$, which implies $\forall \Box_M \psi \in \Sigma_\phi, (\Box_M \psi \in v \Rightarrow \Box_M \psi \in u)$. Thus, $\bar{u} \in R_M^*(\bar{v})$, and $R_M^*(\bar{w}) \subseteq R_M^*(\bar{v})$. The reverse inclusion $R_M^*(\bar{v}) \subseteq R_M^*(\bar{w})$ follows similarly.

It is easy to verify that R_M^* is reflexive, transitive, and symmetric. The case for R_L^* is analogous.

Suppose $w \sim v$. For any $\bar{u} \in f^*(\bar{w}, \psi)$, we have $\forall (\psi > \chi) \in \Sigma_\phi, (\psi > \chi \in w \Rightarrow \chi \in u)$. Since $w \sim v$, we have $\forall \psi \in \Sigma_\phi, (\psi \in w \iff \psi \in v)$, which implies $\forall (\psi > \chi) \in \Sigma_\phi, (\psi > \chi \in v \Rightarrow \chi \in u)$. Given $\Diamond_M \psi, \Diamond_L \psi \in \Sigma_\phi$, we have $\bar{u} \in f^*(\bar{v}, \psi)$, so $f^*(\bar{w}, \psi) \subseteq f^*(\bar{v}, \psi)$. The reverse inclusion follows similarly. $\square$

Lemma 3.18 (Filtration Lemma). *For all $\psi \in \Sigma_\phi$ and all $w \in W^c$:*

$$M^c, w \Vdash \psi \iff M^*, \bar{w} \Vdash \psi$$

Proof. For any $\bar{w} \notin N^*$ and $\psi \in \Sigma_\phi$, the truth condition yields $M^c, w \Vdash \psi \iff w \in V^c(\psi) \iff \bar{w} \in V^*(\psi) \iff M^*, \bar{w} \Vdash \psi$.

For any $\bar{w} \in N^*$, we proceed by induction on the complexity of ψ . The cases for atomic propositions and Boolean connectives are straightforward.

Case $\psi = \Box_M \gamma$:

($\Rightarrow$) Suppose $M^c, w \Vdash \Box_M \gamma$. For any $\bar{v}$ such that $\bar{w} R_M^* \bar{v}$, by the definition of R_M^* , $\Box_M \gamma \in w \iff \Box_M \gamma \in v$. Thus $\Box_M \gamma \in v$. Since M^c is an S5 model, $\gamma \in v$. By IH, $M^*, \bar{v} \Vdash \gamma$. Therefore $M^*, \bar{w} \Vdash \Box_M \gamma$.

($\Leftarrow$) Suppose $M^c, w \nVdash \Box_M \gamma$. In M^c , there exists $v \in N^c$ such that $w R_M^c v$ and $v \nVdash \gamma$. We show that $\bar{w} R_M^* \bar{v}$ holds. For any $\Box_M \chi \in \Sigma_\phi$, if $\Box_M \chi \in w$, then $\chi \in v$ because $w R_M^c v$; by S5 properties, $\Box_M \chi \in v$ also holds. The converse is similar. Thus $\bar{w} R_M^* \bar{v}$ holds. Since $M^c, v \nVdash \gamma$, IH yields $M^*, \bar{v} \nVdash \gamma$, hence $M^*, \bar{w} \nVdash \Box_M \gamma$.

Case $\psi = \Box_L \gamma$: Analogous to the $\Box_M \gamma$ case.

Case $\psi = \alpha > \beta$:

($\Rightarrow$) Suppose $M^c, w \Vdash \alpha > \beta$. For any $\bar{v} \in f^*(\bar{w}, \alpha)$, by the definition of f^* , we have $\forall (\alpha > \chi) \in \Sigma_\phi, (\alpha > \chi \in w \Rightarrow \chi \in v)$. Thus $\beta \in v$. By IH, $M^*, \bar{v} \Vdash \beta$. Therefore $f^*(\bar{w}, \alpha) \subseteq [[\beta]]^*$, i.e., $M^*, \bar{w} \Vdash \alpha > \beta$.

($\Leftarrow$) Suppose $M^c, w \nVdash \alpha > \beta$. In M^c , there exists a world u such that $\{\chi \mid \alpha > \chi \in w\} \subseteq u$ and $\beta \notin u$. Consider the equivalence class $\bar{u}$.

If $\Diamond_M \alpha \in w$: For any $\Box_M \gamma \in \Sigma_\phi$, if $\Box_M \gamma \in w$, then $\alpha > \gamma \in w$ (by SOC-M). From the construction of u , $\gamma \in u$. Since M^c is S5, $\Box_M \gamma \in w \Rightarrow \Box_M \Box_M \gamma \in w \Rightarrow \alpha > \Box_M \gamma \in w \Rightarrow \Box_M \gamma \in u$. The converse $\neg \Box_M \gamma \in w \Rightarrow \neg \Box_M \gamma \in u$ follows similarly. Thus $\bar{w} R_M^* \bar{u}$ holds by the definition of R_M^* , so $\bar{u} \in f^*(\bar{w}, \alpha)$. By IH, $M^*, \bar{u} \nVdash \beta$. Hence $M^*, \bar{w} \nVdash \alpha > \beta$.

If $\Diamond_M \alpha \notin w$ and $\Diamond_L \alpha \in w$: By the properties of the selection function in the canonical model, we have $u \in N^c$, which implies $\bar{u} \in N^*$. It follows directly that $\bar{u} \in f^*(\bar{w}, \alpha)$. By the induction hypothesis, $M^*, \bar{u} \nVdash \beta$, hence $M^*, \bar{w} \nVdash \alpha > \beta$.

If $\Diamond_L \alpha \notin w$: Clearly $\bar{u} \in f^*(\bar{w}, \alpha)$. By IH, $M^*, \bar{u} \nVdash \beta$. Hence $M^*, \bar{w} \nVdash \alpha > \beta$. $\square$

Lemma 3.19. *The filtration model satisfies the semantic constraints: for all $\psi > \chi \in \Sigma_\phi$ and $\bar{w} \in N^*$,*

$$\begin{array}{ll} (\text{modal}) & R_M^*(\bar{w}) \subseteq R_L^*(\bar{w}) \\ (\text{soc-m}) & \text{If } M^*, \bar{w} \Vdash \Diamond_M \psi, \text{ then } f^*(\bar{w}, \psi) \subseteq R_M^*(\bar{w}) \end{array} \quad \begin{array}{ll} (\text{id}) & f^*(\bar{w}, \psi) \subseteq [[\psi]]^* \\ (\text{sic}) & \text{If } M^*, \bar{w} \Vdash \Diamond_L \psi, \text{ then } f^*(\bar{w}, \psi) \subseteq N^* \end{array}$$

Proof. (modal): For any $\bar{v} \in R_M^*(\bar{w})$, we show $\bar{w} R_L^* \bar{v}$, i.e., for all $\Box_L \psi \in \Sigma_\phi$, $\Box_L \psi \in w \iff \Box_L \psi \in v$. Suppose $\Box_L \psi \in w$. Since $\Box_L \psi \in \Sigma_\phi$, we have $\Box_M \Box_L \psi \in \Sigma_\phi$ by Definition 3.14. Since $\bar{w} \in N^*$, w contains at least one MCS $w' \in N^c$. By Proposition 3.8.3, $\models \Box_L \psi \rightarrow \Box_M \Box_L \psi$, so $\Box_M \Box_L \psi \in w'$. By the definition of R_M^* , $\Box_M \Box_L \psi \in v$. By axiom TM and $v \sim v'$, we have $\Box_L \psi \in v'$ and thus $\Box_L \psi \in v$. For the other direction, suppose $\Box_L \psi \in v$. Let $v' \in \bar{v} \cap N^c$. By Axiom $\Box_L \psi \rightarrow \Box_M \Box_L \psi$, we have $\Box_M \Box_L \psi \in v'$. Thus $\Box_M \Box_L \psi \in v$. As R_M^* is symmetric, we have $\Box_M \Box_L \psi \in w$. Let $w' \in \bar{w} \cap N^c$. By Axiom TM, we have $\Box_L \psi \in w'$. Thus $\Box_L \psi \in w$.

(id): Let $\bar{w} \in N^*$. Let $v' \in \bar{w} \cap N^c$. By Axiom ID, $\psi > \psi \in w'$. As $w \sim w'$, we also have $\psi > \psi \in w$. For any $\bar{v} \in U_{\psi, w}$, by the definition of $U_{\psi, w}$, it follows that $\psi \in v$. By the filtration lemma, $M^*, \bar{v} \Vdash \psi$, i.e., $f^*(\bar{w}, \psi) \subseteq [[\psi]]^*$.

(soc-m): Suppose $M^*, \bar{w} \Vdash \Diamond_M \psi$. By Lemma 3.18, $\Diamond_M \psi \in w$. By definition, $f^*(\bar{w}, \psi) \subseteq R_M^*(\bar{w})$.

(sic): Suppose $M^*, \bar{w} \Vdash \Diamond_L \psi$. By Lemma 3.18, $\Diamond_L \psi \in w$. By definition, $f^*(\bar{w}, \psi) \subseteq N^*$. $\square$

Theorem 3.20. *SCP has finite model property and is decidable.*

4 Non-vacuous Semantics (SCP1)

The preceding section established a general semantic framework for **SCP**. While flexible, **SCP** exhibits two theoretical vulnerabilities regarding non-vacuism and logical preservation.

First, **SCP** permits the selection function f to be empty. For an impossible antecedent ϕ , a model can simply assign $f(w, \phi) = \emptyset$, making $\phi > \psi$ vacuously true. If impossible worlds are rich enough, this is counterintuitive and renders the non-vacuiist agenda incompletely realized.

Second, the logical counterpart to (SOC-M), namely (SOC-L): $\Diamond_L \phi \wedge \Box_L \psi \rightarrow (\phi > \psi)$, is invalid in the base system. Consider a model where $N = \{w, v\}$ with disjoint logical clusters $R_L(w) = \{w\}$ and $R_L(v) = \{v\}$. Let $w \in V(\phi) \cap V(\psi)$ and $v \in V(\phi) \setminus V(\psi)$. Clearly, $w \Vdash \Diamond_L \phi \wedge \Box_L \psi$. However, the semantic constraint (sic) only requires $f(w, \phi) \subseteq N$. If the selection yields $f(w, \phi) = \{v\}$, we get $w \not\models \phi > \psi$. This invalidity highlights a flaw in **SCP**: the evaluation of conditionals can cross into disconnected logical clusters, causing established logical necessities ($\Box_L \psi$) to fail.

To provide a more rigorous characterization of counterpossible reasoning, we introduce the extended system **SCP1**. This system augments the semantics with two core constraints: **(ne)** (non-vacuism) ensures f is never empty by evaluating contradictions within a sufficiently rich set of impossible worlds I ; and **(rl)** (universal accessibility) treats R_L as a universal relation over N , reflecting the global consistency of logical laws.

4.1 Semantics

Definition 4.1 (SCP1 Constraints). In addition to Definition 3.5, **SCP1** models must satisfy:

(rl) $R_L(w) = N$ for all $w \in N$.

(ne) $f(w, \phi) \neq \emptyset$ for all $w \in N, \phi \in \mathcal{L}$.

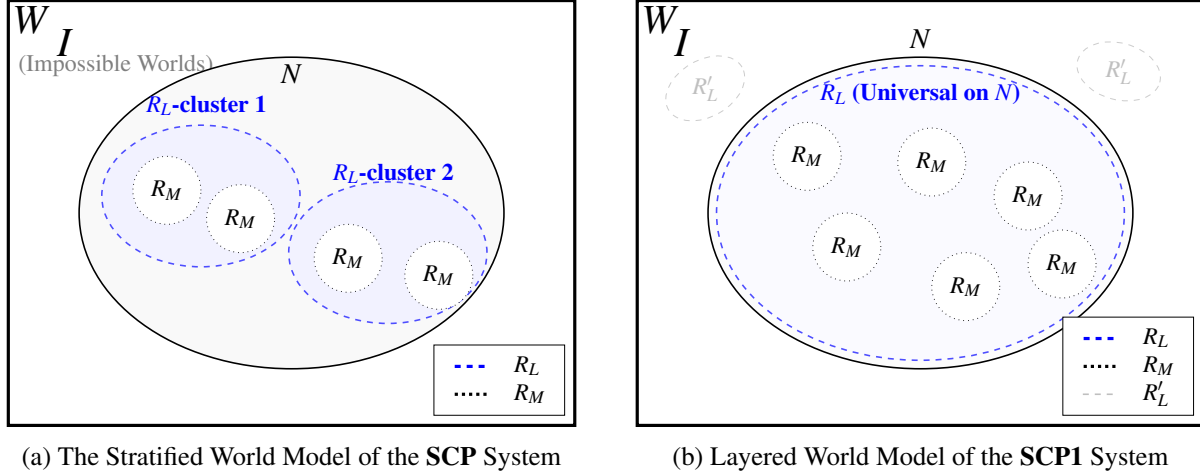
Figure 2: Stratified World Models for **SCP** and **SCP1**

Figure 2b illustrates the distinctions of **SCP1**:

Universal Logical Accessibility (R_L): R_L is defined as a universal relation over N . Since all normal worlds share the same classical laws, they are mutually accessible at the logical level, with metaphysical equivalence classes (R_M) nested within this universal domain.

Evaluation of Contradictions: The I region contains clusters (R'_L) that obey alternative logical rules.⁵ Although these regions are not the focus of the system's deduction, their existence provides a rational explanation for why the **(ne)** constraint can provide concrete evaluation worlds for inconsistent hypotheses.

Let **SCP1** be the class of models satisfying these constraints; $\models_{\text{SCP1}}$ is hereafter abbreviated as $\models$.

4.2 Soundness and completeness

The axiomatic system for **SCP1** is an extension of the base system **SCP**, obtained by adding the following two axioms:

$$\begin{array}{ll}
 \text{(NE)} & \Diamond_L \varphi \rightarrow \neg(\varphi > \neg \varphi) \\
 \text{(SOC-L)} & \Diamond_L \varphi \wedge \Box_L \psi \rightarrow (\varphi > \psi)
 \end{array}$$

Lemma 4.2. $[\varphi] \cap [\neg \varphi] \cap N = \emptyset$.

Proof. On the set of normal worlds N , truth values are determined by the recursive definition in Definition 3.3. According to the condition $M, w \Vdash \neg \varphi \iff M, w \not\Vdash \varphi$, it follows that $[\varphi] \cap [\neg \varphi] \cap N = \emptyset$. $\square$

Theorem 4.3 (Soundness of **SCP1).** For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, if $\Gamma \vdash_{\text{SCP1}} \varphi$, then $\Gamma \models_{\text{SCP1}} \varphi$.

Proof. Refer to Theorem 3.9. We only need to prove the validity of (SOC-L) and (NE).

(SOC-L): Suppose $w \in N$ and $M, w \Vdash \Diamond_L \varphi \wedge \Box_L \psi$ holds. Since $M, w \Vdash \Box_L \psi$, we have $N \subseteq [\psi]$. Since $M, w \Vdash \Diamond_L \varphi$, it follows from the semantic constraints that $f(w, \varphi) \subseteq N$. Thus, $f(w, \varphi) \subseteq [\psi]$, which means $M, w \Vdash \varphi > \psi$.

⁵See [2] on using non-classical closure sets as impossible worlds.

(NE): Suppose $w \in N$ and $M, w \Vdash \Diamond_L \varphi$ holds. This implies $f(w, \varphi) \subseteq N$. By constraint (ne), $f(w, \varphi) \neq \emptyset$; thus, there exists $u \in f(w, \varphi)$. By constraint (id), $f(w, \varphi) \subseteq [\varphi]$. According to Lemma 4.2, $f(w, \varphi) \not\subseteq [\neg\varphi]$. Therefore, $M, w \not\models \varphi > \neg\varphi$, which is equivalent to $M, w \Vdash \neg(\varphi > \neg\varphi)$. $\square$

We construct a generated submodel M^{cg} based on the canonical model M^c defined in Definition 3.10, except that MCSs are based on **SCPI**.

Definition 4.4. Let Γ be an **SCPI**-consistent set, and let $w_\Gamma \in N^c$ be a maximal consistent set (MCS) containing Γ . We construct the generated submodel $M^{cg} = (W^{cg}, N^{cg}, R_M^{cg}, R_L^{cg}, f^{cg}, V^{cg})$ as follows:

- $W^{cg} = \mathcal{P}(\mathcal{L})$
 - $N^{cg} = N^c \cap R_L^c(w_\Gamma)$
 - $I^{cg} = W^c - N^{cg}$
- $R_M^{cg} = R_M^c \cap (R_L^c(w_\Gamma) \times R_L^c(w_\Gamma))$
- $R_L^{cg} = R_L^c \cap (R_L^c(w_\Gamma) \times R_L^c(w_\Gamma))$
- $f^{cg}(w, \varphi) = \begin{cases} \{x \in N^{cg} \mid \{\psi \mid \varphi > \psi \in w\} \subseteq x\} & \text{if } \Diamond_L \varphi \in w \\ \{x \in W^{cg} \mid \{\psi \mid \varphi > \psi \in w\} \subseteq x\} & \text{if } \Diamond_L \varphi \notin w \end{cases}$
- $V^{cg}(\varphi) = V^c(\varphi)$

Let $[[\varphi]]^{cg} := \{w \in W^{cg} \mid M^{cg}, w \Vdash \varphi\}$.

Through the construction of the generated submodel, we first verify whether it satisfies the universal relation requirement for logical accessibility in **SCPI**.

Lemma 4.5. R_L^{cg} is a universal relation on N^{cg} .

Proof. Since $N^{cg} = N^c \cap R_L^c(w_\Gamma)$ and $R_L^c(w_\Gamma) \subseteq N^c$, it follows that $R_L^c(w_\Gamma) = N^{cg}$. Thus, $R_L^{cg}(w_\Gamma) = R_L^c(w_\Gamma)$.

We show that for all $x, y \in R_L^{cg}(w_\Gamma)$, $xR_L^{cg}y$ holds. Suppose $\Box_L \varphi \in x$; we must show $\varphi \in y$. By axiom 4L, $\Box_L \Box_L \varphi \in x$. Suppose for contradiction that $\Box_L \Box_L \varphi \notin w_\Gamma$. Then $\Diamond_L \Diamond_L \neg\varphi \in w_\Gamma$. By axiom 5L, $\Box_L \Diamond_L \Diamond_L \neg\varphi \in w_\Gamma$, which implies $\Diamond_L \Diamond_L \neg\varphi \in x$, a contradiction. Therefore, $\Box_L \Box_L \varphi \in w_\Gamma$. By axiom TL, $\Box_L \varphi \in w_\Gamma$, which implies $\varphi \in y$ since $w_\Gamma R_L^c y$. $\square$

Lemma 4.6 (Truth Lemma). For all $w \in W^{cg}$, $M^{cg}, w \Vdash \alpha \iff \alpha \in w$.

Proof. For any $w \in I^{cg}$ and any $\alpha \in \mathcal{L}$, the truth conditions for anti-logical worlds give $M^{cg}, w \Vdash \alpha \iff w \in V^{cg}(\alpha) \iff \alpha \in w$.

For any $w \in N^{cg}$, we proceed by induction on the complexity of α . The cases for atomic propositions and Boolean connectives are standard.

Case $\alpha = \Box_M \varphi$:

($\Leftarrow$) Suppose $\Box_M \varphi \in w$. For all $x \in R_M^{cg}(w)$, since $R_M^{cg}(w) \subseteq R_M^c(w)$, we have $\varphi \in x$. By IH, $M^{cg}, x \Vdash \varphi$. Thus $M^{cg}, w \Vdash \Box_M \varphi$.

($\Rightarrow$) Suppose $\Box_M \varphi \notin w$. In the canonical model, there exists an MCS $u \in N^c$ such that $wR_M^c u$ and $\neg\varphi \in u$. We must strictly show $u \in N^{cg}$. Since $w \in N^{cg} = R_L^c(w_\Gamma)$, we have $w_\Gamma R_L^c w$. By the axiom $\Box_L \psi \rightarrow \Box_M \psi$, $R_M^c \subseteq R_L^c$, so $wR_L^c u$. S5 logic guarantees R_L^c is transitive, hence $w_\Gamma R_L^c u$. This implies $u \in R_L^c(w_\Gamma)$, so $u \in N^{cg}$. Since $w, u \in N^{cg}$ and $wR_M^c u$, we have $wR_M^{cg} u$. By IH, $M^{cg}, u \not\models \varphi$, therefore $M^{cg}, w \not\models \Box_M \varphi$.

Case $\alpha = \Box_L \varphi$:

($\Leftarrow$) Suppose $\Box_L \varphi \in w$. By Lemma 4.5, R_L^{cg} is a universal relation on N^{cg} . For any $x \in N^{cg}$, we have $wR_L^{cg}x$, which implies $\varphi \in x$. By the induction hypothesis, $M^{cg}, x \Vdash \varphi$, thus $M^{cg}, w \Vdash \Box_L \varphi$.

($\Rightarrow$) Suppose $\Box_L \varphi \notin w$. There exists an MCS $u \in N^c$ such that $wR_L^c u$ and $\neg \varphi \in u$. Since $w \in N^{cg}$, $w \Vdash R_L^c$. By the transitivity of R_L^c , $w \Vdash R_L^c u$, which perfectly places $u \in N^{cg}$. Thus $wR_L^{cg} u$. By IH, $M^{cg}, u \not\Vdash \varphi$, hence $M^{cg}, w \not\Vdash \Box_L \varphi$.

Case $\alpha = \varphi > \psi$:

($\Leftarrow$) Suppose $\varphi > \psi \in w$. By the definition of f^{cg} , for any $x \in f^{cg}(w, \varphi)$, $\psi \in x$. By the induction hypothesis, $M^{cg}, x \Vdash \psi$. Thus $M^{cg}, w \Vdash \varphi > \psi$.

($\Rightarrow$) Suppose $\varphi > \psi \notin w$.

If $\Diamond_L \varphi \notin w$: Construct $u = \{\chi \mid \varphi > \chi \in w\}$. Since $\varphi > \psi \notin w$, $\psi \notin u$. By the definition of f^{cg} , $u \in f^{cg}(w, \varphi)$. By IH, $M^{cg}, u \not\Vdash \psi$, hence $M^{cg}, w \not\Vdash \varphi > \psi$.

If $\Diamond_L \varphi \in w$: We must construct a world $u \in N^{cg}$ such that $u \in f^{cg}(w, \varphi)$ and $\psi \notin u$. We first verify that the set $\Gamma' = \{\chi \mid \varphi > \chi \in w\} \cup \{\neg \psi\}$ is consistent. If not, $\vdash (\chi_1 \wedge \dots \wedge \chi_n) \rightarrow \psi$. By the (SIC) rule, $\vdash \Diamond_L \varphi \rightarrow (((\varphi > \chi_1) \wedge \dots \wedge (\varphi > \chi_n)) \rightarrow (\varphi > \psi))$. Since $\Diamond_L \varphi \in w$ and $\varphi > \chi_i \in w$, this forces $\varphi > \psi \in w$, yielding a contradiction.

By Lindenbaum's Lemma, Γ' can be extended to an MCS $u \in N^c$. To prove $u \in N^{cg}$, we must show $u \in R_L^c(w_\Gamma)$. For any $\Box_L \gamma \in w_\Gamma$, by Axiom 4L, $\Box_L \Box_L \gamma \in w_\Gamma$. Since $w \in R_L^c(w_\Gamma)$, we have $\Box_L \gamma \in w$. By the (SOC-L) axiom, $\Diamond_L \varphi \wedge \Box_L \gamma \rightarrow (\varphi > \gamma)$. Since $\Diamond_L \varphi \in w$, it follows that $\varphi > \gamma \in w$, meaning $\gamma \in \Gamma' \subseteq u$. Thus $w_\Gamma R_L^c u$, which confirms $u \in N^{cg}$. Since $\{\chi \mid \varphi > \chi \in w\} \subseteq u$, we have $u \in f^{cg}(w, \varphi)$. Because $\neg \psi \in u$, $\psi \notin u$. By IH, $M^{cg}, u \not\Vdash \psi$. Therefore, $M^{cg}, w \not\Vdash \varphi > \psi$. $\square$

Finally, we verify that M^{cg} satisfies the specific constraints of **SCPI**.

Lemma 4.7. *The canonical model M^{cg} satisfies the following model constraints for all $w \in N^{cg}$:*

$$\begin{array}{ll}
 (rl) & R_L^{cg}(w) = N^{cg} \\
 (ne) & f^{cg}(w, \varphi) \neq \emptyset \\
 (modal) & R_M^{cg}(w) \subseteq R_L^{cg}(w) \\
 (id) & f^{cg}(w, \varphi) \subseteq [[\varphi]]^{cg} \\
 (soc-m) & \text{If } w \in [[\Diamond_M \varphi]], \text{ then } f^{cg}(w, \varphi) \subseteq R_M^{cg}(w) \\
 (sic) & \text{If } w \in [[\Diamond_L \varphi]], \text{ then } f^{cg}(w, \varphi) \subseteq N^{cg}
 \end{array}$$

Proof. **(rl):** Proven in Lemma 4.5.

(ne): If $\Diamond_L \varphi \notin w$, let $u = \{\psi \mid \varphi > \psi \in w\}$. Since $W^{cg} = \wp(\mathcal{L})$, u is a valid world in W^{cg} , ensuring $f^{cg}(w, \varphi) \neq \emptyset$. If $\Diamond_L \varphi \in w$, by axiom (NE), $\neg(\varphi > \neg \varphi) \in w$. By the Truth Lemma on M^{cg} , $M^{cg}, w \not\Vdash \varphi > \neg \varphi$, which strictly implies $f^{cg}(w, \varphi) \not\subseteq [[\neg \varphi]]^{cg}$. Therefore, $f^{cg}(w, \varphi)$ cannot be empty.

(modal): By definition, $R_M^{cg} = R_M^c \cap (R_L^c(w_\Gamma) \times R_L^c(w_\Gamma))$ and $R_L^{cg} = R_L^c \cap (R_L^c(w_\Gamma) \times R_L^c(w_\Gamma))$. Since $R_M^c(x) \subseteq R_L^c(x)$ holds for all $x \in N^c$ in the canonical model, it algebraically follows that $R_M^{cg}(w) \subseteq R_L^{cg}(w)$ for any $w \in N^{cg}$.

(id): For any $u \in f^{cg}(w, \varphi)$, by the axiom (ID), $\varphi > \varphi \in w$. By the definition of f^{cg} , this requires $\varphi \in u$. By the Truth Lemma on M^{cg} , $u \in [[\varphi]]^{cg}$. Thus $f^{cg}(w, \varphi) \subseteq [[\varphi]]^{cg}$.

(soc-m): Suppose $w \in [[\Diamond_M \varphi]]^{cg}$. By the Truth Lemma, $\Diamond_M \varphi \in w$. Since $\vdash \Diamond_M \varphi \rightarrow \Diamond_L \varphi$, we have $\Diamond_L \varphi \in w$. Therefore, by the definition of f^{cg} , $f^{cg}(w, \varphi) \subseteq N^{cg}$. Now, assume for contradiction there exists $u \in f^{cg}(w, \varphi)$ such that $u \notin R_M^{cg}(w)$. Since $u \in N^{cg}$, it must be that $u \notin R_M^c(w)$. By the definition of R_M^c , there exists ψ such that $\Box_M \psi \in w$ but $\psi \notin u$. By the (SOC-M) axiom, $\varphi > \psi \in w$, which requires $\psi \in u$ according to the definition of f^{cg} , establishing a contradiction. Thus $f^{cg}(w, \varphi) \subseteq R_M^{cg}(w)$.

(sic): Suppose $w \in [[\Diamond_L \varphi]]^{cg}$. By the Truth Lemma, $\Diamond_L \varphi \in w$. By the precise definition of the selection function in M^{cg} for this condition, $f^{cg}(w, \varphi) \subseteq N^{cg}$. $\square$

Theorem 4.8 (Completeness of **SCPI**). *If $\Gamma \models_{\text{SCPI}} \varphi$, then $\Gamma \vdash_{\text{SCPI}} \varphi$.*

4.3 Decidability

We prove that the system remains decidable despite the introduction of the non-emptiness constraint and the universal logical accessibility relation.

The subformula closure Σ_ϕ for **SCP1** follows Definitions 3.14 and 3.15, except that the expansion $\Box_M \Box_L \psi$ is no longer required for the **SCP1** closure set.

The filtration model for **SCP1** is constructed from a generated submodel of the canonical model. While the overall approach is consistent with standard methods, the definition of R_L^* is modified to accommodate the specific constraints of **SCP1**.

Definition 4.9 (Filtration Model). Given a generated submodel $M^{cg} = (W^{cg}, N^{cg}, R_M^{cg}, R_L^{cg}, f^{cg}, V^{cg})$, the filtration model targeting Σ_ϕ is defined as $M^* = (W^*, N^*, R_M^*, R_L^*, f^*, V^*)$, where:

- $W^* = \{\bar{w} \mid w \in W^{cg}\};$
- $N^* = \{\bar{w} \mid w \in N^{cg}\};$
- For any $\bar{w}, \bar{v} \in N^*$, $\bar{w} R_M^* \bar{v} \iff \forall \Box_M \psi \in \Sigma_\phi, (\Box_M \psi \in w \iff \Box_M \psi \in v);$
- For any $\bar{w}, \bar{v} \in N^*$, $\bar{w} R_L^* \bar{v} \iff \exists w' \in \bar{w}, \exists v' \in \bar{v}$ such that $w' R_L^{cg} v';$
- For any $\psi > \chi \in \Sigma_\phi$, let $U_{\psi, w} = \{\bar{v} \mid \forall (\psi > \chi) \in \Sigma_\phi, (\psi > \chi \in w \Rightarrow \chi \in v)\}$. Let

$$f^*(\bar{w}, \psi) = \begin{cases} U_{\psi, w} \cap R_M^*(\bar{w}) & \text{if } \Diamond_M \psi \in w \\ U_{\psi, w} \cap N^* & \text{if } \Diamond_M \psi \notin w \text{ and } \Diamond_L \psi \in w \\ U_{\psi, w} \cap W^* & \text{if } \Diamond_L \psi \notin w \end{cases}$$

- For other ψ , let

$$f^*(\bar{w}, \psi) = \begin{cases} [[\psi]]^* \cap R_M^*(\bar{w}) & \text{if } M^*, \bar{w} \Vdash \Diamond_M \psi \\ [[\psi]]^* \cap N^* & \text{if } M^*, \bar{w} \not\Vdash \Diamond_M \psi \text{ and } M^*, \bar{w} \Vdash \Diamond_L \psi, \\ [[\psi]]^* & \text{otherwise} \end{cases}$$

where $[[\psi]]^* := \{w \in W^* \mid M^*, w \Vdash \psi\};$

- $V^*(\psi) = \{\bar{w} \mid w \in V^{cg}(\psi)\}.$

Lemma 4.10. *The model is well-defined. For any $\bar{w}, \bar{v} \in N^*$ and any $(\psi > \chi) \in \Sigma_\phi$:*

1. *If $w \sim v$, then $R_M^*(\bar{w}) = R_M^*(\bar{v})$;*
2. *R_M^* is an equivalence relation on N^* ;*
3. *R_L^* is a universal relation on N^* ;*
4. *If $w \sim v$, then $R_L^*(\bar{w}) = R_L^*(\bar{v})$;*
5. *If $w \sim v$, then $f^*(\bar{w}, \psi) = f^*(\bar{v}, \psi)$.*

Proof. We only need to prove Items 3, 4, and 5. Items 1 and 2 follow analogously to Lemma 3.17, as the logical structure of R_M^* relies on $\Box_M$ -formulas which behave identically in the closure Σ_ϕ of both systems.

Item 3: Suppose $\bar{w}, \bar{v} \in N^*$. By the definition of N^* , there exist specific representatives $w' \in \bar{w}$ and $v' \in \bar{v}$ such that $w', v' \in N^{cg}$. As proven in Lemma 4.5, R_L^{cg} is a universal relation on N^{cg} , meaning $w' R_L^{cg} v'$ inherently holds. According to the existential definition of R_L^* in the filtration model, the presence of such w' and v' directly yields $\bar{w} R_L^* \bar{v}$. Thus, R_L^* is strictly a universal relation on N^* .

Item 4: Since R_L^* is a universal relation on N^* (as established in Point 4), the relational image of any element in N^* is the entire set N^* . Therefore, if $w \sim v$, both $\bar{w}$ and $\bar{v}$ designate elements in N^* , trivially yielding $R_L^*(\bar{w}) = N^* = R_L^*(\bar{v})$.

Item 5: Suppose $w \sim v$. For any $(\psi > \chi) \in \Sigma_\phi$, since $w \sim v$, we have $\psi > \chi \in w \iff \psi > \chi \in v$. This immediately ensures that the base sets are identical: $U_{\psi,w} = U_{\psi,v}$. Crucially, by the definition of the subformula closure Σ_ϕ , the presence of $(\psi > \chi) \in \Sigma_\phi$ guarantees that both modal preconditions $\Diamond_M \psi \in \Sigma_\phi$ and $\Diamond_L \psi \in \Sigma_\phi$ hold. Since $w \sim v$, w and v must categorically agree on the presence of $\Diamond_M \psi$ and $\Diamond_L \psi$. This logical synchronization guarantees that w and v trigger the exact same conditional branch in the piecewise definition of f^* . Because $U_{\psi,w} = U_{\psi,v}$, $R_M^*(\bar{w}) = R_M^*(\bar{v})$ (from Point 1), and N^* as well as W^* are model constants, the subsequent intersection operations across any of the three branches will mathematically yield identical results. Therefore, $f^*(\bar{w}, \psi) = f^*(\bar{v}, \psi)$. $\square$

Lemma 4.11 (Filtration Lemma). *For all $\psi \in \Sigma_\phi$ and all $w \in W^{cg}$:*

$$M^{cg}, w \Vdash \psi \iff M^*, \bar{w} \Vdash \psi$$

Proof. For any $\bar{w} \notin N^*$ and $\psi \in \Sigma_\phi$, by truth conditions, $M^{cg}, w \Vdash \psi \iff w \in V^{cg}(\psi) \iff \bar{w} \in V^*(\psi) \iff M^*, \bar{w} \Vdash \psi$.

For $\bar{w} \in N^*$, we proceed by induction on the complexity of ψ . We only prove the cases $\psi = \Box_L \gamma$ and $\psi = \alpha > \beta$; other cases refer to Lemma 3.18.

Case $\psi = \Box_L \gamma$:

($\Rightarrow$) Suppose $M^{cg}, w \Vdash \Box_L \gamma$. Since R_L^* is a universal relation on N^* (by Lemma 4.10), we must show that for any $\bar{v} \in N^*$, $M^*, \bar{v} \Vdash \gamma$. Because $\Box_L \gamma \in w$ and R_L^{cg} is a universal relation on N^{cg} , for any $v \in N^{cg}$, $\gamma \in v$. Given $N^* = \{\bar{w} \mid w \in N^{cg}\}$ and IH, for any $\bar{v} \in N^*$, $M^*, \bar{v} \Vdash \gamma$.

($\Leftarrow$) Suppose $M^{cg}, w \nVdash \Box_L \gamma$. Then there exists $v \in N^{cg}$ such that $v \nVdash \gamma$. Since $\bar{v} \in N^*$ and R_L^* is a universal relation, $\bar{w} R_L^* \bar{v}$ holds. By IH, $M^*, \bar{v} \nVdash \gamma$. Thus, $M^*, \bar{w} \nVdash \Box_L \gamma$.

Case $\psi = \alpha > \beta$:

($\Rightarrow$) Suppose $M^{cg}, w \Vdash \alpha > \beta$. By the Truth Lemma on M^{cg} , $\alpha > \beta \in w$. We need to show that for any $\bar{v} \in f^*(\bar{w}, \alpha)$, $M^*, \bar{v} \Vdash \beta$. By the definition of f^* in the filtration model, regardless of whether $\Diamond_M \alpha$ or $\Diamond_L \alpha$ belongs to w , it must hold that $\bar{v} \in U_{\alpha,w}$. By the definition of $U_{\alpha,w}$, for all $(\alpha > \chi) \in \Sigma_\phi$, if $\alpha > \chi \in w$, then $\chi \in v$. Since $\alpha > \beta \in \Sigma_\phi$ and $\alpha > \beta \in w$, we necessarily have $\beta \in v$. By the induction hypothesis, $M^*, \bar{v} \Vdash \beta$. Thus, $f^*(\bar{w}, \alpha) \subseteq [[\beta]]^*$, which implies $M^*, \bar{w} \Vdash \alpha > \beta$.

($\Leftarrow$) Suppose $M^{cg}, w \nVdash \alpha > \beta$. By the Truth Lemma on M^{cg} , $\alpha > \beta \notin w$. Based on the definition of the selection function f^{cg} in the generated submodel M^{cg} , there exists a world $u \in f^{cg}(w, \alpha)$ such that $\beta \notin u$. We consider its equivalence class $\bar{u}$ and prove $\bar{u} \in f^*(\bar{w}, \alpha)$ by examining three cases according to the definition of f^* :

If $\Diamond_M \alpha \in w$: According to the (soc-m) constraint in Lemma 4.7, $f^{cg}(w, \alpha) \subseteq R_M^{cg}(w)$. Since $u \in f^{cg}(w, \alpha)$, it follows that $u \in R_M^{cg}(w)$. Because R_M^{cg} is an equivalence relation on N^{cg} , w and u agree on all $\Box_M$ -formulas. Therefore, for any $\Box_M \gamma \in \Sigma_\phi$, $\Box_M \gamma \in w \iff \Box_M \gamma \in u$. By definition, this means $\bar{w} R_M^* \bar{u}$. Meanwhile, since $\chi \in u$ holds for all $\alpha > \chi \in w$, we have $\bar{u} \in U_{\alpha,w}$. Consequently, $\bar{u} \in U_{\alpha,w} \cap R_M^*(\bar{w}) = f^*(\bar{w}, \alpha)$.

If $\Diamond_M \alpha \notin w$ and $\Diamond_L \alpha \in w$: According to the properties of f^{cg} and the (sic) constraint in Lemma 4.7, $f^{cg}(w, \alpha) \subseteq N^{cg}$. Since $u \in f^{cg}(w, \alpha)$, we immediately obtain $u \in N^{cg}$. By the definition of N^* , it follows that $\bar{u} \in N^*$. Combined with $\bar{u} \in U_{\alpha,w}$, we get $\bar{u} \in U_{\alpha,w} \cap N^* = f^*(\bar{w}, \alpha)$.

If $\Diamond_L \alpha \notin w$: Since $u \in W^{cg}$, we have $\bar{u} \in W^*$. Combined with $\bar{u} \in U_{\alpha,w}$, we get $\bar{u} \in U_{\alpha,w} \cap W^* = f^*(\bar{w}, \alpha)$.

In all cases, we have established $\bar{u} \in f^*(\bar{w}, \alpha)$. Because $\beta \notin u$, IH yields $M^*, \bar{u} \not\models \beta$. Therefore, $M^*, \bar{w} \not\models \alpha > \beta$. $\square$

Lemma 4.12. *The filtration model satisfies the model constraints; that is, for all $\psi > \chi \in \Sigma_\phi$ and $\bar{w} \in N^*$:*

$$\begin{array}{llll} (rl) & R_L^*(\bar{w}) = N^* & (id) & f^*(\bar{w}, \psi) \subseteq [[\psi]]^* \\ (ne) & f^*(\bar{w}, \psi) \neq \emptyset & (soc-m) & \text{If } M^*, \bar{w} \Vdash \Diamond_M \psi, \text{ then } f^*(\bar{w}, \psi) \subseteq R_M^*(\bar{w}) \\ (modal) & R_M^*(\bar{w}) \subseteq R_L^*(\bar{w}) & (sic) & \text{If } M^*, \bar{w} \Vdash \Diamond_L \psi, \text{ then } f^*(\bar{w}, \psi) \subseteq N^* \end{array}$$

Proof. **(rl):** As proved in Lemma 4.10, $R_L^*(\bar{w}) = N^*$.

(id): For any of the three cases in the definition of f^* , it algebraically holds that $f^*(\bar{w}, \psi) \subseteq U_{\psi, w}$. Since $\psi > \chi \in \Sigma_\phi$, for any $\bar{u} \in U_{\psi, w}$, the construction condition implies $\psi \in u$. By the Filtration Lemma, $M^*, \bar{u} \Vdash \psi$, meaning $U_{\psi, w} \subseteq [[\psi]]^*$. Therefore, $f^*(\bar{w}, \psi) \subseteq [[\psi]]^*$.

(ne): Since $f^{cg}(w, \psi) \neq \emptyset$, there exists $v \in f^{cg}(w, \psi)$. By the definition of the canonical selection function, for every $(\psi > \chi) \in \Sigma_\phi$, we have $\psi > \chi \in w \implies \chi \in v$, hence $\bar{v} \in U_{\psi, w}$. We then consider three cases:

If $\Diamond_M \psi \in w$, then $f^{cg}(w, \psi) \subseteq R_M^{cg}(w)$, so $v \in R_M^{cg}(w)$. Since R_M^{cg} is an equivalence relation on N^{cg} , w and v agree on all $\Box_M$ -formulas in Σ_ϕ , hence $\bar{v} \in R_M^*(\bar{w})$. Therefore $\bar{v} \in U_{\psi, w} \cap R_M^*(\bar{w}) = f^*(\bar{w}, \psi)$.

If $\Diamond_M \psi \notin w$ and $\Diamond_L \psi \in w$, then $f^{cg}(w, \psi) \subseteq N^{cg}$. Thus $v \in N^{cg}$, and by definition $\bar{v} \in N^*$. Therefore $\bar{v} \in U_{\psi, w} \cap N^* = f^*(\bar{w}, \psi)$.

If $\Diamond_L \psi \notin w$, then $f^*(\bar{w}, \psi) = U_{\psi, w} \cap W^*$. Since $\bar{v} \in W^*$, $\bar{v} \in U_{\psi, w} \cap W^* = f^*(\bar{w}, \psi)$. In all cases, we find $\bar{v} \in f^*(\bar{w}, \psi)$, meaning $f^*(\bar{w}, \psi) \neq \emptyset$.

(soc-m): Suppose $M^*, \bar{w} \Vdash \Diamond_M \psi$. By the Filtration Lemma, $\Diamond_M \psi \in w$. According to the first case of the definition of f^* , $f^*(\bar{w}, \psi) = U_{\psi, w} \cap R_M^*(\bar{w})$. It directly follows that $f^*(\bar{w}, \psi) \subseteq R_M^*(\bar{w})$.

(modal): Since $R_M^*(\bar{w})$ is defined as an equivalence relation on N^* , we naturally have $R_M^*(\bar{w}) \subseteq N^*$. As proven in Lemma 4.10, $R_L^*(\bar{w}) = N^*$, thus $R_M^*(\bar{w}) \subseteq R_L^*(\bar{w})$.

(sic): Suppose $M^*, \bar{w} \Vdash \Diamond_L \psi$. By the Filtration Lemma, $\Diamond_L \psi \in w$. We consider the two applicable cases in the definition of f^* : If $\Diamond_M \psi \in w$, then $f^*(\bar{w}, \psi) = U_{\psi, w} \cap R_M^*(\bar{w})$. Since $R_M^*(\bar{w}) \subseteq N^*$, we immediately have $f^*(\bar{w}, \psi) \subseteq N^*$. If $\Diamond_M \psi \notin w$, then $f^*(\bar{w}, \psi) = U_{\psi, w} \cap N^*$, which obviously implies $f^*(\bar{w}, \psi) \subseteq N^*$. In both scenarios, the constraint is perfectly satisfied. The case when ψ is not an antecedent of a conditional in Σ_ϕ can be proved analogously. $\square$

Theorem 4.13. *SCP1 has finite model property and is decidable.*

5 Application

To clearly demonstrate the theoretical advantages of the stratified semantic framework, this section uses specific case analyses to illustrate how the **SCP** and **SCP1** systems formally and semantically distinguish between ordinary counterfactuals, anti-metaphysical (anti-essential) conditionals, and anti-logical conditionals. Furthermore, it highlights the necessity of **SCP1** in resolving the problem of vacuous truth.

Let q be a metaphysical necessity, such as “water is H_2O ”. In the actual world $w \in N$, $\Box_M q$ is true.

Case 1: Ordinary Counterfactuals Let p_1 be “I missed the bus”. This is a metaphysically possible hypothesis, meaning $M, w \Vdash \Diamond_M p_1$.

According to the **SCP** semantic constraint (soc-m), if $w \in [\Diamond_M p_1]$, the selection function is restricted to the metaphysically accessible domain, i.e., $f(w, p_1) \subseteq R_M(w)$. Since $\Box_M q$ holds, water is H_2O in

all worlds within $R_M(w)$. Therefore, $f(w, p_1) \subseteq [q]$ necessarily holds, and the conditional $p_1 > q$ (“If I missed the bus, water would still be H_2O ”) is true. Metaphysical necessity is strictly preserved under ordinary counterfactual hypotheses.

Case 2: Anti-Metaphysical Conditionals Let p_2 be “water is XYZ”. This is a metaphysically impossible but logically possible hypothesis, meaning $M, w \Vdash \neg \Diamond_M p_2 \wedge \Diamond_L p_2$.

Here, $\Diamond_M p_2$ is false, so the constraint (soc-m) is not triggered, and the selection function $f(w, p_2)$ must reach beyond $R_M(w)$. However, because the antecedent is logically possible ($\Diamond_L p_2$ is true), the constraint (sic) is activated, requiring $f(w, p_2) \subseteq N$. The selection function extends beyond the metaphysical domain into the logic-normal world N , where specific metaphysical laws may fail. In this broader space, q is no longer universally true. This precisely corresponds to the **SCP** proposition $\Box_M \psi \not\models \Diamond_L \phi \rightarrow (\phi > \psi)$. The system successfully allows us to suspend metaphysical laws while still using classical logic for meaningful deduction within N .

Case 3: Anti-Logical Conditionals Let p_3 be “Hobbes squared the circle”, which is a logical impossibility, meaning $M, w \Vdash \neg \Diamond_L p_3$. Let r be “sick children in Afghanistan would be cured”.

Since $\Diamond_L p_3$ is false, the constraint (sic) protecting classical logical deduction is deactivated. The selection function $f(w, p_3)$ cannot find a world in N where the antecedent is true, and thus must point to the anti-logical world set I . In Case 2, although the evaluation deviates from actuality, the worlds remain in N , enjoying the recursively defined truth conditions of classical logic. In Case 3, however, the worlds fall into I , where the truth values of complex formulas (such as conjunctions and disjunctions) are no longer evaluated recursively. Logical laws completely collapse here.

Although **SCP** establishes a refined stratified structure $N/R_M/I$, as a foundational framework, it permits the selection function f to be empty. This exposes a theoretical vulnerability when handling Case 3. If we construct a specific **SCP** model where no world in the anti-logical domain I satisfies “Hobbes squared the circle” (i.e., $f(w, p_3) = \emptyset$), then since the empty set is trivially a subset of any set, the conditional $p_3 > r$ would be vacuously true simply because its antecedent cannot be satisfied. This means **SCP** might still compromise with vacuism in certain local models.

The extended system **SCP1** introduces the non-emptiness constraint (ne), forcing $f(w, \phi) \neq \emptyset$ for all normal worlds w and formulas ϕ . When faced with p_3 , **SCP1** forces the model to presuppose a sufficiently rich domain I and select at least one world containing this logical contradiction for evaluation. In these selected anti-logical worlds, due to the lack of recursive logical constraints, $f(w, p_3)$ will not automatically be a subset of $[r]$ unless there is a substantive connection between “squaring the circle” and “curing Afghan children” in the specific valuation of the model. Consequently, **SCP1** can robustly evaluate $p_3 > r$ as false.

6 Conclusion

This paper develops a formal theory to distinguish impossibility through the stratified counterpossible logic **SCP** and its extension **SCP1**. We establish the soundness, completeness, finite model property, and decidability for both axiomatic systems.

To address the problem of vacuous truth, **SCP1** requires the selection function to be non-empty for all antecedents, including logical contradictions. By presupposing a robust set of anti-logical worlds I , the system ensures that the truth values of counterpossibles depend on the substantive connection between the antecedent and the consequent.

The hierarchical framework proposed here is generalizable beyond metaphysical and logical modalities. The relations R_M and R_L can be applied to other modal hierarchies, such as distinguishing primary from secondary obligations in deontic logic, or modeling cognitive blind spots in epistemic logic. This layered approach offers a semantic path for resolving modal conflicts and characterizing cognitive limitations.

Future research should further refine the internal structure of impossible worlds. While this paper treats the anti-logical domain I as a space with minimal constraints, future work could classify I using specific non-classical logics, such as paraconsistent systems. This would allow for a more precise characterization of the intermediate domain between metaphysical impossibility and absolute logical inconsistency.

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Pointed Modal Abelian Logic, Algebraically

Filip Jankovec 

Institute of Computer Science of the Czech Academy of Sciences
Department of Algebra, Faculty of Mathematics and Physics, Charles University
Prague, The Czech Republic
jankovec@cs.cas.cz

Wolfgang Poiger 

Institute of Computer Science of the Czech Academy of Sciences
Prague, The Czech Republic.
poiger@cs.cas.cz

In this article, we investigate the pointed modal logic of reals. We first establish its relational (Kripke) semantics with bounded valuations in the Abelian ℓ -group of real numbers with the distinguished negative constant -1 . To study this logic algebraically, we introduce the variety of negatively pointed modal Abelian ℓ -groups, in particular we focus on the strongly pointed members thereof. Constructing complex algebras and canonical frames, we establish a Truth Lemma connecting the relational and algebraic frameworks. Since finitary axiomatizations cannot fully capture the Kripke validities of the reals we introduce further algebraic constraints, in particular including an infinitary Archimedean-style rule. Finally, we prove a corresponding ‘infinitary algebraic completeness’ result for pointed modal Abelian logic with respect to the variety of pointed modal Abelian ℓ -groups.

1 Introduction

Abelian logic, independently introduced by Meyer & Slaney [19] and Casari [7], is a well-known contra-classical, paraconsistent logic. It is also referred to as the *logic of Abelian ℓ -groups* [6] or *Abelian Group Logic* [20] since the matrix models of Abelian logic are based on Abelian ℓ -groups, their positive cones being the filters of designated elements. A primary motivation for studying Abelian logic is its connection to *Łukasiewicz logic*, since there exists a well-known categorical equivalence between the categories of strongly pointed Abelian ℓ -groups and MV-algebras, the algebras of Łukasiewicz logic [8].

Recently, *modal Abelian logic* has also gained some attention, particularly from the perspective of its real-valued Kripke semantics [11, 12, 18]. Following this, we investigate *pointed real-valued modal Abelian logic* from an algebraic perspective. Our motivation for this focus is two-fold. First, it structurally parallels real-valued modal Łukasiewicz logic. Exploiting this connection, *pointed* modal Abelian logic (albeit not explicitly named therein) is already used in [12, Sect. 2.3] to directly analyze real-valued modal Łukasiewicz logic [15]. Specifically, their analysis requires the inclusion of a designated constant c to act as a fixed point under the $\Box$ -modality (i.e., $\Box c = c$), which serves as the falsum for Łukasiewicz logic. Furthermore, by substituting bounded MV-algebras for Abelian ℓ -groups, our framework lifts the traditional restriction to the bounded interval $[0, 1]$, allowing formulas to instead be evaluated over $\mathbb{R}$. Second, this framework allows us to introduce a modal extension of *Łukasiewicz unbound logic*. Originally defined in [10], this logic is strongly complete with respect to the Abelian ℓ -group of real numbers equipped with the designated element -1 . Consequently, by equipping our real-valued Kripke models with the same constant -1 , we provide a natural modal semantics for Łukasiewicz unbound logic. Despite these connections, the *algebraic* aspects of pointed modal Abelian logic have hitherto remained unexplored.

In this article we close this gap, introducing the variety of (negatively) *pointed modal Abelian ℓ -groups* $\mathbf{mAb}_p^-$ (Definition 3.4) and explore its relationship to real-valued Kripke semantics of pointed modal Abelian logic (with constant -1). We do this by constructing complex algebras in $\mathbf{mAb}_p^-$ from Kripke frames (entailing an algebraic soundness result, Proposition 3.9) and vice versa. Our first main result is a Truth Lemma for *strongly* negatively pointed modal Abelian ℓ -groups (Corollary 4.6). Using this, we characterize the validities of pointed modal Abelian logic via an equational condition (A_∞) on the variety $\mathbf{mAb}_p^-$, which may be seen as a form of (infinitary) *algebraic completeness* (Theorem 4.9).

The primary technical challenges to overcome here stem from the algebraic behavior of strong units and the non-trivial intersection of maximal ℓ -ideals. In general, a non-zero element of a negatively pointed modal Abelian ℓ -group may belong to every maximal ℓ -ideal. This effectively prevents us from using such an element as a falsifier for a formula, since no world in the corresponding canonical frame would evaluate it as non-zero. The rule (A_∞) ensure this does not happen. Furthermore, this rule allows us to argue via *strongly* pointed algebras in our proof of Theorem 4.9. While it is a well-known fact that the property of possessing a strong unit is not first-order definable, enforcing a strong unit condition is justified within our framework since it corresponds directly to the restriction to *bounded* valuations in real-valued Kripke models (required to guarantee the semantics of the $\Box$ -modality is well-defined). We leave it up for future research to determine whether these issues could be circumvented in other ways.

- **Related Work.** Prior research on *real-valued modal logic* [12, 11] and *modal Łukasiewicz logic* [15] has been a main inspiration for the current paper. More specifically, we build on the definition of (pointed) real-valued modal logic of the former and introduce an infinitary rule similar to the latter (here, let us also point out [2] about *continuous modal logic* and [17] about *modal Riesz spaces*). More generally, our work stands in the research tradition on *many-valued modal logic* (e.g. [13, 4, 15, 5, 21]), note that we are using crisp accessibility relations (like [21] for Gödel and [15] for Łukasiewicz modal logic). Naturally the current paper is also related to the *algebraic* theory of *lattice-ordered Abelian groups* and *MV-algebras* [8]. In particular, for more context about *pointed* Abelian ℓ -groups and their logic see [10, 23, 16].

- **Structure of the article.** In Section 2, we provide some preliminaries about modal Abelian ℓ -groups and proving some helpful results about ℓ -ideals. In Section 3, we introduce the real-valued relational semantics of pointed modal Abelian logic (Subsection 3.1) as well as the variety of pointed modal Abelian ℓ -groups (Subsection 3.2). We also explain how to relate these two via complex algebras and canonical frames (Subsection 3.3). Section 4 contains the main results of the article, namely a Truth Lemma for strongly pointed algebras (Subsection 4.1) and the infinitary algebraic completeness derived thereof (Subsection 4.2). We discuss potential directions for future research in the concluding Section 5.

2 Pointed Abelian ℓ -groups

In this (mostly preliminary) section, we recall some important facts about pointed Abelian ℓ -groups, focusing in particular on structures equipped with a negative strong unit (the reader should feel free to skip some of the more technical proofs here until we refer back to the corresponding results later on). We give a brief exposition of ℓ -ideal theory, and analyze the structure of ℓ -ideals in the presence of such a designated unit. Adapting techniques from the theory of MV-algebras, we characterize the radical, the intersection of all maximal ℓ -ideals, via infinitesimal elements. Lastly, we establish a pivotal correspondence between the maximal ℓ -ideals of a strongly negatively pointed Abelian ℓ -group and its homomorphisms into the pointed Abelian ℓ -group of reals $\mathbf{R}_{-1}$, which will be an important ‘bridge’ between the abstract algebraic syntax and our concrete real-valued semantics.

Abelian logic (e.g. see [19, 7, 10]) can be elegantly axiomatized as the negation-free fragment of FL_e (i.e. Full Lambek calculus with exchange) extended by the axiom of relativity $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow \varphi$. Due to our algebraic focus, here we only treat Abelian logic via its algebraic models, i.e. Abelian ℓ -groups. Nevertheless, let us mention that the logics corresponding to the varieties defined in this paper inherit similar axiomatizations, since they are superabelian (see [10]).

Definition 2.1 (Abelian ℓ -group). An algebra $\mathbf{A} = \langle A, +, -, \vee, \wedge, 0 \rangle$ is an *Abelian ℓ -group* if $\langle A, +, -, 0 \rangle$ is an Abelian group, $\langle A, \vee, \wedge \rangle$ is a lattice and $\mathbf{A}$ satisfies the monotonicity condition $x \leq y \Rightarrow x + z \leq y + z$.

When brackets are omitted the lattice operations take precedence over the group operations, e.g. $x + y \wedge z$ is understood as $x + (y \wedge z)$. For $a \in A$ we define the *absolute value* $|a| := a \vee -a$ and we let $n \cdot a := a + \dots + a$ denote the n -fold sum for every $n \in \mathbb{N}$. For $z \in \mathbb{Z} \setminus \mathbb{N}$ we set $z \cdot a := (-z) \cdot (-a)$.

Definition 2.2 (Strong Unit). Let $\mathbf{A}$ be an Abelian ℓ -group. We say that $p \in A$ is a *strong unit* of $\mathbf{A}$ if

$$\forall a \in A: \exists z \in \mathbb{Z}: z \cdot p \geq a. \quad (\text{SU})$$

Note that for a strictly negative strong unit $p < 0$ the condition (SU) simplifies to

$$\forall a \in A: \exists n \in \mathbb{N}: n \cdot p \leq a. \quad (\text{SU-})$$

An algebra $\mathbf{A}_p = \langle A, +, -, \vee, \wedge, 0, p \rangle$ is a *pointed Abelian ℓ -group* if $\langle A, +, -, \vee, \wedge, 0 \rangle$ is an Abelian ℓ -group and $p \in A$. It is *strongly pointed* if p is a strong unit and *negatively pointed* if $p \leq 0$.

Remark 2.3. Traditionally, a strong unit is required to be strictly positive (e.g. see [3, p. 300]). Note that we use $\mathbb{Z}$ in (SU), which allows negative strong units. This distinction purely serves for notational convenience: since we primarily work in the negative cone, it avoids a proliferation of minus signs in our proofs. While our definition differs from the standard one, the structural role is identical: $p < 0$ is a strong unit in our sense if and only if its additive inverse $-p > 0$ is a standard strong unit.

As a standard example, let $\mathbf{R}_{-1} = \langle \mathbb{R}, +, -, \vee, \wedge, 0, -1 \rangle$ be the Abelian group of real numbers with addition together with its linear order and the designated point -1 . This clearly is a strongly negatively pointed Abelian ℓ -group (in fact, it is our *algebra of truth-degrees* later on).

The following properties of Abelian ℓ -groups are immediate consequences of the well-known fact that the variety of Abelian ℓ -groups is generated by the ℓ -group of integers $\mathbf{Z}$ (see [1, Thm. 6.1]).

Lemma 2.4. Let $\mathbf{A} = \langle A, +, -, \vee, \wedge, 0 \rangle$ be an Abelian ℓ -group and $a, b \in A$.

- (i) $a = a \wedge 0 + a \vee 0$,
- (ii) $((a - b) \vee 0) \wedge ((b - a) \vee 0) = 0$,
- (iii) $(a \vee b) + c = (a + c) \vee (b + c)$,
- (iv) $(a - b) \vee 0 = a - a \wedge b$,
- (v) $|a + b| \leq |a| + |b|$.

Definition 2.5 (ℓ -ideal). An ℓ -ideal of an Abelian ℓ -group $\mathbf{A}$ is an ℓ -subgroup $I \subseteq \mathbf{A}$ that is *convex* (equivalently $0 \leq |x| \leq |i|$ for some $i \in I$ implies $x \in I$). An ℓ -ideal I is *prime* if it satisfies

$$a \wedge b \in I \Rightarrow a \in I \text{ or } b \in I. \quad (\text{PI})$$

First, let us note that for an ℓ -ideal I and $a \in A$ by definition it holds that $a \in I$ iff $|a| \in I$ iff $-|a| \in I$. One can easily prove (using Zorn's Lemma) that for each element $a \in A \setminus \{0\}$ there exists an ℓ -ideal which is maximal among all the ℓ -ideals which do not contain a . Every such ℓ -ideal is prime and we refer to them as *values* of a [22, Thm. 2.4.2].

It is well known that *maximal* ℓ -ideals are prime (again see [22, Thm. 2.4.2]), although in general an Abelian ℓ -group $\mathbf{A}$ need not have any maximal ideals. However, if $p \in A$ is a strong unit, the maximal ℓ -ideals correspond to values of p (since any ℓ -ideal containing p is improper). In fact, in the presence of a strong unit every proper ℓ -ideal $I \subseteq \mathbf{A}$ can be extended to a maximal ℓ -ideal $M \supseteq I$ (which is again easily shown via Zorn's Lemma).

Let $\mathbf{A}$ be an Abelian ℓ -group and $S \subseteq A$. By $C(S)$ we denote the ℓ -ideal generated by S . It is well-known [22, Thm. 2.2.4 (b)] that

$$C(S) := \{x \in A \mid |x| \leq |b_1| + \cdots + |b_n| \text{ for some } b_1, \dots, b_n \in S\}. \quad (\text{IG})$$

Let us also recall from [22, Thm. 2.2.4 (d)], that for two sets $S, T \subseteq A$ we have

$$C(S) \vee C(T) = C(S \cup T) = \{x \in A \mid |x| \leq |b_1| + \cdots + |b_n| \text{ for some } b_1, \dots, b_n \in S \cup T\}.$$

Therefore, for a principal ℓ -ideal $J = \langle a \rangle = \{x \mid \exists n \in \mathbb{N}: |x| \leq n \cdot |a|\}$, we obtain

$$I \vee J = I \vee \{x \mid \exists n \in \mathbb{N}: |x| \leq n \cdot |a|\} = \{x \mid \exists n \in \mathbb{N} \exists i \in I: |x| \leq i + n \cdot |a|\}.$$

In particular, this implies that for every maximal ℓ -ideal I of an Abelian ℓ -group $\mathbf{A}$ and $a \notin I$:

$$A = \{x \mid \exists n \in \mathbb{N}: \exists i \in I: |x| \leq i + n \cdot |a|\}. \quad (\text{IG*})$$

For a strongly pointed Abelian ℓ -group $\mathbf{A}_p$ we define the *radical* of $\mathbf{A}_p$ as the intersection of all maximal ideals of $\mathbf{A}_p$ and denote it by $\text{rad}(\mathbf{A}_p)$. Equivalently, $\text{rad}(\mathbf{A}_p)$ is the intersection of all values of p . We call $a \in A$ is *infinitely small* (or *infinitesimal*) if

$$\forall n \in \mathbb{N}: n \cdot |a| \leq |p|. \quad (\text{INF})$$

In the case where $a, p \leq 0$, Equation (INF) simplifies to

$$\forall n \in \mathbb{N}: n \cdot a - p \geq 0. \quad (\text{INF-})$$

One can easily check that this is also equivalent to $\forall k \in \mathbb{N}: 2^k \cdot a - p \geq 0$.

We now show that the radical of a strongly pointed Abelian ℓ -group consists exactly of its infinitesimal elements. The proof of the following is similar to the one of [8, Prop. 3.6.4].

Lemma 2.6. *Let $\mathbf{A}_p$ be a strongly pointed Abelian ℓ -group. Then $a \in \text{rad}(\mathbf{A}_p)$ iff a is infinitely small.*

Proof. If $a \notin \text{rad}(\mathbf{A}_p)$ there is a maximal ℓ -ideal I such that $a \notin I$. Consequently, p is in the ℓ -ideal generated by $\{a\} \cup I$ and thus by (IG*) there are $n \in \mathbb{N}$ and $i \in I$ such that $0 < |p| \leq n \cdot |a| + i$, which is equivalent to $|p| - n \cdot |a| \leq i$. Assume towards contradiction that a is infinitely small. Then $|p| - (n+1) \cdot |a| \geq 0$ for each $n \in \mathbb{N}$, equivalently $|p| - n \cdot |a| \geq |a|$ for each $n \in \mathbb{N}$. Thus altogether we have $0 \leq |a| \leq |p| - n \cdot |a| \leq i$, which by convexity of the ℓ -ideal I implies $a \in I$, a contradiction.

Conversely, assume $a \in A$ is not infinitely small. Then there is $n \in \mathbb{N}$ such that $n \cdot |a| - |p| \not\leq 0$, which implies $(n \cdot |a| - |p|) \vee 0 > 0$. Let P be a value of $(n \cdot |a| - |p|) \vee 0$. By Lemma 2.4 (ii) we have

$$((n \cdot |a| - |p|) \vee 0) \wedge (|p| - n \cdot |a| \vee 0) = 0.$$

Since P is prime, it follows that $(|p| - n \cdot |a|) \vee 0 \in P$. Let M be a maximal ℓ -ideal containing P and towards contradiction assume that $a \in M$. Then $(n \cdot |a|) \vee 0 \in M$ and

$$(n \cdot |a|) \vee 0 \geq (n \cdot |a| - |p|) \vee 0 > 0.$$

Therefore, $(n \cdot |a| - |p|) \vee 0 \in M$. Consequently, by Lemma 2.4(i) we find

$$(|p| - n \cdot |a|) \vee 0 - (n \cdot |a| - |p|) \vee 0 = (|p| - n \cdot |a|) \vee 0 + (|p| - n \cdot |a|) \wedge 0 = (|p| - n \cdot |a|) \in M.$$

Since $p \notin M$ we have $|p| \notin M$ and thus also $n \cdot |a| \notin M$. However, this implies $|a| \notin M$, and thus lastly $a \notin M$, a contradiction our assumption $a \in M$ which shows $a \notin \text{rad}(\mathbf{A}_p)$. $\square$

Lemma 2.7. *Let $\mathbf{A}_p$ be a strongly negatively pointed Abelian ℓ -group and I an ℓ -ideal of $\mathbf{A}$. Then I is maximal if and only if $I = v^{-1}(0)$ for some homomorphism $v \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$.*

Proof. $(\Rightarrow)$: Let I be maximal. By Hölder's Theorem [22, Thm. 2.3.10] $\mathbf{A}_p/I$ embeds into $\mathbf{R}_r$ for some $r \in \mathbb{R}$. This gives us $v \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_r)$. Since p is a strong unit of $\mathbf{A}$, $p \notin I$ and $v(p) \neq 0$. Since p is negative, also $v(p) \leq 0$. Therefore $v(p) = r < 0$. Lastly, since $\mathbf{R}_{r_1} \cong \mathbf{R}_{r_2}$ for any two negative $r_1, r_2 \in \mathbb{R}$, we can assume without loss of generality that $v \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$.

$(\Leftarrow)$: For every $v \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$, the subset $v^{-1}(0)$ is an ℓ -ideal of $\mathbf{A}_p$ and any ℓ -subgroup of $\mathbf{R}_{-1}$ is simple, which ensures that the ideal $v^{-1}(0)$ has to be maximal. $\square$

This correspondence between infinitely small elements, maximal ℓ -ideals and homomorphisms into $\mathbf{R}_{-1}$ will be important later on, since $\mathbf{R}_{-1}$ is the algebra of truth-degrees for pointed modal Abelian logic introduced in the next section (in particular, *canonical frames* are structures based on the collections of homomorphisms $\text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$).

3 Pointed Modal Abelian Logic: Frames and Algebras

We first introduce relational (Kripke) semantics for pointed modal Abelian logic similar to the real-valued logic of [12] and other many-valued modal logics on crisp frames (Subsection 3.1). Next, we introduce the variety of pointed modal Abelian ℓ -groups and establish some basic identities therein (Subsection 3.2). Lastly, we explain how to 'shift' between these two frameworks constructing complex algebras (in particular showing algebraic soundness) and canonical frames (Subsection 3.3). This sets up the scene for our main results which we prove in the subsequent Section 4.

3.1 Relational Semantics

The *language* $\mathcal{L}_p$ of pointed modal Abelian logic is defined inductively from a countable set $\text{Prop} = \{x_1, x_2, x_3, \dots\}$ of *propositional variables*, the signature $\langle +, -, \vee, \wedge, 0, p \rangle$ of pointed Abelian ℓ -groups and the unary modality $\Box$. Let $\text{Fm}_{\mathcal{L}_p}$ denote the set of *formulas* over the language $\mathcal{L}_p$.

Definition 3.1 (Model). A *model* $\mathfrak{M} = \langle W, R, \mathcal{V} \rangle$ consists of a *Kripke frame* $\langle W, R \rangle$ (i.e. a non-empty set W with a binary relation $R \subseteq W^2$), together with a *bounded propositional valuation*, by which we mean a map $\mathcal{V} : W \times \text{Prop} \rightarrow \mathbb{R}$ such that for every propositional variable x_i there exists a positive real number r_i with $\{\mathcal{V}(u, x_i) \mid u \in W\} \subseteq [-r_i, r_i]$.

Note that this slightly differs from [12], where a propositional valuation is itself bounded. However, it is clear that our condition suffices to ensure that the following interpretation of $\Box$ is (still) well-defined.

Definition 3.2 (Satisfaction & Validity). Let $\mathfrak{M} = \langle W, R, \mathcal{V} \rangle$ be a model and $u \in W$. We inductively extend the valuation to $\mathcal{V} : W \times Fm_{\mathcal{L}_p} \rightarrow \mathbb{R}$ by interpreting the non-modal connectives in the algebra $\mathbf{R}_{-1}$ in the standard way. Modal formulas are interpreted as follows:

$$\mathcal{V}(u, \Box \varphi) = \bigwedge_{uRv} \mathcal{V}(v, \varphi).$$

(In particular, here we set $\bigwedge \emptyset := 0$.) A formula φ is *satisfied* at $u \in W$ if and only if

$$\mathcal{V}(u, \varphi) \geq 0$$

and the formula is *valid*, denoted $\models \varphi$, if this holds at every state of every model.

The following properties correspond to our axiomatization of pointed modal Abelian ℓ -groups later on (Definition 3.4) and essentially yield (algebraic) *soundness* later on.

Lemma 3.3. For $\mathfrak{M} = \langle W, R, \mathcal{V} \rangle$ a model, $u \in W$, and $\varphi, \psi \in Fm_{\mathcal{L}_p}$, the following hold:

- (i) $\mathcal{V}(u, \Box(\psi - \varphi)) \leq \mathcal{V}(u, \Box\psi) - \mathcal{V}(u, \Box\varphi)$,
- (ii) $\mathcal{V}(u, \Box\varphi) + \mathcal{V}(u, \Box\varphi) = \mathcal{V}(u, \Box(\varphi + \varphi))$,
- (iii) $\mathcal{V}(u, \Box\varphi) \vee 0 = \mathcal{V}(u, \Box(\varphi \vee 0))$,
- (iv) $\mathcal{V}(u, \Box\varphi) \wedge \mathcal{V}(u, \Box\psi) = \mathcal{V}(u, \Box(\varphi \wedge \psi))$,
- (v) $\mathcal{V}(u, |\Box p - p| \wedge |\Box\varphi|) = 0$,
- (vi) $\mathcal{V}(u, -\Box p) = \mathcal{V}(u, \Box -p)$.

Proof. (i): By substituting $\chi = \psi - \varphi$ and adding $\mathcal{V}(u, \Box\varphi)$ to both sides of the inequation, it suffices to prove $\mathcal{V}(u, \Box\chi) + \mathcal{V}(u, \Box\varphi) \leq \mathcal{V}(u, \Box(\chi + \varphi))$, or equivalently

$$\bigwedge_{uRv} \mathcal{V}(v, \chi) + \bigwedge_{uRv} \mathcal{V}(v, \varphi) \leq \bigwedge_{uRv} \mathcal{V}(v, \chi + \varphi).$$

This follows, since we have $\bigwedge_{uRv} \mathcal{V}(v, \chi) + \bigwedge_{uRv} \mathcal{V}(v, \varphi) \leq \mathcal{V}(v, \chi + \varphi)$ for each uRv .

(ii): Directly follows from the identity $\bigwedge_{i \in I} r_i + \bigwedge_{i \in I} r_i = \bigwedge_{i \in I} r_i + r_i$ of $\mathbf{R}_{-1}$.

(iii): Directly follows from the infinite distributivity law of ℓ -groups, see [1, Eq. 1.1.11].

(iv): Directly follows from the law $\bigwedge_{i \in I} r_i \wedge \bigwedge_{i \in I} s_i = \bigwedge_{i \in I} r_i \wedge s_i$ of $\mathbf{R}_{-1}$.

(v) & (vi): We distinguish cases (a) $\exists v: uRv$ and (b) $\nexists v: uRv$. In case (a) note $\mathcal{V}(u, \Box p) = -1 = \mathcal{V}(u, p)$ and thus $\mathcal{V}(u, |\Box p - p| \wedge |\Box\varphi|) = 0$ and $\mathcal{V}(u, -\Box p) = 1 = \mathcal{V}(u, \Box -p)$. In case (b) we have $\mathcal{V}(u, \Box\varphi) = \bigwedge \emptyset = 0$, thus $\mathcal{V}(u, |\Box p - p| \wedge |\Box\varphi|) = 0$. Furthermore, $\mathcal{V}(u, -\Box p) = 0 = \mathcal{V}(u, \Box -p)$. $\square$

3.2 Modal Abelian ℓ -groups

We aim to relate the relational semantics introduced in the previous subsection to the variety of pointed Abelian ℓ -groups with unary operators defined as follows.

Definition 3.4 (Modal Abelian ℓ -group). An algebra $\langle \mathbf{A}, \Box \rangle$ is a *modal Abelian ℓ -group* if $\mathbf{A}$ is an Abelian ℓ -group and $\Box : A \rightarrow A$ is a unary operation which satisfies for each $x, y \in A$ the following:

- (A1) $\Box(y - x) \leq \Box y - \Box x$,
- (A2) $\Box(x + x) = \Box x + \Box x$,
- (A3) $\Box(x \vee 0) = \Box x \vee 0$,
- (A4) $\Box(x \wedge 0) = \Box x \wedge 0$.

If $\mathbf{A}_p$ is a pointed Abelian ℓ -group, we say that $\langle \mathbf{A}_p, \Box \rangle$ is a *pointed modal Abelian ℓ -group* if it furthermore satisfies the following:

$$(A5) \quad |\Box x| \wedge |\Box p - p| = 0,$$

$$(A6) \quad \Box -p = -\Box p.$$

We denote the varieties of modal Abelian ℓ -groups and of negatively pointed modal Abelian ℓ -groups by $\mathbf{mAb}$ and $\mathbf{mAb}_p^-$, respectively. A pointed modal Abelian ℓ -group $\langle \mathbf{A}_p, \Box \rangle$ is *strongly negatively pointed* if $\mathbf{A}_p$ is strongly negatively pointed.

Example 3.5. Let $\mathbf{A}_p$ be a pointed Abelian ℓ -group. There are always at least two ‘trivial’ ways to expand this structure to a modal pointed Abelian ℓ -group, namely the identity $\Box_1 x := x$ and the constant zero $\Box_2 x := 0$. It is straightforward to verify that both $\langle \mathbf{A}_p, \Box_1 \rangle$ and $\langle \mathbf{A}_p, \Box_2 \rangle$ satisfy axioms (A1)–(A6).

Clearly $\mathbf{mAb}_p^-$ is a conservative expansion of $\mathbf{mAb}$, as in every $\mathbf{A} \in \mathbf{mAb}$ one can interpret the additional constant symbol as $p^{\mathbf{A}} := 0^{\mathbf{A}}$. Then Axioms (A5)–(A6) are trivially satisfied by (i) in the following.

Lemma 3.6. *The following hold in all modal Abelian ℓ -groups.*

$$(i) \quad \Box 0 = 0,$$

$$(iii) \quad x \geq 0 \Rightarrow \Box x \geq 0,$$

$$(ii) \quad \Box(2^k \cdot x) = 2^k \cdot \Box x \text{ for each } k \in \mathbb{N},$$

$$(iv) \quad x \leq y \Rightarrow \Box x \leq \Box y.$$

Moreover, the following hold in all pointed modal Abelian ℓ -groups.

$$(v) \quad \begin{aligned} \Box(x + p) &= \Box x + \Box p \text{ and} \\ \Box(x - p) &= \Box x - \Box p, \end{aligned}$$

$$(vii) \quad \Box(x \vee p) = \Box x \vee \Box p,$$

$$(viii) \quad p \leq \Box p,$$

$$(vi) \quad \Box(x + z \cdot p) = \Box x + z \cdot \Box p \text{ for each } z \in \mathbb{Z},$$

$$(ix) \quad \Box(-k \cdot p) = -\Box(k \cdot p) \text{ for each } k \in \mathbb{N}.$$

Proof. (i): Follows from (A2) with $x = 0$.

(ii): Follows from iterating (A2).

(iii): Follows from (A3).

(iv): If $x \leq y$, we have $0 \leq y - x$ and thus $0 \leq \Box(y - x)$ by the previous item. Therefore $0 \leq \Box(y - x) \leq \Box y - \Box x$ by (A1), which implies $\Box x \leq \Box y$.

(v): Using (A1) and (A6) we compute

$$\Box(x + p) \leq \Box x - \Box(-p) = \Box x + \Box p = \Box(x + p - p) + \Box p \leq \Box(x + p) - \Box p + \Box p = \Box(x + p).$$

Therefore, $\Box(x + p) = \Box x + \Box p$ for each $x \in A$. Consequently by substitution, $\Box x = \Box(x - p + p) = \Box(x - p) + \Box p$ for each $x \in A$.

(vi): Follows from iterating the previous item.

(vii): Using (A3), (vi) and Lemma 2.4 (iii) we compute

$$\begin{aligned} \Box(x \vee p) &= \Box(((x - p) \vee 0) + p) && \text{(Lemma 2.4(iii))} \\ &= \Box((x - p) \vee 0) + \Box p && \text{(Item (v))} \\ &= (\Box(x - p) \vee 0) + \Box p && \text{((A3) \& Item (v))} \\ &= (\Box(x - p) + \Box p) \vee \Box p = \Box x \vee \Box p. && \text{(Lemma 2.4(iii) \& Item (v))} \end{aligned}$$

(viii): Due to Lemma 2.4 (iv) & (v) we compute that $(|\Box p| - |\Box p - p|) \leq |p|$ and furthermore that $|\Box p| - |\Box p| \wedge |\Box p - p| = (|\Box p| - |\Box p - p|) \vee 0$. Therefore by (A5) we obtain

$$|\Box p| = |\Box p| - |\Box p| \wedge |\Box p - p| = (|\Box p| - |\Box p - p|) \vee 0 \leq |p|.$$

Consequently, by applying (A6),(vii) and $p \leq 0$ compute

$$|\Box p| = -\Box p \vee \Box p = \Box(-p) \vee \Box p = \Box(-p \vee p) = \Box|p| = \Box(-p) = -\Box p$$

which shows $p \leq \Box p$ as desired.

(ix): Is a straightforward consequence of (i) and (vi) with $x = 0$. $\square$

Remark 3.7. As already discussed in Lemma 3.3, by applying subtraction and substitution, it is easy to see that axiom (A1) can be equivalently replaced with

$$\Box x + \Box y \leq \Box(x + y). \quad (\text{A1*})$$

Let us also note that we could replace (A5) and (A6) with a single axiom

$$|\Box x| \wedge |\Box(x + x - p) - \Box x + \Box x - p| = 0 \quad (\text{A5*})$$

Although this axiom is harder to verify, together with (A2) it provides an equational counterpart to the modal MV-algebra axioms $\Box(x \oplus x) = \Box x \oplus \Box x$ and $\Box(x \odot x) = \Box x \odot \Box x$ used in [15].

3.3 Complex Algebras and Canonical Frames

As usual, to ‘shift’ between the relational and algebraic frameworks introduced in the previous two subsections, we define complex algebras and canonical frames, starting with the former.

Definition 3.8 (Complex Algebra). Let $\langle W, R \rangle$ be a Kripke frame. The corresponding *complex algebra* is given by $\mathfrak{C}_{\langle W, R \rangle} = \langle b\mathbf{R}_{-1}^W, \Box_R \rangle$ where $b\mathbf{R}_{-1}^W \leq \mathbf{R}_{-1}^W$ denotes the subalgebra of the product $\mathbf{R}_{-1}^W$ defined by the subuniverse of *bounded maps* $\{f: W \rightarrow \mathbb{R} \text{ bounded}\}$ and the operator $\Box_R$ is defined by

$$(\Box_R f)(u) = \bigwedge_{uRv} f(v).$$

(Note that $\Box f$ is a well-defined bounded map if f is bounded.)

It is clear from Lemma 3.3 that all complex algebras are pointed modal Abelian ℓ -groups. Furthermore, the map of constant value -1 is a strong unit. From this we get the following.

Proposition 3.9 (Algebraic Soundness). *If $\text{mAb}_p^- \models \varphi \geq 0$ then $\models \varphi$.*

Proof. If $\not\models \varphi$ then there is a model $\langle W, R, \mathcal{V} \rangle$ such that $\mathcal{V}(u, \varphi) < 0$. Then $e_{\mathcal{V}}: \text{Prop} \rightarrow \mathfrak{C}_{\langle W, R \rangle}$ defined by $e_{\mathcal{V}}(x_i) = \mathcal{V}(-, x_i)$ is an assignment under which $\varphi \geq 0$ fails in the complex algebra $\mathfrak{C}_{\langle W, R \rangle}$. $\square$

Similarly, we move from mAb_p^- -algebras to Kripke frames via the following construction based on the set of all homomorphisms into our algebra of truth-values $\mathbf{R}_{-1}$.

Definition 3.10 (Canonical Frame). Let $\langle \mathbf{A}_p, \Box \rangle$ be a strongly negatively pointed modal Abelian ℓ -group and let $u, v \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ be homomorphisms. We define

$$uR_{\Box}v \quad \text{if and only if} \quad u(\Box a) \geq 0 \Rightarrow v(a) \geq 0 \text{ for all } a \in \mathbf{A}_p.$$

We call $\mathfrak{F}_{\langle \mathbf{A}_p, \Box \rangle} := \langle \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1}), R_{\Box} \rangle$ the *canonical frame* of $\langle \mathbf{A}_p, \Box \rangle$.

Note that from any *algebraic evaluation* $e: \text{Prop} \rightarrow \mathbf{A}_p$ we can define a propositional valuation $\mathcal{V}_e: \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1}) \times \text{Prop} \rightarrow \mathbb{R}$ on the canonical frame $\mathfrak{F}_{\langle \mathbf{A}_p, \Box \rangle}$ via $\mathcal{V}_e(u, x_i) = u(e(x_i))$. However, these are *not* necessarily bounded valuations, that is, they do not necessarily define models according to Definition 3.1. However, note that these valuations are bounded if p is a *negative strong unit*, since then $e(x_i) \in \mathbf{A}_p$ is bounded by $n \cdot p \leq e(x_i) \leq -n \cdot p$ for some $n \in \mathbb{N}$. Therefore, every homomorphism $u: \mathbf{A}_p \rightarrow \mathbf{R}_{-1}$ satisfies $u(e(x_i)) \in [-n, n]$. In the next section, we prove that the corresponding Truth Lemma also holds for strongly negatively pointed modal Abelian ℓ -groups.

4 Pointed Modal Abelian Logic, Algebraically

In Subsection 4.1, we establish the Truth Lemma with respect to the constructions of the previous subsection for *strongly* negatively pointed modal Abelian ℓ -groups (Corollary 4.6). For this, we introduce some terms t_d which can ‘separate’ elements of $\mathbf{R}_{-1}$.

In Subsection 4.2, we introduce an infinitary derivability relation with respect to $\mathbf{mAb}_p^-$ and prove completeness of real-valued modal logic with respect to this relation (Theorem 4.9).

4.1 The Truth Lemma

First let us note that the following is easily derived from (A5) and the fact that $\mathbf{R}_{-1}$ is totally ordered.

Lemma 4.1. *Let $\langle \mathbf{A}_p, \square \rangle$ be a pointed modal Abelian ℓ -group and let $u \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ be a homomorphism. Then $u(\square a) = 0$ for each $a \in \mathbf{A}_p$ or $u(\square p) = u(p) = -1$.*

The set $D := \{\frac{z}{2^k} \mid z \in \mathbb{Z}, k \in \mathbb{N}\}$ is a dense subset of $\mathbb{R}$. For each $d \in D$ we fix some representation $d = \frac{z}{2^k}$ and define t_d to be a unary term

$$t_d(x) = (2^k \cdot x + z \cdot p) \wedge 0.$$

The following summarizes the main properties of these terms for our purposes.

Lemma 4.2. *The terms t_d defined above satisfy the following.*

- (i) $t_d^{\mathbf{A}_p}(x) \leq 0$ for every $x \in \mathbf{A}_p$.
- (ii) $t_d^{\mathbf{R}_{-1}}(r) = 0$ iff $r \geq d$.
- (iii) Assume $u(\square y) \neq 0$ for some $y \in \mathbf{A}_p$. Then $u(t_d(\square x)) = u(\square t_d(x))$.

Proof. Let $d = \frac{z}{2^k}$ be the representation chosen in the definition of t_d . By definition, (i) is obvious. (ii): Observe that for each $r \in \mathbb{R}$ we have $t_d(r) \geq 0$ iff $2^k \cdot r + z \cdot (-1) \geq 0$, which is equivalent to $r - d \geq 0$. (iii): First observe that by Lemma 4.1 and the assumption we obtain $u(\square p) = u(p)$. Compute

$$\begin{aligned}
 u(t_d(\square x)) &= u((2^k \cdot \square x + z \cdot p) \wedge 0) && \text{(Def. } t_d) \\
 &= (u(2^k \cdot \square x) + z \cdot u(p)) \wedge 0 && (u \text{ homomorphism}) \\
 &= (u(2^k \cdot \square x) + z \cdot u(\square p)) \wedge 0 && (u(\square p) = p) \\
 &= u((2^k \cdot \square x + z \cdot \square p) \wedge 0) && (u \text{ homomorphism}) \\
 &= u(\square(2^k \cdot x + z \cdot p) \wedge 0) && \text{(Lemma 3.6(ii))} \\
 &= u(\square(2^k \cdot x + z \cdot p) \wedge 0) && \text{(Lemma 3.6(vi))} \\
 &= u(\square((2^k \cdot x + z \cdot p) \wedge 0)) = u(\square t_d(x)) && \text{(Axioms (A1), (A4) \& Def. } t_d)
 \end{aligned}$$

as desired, finishing the proof. $\square$

Among others, we use these terms to prove the following (recall the Definition 3.10 of the canonical frame $\langle \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1}), R_\square \rangle$).

Lemma 4.3. *Let $\langle \mathbf{A}_p, \square \rangle \in \mathbf{mAb}_p^-$ be strongly negatively pointed and $u, v \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$. The following conditions are equivalent:*

- (1) $u R_\square v$,

- (2) $u(\Box a) = 0 \Rightarrow v(a) = 0$ for all $a \leq 0$,
- (3) $\Box^{-1}u^{-1}(0) \cap \{a \in A : a \leq 0\} \subseteq v^{-1}(0)$,
- (4) $u(\Box a) \leq v(a)$ for all $a \in A$.

Proof. (1) $\Rightarrow$ (2): Let $a \leq 0$, and $u(\Box a) = 0$. The former yields $v(a) \leq 0$ and the latter implies together with $uR_{\Box}v$ that $v(a) \geq 0$.

(2) $\Leftrightarrow$ (3) is obvious.

(2) $\Rightarrow$ (4): We proceed by contraposition. Assume there is a such that $v(a) < u(\Box a)$. Then there is $d \in D$ such that $v(a) < d \leq u(\Box a)$. By Lemma 4.2(ii) we have $t_d^{\mathbf{R}^{-1}}(v(a)) = v(t_d(a)) < 0$ and $t_d^{\mathbf{R}^{-1}}(u(\Box a)) = 0$. By Lemma 4.2(iii) this implies $u(\Box t_d(a)) = 0$. Also, by Lemma 4.2(i) we have $t_d(a) \leq 0$.

(4) $\Rightarrow$ (1): If $u(\Box a) \leq v(a)$ for each $a \in A$ then clearly whenever $0 \leq u(\Box a)$ then also $0 \leq v(a)$. $\square$

We show one more technical result before proving the main theorem of this subsection.

Lemma 4.4. *Let $\langle \mathbf{A}_p, \Box \rangle$ be a strongly negatively pointed modal Abelian ℓ -group, $u \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$, and let I be the ℓ -ideal generated by the set $S := \{x \in A \mid x \leq 0\} \cap \Box^{-1}u^{-1}(0)$. Let $a \in I$ such that $a \leq 0$. Then $a \in \Box^{-1}u^{-1}(0)$.*

Proof. Since a is in the ℓ -ideal generated by the set S , due to (IG), there exist $b_1, \dots, b_n \in S$ such that $\sum_{i=1}^n b_i \leq a \leq 0$. Since by Lemma 3.6 (iv) the operator $\Box$ is order preserving, by iterating Equation (A1*) we have the following in $\mathbf{A}_p$:

$$\Box b_1 + \dots + \Box b_n \leq \Box(b_1 + \dots + b_n) \leq \Box a \leq \Box 0 = 0.$$

By applying u , we obtain:

$$0 = u(\Box b_1 + \dots + \Box b_n) \leq u(\Box(b_1 + \dots + b_n)) \leq u(\Box a) \leq u(\Box 0) = 0.$$

Thus, $u(\Box a) = 0$, which completes the proof. $\square$

The following is crucial, essentially being the clause of modal formulas in the Truth Lemma.

Theorem 4.5. *Let $\langle \mathbf{A}_p, \Box \rangle$ be a strongly negatively pointed modal Abelian ℓ -group. Then $u(\Box a) = \bigwedge_{uR_{\Box}v} v(a)$ holds for every homomorphism $u \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$.*

Proof. We already know by Lemma 4.3(4) that $u(\Box a) \leq \bigwedge_{uR_{\Box}v} v(a)$ (from this we also see that the meet exists). First, we consider the case $u(\Box a) = 0$ for every $a \in \mathbf{A}_p$. Since for any $v \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ we have $v(p) = -1$, it follows there is no v such that $uR_{\Box}v$. Consequently, $\bigwedge_{uR_{\Box}v} v(a) = 0 = u(\Box a)$.

For the remaining case, by Lemma 4.1 we know that $u(\Box p) = -1$. Towards contradiction, assume $u(\Box a) < \bigwedge_{uR_{\Box}v} v(a)$. Since D is dense there exists $d \in D$ such that $u(\Box a) < d \leq \bigwedge_{uR_{\Box}v} v(a)$. Therefore, by Lemma 4.2(ii) & (iii) we have $t_d^{\mathbf{R}^{-1}}(u(\Box a)) = u(t_d(\Box a)) = u(\Box t_d(a)) < 0$ and $t_d^{\mathbf{R}^{-1}}(v(a)) = v(t_d(a)) = 0$ for each $uR_{\Box}v$. By Lemmata 4.3 & 2.7 each maximal ideal of $\mathbf{A}_p$ containing $\{x \leq 0\} \cap \Box^{-1}u^{-1}(0)$ has to contain $t_d(a)$. Let I be the ℓ -ideal generated by the set $\{x \leq 0\} \cap \Box^{-1}u^{-1}(0)$. Now $[t_d(a)]_I \in \text{rad}(\mathbf{A}_p/I)$ and thus by Lemma 2.6 we obtain $2^k \cdot [t_d(a)]_I - [p]_I \geq [0]_I$ for each $k \in \mathbb{N}$. Therefore, there exists $x \in I$ such that $2^k \cdot t_d(a) - p \geq x$ and without loss of generality we can assume $x \leq 0$ (since otherwise we can replace it by $x \wedge 0 \in I$). From Lemma 4.4 we get $x \in \Box^{-1}u^{-1}(0)$ and thus we have by Lemma 3.6(ii),(iv) & (v) that

$$0 = u(\Box x) \leq u(\Box(2^k \cdot t_d(a) - p)) = u(\Box(2^k \cdot t_d(a))) - u(\Box p) = 2^k \cdot u(\Box t_d(a)) + 1$$

for each $k \in \mathbb{N}$. Since -1 is a negative strong unit in $\mathbf{R}_{-1}$ this means that $u(\Box t_d(a))$ is infinitesimal in $\mathbf{R}_{-1}$. However, since the algebra $\mathbf{R}_{-1}$ is simple this implies $u(\Box t_d(a)) = 0$, a contradiction. $\square$

The following Truth Lemma is a standard consequence. Note that the set of formulas $Fm_{\mathcal{L}_p}$ can be seen as the absolutely free algebra in the language of pointed modal Abelian ℓ -groups and we identify algebraic evaluations $e: Prop \rightarrow \langle \mathbf{A}_p, \square \rangle$ with the corresponding homomorphisms $e: \mathcal{L}_p \rightarrow \langle \mathbf{A}_p, \square \rangle$.

Corollary 4.6 (Truth Lemma). *Let $\langle \mathbf{A}_p, \square \rangle \in \mathbf{mAb}_p^-$ be strongly negatively pointed, $u \in \text{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ and $e: Prop \rightarrow \mathbf{A}_p$ an algebraic evaluation. Let $\mathcal{V}_e$ be the propositional valuation on $\mathfrak{F}_{\langle \mathbf{A}_p, \square \rangle}$ defined by $\mathcal{V}_e(u, x) = u(e(x))$. Then the property $\mathcal{V}_e(u, \varphi) = u(e(\varphi))$ extends to every formula φ .*

Proof. Induction on the formula, where the case $\square\varphi$ follows from Theorem 4.5:

$$\mathcal{V}_e(u, \square\varphi) = \bigwedge_{uR_{\square}v} \mathcal{V}_e(v, \varphi) = \bigwedge_{uR_{\square}v} v(e(\varphi)) = u(e(\square\varphi)).$$

All other cases are clear (since u is a homomorphism with respect to the non-modal language). $\square$

4.2 An Infinitary Algebraic Completeness

Similar to the case of modal MV-algebras [15], the Truth Lemma of the previous section does not immediately yield algebraic completeness. In particular, if the inequation $\varphi \geq 0$ does not hold in some $\langle \mathbf{A}_p, \square \rangle \in \mathbf{mAb}_p^-$ we have yet to obtain the *witnessing homomorphism* which shows that φ is not valid in the corresponding canonical frame $\mathfrak{F}_{\langle \mathbf{A}_p, \square \rangle}$. To construct such a homomorphism, we will use Lemma 2.6 and introduce an infinitary rule (A ∞) based on (INF-) to capture that the corresponding term is not evaluated inside the radical.

Moreover, we must ensure that a potential counterexample is witnessed in a well-defined model, that is, with a *bounded valuation* (recall Definition 3.1). We observed in Subsection 3.3 that this is ensured for canonical frames of *strongly pointed* modal Abelian ℓ -groups.

To formalize the requirement that it suffices to check φ within these bounds, let us first set the following notation. For a variable x_i and $k \in \mathbb{N}$, we define terms

$$x_i^k := (x_i \vee k \cdot p) \wedge k \cdot (-p).$$

For a formula $\varphi(x_1, \dots, x_n)$ and $k \in \mathbb{N}$ we let φ^k denote the substitution instance

$$\varphi^k(x_1, \dots, x_n) = \varphi(x_1^k, \dots, x_n^k).$$

Now the *bounded valuations* rule is expressed as follows.

$$\mathbf{mAb}_p^- \models \varphi^k \geq 0 \text{ for all } k \in \mathbb{N} \implies \mathbf{mAb}_p^- \models \varphi \geq 0 \quad (\text{BV})$$

Note that the other implication trivially holds. Let us emphasize that rule (BV) is *not* a single infinitary rule in the standard sense. Because the syntactic substitutions happen *within* the formula itself, this property cannot be captured by one generalized quasi-equation. Instead, (BV) represents an infinite family of generalized quasi-equations, one for every formula $\varphi \in Fm_{\mathcal{L}_p}$.

As previously mentioned, we need to combine this with (INF-) to ensure the existence of homomorphisms witnessing non-validity. Thus, we end up with the following.

Definition 4.7 (Infinitary Algebraic Derivability). We define

$$\vdash_{\infty} \varphi \iff \mathbf{mAb}_p^- \models n \cdot (\varphi^k \wedge 0) - p \geq 0 \text{ for every } n, k \in \mathbb{N}. \quad (\text{A}\infty)$$

Recall that our Truth lemma was only proved for strongly negatively pointed modal Abelian ℓ -groups. The following lemma will allow us to ‘move’ to such an algebra in the proof of Theorem 4.9 later on.

Lemma 4.8. *Let $\langle \mathbf{A}_p, \square \rangle$ be a negatively pointed modal Abelian ℓ -group with $p \neq 0$ and let $\mathbf{B}$ be the ℓ -ideal of $\mathbf{A}$ generated by p . Then $\langle \mathbf{B}_p, \square \rangle$ is a strongly negatively pointed modal Abelian ℓ -group.*

Proof. We only need to verify that $\mathbf{B}_p$ is closed under $\square$. Let $b \in \mathbf{B}_p$. We have $|b| \leq n \cdot |p|$ and hence we obtain $n \cdot p \leq b \leq -n \cdot p$. By Lemma 3.6 (iv) we know that the operator $\square$ is monotone, from which we get (using $-\square(n \cdot p) = \square(-n \cdot p)$) that $\square(n \cdot p) \leq \square b \leq \square(-n \cdot p)$ and $\square(n \cdot p) \leq -\square b \leq \square(-n \cdot p)$. Therefore, we showed $|\square b| \leq |n \cdot p|$ and thus $\square b \in \mathbf{B}_p$. $\square$

We are now ready to prove the main result of this section, completeness of pointed modal Abelian logic with respect to the derivability relation of Definition 4.7.

Theorem 4.9 (Infinitary Algebraic Completeness). *Let φ be a formula. Then $\models \varphi$ if and only if $\vdash_\infty \varphi$.*

Proof. For the implication $(\Rightarrow)$, assume $\not\models_\infty \varphi$. By definition, there exist $n, k \in \mathbb{N}$ such that $\text{mAb}_p^- \not\models n \cdot (\varphi^k \wedge 0) - p \geq 0$. Let $\mathbf{A}_p \in \text{mAb}_p^-$ and e be an evaluation on $\mathbf{A}_p$ such that $n \cdot (e(\varphi^k) \wedge 0) - p \not\geq 0$. Note that $\mathbf{A}_p$ satisfies $p \neq 0$, since otherwise it would satisfy $e(x_i^k) = (e(x_i) \vee 0) \wedge 0 = 0$ for each x_i and thus $n \cdot (e(\varphi^k) \wedge 0) - p = 0$. Let $\mathbf{B}$ be the ℓ -ideal of $\mathbf{A}$ generated by p . By Lemma 4.8 we know that $\mathbf{B}_p \in \text{mAb}_p^-$ is strongly negatively pointed. We define an evaluation $\hat{e}$ on $\mathbf{B}_p$ by $\hat{e}(x) = e(x^k)$. By the above, it holds that $n \cdot (\hat{e}(\varphi) \wedge 0) - p \not\geq 0$. From Lemma 2.6 we obtain that $\hat{e}(\varphi \wedge 0) \notin \text{rad}(\mathbf{B}_p)$. Consequently, there exists a homomorphism $u \in \text{Hom}(\mathbf{B}_p, \mathbf{R}_{-1})$ such that $u(\hat{e}(\varphi \wedge 0)) < 0$. Let $\mathcal{V}_{\hat{e}}$ be the valuation on the canonical frame of $\mathbf{B}_p$ defined by $\mathcal{V}_{\hat{e}}(v, x_i) = v(\hat{e}(x_i))$ for all $v \in \text{Hom}(\mathbf{B}_p, \mathbf{R}_{-1})$. The Truth Lemma yields

$$\mathcal{V}_{\hat{e}}(u, \varphi) \wedge 0 = \mathcal{V}_{\hat{e}}(u, \varphi \wedge 0) = u(\hat{e}(\varphi \wedge 0)) < 0$$

and therefore clearly $\mathcal{V}_{\hat{e}}(u, \varphi) < 0$, which witnesses $\not\models \varphi$.

For the implication $(\Leftarrow)$, assume $\not\models \varphi$ and let $\langle W, R, \mathcal{V} \rangle$ be a model such that $\mathcal{V}(u, \varphi) < 0$ for some $u \in W$. Since $\mathcal{V}$ is a bounded valuation there is some $k \in \mathbb{N}$ such that $\mathcal{V}(u, \varphi^k) < 0$ also holds. Let e be the evaluation on the complex algebra $\mathfrak{C}_{\langle W, R \rangle}$ defined by $e(x_i) = \mathcal{V}(-, x_i)$. We have $e(\varphi^k) = \mathcal{V}(-, \varphi^k)$. Since $e(\varphi^k)(u) = \mathcal{V}(u, \varphi^k) < 0$ it follows that $e(\varphi^k \wedge 0) < 0$. Since $\{f \in \mathfrak{C}_{\langle W, R \rangle} : f(u) = 0\}$ is a maximal ℓ -ideal of $\mathfrak{C}_{\langle W, R \rangle}$ (because it is the preimage $\text{ev}_u^{-1}(0)$ of the homomorphism $\text{ev}_u : \mathfrak{C}_{\langle W, R \rangle} \rightarrow \mathbf{R}_{-1}$ evaluating at u) and $e(\varphi^k \wedge 0) \notin \{f \in \mathfrak{C}_{\langle W, R \rangle} : f(u) = 0\}$ by Lemma 2.6 it follows there is $n \in \mathbb{N}$ such that $e(n \cdot (\varphi^k \wedge 0) - p) \not\geq 0$, which by definition shows $\not\models_\infty \varphi$. This completes the proof. $\square$

5 Conclusion

In this paper, we introduced the variety of negatively pointed modal Abelian ℓ -groups and established its connection to real-valued modal logic, culminating in Theorem 4.9 characterizing modal validity via the rule (A ∞). We take this as evidence that the variety mAb_p^- deserves further investigation in the future.

Naturally, this also applies to the study of modal Abelian ℓ -groups *without* a designated point. The crucial role played by the non-zero designated element was highlighted throughout the present work. Without the constant symbol p in the language, every term function $t(x_1, \dots, x_n)$ over any modal Abelian ℓ -group trivially satisfies $t(0, \dots, 0) = 0$, which effectively prevents the construction of the separating terms t_d of Lemma 4.2. Moreover, the strong unit, the radical, and the boundedness of valuations are all dependent on the assumption of pointedness. Nevertheless, in future work we aim to establish a similar algebraic completeness result for (point-free) real-valued modal logic with respect to the variety mAb .

It is common in many-valued modal logic to not only consider crisp, but *labeled accessibility relations* (e.g. see [4, 5] for the case of finitely-valued modal Łukasiewicz logic). In (pointed) modal Abelian logic, one may consider *bounded* labeled accessibility relations $R: W^2 \rightarrow [-r, r] \subseteq \mathbb{R}$ with the modality interpreted via $\mathcal{V}(u, \Box \varphi) = \bigwedge_{v \in W} \mathcal{V}(v, \varphi) - R(u, v)$. Studying this logic and the relationship to modal Łukasiewicz logic would clearly be an interesting (albeit challenging) future direction.

There is a well-known categorical equivalence between the class of strongly pointed Abelian ℓ -groups and the class of MV-algebras established by the *Mundici functor* (e.g. see [8]). Similarly, we plan to investigate the category-theoretic relationship between strongly negatively pointed modal Abelian ℓ -groups and the category MMV of modal MV-algebras of [15] (let us note here that the Mundici equivalence was extended to *monadic* MV-algebras in [9]). As a first step we can show that, given a negatively pointed modal Abelian ℓ -group, its interval $[p, 0]$ is a modal MV-algebra. The construction in the reverse direction appears to be less straightforward.

Another open question is whether we can replace rule $(A\infty)$ with a *single* generalized quasi-equation. As discussed previously, rule (BV) corresponds to an infinite family of generalized quasi-equations (one for each formula φ) and consequently defines a generalized quasi-variety, and we want to investigate whether there exists a single generalized quasi-equation defining it.

Finally, we also want to address questions regarding computational *complexity*. To this end, we plan to generalize and apply the approach presented in [14] for unbounded Łukasiewicz logic to the setting of strongly pointed modal Abelian ℓ -groups.

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Fusions of One-Variable First-Order Modal Logics

Roman Kontchakov

Birkbeck, University of London, UK
r.kontchakov@bbk.ac.uk

Dmitry Shkatov

University of the Witwatersrand,
Johannesburg, South Africa
dmitry.shkatov@wits.ac.za

Frank Wolter

University of Liverpool, UK
wolter@liverpool.ac.uk

We investigate preservation results for the independent fusion of one-variable first-order modal logics. We show that, without equality, Kripke completeness and decidability of global and local consequence relations are preserved, under both expanding and constant domain semantics. By contrast, Kripke completeness and decidability are not preserved for fusions with equality and non-rigid constants (or, equivalently, counting up to one), again for the global and local consequence and under both expanding and constant domain semantics. This result is shown by encoding Diophantine equations. Even without equality, the finite model property is preserved only in the local case. Finally, we view fusions of one-variable modal logics as fusions of propositional modal logics sharing an **S5** modality and provide a general sufficient condition for transfer of Kripke completeness and decidability (but not of finite model property).

1 Introduction

The study of combined modal logics has been an active area of research for many years [10, 23, 35]. Only a few methods of combining modal systems have proved to be robust in the sense that key properties of the component logics are always inherited by the resulting combined logic. A central example of such a robust combination is fusion: a basic method in which the component logics are combined without introducing axioms that mix their modal operators, and, correspondingly, without imposing any semantic constraints that relate the distinct accessibility relations. In fact, properties such as Kripke completeness, decidability, finite model property, and Craig interpolation are all preserved under forming the fusion of two (or more) classical propositional modal logics, for both the local and global consequence relations [31, 21, 9, 33, 4, 14, 2, 6].

The aim of this contribution is to initiate a systematic investigation into the extent to which these preservation results for fusions of propositional modal logics can be lifted to (fragments of) first-order modal logics and corresponding many-dimensional modal logics. We focus in this paper on one-variable first-order modal logics, either without equality or with equality and non-rigid constants, and on corresponding commutators and products of propositional modal logics. These logics have been studied extensively [11, 19, 24, 5, 28] as they are often computationally and semantically better behaved than full first-order modal logics. For fusions without equality our main result is as follows:

Theorem A (Equality-Free Fusion). *Kripke completeness and decidability are preserved under fusions of Kripke complete one-variable first-order modal logics, for both the local and global consequence and under both expanding and constant domain semantics. The finite model property is preserved for the local consequence but not for the global consequence. Moreover, the finite model property fails for the global consequence for all non-trivial fusions under both expanding and constant domain semantics.*

The proof builds on the cactus model construction [16], extending it to one-variable modal logic via the quasimodel technique [25]. For one-variable first-order modal logics with equality, the presence of equality can only have a substantial impact if one also allows non-rigid constants or, equivalently, counting quantifiers such as $\exists^=1 x P(x) \equiv \exists x [P(x) \wedge \forall y (P(y) \rightarrow (x = y))]$, which expresses that

predicate P is interpreted as a singleton. In this setting, drawing on known decidability results for the one-variable first-order extension of the modal logic of inequality (also known as the ‘elsewhere’ logic) and for one-variable temporal logics [19, 17, 12] and by encoding Diophantine equations and Minsky machines to show undecidability, we establish the following non-transfer result:

Theorem B (Fusion with Equality). *Decidability and recursive axiomatisability are not preserved for fusions of one-variable first-order modal logics with equality, for both the local and global consequence and under both expanding and constant domain semantics. Moreover, the global consequence is undecidable for all non-trivial fusions for constant domain semantics.*

We stress that it would be a mistake to infer that fusions of one-variable modal logics with equality are broadly undecidable. In fact, when based on standard modal systems such as polymodal **K** or **S5**, one-variable modal logics with equality remain decidable, even in CONEXPTIME [1].

Since the one-variable fragment of first-order logic without equality can be identified with propositional modal logic **S5**, one-variable first-order modal logics can often be viewed as propositional multimodal logics equipped with an **S5** modality. Working within this perspective, we provide a sufficient condition for the transfer of Kripke completeness and decidability in fusions of propositional modal logics that share an **S5** modality. We say that a propositional modal logic admits E -homogeneous models for an equivalence relation E interpreting an **S5** modality $\Box_E$ if, for sufficiently large infinite cardinals κ , it is complete for Kripke models in which, for every E -equivalence class W of worlds, each formula is either not satisfied anywhere in W or is satisfied in κ -many worlds of W .

Theorem C (Propositional Fusion with Shared **S5**). *Kripke completeness and decidability are preserved for fusions of propositional modal logics admitting E -homogeneous models for a shared **S5** modality $\Box_E$, for both the local and global consequence.*

The conditions of Theorem C hold, for example, for Kripke complete (semi)commutators with **S5** and for (expanding) product modal logics with **S5**. Hence, it provides transfer results for corresponding one-variable modal logics. We note that, in contrast to the proof of Theorem A that uses quasimodels, one cannot directly prove transfer of the finite model property based on homogeneous models. Theorem C addresses a problem posed by Baader, Ghilardi and Tinelli [2].

2 Preliminaries

We consider first-order (multi-)modal logics in languages with a single, fixed, individual variable. A language $\mathcal{L}_M$ contains countably many unary predicate letters, the individual variable x , the Booleans $\neg$ and $\wedge$, the quantifier symbol $\exists$, and a finite set M of unary (necessity-like) modal operators, denoted by $\Box$, possibly with subscripts. Thus, in $\mathcal{L}_M$, atomic formulas have the form $P(x)$, where P is a predicate letter. Formulas of $\mathcal{L}_M$, denoted by $\varphi, \psi, \dots$, are built from atomic ones in the usual way, the only allowed quantifier being $\exists x$. The symbols $\perp, \rightarrow, \vee, \forall$, and $\Diamond$ are standard abbreviations. We also denote $\Box^{\leq n} \varphi = \varphi \wedge \Box \varphi \wedge \dots \wedge \Box^n \varphi$. Proposition letters $p, q, \dots$ are abbreviations for $\exists x P(x), \exists x Q(x)$, etc. We also consider languages with equality: $\mathcal{L}_M^=$ contains, in addition to the symbols of $\mathcal{L}_M$, the binary predicate symbol $=$ and constant symbols, denoted by $c_1, c_2, \dots$. Thus, in $\mathcal{L}_M^=$, a term t is either x or a constant c_i , and atomic formulas have the form $P(t)$ or $t_1 = t_2$. We usually identify languages with sets of their formulas. The set of subformulas of a formula φ is denoted by $\text{sub } \varphi$.

By a *one-variable (first-order) M -logic*, we mean a set of $\mathcal{L}_M$ -formulas that includes the one-variable fragment of the classical first-order logic **FO** and formulas $\Box(P(x) \rightarrow Q(x)) \rightarrow (\Box P(x) \rightarrow \Box Q(x))$, for all $\Box \in M$, and is closed under Modus Ponens (MP), Generalisation ($\varphi / \forall x \varphi$), Necessitation ($\varphi / \Box \varphi$, for

each $\Box \in M$), and Substitution (of arbitrary one-variable formulas for formulas of the form $P(x)$).¹ If L is a one-variable M -logic and $\Gamma \subseteq \mathcal{L}_M$, then $L + \Gamma$ denotes the smallest one-variable M -logic including $L \cup \Gamma$; we write $L + \varphi$ for $L + \{\varphi\}$. We note that every one-variable M -logic contains the converse Barcan formula $CBF_\Box = \Box \forall x P(x) \rightarrow \forall x \Box P(x)$, for each $\Box \in M$.

Let L be a one-variable M -logic and $\varphi, \psi \in \mathcal{L}_M$. We write $\varphi \vdash_L \psi$ and say that ψ is a *local consequence* of φ in L if ψ belongs to the smallest set including $L \cup \{\varphi\}$ and closed under MP and Generalisation. By the deduction theorem, L is decidable iff $\vdash_L$ is decidable. We write $\varphi \vdash_L^* \psi$ and say that ψ is a *global consequence* of φ in L if ψ belongs to the smallest set including $L \cup \{\varphi\}$ and closed under MP, Generalisation, and Necessitation for all $\Box \in M$. We call L *globally decidable* if the global consequence in L is decidable.

The *fusion* of a one-variable M_1 -logic L_1 and a one-variable M_2 -logic L_2 , where $M_1 \cap M_2 = \emptyset$, is the smallest one-variable $(M_1 \cup M_2)$ -logic $L_1 \otimes L_2$ that includes $L_1 \cup L_2$.

We are mostly concerned with logics defined in terms of Kripke semantics. A *Kripke M -frame* is a tuple $\mathfrak{F} = (W, \{R_\Box\}_{\Box \in M})$, where W is a non-empty set of *worlds* and $R_\Box \subseteq W \times W$, for all $\Box \in M$. We write $R_\Box(w)$ for $\{w' \in W \mid wR_\Box w'\}$. We say that $\mathfrak{F}$ is *non-trivial* if there are distinct $w, v \in W$ with $wR_\Box v$, for some $\Box \in M$. A *Kripke cd -model* based on $\mathfrak{F}$ is a pair $\mathfrak{M} = (\mathfrak{F}, I)$, where $I = \{\langle \Delta, I_w \rangle \mid w \in W\}$ is a family of classical first-order models. A *Kripke xd -model* based on $\mathfrak{F}$ is a pair $\mathfrak{M} = (\mathfrak{F}, I)$, where $I = \{\langle \Delta_w, I_w \rangle \mid w \in W\}$ is a family of classical first-order models satisfying the *expanding domains condition*: $\Delta_w \subseteq \Delta_v$, for all $w, v \in W$ with $wR_\Box v$ and all $\Box \in M$. For languages with equality and constants, the classical models associated with worlds are required to be *normal* (i.e., $=$ is interpreted as identity); the resultant Kripke models are called *Kripke d -models with equality*, for $d \in \{xd, cd\}$; note that rigid interpretation of constants, i.e., $I_w(c_k) = I_v(c_k)$, for all $w, v \in W$, is *not* assumed. We say that a Kripke model is *finite* if its set of worlds is finite and every classical model associated with its worlds is finite.

Since cd -models are a special case of xd -models, we define truth and validity only for the latter. Let $\mathfrak{M} = (W, \{R_\Box\}_{\Box \in M}, \langle \Delta_w, I_w \rangle_{w \in W})$ be an xd -model. An *assignment* in $\mathfrak{M}$ is a map α sending x to an element of the set $\bigcup_{w \in W} \Delta_w$. The *truth* of one-variable $\mathcal{L}_M$ - and $\mathcal{L}_M^-$ -formulas at world w of $\mathfrak{M}$ under assignment α is defined by recursion on the formula construction in the standard way, in particular:

- $\mathfrak{M}, w \models^\alpha \exists x \varphi$ if $\mathfrak{M}, w \models^{\alpha'} \varphi$, for some α' such that $\alpha'(x) \in \Delta_w$;
- $\mathfrak{M}, w \models^\alpha t_1 = t_2$ if $|t_1|_w^\alpha = |t_2|_w^\alpha$, where $|t|_w^\alpha$ denotes the value of the term t at w under α .

We say that φ is *true at $w \in W$* , and write $\mathfrak{M}, w \models \varphi$, if $\mathfrak{M}, w \models^\alpha \varphi$ holds for every α with $\alpha(x) \in \Delta_w$. We also say that φ is *true in $\mathfrak{M}$* , and write $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, w \models \varphi$ holds for every $w \in W$.

Let $d \in \{xd, cd\}$. We write $\mathfrak{F} \models_d \varphi$ if φ is true in every d -model based on $\mathfrak{F}$. Then $\mathfrak{F}$ is a *d -frame* for a logic L , in symbols $\mathfrak{F} \models_d L$, if $\mathfrak{F} \models_d \varphi$ for all $\varphi \in L$. An $\mathcal{L}_M$ -formula φ is *d -valid* on a class $\mathcal{C}$ of Kripke M -frames (written as $\mathcal{C} \models_d \varphi$) if $\mathfrak{F} \models_d \varphi$, for all $\mathfrak{F} \in \mathcal{C}$. Analogous notions are defined for formulas and models with equality. We define the following sets of formulas:

$$\text{Log}_d \mathcal{C} = \{\varphi \in \mathcal{L}_M \mid \mathcal{C} \models_d \varphi\} \quad \text{and} \quad \text{Log}_d^- \mathcal{C} = \{\varphi \in \mathcal{L}_M^- \mid \mathcal{C} \models_d \varphi\}.$$

Note that, for every $\mathcal{C}$, we have $\mathcal{C} \models_{cd} \forall x \Box P(x) \rightarrow \Box \forall x P(x)$ (the Barcan formula, $BF_\Box$), for all $\Box \in M$.

Let L be a one-variable M -logic, $\mathcal{C}$ a class of M -frames, and $d \in \{xd, cd\}$. We say L is *d -complete* for $\mathcal{C}$ if $L = \text{Log}_d \mathcal{C}$, and that L is *globally d -complete* for $\mathcal{C}$ in case $\varphi \vdash_L^* \psi$ iff $\mathfrak{M} \models \varphi$ implies $\mathfrak{M} \models \psi$, for every d -model $\mathfrak{M}$ based on a frame from $\mathcal{C}$. Note that global d -completeness implies d -completeness,

¹One-variable fragments of first-order modal logics are one-variable logics in our sense, but not every one-variable logic is a one-variable fragment of a first-order modal logic; see, e.g., [29, Section 3].

but the converse is not always true. We say that L has the *finite d-model property* (*d-fmp*) if, for every $\varphi \notin L$, there exists a finite d -model $\mathfrak{M}$ such that $\mathfrak{M} \models L$ and $\mathfrak{M} \not\models \varphi$. We say that L has the *global finite d-model property* (*global d-fmp*) if, for every φ and ψ with $\varphi \not\vdash_L^* \psi$, there exists a finite d -model $\mathfrak{M}$ such that $\mathfrak{M} \models L \cup \{\varphi\}$ and $\mathfrak{M} \not\models \psi$.

The *fusion* of Kripke frames $\mathfrak{F}_1 = (W, \{R_\square\}_{\square \in M_1})$ and $\mathfrak{F}_2 = (W, \{R_\square\}_{\square \in M_2})$, where $M_1 \cap M_2 = \emptyset$, is the frame $\mathfrak{F}_1 \otimes \mathfrak{F}_2 = (W, \{R_\square\}_{\square \in M_1 \cup M_2})$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of Kripke frames, then $\mathcal{C}_1 \otimes \mathcal{C}_2$ denotes the class of frames comprising $\mathfrak{F}_1 \otimes \mathfrak{F}_2$, for all $\mathfrak{F}_1 \in \mathcal{C}_1$ and $\mathfrak{F}_2 \in \mathcal{C}_2$ with the same set of worlds.

We will also consider propositional modal logics that can be viewed as fragments of the one-variable modal logics introduced above. Our notation and definitions are standard, following the conventions established earlier; see also the full version [20]. The *fusion* of a propositional M_1 -logic L_1 and a propositional M_2 -logic L_2 , where $M_1 \cap M_2 = \emptyset$, is the smallest propositional $(M_1 \cup M_2)$ -logic $L_1 \otimes L_2$ that includes $L_1 \cup L_2$. The *semicommutator*² and the *commutator* [25, 23] of a propositional M -logic L with the propositional monomodal logic **S5** are the propositional $(M \cup \{\square_E\})$ -logics

$$[L, \mathbf{S5}]^{xd} = L \otimes \mathbf{S5} + \{(lcom_\square) \mid \square \in M\} \quad \text{and} \quad [L, \mathbf{S5}]^{cd} = [L, \mathbf{S5}]^{xd} + \{(rcom_\square) \mid \square \in M\},$$

where $(lcom_\square)$ and $(rcom_\square)$ are the following formulas with their Kripke frame correspondents:

$$\begin{aligned} \square \square_E p \rightarrow \square_E \square p, & \quad E \circ R \subseteq R \circ E, & (lcom_\square) \\ \square_E \square p \rightarrow \square \square_E p, & \quad R \circ E \subseteq E \circ R. & (rcom_\square) \end{aligned}$$

There exists a well-known translation of one-variable M -formulas into propositional $(M \cup \{\square_E\})$ -formulas: $P(x)^* = p$; $(\neg \varphi)^* = \neg \varphi^*$; $(\varphi_1 \wedge \varphi_2)^* = \varphi_1^* \wedge \varphi_2^*$; $(\square \varphi)^* = \square \varphi^*$, for all $\square \in M$; $(\exists x \varphi)^* = \diamond_E \varphi^*$. We note that $CBF_\square^* = (lcom_\square)$ and $BF_\square^* = (rcom_\square)$. If $\Gamma \subseteq \mathcal{L}_M$, then $\Gamma^* = \{\varphi^* \mid \varphi \in \Gamma\}$.

If L is a propositional M -logic, then the smallest one-variable M -logic containing all the substitution instances of formulas from L (here, we substitute arbitrary one-variable formulas for proposition letters) is denoted by $\mathcal{Q}L$.

Lemma 2.1. *If L is a propositional M -logic, then $\mathcal{Q}L^* = [L, \mathbf{S5}]^{xd}$ and $(\mathcal{Q}L + BF)^* = [L, \mathbf{S5}]^{cd}$.*

Proof. Let $L' = [L, \mathbf{S5}]^{xd}$ and let $\mathbf{FO}_1$ be the one-variable fragment of the classical first-order logic. It is easy to check that $\vdash_{L'} \square \diamond_E p \rightarrow \square_E \diamond p (= chr_\diamond)$, for each $\diamond$ corresponding to $\square \in M$. We also rely on the well-known [32] fact that $\mathbf{FO}_1^* = \mathbf{S5}$.

First, we prove that the translation $\cdot^*$ commutes with substitution: for all one-variable formulas $\varphi, \psi_1, \dots, \psi_n$,

$$(\varphi)^*[\psi_1, \dots, \psi_n/P_1(x), \dots, P_n(x)] = \varphi^*[\psi_1^*, \dots, \psi_n^*/p_1, \dots, p_n]. \quad (*)$$

The proof is by induction on φ ; we only consider the case $\varphi = \exists x \theta$. Let $\sigma = [\psi_1, \dots, \psi_n/P_1(x), \dots, P_n(x)]$ and $\sigma^* = [\psi_1^*, \dots, \psi_n^*/p_1, \dots, p_n]$. Since, by IH, $(\theta \sigma)^* = \theta^* \sigma^*$, it follows that

$$((\exists x \theta) \sigma)^* = (\exists x (\theta \sigma))^* = \diamond_E (\theta \sigma)^* = \diamond_E (\theta^* \sigma^*) = (\diamond_E \theta^*) \sigma^* = (\exists x \theta)^* \sigma^*.$$

We now prove that $L' \subseteq \mathcal{Q}L^*$. It should be clear that $L \subseteq \mathcal{Q}L^*$, and that $\mathbf{S5} = \mathbf{FO}_1^* \subseteq \mathcal{Q}L^*$. Since $CBF_\square$ belongs to every one-variable modal logic and $CBF_\square^* = (lcom_\square)$, it follows that $(lcom_\square) \in \mathcal{Q}L^*$. Now, the claim follows from $(*)$ and the closure of $\mathcal{Q}L^*$ under MP, Necessitation, and Generalisation.

²In general, semicommutators are also required to contain confluence axioms for commuting modalities; these are derivable in semicommutators with **S5**.

We next prove that $\mathcal{Q}L^* \subseteq L'$. Here, we use the fact that $\mathbf{FO}_1$ is obtained by adding to the classical propositional logic the axiom $\forall x P(x) \rightarrow P(x)$ and the rule $\psi \rightarrow \varphi(x) / \psi \rightarrow \forall x \varphi(x)$, where x is not free in ψ . In view of (*), the only non-trivial part is to show that $\vdash_{L'} (\psi \rightarrow \varphi)^*$ implies $\vdash_{L'} (\psi \rightarrow \forall x \varphi)^*$ provided that x is not free in ψ . To this end, we show that, if γ is a propositional $(M \cup \{\Box_E\})$ -formula and β is a propositional $(M \cup \{\Box_E\})$ -formula such that all occurrences of proposition letters occur within the scope of $\Box_E$, then $\vdash_{L'} \beta \rightarrow \gamma$ implies $\vdash_{L'} \beta \rightarrow \Box_E \gamma$. Since $\Box_E$ is monotonic, it is enough to show that $\vdash_{L'} \beta \rightarrow \Box_E \beta$, for all β of the assumed form.

Due to our assumption about its form, we may rewrite β with $\neg$, $\wedge$, $\vee$, $\Box_E$, $\Box$, and $\Diamond$ so that all negations occur either within the scope of $\Box_E$ or immediately precede $\Box_E$ (to that end, we push all the negations outside of any $\Box_E$ down until they reach some $\Box_E$). Thus, β is built from formulas of the form $\Box_E \delta$ and $\neg \Box_E \delta$ using $\wedge$, $\vee$, $\Box$, and $\Diamond$. We proceed by induction on the structure of such β .

- If $\beta = \Box_E \gamma$ or $\beta = \neg \Box_E \gamma$, then the claim holds because $\Box_E$ is an **S5** modality, and therefore $\vdash_{L'} \Box_E \gamma \rightarrow \Box_E \Box_E \gamma$ and $\vdash_{L'} \neg \Box_E \gamma \rightarrow \Box_E \neg \Box_E \gamma$.
- The cases of $\wedge$ and $\vee$ are straightforward.
- If $\beta = \Box \gamma$ and $\vdash_{L'} \gamma \rightarrow \Box_E \gamma$, then, by monotonicity of $\Box$ and $(lcom_\Box)$, $\vdash_{L'} \Box \gamma \rightarrow \Box_E \Box \gamma$.
- If $\beta = \Diamond \gamma$ and $\vdash_{L'} \gamma \rightarrow \Box_E \gamma$, then, by monotonicity of $\Diamond$ and $(chr_\Box)$, $\vdash_{L'} \Diamond \gamma \rightarrow \Box_E \Diamond \gamma$.

To prove the claim of the lemma for $[L, \mathbf{S5}]^{cd}$, it suffices to recall that $BF_\Box^* = (rcom_\Box)$. $\square$

3 Fusions of Equality-Free One-Variable Logics

In this section, we prove a precise version of Theorem A.

Theorem 3.1. *Let $\mathcal{C}_1, \mathcal{C}_2$ be non-empty classes of frames closed under disjoint unions, $d \in \{xd, cd\}$, and L_i be one-variable modal logics (globally) d -complete for $\mathcal{C}_i$, for both $i = 1, 2$. Then*

- $L_1 \otimes L_2$ is (globally) d -complete for $\mathcal{C}_1 \otimes \mathcal{C}_2$;
- $L_1 \otimes L_2$ is (globally) decidable if both L_1 and L_2 are (globally) decidable;
- $L_1 \otimes L_2$ has the d -fmp if both L_1 and L_2 have the d -fmp.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ contain non-trivial frames, then $L_1 \otimes L_2$ does not have the global d -fmp.

We begin the proof of these results by defining a general structure of frames for fusions, which follows Goranko and Passy's cactus model construction [16].

Definition 3.2. An *i-cactus frame* (W, R_1, R_2) with thorns $V \subseteq W$ is defined inductively.

basis A 1-frame $(W^0, R_1^0) \in \mathcal{C}_1$ gives rise to a 2-cactus frame $(W^0, R_1^0, \emptyset)$ with thorns W^0 . A 1-cactus frame is defined symmetrically.

induction step Let (W, R_1, R_2) be a 2-cactus frame with thorns V and $(W^v, R_2^v) \in \mathcal{C}_2$, for $v \in V$, a collection of 2-frames such that, for each $v \in V$, the following conditions hold:

$$W \cap W^v = \{v\} \quad \text{and} \quad W^v \cap W^{v'} = \emptyset, \text{ for all } v' \in V \text{ with } v' \neq v. \quad (1)$$

Then $(W', R_1, R_2 \cup \bigcup_{v \in V} R_2^v)$ is a 1-cactus frame with thorns $W' \setminus W$, where $W' = W \cup \bigcup_{v \in V} W^v$.

Given a 1-cactus frame and a collection of 1-frames, a 2-cactus frame is defined symmetrically.

A *limit i-cactus frame* is obtained as the limit of the sequence of cactus frames starting from an *i*-frame.

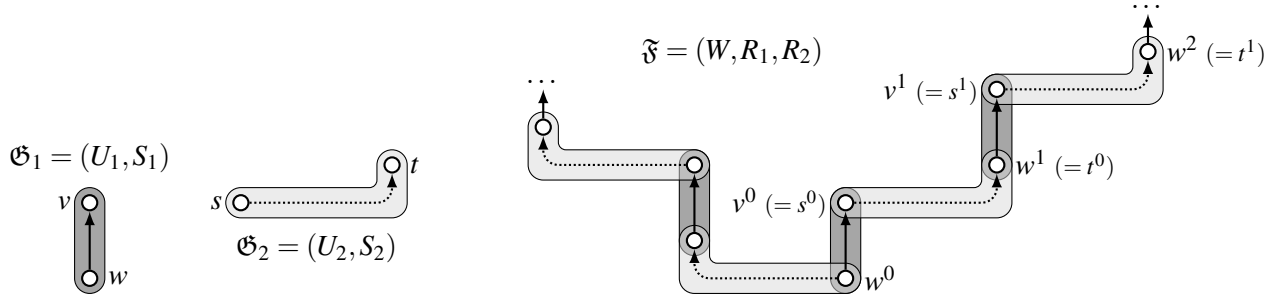


Figure 1: Sample non-trivial frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$ and resulting limit 1-cactus frame $\mathfrak{F}$: solid arrows with dark background for R_1/S_1 -components and dotted arrows with light background for R_2/S_2 -components.

The following property of limit i -cactus frames is immediate from (1):

Proposition 3.3. *A limit i -cactus frame is a frame from $\mathcal{C}_1 \otimes \mathcal{C}_2$.*

With these definition and property at hand, we can establish the general non-transfer of the global d -fmp claimed in Theorem 3.1.

Lemma 3.4. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of frames closed under disjoint unions containing non-trivial frames, $d \in \{xd, cd\}$, and $L_i = \text{Log}_d \mathcal{C}_i$, for $i = 1, 2$. Then $L_1 \otimes L_2$ does not enjoy the global d -fmp.*

Proof. We define the modal operators

$$\Diamond_p \psi = \Diamond_1(\neg p \wedge \Diamond_2(p \wedge \psi)) \quad \text{and} \quad \Box_p \psi = \neg \Diamond_p \neg \psi,$$

where p is a proposition letter not in ψ , and consider the formula (‘every dog has its day’)

$$\varphi = p \rightarrow [\exists x (\neg Q(x) \wedge \Diamond_p Q(x)) \wedge \forall x (Q(x) \rightarrow \Box_p Q(x))].$$

We show that $\varphi \not\models_{L_1 \otimes L_2}^* \neg p$ and that this cannot be witnessed by a finite model.

First, we note that, if $\mathfrak{M}$ is a model based on a frame $\mathfrak{F} = (W, R_1, R_2)$ for $L_1 \otimes L_2$ and $w^0 \in W$ such that $\mathfrak{M}, w^0 \models p$ and $\mathfrak{M} \models \varphi$, then $\mathfrak{M}$ is infinite: such a model $\mathfrak{M}$ contains an infinite $R_1 R_2$ -path of worlds, starting at w^0 , with the ever-increasing, non-universal extension of Q , all of which make p true.

Next, to show that $\varphi \not\models_{L_1 \otimes L_2}^* \neg p$, we define an $L_1 \otimes L_2$ -model $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$, but $\mathfrak{M}, w^0 \not\models p$, for some world w^0 . Let $\mathfrak{G}_1 = (U_1, S_1)$ and $\mathfrak{G}_2 = (U_2, S_2)$ be non-trivial frames from, respectively, $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus, $\mathfrak{G}_1$ contains worlds w and v such that $w \neq v$ and $w S_1 v$; similarly, $\mathfrak{G}_2$ contains worlds s and t such that $s \neq t$ and $s S_2 t$; see Fig. 1 for a simple example. Let $\mathfrak{F} = (W, R_1, R_2)$ be the limit 1-cactus frame constructed from the sequence $\mathfrak{G}_1^i = (U_1^i, S_1^i)$, $i \in \mathbb{N}$, of disjoint copies of $\mathfrak{G}_1$ and the sequence $\mathfrak{G}_2^i = (U_2^i, S_2^i)$, $i \in \mathbb{N}$, of disjoint copies of $\mathfrak{G}_2$, starting with $\mathfrak{G}_1^0$; we identify v^i with s^i and t^i with w^{i+1} , for all $i \in \mathbb{N}$, where u^i denotes the copy of u from U_k^i , for $i \in \mathbb{N}$. By Proposition 3.3, $\mathfrak{F} \in \mathcal{C}_1 \otimes \mathcal{C}_2$, and so $\mathfrak{F}$ is a frame for $L_1 \otimes L_2$. Let $W^0 = \{w^0\}$ and, for all $i \in \mathbb{N}$, let $V^i = R_2(W^i)$ and $W^{i+1} = R_1(V^i)$. Now, define a model $\mathfrak{M} = (\mathfrak{F}, \langle \mathbb{N}, I_w \rangle_{w \in W})$ by putting $I_w(Q) = \{j \in \mathbb{N} \mid j < i\}$ if $w \in W^i$, and $I_w(Q) = \emptyset$ otherwise, and by making p true at $w \in W$ iff $w \in \bigcup_{i \in \mathbb{N}} W^i$. Then $\mathfrak{M} \models \varphi$ but $\mathfrak{M}, w^0 \not\models \neg p$. $\square$

The proofs of the remaining transfer results claimed in Theorem 3.1 lift the cactus model construction to the one-variable modal logic using the quasimodel technique; see, e.g., [25]. We require two additional main ingredients: (a) types and quasistates for a finite representation of any number of domain elements and (b) surrogates for separating the two components of the fusion.

In the sequel, we denote the language of $L_1 \otimes L_2$ by $\mathcal{L}$ and the language of L_i by $\mathcal{L}_i$, for $i = 1, 2$. For simplicity, we assume that L_1 and L_2 are monomodal logics with modalities $\Box_1$ and $\Box_2$, respectively. Let Φ be a finite set of $\mathcal{L}$ -formulas closed under subformulas. A Φ -type t is a maximal Boolean-consistent subset of $\{\psi, \neg\psi \mid \psi \in \Phi\}$, that is, $\neg\psi \in t$ iff $\psi \notin t$, for each $\psi \in \Phi$, and $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$, for each $\psi_1 \wedge \psi_2 \in \Phi$. The number of Φ -types is bounded by $2^{|\Phi|}$ (for $\neg\psi \in \Phi$, a type contains either $\neg\psi$ or both $\neg\neg\psi$ and ψ). A Φ -quasistate is a non-empty set q of Φ -types such that

$$\exists x \psi \in t \quad \text{iff} \quad \text{there is } t' \in q \text{ with } \psi \in t', \quad \text{for each } \exists x \psi \in \Phi \text{ and } t \in q.$$

With a Φ -quasistate q , we associate the following *realisability* $\mathcal{L}$ -sentence:

$$\hat{q} = \forall x \bigvee_{t \in q} t(x) \wedge \bigwedge_{t \in q} \exists x t(x),$$

where $t(x)$ denotes the conjunction of formulas in t . If $\mathcal{Q}$ is a set of Φ -quasistates, then $\hat{\mathcal{Q}} = \bigvee \{\hat{q} \mid q \in \mathcal{Q}\}$.

Let φ be an $\mathcal{L}$ -formula. For each $\Box_i \psi \in \text{sub } \varphi$, we introduce a fresh unary predicate letter $P_{\Box_i \psi}$ and define the *surrogate* of $\Box_i \psi$ by taking $P_{\Box_i \psi}(x)$ if ψ has a free variable and $\forall x P_{\Box_i \psi}(x)$ if ψ is closed. Denote by φ^1 the $\mathcal{L}_1$ -formula obtained from φ by replacing $\Box_2 \psi$ not in the scope of another $\Box_2$ with its surrogate; the $\mathcal{L}_2$ -formula φ^2 is defined symmetrically.

As has already been observed in the propositional setting [21, 2], the analysis of local consequence is substantially more intricate than that of global consequence. We begin our proof with the latter case and show the following lemma which states transfer of *xd*-completeness of the global consequence and from which the transfer of decidability of global consequence follows by a simple enumeration of all possible Φ -quasistates.

Lemma 3.5. *Let $\mathcal{C}_1, \mathcal{C}_2$ be non-empty classes of frames closed under disjoint unions and L_i be one-variable modal logics globally *xd*-complete for $\mathcal{C}_i$, for both $i = 1, 2$. Let φ and ψ be $\mathcal{L}$ -formulas. Denote $\Phi = \text{sub } \varphi \cup \text{sub } \psi$. Then the following are equivalent:*

- (G1) $\varphi \Vdash_{L_1 \otimes L_2}^* \psi$;
- (G2) *there is an *xd*-model $\mathfrak{M}$ based on a frame from $\mathcal{C}_1 \otimes \mathcal{C}_2$ such that $\mathfrak{M} \models \varphi$ but $\mathfrak{M} \not\models \psi$;*
- (G3) *there is a set $\mathcal{Q}$ of Φ -quasistates such that*
 - (G3.1) $\varphi^1 \wedge \hat{\mathcal{Q}}^1 \Vdash_{L_1}^* \psi^1$,
 - (G3.2) $\varphi^1 \wedge \hat{\mathcal{Q}}^1 \Vdash_{L_1}^* \neg \hat{q}_i^1$, for each $q_i \in \mathcal{Q}$,
 - (G3.3) $\hat{\mathcal{Q}}^2 \Vdash_{L_2}^* \neg \hat{q}_i^2$, for each $q_i \in \mathcal{Q}$.

Proof. Implication (G2) $\Rightarrow$ (G1) follows from soundness of $\vdash_{L_1 \otimes L_2}^*$.

(G1) $\Rightarrow$ (G3) Suppose $\varphi \Vdash_{L_1 \otimes L_2}^* \psi$ and consider the following set $\mathcal{Q}$ of Φ -quasistates:

$$\mathcal{Q} = \{q \text{ a } \Phi\text{-quasistate} \mid \varphi \Vdash_{L_1 \otimes L_2}^* \neg \hat{q}\}.$$

Clearly, $\bigvee_q \hat{q}$, for all Φ -quasistates q , is an $L_1 \otimes L_2$ -tautology, and so $\varphi \vdash_{L_1 \otimes L_2}^* \bigvee_q \hat{q}$. Observe that $\hat{\mathcal{Q}}$ consists of the disjuncts left after removing from $\bigvee_q \hat{q}$ all q with $\varphi \vdash_{L_1 \otimes L_2}^* \neg \hat{q}$. Thus, we obtain $\varphi \vdash_{L_1 \otimes L_2}^* \hat{\mathcal{Q}}$. Next, $\varphi \Vdash_{L_1 \otimes L_2}^* \psi$ implies $\varphi \wedge \hat{\mathcal{Q}} \Vdash_{L_1 \otimes L_2}^* \psi$. Moreover, by definition, we also have $\hat{\mathcal{Q}} \Vdash_{L_1 \otimes L_2}^* \neg \hat{q}_i$ and therefore $\varphi \wedge \hat{\mathcal{Q}} \Vdash_{L_1 \otimes L_2}^* \neg \hat{q}_i$, for every $q_i \in \mathcal{Q}$. Then (G3.1)–(G3.3) follow by contraposition: for example, if we assume $\varphi \wedge \hat{\mathcal{Q}} \Vdash_{L_1 \otimes L_2}^* \psi$ but $\varphi^1 \wedge \hat{\mathcal{Q}}^1 \not\vdash_{L_1}^* \psi^1$, then the latter also provides an inference in $L_1 \otimes L_2$, whence, as a substitution instance, we obtain $\varphi \wedge \hat{\mathcal{Q}} \vdash_{L_1 \otimes L_2}^* \psi$ contrary to our assumption.

(G3) $\Rightarrow$ (G2) Suppose there is a set $\mathcal{Q}$ of Φ -quasistates such that (G3.1)–(G3.3) hold. To describe the construction of the witnessing *xd*-model $\mathfrak{M}$, we require the following set of definitions.

Definition 3.6. A Φ -run r through a frame (W, R_i) is a function from an R_i -closed subset of W , denoted by $\text{dom } r$, to Φ -types. For $w \in \text{dom } r$, a run r is called

- *i-coherent at w* if $\Box_i \psi \in r(w)$ implies $\psi \in r(w')$, for all $w' \in R_i(w)$;
- *i-saturated at w* if $\Box_i \psi \in r(w)$ whenever $\Box_i \psi \in \Phi$ and $\psi \in r(w')$, for all $w' \in R_i(w)$.

We say that r is *i-coherent on $W' \subseteq W$* if r is *i-coherent at each $w \in \text{dom } r \cap W'$* and *i-coherent* if r is *i-coherent on W* ; similarly for r being *i-saturated*.

An *i-quasimodel* $\mathfrak{Q}$ for Φ is a tuple of the form $((W, R_i), w_0, \mathfrak{R})$, where $(W, R_i) \in \mathcal{C}_i$, $w_0 \in W$ and $\mathfrak{R}$ is a set of Φ -runs through (W, R_i) such that each $r \in \mathfrak{R}$ is *i-coherent* and *i-saturated*, and each $\mathfrak{R}(w)$ is a Φ -quasistate, for $w \in W$, where we denote $\mathfrak{R}_w = \{r \in \mathfrak{R} \mid w \in \text{dom } r\}$ and $\mathfrak{R}(w) = \{r(w) \mid r \in \mathfrak{R}_w\}$.

Definition 3.7. An *i-cactus* $((W, R_1, R_2), w_0, \mathfrak{R})$ for Φ with thorns $V \subseteq W$ is defined inductively.

basis A 1-quasimodel $((W^0, R_1^0), w_0, \mathfrak{R}^0)$ for Φ with a 1-frame $(W^0, R_1^0) \in \mathcal{C}_1$ gives rise to a 2-cactus $((W^0, R_1^0, \emptyset), w_0, \mathfrak{R}^0)$ for Φ with thorns W^0 . Note that each $r \in \mathfrak{R}^0$ is vacuously 2-coherent, but not necessarily 2-saturated. A 1-cactus is defined symmetrically.

induction step Let $((W, R_1, R_2), w_0, \mathfrak{R})$ be a 2-cactus for Φ with thorns V and $\mathfrak{Q}^v$, for $v \in V$, a collection of 2-quasimodels $\mathfrak{Q}^v = ((W^v, R_2^v), v, \mathfrak{R}^v)$ for Φ such that, for each $v \in V$, we have (1) and

$$\mathfrak{R}(v) = \mathfrak{R}^v(v).$$

Then $((W', R_1, R_2 \cup \bigcup_{v \in V} R_2^v), w_0, \mathfrak{R}')$ is a 1-cactus for Φ with thorns $W' \setminus W$, where $W' = W \cup \bigcup_{v \in V} W^v$,

$$\mathfrak{R}' = \bigcup_{r \in \mathfrak{R}} \{r \cup \bigcup_{v \in \text{dom } r \cap V} r^v \mid r^v \in (\mathfrak{R}^v)_v \text{ with } r(v) = r^v(v), \text{ for } v \in \text{dom } r \cap V\} \cup \bigcup_{v \in V} (\mathfrak{R}^v \setminus (\mathfrak{R}^v)_v),$$

and $f \cup \bigcup_{v \in V'} f^v$ denotes the partial function that coincides with f on $\text{dom } f$ and with f^v on $\text{dom } f^v$, for each $v \in V'$; clearly, the Φ -runs in $\mathfrak{R}'$ are well-defined. We say that the $\mathfrak{Q}^v$ are *grafted* onto the 2-cactus. It can be seen that each $r' \in \mathfrak{R}'$ is 1- and 2-coherent as well as 2-saturated on W' , and is 1-saturated on W but not necessarily on $W' \setminus W$. Given a 1-cactus for Φ and a collection of 1-quasimodels, a 2-cactus for Φ is defined symmetrically.

A *limit i-cactus* for Φ is obtained as the limit of the sequence of cacti starting from a *i-quasimodel*.

We construct a limit 1-cactus for Φ by using the following procedure. We first observe that, by (G3.2) and global Kripke completeness of L_1 , we have $\varphi^1 \in t^1$ and so $\varphi \in t$ and thus $\varphi^2 \in t^2$, for every $t \in \mathbf{q}$ and $\mathbf{q} \in \mathbf{Q}$. We then begin, by (G3.1) and global Kripke completeness of L_1 , with an *xd-model* $\mathfrak{M}^0$ based on $(W^0, R_1^0) \in \mathcal{C}_1$ such that $\mathfrak{M}^0, w \models \varphi^1$ and $\mathfrak{M}^0, w \models \hat{\mathbf{Q}}^1$, for all $w \in W^0$, but $\mathfrak{M}^0, w_0 \not\models \psi^1$, for some $w_0 \in W^0$. With each $w \in W^0$ and $e \in \Delta_w^0$, we associate a uniquely determined Φ -type $t_{w,e}$ such that

$$\mathfrak{M}^0, w \models t_{w,e}^1[e].$$

Let $\mathfrak{R}^0$ be the set of functions r_e , for $e \in \bigcup_{w \in W^0} \Delta_w^0$, that map each $w \in W^0$ with $e \in \Delta_w^0$ to the type $t_{w,e}$. It follows that each r_e is a 1-coherent and 1-saturated Φ -run. The 1-quasimodel $((W^0, R_1^0), w_0, \mathfrak{R}^0)$ for Φ gives rise to our initial 2-cactus for Φ with thorns W^0 . Observe that, for every $\chi \in \Phi$ and $i = 1, 2$, we have

$$\mathfrak{M}^0, w \models \chi^i[e] \quad \text{iff} \quad \chi^i \in t_{w,e}^i, \quad \text{for all } w \in W^0 \text{ and } e \in \Delta_w^0.$$

Let $\mathfrak{C} = ((W, R_1, R_2), w_0, \mathfrak{R})$ be a 2-cactus for Φ with thorns $V \subseteq W$. Take any $v \in V$ and consider the Φ -quasistate $\mathfrak{R}(v) = \mathbf{q}^v \in \mathbf{Q}$ selected for v . By (G3.3) and global Kripke completeness of L_2 , there is an

xd -model $\mathfrak{M}^v$ based on $(W^v, R_2^v) \in \mathcal{C}_2$ such that $\mathfrak{M}^v, v \models (\hat{q}^v)^2$ and $\mathfrak{M}^v, w \models \hat{Q}^2$, for all $w \in W^v$. Similarly to the previous step, with each $w \in W^v$ and $e \in \Delta_w^v$, we associate a uniquely determined Φ -type $t_{w,e}^v$ such that

$$\mathfrak{M}^v, w \models (t_{w,e}^v)^2[e].$$

Let $\mathfrak{R}^v$ be the set of functions r_e^v , for $e \in \bigcup_{w \in W^v} \Delta_w^v$, that map each $w \in W^v$ with $e \in \Delta_w^v$ to the type $t_{w,e}^v$. We have $\mathfrak{R}^v(v) = \mathbf{q}^v = \mathfrak{R}(v)$. As before, $\mathfrak{Q}^v = ((W^v, R_2^v), v, \mathfrak{R}^v)$, for each thorn $v \in V$, is a 2-quasimodel for Φ . We can assume that the W^v satisfy (1), and so we extend the 2-cactus $\mathfrak{C}$ to a 1-cactus $\mathfrak{C}'$ for Φ by grafting 2-quasimodels $\mathfrak{Q}^v$ onto $\mathfrak{C}$.

By repeating this procedure, we obtain a limit 1-cactus $\mathfrak{C} = ((W, R_1, R_2), w_0, \mathfrak{R})$ for Φ . Since the $\mathfrak{M}^v$ realise the respective quasimodels, for every $\chi \in \Phi$, $i = 1, 2$ and $v \in W$, we have

$$\mathfrak{M}^v, w \models \chi^i[e] \quad \text{iff} \quad \chi^i \in (r_e(w))^i, \quad \text{for all } w \in W^v \text{ and } e \in \Delta_w^v.$$

We now construct a Kripke model $\mathfrak{M}$ from the limit 1-cactus $\mathfrak{C}$ for Φ . The frame is taken from $\mathfrak{C}$ (by assumption, both $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed under disjoint unions), the domain $\Delta_w = \mathfrak{R}_w$, for each $w \in W$, and $\mathfrak{M}, w \models P[r]$ iff $P(x) \in r(w)$, for every unary predicate letter P , $w \in W$ and $r \in \mathfrak{R}_w$. By induction on the structure of $\chi \in \Phi$, we show that, for each $w \in W$ and $r \in \mathfrak{R}_w$,

$$\mathfrak{M}, w \models \chi[r] \quad \text{iff} \quad \chi^1 \in (r(w))^1 \quad \text{iff} \quad \chi^2 \in (r(w))^2.$$

The basis of induction is by definition. The cases of the Boolean connectives and quantifiers follow from the definition of quasistates. The cases of the modalities follow from the fact that the Φ -runs in $\mathfrak{R}$ are i -coherent and i -saturated. Thus, $\mathfrak{M}, w \models \varphi$, for all $w \in W$, but $\mathfrak{M}, w_0 \not\models \psi$. $\square$

The claims in Theorem 3.1 for cd -models are obtained by a straightforward modification of the construction in Lemma 3.5 (the runs are required to be total functions on the sets of worlds).

We now turn to the case of local consequence, which is technically more elaborate. The construction is also inspired by the cactus (quasi)models, but instead of starting from quasimodels and grafting more quasimodels to the cactus to obtain the required (quasi)model in the limit, the proof for the local consequence proceeds in the opposite direction: it starts from the ‘leaves’ and grafts smaller cacti onto a quasimodel. In fact, this proof requires only a finite number of steps, which depends on the so-called alternation depth (which is formally defined below, but, intuitively, reflects the number of switches between the modalities in the formula), while the ‘relevant’ part of types in quasimodels gets smaller and smaller as we move away from the world where the given formula is to be satisfied.

For an $\mathcal{L}$ -formula φ , consider the following set of subformulas:

$$\Theta^1(\varphi) = \{P(x) \mid P(x) \in \text{sub } \varphi\} \cup \{\chi \in \text{sub } \Box_2 \psi \mid \Box_2 \psi \in \text{sub } \varphi\}.$$

It can be thought of as removing the outermost ‘layer’ of Booleans, quantifiers and $\Box_1$ -operators. The set $\Theta^2(\varphi)$ is defined symmetrically. For a set Φ of $\mathcal{L}$ -formulas, let $\Theta^i(\Phi) = \bigcup_{\varphi \in \Phi} \Theta^i(\varphi)$.

We now define the notion of *alternation depth* of a formula using a converse operation to Θ^i . More precisely, for an $\mathcal{L}$ -formula ψ with surrogates for $\Box_2$ -subformulas, we denote

$$\psi \uparrow^1 = \psi[(\Box_2 \chi)^1 / (\Box_2 \chi)^2 \mid (\Box_2 \chi)^1 \in \text{sub } \psi].$$

In other words, operation $\cdot \uparrow^1$ restores the ‘meaning’ of the surrogates by re-introducing their $\Box_2$ -modalities, but the replacement subformulas are built from surrogates for their own $\Box_1$ -subformulas instead

(and so contain no $\Box_1$ -modalities themselves); $\psi \uparrow^2$ is defined symmetrically. It should be clear that, if these operations are applied repeatedly, alternating between $\cdot \uparrow^1$ and $\cdot \uparrow^2$, then we arrive at the original formula without surrogates: more precisely, for any $\mathcal{L}$ -formula φ , we have $\varphi = (\varphi^1) \uparrow^1 \uparrow^2 \dots \uparrow^i$, for some $i \in \{1, 2\}$. Denote by $\text{adp}^1(\varphi)$ the minimum length of such a sequence; symmetrically, $\text{adp}^2(\varphi)$ is the smallest number of $\cdot \uparrow^2$ and $\cdot \uparrow^1$ operations such that $\varphi = (\varphi^2) \uparrow^2 \uparrow^1 \dots \uparrow^i$. For example, $\text{adp}^1(\Box_1 \Box_2 p) = 1$ and $\text{adp}^1(\Box_2 \Box_2 \Box_1 r) = 2$; thus, $\text{adp}^1(\Box_1 \Box_2 p \wedge \Box_2 \Box_2 \Box_1 r) = 2$. It follows from the definitions that

$$|\text{adp}^1(\varphi) - \text{adp}^2(\varphi)| \leq 1,$$

depending on the leads of the longest alternation sequences in φ . Denote $\text{adp}^i(\Phi) = \text{adp}^i(\bigwedge_{\varphi \in \Phi} \varphi)$. Let $\text{adp}(\varphi) = \min(\text{adp}^1(\varphi), \text{adp}^2(\varphi))$ and $\text{adp}(\Phi) = \text{adp}(\bigwedge_{\varphi \in \Phi} \varphi)$.

Observe that, for any set Φ of $\mathcal{L}$ -formulas and $i = 1, 2$ with $\text{adp}(\Phi) = \text{adp}^i(\Phi)$, we have

$$\text{adp}(\Theta^i(\Phi)) = \max\{0, \text{adp}(\Phi) - 1\} \quad \text{and} \quad \text{adp}(\Theta^i(\Phi)) = \text{adp}^{3-i}(\Theta^i(\Phi)). \quad (\dagger)$$

In addition to adp , we define the *modal 1-depth* $md^1(\varphi)$ of φ as the maximum number of nested $\Box_1$ (ignoring $\Box_2$); the *modal 2-depth* $md^2(\varphi)$ is defined symmetrically. We also set $md^i(\Phi) = md^i(\bigwedge_{\varphi \in \Phi} \varphi)$. The modal i -depth will be used to describe the decreasing number of ‘relevant’ formulas as we move away from the world witnessing the formula, while the alternation depth will be used to count the number of requires R_1 - and R_2 -components along the paths.

Finally, for an $\mathcal{L}$ -formula φ , consider

$$\mathcal{Q}_i(\varphi) = \{q \text{ a } \Theta^i(\varphi)\text{-quasistate such that } \not\models_{L_1 \otimes L_2} \neg \hat{q}\},$$

and, as before, let $\hat{\mathcal{Q}}_i(\varphi)$ denote the disjunction of all $\hat{q}$, for $q \in \mathcal{Q}_i(\varphi)$.

With these definitions at hand, we can formulate our criterion for deciding local consequence, as claimed in Theorem 3.1.

Lemma 3.8. *Let $\mathcal{C}_1, \mathcal{C}_2$ be non-empty classes of frames closed under disjoint unions and $L_i = \text{Log}_{\text{xd}} \mathcal{C}_i$, for both $i = 1, 2$. For all $\mathcal{L}$ -formulas φ and $i = 1, 2$ with $\text{adp}(\varphi) = \text{adp}^i(\varphi)$, the following are equivalent:*

- (L1) $\not\models_{L_1 \otimes L_2} \varphi$;
- (L2) *there is an xd-model $\mathfrak{M}$ based on a $\mathcal{C}_1 \otimes \mathcal{C}_2$ -frame such that $\mathfrak{M} \not\models \varphi$;*
- (L3) $\not\models_{L_i} \Box_i^{\leq md^i(\varphi)} (\hat{\mathcal{Q}}_i(\varphi))^i \rightarrow \varphi^i$.

Completeness of $L_1 \otimes L_2$ for $\mathcal{C}_1 \otimes \mathcal{C}_2$ claimed in Theorem 3.1 then follows immediately. For decidability, observe that, while the definition of $\mathcal{Q}_i(\varphi)$ refers to $\vdash_{L_1 \otimes L_2}$ and thus may appear circular, by property $(\dagger)$, the $\Theta^i(\varphi)$ -formulas have smaller alternation depth, and so we can apply the criterion in Lemma 3.8 recursively to the simpler formulas to decide whether the respective q should be included in the set $\mathcal{Q}_i(\varphi)$. This recursion terminates as we eventually reach formulas of alternation depth 0, which, by definition, belong to either $\mathcal{L}_1$ or $\mathcal{L}_2$, where, by assumption, the consequence relation is decidable.

Definition 3.9. Let Φ be a set of $\mathcal{L}$ -formulas closed under subformulas, For $i = 1, 2$ and $h \leq md^i(\Phi)$, denote $\Gamma_h^i(\Phi) = \{\psi \in \Phi \mid md^i(\psi) \leq h\}$. Since $\Gamma_h^i(\Phi) = \emptyset$ for $h < 0$, there is only one such $\Gamma_h^i(\Phi)$ -type, $\emptyset$. For a type t , let $t|_\Phi$ be the restriction of t to Φ , that is, $t \cap \{\psi, \neg \psi \mid \psi \in \Phi\}$, and denote $q|_\Phi = \{t|_\Phi \mid t \in q\}$.

Let $(W, R_i) \in \mathcal{C}_i$. For $w, w' \in W$, let $\text{dist}_{R_i}(w, w')$ be the length of the shortest R_i -path from w to w' . A *tapered Φ -run through* $((W, R_i), w_0)$ is a function r from an R_i -closed subset of W , denoted by $\text{dom } r$, such that each $r(w)$ is a $\Gamma_{h(w)}^i(\Phi)$ -type, where $h(w) = md^i(\Phi) - \text{dist}_{R_i}(w_0, w)$. For $w \in \text{dom } r$, run r is

- *i-coherent at w* if $\Box_i \psi \in r(w)$ implies $\psi \in r(w')$, for all $w' \in R_i(w)$;

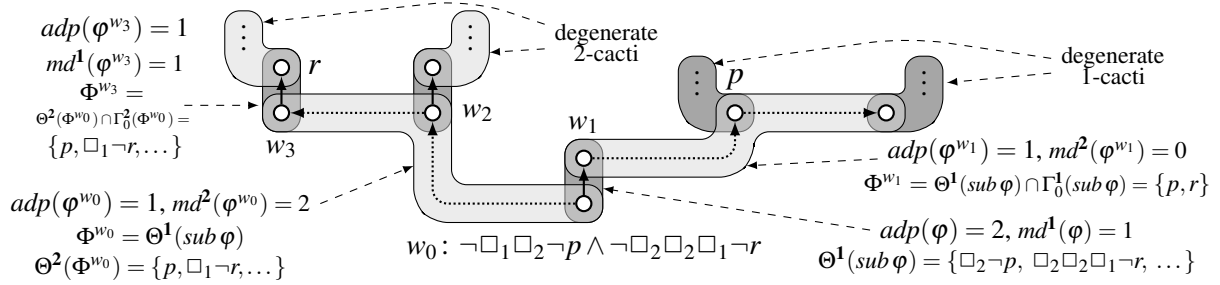


Figure 2: A fully grafted tapered 1-cactus satisfying $\phi = \Diamond_1 \Diamond_2 p \wedge \Diamond_2 \Diamond_2 \Diamond_1 r$: solid arrows with dark background for R_1 -components and dotted arrows with light background for R_2 -components.

- *i-saturated at w* if $\neg \Box_i \psi \notin r(w)$ whenever $\psi \in r(w')$, for all $w' \in R_i(w)$.³

We say that r is *i-coherent on* $W' \subseteq W$ if r is *i-coherent* at each $w \in \text{dom } r \cap W'$ and *i-coherent* if r is *i-coherent* on W ; similarly for r being *i-saturated*.

A *tapered i-quasimodel* Ω for Φ is a tuple of the form $((W, R_i), w_0, \mathfrak{R})$, where $(W, R_i) \in \mathcal{C}_i$, $w_0 \in W$, and $\mathfrak{R}$ is a set of tapered Φ -runs through $((W, R_i), w_0)$ such that each $r \in \mathfrak{R}$ is *i-coherent* and *i-saturated* and each $\mathfrak{R}(w)$ is a Φ -quasistate, for $w \in W$, with $\mathfrak{R}_w = \{r \in \mathfrak{R} \mid w \in \text{dom } r\}$ and $\mathfrak{R}(w) = \{r(w) \mid r \in \mathfrak{R}_w\}$.

Definition 3.10. We inductively define a *tapered i-cactus* $\mathfrak{C}$ for Φ as a tuple of the form $((W, R_1, R_2), w_0, \mathfrak{R})$, where $(W, R_i) \in \mathcal{C}_i$ and $(W \setminus \{w_0\}, R_{3-i}) \in \mathcal{C}_{3-i}$ with a distinguished world $w_0 \in W$, which is not in the domain or range of R_{3-i} , and $\mathfrak{R}$ is a set of tapered Φ -runs that are *i-* and $(3-i)$ -coherent as well as *i-saturated* on W but $(3-i)$ -saturated only on $W \setminus \{w_0\}$; see Fig. 2, which shows the frame structure.

basis: $\Phi = \emptyset$. A tapered ‘degenerate’ 1-cactus $((W, R_1, R_2), w_0, \mathfrak{R})$ for $\emptyset$ is constructed by first taking sufficiently large disjoint unions (W, R_1) and (W', R'_2) of copies of $\mathfrak{F}_1 \in \mathcal{C}_1$ and $\mathfrak{F}_2 \in \mathcal{C}_2$, respectively, such that $|W| - 1 = |W'|$. We then take any $w_0 \in W$ and obtain the required (W, R_1, R_2) by applying to R'_2 the bijection between $W \setminus \{w_0\}$ and W' . Finally, $\mathfrak{R}$ consists of a single tapered $\emptyset$ -run r with $r(w) = \emptyset$, for all $w \in W$, which is trivially 1- and 2-coherent and 1- and 2-saturated.

induction step: $\Phi \neq \emptyset$. Let $\Omega = ((W^0, R_1^0), w_0, \mathfrak{R}^0)$ be a tapered 1-quasimodel for Φ . For each $v \in W^0$, denote

$$\Phi^v = \Theta^1(\Phi) \cap \Gamma_{h(v)}^1(\Phi), \quad \text{where } h(v) = md^1(\Phi) - dist_{R_1^0}(w_0, v).$$

For each $v \in W^0 \setminus \{w_0\}$, take a tapered 2-cactus $\mathfrak{C}^v = ((W^v, R_1^v, R_2^v), v, \mathfrak{R}^v)$ for Φ^v that is *compatible* with Ω :

$$W^0 \cap W^v = \{v\} \quad \text{and} \quad \mathfrak{R}^0(v)|_{\Theta^1(\Phi)} = \mathfrak{R}^v(v).$$

We also assume that the W^v are disjoint. Then $((W, R_1, R_2), w_0, \mathfrak{R})$ is a *tapered 1-cactus* $\mathfrak{C}$ for Φ if it is obtained by grafting the $\mathfrak{C}^v$, for $v \in W^0 \setminus \{w_0\}$, onto Ω . More precisely, we say that $((W^0, R_1^0, \emptyset), w_0, \mathfrak{R}^0)$ is an initial *stock* and W^0 its *thorns*. Then, given a stock $((W, R_1, R_2), w_0, \mathfrak{R})$ and a thorn $v \in W$, the *grafting of* $\mathfrak{C}^v$ *onto* v is a stock $((W \cup W^v, R_1 \cup R_1^v, R_2 \cup R_2^v), w_0, \mathfrak{R}')$, where

$$\mathfrak{R}' = \mathfrak{R} \setminus \mathfrak{R}_v \cup \{r \cup r^v \mid r \in \mathfrak{R}_v \text{ and } r^v \in (\mathfrak{R}^v)_v \text{ with } r(v)|_{\Theta^1(\Phi)} = r^v(v)\} \cup \mathfrak{R}^v \setminus (\mathfrak{R}^v)_v.$$

Observe that, as v in $\mathfrak{C}^v$ is not R_1 -connected, the R_1 -components in Ω and $\mathfrak{C}^v$ are disjoint, and so the frame in the resulting stock belongs to $\mathcal{C}_1 \otimes \mathcal{C}_2$. The resulting runs are 1- and 2-coherent and 1-saturated on $W \cup W^v$. The runs are 2-saturated on $W' \cup W^v$ if they are 2-saturated on $W' \subseteq W$.

³We use $\neg \Box_i \psi \notin r(w)$ as a convenient replacement for either $\Box_i \psi \notin \Gamma_{h(w)}^i(\Phi)$ or $\Box_i \psi \in r(w)$.

A fully grafted tapered i -cactus $((W, R_1, R_2), \mathfrak{R})$ for Φ results from grafting a compatible tapered $(3-i)$ -cactus $\mathfrak{C}^{w_0}$ for $\Phi^{w_0} = \Theta^i(\Phi)$ onto the world w_0 of a tapered i -cactus $((W^0, R_1^0, R_2^0), w_0, \mathfrak{R}^0)$ for Φ . Runs in a fully grafted tapered cactus are i -coherent and i -saturated for both $i = 1, 2$ on the whole W .

Claim 3.11. For all $\mathcal{L}$ -formulas φ and $i = 1, 2$ with $\text{adp}(\varphi) = \text{adp}^i(\varphi)$, each of **(L1)–(L3)** holds iff **(L4)** there is a (fully grafted) tapered i -cactus $((W, R_1, R_2), \mathfrak{R})$ for $\text{sub } \varphi$ with $\varphi \notin r(w_0)$, for some $r \in \mathfrak{R}$.

Proof. We proceed by induction on alternation depth. The basis, $\text{adp}(\varphi) = 0$, is trivial as the formula belongs to $\mathcal{L}_1$ or $\mathcal{L}_2$ (or both, if it has no modalities). For the inductive step, consider an $\mathcal{L}$ -formula φ and suppose that $\text{adp}(\varphi) = k > 0$. Assume that $\text{adp}^1(\varphi) = k$ (the other case is symmetrical) and that the claim has been shown for all formulas ψ with $\text{adp}^2(\psi) < k$, which, by $(\dagger)$, includes all $\psi \in \Theta^1(\varphi)$.

While **(L2)** $\Rightarrow$ **(L1)** follows from soundness of $\vdash_{L_1 \otimes L_2}$, **(L1)** $\Rightarrow$ **(L3)** follows from closure under substitutions of $\vdash_{L_i}$ (cf. **(G1)** $\Rightarrow$ **(G3)** in the proof of Lemma 3.5).

(L3) $\Rightarrow$ **(L4)** By Kripke completeness of L_1 , there is an xd -model $\mathfrak{M}^0$ based on $(W^0, R_1^0) \in \mathcal{C}_1$ such that $\mathfrak{M}^0, w_0 \models \Box_1^{<\text{adp}^1(\varphi)}(\hat{\mathcal{Q}}_1(\varphi))^1$ and $\mathfrak{M}^0, w_0 \not\models \varphi^1$ for some $w_0 \in W^0$. Let $\Phi = \text{sub } \varphi$. For each $v \in W^0$, let $h(v) = \text{md}^1(\Phi) - \text{dist}_{R_1^0}(w_0, v)$. Construct a set $\mathfrak{R}^0$ of tapered Φ -runs through $((W^0, R_1^0), w_0)$: for each $e \in \bigcup_{v \in W^0} \Delta_v^0$, consider r_e that maps each $v \in W^0$ with $e \in \Delta_v^0$ to the $\Gamma_{h(v)}^1(\Phi)$ -type $t_{v,e}$ with $\mathfrak{M}, v \models t_{v,e}^1[e]$. Thus, $\mathfrak{Q} = ((W^0, R_1^0), w_0, \mathfrak{R}^0)$ is a tapered 1-quasimodel for Φ with $\varphi \notin r_e(w_0)$, for some $e \in \Delta_{w_0}^0$.

For each $v \in W^0$ with $h(v) \geq 0$, take the $\Theta^1(\Phi)$ -quasistate $\mathbf{q}^v \in \mathcal{Q}_1(\varphi)$ with $\mathfrak{M}^0, v \models (\hat{\mathbf{q}}^v)^1$. Denote $\Phi^v = \Theta^1(\Phi) \cap \Gamma_{h(v)}^1(\Phi)$ and consider $\mathbf{q}^v|_{\Gamma_{h(v)}^1(\Phi)}$, which is a Φ^v -quasistate with $\mathbf{q}^v|_{\Gamma_{h(v)}^1(\Phi)} = \mathfrak{R}^0(v)|_{\Theta^1(\Phi)}$. Let φ^v be its realisability sentence. As $\text{adp}^2(\varphi^v) < k$, we apply IH to φ^v and obtain a tapered 2-cactus $\mathfrak{C}^v = ((W^v, R_1^v, R_2^v), v, \mathfrak{R}^v)$ for Φ^v such that $\mathbf{q}^v|_{\Gamma_{h(v)}^1(\Phi)} = \mathfrak{R}^v(v)$. For each $v \in W^0$ with $h(v) < 0$, take a tapered 2-cactus $\mathfrak{C}^v$ for $\emptyset$. So, for each $v \in W^0 \setminus \{w_0\}$, we graft $\mathfrak{C}^v$ onto v in $\mathfrak{Q}$ to obtain a tapered 1-cactus $\mathfrak{C}$. A fully grafted tapered 1-cactus can be obtained by grafting, in addition, $\mathfrak{C}^{w_0}$ onto w_0 in $\mathfrak{C}$.

(L4) $\Rightarrow$ **(L2)** Let $\mathfrak{C} = ((W, R_1, R_2), \mathfrak{R})$ be a fully grafted tapered i -cactus for $\text{sub } \varphi$ with $\varphi \notin r(w_0)$, for some $r \in \mathfrak{R}$. We construct $\mathfrak{M}$ based on a $\mathcal{C}_1 \otimes \mathcal{C}_2$ -frame (W, R_1, R_2) and domains $\Delta_w = \mathfrak{R}_w$, for $w \in W$: we set $\mathfrak{M}, w \models P[r]$ iff $P(x) \in r(w)$, for each unary predicate letter P , $w \in W$ and $r \in \mathfrak{R}_w$. By induction on the structure of formulas, we show that, for each $w \in W$, $r \in \mathfrak{R}_w$, $\chi \in \{\psi, \neg\psi \mid \psi \in r(w)\}$, and $i = 1, 2$,

$$\mathfrak{M}, w \models \chi[r] \quad \text{iff} \quad \chi^i \in (r(w))^i.$$

The basis of induction is by definition. The cases of the Boolean connectives and quantifiers are immediate from the definition of types and quasistates. The cases of the modalities follow from the fact that the runs in $\mathfrak{C}$ are i -coherent and i -saturated. Thus, $\mathfrak{M}, w_0 \not\models \varphi$. $\square$

The claims on local consequence in Theorem 3.1 for cd -models are obtained by a straightforward modification of the construction in Lemma 3.8 (the tapered runs are required to be total functions on the sets of worlds). Finally, the transfer of the d -fmp follows from the proof of Claim 3.11 (the fully grafted cactus is finite).

4 Fusions of One-Variable Logics with Equality

We show that once equality is allowed in one-variable modal logics, the transfer results established in Section 3 no longer hold. Remarkably, the counterexamples we introduce next arise quite naturally. The propositional modal logic **Diff** of the ‘elsewhere’ operator was introduced by Von Wright [34] and is determined by the class $\mathcal{D}$ of all disjoint unions of frames (W, R) with $R = \{(w, v) \in W \times W \mid w \neq v\}$; see

also [27]. We are interested here in the one-variable modal logics $\text{Log}_{cd}^{\bar{}} \mathcal{D}$ and $\text{Log}_{cd}^{\bar{}} \mathcal{D}_{\text{fin}}$ with $\mathcal{D}_{\text{fin}}$ the class of disjoint unions of *finite* frames in $\mathcal{D}$. Note that the propositional logic **Diff** has the fmp, and so is complete for both $\mathcal{D}$ and $\mathcal{D}_{\text{fin}}$. On the other hand, while one can easily prove the fmp for $\text{Log}_{cd}^{\bar{}} \mathcal{D}_{\text{fin}}$, the logic $\text{Log}_{cd}^{\bar{}} \mathcal{D}$ does not enjoy the fmp, and so $\text{Log}_{cd}^{\bar{}} \mathcal{D} \subsetneq \text{Log}_{cd}^{\bar{}} \mathcal{D}_{\text{fin}}$ [18]. It has been observed [17, 18] that both logics can be reduced in polytime to the two-variable fragment of first-order logic with counting on arbitrary and finite domains, respectively, and so they are decidable in CONEXPTIME.

Theorem 4.1. *Both $\text{Log}_{cd}^{\bar{}} \mathcal{D}$ and $\text{Log}_{cd}^{\bar{}} \mathcal{D}_{\text{fin}}$ are decidable but $\text{Log}_{cd}^{\bar{}}(\mathcal{D} \otimes \mathcal{D}_{\text{fin}})$ is undecidable.*

Proof. The proof is by reduction of the problem of solving Diophantine equations. For arbitrary polynomials g, h with coefficients from $\mathbb{N} \setminus \{0\}$, it is undecidable whether $g = h$ has a solution in $\mathbb{N} \setminus \{0\}$ [7]. Each polynomial equation can be rewritten into a set of elementary equations $y = n$, $y = z_1 + z_2$, and $y = z_1 \times z_2$, with variables y, z_1, z_2 and $n \in \mathbb{N} \setminus \{0\}$. So we assume that a set $E = \{e_1, \dots, e_n\}$ of such equations is given. We construct a formula φ_E which is satisfiable iff E has a solution in $\mathbb{N} \setminus \{0\}$. Let the modal operators $\Box$ and $\Box_C$ be interpreted over frames with accessibility relations R in $\mathcal{D}$ and R_C in $\mathcal{D}_{\text{fin}}$, respectively. We set $\Box^+ \varphi = \varphi \wedge \Box \varphi$, and likewise for $\Box_C$. Formula φ_E will hardly need any alternation between modalities, the maximal alternation being of the form $\Box \Box_C \chi$. In fact, the only contribution of $\mathcal{D}_{\text{fin}}$ is to supply the required *finite* cardinality of the connected components of its frames.

When discussing the construction of φ_E , we assume a world w in the set W of worlds reachable along R . To W attached are the models for $\Box_C$ with the R_C -connected component of w denoted by W_w . For a unary predicate letter P and $w \in W$, we denote by $|P|_w$ the cardinality of the extension of P at w . In the sequel, for a closed formula I and a unary predicate P , we use formula $\text{card}(I, P)$ defined as follows:

$$\Box^+ [I \rightarrow \Box_C^+ \forall x ((x = c) \rightarrow \Box_C(x \neq c)) \wedge \forall x (P(x) \leftrightarrow \Diamond_C^+(x = c))],$$

where c is a fresh constant. Formula $\text{card}(I, P)$ ensures that, at any world w where I holds, the number of worlds in the R_C -connected component of w coincides with the number of domain elements in the extension of P at w , that is, $|W_w| = |P|_w$; see Fig. 3 a), where the nodes representing the same domain element in different worlds are shown connected by the R - and R_C -relations.

For each variable y used on the left- or right-hand side of equations in E , we take unary predicate letters P_y and C_y . Intuitively, $|P_y|_w$ encodes the value of y , while C_y is used to ensure that the value of y is finite. To this end, φ_E contains the conjunct

$$\Diamond^+(I_y \wedge \Box \neg I_y), \quad \text{where } I_y = \exists x C_y(x), \quad (2)$$

to mark a single world $w \in W$ with I_y . We add the conjunct

$$\text{card}(I_y, P_y) \quad (3)$$

to φ_E to ensure that $|P_y|_w = |W_w|$. Note that $|P_y|_w$ is finite. The following conjuncts guarantee that each P_y is non-empty (only solutions in $\mathbb{N} \setminus \{0\}$ are admitted) and constant on W :

$$\exists x P_y(x) \quad \text{and} \quad \Box^+ \forall x (P_y(x) \rightarrow \Box P_y(x)). \quad (4)$$

- If $e \in E$ is of the form $y = n$, then we take n constants $c_1^e, \dots, c_n^e$ interpreted differently and make the set of their interpretations the extension of P_y :

$$\forall x (P_y(x) \leftrightarrow (x = c_1^e) \vee \dots \vee (x = c_n^e)) \wedge \bigwedge_{i \neq j} (c_i^e \neq c_j^e). \quad (5)$$

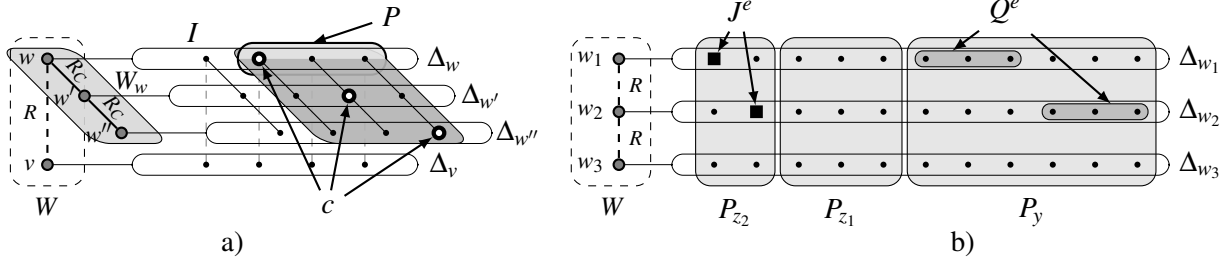


Figure 3: a) Formula $\text{card}(I, P)$. b) Equation e of the form $y = z_1 \times z_2$ with $z_1 = 3$ and $z_2 = 2$.

- If $e \in E$ is of the form $y = z_1 + z_2$, we introduce unary predicate letters Q_1^e and Q_2^e and add formulas

$$\text{card}(I_{z_i}, Q_i^e) \quad \text{for } i = 1, 2, \quad (6)$$

which ensure $|P_{z_i}|_{w_i} = |Q_i^e|_{w_i}$ at the world w_i marked with I_{z_i} . Next, we state that Q_1^e and Q_2^e are disjoint at all worlds $w \in W$ and empty at worlds $w \in W$ where I_{z_i} does not hold:

$$\Box^+ \forall x (Q_1^e(x) \rightarrow \Box^+ \neg Q_2^e(x)), \quad \Box^+ (\exists x Q_1^e(x) \rightarrow I_{z_1}) \quad \text{and} \quad \Box^+ (\exists x Q_2^e(x) \rightarrow I_{z_2}). \quad (7)$$

Finally, we add the conjunct that ensures that P_y is the union of Q_1^e and Q_2^e :

$$\forall x (P_y(x) \leftrightarrow \Diamond^+ Q_1^e(x) \vee \Diamond^+ Q_2^e(x)), \quad (8)$$

which implies that equation e holds in the sense that $|P_y|_w = |P_{z_1}|_w + |P_{z_2}|_w$, for all worlds $w \in W$.

- If $e \in E$ is of the form $y = z_1 \times z_2$, then we introduce a unary predicate letter C^e and a constant c^e and let $J^e(x) = C^e(x) \wedge (c^e = x)$. First, we ensure $|P_{z_2}|_w$ is equal to the number of worlds in W satisfying $\exists x J^e(x)$:

$$\forall x (P_{z_2}(x) \leftrightarrow \Diamond^+ J^e(x)) \quad \text{and} \quad \Box^+ \forall x (J^e(x) \rightarrow \Box \neg J^e(x)), \quad (9)$$

where, due to the second conjunct, $J^e(x)$ cannot hold on the same domain element more than once in W . Next, we introduce a unary predicate letter Q^e and add formulas

$$\text{card}(\exists x J^e(x), P_{z_1}) \quad \text{and} \quad \text{card}(\exists x J^e(x), Q^e) \quad (10)$$

to ensure that $|P_{z_1}|_w = |Q^e|_w$ for all worlds $w \in W$ with $\exists x J^e(x)$. Then we state that the Q^e are mutually disjoint at different worlds $w \in W$ and empty at worlds $w \in W$ where $\exists x J^e(x)$ does not hold:

$$\Box^+ \forall x (Q^e(x) \rightarrow \Box \neg Q^e(x)) \quad \text{and} \quad \Box^+ (\exists x Q^e(x) \rightarrow \exists x J^e(x)). \quad (11)$$

We have $|P_{z_1}|_w$ (encoded as $|Q^e|_w$) in exactly $|P_{z_2}|_w$ -many worlds and take the sum by adding the conjunct

$$\forall x (P_y(x) \leftrightarrow \Diamond^+ Q^e(x)), \quad (12)$$

which implies that equation e holds in the sense that $|P_y|_w = |P_{z_1}|_w \times |P_{z_2}|_w$, for all $w \in W$; see Fig. 3 b), where only worlds in W and their domains are shown, and $|W_{w_1}| = |W_{w_2}| = 2$.

So, φ_E contains (2)–(4), for each variable y in equations in E , and one of (5), (6)–(8) or (9)–(12), for each equation $e \in E$. It should be clear that φ_E is as required. $\square$

Theorem 4.1 does not depend on the constant domain semantics since $\mathcal{D}$ and $\mathcal{D}_{\text{fin}}$ do not feel the difference between expanding and constant domains. Note also that recursive axiomatisability is not preserved under fusions since we have shown that $\text{Log}_{\text{cd}}^-(\mathcal{D} \otimes \mathcal{D}_{\text{fin}})$ is not recursively enumerable.

For the global consequence, one can prove an even stronger non-transfer result.

Theorem 4.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of Kripke frames that are closed under disjoint unions and contain non-trivial frames. Then the global consequence for $\text{Log}_{\text{cd}}^-(\mathcal{C}_1 \otimes \mathcal{C}_2)$ is undecidable.*

The proof is given in the full version of this paper available at [20]. It is by reduction of Minsky machines and very similar to the known undecidability proof for temporal first-order logic with equality [8].

We conclude this section with two remarks. First, note that neither $\mathcal{D}$ nor $\mathcal{D}_{\text{fin}}$ are propositionally modally definable classes of frames. In fact, as **Diff** is axiomatised by $p \rightarrow \Box \Diamond p$ and $\Diamond p \rightarrow \Box(p \vee \Diamond p)$, any frame (W, E) with an equivalence relation E validates **Diff** (in other words, **Diff** $\subseteq$ **S5**). The question arises whether our non-preservation result continues to hold when one restricts the components to propositionally modally definable frame classes. We conjecture that it does, but at present we only have a proof for expanding domain semantics and under the assumption that polymodal logics are permitted as components. To demonstrate this, let $\mathcal{C}_{\text{diff}}$ be the class of *all* frames validating **Diff** and $\mathcal{C}_{\text{ftime}}$ the class of all frames validating the bimodal logic **LTL_f** of all frames (W, S, R) with R a finite strict linear order and S its successor relation. **LTL_f** is axiomatised by taking the fusion of **GL3** and **Alt₁** and adding the standard induction axioms [15].

Theorem 4.3. *$\text{Log}_{\text{xd}}^-(\mathcal{C}_{\text{diff}})$ and $\text{Log}_{\text{xd}}^-(\mathcal{C}_{\text{ftime}})$ are decidable but $\text{Log}_{\text{xd}}^-(\mathcal{C}_{\text{diff}} \otimes \mathcal{C}_{\text{diff}} \otimes \mathcal{C}_{\text{ftime}})$ is undecidable.*

Proof. Decidability of $\text{Log}_{\text{xd}}^-(\mathcal{C}_{\text{diff}})$ is shown in [18]. For decidability (and Ackermann-hardness) of $\text{Log}_{\text{xd}}^-(\mathcal{C}_{\text{ftime}})$, see [19, 12]. Undecidability of $\text{Log}_{\text{xd}}^-(\mathcal{C}_{\text{diff}} \otimes \mathcal{C}_{\text{diff}} \otimes \mathcal{C}_{\text{ftime}})$ can be shown by adapting the reduction of Diophantine equations: first, observe that one can enforce that a frame in $\mathcal{C}_{\text{diff}}$ is actually in $\mathcal{D}$ using the formula

$$\Box^+ \forall x ((x = c) \rightarrow \Box(x \neq c)),$$

which can only be satisfied if all worlds in W are irreflexive. The fusion with $\mathcal{C}_{\text{ftime}}$ is used to enforce a *finite* first-order domain for worlds in W . This can be achieved as follows: for another fresh constant c' and $\Box_L$ the modal operator interpreted by the strict order in $\mathcal{C}_{\text{ftime}}$, if the sentence $\forall x \Box_L^+(x = c')$ is true at w_0 , then the domain at w_0 cannot be larger than the length of the finite strict order seen from w_0 . It is now straightforward to adapt the remaining steps of the undecidability proof. $\square$

We conjecture that using polymodal logics here is not essential. For instance, the standard reduction of polymodal logics to logics with one modality might be applicable [22].

Second, the undecidability proof can be adapted to resolve an open problem concerning monodic fragments of first-order modal logics [25, 1]. A key ingredient of the encoding of Diophantine equations is that we can enforce equal cardinality $|P|_w = |Q|_w$ of the extensions of predicates P and Q at world w . This can also be achieved directly using a bijective function in the extension of one-variable modal logic to, for instance, the monodic fragment of modal first-order logic over the two-variable fragment of **FO** with counting. Hence, it can be shown that this monodic fragment is undecidable over the class $\mathcal{D}_{\text{fin}}$. This shows that decidability does not always transfer from one-variable modal logic with equality to monodic fragments based on decidable fragments of first-order logic (recall that the two-variable fragment with counting is decidable [26]). Note that sufficient conditions for this kind of preservation from one-variable modal logics to monodic fragments are shown in [13].

Theorem 4.4. *$\text{Log}_{\text{cd}}^-(\mathcal{D}_{\text{fin}})$ is decidable, but the monodic fragment of the two-variable fragment with counting over $\mathcal{D}_{\text{fin}}$ is undecidable.*

5 Fusions of Propositional Modal Logics Sharing an S5 Modality

In this section, we generalise the fusion construction for one-variable first-order modal logics without equality to fusions of propositional modal logics sharing an **S5** modality. We begin by establishing a transfer theorem for logics admitting models that are homogeneous for the shared modality, and then apply it to (semi)commutators. In contrast to the transfer proof in Lemma 3.5, in the global case we do not construct cactus models; instead, we merge the respective models of the component logics in a single step to obtain models of the fusion. This approach works provided we can ensure that the extensions of the relevant types in models of the component logics have matching cardinalities. A closely related technique was first used in [33] to obtain transfer results via algebraic semantics rather than Kripke semantics, thereby avoiding any reliance on the Kripke completeness of the component logics.

Consider a propositional modal language with **S5** operator $\Box_E$ interpreted in Kripke models by the equivalence relation E . The E -equivalence class $[w]_E$ of $w \in W$ is defined as usual by setting $[w]_E = E(w)$. Let κ be an infinite cardinal. Then $\mathfrak{M}$ is called (E, κ) -homogeneous if, for every E -equivalence class U in $\mathfrak{F}$ and every modal formula φ , $|\{w \in U \mid \mathfrak{M}, w \models \varphi\}| \in \{0, \kappa\}$.

Definition 5.1. A propositional modal logic L with **S5** operator $\Box_E$ admits E -homogeneous models (for global consequence) if there exists an infinite cardinal κ such that, for all $\kappa' \geq \kappa$, whenever $\varphi \notin L$ (or, respectively, $\varphi \not\vdash_L^* \psi$), there exists an (E, κ') -homogeneous model $\mathfrak{M}$ based on an L -frame with $\mathfrak{M} \not\models \varphi$ (or, respectively, with $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \not\models \psi$).

Note that modal logics that admit homogeneous models (for global consequence) are (globally) Kripke complete, by definition. To formulate our main results, assume for simplicity that L_i is a bi-modal logic in the propositional modal language $\mathcal{L}_{i,E}$ with modal operators $\Box_i$ and $\Box_E$, for $i = 1, 2$. Then $L_1 \otimes_E L_2$ denotes the smallest modal logic containing $L_1 \cup L_2$ in the language $\mathcal{L}_{1,2,E}$ with modal operators $\Box_1, \Box_2$, and $\Box_E$.

Theorem 5.2. *Let L_1 and L_2 admit E -homogeneous models (for global consequence). Then*

- $L_1 \otimes_E L_2$ is (globally) Kripke complete;
- $L_1 \otimes_E L_2$ is (globally) decidable if L_1 and L_2 are both (globally) decidable.

We give a detailed proof for the global consequence. The local case can be shown by a rather straightforward combination of the approaches to the local case in Section 3 and the global case in this section, see [20] for details. Assume L_1 and L_2 admit E -homogeneous models for global consequence.

Modulo replacing $\forall x$ with $\Box_E$, we use the same notation as in Section 3. For instance, if Φ is a finite set of $\mathcal{L}_{1,2,E}$ -formulas closed under subformulas, then a Φ -type is a maximal Boolean-consistent subset of $\{\psi, \neg\psi \mid \psi \in \Phi\}$, and a Φ -quasistate is a non-empty set of Φ -types such that

$$\Box_E \psi \in t \quad \text{iff} \quad \psi \in t', \text{ for all } t' \in \mathbf{q}, \quad \text{for each } \Box_E \psi \in \Phi \text{ and } t \in \mathbf{q}.$$

We set

$$\hat{\mathbf{q}} = \Box_E \bigvee_{t \in \mathbf{q}} t \wedge \bigwedge_{t \in \mathbf{q}} \Diamond_E t,$$

where, for simplicity, we use t to denote the conjunction of all formulas in Φ -type t . Again, $\hat{\mathbf{Q}}$ denotes $\bigvee \{\hat{\mathbf{q}} \mid \mathbf{q} \in \mathbf{Q}\}$ for a set $\mathbf{Q}$ of Φ -quasistates. We also define surrogates in the obvious way by introducing for each formula of the form $\Box_i \psi$, a fresh proposition letter $p_{\Box_i \psi}$ called the *surrogate* of $\Box_i \psi$. To establish the claimed results in the global case, it suffices to show the following lemma.

Lemma 5.3. *Let φ, ψ be $\mathcal{L}_{1,2,E}$ -formulas. Denote $\Phi = \text{sub } \varphi \cup \text{sub } \psi$. Then the following are equivalent:*

(G_E1) $\varphi \not\models_{L_1 \otimes_E L_2}^* \psi$;

(G_E2) *there is a model $\mathfrak{M}$ based on an $L_1 \otimes_E L_2$ -frame such that $\mathfrak{M} \models \varphi$ but $\mathfrak{M} \not\models \psi$;*

(G_E3) *there is a set $\mathcal{Q}$ of Φ -quasistates such that*

(G_E3.1) $\varphi^1 \wedge \hat{\mathcal{Q}}^1 \not\models_{L_1}^* \psi^1$,

(G_E3.2) $\varphi^1 \wedge \hat{\mathcal{Q}}^1 \not\models_{L_1}^* \neg \hat{q}_i^1$, for each $q_i \in \mathcal{Q}$,

(G_E3.3) $\hat{\mathcal{Q}}^2 \not\models_{L_2}^* \neg \hat{q}_i^2$, for each $q_i \in \mathcal{Q}$.

Proof. (G_E2) $\Rightarrow$ (G_E1) and (G_E1) $\Rightarrow$ (G_E3) are shown similar to the proof of Lemma 3.5.

(G_E3) $\Rightarrow$ (G_E2) Let $\mathcal{Q}$ be a set of Φ -quasistates such that (G_E3.1)–(G_E3.3) hold. Choose an infinite κ such that if $\chi \not\models_{L_i}^* \chi'$, then for every $\kappa' \geq \kappa$ there exists an (E, κ') -homogeneous model $\mathfrak{M}_i$ based on an L_i -frame with $\mathfrak{M}_i \models \chi$ and $\mathfrak{M}_i \not\models \chi'$, for $i = 1, 2$. By choosing $\kappa' \geq \kappa$ sufficiently large and taking appropriate disjoint unions of models, it follows from (G_E3.1)–(G_E3.3) that, for each $i = 1, 2$, we can find an (E_i, κ') -homogeneous model $\mathfrak{M}_i = (\mathfrak{F}_i, V_i)$ with $\mathfrak{F}_i = (W_i, R_i, E_i)$ such that $\mathfrak{F}_i \models L_i$ and the following conditions hold:

1. $\mathfrak{M}_1 \models \varphi^1 \wedge \hat{\mathcal{Q}}^1$, $\mathfrak{M}_1 \not\models \psi^1$, and $|\{[w]_{E_1} \mid \mathfrak{M}_1, w \models \hat{q}^1\}| = \kappa'$ for all $q \in \mathcal{Q}$;
2. $\mathfrak{M}_2 \models \hat{\mathcal{Q}}^2$ and $|\{[w]_{E_2} \mid \mathfrak{M}_2, w \models \hat{q}^2\}| = \kappa'$ for all $q \in \mathcal{Q}$.

It follows that we find a bijection $F: W_1 \rightarrow W_2$ such that, for all $w \in W_1$:

1. there is a (uniquely determined) Φ -type t such that $\mathfrak{M}_1, w \models t^1$ and $\mathfrak{M}_2, f(w) \models t^2$;
2. $[f(w)]_{E_2} = \{f(w') \mid w' \in [w]_{E_1}\}$.

Now construct a new model $\mathfrak{M} = (\mathfrak{F}, V)$ by taking $\mathfrak{M}_1$, interpreting $\Box_2$ by $f^{-1}[R_2] = \{(w, w') \in W_1 \mid (f(w), f(w')) \in R_2\}$, and setting $V(p) = f^{-1}(V_2(p))$ for all variables p introduced as surrogates of subformulas of the form $\Box_2 \chi$. Then, for $\mathfrak{F} = (W_1, R_1, f^{-1}[R_2], E_1)$, we have $\mathfrak{F} \models L_1 \otimes_E L_2$ by definition, and one can show by induction for all formulas $\chi \in \Phi$ and $w \in W_1$:

$$\mathfrak{M}, w \models \chi \quad \text{iff} \quad \mathfrak{M}_1, w \models \chi^1 \quad \text{iff} \quad \mathfrak{M}_2, f(w) \models \chi^2.$$

It follows that $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \not\models \psi$, as required. $\square$

The following lemma shows that the conditions of Theorem 5.2 are satisfied for (globally) Kripke complete (semi)commutators with **S5**.

Lemma 5.4. *If a logic of the form $[L, \mathbf{S5}]^{xd}$ or $[L, \mathbf{S5}]^{cd}$ is (globally) Kripke complete, then it admits E -homogeneous models (for global consequence).*

Proof. We give the proof for the local case, the proof in the global case is the same. Assume first that $[L, \mathbf{S5}]^{xd}$ is Kripke complete. Let κ be any infinite cardinal such that, for every $\varphi \notin [L, \mathbf{S5}]^{xd}$, there exists a model refuting φ based on an $[L, \mathbf{S5}]^{xd}$ -frame of size at most κ . We show that κ is as required. Assume $\kappa' \geq \kappa$ and $\varphi \notin [L, \mathbf{S5}]^{xd}$. By definition of κ , there exists a model $\mathfrak{M} = (\mathfrak{F}, V)$ with $\mathfrak{F} = (W, R, E)$, $\mathfrak{M} \not\models \varphi$, $\mathfrak{F} \models [L, \mathbf{S5}]^{xd}$, and W of size κ . Then define $\mathfrak{M}_{\kappa'} = (\mathfrak{F}_{\kappa'}, V_{\kappa'})$ with $\mathfrak{F}_{\kappa'} = (W', R', E')$ by setting

- $W' = W \times \kappa'$;
- $(w, n)R'(w', n')$ if wRw' and $n = n'$, for all $n, n' \in \kappa'$;
- $(w, n)E'(w', n')$ if wEw' , for all $n, n' \in \kappa'$;
- $V_{\kappa'}(p) = V(p) \times \kappa'$, for all proposition letters p .

Claim. $\mathfrak{F}_{\kappa'} \models [L, \mathbf{S5}]^{xd}$, $\mathfrak{M}_{\kappa'} \not\models \varphi$, and $\mathfrak{M}_{\kappa'}$ is an (E, κ') -homogeneous model.

Proof of Claim. By closure under disjoint unions, (W', R') validates L . Also, E' is easily seen to be an equivalence relation. We next show that (W', R', E') satisfies $(lcom_{R,E})$. Assume $(w, n)E'(w', m)$ and $(w', m)R'(w'', k)$. By definition, wEw' , $m = k$, and $w'Rw''$. Then, as (W, R, E) satisfies $(lcom_{R,E})$, we have a v' with wRv' and $v'Ew''$. But then $(w, n)R'(v', n)$ and $(v', n)E'(w'', k)$, as required.

Let χ be any formula. Then one can easily show by induction that $\mathfrak{M}, w \models \chi$ iff $\mathfrak{M}_{\kappa'}, (w, n) \models \chi$ for all $n \in \kappa'$. It follows that $\mathfrak{M}_{\kappa'} \not\models \varphi$ and $\mathfrak{M}_{\kappa'}$ is an (E, κ') -homogeneous model. This finishes the proof of the claim and also the proof for logics of the form $[L, \mathbf{S5}]^{xd}$.

Now assume a logic of the form $[L, \mathbf{S5}]^{cd}$ is given and Kripke complete. The construction of $\mathfrak{M}_{\kappa'}$ is the same, using a model $\mathfrak{M} = (\mathfrak{F}, V)$ with $\mathfrak{F}$ also satisfying $(rcom_{R,E})$. So it suffices to show that $\mathfrak{M}_{\kappa'}$ then also satisfies $(rcom_{R,E})$: assume $(w, n)R'(w', m)$ and $(w', m)E'(w'', k)$. By definition, $n = m$, wRw' , and $w'Ew''$. Then, as (W, R, E) satisfies $(rcom_{R,E})$, we have a v' with wEv' and $v'Rw''$. But then $(w, n)E'(v', k)$ and $(v', k)R'(w'', k)$, as required. $\square$

We obtain the following transfer result for (semi)commutators.

Theorem 5.5. *Let $d \in \{xd, cd\}$ and L_1 and L_2 be consistent propositional modal logics such that $[L_i, \mathbf{S5}]^d$ is (globally) Kripke complete, for both $i = 1, 2$. Then*

- $[L_1, \mathbf{S5}]^d \otimes_E [L_2, \mathbf{S5}]^d$ is (globally) Kripke complete;
- $[L_1, \mathbf{S5}]^d \otimes_E [L_2, \mathbf{S5}]^d$ is (globally) decidable iff $[L_1, \mathbf{S5}]^d$ and $[L_2, \mathbf{S5}]^d$ are (globally) decidable.

6 Discussion

We have established preservation and non-preservation results for one-variable first-order modal logics and for related propositional modal logics. Many questions remain open:

- The behaviour of fusions of full first-order modal logic is still largely unexplored. A transfer theorem for canonicity is given in [30], but, for example, the transfer of completeness is open.
- Monodic fragments extend one-variable modal logics by allowing more expressive first-order fragments while restricting modal operators to formulas with at most one free variable [25, 1]. We have shown that decidability does not transfer from the one-variable fragment with equality to the monodic fragment over the two-variable fragment of FO with counting. Positive preservation results are known [13], and it is straightforward to adapt our proofs to, for instance, the monodic fragments based on the two-variable fragment of **FO** without equality or description logics such as *ALC*, *ALCI*, and *SHI* with the universal role [3, 25]. However, a systematic study of fusions of monodic fragments is still missing.
- It would be valuable to identify sufficient conditions under which Kripke completeness and decidability transfer to fusions of one-variable modal logics with equality. We note that many standard polymodal fusions with equality such as polymodal **K** and **S5** are in fact decidable [1].
- We have seen that, in many cases, completeness and decidability are preserved under fusions of propositional modal logics sharing an **S5** operator. Can this be extended to other modal logics?
- Our analysis focused on Kripke complete modal logics. Since first-order modal logics are often incomplete under standard Kripke semantics, it would be interesting to obtain preservation results that do not presuppose Kripke completeness. Techniques from [33, 2] may be helpful here.

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On the Expressive Power of Inquisitive Team Logic and Inquisitive First-Order Logic

Juha Kontinen

University of Helsinki
Helsinki, Finland

juha.kontinen@helsinki.fi

Ivano Ciardelli

University of Padua
Padua, Italy

ivano.ciardelli@unipd.it

Inquisitive team logic is a variant of inquisitive logic interpreted in team semantics, which has been argued to provide a natural setting for the regimentation of dependence claims. With respect to sentences, this logic is known to be expressively equivalent with first-order logic. In this article we show that, on the contrary, the expressive power of open formulas in this logic properly exceeds that of first-order logic. On the way to this result, we show that if inquisitive team logic is extended with the range-generating universal quantifier adopted in dependence logic, the resulting logic can express finiteness; as a consequence, this logic is not compact and has non-arithmetic complexity. We further extend our results to standard inquisitive first-order logic, showing that some sentences of this logic express non first-order properties of models, thus settling an open problem posed in [5].

1 Introduction

The last two decades have seen the rise of many logics based on team semantics [25], a mathematical framework for studying concepts and phenomena that arise in contexts involving a plurality of data such as databases or probability distributions. Logics based on team semantics have found applications in various fields, including database theory [28, 21], Bayesian networks and probabilistic dependencies [10, 24], quantum foundations [11, 1, 3], as well as formal semantics of natural language [9, 23, 4].

In team semantics, formulas are interpreted over a single first-order structure together with *a set of assignments* (aka *a team*) rather than single assignment as in Tarskian semantics. A prominent example of a team-based logic is dependence logic, introduced in [30], which may be seen as an extension of standard first-order logic with dependence atoms $=(\vec{x}, y)$, expressing the fact that the values of the variables $\vec{x}$ functionally determine the value of y . Inclusion logic [15] and independence logic [20] are two other extensively studied team-based logics that extend first-order logic with inclusion and independence, corresponding to inclusion and embedded multivalued dependencies in database theory [14].

Since the introduction of dependence logic, the expressivity and complexity properties of logics in team semantics have been extensively studied (see, e.g., [22, 12, 13]). In particular, the interesting discrepancy between sentences and open formulas of dependence logic is now well understood [29]. With respect to sentences, dependence logic is equi-expressive with existential second-order logic (ESO); however, for open formulas the correspondence is not as simple. In particular, formulas $\phi(x_1, \dots, x_n)$ of dependence logic can be translated to ESO-sentences $\psi(R)$ which can refer to a team over $\{x_1, \dots, x_n\}$ via an extra n -ary relation symbol R occurring *only negatively* in $\psi(R)$. The converse also holds, i.e., for any such ESO-sentence $\psi(R)$ there exists a formula $\phi(x_1, \dots, x_n)$ of dependence logic such that for all first-order models $\mathcal{M}$ and non-empty teams (i.e., sets of assignments) X over the variables $\{x_1, \dots, x_n\}$:

$$\mathcal{M} \models_X \phi(x_1, \dots, x_n) \iff (\mathcal{M}, X[\vec{x}]) \models \psi(R), \quad (1)$$

where $X[\vec{x}] = \{(s(x_1), \dots, s(x_n)) \mid s \in X\}$ is the relation encoding the team X .

Our primary focus in this paper is an interesting team-based logic, closely related to inquisitive logic [9, 5], a research program that aims to extend the scope of logic to include not only statements, but also questions. The standard system of inquisitive first-order logic, called InqBQ [5],¹ can be seen as an extension of standard first-order logic with a question-forming disjunction, $\vee$, and a question-forming existential, $\exists$. Using these operators, one can express, in addition to statements like “all objects are P ” (expressed as usual by $\forall x Px$) also questions such as “whether or not all objects are P ” ($\forall x Px \vee \neg \forall x Px$), “which objects are P ” ($\forall x (Px \vee \neg Px)$), and “what is one example of a P ” ($\exists x Px$).

The logic InqBQ is interpreted in terms of an intensional semantics involving multiple possible worlds, each capturing a state of affairs and corresponding formally to a standard relational structure for predicate logic. However, the very same language can also be interpreted relative to teams, leading to a system known as *inquisitive team logic*, InqBT [5, 7].² In this system, one may express questions about the values of variables, including in particular the question “what is the value of x ?”, which may be expressed by the formula $\mu x := \forall y (y = x \vee y \neq x)$ (or, equivalently, by the formula $\lambda x := \exists y (y = x)$). Dependencies can be expressed in this system as implications among questions: in particular, the dependence atom $=(\vec{x}, y)$ can be defined in InqBT by the formula $\mu x_1 \wedge \dots \wedge \mu x_n \rightarrow \mu y$. While the logics InqBQ and InqBT are different, they are related by entailment-preserving translations that often allow a transfer of results between them [7].

In spite of many investigations, which have led to important results ([16, 17, 18, 8, 19]), some key questions about the meta-theoretic properties of the (semantically defined) inquisitive logics InqBT and InqBQ remain open. It is not known, e.g., whether the sets of validities of these logics are recursively enumerable (and, thus, if a complete axiomatization is possible), nor whether these logics are entailment-compact, in the sense that a conclusion follows from a set of premises only if it follows from a finite subset of these premises.

In this paper, we investigate the expressive power of the logics InqBT and InqBQ. In the case of InqBT, a simple argument going back to [31] shows that with respect to sentences, the expressive power of this system coincides with that of first-order logic. The question about the expressive power of open formulas, however, is currently open. In particular, it is not known whether open formulas of InqBT can be systematically translated to sentences of first-order logic in a language extended with a relation symbol referring to the team.

- **Open Question 1.** Given a formula $\phi(x_1, \dots, x_n)$ of InqBT, is there always a first-order sentence $\psi(R)$, in a signature extended with an n -ary relation symbol R , which is equivalent to $\phi(x_1, \dots, x_n)$ in the sense of equation (1) above?

A wide range of particular examples admitting a first-order translation may suggest a positive answer to this question. We show that, on the contrary, the answer is negative: there are open formulas of InqBT that do not have a first-order counterpart—expressing genuinely second-order properties.

On the way to this result, we consider a richer logic, denoted InqBT + $[x]$. In InqBT, the two quantifiers $\forall x$ and $\exists x$ have a simple semantics: they explore different ways of assigning to x a constant value throughout the team (in the team semantics literature, these quantifiers are standardly denoted by means of the notation $\forall^1 x$ and $\exists^1 x$, see for instance [29]). In the logic InqBT + $[x]$, we add to these quantifiers a third quantifier, $[x]$, corresponding to the universal quantifier adopted in dependence logic (and denoted in that context simply as $\forall x$): semantically, this quantifier expands the team with all possible values for x .

¹In the inquisitive semantics literature, the so-called “basic” system of inquisitive propositional logic is denoted InqB; the name InqBQ reflects the fact that this system is obtained by enriching InqB with Quantifiers.

²This system was first considered by Yang [31] under the name WID (for weak intuitionistic dependence logic).

The reasons to be interested in the logic $\text{InqBT} + [x]$ are not merely technical. In recent work [6], it has been argued that the quantifiers $\forall x$ and $[x]$ reflect two different ways to make general claims in natural language: an “extensional” way, corresponding to universal claims like (a) below, and an “intensional” way, corresponding to generic claims like (b).

- (a) Every triangle has angles that sum to 360°
- (b) A triangle has angles that sum to 360°

The basic idea is that, while (a) is evaluated by considering many particular triangles, (b) is evaluated by considering a single but completely generic triangle. In [6], the relationship between these two ways of expressing generality has been studied systematically in the context of a logic, $\text{InqWT} + [x]$, which is nothing but the $\exists$ -free fragment of $\text{InqBT} + [x]$. A central result of [6] is that every sentence of this logic can be translated (albeit in a rather complex way) into an equivalent sentence of standard predicate logic. It is natural to wonder whether such a result extends to the richer system $\text{InqBT} + [x]$, including the inquisitive existential quantifier $\exists$.

• **Open Question 2.**

Is every sentence of $\text{InqBT} + [x]$ equivalent to some sentence of standard first-order logic?

Again, in this paper we answer the question in the negative. We show that in $\text{InqBT} + [x]$ we can write a sentence expressing the fact that the model is finite (or infinite)—a property which is famously not expressible in first-order logic. Using this result, we show that the logic $\text{InqBT} + [x]$ is not compact (even in the weaker sense of satisfiability), and that it is not recursively axiomatizable (indeed, we show that the set of its validities is not even arithmetical).

The sentence of $\text{InqBT} + [x]$ which is shown to express finiteness has the form $[x][y]\phi(x,y)$, where $\phi(x,y)$ is an open formula of InqBT . It is this formula $\phi(x,y)$ which we show not to be expressible in first-order logic augmented with a binary predicate R for the team.

Finally, we adapt our results to settle a related open question about the expressive power of the standard system of inquisitive first-order logic, InqBQ . A model $M = (W, D, I)$ for InqBQ consists of a domain W of possible worlds, a domain D of individuals, and a world-relative evaluation function I , which assigns to each world w a standard relational structure M_w over D . Such a model can be encoded as a two-sorted structure $M^* = (W, D, I^*)$ for a signature in which each function or relation symbols is expanded with an extra argument for the world of evaluation. A natural question is whether, modulo this encoding, there exists a translation from InqBQ into two-sorted first-order logic, in the following sense.

- **Open Question 3.** Given a sentence ϕ of InqBQ , is there always a corresponding sentence ϕ^* of two-sorted first-order logic (over a suitably translated signature incorporating world-dependency into the predicate and function symbols) such that for every model M , M satisfies ϕ in InqBQ if and only if M^* satisfies ϕ^* in predicate logic?

This question is explicitly posed as an open problem in [5]. A positive answer is known for two important fragments of InqBQ : the *classical antecedent (clant) fragment* [18, 5], where antecedents are restricted to formulas of standard first-order logic, and the *restricted existential (rex) fragment* [8], where inquisitive existentials can occur only within conditional antecedents. Using an adaptation of the proof we develop for InqBT , we show that in the general case, the answer to the question is negative: there are formulas of InqBQ which have no first-order counterpart, expressing genuinely second-order properties.

Organization. The paper is structured as follows. In §2 we cover the necessary background on InqBT and $\text{InqBT} + [x]$. In §3 we prove the novel results about these logics, settling Open Questions 1 and 2. §4 extends the scope to InqBQ , settling Open Question 3. §5 concludes with a summary and open problems.

2 Preliminaries

In this section, we introduce the inquisitive team logic InqBT and its extension InqBT + $[x]$ which are the subjects of this paper. We also mention some of their basic properties. We omit the proofs of the results in this section, since they are straightforward and standard in the literature (see, e.g., [5]).

The syntax of InqBT and InqBT + $[x]$, and the classical fragment of these logics, are given by the following BNF definitions, where p denotes an atom of predicate logic which, as usual, can have the form $(t_1 = t_2)$ or the form $P(t_1, \dots, t_n)$, where P is an n -ary predicate and $t_1, t_2, \dots, t_n$ are terms.

Syntax of InqBT	$\phi ::= p \mid \perp \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \forall x \phi \mid \exists x \phi$
Syntax of InqBT + $[x]$	$\phi ::= p \mid \perp \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \forall x \phi \mid \exists x \phi \mid [x] \phi$
Classical formulas	$\phi ::= p \mid \perp \mid (\phi \wedge \phi) \mid (\phi \rightarrow \phi) \mid \forall x \phi$

The set of classical formulas can be identified with the standard language of first-order predicate logic, with a particular choice of primitives. The remaining operators can be defined in a standard way by letting $\neg \phi := (\phi \rightarrow \perp)$; $\phi \vee \psi := \neg(\neg \phi \wedge \neg \psi)$; $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$; $\exists x \phi := \neg \forall x \neg \phi$. In addition, it is standard in inquisitive logic to define a question-mark operator as $?\phi := (\phi \vee \neg \phi)$.

The semantics of these systems is given in terms of a relation of *support*, which is defined relative to a model and a team. A model is defined, as usual in predicate logic, as a pair $\mathcal{M} = (D, I)$ consisting of a non-empty domain D and an interpretation function I assigning to each predicate or function symbol in the vocabulary a relation or function over D with the corresponding arity. A team is defined as a set of assignments defined over a common domain of variables; we use the meta-variables $X, Y, \dots$ for teams, and indicate by $\text{dom}(X), \text{dom}(Y), \dots$ the corresponding domains, i.e., the sets of variables on which the assignments in the teams are defined. If $\{x_1, \dots, x_n\} \subseteq \text{dom}(X)$, we denote by $X[x_1, \dots, x_n]$ the n -ary relation on D defined by

$$X[x_1, \dots, x_n] = \{(g(x_1), \dots, g(x_n)) \mid g \in X\}.$$

Note that, if $x_1, \dots, x_n$ is a finite sequence of variables, there is a one-to-one correspondence between teams defined on the set $\{x_1, \dots, x_n\}$ and n -ary relations on D , given by the map $X \mapsto X[x_1, \dots, x_n]$.

To state the semantics, we will make use of two operations on teams. First, if g is an assignment, x is a variable, and d an object from the domain D of the model, we denote by $g[x \mapsto d]$ the assignment over the variables $\text{dom}(g) \cup \{x\}$ which maps x to d and agrees with g on the remaining variables in $\text{dom}(g)$. Now if X is a team, x a variable, and $d \in D$ an object, we denote by $X[x \mapsto d]$ the team over the domain $\text{dom}(X) \cup \{x\}$ obtained by assigning to x the constant value d through the team:

$$X[x \mapsto d] = \{g[x \mapsto d] \mid g \in X\}.$$

Similarly, if X is a team, x a variable, and $A \subseteq D$ a set of objects, we denote by $X[x \mapsto A]$ the team over $\text{dom}(X) \cup \{x\}$ obtained by assigning to x each possible value from A , in combination with all values for the remaining variables represented in X :

$$X[x \mapsto A] = \{g[x \mapsto d] \mid g \in X, d \in A\} = \bigcup_{d \in A} X[x \mapsto d].$$

If t is a term, its denotation $[t]_{\mathcal{M}}^g$ relative to a model $\mathcal{M}$ and an assignment g defined on the variables occurring in t is defined in the usual way. We can now state the semantics of our logics as follows.

Definition 2.1. The relation of *support* $\mathcal{M} \models_X \phi$ between a model $\mathcal{M}$, a team X , and a formula ϕ whose free variables are included in $\text{dom}(X)$ is defined inductively as follows:

- $\mathcal{M} \models_X p \iff \forall g \in X : \mathcal{M} \models_g p$ in standard Tarskian semantics if p is an atom
- $\mathcal{M} \models_X \perp \iff X = \emptyset$
- $\mathcal{M} \models_X \phi \wedge \psi \iff \mathcal{M} \models_X \phi \text{ and } \mathcal{M} \models_X \psi$
- $\mathcal{M} \models_X \phi \vee \psi \iff \mathcal{M} \models_X \phi \text{ or } \mathcal{M} \models_X \psi$
- $\mathcal{M} \models_X \phi \rightarrow \psi \iff \forall Y \subseteq X : \mathcal{M} \models_Y \phi \text{ implies } \mathcal{M} \models_Y \psi$
- $\mathcal{M} \models_X \forall x \phi \iff \forall d \in D : \mathcal{M} \models_{X[x \rightarrow d]} \phi$
- $\mathcal{M} \models_X \exists x \phi \iff \exists d \in D : \mathcal{M} \models_{X[x \rightarrow d]} \phi$
- $\mathcal{M} \models_X [x] \phi \iff \mathcal{M} \models_{X[x \rightarrow D]} \phi$

For a detailed discussion of the clauses for atoms and connectives, we refer to [5]. As for the quantifiers, $\forall x$ and $\exists x$ test different ways of assigning to the variable x a constant value, and check whether the embedded formula is supported in all/some of the resulting teams. By contrast, $[x]$ checks the support of its argument relative to a single team, obtained by assigning to x all possible values. For discussion of the philosophical difference between the kinds of generality achieved by $\forall x$ and $[x]$, see [6]. The following general properties are familiar from the literature on inquisitive logic and dependence logic.

Proposition 2.2 (Persistence). *For any model $\mathcal{M}$, teams X, Y , and formulas ϕ of $\text{InqBT} + [x]$, if $\mathcal{M} \models_X \phi$ and $Y \subseteq X$ then $\mathcal{M} \models_Y \phi$.*

Proposition 2.3 (Empty team property). *For any model $\mathcal{M}$, $\mathcal{M} \models_\emptyset \phi$ for all formulas ϕ of $\text{InqBT} + [x]$.*

Proposition 2.4 (Locality). *For any model $\mathcal{M}$, any formula ϕ of $\text{InqBT} + [x]$, and all teams X, Y whose domain includes the set $FV(\phi)$ of free variables in ϕ :*

$$X|_{FV(\phi)} = Y|_{FV(\phi)} \text{ implies } \mathcal{M} \models_X \phi \iff \mathcal{M} \models_Y \phi,$$

where $X|_{FV(\phi)}$ denotes the team obtained by restricting each assignment in X to $FV(\phi)$, and likewise for Y .

Note that for a sentence ϕ , all nonempty teams have the same restriction to $FV(\phi) = \emptyset$, namely, the singleton of the empty assignment. So by Locality, all nonempty teams agree about whether or not ϕ is supported. We can thus say that a model $\mathcal{M}$ satisfies ϕ (written $\mathcal{M} \models \phi$) in case $\mathcal{M} \models_X \phi$ holds relative to an arbitrary nonempty team X ; if $\mathcal{M} \not\models \phi$ we say that $\mathcal{M}$ falsifies ϕ . Classical propositional connectives behave standardly with respect to the relation of satisfaction, for instance, $\mathcal{M} \models \neg \phi \iff \mathcal{M} \not\models \phi$.

The logical notions of entailment, validity, and equivalence are defined in the obvious way: a set of formulas Φ entails a formula ψ (denoted $\Phi \models \psi$), if in every model, any team that is defined on all the free variables in $\Phi \cup \{\psi\}$ and supports all $\phi \in \Phi$ also supports ψ ; a formula ϕ is valid ($\models \phi$) if in every model it is supported by any team whose domain includes its free variables; two formulas ϕ, ψ are equivalent ($\phi \equiv \psi$) if they entail each other. As for the notion of satisfiability, given the empty state property, it must be defined with respect to nonempty teams only: ϕ is satisfiable if $\mathcal{M} \models_X \phi$ for some model $\mathcal{M}$ and some nonempty team X ; note that if ϕ is a sentence, satisfiability simply means that ϕ is satisfied in some model, as usual.

The connection with standard first-order logic is made through an important semantic property called *flatness* (or *truth-conditionality* in the inquisitive literature). We say that a formula ϕ is *flat* if support for ϕ relative to a team reduces to support at each singleton sub-team.

Definition 2.5. A formula ϕ is flat if for all models $\mathcal{M}$ and teams X with $\text{dom}(X) \supseteq FV(\phi)$ we have

$$\mathcal{M} \models_X \phi \iff \forall g \in X : \mathcal{M} \models_{\{g\}} \phi.$$

The following proposition shows that with respect to the classical fragment, our team semantics is essentially equivalent to the standard Tarskian semantics.

Proposition 2.6. *For classical formulas, support relative to a singleton team $\{g\}$ coincides with truth relative to g in Tarskian semantics. Moreover, all classical formulas are flat. Thus, for a classical formula, support at a team X coincides with truth relative to each $g \in X$ in Tarskian semantics. In particular, for a classical sentence, our definition of satisfaction in a model coincides with the one given by Tarskian semantics.*

Using this fact, it is easy to show that our logics are, in a precise sense, conservative extensions of classical first-order logic: for classical formulas, the notions of entailment, validity, and equivalence determined by our semantics coincide with those of classical first-order logic.

The property of flatness extends beyond the classical fragment to formulas that include $[x]$ but not the inquisitive operators $\vee$ and $\exists$: indeed, any $\{\vee, \exists\}$ -free formula is equivalent to a classical one, since when α is flat, $[x]\alpha \equiv \forall x\alpha$. On the other hand, formulas including the inquisitive operators $\vee$ and $\exists$ are typically not flat: they express global properties of a team which do not reduce to local properties of the individual assignments in the team.

For a simple example, suppose α is a classical formula and consider the formula $? \alpha$ ($:= \alpha \vee \neg \alpha$). It is easy to check that we have:

$$\mathcal{M} \models_X ? \alpha \iff \alpha \text{ has the same truth value relative to each } g \in X,$$

where the right-hand side refers to the standard truth-value in standard Tarskian semantics. Intuitively, the formula $? \alpha$ is regarded as capturing the question whether α is true or false; it is supported if all assignments in the team agree on the answer to this question. Obviously, $? \alpha$ is supported relative to any singleton team, but (typically) not relative to an arbitrary team, thus violating truth-conditionality.

Another example of an inquisitive formula, which will play an important role for our purposes, is the following. For a variable term t , let us define

$$\mu t := \forall x ?(x = t),$$

where x is an arbitrary variable not occurring in t . It is easy to check that μt is supported by a team X just in case the value of t is constant throughout X :

$$\mathcal{M} \models_X \mu t \iff \forall g, g' \in X : [t]_M^g = [t]_M^{g'}.$$

Intuitively, μt expresses the question “what is t ”. We refer to formulas of the form μt as *value questions*.³ Using a strategy that goes back to [2], we can then express dependencies in terms of value questions and implication. We have:

$$\mathcal{M} \models_X \mu x_1 \wedge \cdots \wedge \mu x_n \rightarrow \mu y \iff \forall g, g' \in X : \text{if } g(x_i) = g'(x_i) \text{ for } 1 \leq i \leq n \\ \text{then } g(y) = g'(y).$$

This is exactly the semantics of a dependence atom $=(\vec{x}, y)$ in dependence logic (where $\vec{x} = x_1 \dots x_n$). For convenience, we will adopt the notation $=(\vec{x}, y)$ as an abbreviation, defined as follows:

$$=(\vec{x}, y) := \mu x_1 \wedge \cdots \wedge \mu x_n \rightarrow \mu y.$$

³It is easy to check that the same result can be obtained by the formula $\lambda t := \exists x(x = t)$. This was indeed the strategy adopted in some previous work [5]. Here, we prefer to make use of the formulas μt (defined in terms of $\forall$ and $?$) rather than λt (defined in terms of $\exists$) since the former option allows us to obtain formulas expressing dependency that fit within the $\exists$ -free fragment of InqBT, which is known to be especially well-behaved: as shown in [8] and [7], this fragment is entailment-compact and recursively axiomatizable.

3 Results on InqBT and InqBT + [x]

In this section we present our results for the logics InqBT + [x] and InqBT.

Theorem 3.1. *There is a sentence ψ of InqBT + [x] in the empty vocabulary which defines finiteness, i.e., such that for every model $\mathcal{M}$: $\mathcal{M} \models \psi \iff \mathcal{M}$ is finite.*

Proof. We utilize the fact that a set A is finite if and only if every injection $f: A \rightarrow A$ is also a surjection. Recall that we write $=(x, y)$ to abbreviate the formula $\mu x \rightarrow \mu y$, that is, $\forall z?(z = x) \rightarrow \forall z?(z = y)$, and similarly for $=(y, x)$. Define:

$$\phi(x, y) := (=(x, y) \wedge =(y, x) \wedge \exists z(z \neq y) \rightarrow \exists u(u \neq x)), \quad (2)$$

$$\psi := [x][y]\phi(x, y). \quad (3)$$

A model $\mathcal{M} = (D, I)$ falsifies ψ just in case the team X that consists of all assignments of values to x and y fails to support the formula $\phi(x, y)$. By the semantics of implication, this holds iff there is a sub-team $Y \subseteq X$ which supports the antecedent but not the consequent. As discussed above, a team Y with domain $\{x, y\}$ can be identified with a binary relation $R = Y[x, y] = \{(g(x), g(y)) \mid g \in Y\}$. Spelling out the semantics, we find that:

- $\mathcal{M} \models_Y =(x, y) \iff R$ is a function;
- $\mathcal{M} \not\models_Y \exists u(u \neq x) \iff \text{dom}(R) = D$;
- $\mathcal{M} \models_Y =(y, x) \iff R$ is injective;
- $\mathcal{M} \models_Y \exists z(z \neq y) \iff \text{ran}(R) \neq D$.

Putting things together, then, ψ is falsified in a model $\mathcal{M} = (D, I)$ just in case there exists a function defined on D which is injective but not surjective, that is, just in case D is infinite. Equivalently, ψ is satisfied in a model $\mathcal{M}$ just in case $\mathcal{M}$ is finite. $\square$

Since negation behaves classically at the level of sentences, our theorem also implies that the negation $\neg\psi$ of the sentence defined in 3 defines infinity, i.e., it is satisfied exactly by the infinite models. Since finiteness and infinity of the model are famously not expressible in first-order logic, our theorem immediately yields a negative answer to Open Question 2 in the introduction.

Corollary 3.2. *Some sentences of InqBT + [x] are not equivalent to any sentence of standard first-order logic.*

Using the possibility of expressing finiteness in InqBT + [x] we can show that this logic, like second-order logic, violates compactness and has a non-recursively enumerable (and, indeed, non-arithmetical) set of validities. First, recall that in the setting of inquisitive logic and team-based logics, it makes sense to distinguish between two notions of compactness.

Definition 3.3. A logic L is said to be:

- *satisfiability compact*, if for all sets of L -formulas Φ the following holds: if every finite $\Phi_0 \subset \Phi$ is satisfiable, then Φ is satisfiable;
- *entailment compact*, if for all sets of L -formulas Φ and any formula ψ , the following holds: if $\Phi \models_L \psi$, then $\Phi_0 \models_L \psi$, for some finite $\Phi_0 \subseteq \Phi$.

For a team-based logic including $\perp$, entailment compactness implies satisfiability compactness (since the satisfiability of Φ reduces to the non-entailment $\Phi \not\models \perp$) but, in general, not the other way around; for instance, dependence logic is satisfiability compact, but not entailment compact as there is a sentence of dependence logic satisfied exactly by the infinite models [30]. For the logic InqBT + [x], by contrast, even satisfiability-compactness fails.

Theorem 3.4. *The logic $\text{InqBT} + [x]$ is not satisfiability-compact (and, thus, not entailment-compact).*

Proof. Define $\Gamma = \{\phi_n \mid n \in \mathbb{N}\}$, where ϕ_n is a classical sentence expressing the existence of at least n distinct objects:

$$\exists x_1 \dots \exists x_n \bigwedge_{i \neq j} (x_i \neq x_j).$$

Using Theorem 3.1 and Proposition 2.6, we can see that the set $\Gamma \cup \{\psi\}$, where ψ is defined as in 3, is not satisfiable, but any finite subset of this set is satisfiable. $\square$

Theorem 3.5. *The set of valid formulas of $\text{InqBT} + [x]$ has non-arithmetic complexity.*

Proof. We first show that the standard model of arithmetic $\mathcal{N} = (\mathbb{N}, +, \times, \leq, 0, 1)$ can be characterized up to isomorphisms by a single sentence of $\text{InqBT} + [x]$. In order to define such a sentence, we take the conjunction of the usual Peano axioms without the induction scheme Φ_{PA} and conjoin it with a sentence that says that every element of the model has only finitely many predecessors. This can be expressed by adapting the sentence ψ from (3) in the following way:

$$\psi' := \forall z[x][y](x \leq z \wedge y \leq z \wedge =(x,y) \wedge =(y,x) \wedge \exists u(u \leq z \wedge u \neq y) \rightarrow \exists t(t \leq z \wedge t \neq x)).$$

Notice that for any classical sentence θ it now holds that

$$\mathcal{N} \models \theta \iff (\Phi_{PA} \wedge \psi' \rightarrow \theta) \text{ is valid in } \text{InqBT} + [x].$$

Thus, the set of arithmetic truths reduces to the set of $\text{InqBT} + [x]$ -validities through the computable map $\theta \mapsto (\Phi_{PA} \wedge \psi' \rightarrow \theta)$. Since the former set has non-arithmetic complexity, so does the latter. $\square$

Let us now turn to the inquisitive team logic InqBT . Without the operator $[x]$, it is no longer possible to express the finiteness/infiniteness of the model; indeed, as we mentioned in the beginning, with respect to sentences InqBT is no more expressive than standard first-order logic [31]. However, if we start from the sentence $[x][y]\phi(x,y)$ that we used to express finiteness and remove the prefix $[x][y]$, we obtain an open formula $\phi(x,y)$ of InqBT . We will now see that this formula allows us to give a negative answer to Open Question 1 in the introduction, concerning the expressive power of open formulas in InqBT .

Theorem 3.6. *There is an open formula $\phi(x,y)$ of InqBT in the empty vocabulary that cannot be expressed in first-order logic over models with vocabulary $\{R\}$, where R is a binary relation.*

Proof. Define $\phi(x,y)$ as in 2 above. We know from the proof of Theorem 3.1 that, for any model $\mathcal{M} = (D, I)$, if X_D is the maximal team over D defined on the variables x and y , we have

$$\mathcal{M} \models_{X_D} \phi(x,y) \iff D \text{ is finite.} \quad (4)$$

Assume, for a contradiction, that there exists a first-order sentence $\phi^*(R)$, in the vocabulary comprising only the relation symbol R , satisfying

$$\mathcal{M} \models_X \phi(x,y) \iff (\mathcal{M}, X[x,y]) \models \phi^*(R), \quad (5)$$

for all models $\mathcal{M}$ and teams X . In particular, for any model $\mathcal{M}$ with domain D , if we instantiate X to the maximal team X_D for the variables x, y , then given that $X_D[x,y] = D^2$, from (4) and (5) we obtain:

$$(\mathcal{M}, D^2) \models \phi^*(R) \iff D \text{ is finite.} \quad (6)$$

Now let k be the quantifier-rank of $\phi^*(R)$ and let $K = \{1, 2, \dots, k\}$. We can show by a simple Ehrenfeucht-Fraïssé game argument that for all first-order sentences θ of quantifier-rank up to k we have:

$$(K, K^2) \models \theta \iff (\mathbb{N}, \mathbb{N}^2) \models \theta. \quad (7)$$

Indeed, a winning strategy for the Duplicator in the k -round game over the models (K, K^2) and $(\mathbb{N}, \mathbb{N}^2)$ consists in maintaining the condition that

$$a_i = a_j \iff b_i = b_j,$$

for all elements $a_1, \dots, a_h \in K$ and $b_1, \dots, b_h \in \mathbb{N}$ chosen at any given stage in the game. This condition can be maintained for all $h \leq k$ —and, so, until the end of the game—since the cardinality of both models is at least k , and so at any stage in the game, Duplicator can always find a fresh element if needed.

Since the quantifier-rank of $\phi^*(R)$ is k , the equivalence (7) holds for it. But this is a contradiction, since (6) implies that $(K, K^2) \models \phi^*(R)$ but $(\mathbb{N}, \mathbb{N}^2) \not\models \phi^*(R)$. $\square$

To conclude our investigation of InqBT, we examine its expressive power over finite structures. We show that InqBT can express properties of finite structures and teams that cannot be expressed in first-order logic. This is established by constructing an InqBT formula whose model checking problem is provably harder than that of any first-order formula.

Let 3SAT (respectively, 3UNSAT) denote the set of satisfiable (respectively, unsatisfiable) Boolean formulas in conjunctive normal form with clauses of size three. It is known that 3UNSAT is coNP-complete, and that the model checking problem of any first-order formula has a strictly lower computational complexity (see, e.g., [26]).

Theorem 3.7. *There exists a formula φ of InqBT such that the problem of deciding whether a given finite structure and a team satisfies φ is coNP-complete.*

Proof. We give a reduction from 3UNSAT. Let

$$\Theta := \bigwedge_i (l_{i_1} \vee l_{i_2} \vee l_{i_3})$$

be a Boolean formula, where each l_{i_j} is either a propositional symbol (p_k) or its negation ($\neg p_k$). We use an encoding of Θ as a finite structure and a team that is similar to the one used for showing the NP-completeness of the model checking of certain simple quantifier-free formulas of dependence logic [27].

Define a team X over the domain $\{z, u, x, y\}$ as follows. Each clause $c_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3})$ of Θ is represented by three assignments:

z = clause	u = position	x = variable	y = parity
i	a	p_{i_1}	ε_{i_1}
i	b	p_{i_2}	ε_{i_2}
i	c	p_{i_3}	ε_{i_3}

Here $\varepsilon_{i_j} = +$ if the literal l_{i_j} is positive and $\varepsilon_{i_j} = -$ if it is negated. Let X be the union of all these assignments. The domain M of the associated structure $\mathcal{M}_\Theta$ consists exactly of the elements occurring as values of variables in X . We assume that the possible values of different variables in X partition the domain of M and we use unary relations V and C to distinguish the elements encoding variables and clauses of Θ from the rest of the elements of M .

We use the following idea: any subteam $Y \subseteq X$ that satisfies $=(x, y)$ and includes, for each clause c_i of Θ , at least one assignment s such that $s(z) = i$ encodes a (set of) Boolean assignments satisfying Θ . One of them can be defined as follows:

$$f(p_i) = 1 \iff \exists s \in Y \text{ with } s(x) = p_i \text{ and } s(y) = +.$$

The dependence atom $=(x, y)$ guarantees that f is well-defined. Furthermore, it is easy to check that any satisfying assignment f for Θ gives rise to a subset $Y \subseteq X$ that satisfies $=(x, y)$ and includes at least one assignment for each clause. Therefore, in order to express that Θ is unsatisfiable, it suffices to express that any subset $Y \subseteq X$ satisfying $=(x, y)$ necessarily excludes, for some i , all s such that $s(z) = i$. This can be expressed by the following formula:

$$\varphi := (= (x, y) \rightarrow \exists w (C(w) \wedge w \neq z)).$$

It is now straightforward to verify that indeed Θ is unsatisfiable, if and only if, $\mathcal{M}_\Theta \models_X \varphi$. Furthermore, since the model $\mathcal{M}_\Theta$ and the team X can be constructed in polynomial time in the size of Θ , the model checking problem $\mathcal{M} \models_Y \varphi$ is coNP-hard. Membership in coNP follows from the fact that φ can be easily rewritten in second-order logic using universal quantification over relations assuming the team X is available as an auxiliary relation $X[z, u, x, y]$. $\square$

4 Extension to InqBQ

In this section we turn our attention to the standard system of inquisitive first-order logic, InqBQ. We will show that a simple adaptation of the proof we gave above for InqBT allows us to settle Open Question 3 about the expressive power of InqBQ in the negative.

Basic notions. The language of InqBQ is the same as for InqBT. However, InqBQ is interpreted over *first-order information models*, which are triples $M = (W, D, I)$, where W is a non-empty set whose elements are called *worlds*, D is a non-empty set whose elements are called *individuals*, and I is a world-relative interpretation function that gives for each $w \in W$ a map I_w assigning to each n -ary relation symbol R an n -ary relation $I_w(R)$ over D , and to each n -ary function symbol f an n -ary function $I_w(f)$ over D (instead of $I_w(R)$ and $I_w(f)$ we also write R_w and f_w). Thus, with each world w we can associate a standard relational structure $\mathcal{M}_w = (D, I_w)$, and the model M as a whole can be regarded as a family $(\mathcal{M}_w)_{w \in W}$ of relational structures over a common domain.

An *information state* (or simply *state*) in M is a subset $s \subseteq W$. An assignment is a function g from the set of variables into D . Terms t from the language are assigned a denotation $[t]_w^g$ relative to a world w and an assignment g inductively by letting $[x]_w^g = g(x)$ and $[f(t_1, \dots, t_n)]_w^g = f_w([t_1]_w^g, \dots, [t_n]_w^g)$. The semantics is given in terms of a relation of *support* $M, s \models_g \phi$ that interprets a formula ϕ relative to a model M , a state s , and an assignment g . This relation is defined by the following clauses:

- $M, s \models_g R(t_1, \dots, t_n) \iff \forall w \in s : ([t_1]_w^g, \dots, [t_n]_w^g) \in R_w$
- $M, s \models_g (t_1 = t_2) \iff \forall w \in s : [t_1]_w^g = [t_2]_w^g$ ⁴

⁴The clause we consider here is a special case: in general, the identity symbol can be treated in InqBQ as a world-dependent congruence relation [5]. Models where ‘=’ is interpreted as meta-language identity are called id-models. For our purposes, we can restrict to this special class of models: if a formula ϕ does not have a first-order counterpart relative to a more restricted class of models, *a fortiori* it does not have such a counterpart relative to a more general class of models.

- $M, s \models_g \perp \iff s = \emptyset$
- $M, s \models_g \phi \wedge \psi \iff M, s \models_g \phi \text{ and } M, s \models_g \psi$
- $M, s \models_g \phi \vee \psi \iff M, s \models_g \phi \text{ or } M, s \models_g \psi$
- $M, s \models_g \phi \rightarrow \psi \iff \forall t \subseteq s : M, t \models_g \phi \text{ implies } M, t \models_g \psi$
- $M, s \models_g \forall x \phi \iff \forall d \in D : M, s \models_{g[x \rightarrow d]} \phi$
- $M, s \models_g \exists x \phi \iff \exists d \in D : M, s \models_{g[x \rightarrow d]} \phi$

We write simply $M \models_g \phi$ in case $M, W \models_g \phi$. As usual, for sentences the assignment does not matter and we can drop reference to it.

Clearly, the semantics of InqBQ is strictly parallel to that of InqBT, where states now play the role of teams. As a consequence, these logics share many properties. In particular, support in InqBQ is persistent (i.e., preserved under substates), and the empty state trivially supports every formula. Classical formulas are flat, i.e., they are supported at a state s just in case they are supported at each singleton $\{w\} \subseteq s$. Moreover, for a classical formula support at a singleton $\{w\}$ coincides with truth at the relational structure $\mathcal{M}_w = (D, I_w)$ under Tarskian semantics.

Relational encoding. A first-order information model $M = (W, D, I)$ for a signature Σ can be faithfully encoded as a relational model $M^* = (W, D, I^*)$ with two sorts, w for worlds and e for individuals, for a modified signature Σ^* defined as follows:

- for every n -ary predicate symbol $P \in \Sigma$, Σ^* contains a predicate symbol P^* of arity $n + 1$ where the first argument is of sort w and the remaining n arguments are of sort e ;
- for every n -ary function symbol $f \in \Sigma$, Σ^* contains a function symbol f^* of arity $n + 1$ where the first argument is of sort w and the remaining n arguments as well as the output are of sort e .

Given a first-order information model $M = (W, D, I)$ for the signature Σ , its encoding $M^* = (W, D, I^*)$ is the two-sorted model over the signature Σ^* defined in the following way:

$$(w, \vec{d}) \in I^*(R^*) \iff \vec{d} \in I_w(R), \quad I^*(f^*)(w, \vec{d}) = I_w(f)(\vec{d}).$$

We are now in a position to formulate Open Question 3 from the introduction in a fully precise way.

- **Open Question 3.** Given a sentence ϕ of InqBQ in a signature Σ , is there always a corresponding sentence ϕ^* of two-sorted predicate logic in the signature Σ^* such that for any first-order information model M we have $M \models \phi \iff M^* \models \phi^*$?

To get the intuitive idea, the table below includes some InqBQ sentences and corresponding first-order sentences which are equivalent to them in the relevant sense. We consider a signature Σ with a unary predicate P and a constant symbol a , and use w as a first-order variable of sort w .

InqBQ sentence	first-order sentence
$P(a)$	$\forall w P^*(w, a)$
$?P(a)$	$\forall w P^*(w, a) \vee \forall w \neg P^*(w, a)$
$\exists x P(x)$	$\exists x \forall w P^*(w, x)$

Novel result. Consider a signature Σ containing two constant symbols, a and b . A first-order information model for Σ is a triple $M = (W, D, I)$ where relative to a world $w \in W$, the function I assigns an extension for each constant, $a_w \in D$ and $b_w \in D$. Now to each information state $s \subseteq W$ we can associate a corresponding binary relation $R_s \subseteq D^2$ defined by $R_s = \{(a_w, b_w) \mid w \in s\}$. We say that a model $M = (W, D, I)$ is *full* in case $R_W = D^2$. Now consider the following formula, obtained by taking the formula $\phi(x, y)$ defined in 2 and replacing the variables x, y with the constants a, b :

$$\phi(a, b) := (=(a, b) \wedge =(b, a) \wedge \exists z(z \neq b) \rightarrow \exists u(u \neq a)). \quad (8)$$

We prove that, in restriction to the class of full models, $\phi(a, b)$ defines finiteness of the domain.

Proposition 4.1. *If $M = (W, D, I)$ is a full model then $M \models \phi(a, b) \iff D$ is finite.*

Proof. We prove the equivalent claim that if M is full, $M \not\models \phi(a, b) \iff D$ is infinite. We have that $M \not\models \phi(a, b)$ if there is an $s \subseteq W$ that supports the antecedent but not the consequent. As we discussed above, this state is associated with a relation $R_s = \{(a_w, b_w) \mid w \in s\}$. Spelling out the semantics, we find:

- $M, s \models =(a, b) \iff R_s \text{ is a function}$
- $M, s \models =(b, a) \iff R_s \text{ is injective}$
- $M, s \not\models \exists u(u \neq a) \iff \text{dom}(R_s) = D$
- $M, s \models \exists z(z \neq b) \iff \text{ran}(R_s) \neq D$

So, a state s falsifies the conditional $\phi(a, b)$ just in case the associated relation R_s is an injective function on D which is not surjective. In a full model, every binary relation on D is represented as R_s for some state $s \subseteq W$. So, in a full model, a state falsifying the conditional exists iff there exists an injective function defined on D which is not surjective, i.e., iff D is infinite. $\square$

We can use this result to give a negative answer to Open Question 3.

Theorem 4.2. *For some sentence ϕ of InqBQ in a signature Σ , there is no sentence ϕ^* of two-sorted first-order logic in the signature Σ^* such that for every model M of InqBQ we have $M \models \phi \iff M^* \models \phi^*$.*

Proof. We can take ϕ to be the formula $\phi(a, b)$ defined in 8, in the signature Σ including only the constants a, b . Towards a contradiction, suppose there is a formula ϕ^* of two-sorted first-order logic in the signature Σ^* that is equivalent to ϕ in the relevant sense.

Let q be the quantifier rank of ϕ^* , let $k = q^2$, and let $K = \{1, \dots, k\}$. Now consider the two models $M_K = (K^2, K, I_K)$ and $M_{\mathbb{N}} = (\mathbb{N}^2, \mathbb{N}, I_{\mathbb{N}})$, where both I_K and $I_{\mathbb{N}}$ assign to a world (n, m) the extensions given by $a_{(n, m)} = n$ and $b_{(n, m)} = m$. Clearly, these models are full, so by the previous proposition we have $M_K \models \phi$ but $M_{\mathbb{N}} \not\models \phi$. By our assumption on ϕ^* , it follows that $M_K^* \models \phi^*$ and $M_{\mathbb{N}}^* \not\models \phi^*$. If we make explicit the translated structures M_K^* and $M_{\mathbb{N}}^*$, we find that they are:

$$M_K^* = (K^2, K, \pi_1^K, \pi_2^K) \quad M_{\mathbb{N}}^* = (\mathbb{N}^2, \mathbb{N}, \pi_1^{\mathbb{N}}, \pi_2^{\mathbb{N}}),$$

where π_1^K and π_2^K are the projection functions from K^2 to K , and similarly for $\pi_1^{\mathbb{N}}, \pi_2^{\mathbb{N}}$. A standard argument shows that these structures are equivalent in a q -move Ehrenfeucht-Fraïssé game: the basic idea is that within q moves, Spoiler can only pick up to q distinct elements of $\mathbb{N}^2 \cup \mathbb{N}$, corresponding to at most $q^2 = k$ distinct natural numbers; Duplicator can then identify these numbers with corresponding numbers from $\{1, \dots, k\}$ and make the counterpart of Spoiler's moves under this identification. So, M_K^* and $M_{\mathbb{N}}^*$ satisfy the same formulas of quantifier rank up to q . As a consequence, they must agree on ϕ^* , contrary to our assumption. $\square$

We have thus established that some sentences of InqBQ express genuine second-order properties of models: in particular, the formula $\phi(a, b)$ defined above is an example.

It is interesting to comment on the syntactic form of this example. Recall from the introduction that, in contrast to our result, two broad fragments of InqBQ allow for a translation to two-sorted first-order logic: the classical antecedent (clant) fragment [18, 5], where antecedents are restricted to classical formulas, and the restricted existential (rex) fragment [8], where the inquisitive existential quantifier is restricted to occur only in conditional antecedents. This implies that any example of an InqBQ sentence expressing a non first-order property of models must have two features: it must involve (i) an inquisitive antecedent and (ii) an inquisitive existential which is not in a conditional antecedent. Our example sentence $\phi(a, b)$ respects both conditions in a relatively minimal way: it is a conditional with an inquisitive antecedent and an inquisitive existential in the consequent.

5 Conclusion

We have shown that there is a sentence of $\text{InqBT} + [x]$ whose models are exactly the finite structures and that, therefore, $\text{InqBT} + [x]$ fails compactness and cannot be axiomatized. We also showed that there are open formulas of InqBT which express non first-order properties of teams, and sentences of InqBQ that express non first-order properties of models, viewed as two-sorted relational structures. Some important problems in this area remain open, including the problem of characterizing the exact expressive power of the logics InqBT and InqBQ in relation to fragments of second-order logic, and the problem of whether these logics satisfy entailment compactness and admit a recursive axiomatization.⁵

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⁵In fact, after we completed this paper, we discovered that the ideas presented here can be used to settle these long-standing problems in the negative. A paper on this is in preparation; a first draft is available at <https://arxiv.org/abs/2603.20845>.

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Possibly Relevant Translations

Søren Brinck Knudstorp

ILLC and Philosophy
University of Amsterdam
Amsterdam, the Netherlands
s.b.knudstorp@uva.nl

We develop translations from relevant logics into normal modal logics, and use them to clarify structural connections between relevant and modal logic, obtain a few corollary results, and raise questions for future work.

The past century has witnessed the flourishing of many a new logical tradition, each developing along its own lines, yet with analogies identified and formalized through translation.

A classic example is the Gödel–McKinsey–Tarski translation of intuitionistic propositional logic into classical modal logic, which has fostered a rich exchange of ideas, results, and research questions [6]; its original intent of demonstrating how intuitionistic logic can be interpreted classically now seems the least of its impact.

The present paper pursues a similar, though more modest, aim for another pair of traditions: relevant logic as conceived in the Anderson–Belnap lineage and modal logic as studied in the theory of normal modal logic.

Our main contributions are full and faithful translations from relevant logics into normal modal logics. We leave aside discussion of philosophical differences and motivations, though we hope that the translations shall as well be conceptually clarifying. We also use the translations to sample a few corollary results and raise some questions, but leave a fuller analysis for future work.

The paper proceeds by developing a series of translations, starting with the relevant vocabulary $\{\wedge, \vee, \rightarrow\}$ and gradually adding $\mathbf{t}$, $\circ$, and $\neg$.

1 A basic positive translation

1.1 Preliminaries

We begin by presenting relevant logics axiomatically.¹ As we, at first, will be concerned with translating the $\{\wedge, \vee, \rightarrow\}$ -fragment of the relevant language, sometimes called the positive fragment, we set out only the axioms and rules pertaining to this language.

The weakest logic of concern will be the basic relevant logic **B**. It stands to relevant logic much like how **K** stands to modal logic – a point which our first translations will help refine. Its $\{\wedge, \vee, \rightarrow\}$ -fragment, often denoted **B**⁺, admits the following Hilbert-style axiomatization. (We follow the convention that $\wedge, \vee$ bind tighter than $\rightarrow$.)

¹The relevant-logic overview presented here is brief and selective. For more comprehensive expositions, I invite the reader to consult [1, 2, 9, 27].

Axiom schemes

1. $\varphi \rightarrow \varphi$,
2. $\varphi \wedge \psi \rightarrow \varphi$ and $\varphi \wedge \psi \rightarrow \psi$,
3. $\varphi \rightarrow \varphi \vee \psi$ and $\psi \rightarrow \varphi \vee \psi$,
4. $[(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)] \rightarrow [\varphi \rightarrow \psi \wedge \chi]$,
5. $[(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)] \rightarrow [\varphi \vee \psi \rightarrow \chi]$,
6. $[\varphi \wedge (\psi \vee \chi)] \rightarrow [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$.

Rules

$$\begin{array}{c}
\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \text{ modus ponens} \\
\\
\frac{\varphi \rightarrow \psi}{(\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)} \text{ prefixing}
\end{array}
\qquad
\begin{array}{c}
\frac{\varphi \quad \psi}{\varphi \wedge \psi} \text{ adjunction} \\
\\
\frac{\varphi \rightarrow \psi}{(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)} \text{ suffixing}
\end{array}$$

The following axiom schemes serve to axiomatize many of the more common stronger relevant logics.

7. $(\varphi \rightarrow \psi) \rightarrow [(\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)]$ (prefixing)
8. $(\varphi \rightarrow \psi) \rightarrow [(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)]$ (suffixing)
9. $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$ (hypothetical syllogism)
10. $\varphi \wedge (\varphi \rightarrow \psi) \rightarrow \psi$ (pseudo-modus ponens)
11. $[\varphi \rightarrow (\varphi \rightarrow \psi)] \rightarrow (\varphi \rightarrow \psi)$ (contraction)
12. $\varphi \rightarrow [(\varphi \rightarrow \psi) \rightarrow \psi]$ (assertion)

For instance, all of these are theorems of the $\{\wedge, \vee, \rightarrow\}$ -fragment of $\mathbf{R}$, denoted $\mathbf{R}^+$, and already $\{8., 11., 12.\}$ axiomatize $\mathbf{R}^+$ relative to $\mathbf{B}^+$; that is,

$$\mathbf{R}^+ = \mathbf{B}^+ \oplus \{7., \dots, 12.\} = \mathbf{B}^+ \oplus \{8., 11., 12.\},$$

where we, for a set of formulas Φ , by the notation $\mathbf{B}^+ \oplus \Phi$ mean the result of adding Φ (as axiom schemes) to $\mathbf{B}^+$ and closing under the above rules. Note that in the presence of the prefixing and suffixing axioms (7. and 8., respectively), the namesake rules become redundant; for example, only the rules of modus ponens and adjunction are needed to axiomatize $\mathbf{R}^+$. The same holds for each of the following relevant logics:

$$\begin{array}{lll}
\mathbf{TW}^+ := \mathbf{B}^+ \oplus \{7., 8.\} & \mathbf{TWJ}^+ := \mathbf{B}^+ \oplus \{7., 8., 9.\} & \mathbf{C}^+ := \mathbf{B}^+ \oplus \{7., 8., 10.\} \\
\mathbf{T}^+ := \mathbf{B}^+ \oplus \{7., 8., 11.\} & \mathbf{RW}^+ := \mathbf{B}^+ \oplus \{7., 8., 12.\} &
\end{array}$$

One major relevant logic not axiomatizable via 7.-12. is Anderson and Belnap's logic of entailment $\mathbf{E}$. It is most perspicuously presented when additional vocabulary is available. Here, we make do by setting out $\mathbf{E}^+$ as $\mathbf{T}^+$ plus the less than elegant pair:

$$((\varphi \rightarrow \varphi) \rightarrow \psi) \rightarrow \psi, \quad [((\varphi \rightarrow \varphi) \rightarrow \varphi) \wedge ((\psi \rightarrow \psi) \rightarrow \psi)] \rightarrow [(\varphi \wedge \psi \rightarrow \varphi \wedge \psi) \rightarrow \varphi \wedge \psi].^2$$

Both are theorems of $\mathbf{R}^+$, so $\mathbf{E}^+$ is a sublogic of $\mathbf{R}^+$. More generally, the introduced logics are ordered by strength as presented in Figure 1.

²If one abbreviates $\Box\alpha := (\alpha \rightarrow \alpha) \rightarrow \alpha$, the latter is shortened to $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$. And when the language includes the Ackermann constant t , the pair can be traded for the much simpler $(t \rightarrow \varphi) \rightarrow \varphi$.

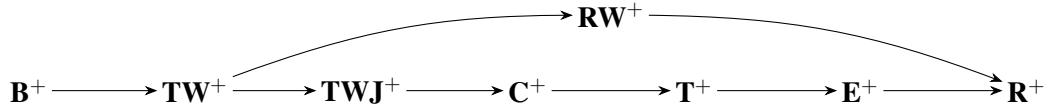


Figure 1: Arrows indicate strict inclusion $\subsetneq$. Additionally, $\text{TWJ}^+ \not\subseteq \text{RW}^+ \not\subseteq \text{E}^+$.

To situate the relevant family within a bigger logical realm, let us note that the $\{\wedge, \vee, \rightarrow\}$ -fragment of intuitionistic propositional logic **IPL** results from adding to $\mathbf{R}^+$ the axiom schema $\varphi \rightarrow (\psi \rightarrow \varphi)$.³ If one further adds Peirce's law $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$, one gets the $\{\wedge, \vee, \rightarrow\}$ -fragment of classical propositional logic **CPL**; in fact, over $\mathbf{R}^+$, from Peirce's law we already derive $\varphi \rightarrow (\psi \rightarrow \varphi)$ [20].⁴

From the above axiomatic presentation, it is not at all clear what affinities relevant logic shares with modal logic (unless we count their equally confusing nomenclature). This is made somewhat clearer by their relational semantics.

Formulas of the positive relevant language are interpreted on *Routley–Meyer frames and models*:

Definition 1.1 (Models and frames). A *Routley–Meyer frame* is a triple $\mathfrak{F} = (K, R, N)$ where $R \subseteq K \times K \times K$ is a ternary relation and $N \subseteq K$ a set whose elements are denoted *normal points*, and which is subject to the conditions, writing $x \leq y$ for $\exists n \in N(Rnxy)$:

- (preorder) $\leq$ is a preorder,
- (upset) if $n \in N$ and $n \leq m$, then $m \in N$,
- (down-down-up) if $Rxyz$, $x^- \leq x$, $y^- \leq y$ and $z \leq z^+$, then $Rx^-y^-z^+$.

A *Routley–Meyer model* $\mathfrak{M} = (\mathfrak{F}, V)$ is a Routley–Meyer frame $\mathfrak{F}$ with a *persistent valuation* V : if $x \in V(p)$ and $x \leq y$, then $y \in V(p)$.

Definition 1.2 (Semantic clauses). Given a model $\mathfrak{M} = (K, R, N, V)$, a point $x \in K$, and a formula φ , we say that x *satisfies* φ and write $\mathfrak{M}, x \Vdash \varphi$, or simply $x \Vdash \varphi$, based on the clauses:

$\mathfrak{M}, x \Vdash p$	iff	$x \in V(p)$,
$\mathfrak{M}, x \Vdash \varphi \wedge \psi$	iff	$\mathfrak{M}, x \Vdash \varphi$ and $\mathfrak{M}, x \Vdash \psi$,
$\mathfrak{M}, x \Vdash \varphi \vee \psi$	iff	$\mathfrak{M}, x \Vdash \varphi$ or $\mathfrak{M}, x \Vdash \psi$,
$\mathfrak{M}, x \Vdash \varphi \rightarrow \psi$	iff	for all $y, z \in K$: if $Rxyz$ and $\mathfrak{M}, y \Vdash \varphi$, then $\mathfrak{M}, z \Vdash \psi$.

Validity is defined with respect to normal points: φ is valid on a Routley–Meyer frame $\mathfrak{F} = (K, R, N)$, written $\mathfrak{F} \Vdash \varphi$, if for all persistent valuations V and normal points $n \in N$, $(\mathfrak{F}, V), n \Vdash \varphi$; and φ is valid on a class of frames $\mathcal{C}$, written $\mathcal{C} \Vdash \varphi$, if φ is valid on each frame $\mathfrak{F} \in \mathcal{C}$. A handy fact, readily proven using persistence ($\mathfrak{M}, x \Vdash \varphi$ and $x \leq y$ imply $\mathfrak{M}, y \Vdash \varphi$) and reflexivity of $\leq$, is that for any frame $\mathfrak{F} = (K, R, N)$ and implication $\varphi \rightarrow \psi$,

$$\mathfrak{F} \Vdash \varphi \rightarrow \psi \quad \text{iff} \quad \text{for all } x \in K \text{ and persistent valuations } V: \text{ if } (\mathfrak{F}, V), x \Vdash \varphi, \text{ then } (\mathfrak{F}, V), x \Vdash \psi. \quad (1)$$

³Relevance, at least over the signature $\{\wedge, \vee, \rightarrow\}$, has close ties to constructivism; see [30] for points of contact.

⁴Algebraically, the $\{\wedge, \vee, \rightarrow\}$ -fragment of **IPL** corresponds to the well-studied variety of Brouwerian algebras (only that the latter include a designated element $\top$ for the term definable $p \rightarrow p$). The subvariety axiomatized by Peirce's law is the variety of generalized Boolean algebras, corresponding to the $\{\wedge, \vee, \rightarrow\}$ - (or $\{\wedge, \vee, \rightarrow, \top\}$ -) fragment of **CPL**. Let us also mention that minimal logic [11] is like the $\{\wedge, \vee, \rightarrow\}$ -fragment of **IPL** but with a constant symbol $\perp$ for which no axioms or rules are specified.

The set of validities on all Routley–Meyer frames is the basic relevant logic $\mathbf{B}^+$. For the axioms 7.–12., we list their frame correspondents below, which facilitate completeness for other relevant logics with respect to the corresponding classes of frames (as the formulas are canonical). There, we use a convenient equivalence between a ternary relation $R \subseteq K^3$ and a function $\cdot : K^2 \rightarrow \mathcal{P}(K)$ defined by $z \in x \cdot y$ iff $Rxyz$ and lifted to arbitrary sets $X, Y \subseteq K$ by $X \cdot Y := \bigcup_{x \in X, y \in Y} x \cdot y$.

Proposition 1.3 (Relevant correspondences). *Let $\mathfrak{F} = (K, R, N)$ be a Routley–Meyer frame. Then:*

$$\begin{array}{ll}
\mathfrak{F} \Vdash (p \rightarrow q) \rightarrow [(r \rightarrow p) \rightarrow (r \rightarrow q)] & \text{iff } (x \cdot y) \cdot z \subseteq x \cdot (y \cdot z) & 7. \\
\mathfrak{F} \Vdash (p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)] & \text{iff } (x \cdot y) \cdot z \subseteq y \cdot (x \cdot z) & 8. \\
\mathfrak{F} \Vdash (p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r) & \text{iff } x \cdot y \subseteq x \cdot (x \cdot y) & 9. \\
\mathfrak{F} \Vdash p \wedge (p \rightarrow q) \rightarrow q & \text{iff } x \in x \cdot x & 10. \\
\mathfrak{F} \Vdash [p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q) & \text{iff } x \cdot y \subseteq (x \cdot y) \cdot y & 11. \\
\mathfrak{F} \Vdash p \rightarrow [(p \rightarrow q) \rightarrow q] & \text{iff } x \cdot y \subseteq y \cdot x & 12.
\end{array}$$

With this, we turn to the target modal logic.⁵ The language is the modal language with a single binary modality $\Diamond$; that is, the language generated by the grammar

$$\varphi ::= \perp \mid \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond(\varphi, \varphi).$$

Conjunction ‘ $\wedge$ ’ and material implication ‘ $\supset$ ’ are defined in the usual way.

Semantics for the modal language are provided by *Kripke frames* – pairs $\mathbb{F} = (W, R)$ such that $R \subseteq W^3$ – and *Kripke models* $\mathbb{M} = (\mathbb{F}, V)$, which are frames $\mathbb{F} = (W, R)$ with a *valuation* $V : \text{Prop} \rightarrow \mathcal{P}(W)$. The semantic clauses are the usual ones. As it will make later proofs simpler, we follow the convention that the modality is evaluated at the first point of a triple $(w, v, u) \in R$; i.e., given a model $\mathbb{M} = (W, R, V)$ and a point $w \in W$,

$$\mathbb{M}, w \models \Diamond(\varphi, \psi) \quad \text{iff} \quad \text{there exist } v, u \in W \text{ such that } R w v u, \mathbb{M}, v \models \varphi, \text{ and } \mathbb{M}, u \models \psi.$$

Having in mind the Routley–Meyer clause for implication, it is particularly suggestive to observe that

$$\mathbb{M}, w \models \neg\Diamond(\varphi, \neg\psi) \quad \text{iff} \quad \text{for all } v, u \in W: \text{ if } R w v u \text{ and } \mathbb{M}, v \models \varphi, \text{ then } \mathbb{M}, u \models \psi. \quad (2)$$

The set of validities on all frames is the least normal modal logic $\mathbf{K}_2$, the analog of $\mathbf{K}$ in the language with a binary, rather than unary, diamond. Given a set of modal formulas Φ , we write $\mathbf{K}_2 \oplus \Phi$ for the least normal modal logic containing Φ .

Notation. We maintain a notational distinction between our two semantic frameworks: we reserve the symbol ‘ $\Vdash$ ’ and fraktur letters ($\mathfrak{M}$) for Routley–Meyer models, and the symbol ‘ $\models$ ’ and blackboard bold letters ($\mathbb{M}$) for Kripke models. Also, given a formula φ in the relevant language and a class of Routley–Meyer frames $\mathcal{C}$, we write $\mathcal{C} \Vdash \varphi$ if φ is valid on $\mathcal{C}$; and for the analogous notion in the modal setting, we write $\mathcal{C} \models \varphi$. However, in both frameworks ‘ $\vdash$ ’ denotes derivability (or theoremhood) and subscripts indicate the logic, i.e., for a relevant logic $\mathbf{L}$, we write $\vdash_{\mathbf{L}} \varphi$ if φ is a theorem of $\mathbf{L}$; and for modal logics $\mathbf{M}$, we likewise write $\vdash_{\mathbf{M}} \varphi$.

⁵The paper assumes basic familiarity with normal modal logic. For a more detailed treatment of the subject, the reader is referred to standard textbooks such as [5, 6].

1.2 Proof outline

The first goal is to define a translation * such that for all formulas φ, ψ of the positive relevant language,

$$\vdash_{\mathbf{B}^+} \varphi \rightarrow \psi \quad \text{iff} \quad \vdash_{\mathbf{K}_2} \varphi^* \supset \psi^*. \quad (3)$$

Afterwards, we extend the translation across further relevant logics and then to arbitrary formulas in the positive language (not only those with main connective $\rightarrow$). In a later section, we will also translate relevant logics in languages with additional relevant vocabulary.

We proceed to outline the proof strategy but defer most proof steps to the appendix. The proof will be carried out semantically by defining a function F that takes *pointed Kripke models* (model-world pairs $\mathbb{M}, w$) and returns *pointed Routley–Meyer models* (model-point pairs $\mathfrak{M}, x$), subject to the following adjunction pattern:

$$F(\mathbb{M}, w) \Vdash \alpha \quad \text{iff} \quad \mathbb{M}, w \models \alpha^*.^6 \quad (4)$$

This ensures left-to-right of (3) (fullness of the translation): if $\mathbb{M}, w \models \varphi^*$, then $F(\mathbb{M}, w) \Vdash \varphi$, whence $\vdash_{\mathbf{B}^+} \varphi \rightarrow \psi$ (and soundness for $\mathbf{B}^+$) imply $F(\mathbb{M}, w) \Vdash \psi$ (cf. (1)), so we have $\mathbb{M}, w \models \psi^*$, which suffices by completeness for $\mathbf{K}_2$.

For the converse direction (faithfulness, or conservativity, of the translation), it suffices to show that F is essentially surjective, up to satisfaction equivalence; i.e., that for all pointed Routley–Meyer models $(\mathfrak{M}, x)$, there is a pointed Kripke model $(\mathbb{M}, w)$ such that for all α ,

$$\mathfrak{M}, x \Vdash \alpha \quad \text{iff} \quad F(\mathbb{M}, w) \Vdash \alpha.$$

It will, however, be more convenient (and equivalent) to directly define a function G that takes pointed Routley–Meyer models and returns pointed Kripke models, subject to the following:

$$\mathfrak{M}, x \Vdash \alpha \quad \text{iff} \quad G(\mathfrak{M}, x) \models \alpha^*. \quad (5)$$

Then, whenever $\mathfrak{M}, x \Vdash \varphi$, we also have $G(\mathfrak{M}, x) \models \varphi^*$, so from $\vdash_{\mathbf{K}_2} \varphi^* \supset \psi^*$ (and soundness for $\mathbf{K}_2$) we get $G(\mathfrak{M}, x) \models \psi^*$, hence $\mathfrak{M}, x \Vdash \psi$, which, by (1) and completeness for $\mathbf{B}^+$, is as required.

In sum, we shall define a triple consisting of a translation of formulas * and two model transformations $F : \{(\mathbb{M}, w)\} \rightleftharpoons \{(\mathfrak{M}, x)\} : G$ fulfilling (4) and (5).

1.3 Basic translation

We define * as follows, using an abbreviated operator in the modal language $\cdot \rightarrow \cdot := \neg \Diamond(\cdot, \neg \cdot)$ (recall (2)).

$$\begin{array}{llll} p^* & := & p & (\varphi \vee \psi)^* & := & \varphi^* \vee \psi^* \\ (\varphi \rightarrow \psi)^* & := & \varphi^* \rightarrow \psi^* & (\varphi \wedge \psi)^* & := & \varphi^* \wedge \psi^*. \end{array}$$

Theorem 1.4. *For all formulas φ, ψ of the positive relevant language,*

$$\vdash_{\mathbf{B}^+} \varphi \rightarrow \psi \quad \text{iff} \quad \vdash_{\mathbf{K}_2} \varphi^* \supset \psi^*.$$

⁶It may be worth footnoting that this is an instance of a *Chu transform* [7] between satisfaction relations, i.e. a morphism of Chu spaces over 2. Within situation semantics and formal concept analysis, this is also known as an *infomorphism*. See [3] for more on Chu transforms and translations.

Proof. (4) is proven in Lemma A.1 and (5) in Lemma A.2. $\square$

Next, we show how the $\star$ -translation extends, and fails to extend, to further relevant logics. For this, we use a modal version of Proposition 1.3 and the earlier identification of ternary relations $R \subseteq W^3$ with functions $\cdot : W^2 \rightarrow \mathcal{P}(W)$.

Proposition 1.5 (Modal correspondences). *Let $\mathbb{F} = (W, R)$ be a Kripke frame. Then:*

$$\begin{array}{ll} \mathbb{F} \models (p \rightarrow q)^\star \supset [(r \rightarrow p) \rightarrow (r \rightarrow q)]^\star & \text{iff } (x \cdot y) \cdot z \subseteq x \cdot (y \cdot z) \\ \mathbb{F} \models (p \rightarrow q)^\star \supset [(q \rightarrow r) \rightarrow (p \rightarrow r)]^\star & \text{iff } (x \cdot y) \cdot z \subseteq y \cdot (x \cdot z) \\ \mathbb{F} \models [(p \rightarrow q) \wedge (q \rightarrow r)]^\star \supset (p \rightarrow r)^\star & \text{iff } x \cdot y \subseteq x \cdot (x \cdot y) \\ \mathbb{F} \models [p \wedge (p \rightarrow q)]^\star \supset q^\star & \text{iff } x \in x \cdot x \\ \mathbb{F} \models [p \rightarrow (p \rightarrow q)]^\star \supset (p \rightarrow q)^\star & \text{iff } x \cdot y \subseteq (x \cdot y) \cdot y \\ \mathbb{F} \models p^\star \supset [(p \rightarrow q) \rightarrow q]^\star & \text{iff } x \cdot y \subseteq y \cdot x \end{array}$$

Proof. Routine exercise, paralleling the proofs from relevant logic. $\square$

Moreover, by again mirroring proofs from relevant logic, one can use the right-hand side conditions to show that the left-hand side translated formulas are canonical. Consequently, each modal logic axiomatized by any subset of the listed translated formulas is complete with respect to the corresponding class of frames. This observation is helpful in extending Theorem 1.4 to stronger relevant logics. In particular, we have the following, where we, given a set of relevant formulas Φ , write $\Phi^\star := \{\alpha^\star \supset \beta^\star \mid \alpha \rightarrow \beta \in \Phi\}$.

Theorem 1.6. *Let Φ be a subset of the formulas 7.-10. Then for all formulas φ, ψ of the positive relevant language,*

$$\vdash_{\mathbf{B}^+ \oplus \Phi} \varphi \rightarrow \psi \quad \text{iff} \quad \vdash_{\mathbf{K}_2 \oplus \Phi^\star} \varphi^\star \supset \psi^\star. \quad (6)$$

Proof. It suffices to verify that the model transformations $F : \{(\mathbb{M}, w)\} \rightleftharpoons \{(\mathbb{M}, x)\} : G$ from Lemma A.1 and A.2 take Kripke frames for $\mathbf{K}_2 \oplus \Phi^\star$ to Routley–Meyer frames for $\mathbf{B}^+ \oplus \Phi$, and vice versa.

The latter is immediate by the frame correspondents of Proposition 1.3 and 1.5, as given a pointed Routley–Meyer model over a frame $\mathfrak{F} = (K, R, N)$, we simply considered the pointed Routley–Meyer model as a pointed Kripke model over the frame (K, R) .

Similarly, for the former, one needs to check that a Kripke frame (W, R) is a $\mathbf{K}_2 \oplus \Phi^\star$ frame only if the Routley–Meyer frame $(W \cup \{0\}, R', \{0\})$ of Lemma A.1 is a $\mathbf{B}^+ \oplus \Phi$ frame. While not a direct consequence of Proposition 1.3 and 1.5, it is still readily verified. $\square$

Thus the $\star$ -translation not only ties together the basic relevant logic $\mathbf{B}^+$ with the basic modal logic $\mathbf{K}_2$, it also translates a greater family of relevant reasoning systems that includes $\mathbf{TW}^+$, $\mathbf{TWJ}^+$, and $\mathbf{C}^+$. However, as we shall see, it fails to meet the translation criterion of (6) for other sets of relevant axioms Φ ; notably, Theorem 1.6 states $\Phi \subseteq \{7., \dots, 10.\}$, excluding 11. and 12. (contraction and assertion). We will get back to this below, but let us first ameliorate another shortcoming of the translation as so far proposed: it only translates formulas with main connective $\rightarrow$ (cf. (3)). This will be particularly instructive in elucidating a subtlety in how relevant implications behave differently when nested.

1.4 Nested implication, nested translation

To extend Theorem 1.4 to formulas with main connectives other than $\rightarrow$, we define a second-layer translation $*$, using the abbreviated modal operator $\Box_{\rightarrow} \varphi := \top \rightarrow \varphi$.

$$\begin{array}{llll} p^* & := & p & (\varphi \vee \psi)^* & := & \varphi^* \vee \psi^* \\ (\varphi \rightarrow \psi)^* & := & \Box_{\rightarrow}(\varphi^* \supset \psi^*) & (\varphi \wedge \psi)^* & := & \varphi^* \wedge \psi^*. \end{array}$$

As we now prove, the $*$ -translation is full and faithful for $\mathbf{B}^+$ and $\mathbf{K}_2$, thereby tightening the connection between the two: $\mathbf{B}^+$ is realized as a syntactic fragment of $\mathbf{K}_2$.

Theorem 1.7. *For all formulas φ of the positive relevant language,*

$$\vdash_{\mathbf{B}^+} \varphi \quad \text{iff} \quad \vdash_{\mathbf{K}_2} \varphi^*.$$

Proof. The proof is much the same as that of Theorem 1.4, relying on $F : \{(\mathbb{M}, w)\} \rightleftharpoons \{(\mathfrak{M}, x)\} : G$ from Lemma A.1 and A.2. However, now, we also think of F (or call it F^*) as taking a pointed Kripke model $(\mathbb{M}, w)$ and returning a Routley–Meyer model $(\mathfrak{M}, n)$ pointed at a normal point n such that

$$F^*(\mathbb{M}, w) \Vdash \alpha \quad \text{only if} \quad \mathbb{M}, w \models \alpha^*. \quad (7)$$

The details and proof are found in Lemma A.3.

Conversely, in Lemma A.4, we show that when G is applied to a Routley–Meyer model $(\mathfrak{M}, n)$ pointed at a normal point n , then

$$\mathfrak{M}, n \Vdash \alpha \quad \text{if} \quad G(\mathfrak{M}, n) \models \alpha^*. \quad (8)$$

(7) yields fullness of $*$ and (8) yields faithfulness, so combined they establish the theorem.⁷ $\square$

The translation makes precise a sense in which relevant logic draws a qualitative distinction between conditionals within the scope of another conditional (where the $*$ -translation is in force) and those outside it (where $*$ is in force). Note that the latter resemble intuitionistic conditionals – which perhaps becomes particularly evident upon recalling the Gödel–McKinsey–Tarski translation of **IPL** into the modal logic **S4** – and both traditions invalidate $(p \rightarrow q) \vee (q \rightarrow p)$ in much the same manner.⁸

As with the $*$ -translation, $*$ can be extrapolated along the lines of Theorem 1.6:

Corollary 1.8. *Let Φ be a subset of the formulas 7.-10. Then for all formulas φ of the positive relevant language,*

$$\vdash_{\mathbf{B}^+ \oplus \Phi} \varphi \quad \text{iff} \quad \vdash_{\mathbf{K}_2 \oplus \Phi^*} \varphi^*.$$

Proof. Immediate from the proofs of Theorem 1.6 and 1.7. $\square$

Of course, one might here prefer to replace Φ^* with $\Phi^* := \{\alpha^* \mid \alpha \in \Phi\}$. The issue then is that the correspondences of Proposition 1.5 fail for 7.*-10.* There are ways to get around this hurdle, most easily by swapping $\Box_{\rightarrow}$ for the global box of modal logic in the $*$ -translation. Rather than detailing this here, we will set out and examine yet another translation. This is in part due to the nature of this paper – being exploratory and broad rather than deep – and in part due to some limitations of the translations proposed so far, as we now turn to discuss.

⁷We footnote that (7) isn't a biimplication (hence not a Chu transform/infomorphism), nor is (8). But neither biimplication is needed.

⁸In addition to the links between constructivism and relevance drawn in [30], we point the reader to [18] for a translational embedding of **IPL** into **R**, and to [8] for an embedding of the Gödel–Dummett intermediate logic into the relevant logic **RM**.

1.5 Lost and found in translation

On translating the relevant family. Theorem 1.6 (and Corollary 1.8) collect a few relevant logics – like $\mathbf{TW}^+$, $\mathbf{TWJ}^+$, and $\mathbf{C}^+$ – for which our translation naturally extends, in the sense of (6). However, many more relevant logics lie outside the scope of Theorem 1.6. For good reason: the theorem need not apply to them. For example, over $\mathbf{B}^+$, contraction (11.) derives pseudo-modus ponens (10.), but the corresponding derivation over $\mathbf{K}_2$ fails for their translations:

$$\vdash_{\mathbf{B}^+ \oplus [p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)} [p \wedge (p \rightarrow q)] \rightarrow q \quad \text{but} \quad \not\vdash_{\mathbf{K}_2 \oplus [p \rightarrow (p \rightarrow q)]^* \supset (p \rightarrow q)^*} [p \wedge (p \rightarrow q)]^* \supset q^*,$$

as witnessed by, e.g., $x \not\models [p \wedge (p \rightarrow q)]^* \supset q^*$ in the $\mathbf{K}_2 \oplus [p \rightarrow (p \rightarrow q)]^* \supset (p \rightarrow q)^*$ -model with $W = \{0, x, y\}$, $V(p) = \{x, y\}$, $V(q) = \emptyset$, and $R = \{(0, 0, 0), (0, x, x), (0, y, y), (0, x, y), (y, x, x), (y, x, y), (y, y, y)\}$.

Model-theoretically, the issue is that the simpler pairs (W, R) of Kripke modeling lack a means to account for relevant inferences relying on the $\leq$ -conditions of Routley–Meyer frames. Specifically, although the relevant frame correspondent of $[p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)$ and the modal frame correspondent of $[p \rightarrow (p \rightarrow q)]^* \supset (p \rightarrow q)^*$ both are $x \cdot y \subseteq (x \cdot y) \cdot y$, and the relevant frame correspondent of $[p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)$ and the modal frame correspondent of $[p \rightarrow (p \rightarrow q)]^* \supset (p \rightarrow q)^*$ likewise both are $x \in x \cdot x$, only for Routley–Meyer frames does the former correspondent imply the latter, as it crucially relies on the down-down-up condition (recall Definition 1.1). This is exemplified by the above Kripke model, which, if we follow the suggestive notation and let $N = \{0\}$, fails to define a Routley–Meyer frame as we then have $x \leq y$ and $Ryxx$ yet $\bar{R}xxx$.⁹

Similarly, for assertion (12.) and one of the characteristic axioms of $\mathbf{E}^+$, we have

$$\vdash_{\mathbf{B}^+ \oplus p \rightarrow [(p \rightarrow q) \rightarrow q]} [(p \rightarrow p) \rightarrow q] \rightarrow q \quad \text{but} \quad \not\vdash_{\mathbf{K}_2 \oplus p^* \supset [(p \rightarrow q) \rightarrow q]^*} [(p \rightarrow p) \rightarrow q]^* \supset q^*,$$

where the latter can be readily witnessed by a Kripke frame with $R = \emptyset$; here, the Kripke modeling is unable to account for relevant inferences validated via reflexivity of $\leq$ in Routley–Meyer frames.

To make matters more pressing, take $\mathbf{R}^+$. On the one hand, recall that $\mathbf{R}^+ = \mathbf{B}^+ \oplus \{8., 11., 12.\}$ and that $\vdash_{\mathbf{R}^+} p \wedge (p \rightarrow q) \rightarrow q$. On the other, we have $\not\vdash_{\mathbf{K}_2 \oplus \{8., 11., 12.\}^*} [p \wedge (p \rightarrow q)]^* \supset q^*$ – so by completeness, not even the frame-class for $\{8., 11., 12.\}^*$ validates $[p \wedge (p \rightarrow q)]^* \supset q^*$ – which too can be shown by considering a Kripke frame with $R = \emptyset$. Regrettably, the translation falls short of $\mathbf{R}^+$, and for a deeper structural relation between relevant logic and modal logic, we should, and will, look elsewhere.

Import, export, transport. It is a staple of logical practice to transfer meta-logical properties along translations. We shall here focus on but two: computability and interpolation.

First, we note that, e.g., $\mathbf{B}^+$ inherits decidability as well as the finite-model property from $\mathbf{K}_2$ through translation.¹⁰ But more interesting, and promising, seem the transfer of undecidability. In seminal work,

⁹Proof-theoretically, an issue has to do with the translation of the rules. To illustrate, for implications $\varphi = \alpha \rightarrow \beta$, $\psi = \gamma \rightarrow \delta$ the translated rule of modus ponens reads

$$\frac{\alpha^* \supset \beta^* \quad (\alpha^* \rightarrow \beta^*) \supset (\gamma^* \rightarrow \delta^*)}{\gamma^* \supset \delta^*},$$

which should be admissible in the target logic; however, as the translation highlights, there is a compositional tension with nested versus non-nested conditionals.

¹⁰One may think that only the $*$ -translation imparts decidability on $\mathbf{B}^+$, as $*$ operates on formulas with implication as main connective. However, $\mathbf{B}^+$, like most other positive relevant logics, enjoys the disjunction property, hence the problem of deciding $\vdash_{\mathbf{B}^+} \varphi$ reduces to the problem of deciding $\vdash_{\mathbf{B}^+} \alpha \rightarrow \beta$. Proofs of the disjunction property for relevant logics typically go via *metacompleteness* [19, 26].

Urquhart [28] proved that a great many relevant logics are undecidable, not least $\mathbf{R}^+$. While the $*$ -translation fell short of $\mathbf{R}^+$, there may be, and still are, results to be had. First off, all we have shown is that $*$ isn't full for $\mathbf{R}^+$ and $\mathbf{K}_2 \oplus \{8., 11., 12.\}^*$. This still leaves considerable room for variation; for example, one may append to $\{8., 11., 12.\}^*$ one or more axioms, and thereby have the undecidability of $\mathbf{R}^+$ transfer. This remains to be explored.¹¹ Even if this were to fail, we can, with no further work, collect other undecidability results. The result of [28] applies to $\mathbf{C}^+$ as well; and in recent work, undecidability is also shown for the weaker $\mathbf{TWJ}^+$ [14]. Since the proof of [14] is for formulas whose principal connective is $\rightarrow$, we obtain the following theorem.

Theorem 1.9. $\mathbf{K}_2 \oplus \{7., 8., 9.\}^*$ and $\mathbf{K}_2 \oplus \{7., 8., 10.\}^*$ are undecidable. That is, the modal logic of the class of frames where

$$(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z), \quad (x \cdot y) \cdot z \subseteq y \cdot (x \cdot z), \quad x \cdot y \subseteq x \cdot (x \cdot y)$$

is undecidable; and so is the modal logic of frames where

$$(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z), \quad (x \cdot y) \cdot z \subseteq y \cdot (x \cdot z), \quad x \in x \cdot x.$$

The theorem supplements related results from [15, 16, 13] by, in comparison, pertaining to some sort of rotated setting yet without full associativity and using only a fragment of the language.¹²

Next, say that a logic $\mathbf{L}$ has the Craig interpolation property (CIP) for a binary connective $\rightsquigarrow$ if whenever $\vdash_{\mathbf{L}} \varphi \rightsquigarrow \psi$, there is a formula θ (an interpolant) such that $\vdash_{\mathbf{L}} \varphi \rightsquigarrow \theta$ and $\vdash_{\mathbf{L}} \theta \rightsquigarrow \psi$ and $\text{Var}(\theta) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$, where $\text{Var}(\alpha)$ denotes the set of propositional variables occurring in α . For normal modal logics, and other classical systems, the connective of concern tends to be the material implication $\supset$, and for relevant logics, the relevant implication $\rightarrow$.

Intriguingly, although all relevant logics between $\mathbf{C}^+$ and $\mathbf{R}^+$ were shown to lack interpolation already in the 1990s [29], it remains open whether $\mathbf{B}^+$ enjoys it, cf. [27]. In contrast, on the one hand, $\mathbf{K}_2$ is known to have the CIP, while on the other, if the diamond of $\mathbf{K}_2$ takes only diagonal entries $\Diamond(\varphi, \varphi)$, one obtains a weakly aggregative modal logic (WAML) which lacks the CIP [17]. Since our $*$ -translation sends formulas with main connective $\rightarrow$ to formulas with main connective $\supset$, this raises a number of questions. Can the proof of the CIP for $\mathbf{K}_2$ be adapted to work for $\mathbf{B}^+$? Or can the argument showing that WAML lacks the CIP inspire a corresponding failure for $\mathbf{B}^+$? In case of the latter, since the translation establishes that $\mathbf{B}^+$ -validities $\varphi \rightarrow \psi$ find interpolants within $\mathbf{K}_2$, what additional vocabulary would need to be added to $\mathbf{B}^+$ to obtain the CIP?

2 Translating beyond the basics

2.1 Truth $\mathbf{t}$ and fusion $\circ$

Sometimes relevant languages include the *Ackermann truth constant* $\mathbf{t}$ and *fusion* $\circ$, residuated by $\rightarrow$. We should like to translate these too.

Adding $\mathbf{t}$ and $\circ$, we denote by $\mathcal{L}_{\circ, \mathbf{t}}^+$ the relevant language with the grammar

$$\varphi ::= p \mid \mathbf{t} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \varphi \circ \varphi.$$

¹¹One place to look for inspiration is in the work on classical relevant logic and simplified semantics [21, 23, 24, 25]. Both bear some similarity to the present enterprise, inasmuch as translations also produce semantics – fullness as soundness, faithfulness as completeness.

¹²More generally, the undecidability techniques of [13, 14] have found somewhat broad application [12, 10], and comparing these results via translation could serve in mapping the border between the solvable and the unsolvable.

The $\mathcal{L}_{\circ, \mathbf{t}}^+$ -fragment of the basic relevant logic is axiomatized by adding to the axiomatization of section 1.1, the formula $\mathbf{t}$ as an axiom and the rules

$$\frac{\varphi \rightarrow (\psi \rightarrow \chi)}{(\varphi \circ \psi) \rightarrow \chi} \quad \frac{(\varphi \circ \psi) \rightarrow \chi}{\varphi \rightarrow (\psi \rightarrow \chi)} \quad \frac{\varphi}{\mathbf{t} \rightarrow \varphi}$$

Formulas of the extended language are still interpreted on the Routley–Meyer frames and models of Definition 1.1; the clauses for $\mathbf{t}$ and $\circ$ are as in the later presented (10) and (11). The $\mathcal{L}_{\circ, \mathbf{t}}^+$ -validities on the class of all Routley–Meyer frames are precisely the $\mathcal{L}_{\circ, \mathbf{t}}^+$ -theorems of the basic relevant logic.

To translate the augmented language, we augment the modal language accordingly:

$$\varphi ::= \perp \mid \top \mid p \mid \mathbf{t} \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond(\varphi, \varphi) \mid \varphi \circ \varphi,$$

where $\mathbf{t}$ is a nullary modality and $\circ$ is a binary modality.¹³

With two binary modalities and one nullary modality, Kripke frames are now tuples (W, R, S, N) where $N \subseteq W$ is a set – as $\mathbf{t}$ is nullary – and $S \subseteq W^3$ is another ternary relation; however, like in temporal logic, triples (W, R, N) will suffice, as $\circ$ is residuated by $\rightarrow$, which is ensured by requiring the adjunction

$$(\varphi \circ \psi) \supset \chi \text{ is a theorem} \quad \text{iff} \quad \varphi \supset (\psi \rightarrow \chi) \text{ is a theorem,}$$

or, equivalently, by including the following two formulas as axioms:

$$[(p \rightarrow q) \circ p] \supset q \quad \text{and} \quad p \supset [q \rightarrow (p \circ q)]. \quad (9)$$

The clauses for $\mathbf{t}$ and $\circ$ on Kripke models (W, R, N) are then just as on Routley–Meyer models:

$$u \models \mathbf{t} \quad \text{iff} \quad u \in N, \quad (10)$$

$$u \models \varphi \circ \psi \quad \text{iff} \quad \text{there exist } w, v \text{ such that } R w v u, w \models \varphi, \text{ and } v \models \psi. \quad (11)$$

A pleasant benefit of translating with $\mathbf{t}$ and $\circ$ is that now the modal language can define Routley–Meyer frames via correspondence theory. (For a frame (W, R, N) , we write $x \leq y$ iff $\exists n \in N(Rnxy)$.)

Proposition 2.1 (Modal correspondences). *Let $\mathbb{F} = (W, R, N)$ be a Kripke frame. Then*

$$\begin{array}{lll} \mathbb{F} \models p \supset \mathbf{t} \circ p & \text{iff} & \leq \text{ is reflexive,} \\ \mathbb{F} \models \mathbf{t} \circ (\mathbf{t} \circ p) \supset \mathbf{t} \circ p & \text{iff} & \leq \text{ is transitive,} \\ \mathbb{F} \models \mathbf{t} \circ \mathbf{t} \supset \mathbf{t} & \text{iff} & N \text{ is an } \leq\text{-upset,} \\ \mathbb{F} \models (\mathbf{t} \circ p) \circ q \supset p \circ q & \text{iff} & Rxyz, x^- \leq x \Rightarrow Rx^-yz, \\ \mathbb{F} \models p \circ (\mathbf{t} \circ q) \supset p \circ q & \text{iff} & Rxyz, y^- \leq y \Rightarrow Rxy^-z, \\ \mathbb{F} \models \mathbf{t} \circ (p \circ q) \supset p \circ q & \text{iff} & Rxyz, z \leq z^+ \Rightarrow Rxyz^+. \end{array}$$

In sum, a Kripke frame $\mathbb{F} = (W, R, N)$ is a Routley–Meyer frame iff it validates the listed axioms.

Proof. Follows by Sahlqvist correspondence, but see the appendix (2.1) for representative proofs. $\square$

This suggests another translation, into a modal version of **B**.

¹³Note that nullary modalities are normal modalities in the technical sense, and are used in e.g. arrow logic. Also note that we use infix notation for $\circ$ and prefix for $\Diamond$. This is perhaps somewhat odd, but it's just a matter of convention (aligning partly with relevant conventions, and partly with modal conventions).

Definition 2.2. Let $\mathbf{Mod.B}^+$ (Modal $\mathbf{B}^+$) denote the least normal modal logic containing the residuation axioms (9) and the RM-frame axioms of Proposition 2.1.¹⁴

Observe that $\mathbf{Mod.B}^+$ is complete for the Kripke frames (W, R, N) that are Routley–Meyer frames in the sense of Proposition 2.1. This follows from Sahlqvist canonicity for those formulas, and from the canonicity of the residuation axioms (9), which can be readily verified.¹⁵

We continue by defining the translation $^\circ$, a variant of $*$ available in the modal language with $\circ$ and t ,

$$\begin{array}{llll} p^\circ & := & t \circ p & (\varphi \vee \psi)^\circ & := & \varphi^\circ \vee \psi^\circ \\ (\varphi \rightarrow \psi)^\circ & := & \varphi^\circ \rightarrow \psi^\circ & (\varphi \wedge \psi)^\circ & := & \varphi^\circ \wedge \psi^\circ \\ t^\circ & := & t & (\varphi \circ \psi)^\circ & := & \varphi^\circ \circ \psi^\circ. \end{array}$$

We have the following lemmas, auxiliary to proving full- and faithfulness of the $^\circ$ -translation.

Lemma 2.3. *Let (W, R, N, V) be a Kripke model with (W, R, N) a $\mathbf{Mod.B}^+$ -frame. If $V(p)$ is a $\leq$ -upset, then for all $w \in W$,*

$$w \models p \quad \text{iff} \quad w \models p^\circ.$$

Proof. By reflexivity, we have that $w \models p$ only if $w \models t \circ p$ iff $w \models p^\circ$.

Conversely, if $V(p)$ is a $\leq$ -upset, then $w \models t \circ p$ only if $w \models p$,

as the p -witness for $w \models t \circ p$ is $\leq$ -below w . □

Lemma 2.4. *For all Kripke models (W, R, N, V) with (W, R, N) a $\mathbf{Mod.B}^+$ -frame, $\|p^\circ\|$ is a $\leq$ -upset.*

Proof. If $w \models t \circ p$, then there is $w^- \leq w$ such that $w^- \models p$, so if also $w \leq w^+$, then we have $w^- \leq w^+$ by transitivity (cf. Proposition 2.1), whence $w^+ \models t \circ p$. □

From these lemmas it follows that the translation is full and faithful; see the appendix (2.5) for the proof.

Theorem 2.5. *For all formulas $\varphi \in \mathcal{L}_{\circ, t}^+$,*

$$\vdash_{\mathbf{B}} \varphi \quad \text{iff} \quad \vdash_{\mathbf{Mod.B}^+} t \supset \varphi^\circ.$$

What this translation may lack in simplicity or elegance, it may compensate for with greater mathematical naturality. If we let μ^+ denote the conjunction of the residuation and RM-frame axioms (i.e., the formulas axiomatizing $\mathbf{Mod.B}^+$), an appeal to the proof of Theorem 2.5 shows that for all $\Gamma \subseteq \mathcal{L}_{\circ, t}^+$,

$$\begin{aligned} \mathbf{RMFr}^+(\Gamma) &:= \{\mathfrak{F} \mid \mathfrak{F} \text{ is a Routley–Meyer frame validating each } \varphi \in \Gamma\} \\ &= \{\mathbb{F} \mid \mathbb{F} \text{ is a } \mathbf{Mod.B}^+ \text{ frame validating } t \supset \varphi^\circ \text{ for each } \varphi \in \Gamma\} \\ &= \{\mathbb{F} \mid \mathbb{F} \text{ is a Kripke frame validating } \mu^+ \text{ and } t \supset \varphi^\circ \text{ for each } \varphi \in \Gamma\} =: \mathbf{KFr}^+(\mu^+, t \supset \Gamma^\circ). \end{aligned}$$

Thus, for any class $\mathcal{C}$ of Routley–Meyer frames, the translation is full and faithful, in that for all formulas $\varphi \in \mathcal{L}_{\circ, t}^+$,

$$\mathcal{C} \Vdash \varphi \quad \text{iff} \quad \mathcal{C} \models t \supset \varphi^\circ.$$

It seems plausible that, and certainly interesting whether, this structural connection can be deepened to include frame-incomplete logics as well. It is also expected to facilitate significant transfer of results. We leave this as open questions and close by extending the translation to the language with negation.

¹⁴One might also consider adding further axioms to capture restrictions sometimes imposed on Routley–Meyer frames to obtain (typically) simpler semantics. An example is $t \circ p \supset p$, which is Sahlqvist with correspondent $N \cdot x \subseteq \{x\}$.

¹⁵For example, by observing that a normal modal logic contains $[(p \rightarrow q) \circ p] \supset q$ iff it contains $[(p \rightarrow \neg q) \circ p] \wedge q \supset \perp$, and it contains $p \supset [q \rightarrow (p \circ q)]$ iff it contains $\Diamond(q, \neg(p \circ q)) \wedge p \supset \perp$. So canonicity follows because $[(p \rightarrow \neg q) \circ p] \wedge q \supset \perp$ and $\Diamond(q, \neg(p \circ q)) \wedge p \supset \perp$ are Sahlqvist: $(p \rightarrow \neg q)$ and $\neg(p \circ q)$ are negative formulas; $\perp$ is vacuously positive; and p and q are vacuously boxed atoms.

2.2 Negation $\neg$

In this final section, we translate the relevant negation $\neg$. Proof-theoretically, the basic relevant logic **B** is axiomatized by adding the double-negation elimination scheme together with a rule for contraposition:¹⁶

$$\neg\neg\varphi \rightarrow \varphi \qquad \frac{\varphi \rightarrow \neg\psi}{\psi \rightarrow \neg\varphi} \text{ contraposition}$$

Model-theoretically, Routley–Meyer frames are expanded with a function $*$: $K \rightarrow K$ subject to

$$x \leq y \text{ only if } y^* \leq x^* \quad \text{and} \quad x^{**} = x,$$

so that frames become tuples $(K, R, N, *)$, with this clause for negation: $x \Vdash \neg\varphi$ iff $x^* \nVdash \varphi$. We translate by expanding the modal grammar with a unary modality $\Box$.¹⁷ Although we shall only be concerned with $\circ$, for each translation $\sharp$, we can set $(\neg\varphi)^\sharp := \Box\neg\varphi^\sharp$.

With an extra modality present, Kripke frames $(W, R, N, *)$ now come equipped with an additional binary relation $*$ $\subseteq W \times W$. The next proposition is useful for the purpose of translation.

Proposition 2.6 (Modal $*$ -correspondences). *Let $\mathbb{F}$ be a Kripke frame. Then*

$$\begin{aligned} \mathbb{F} \models \Box p \supset \neg\Box\neg p & \quad \text{iff} \quad \text{the relation } * \text{ is a function,} \\ \mathbb{F} \models p \supset \Box\Box p & \quad \text{iff} \quad * \text{ is an involution.} \end{aligned}$$

In case $\mathbb{F}$ validates the first of the two axioms above, then

$$\mathbb{F} \models \Box p \wedge (t \circ q) \supset t \circ (q \wedge \Box(t \circ p)) \quad \text{iff} \quad x \leq y \Rightarrow y^* \leq x^*.$$

In sum, a Kripke frame $\mathbb{F} = (W, R, N, *)$ is a Routley–Meyer frame iff it validates the axioms of Proposition 2.1 along with the ones listed here.

Proof. All three formulas are Sahlqvist; see the appendix for a proof of the last correspondent (2.6). $\square$

Definition 2.7. Let **Mod.B** (Modal **B**) denote the least normal modal logic containing **Mod.B**⁺ and the axioms of the preceding proposition.

Since **Mod.B**⁺ and the axioms of the preceding proposition are canonical, so is **Mod.B**, hence it is complete for the Kripke frames $(W, R, N, *)$ that are Routley–Meyer frames in the language with negation.

Theorem 2.8. *For all formulas φ :* $\vdash_{\mathbf{B}} \varphi$ iff $\vdash_{\mathbf{Mod.B}} t \supset \varphi^\circ$.

Proof. Same proof as that of Theorem 2.5. $\square$

Letting μ be the conjunction of μ^+ and the formulas of Proposition 2.6, an appeal to the proof strengthens the theorem to show that for any set of relevant formulas Γ ,

$$\begin{aligned} \text{RMFr}(\Gamma) &:= \{\mathfrak{F} \mid \mathfrak{F} \text{ is a Routley–Meyer frame validating each } \varphi \in \Gamma\} \\ &= \{\mathbb{F} \mid \mathbb{F} \text{ is a } \mathbf{Mod.B} \text{ frame validating } t \supset \varphi^\circ \text{ for each } \varphi \in \Gamma\} \\ &= \{\mathbb{F} \mid \mathbb{F} \text{ is a Kripke frame validating } \mu \text{ and } t \supset \varphi^\circ \text{ for each } \varphi \in \Gamma\} =: \text{KFr}(\mu, t \supset \Gamma^\circ). \end{aligned}$$

We end by concluding that for any class $\mathcal{C}$ of Routley–Meyer frames for the language with negation, the translation is full and faithful, in that for all relevant formulas φ ,

$$\mathcal{C} \Vdash \varphi \quad \text{iff} \quad \mathcal{C} \models t \supset \varphi^\circ.$$

¹⁶The weaker logic **BM** is sometimes taken as the fundamental relevant logic with negation (without negation, **B** and **BM** coincide). The results of this section readily extend to **BM** and its extensions. Details are left as an exercise.

¹⁷This makes literal the idea of negation as a modality (see, e.g., [4]). See also [22] where the relevant language is augmented by a *weak-assertion* operator that behaves like $\Box$.

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A Technical appendix

The first lemma gives the model transformation F and proves (4), while the subsequent lemma gives the model transformation G and proves (5).

Lemma A.1. *Let (W, R, V) be a Kripke model. For $0 \notin W$, define $R' := R \cup \{(0, x, x) \mid x \in W \cup \{0\}\}$. Then $(W \cup \{0\}, R', \{0\}, V)$ is a Routley–Meyer model and for all $x \in W$ and formulas α of the positive relevant language,*

$$x \Vdash \alpha \quad \text{iff} \quad x \models \alpha^*.$$

Proof. It is readily verified that $(W \cup \{0\}, R', \{0\}, V)$ is a Routley–Meyer model. The biimplication is proven by induction. The only non-trivial case is the inductive step for implication. There, we have the following for all $x \in W$:

$$\begin{aligned} x \Vdash \beta \rightarrow \gamma & \stackrel{(\text{def})}{\iff} \text{for all } y, z \in W \cup \{0\}: \text{if } R'xyz \text{ and } y \Vdash \beta, \text{ then } z \Vdash \gamma \\ & \stackrel{(i)}{\iff} \text{for all } y, z \in W: \text{if } Rxyz \text{ and } y \Vdash \beta, \text{ then } z \Vdash \gamma \\ & \stackrel{(\text{IH})}{\iff} \text{for all } y, z \in W: \text{if } Rxyz \text{ and } y \models \beta^*, \text{ then } z \models \gamma^* \\ & \stackrel{(\text{def})}{\iff} x \models \beta^* \rightarrow \gamma^* \\ & \stackrel{(\text{def})}{\iff} x \models (\beta \rightarrow \gamma)^*. \end{aligned}$$

Here, (i) holds because when $x \in W$, then $R'xyz$ implies $y, z \in W$ and $Rxyz$. □

Lemma A.2. *Let $(\mathfrak{F}, V)$ with $\mathfrak{F} = (K, R, N)$ be a Routley–Meyer model. Then (K, R, V) is a Kripke model and for all $x \in K$ and formulas α of the positive relevant language,*

$$x \Vdash \alpha \quad \text{iff} \quad x \models \alpha^*.$$

Proof. (K, R, V) is evidently a Kripke model. We prove the biimplication by induction on α . The base case is by definition, and the inductive steps for conjunction and disjunction are immediate by the induction hypothesis. Lastly, for the case where $\alpha = \beta \rightarrow \gamma$, we have the following:

$$\begin{aligned} x \Vdash \beta \rightarrow \gamma & \stackrel{(\text{def})}{\iff} \text{for all } y, z: \text{if } Rxyz \text{ and } y \Vdash \beta, \text{ then } z \Vdash \gamma \\ & \stackrel{(\text{IH})}{\iff} \text{for all } y, z: \text{if } Rxyz \text{ and } y \models \beta^*, \text{ then } z \models \gamma^* \\ & \stackrel{(\text{def})}{\iff} x \models \beta^* \rightarrow \gamma^* \\ & \stackrel{(\text{def})}{\iff} x \models (\beta \rightarrow \gamma)^*, \end{aligned}$$

which shows the required equivalence. □

The following two lemmas build on the preceding two lemmas, in order to establish (7) and (8) needed for Theorem 1.7.

Lemma A.3. *Let (W, R, V) be a Kripke model. For $0 \notin W$, define $R' := R \cup \{(0, x, x) \mid x \in W \cup \{0\}\}$. Then $(W \cup \{0\}, R', \{0\}, V)$ is a Routley–Meyer model and for all $x \in W$ and formulas α of the positive relevant language,*

$$0 \Vdash \alpha \quad \text{only if} \quad x \models \alpha^*.$$

Proof. The proof is by induction on α . The base case holds as $0 \notin V(p)$ for all propositional letters p . The inductive cases for $\wedge, \vee$ are immediate by the induction hypothesis. Lastly, for the case where $\alpha = \beta \rightarrow \gamma$, let $x \in W$ be arbitrary. Using Lemma A.1, we obtain the required as follows.

$$\begin{aligned}
0 \Vdash \beta \rightarrow \gamma & \stackrel{(\text{def})}{\Leftrightarrow} && \text{for all } y, z \in W \cup \{0\}: \text{ if } R'0yz \text{ and } y \Vdash \beta, \text{ then } z \Vdash \gamma \\
& \Leftrightarrow && \text{for all } z \in W \cup \{0\}: \text{ if } z \Vdash \beta, \text{ then } z \Vdash \gamma \\
& \Rightarrow && \text{for all } z \in W: \text{ if } z \Vdash \beta, \text{ then } z \Vdash \gamma \\
& \stackrel{A.1}{\Leftrightarrow} && \text{for all } z \in W: \text{ if } z \models \beta^*, \text{ then } z \models \gamma^* \\
& \Rightarrow && \text{for all } y, z \in W: \text{ if } Rxyz, \text{ then } z \models \beta^* \supset \gamma^* \\
& \Leftrightarrow && \text{for all } y, z \in W: \text{ if } Rxyz \text{ and } y \models \top, \text{ then } z \models \beta^* \supset \gamma^* \\
& \stackrel{(\text{def})}{\Leftrightarrow} && x \models \Box_{\rightarrow}(\beta^* \supset \gamma^*) \\
& \stackrel{(\text{def})}{\Leftrightarrow} && x \models (\beta \rightarrow \gamma)^*.
\end{aligned}$$

□

Lemma A.4. *Let $(\mathfrak{F}, V)$ with $\mathfrak{F} = (K, R, N)$ be a Routley–Meyer model. Then (K, R, V) is a Kripke model and for all $n \in N$ and formulas α of the positive relevant language,*

$$n \Vdash \alpha \quad \text{if} \quad n \models \alpha^*.$$

Proof. Once again, the proof is by induction on α , and only the implication case, $\alpha = \beta \rightarrow \gamma$, is non-trivial. In that case, we have the following:

$$\begin{aligned}
n \Vdash \beta \rightarrow \gamma & \stackrel{(\text{def})}{\Leftrightarrow} && \text{for all } y, z: \text{ if } Rnyz \text{ and } y \Vdash \beta, \text{ then } z \Vdash \gamma \\
& \stackrel{(i)}{\Leftrightarrow} && \text{for all } y, z: \text{ if } Rnyz \text{ and } z \Vdash \beta, \text{ then } z \Vdash \gamma \\
& \stackrel{A.2}{\Leftrightarrow} && \text{for all } y, z: \text{ if } Rnyz \text{ and } z \models \beta^*, \text{ then } z \models \gamma^* \\
& \Leftrightarrow && \text{for all } y, z: \text{ if } Rnyz, \text{ then } z \models \beta^* \supset \gamma^* \\
& \Leftrightarrow && \text{for all } y, z: \text{ if } Rnyz \text{ and } y \models \top, \text{ then } z \models \beta^* \supset \gamma^* \\
& \stackrel{(\text{def})}{\Leftrightarrow} && n \models \Box_{\rightarrow}(\beta^* \supset \gamma^*) \\
& \stackrel{(\text{def})}{\Leftrightarrow} && n \models (\beta \rightarrow \gamma)^*,
\end{aligned}$$

where (i) followed from persistence in Routley–Meyer models: $y \Vdash \beta$ and $y \leq z$ imply $z \Vdash \beta$. □

Next, we provide proof of representative items from Proposition 2.1. Recall that for a frame (W, R, N) , we write $x \leq y$ iff $\exists n \in N(Rnxy)$.

Proposition 2.1 (Restated). *Let $\mathbb{F} = (W, R, N)$ be a Kripke frame. Then*

$$\begin{array}{lll}
 \mathbb{F} \models p \supset t \circ p & \text{iff} & \leq \text{ is reflexive,} \\
 \mathbb{F} \models t \circ (t \circ p) \supset t \circ p & \text{iff} & \leq \text{ is transitive,} \\
 \mathbb{F} \models t \circ t \supset t & \text{iff} & N \text{ is an } \leq\text{-upset,} \\
 \mathbb{F} \models (t \circ p) \circ q \supset p \circ q & \text{iff} & Rxyz, x^- \leq x \Rightarrow Rx^-yz, \\
 \mathbb{F} \models p \circ (t \circ q) \supset p \circ q & \text{iff} & Rxyz, y^- \leq y \Rightarrow Rxy^-z, \\
 \mathbb{F} \models t \circ (p \circ q) \supset p \circ q & \text{iff} & Rxyz, z \leq z^+ \Rightarrow Rxyz^+.
 \end{array}$$

In sum, a Kripke frame $\mathbb{F} = (W, R, N)$ is a Routley–Meyer frame iff it validates the listed axioms.

Proof. We prove the second and the fourth item and leave the rest as an exercise.

First, suppose $\mathbb{F} \models t \circ (t \circ p) \supset t \circ p$ and $x \leq y \leq z$; i.e., that there are $m, n \in N$ s.t. $Rmxy$ and $Rnyz$. We are to find $l \in N$ such that $Rlxz$. Set $V(p) := \{x\}$. Then $z \models t \circ (t \circ p)$, so since $\mathbb{F} \models t \circ (t \circ p) \supset t \circ p$, we have $z \models t \circ p$, which precisely shows that there is $l \in N$ with $Rlxz$.

Conversely, assume $\leq$ is transitive, and let V, z be arbitrary with $z \models t \circ (t \circ p)$. We are to show that $z \models t \circ p$. Since $z \models t \circ (t \circ p)$, there are $m, n \in N$ and $x, y \in W$ with $Rmxy, Rnyz, x \models p$. Thus, $x \leq y \leq z$, so by transitivity $x \leq z$, hence there is $l \in N$ with $Rlxz$, as desired.

Second, suppose $\mathbb{F} \models (t \circ p) \circ q \supset p \circ q$ and $Rxyz, x^- \leq x$; i.e., there is $n \in N$ s.t. Rnx^-x . We are to show that Rx^-yz . Set $V(p) := \{x^-\}$, $V(q) := \{y\}$. Then $z \models (t \circ p) \circ q$, so $z \models p \circ q$, whence Rx^-yz , as required.

Conversely, assume that for all $x^-, x, y, z \in W$: $Rxyz$ and $x^- \leq x$ imply Rx^-yz . Let V, z be arbitrary with $z \models (t \circ p) \circ q$. We are to show that $z \models p \circ q$. Since $z \models (t \circ p) \circ q$, there are $n \in N$ and $x^-, x, y \in W$ such that $Rnx^-x, Rxyz, x^- \models p, y \models q$. But then Rx^-yz , hence $z \models p \circ q$. $\square$

The proofs of Theorem 2.5 and Proposition 2.6 follow next.

Theorem 2.5 (Restated). *For all formulas $\varphi \in \mathcal{L}_{o,t}^+$,*

$$\vdash_{\mathbf{B}} \varphi \quad \text{iff} \quad \vdash_{\mathbf{Mod.B}^+} t \supset \varphi^\circ.$$

Proof. For right-to-left, assume $\vdash_{\mathbf{Mod.B}^+} t \supset \varphi^\circ$ and let $(\mathfrak{F}, V)$ be an arbitrary Routley–Meyer model. Then $(\mathfrak{F}, V)$ is a $\mathbf{Mod.B}^+$ model as well, cf. Proposition 2.1. By Lemma 2.3, for all $x \in \mathfrak{F}$,

$$x \Vdash p \quad \stackrel{(\text{def})}{\text{iff}} \quad x \models p \quad \stackrel{2.3}{\text{iff}} \quad x \models p^\circ.$$

An induction readily extends it to all formulas $\alpha \in \mathcal{L}_{o,t}^+$; i.e., for all $x \in \mathfrak{F}$,

$$x \Vdash \alpha \quad \text{iff} \quad x \models \alpha^\circ.$$

Therefore, if n is a normal point, then $n \models t$, so by soundness for $\mathbf{Mod.B}^+$, we get $n \models \varphi^\circ$, hence $n \Vdash \varphi$, which shows that $\vdash_{\mathbf{B}} \varphi$ by completeness for $\mathbf{B}$.

Conversely, assume $\vdash_{\mathbf{B}} \varphi$ and let (W, R, N, V) be an arbitrary $\mathbf{Mod.B}^+$ model. By completeness of $\mathbf{Mod.B}^+$ for the frames of Proposition 2.1, it is enough to show that for all normal points $n \in N$, $n \models \varphi^\circ$. From Proposition 2.1 it also follows that (W, R, N) is a Routley–Meyer frame. We define a valuation V' over (W, R, N) by setting

$$V'(p) := \Vdash p^\circ \Vdash.$$

This yields a well-defined Routley–Meyer model by Lemma 2.4. So by assumption and soundness for **B**, for all normal points $n \in N$, $n \Vdash \varphi$. An easy induction then shows that for all formulas $\alpha \in \mathcal{L}_{o,t}^+$ and $x \in W$,

$$x \Vdash \alpha \quad \text{iff} \quad x \models \alpha^\circ,$$

which completes the proof. $\square$

Proposition 2.6 (Restated). *Let $\mathbb{F}$ be a Kripke frame. Then*

$$\begin{array}{ll} \mathbb{F} \models \Box p \supset \neg \Box \neg p & \text{iff} \quad \text{the relation } * \text{ is a function,} \\ \mathbb{F} \models p \supset \Box \Box p & \text{iff} \quad * \text{ is an involution.} \end{array}$$

In case $\mathbb{F}$ validates the first of the two axioms above, then

$$\mathbb{F} \models \Box p \wedge (t \circ q) \supset t \circ (q \wedge \Box(t \circ p)) \quad \text{iff} \quad x \leq y \Rightarrow y^* \leq x^*.$$

In sum, a Kripke frame $\mathbb{F} = (W, R, N, *)$ is a Routley–Meyer frame iff it validates the axioms of Proposition 2.1 along with the ones listed here.

Proof. The first two claims are readily proven; we show the third.

Assume $\mathbb{F} \models \Box p \wedge (t \circ q) \supset t \circ (q \wedge \Box(t \circ p))$ and $x \leq y$; i.e., that there is $m \in N$ s.t. $Rmxy$. We are to find $n \in N$ such that Rny^*x^* . Set $V(p) := \{y^*\}$, $V(q) := \{x\}$. Then $y \models \Box p \wedge (t \circ q)$, so since $\mathbb{F} \models \Box p \wedge (t \circ q) \supset t \circ (q \wedge \Box(t \circ p))$, we have $y \models t \circ (q \wedge \Box(t \circ p))$. Since $V(q) := \{x\}$, this entails that $x \models q \wedge \Box(t \circ p)$, whence $x^* \models t \circ p$. Thus there are $n \in N, z \in W$ such that $Rnzx^*$ and $z \models p$. But then $V(p) := \{y^*\}$ implies $z = y^*$, and hence Rny^*x^* , as desired.

Conversely, assume that for all $x, y \in W$: $x \leq y \Rightarrow y^* \leq x^*$. Let V, y be arbitrary with $y \models \Box p \wedge (t \circ q)$. Then $y^* \models p$ and there are $m \in N$ and $x \in W$ with $Rmxy, x \models q$. It suffices to show that $x \models \Box(t \circ p)$. By definition, $m \in N$ and $Rmxy$ mean that $x \leq y$, hence $y^* \leq x^*$, i.e. there is $n \in N$ with Rny^*x^* . So as $y^* \models p$, we have $x^* \models t \circ p$, which entails that $x \models \Box(t \circ p)$, as sufficed. $\square$

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Some Results on Causal Modalities in General Spacetimes

Marco Lewis

Nesta van der Schaaf

Université Paris-Saclay, CNRS, CentraleSupélec, ENS Paris-Saclay, Inria,
Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Causality is one of the fundamental structures of spacetimes, determining the possible behaviour and propagation of physical information. Causal structure can be analysed through the various modal logics it induces. The modal logics for the chronological and causal relations of the archetypal Minkowski spacetime have been classified. However, only partial results have been achieved for the strict variant of the causal relation, known as the *after relation*. Towards classification, it was shown by Shapirovsky and Shehtman that the after modality in Minkowski space satisfies a formula we call the ‘after formula’.

The present work continues this analysis towards arbitrary spacetimes. In particular, we prove that the after modality in any smooth spacetime satisfies the after formula. We introduce a related modal formula that demonstrates that the logic of two-dimensional spacetimes are more expressive than higher-dimensional ones. Lastly, we study the interrelation between the logical properties and physical properties along the causal ladder.

1 Introduction

There has been some research into the modal logic of *Minkowski space*, the spacetime of special relativity (Example 2.1). One of the early works is by Goldblatt [4], who showed that the modal logic of any $1 + n$ -dimensional Minkowski spacetime with the causal relation $\preceq$, moving up to and possibly at the speed of light, is **S4.2**. Shehtman [17] also observed similar results around that time for various **S4** logics. Shapirovsky and Shehtman [15] showed that the chronological relation $\ll$, moving strictly slower than the speed of light, again for any $1 + n$ -dimensional Minkowski spacetime, is given by the modal logic **OI.2**, which is a sublogic of **S4.2**. The general definitions of $\preceq, \ll$ are given in Section 2.

Recent works, such as [5, 6], have been investigating the temporal modal logic, using both past and future modalities, of the strict variant of the causal relation (what is referred to as the *after relation*; see Section 2). However, it has only been shown in two-dimensional Minkowski spacetime that the temporal logic of the after relation is decidable [6]. Whilst the frames of Minkowski spacetime with the chronological and causal relation are complete with respect to their logics, it is unknown what the modal logic of Minkowski spacetime with the after relation is, and whether it is complete, although previous works have noted it has distinct properties [4, 16]. Mainly, it was proved that the modal logic of the after relation satisfies a formula we call the *after formula* (Section 4.1).

To the best of our knowledge, the modal logics of general spacetimes has not been investigated.¹ What is the modal logic of a general spacetime? Some properties can be derived from the general differential geometric setup, such as that $\ll$ is transitive and 2-dense (**OI**) and $\preceq$ is a preorder (**S4**). One of the main difficulties in providing a more refined characterisation lies in the fact that, even though spacetimes locally behave like Minkowski space (locally inheriting the logic), general spacetimes can vary greatly in global topology and causal structure. Our main result (Theorem 4.6) proves that the after relation of *any* spacetime satisfies the after formula. Moreover, we study how the modal logics of the spacetime vary along where it sits on the so-called causal ladder (Sections 2.2 and 6).

¹Though there are related approaches such as via domain theory [10] or orthomodular lattices [2].

2 Spacetimes and Causal Structure

We recall the basic definitions of *causal structure* in smooth spacetimes [8, 13]. Formally, a *space-time* is a (real second countable Hausdorff) smooth manifold M equipped with a *Lorentzian metric* g and a *time orientation* T . The metric defines at each point $x \in M$ a nondegenerate symmetric bilinear form $g_x: T_x M \times T_x M \rightarrow \mathbb{R}$ on the tangent space, to be thought of as a generalised inner product. The difference is that g_x need not be positive definite, meaning non-zero vectors may have zero or even negative length. The Lorentzian signature is interpreted as one *time dimension* and n *spatial dimensions*. The time orientation T is a global smooth vector field that determines a consistent notion of future.

The fundamental *causal relations* are defined in terms of certain classes of curves induced by the metric (see Figure 1). For our purposes a *curve* is a continuous (piecewise) smooth function $\gamma: I \rightarrow M$ defined on an interval $I \subseteq \mathbb{R}$, and its *image* is the set $\text{im}(\gamma) = \{\gamma(t) : t \in I\}$. For $x, y \in M$ we define the *causal(ity)*, *chronology* and *horismos relation* as follows, respectively:

- $x \preceq y$ iff there exists a causal curve (*moving up to lightspeed*) from x to y , or $x = y$;
- $x \ll y$ iff there exists a timelike curve (*moving slower than lightspeed*) from x to y ;
- $x \rightarrow y$ iff there exists a lightlike curve (*moving at lightspeed*) from x to y , or $x = y$ (i.e., $x \preceq y$ and not $x \ll y$).

For more detailed definitions see [11, §1.11]. Additionally, we define the *after* (or *strictly causal*) and *reflexive chronological relation* as follows, respectively:

- $x \alpha y$ iff there exists a causal curve from x to y (read as “ y happens after x ”);
- $x \leq y$ iff there exists a timelike curve from x to y , or $x = y$.

The after relation, introduced in [14], is the ‘strict’ version of $\preceq$, since causal curves are by definition non-degenerate, only allowing a reflexive pair $x \alpha x$ when there exists a causal loop from x to x . We say x and y are *spacelike separated* if not $x \preceq y$ and not $y \preceq x$.

Example 2.1 (Minkowski spacetime). An important spacetime is $1 + n$ -dimensional *Minkowski space*, based on the manifold $\mathbb{R}^{1+n}$. We denote a point $x \in \mathbb{R}^{1+n}$ in components as $(x^0, \vec{x}) = (x^0, x^1, \dots, x^n)$. The metric and induced causal relations are then as follows (see Figure 1 for intuition):

$$\eta(x, y) = -x^0 y^0 + \sum_{i=1}^n x^i y^i, \quad \text{and} \quad \begin{array}{ll} x \ll y & \text{iff } y^0 > x^0 + \|\vec{y} - \vec{x}\|, \\ x \preceq y & \text{iff } y^0 \geq x^0 + \|\vec{y} - \vec{x}\|, \\ x \rightarrow y & \text{iff } y^0 = x^0 + \|\vec{y} - \vec{x}\|. \end{array} \quad (1)$$

2.1 Properties of the causal relations

A central theme of this work is characterising the (modal) properties of the causal relations of a space-time. We first collect some well-established results from the literature [7, 13].

Definition 2.2. A relation $\triangleleft \subseteq S \times S$ is called *semi-full* if:

- (seriality) $\forall x \in S$, there exists $y \in S$ such that $x \triangleleft y$;
- (2-dense) if $x \triangleleft y_1, y_2$, there exists $z \in S$ such that $x \triangleleft z \triangleleft y_1, y_2$.

Lemma 2.3. We have the following relations and properties of the causal orders:

- $x \ll y \implies x \alpha y \implies x \preceq y$;
- if $x \neq y$ then $x \alpha y$ iff $x \ll y$ or $x \rightarrow y$;
- $\ll$ is semi-full, α is serial;
- $\ll, \leq, \alpha, \preceq$ are transitive;
- $\preceq, \leq, \rightarrow$ are reflexive.

Proposition 2.4 (Push-up Rule). *We have that for $x, y, z \in M$:*

$$x \ll y \preceq z \implies x \ll z \quad \text{and} \quad x \preceq y \ll z \implies x \ll z.$$

2.2 The Causal Ladder

The causal ladder [11] provides a classification of spacetimes based on the properties of their causal relations. A basic version of it, sufficient for the present work, is depicted in Figure 2 (with the weakest property at the top). Below we define the properties needed in this work (see Section 6); for more disambiguated versions see *e.g.*, [11, Figure 8]. Let $J^\pm$ and $I^\pm$ denote the up/down closures of $\preceq$ and $\ll$, respectively. So $J^+(x) := \{y \in M : x \preceq y\}$ and $J^-(x) := \{z \in M : z \preceq x\}$, and similarly for $I^\pm$.

Definition 2.5. A spacetime is called:

- *totally vicious* if $x \ll x$ for all $x \in M$ (*i.e.*, $\ll$ is reflexive). A spacetime is *non-totally vicious* (NTV) if there is some $x \in M$ such that not $x \ll x$ (*i.e.*, $\ll$ is not reflexive);
- *chronological* if not $x \ll x$ for all $x \in M$ (*i.e.*, $\ll$ is irreflexive);
- *causal* if $x \preceq y$ and $y \preceq x$ imply $x = y$ (*i.e.*, $\preceq$ is anti-symmetric);
- *past-distinguishing* (resp. *future-distinguishing*) if $I^-(x) = I^-(y)$ (resp. $I^+(x) = I^+(y)$) implies $x = y$. If a spacetime is both past- and future-distinguishing it is called *distinguishing*;
- *globally hyperbolic* if the *causal diamonds* $J^+(x) \cap J^-(y)$ are compact in the manifold topology for all $x, y \in M$.

For proofs of implication between the different properties, see [11, 12]. We note that the implications are strict, *e.g.*, there are chronological spacetimes that are not causal. The study of the causal ladder and the underlying dependencies is one of the cornerstones of modern mathematical relativity theory.

We provide two examples of spacetime that reside on different levels of the causal ladder, but we will see other examples throughout the paper.

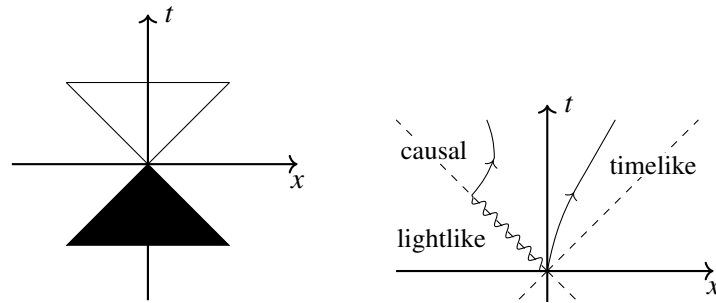


Figure 1: Two-dimensional Minkowski spacetime and its causal structure. Timelike curves move strictly inside of the cone, with tangent strictly greater than 45° from the horizontal axis at every point. Lightlike curves move precisely at 45° , while causal curves are a mixture of the two.

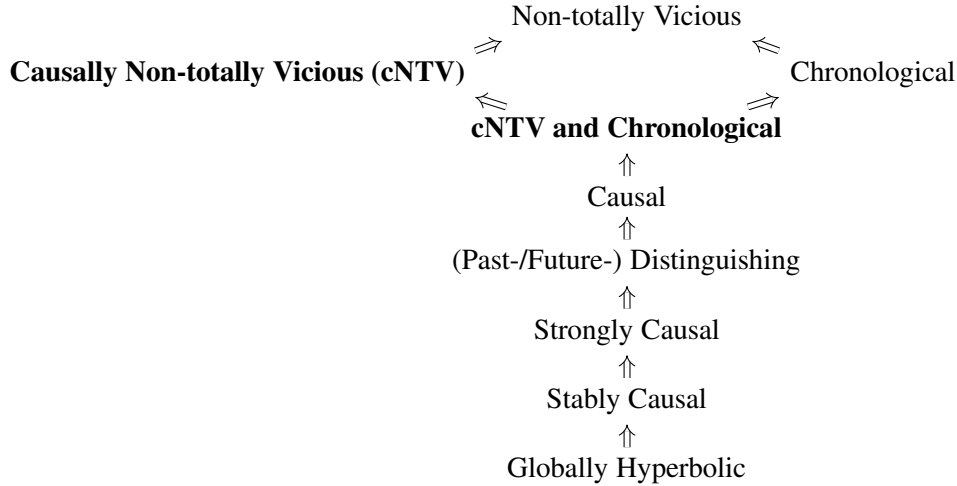


Figure 2: Modified Causal Ladder (our additions in bold).

Example 2.6. Minkowski space is globally hyperbolic.

Example 2.7. The spacetime depicted in Figure 3b is a non-totally vicious spacetime. The manifold is $\mathbb{R} \times S^1$, the surface of an infinitely long unit cylinder, and the lightcones are tilted from 90° to being upright at 0° . This is known as the *Misner spacetime*, see [11, Example 4.28] for details. It is non-totally vicious because points on the dotted line, $(0, \theta)$, cannot chronologically reach themselves (even though causally they can).

2.3 Causally Non-Totally Vicious: A New Property for the Causal Ladder

Causal ladder properties can have multiple and equivalent definitions. In particular, the causal step has an interesting equivalent condition.

Lemma 2.8. *Let M be a spacetime. Then M is causal, i.e., $\preceq$ is anti-symmetric, iff α is irreflexive.*

Both the chronological and causal properties on the causal ladder are affecting the spacetime in a similar way, but affect different relations. With the chronological relation, $\ll$, there is a difference between losing reflexivity (being non-totally vicious) and gaining irreflexivity (being chronological). However, the causal property in the ladder combines the loss of reflexivity and gaining of irreflexivity on α into one ladder property. By changing the perspective of the causal property affecting α , we can introduce a class of spacetimes that separates the notion of losing reflexivity and gaining irreflexivity. Thus, we get a causal analogue of non-total vicious.

Definition 2.9. A spacetime is *causally non-totally vicious (cNTV)* iff $\exists x \in M$ such that not $x \alpha x$.

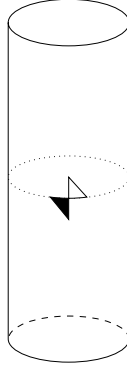
The cNTV property makes α lose reflexivity and now the causal property only gives α the irreflexive property, separating the relational properties into individual steps on the causal ladder. There is an obvious question to ask: where does cNTV sit on the causal ladder?

Proposition 2.10. *A causal spacetime is cNTV, and a cNTV spacetime is NTV.*

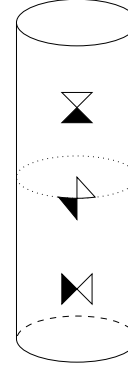
The proof of these implications is simply based on the definitions of the properties and is obvious.

Then, what is the relation between a spacetime being cNTV and it being chronological?

Theorem 2.11. *We have that*



(a) A chronological (non-causal) spacetime that is not cNTV. Every point acts lightlike on a slice of the cylinder allowing every point to (causally) loop around to itself, but a point cannot reach itself in a timelike way.



(b) A cNTV (non-causal) spacetime that is not chronological [11, Fig. 10]. The points above the boundary are irreflexive in α but the points below the boundary are reflexive in α and $\ll$.

Figure 3: Spacetimes for Lemma 2.11.

- (i) *there are chronological spacetimes that are not cNTV (Figure 3a);*
- (ii) *there are cNTV spacetimes that are not chronological (Figure 3b).*

Proof. The proof of both can be observed in the spacetimes depicted in Figure 3, where each has one property but not the other. Both spacetimes use the same manifold, $\mathbb{R} \times S^1$. The metric of Figure 3a is just quotiented Minkowski space, and Figure 3b is Misner spacetime from Example 2.7. $\square$

Remark 2.12. It is clear that a causal spacetime is cNTV and chronological, but we further have that a spacetime that is cNTV and chronological is not necessarily causal. A counterexample can be observed by taking the spacetime in Figure 3a and removing a point from it.

The cNTV property changes one end of the causal ladder to look different to the original ladder. Our additions to the causal ladder in Figure 2 are in bold. In Section 6, we will see how introducing the cNTV property into the causal ladder will make a clearer distinction of the modal logic of a spacetime to its position on the causal ladder.

3 Modal Logic

Now we introduce modal logic, which includes the standard logical connectives (conjunction, disjunction, *etc.*), atomic propositions, and modal operators ($\Box, \Diamond$). Throughout, we assume a finite set of atomic propositions, $AP = \{p_1, p_2, \dots\}$. We refer to [1, 19] for common definitions and theorems. Throughout, we make use of Kripke frames, $\langle M, \triangleleft \rangle$, using M to denote the set of points/worlds and $\triangleleft$ to denote the relation. For models we use $V : AP \rightarrow 2^M$ to denote the evaluation function. For a relation, $\triangleleft$, let $\triangleleft^{-1} = \{(y, x) : (x, y) \in \triangleleft\}$. For $x \in M$, let $\triangleleft(x) = \{y \in M : x \triangleleft y\}$; and for $S \subseteq M$, let $\triangleleft(S) = \bigcup_{x \in S} \triangleleft(x)$. We use the standard grammar of modal logic and entailment for a model $(\langle M, \triangleleft, V \rangle, x \models \phi)$.

We denote the Necessitation rule ($\frac{A}{\Box A}$) as (Nec) and we have the modal formula $K := \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ that holds in all frames. Other common modal formula are defined along with their corre-

sponding properties of frames:

$$\begin{aligned} a4 &:= \Diamond\Diamond A \rightarrow \Diamond A, \text{ transitivity}; & aT &:= A \rightarrow \Diamond A, \text{ reflexivity}; \\ aD &:= \Diamond \top, \text{ seriality}; & ad &:= \Diamond A \rightarrow \Diamond\Diamond A, \text{ density}; \\ ad_2 &:= \Diamond A \wedge \Diamond B \rightarrow \Diamond(\Diamond A \wedge \Diamond B), \text{ 2-density}; & a2 &:= \Diamond\Box A \rightarrow \Box\Diamond A, \text{ confluence.}^2 \end{aligned}$$

The smallest modal logic is **K**, which contains the standard propositional logic axioms, *modus ponens*, *K*, and (Nec). A normal modal logic is the smallest modal logic that consists of **K**; any extra (valid) formulae $a_1, a_2, \dots$; and any formula that can be obtained from applying the rules in **K** on the formulae given. The respective logic is denoted $\mathbf{K} + a_1 + a_2 + \dots$.

We denote some normal modal logics derived from the formulas given above:

$$\begin{aligned} \mathbf{K4} &:= \mathbf{K} + a4, & \mathbf{T} &:= \mathbf{K} + aT, \\ \mathbf{D4} &:= \mathbf{K4} + aD, & \mathbf{S4} &:= \mathbf{K4} + aT, \\ \mathbf{OI} &:= \mathbf{D4} + ad_2, & \mathbf{\Lambda.2} &:= \mathbf{\Lambda} + a2; \end{aligned}$$

where $\mathbf{\Lambda}$ is an arbitrary normal modal logic. Each of these has a combination of the properties of the corresponding modal formula.

As standard, for a set of formulae, Γ , and a formula, ϕ , let $\Gamma \vdash \phi$ mean that ϕ can be logically derived from Γ ($\phi \in \mathbf{K} + \Gamma$) and $\Gamma \not\vdash \phi$ iff $\Gamma \vdash \neg\phi$. For a frame, $F = \langle M, \triangleleft \rangle$, we write $\mathcal{L}(F)$ to mean the set of formula that are satisfied under all possible evaluations of F . This set of formula could be some logic, $\mathbf{\Lambda}$, and we may write $\mathcal{L}(F) = \mathbf{\Lambda}$.

We introduce Gabbay's Irreflexive Rule [3]. The rule is

$$(\text{Gabb}) \frac{\neg(p \rightarrow \Diamond p) \rightarrow \phi}{\phi} \text{ if } p \text{ does not occur in } \phi.$$

For a normal modal logic $\mathbf{\Lambda}$ without reflexivity (aT), let $\mathbf{\Lambda} + (\text{Gabb})$ be the smallest modal logic of $\mathbf{\Lambda}$ with the addition of being closed under (Gabb). A frame whose logic is $\mathbf{\Lambda} + (\text{Gabb})$ is irreflexive.

4 Spacetime and Modal Logic

We begin by looking at modal logic within the context of a generic spacetime. No assumptions are made on where the spacetime lies on the causal ladder. We analyse the normal modal logics that act as a subset for the modal logic of spacetimes.

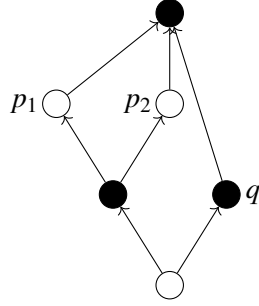
Given the definitions of the various causal relations, the relational properties they have (known from Lemma 2.3), and the properties on frames that normal modal logics describe; we have the following results on all spacetimes.

Corollary 4.1. *Let M be a spacetime. Then, we have that*

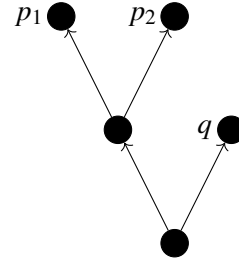
- (i) $\mathbf{S4} \subseteq \mathcal{L}(\langle M, \leq \rangle), \mathcal{L}(\langle M, \preceq \rangle);$
- (ii) $\mathbf{OI} \subseteq \mathcal{L}(\langle M, \ll \rangle);$
- (iii) $\mathbf{D4} + ad \subseteq \mathcal{L}(\langle M, \alpha \rangle).$

This can be seen simply from the fact that the normal modal logics yield frames that have the associated properties and the causal relations have those properties. For instance, **S4** yields frames that are transitive and reflexive, and $\preceq$ is a transitive and reflexive relation.

²Confluence is also referred to as the Church-Rosser property, the diamond property, or being weakly directed.



(a) A frame that is transitive, serial, dense, and confluent (**D4.2** + *ad*).



(b) A reflexive frame where the relation is non-transitive (**T**).

Figure 4: Counter models to $\alpha\alpha f$ in specific normal modal logics.

4.1 The After Formula

In [16], it was shown that the formula, which we refer to as the *after formula*,

$$\alpha\alpha f := \Diamond(\Diamond(p_1 \wedge \neg p_2 \wedge \Box \neg p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \Box \neg p_1)) \wedge \Diamond q \rightarrow \Diamond(\Diamond p_1 \wedge \Diamond q) \vee \Diamond(\Diamond p_2 \wedge \Diamond q),$$

holds in the modal logic of any $1 + n$ -dimensional Minkowski spacetime with the α relation. This formula corresponds to the first-order property:

$$\begin{aligned} \forall x, y, y_1, y_2, z \Big[& (x \triangleleft y \wedge y \triangleleft y_1 \wedge y \triangleleft y_2 \wedge x \triangleleft z \wedge y_1 \neq y_2 \wedge \neg(y_1 \triangleleft y_2) \wedge \neg(y_2 \triangleleft y_1)) \\ & \rightarrow \exists t (x \triangleleft t \wedge t \triangleleft z \wedge (t \triangleleft y_1 \vee t \triangleleft y_2)) \Big].^3 \end{aligned} \quad (2)$$

Observe that $\alpha\alpha f$ is a modified version of 3-density: if two (unrelated) points are related by a common point from the root and there is a third point that can be reached from the root, then at least one of the two points and the third point must share a common point from the root. In fact, $\alpha\alpha f$ is a Sahlqvist formula (see [1]), which provides properties such as canonicity, first-order correspondence (as already demonstrated in Equation (2)), etc.

We make the following observation of $\alpha\alpha f$ within the scope of normal modal logics:

Lemma 4.2. *We have (i) $\mathbf{D4.2} + ad \not\models \alpha\alpha f$; (ii) $\mathbf{T} \not\models \alpha\alpha f$; (iii) $\mathbf{K4} + ad_2 \vdash \alpha\alpha f$.*

Proof. Claims (i) and (ii) are proven by models that act as counterexamples given in Figures 4a and 4b respectively.

For (iii), let $\mathcal{M} = \langle M, \triangleleft, V \rangle$ be a $\mathbf{K4} + ad_2$ model and let $x \in M$. If we have that

$$\mathcal{M}, x \models \Diamond(\Diamond(p_1 \wedge \neg p_2 \wedge \Box \neg p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \Box \neg p_1)) \wedge \Diamond q;$$

then $\mathcal{M}, x \models \Diamond p_1$ (by transitivity and properties of $\wedge$). Since we have $\mathcal{M}, x \models \Diamond p_1 \wedge \Diamond q$, then by 2-density we must have that $\mathcal{M}, x \models \Diamond(\Diamond p_1 \wedge \Diamond q)$ and therefore $\mathcal{M}, x \models \alpha\alpha f$. Since $x, \mathcal{M}$ are generic, then $\mathbf{K4} + ad_2 \vdash \alpha\alpha f$. $\square$

³Note $y_1 \neq y_2 \wedge \neg(y_1 \triangleleft y_2) \wedge \neg(y_2 \triangleleft y_1) \equiv y_1 \triangleleft^\infty y_2$ in [16].

Note that reflexivity implies 2-density. Additionally, since $\mathbf{D4.2} + ad \not\models a\alpha f$, then any sub-logics of $\mathbf{D4.2} + ad$ ($\mathbf{D4}$, $\mathbf{K4}$, ...) do not contain $a\alpha f$. Conversely, any logics that contain $\mathbf{K4} + ad_2$ as a sub-logic, such as $\mathbf{S4}$, contain $a\alpha f$. Therefore, since $a\alpha f$ holds in transitive and 2-dense (or reflexive) frames, then it holds in frames that use a spacetime and one of $\ll$, $\leq$ or $\preceq$.

Let $\mathbf{D4da} := \mathbf{D4} + ad + a\alpha f$.⁴ Since ad and $a\alpha f$ are Sahlqvist formula, then by the Sahlqvist Theorem [1] canonicity is achieved for the formulas and $\mathbf{D4da}$.

Corollary 4.3. *The logic $\mathbf{D4da}$ is canonical.*

4.2 After Formula in Spacetime

As mentioned previously, it was shown in [16] that the after formula is satisfied in the after modality of any Minkowski space.

Proposition 4.4 ([16]). *Consider 1 + n-dimensional Minkowski space $\mathbb{R}^{1+n}$ with its after relation α . Then*

$$\langle \mathbb{R}^{1+n}, \alpha \rangle \models a\alpha f.$$

Now we generalise this result, showing that $a\alpha f$ holds in *any* spacetime. The main technical step towards proving this is to show the following property: if $x \alpha y \alpha y_1, y_2$ and $y_1 \neq y_2$ are spacelike separated, then $x \ll y_i$ for some i . First we give an elementary proof in Minkowski space (Lemma A.3 and Corollary A.4), which is then lifted to arbitrary spacetimes using geodesic normal coordinates (Theorem A.1). The key intuition is that in the extreme scenario that $x \alpha y \alpha y_j$ represents a lightline, the fact that y_i is spacelike separated from y_j means that it must necessarily move away from this lightline, and so must be in the interior cone of x . This behaviour lifts from Minkowski space to general spacetimes.

Lemma 4.5. *In any spacetime, if $x \alpha y \alpha y_1, y_2$ and $y_1 \neq y_2$ are spacelike separated, then $x \ll y_i$ for some i .*

Note that the proof (given in Appendix A.2) still works if $x = y$, in which case from $x \alpha x$ we get a causal curve from x to itself, which is all that is needed for the proof. A visualisation of the proof of Lemma 4.5 is in Figure 5.

Theorem 4.6. *Let M be a spacetime and α its after relation. Then*

$$\langle M, \alpha \rangle \models a\alpha f.$$

Proof. Take $x, y, y_1, y_2, z \in M$ such that $x \alpha y, y_1, y_2, z$ and $y \alpha y_1, y_2$, where $y_1 \neq y_2$ are unrelated by α . This implies y_1 and y_2 are spacelike separated. We need to find $t \in M$ such that $x \alpha t \alpha y_i, z$, for some i . First, by Lemma 4.5 we can assume without loss of generality that $x \ll y_1$. In the case $x = z$ the result is trivial (pick $t = x$), so assume $x \neq z$. Thus we have a case distinction of whether $x \ll z$ or $x \rightarrow z$ (Lemma 2.3). In the first case, the existence of $t \in M \setminus \{x\}$ with $x \ll t \ll y_1, z$ follows immediately from 2-density of $\ll$.

Instead, suppose that $x \rightarrow z$. Let γ be the timelike curve connecting $x \ll y_1$ and let δ be the lightcurve connecting $x \rightarrow z$. Take a point $p \in \text{im}(\gamma) \setminus \{x\}$ so that $x \ll p \ll y_1$. Then $x \in I^-(p)$, and since chronological cones are open there must exist a point $t \in I^-(p) \cap \text{im}(\delta) \setminus \{x\}$, which hence satisfies $x \alpha t \alpha y_1$ and $x \alpha t \alpha z$, as desired. $\square$

Finally, we build on Corollary 4.1 using Theorem 4.6 and Lemma 4.2.

⁴In [16], the logic $\mathbf{La}_0 = \mathbf{D4da.2} = \mathbf{D4.2} + ad + a\alpha f$ is used for an analysis on Minkowski spacetime.

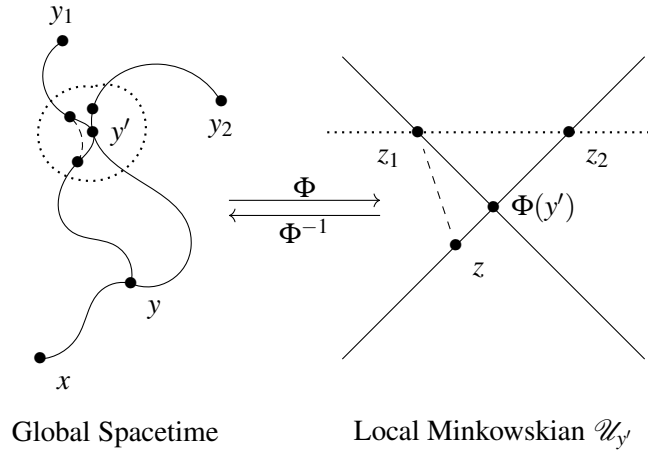


Figure 5: Visualisation of the proof for Lemma 4.5 using two-dimensional Minkowski spacetime. Dashed (solid) lines represent chronological (causal) curves. The dotted region on the left denotes $U_{y'}$, and the dotted line on the right depicts the spacelike hypersurface defined by $x^0 = \varepsilon$.

Corollary 4.7. *For any spacetime M we have that $\mathbf{D4da} \subseteq \mathcal{L}(\langle M, \alpha \rangle)$.*

What this section shows is that there is a connection between the push-up rule, which naturally occurs in spacetimes, and $\alpha\alpha f$. Many of the proofs that we have used for $\alpha\alpha f$ are reliant on the push-up rule. An exploration of the push-up rule and its link to Kripke frames (possibly with multiple relations) is warranted, but left to future work.

5 Two-dimensional Spacetimes

It remains an open problem as to what the modal logic of Minkowski spacetime with α is, but it is known that the logic of two-dimensional Minkowski spacetime is unique to any $1+n$ -dimensional Minkowski spacetime ($n > 2$) [16].

5.1 After Formula for Two Dimensions

Consider the formula,

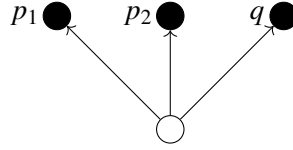
$$\alpha\alpha_2f := \Diamond(p_1 \wedge \neg p_2 \wedge \Box \neg p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \Box \neg p_1) \wedge \Diamond q \rightarrow \Diamond(\Diamond p_1 \wedge \Diamond q) \vee \Diamond(\Diamond p_2 \wedge \Diamond q),$$

which is $\alpha\alpha f$ without the first diamond.⁵ It corresponds to the following first order property:

$$\begin{aligned} \forall x, y_1, y_2, z \Big[& [x \triangleleft y_1 \wedge x \triangleleft y_2 \wedge x \triangleleft z \wedge y_1 \neq y_2 \wedge \neg(y_1 \triangleleft y_2) \wedge \neg(y_2 \triangleleft y_1)] \\ & \rightarrow \exists t (x \triangleleft t \wedge t \triangleleft z \wedge (t \triangleleft y_1 \vee t \triangleleft y_2)) \Big]. \end{aligned}$$

This formula is a Sahlqvist formula and, therefore, any canonical normal modal logic that adds $\alpha\alpha_2f$ is canonical. Even though $\alpha\alpha_2f$ is a simple modification to $\alpha\alpha f$, we can show that it is strictly more expressive within the context of spacetimes.

⁵Read as the after-2 formula/axiom.

Figure 6: A finite **D4da** model that does not have $a\alpha_2f$ on the root.

Theorem 5.1. *We have that (i) $\mathbf{K4} + a\alpha_2f \vdash a\alpha f$; (ii) $\mathbf{D4da} \not\vdash a\alpha_2f$.*

Proof. (i): Let $\mathcal{M} = \langle M, \triangleleft, V \rangle$ be a $\mathbf{K4} + a\alpha_2f$ model and $x \in M$. Assume that $\mathcal{M}, x \models \Diamond(\Diamond(p_1 \wedge \neg p_2 \wedge \neg \Box p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \neg \Box p_1)) \wedge \Diamond q$. Let $x \triangleleft y$ such that $\mathcal{M}, y \models \Diamond(p_1 \wedge \neg p_2 \wedge \neg \Box p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \neg \Box p_1)$, then $\mathcal{M}, x \models \Diamond\Diamond(p_1 \wedge \neg p_2 \wedge \neg \Box p_2) \wedge \Diamond\Diamond(p_2 \wedge \neg p_1 \wedge \neg \Box p_1)$.

Since $\mathcal{M}$ is transitive, then we have $\mathcal{M}, x \models \Diamond(p_1 \wedge \neg p_2 \wedge \neg \Box p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \neg \Box p_1) \wedge \Diamond q$. As $\mathcal{M}$ is a $a\alpha_2f$ model, we have $\mathcal{M}, x \models \Diamond(\Diamond p_1 \wedge \Diamond q) \vee \Diamond(\Diamond p_2 \wedge \Diamond q)$ and therefore $\mathcal{M}, x \models a\alpha f$. Because x and $\mathcal{M}$ are general, then $\mathbf{K4} + a\alpha_2f \vdash a\alpha f$.

(ii): A counterexample is provided by the finite model given in Figure 6. The frame has all the properties of **D4da**, but cannot express $a\alpha_2f$. One of the points p_1 or p_2 must share a common point with q after the root but they do not. $\square$

Remark 5.2. For finite (spacetime) frames, $a\alpha_2f$ says that if there are successor clusters from an irreflexive point, there are at most two of them each possibly with their own unique chains of clusters (light lines).

5.2 Separability of Two-Dimensional Spacetimes

With the after relation, we can see that $a\alpha_2f$ holds in two-dimensional Minkowski spacetime but does not hold in any other $1 + n$ -dimensional Minkowski spacetime.

Lemma 5.3. *Let $M = \mathbb{R}^{1+n}$ and α be the after relation on Minkowski spacetime. We have that*

- (i) $\langle \mathbb{R}^2, \alpha \rangle \models a\alpha_2f$
- (ii) $\langle \mathbb{R}^{1+n}, \alpha \rangle \not\models a\alpha_2f$ for $n \geq 2$

Proof. The argument is similar to [16, Lemma 6.2]. $\square$

It turns out that $a\alpha_2f$ holds in any two-dimensional spacetime.

Theorem 5.4. *Let M be a two-dimensional spacetime. Then,*

$$\langle M, \alpha \rangle \models a\alpha_2f.$$

Denote $\mathbf{D4da}_2 := \mathbf{D4} + ad + a\alpha_2f$. It is clear from Theorem 5.1 that $\mathbf{D4da} \subset \mathbf{D4da}_2$.

Corollary 5.5. *Let M be a 2-dimensional spacetime. Then $\mathbf{D4da}_2 \subseteq \mathcal{L}(\langle M, \alpha \rangle)$.*

In Lemma 5.3, we were shown how $a\alpha_2f$ only holds in two dimensions, providing a separation from $1 + n$ -dimensional Minkowski spacetime. However, to show a full separation of the logics of 2-dimensional spacetimes from higher dimensional ones, it needs to be shown that $a\alpha_2f$ does not hold in higher dimensional spacetimes. We note that non-cNTV spacetimes are reflexive in α , meaning that $a\alpha_2f$ holds in those spacetimes. However, for cNTV spacetimes of dimension greater than 3, it is likely that $a\alpha_2f$ does not hold. Proving this requires that any high dimensional, cNTV spacetime has at least 3 different light lines from a point irreflexive in α .

We end this section by forming the following conjecture.

Conjecture 5.6. *Let $n \geq 3$. For any cNTV n -dimensional spacetime, M , we have that $\langle M, \alpha \rangle \not\models a\alpha_2f$ ($\mathbf{D4da}_2 \not\subseteq \mathcal{L}(\langle M, \alpha \rangle)$).*

6 Effects of Causal Ladder Properties on Modal Logics

We now analyse the modal logic of spacetimes on different levels of the causal ladder. The proofs are fairly obvious throughout, simply relying on the properties of the relevant causal relation.

We have the following obvious result.

Corollary 6.1. *Let M be a spacetime and $\ll$ and α be the respective causal relations, then:*

- (i) *if $\ll$ is reflexive, $\mathbf{S4} \subseteq \mathcal{L}(\langle M, \ll \rangle)$;*
- (ii) *if α is reflexive, $\mathbf{S4} \subseteq \mathcal{L}(\langle M, \alpha \rangle)$.*

6.1 Totally and Non-totally Vicious

Totally vicious spacetimes are reflexive in $\ll$ and, consequently, in α . Note that $\ll$ and $\leq$ become indistinguishable from one another ($x \ll y$ iff $x \leq y$), and similarly for α and $\preceq$. What becomes more apparent is that $\ll$ and $\preceq$ are indistinguishable.

Lemma 6.2. *Let M be a totally vicious spacetime and $x, y \in M$. Then $x \ll y$ iff $x \preceq y$.*

Proof. If $x \ll y$, there is a timelike path from x to y , which is also a causal path. Therefore, $x \preceq y$. If $x \preceq y$, then we have that $x \ll y$ since $y \ll y$ and the push-up rule is used to obtain the result. $\square$

This leads to the following result.

Theorem 6.3. *Let M be a totally vicious spacetime. Then,*

$$\mathbf{S4} \subseteq \mathcal{L}(\langle M, \ll \rangle) = \mathcal{L}(\langle M, \leq \rangle) = \mathcal{L}(\langle M, \alpha \rangle) = \mathcal{L}(\langle M, \preceq \rangle).$$

For non-totally vicious spacetimes, the logics for $\ll$ are restricted since reflexivity is lost.

Theorem 6.4. *If M is a non-totally vicious spacetime, then $\mathbf{S4} \not\subseteq \mathcal{L}(\langle M, \ll \rangle)$.*

Note that α and $\preceq$ are not affected by a spacetime being NTV (α is still reflexive), and therefore their modal logics are not affected either.

Corollary 6.5. *If M is NTV and not cNTV, then $\mathbf{S4} \subseteq \mathcal{L}(\langle M, \alpha \rangle) = \mathcal{L}(\langle M, \preceq \rangle)$.*

6.2 Chronological

Irreflexivity of $\ll$ means that the normal modal logic does not fundamentally change from a non-totally vicious spacetime. However, by considering (Gabb), we obtain the following.

Theorem 6.6. *If M is a chronological spacetime, then $\mathbf{OI} + (\text{Gabb}) \subseteq \mathcal{L}(\langle M, \ll \rangle)$.*

Note how the logic of α or the reflexive causal relations have not changed at the levels of non-totally vicious and chronological from a totally vicious spacetime.

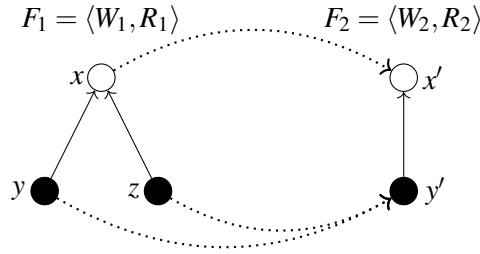


Figure 7: A bisimulation from a past-distinguishing frame to one that is not.

6.3 Causally Non-Totally Vicious (cNTV)

Having introduced the notion of cNTV in Section 2.3, we now see the results of separating the causal condition in the modal logic of α .

Theorem 6.7. *If M is a causally non-totally vicious spacetime, then $\mathbf{S4} \not\subseteq \mathcal{L}(\langle M, \alpha \rangle)$.*

When considering a spacetime that is both chronological and cNTV, the two respective theorems are used together to reveal that both $\ll$ and α have different modal logics from a NTV spacetime.

6.4 Causal

A causal spacetime, in a similar way to a chronological spacetime, does not change the normal modal logics of its causal relations, but the logic under α can be changed if (Gabb) is considered.

Theorem 6.8. *If M is a causal spacetime, then $\mathbf{D4da} + (\text{Gabb}) \subseteq \mathcal{L}(\langle M, \alpha \rangle)$.*

6.5 Distinguishing

We begin first by extending the notion of a spacetime being distinguishing into the Kripke frame setting.

Definition 6.9. Let $F = \langle M, \triangleleft \rangle$. We say F is future-distinguishing if for all $x, y \in M$ whenever $\triangleleft(x) = \triangleleft(y)$, then $x = y$. We say F is past-distinguishing if for all $x, y \in M$ whenever $\triangleleft^{-1}(x) = \triangleleft^{-1}(y)$, then $x = y$. Finally, F is distinguishing if it is both past- and future-distinguishing.

Thus, the spacetime definition of distinguishing is preserved when considering the frame of a spacetime with $\ll$ (since $I^\pm(x) = \triangleleft^{\pm 1}(x)$). Our goal is to find some normal modal logic that can capture distinguishing spacetimes. However, despite being able to maintain definitions across domains, it becomes clear that a past-distinguishing frame has no associated modal formula. The proofs of the following results are by bisimulation, see [1] for more information.

Theorem 6.10. *There is no modal formula, ϕ , such that if $\langle M, \triangleleft \rangle \models \phi$ then $\langle M, \triangleleft \rangle$ is past-distinguishing.*

Proof. The frames in Figure 7 are bisimilar and so act as a counterexample. $\square$

Similarly, future-distinguishing frames have no associated modal formula.

Theorem 6.11. *There is no modal formula, ϕ , such that if $\langle M, \triangleleft \rangle \models \phi$ then $\langle M, \triangleleft \rangle$ is future-distinguishing.*

Proof. A contradiction is obtained by reversing the relation in Figure 7 ($F'_1 = \langle W_1, R_1^{-1} \rangle$, $F'_2 = \langle W_2, R_2^{-1} \rangle$) and observing a similar bisimulation from F'_1 to F'_2 . $\square$

Table 1: Modal logics along the causal ladder. Blank cells use the same logic as the one above.

Spacetime Type (M)	Spacetime Modal Logic Containment ($L = \mathcal{L}(\langle M, \triangleleft \rangle)$)		
	$\ll$	α	$\leq / \preceq$
Totally Vicious	S4 $\subseteq L$	S4 $\subseteq L$	S4 $\subseteq L$
Non-Totally Vicious	OI $\subseteq L$		
Chronological	OI + (Gabb) $\subseteq L$		
Causally Non-Totally Vicious	OI	D4da $\subseteq L$	
cNTV and Chronological	OI + (Gabb) $\subseteq L$		
Casual		D4da + (Gabb) $\subseteq L$	
Distinguishing			
$\vdots$		$\vdots$	
Globally hyperbolic	OI + (Gabb) $\subseteq L$	D4da + (Gabb) $\subseteq L$	S4 $\subseteq L$

Corollary 6.12. *A frame being distinguishing is not modally definable.*

Proof. In Figure 7 we have that the frame F_1 is (past- and future-)distinguishing, but it is bisimilar to F_2 , which is not (past-)distinguishing. $\square$

7 Discussion

Modal logics of the causal ladder. Table 1 shows the containment of the various modal logics for spacetimes and their various causal relations on different levels of the causal ladder. As explored in Section 5, the modal logic for two-dimensional spacetimes replaces **D4da** with **D4da₂** in Table 1. The later levels of the causal ladder require new rules or additional operations to distinguish between their logics on the ladder.

Other effects on the logic. Note that spacetimes on the same rung of the causal ladder may nevertheless exhibit distinct modal logics. For instance, contrast ordinary two-dimensional Minkowski space with the 2⁻-Minkowski spacetime: $\{(t, x) \in \mathbb{R}^2 : t < 0\}$, inheriting the same causal structure as in Example 2.1. Both are globally hyperbolic but the logics of those spacetimes under the causal (chronological) relation are **S4.2** [4] (**OI.2** [15]) and **S4** (**OI**) [16, Lemma 3.6] respectively.

Conversely, it is clear that the modal structure is not expressive enough to completely characterise the spacetime, since all Minkowski spaces of total dimension greater than three share the same modal logic. More can perhaps be said using a multi-modal logic, where it is possible to quantify over causal and chronological curves, and then employing Malament's theorem [9].

Multiple time dimensions. There is a notion of spacetimes with multiple time dimensions. A n -spatial, t -temporal dimensional (or (n, t) -dimensional) spacetime can then be described as working on a $n + t$ -dimensional manifold. Our universe, $(3, 1)$ -dimensional, can be considered “nice”. By changing the number of spatial or temporal dimensions, the resulting universe could be considered either too simple or too unstable (not “nice”). For instance, Tegmark [18] provides various arguments for why observers cannot exist in (n, t) -dimensional universes where $(n, t) \neq (3, 1)$, but the arguments made are not rigorous.



Figure 8: Confluence on different spacetimes

One fundamental question to ask is whether there is a fundamental logical reason that these universes are not “nice”. In Section 5.2, we showed that modal logic seems expressible enough to distinguish between $(1, 1)$ -dimensional and $(n, 1)$ -dimensional spacetimes ($n \geq 2$), but it is unlikely to distinguish any other (spatial) dimensional spacetime. Extending our definition of $\Box$ and $\Diamond$ on the causal relations to higher temporal dimensions, or using multimodal logic to represent causal relations in different time dimensions, could provide the start towards a logical language to discuss what universes are “nice” or not.

Acknowledgements

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A Proofs for Section 4

A.1 Geodesic Normal Coordinates

In this section we define and collect some technical results from causality theory that are needed to prove the after formula holds in general spacetimes. We refer to [8, §5] for details.

Given any subset $A \subseteq M$, we write $x \ll_A y$ if there exists a fd timelike curve from x to y that lies entirely within A . Clearly $x \ll_A y$ implies $x \ll y$. Define $\preceq_A$ and α_A similarly.

A curve γ is called a *geodesic* if it is length extremising (technically: $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ with respect to the Levi-Civita connection), which is interpreted as the motion of a non-accelerating, free falling body. For the present work it is important to note that geodesics are locally uniquely determined by their starting position and initial velocity. This ensures the existence of the *exponential map*: for $x \in M$ it is the smooth map $\exp_x: \mathcal{V}_x \subseteq T_x M \rightarrow M$ defined on a suitable open neighbourhood $\mathcal{V}_x$ of 0 by $\exp_x(v) := \gamma_v(1)$, where γ_v is the unique geodesic with $\gamma_v(0) = x$ and $\dot{\gamma}_v(0) = v$. The region $\mathcal{V}_x$ can be shrunk to a convex open neighbourhood of 0 such that $\exp_x$ becomes a diffeomorphism onto its image $\exp_x(\mathcal{U}_x) = U_x$, which is called a (*convex*) *normal neighbourhood* of x . This diffeomorphism facilitates what is known as *geodesic normal coordinates*. Under these coordinates, geodesics within the neighbourhood U_x emanating from x correspond precisely to straight line segments in $T_x M$ through the origin.

Theorem A.1. *Every $x \in M$ admits an open convex normal neighbourhood $U_x \subseteq M$ together with a diffeomorphism $\Phi := \exp_x^{-1}: U_x \rightarrow \mathcal{U}_x$ onto an open convex neighbourhood $\mathcal{U}_x$ of the origin in Minkowski space, such that $\Phi(x) = 0$, affinely parametrised geodesics γ in U_x with $\gamma(0) = x$ are transformed into straight lines as $\Phi \circ \gamma(t) = t\dot{\gamma}(0)$, and for every $y \in U_x$ we have:*

$$x \ll_{U_x} y \iff 0 \ll \Phi(y), \quad x \preceq_{U_x} y \iff 0 \preceq \Phi(y), \quad x \alpha_{U_x} y \iff 0 \alpha \Phi(y). \quad (\star)$$

Proof. See [8, Theorem 5.5] and [11, Corollary 2.10]. $\square$

Remark A.2. Of course, the dual of Equation $(\star)$ also holds, where y appears in the past of x . Note however that Φ does not preserve the causal relations between arbitrary points in U_x : only to and from the basepoint x .

A.2 Proof of Theorem 4.6

As mentioned in Section 4.2, the main technical step is to prove: if $x \alpha y \alpha y_1, y_2$ and $y_1 \neq y_2$ are spacelike separated, then $x \ll y_i$ for some i . We first do this in Minkowski space.

Lemma A.3. *In Minkowski space, let $x \rightarrow y \rightarrow y_1, y_2$ and $x \rightarrow y_1, y_2$. Then $y_1 \rightarrow y_2$ or $y_2 \rightarrow y_1$.*

Proof. Take some arbitrary z in the intersection of the lightcones of x and y . Then we get from Equation (1) that $\|\vec{z} - \vec{x}\| = z^0 - x^0$ and $\|\vec{z} - \vec{y}\| = z^0 - y^0$, and hence

$$\|\vec{z} - \vec{x}\| - \|\vec{z} - \vec{y}\| = y^0 - x^0 = \|\vec{y} - \vec{x}\|.$$

This shows that equality is attained in the triangle inequality:

$$\|\vec{z} - \vec{x}\| \leq \|\vec{z} - \vec{y}\| + \|\vec{y} - \vec{x}\|,$$

which occurs precisely when there exists a scalar $\lambda \geq 0$ with $\vec{z} - \vec{y} = \lambda(\vec{y} - \vec{x})$. In particular, for our situation, we get $\lambda_1, \lambda_2 \geq 0$ with $\vec{y}_i = \vec{y} + \lambda_i(\vec{y} - \vec{x})$. Note now by $y \rightarrow y_i$ that

$$y_i^0 = y^0 + \|\vec{y}_i - \vec{y}\| = y^0 + \lambda_i \|\vec{y} - \vec{x}\| = y^0 + \lambda_i(y^0 - x^0),$$

so in particular $y_2^0 - y_1^0 = (\lambda_2 - \lambda_1)(y^0 - x^0)$. Similarly note $\vec{y}_2 - \vec{y}_1 = (\lambda_2 - \lambda_1)(\vec{y} - \vec{x})$, and so we get

$$\|\vec{y}_2 - \vec{y}_1\| = |\lambda_2 - \lambda_1| \|\vec{y} - \vec{x}\| = |\lambda_2 - \lambda_1|(y^0 - x^0) = |y_2^0 - y_1^0|,$$

which is precisely what it means for $y_1 \rightarrow y_2$ or $y_2 \rightarrow y_1$ to hold. $\square$

Corollary A.4. *In Minkowski space, if $x \alpha y \alpha y_1, y_2$ and $y_1 \neq y_2$ are spacelike separated, then $x \ll y_i$ for some i .*

Proof. Transitivity implies $x \alpha y_1, y_2$. So either $x \ll y_i$ for some i , in which case we are done, or $x \rightarrow y_1, y_2$. Moreover, if $x \ll y$ or $y \ll y_i$ for some i we get the desired relation via the push-up rule. We are therefore left to consider the case that $x \rightarrow y \rightarrow y_1, y_2$. However, by Lemma A.3, this would imply $y_1 \rightarrow y_2$ or $y_2 \rightarrow y_1$, contradicting that y_1 and y_2 are spacelike separated. $\square$

The next few lemmas generalise this result to arbitrary spacetimes. We need some more definitions. A subset $A \subseteq M$ is called *achronal* if $I^+(A) \cap A = \emptyset$, and a curve γ is called *achronal* if $\text{im}(\gamma)$ is achronal, where we recall for $\gamma: I \rightarrow M$ that $\text{im}(\gamma) = \{\gamma(t) : t \in I\}$. Explicitly, this means there are no times s, t so that $\gamma(s) \ll \gamma(t)$. A geodesic from p to q is called *maximising* if it has greater or equal length than every other causal curve from p to q . We then have the powerful result [11, Theorem 2.22], saying achronal lightlike geodesics are precisely the maximising lightlike ones.

Theorem A.5. *Let γ be a causal curve connecting p to $q \neq p$. Either: there is a timelike curve σ from p to q whose length is strictly greater than that of γ ; or γ is a maximising geodesic (up to parametrisation). In particular, if there are no timelike curves connecting p to q then γ is an achronal lightlike geodesic (up to parametrisation).*

Lemma A.6. *Let $U \subseteq M$ be open, and take $p, q \in U$. If there exists a causal curve γ in U from p to $q \neq p$ that is not an achronal lightlike geodesic, then $p \ll_U q$.*

Proof. Consider the spacetime $(U, g|_U)$. Applying Theorem A.5 to the given curve γ in U from p to $q \neq p$, either we get a timelike curve σ facilitating $p \ll_U q$, in which case we are done, or γ is a maximising geodesic. If γ is itself timelike we are also done, but if γ is a maximising lightlike geodesic then it is achronal, contradicting the hypothesis. $\square$

Lemma A.7. *Let $x \in M$, and take geodesic normal coordinates $\Phi: U_x \rightarrow \mathcal{U}_x$ from Theorem A.1. Suppose $p, q \in U_x$ satisfy $p \alpha_{U_x} x \alpha_{U_x} q$. If $\Phi(p) \ll \Phi(q)$ in Minkowski space, then $p \ll_{U_x} q$.*

Proof. If $p \ll_{U_x} x$ or $x \ll_{U_x} q$ then the desired result follows from the push-up rule. Suppose therefore that $p \rightarrow_{U_x} x$ and $x \rightarrow_{U_x} q$. Let α and β be the curves facilitating these relations, and let γ be their concatenation. We claim that γ is not a lightlike geodesic, so in particular not an achronal lightlike geodesic. From that, the desired $p \ll_{U_x} q$ will follow by Lemma A.6.

Suppose for the sake of contradiction that γ is a lightlike geodesic, affinely parametrised and so that $\gamma(0) = x$. Then the curve $\Phi \circ \gamma$ is the straight line passing through the origin with velocity $\dot{\gamma}(0)$, which immediately gives $\Phi(p) \rightarrow \Phi(q)$, a contradiction to $\Phi(p) \ll \Phi(q)$. $\square$

Lemma 4.5. *In any spacetime, if $x \alpha y \alpha y_1, y_2$ and $y_1 \neq y_2$ are spacelike separated, then $x \ll y_i$ for some i .*

Proof. Let $\gamma, \delta_1, \delta_2$ denote the causal curves corresponding to $x \alpha y$ and $y \alpha y_1, y_2$, respectively. Without loss of generality, suppose that δ_i are parametrised on the unit interval $[0, 1] \rightarrow M$. Consider the set $S = \delta_1^{-1}(\text{im}(\delta_2))$, which is nonempty since $\delta_1(0) = \delta_2(0)$ and a strict subset of $[0, 1]$ because $y_1 \neq y_2$ are spacelike separated. Since $[0, 1]$ is compact and δ_2 is continuous, $\text{im}(\delta_2)$ is compact in M . Since M is Hausdorff, $\text{im}(\delta_2)$ is thus closed, and by continuity of δ_1 the preimage S is closed. Hence S has a maximum element t_0 , the latest time at which δ_1 overlaps δ_2 . We can further reparametrise δ_2 without loss of generality such that $\delta_1(t_0) = \delta_2(t_0)$, and hence δ_1 and δ_2 never intersect after t_0 . Let $y' := \delta_1(t_0) = \delta_2(t_0)$ be the last intersection point; see the left diagram in Figure 5. Note if $t_0 = 0$ then $y' = y$.

Using Theorem A.1, choose a convex normal neighbourhood $U_{y'} \subseteq M$ and geodesic normal coordinates $\Phi: U_{y'} \rightarrow \mathcal{U}_{y'}$. Since $U_{y'}$ is open and δ_i are continuous with $\delta_i(t_0) = y'$, we can find time intervals $\varepsilon_i > 0$ such that $t_0 + \varepsilon_i \leq 1$ and on $[t_0, t_0 + \varepsilon_i]$ the curves δ_i stay in $U_{y'}$. For any $t \in (t_0, t_0 + \varepsilon_i]$ we then get that $y' \alpha_{U_{y'}} \delta_i(t)$, and by $(\star)$ we have $0 \alpha \Phi \circ \delta_i(t)$ in $\mathcal{U}_{y'}$. From this it follows that $\delta_i(t) \neq y'$ for every $t \in (t_0, t_0 + \varepsilon_i]$, otherwise $y' \alpha_{U_{y'}} y'$ and hence $0 \alpha 0$, which is impossible in Minkowski space. Using this, we see from Equation (1) that the time coordinate $(\Phi \circ \delta_i(t))^0$ must be strictly greater than zero for all $t \in (t_0, t_0 + \varepsilon_i]$. In particular there exists some $\varepsilon > 0$ that is below the endpoints: $\varepsilon < (\Phi \circ \delta_i(t_0 + \varepsilon_i))^0$, meaning that eventually the curves $\Phi \circ \delta_i$ are above the hyperplane $\{x \in \mathbb{R}^{1+n} : x^0 = \varepsilon\}$. By the intermediate value theorem there exists $t_i \in (t_0, t_0 + \varepsilon_i]$ so that the curves are precisely on the hyperplane: $(\Phi \circ \delta_i(t_i))^0 = \varepsilon$. Using this, define the distinct points $p_i := \delta_i(t_i)$, from which we get the distinct points $z_i := \Phi(p_i)$ on the hypersurface, meaning that $z_1 \neq z_2$ are spacelike separated.

We similarly pick a point p in the past of y' . If $t_0 > 0$, find some $\varepsilon_0 > 0$ so that δ_1 remains within $U_{y'}$ on the interval $[t_0 - \varepsilon_0, t_0]$, and define $p := \delta_1(t_0 - \varepsilon_0)$ and $z := \Phi(p)$; see the right diagram in Figure 5. In the case that $t_0 = 0$, so $y' = y$, pick p analogously but on the curve γ instead.

In total, we have the situation in the normal neighbourhood $U_{y'}$ that $p \alpha_{U_{y'}} y' \alpha_{U_{y'}} p_1, p_2$. Using $(\star)$ and its dual, we get the situation in the region $\mathcal{U}_{y'}$ of Minkowski space that $z \alpha 0 \alpha z_1, z_2$ where $z_1 \neq z_2$ are spacelike separated, so by Corollary A.4 we get $z \ll z_k$ for some k . Applying Lemma A.7 we hence get $p \ll_{U_{y'}} p_k$.

Finally, by construction we have $x \alpha p$ and $p_i \alpha y_i$, so globally $x \alpha p \ll p_k \alpha y_k$, and the desired result follows by the push-up rule: $x \ll y_k$. $\square$

B Proofs for Section 5

Theorem 5.4. *Let M be a two-dimensional spacetime. Then,*

$$\langle M, \alpha \rangle \models a \alpha_2 f.$$

Proof. Let $x \alpha y_1, y_2, z$ with $y_1 \neq y_2$ and $y_i \not\alpha y_j$. We need to prove there exists $t \in M$ such that $x \alpha t \alpha z, y_i$ for some i . Again, in the case that $x = y_i$ or $x = z$ simply take $t = x$. Thus assume all points are distinct from x , so α is described via Lemma 2.3.

If $x \ll y_i$ for some i and $x \ll z$, then by 2-density of $\ll$ we are done.

Suppose now that $x \rightarrow y_1, y_2, z$, and let $\delta_1, \delta_2, \gamma$ be the corresponding causal curves. Since $x \rightarrow y_i$, there are no timelike curves from x to y_i , and hence by Theorem A.5 each δ_i is an achronal lightlike geodesic (up to reparametrisation). Similarly, since $x \rightarrow z$, the curve γ is an achronal lightlike geodesic from x to z .

Choose a convex normal neighbourhood U_x of x , and consider the restrictions of $\delta_1, \delta_2, \gamma$ to U_x . In a two-dimensional spacetime there are exactly two future-directed lightlike velocities at x , and hence exactly two future-directed lightlike geodesics emanating from x in U_x . Moreover, since y_1 and y_2 are

spacelike separated, the geodesics δ_1 and δ_2 cannot coincide, and so the velocities of δ_1 and δ_2 at x correspond to the two distinct lightlike directions in $T_x M$. Since γ is also lightlike, its velocity at x must therefore be equal to that of, say, δ_1 . Pick a point $t \in \text{im}(\gamma) \cap \text{im}(\delta_1) \cap U_x \setminus \{x\}$ sufficiently close to x so that t lies on the subcurves from x to z and from x to y_1 , so that $x \alpha t \alpha y_1, z$, as desired.

Finally, consider the scenario $x \rightarrow y_1, y_2$ and $x \ll z$. Let δ_1, δ_2 be the achronal lightlike geodesics from x to y_1, y_2 (as above), and let γ be a timelike curve from x to z . Pick a point $z' \in \text{im}(\gamma) \setminus \{x, z\}$ sufficiently close to x so that $x \ll z' \ll z$. Since $I^-(z')$ is open and contains x , and δ_1 is continuous with $\delta_1(0) = x$, there exists $t \in \text{im}(\delta_1) \setminus \{x\}$ sufficiently close to x such that $t \in I^-(z')$. Then $x \alpha t \alpha y_1$. Moreover $t \ll z'$ implies $t \alpha z' \alpha z$, hence $t \alpha z$. Thus $x \alpha t \alpha y_1, z$, completing the proof. $\square$

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Intuitionistic Justification Logic, Semantically

Sonia Marin

Paaras Padhiar

Ian Shillito*

s.marin@bham.ac.uk pxp367@student.bham.ac.uk i.b.p.shillito@bham.ac.uk

School of Computer Science
University of Birmingham
Birmingham, United Kingdom

Justification logics are explicit versions of modal logic. In the classical setting, this means boxes are refined with explicit proof terms and interact with each other through proof operations. This exercise was extended to intuitionistic modal logic with native diamonds. In this setting, diamonds are refined to satisfier terms and come equipped with additional operations.

Justification logic enjoys a connection to its corresponding modal logic through a realisation theorem. In the classical setting, this is achieved through either proof-theoretic or semantic methodology. So far, intuitionistic justification logic with satisfiers has only been presented syntactically with a proof-theoretic realisation theorem.

We present two classes of semantics for intuitionistic justification logic with soundness and completeness results: basic modular models, which extend possible world semantics for intuitionistic propositional logic; modular models which contain Kripke-style machinery to promote “backwards compatibility” to modal logic. Using modular models, we present a realisation theorem to establish a connection between intuitionistic justification logic and its corresponding intuitionistic modal logic.

1 Introduction

Justification logic makes knowledge explicit by attaching reasons to statements. It refines modalities in standard modal logic: the usual $\Box A$, which can mean “ A is *provable* or *known*”, is replaced with $[\tau]A$ for some proof term τ , interpreted as “ τ is a *proof* or *evidence* of A ”.

Realisation theorem. The first proposed justification logic is the Logic of Proofs (LP), introduced by Artemov [1], which makes the modal logic S4 explicit. This is achieved formally via a *realisation theorem* translating each theorem of S4 into its corresponding theorem in LP by *realising* every $\Box$ modality into a suitable proof term. The realisation theorem was first proved by Artemov using a proof-theoretic (and constructive) methodology [2], and later on via semantic techniques by Fitting [18].

Semantics for justification logic. Originally, the logic of proofs was interpreted using proofs in Peano Arithmetic (PA) [1]. Later, simple and flexible semantic models akin to epistemic (possible-world) models were created, in which justification terms represent evidence rather than explicit arithmetic proofs [32, 18]. These models helped apply justification logic in a more general fashion to the concepts of knowledge and evidence, better understand what justification terms mean, and prove important logical properties like decidability [11] and complexity bounds [25].

Intuitionistic justification logic. Since LP interprets proofs in classical arithmetic, an intuitionistic version of the Logic of Proofs (iLP) was sought to represent proofs in Heyting Arithmetic (HA) [7, 14]. Though iLP is more than just LP over intuitionistic logic: to match the proof interpretation in HA, additional axioms representing admissible rules must be included. The base logic (without these additional

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axioms) was shown to be an explicit justification version of intuitionistic modal logic iS4 [3]. Its semantics was studied in [30], where they provide intuitionistic Kripke-style models and prove completeness, and its proof theory in [24]. Intermediate variants of LP were also studied by Pischke [33] providing in particular an adaptation of the semantic realisation technique.

Intuitionistic diamonds. In classical modal logic, $\Box$ and $\Diamond$ are dual, i.e., $\Diamond A$ can be defined as $\neg\Box\neg A$, making the behaviour of $\Diamond$ automatically determined by the one of $\Box$. However, the weaker negation of intuitionistic logic breaks this duality: $\Diamond$ must be given its own independent semantics and axioms, as it cannot simply be defined in terms of $\Box$. Furthermore, the addition of certain $\Diamond$ -axioms produces logics which are not conservative over the $\Diamond$ -free language [13, 22, 12], which justifies the interest in intuitionistic variants of modal logics with both modalities [15, 36].

Diamonds in justification logic. In justification logic, even intuitionistic variants, $\Diamond$ was never considered as a first-class citizen. The first justification logic which makes the $\Diamond$ modality explicit was postulated by [26] as a justification counterpart to *constructive modal logic* CK [9] with a syntactic realisation result. This work was then extended to provide a justification counterpart to *intuitionistic modal logic* IK [15] syntactically in [29] with a proof-theoretic procedure based on nested sequents.

Contributions. We continue this line of work with a semantic exploration of intuitionistic justification logics (with $\Diamond$). We extend the definition of basic intuitionistic modular models and intuitionistic modular models of [30], and establish soundness and completeness results with respect to these semantics for JIK, the justification counterpart of intuitionistic modal logic IK. Building on these results, our investigation is crowned by an adaptation of the semantic realisation procedure in [33] to JIK.

Outline. In Section 2 we introduce the preliminaries on IK and intuitionistic justification logic JIK, notably providing some key properties of the latter logic our work relies on. In Section 3 we introduce basic intuitionistic modular models which extend [31, 33] to incorporate the $\Diamond$ operator, and prove that they are still enough to ensure completeness with JIK. In Section 4 we expand on these to get intuitionistic modular models. They require more adaptation due to the addition of $\Diamond$ on the one hand, and to the complexity inherent to the semantics for IK on the other hand. Indeed, the confluence conditions traditionally expected in the semantics for IK are superfluous for justification JIK (given that persistence already holds in the basic semantics) but they are needed to ensure backward compatibility with IK in the realisation theorem. In Section 5 we are indeed able to prove the realisation theorem, that IK and JIK formally correspond, by adapting the semantic method [18, 33] to IK and the intuitionistic $\Diamond$. We conclude in Section 6 with some discussions on avenues for taking these ideas further.

2 Preliminaries

2.1 Intuitionistic Modal Logic

Let Prop be a countable set of propositional variables. The language of modal logic $\mathcal{L}_{\Box}$ is defined by

$$G ::= p \in \text{Prop} \mid \perp \mid G \wedge G \mid G \vee G \mid G \rightarrow G \mid \Box G \mid \Diamond G$$

Let Int be a finite set of axioms for IPL, e.g. [37]. We add a superscript I to denote the set of axiom instances of a given set of axioms, e.g. Int^I for the set of axiom instances of Int . We define Mod to be the set of axioms extending Int with the *modal* axioms presented below [34].

$$\begin{array}{lll} k_1 : & \Box(G \rightarrow H) \rightarrow (\Box G \rightarrow \Box H) & k_3 : \Diamond(G \vee H) \rightarrow (\Diamond G \vee \Diamond H) \\ k_2 : & \Box(G \rightarrow H) \rightarrow (\Diamond G \rightarrow \Diamond H) & k_4 : (\Diamond G \rightarrow \Box H) \rightarrow \Box(G \rightarrow H) \end{array} \quad k_5 : \Diamond \perp \rightarrow \perp$$

Definition 2.1. The logic IK is the set of consecutions $\Gamma \vdash G$, where $\Gamma = \{G_1, \dots, G_n\}$ is a finite set of formulas, derivable in the system described by the following rules.

$$\frac{G \in \text{Mod}^I}{\Gamma \vdash G} \text{ ax} \quad \frac{1 \leq i \leq n}{\Gamma \vdash G_i} \text{ id} \quad \frac{\emptyset \vdash G}{\Gamma \vdash \Box G} \text{ nec} \quad \frac{\Gamma \vdash G \quad \Gamma \vdash G \rightarrow H}{\Gamma \vdash H} \text{ mp}$$

If the consecution $\Gamma \vdash G$ is provable in the system for IK, we write $\Gamma \vdash_{\text{IK}} G$. We omit the curly brackets when clear from context and write $G_1, \dots, G_n \vdash_{\text{IK}} G$ for $\{G_1, \dots, G_n\} \vdash_{\text{IK}} G$. For a potentially infinite set of formulas Γ , we write $\Gamma \vdash_{\text{IK}} G$ if there are formulas $G_1, \dots, G_n \in \Gamma$ such that $G_1, \dots, G_n \vdash_{\text{IK}} G$. We write $\Gamma, H \vdash_{\text{IK}} G$ for $\Gamma \cup \{H\} \vdash_{\text{IK}} G$, and $\vdash_{\text{IK}} G$ for $\emptyset \vdash_{\text{IK}} G$.

Remark 2.2. The fact that the premise of the nec rule requires an empty context entails that we capture the *local* modal logic IK, understood as a consequence relation. A benefit of this rule is that the deduction theorem holds [23], a result implicitly used in some of the results below about IK.

With the syntax and axiomatic system of IK defined, we turn to its (bi)relational semantics [15, 16].

Definition 2.3 (Frames). A (birelational) frame is a tuple $(W, \leq, R)$, where W is a non-empty set of worlds, $\leq$ is a preorder on W and R is a binary relation on W such that:

(BC) for all worlds w, v, v' with $wRv \leq v'$, then there exists a world w' with $w \leq w'Rv'$;

(FC) for all worlds w, w', v with $w \leq w'$ and wRv , there exists a world v' with $w'Rv'$ and $v \leq v'$.



Note that *BC* stands for *Backward Confluence*, while *FC* stands for *Forward Confluence*.

Definition 2.4 (Models). A (birelational) model for IK, is a tuple $\mathcal{M} = (W, \leq, R, *)$ with $(W, \leq, R)$ a frame equipped with a map $*$: $\text{Prop} \times W \rightarrow \{0, 1\}$ such that $w \leq v$ and $*(p, w) = 1$ entail $*(p, v) = 1$. We henceforth use the notation p_w^* for $*(p, w)$. For a model $\mathcal{M}$, a point $w \in W$ and a formula G , the *truth of G at w in $\mathcal{M}$* is recursively defined on the structure of G :

$$\begin{aligned} \mathcal{M}, w \Vdash p & \quad \text{iff} \quad p_w^* = 1 \\ \mathcal{M}, w \Vdash \perp & \quad \text{never} \\ \mathcal{M}, w \Vdash G \wedge H & \quad \text{iff} \quad \mathcal{M}, w \Vdash G \text{ and } \mathcal{M}, w \Vdash H \\ \mathcal{M}, w \Vdash G \vee H & \quad \text{iff} \quad \mathcal{M}, w \Vdash G \text{ or } \mathcal{M}, w \Vdash H \\ \mathcal{M}, w \Vdash G \rightarrow H & \quad \text{iff} \quad \text{for all worlds } w' \text{ with } w \leq w', \text{ if } \mathcal{M}, w' \Vdash G \text{ then } \mathcal{M}, w' \Vdash H \\ \mathcal{M}, w \Vdash \Box G & \quad \text{iff} \quad \text{for all worlds } w', v' \text{ with } w \leq w'Rv', \mathcal{M}, v' \Vdash G \\ \mathcal{M}, w \Vdash \Diamond G & \quad \text{iff} \quad \text{there exists world } v \text{ with } wRv, \mathcal{M}, v \Vdash G \end{aligned}$$

We write $\mathcal{M}, w \Vdash \Gamma$ if $\mathcal{M}, w \Vdash G$ for all $G \in \Gamma$. We write $\models G$ if $\mathcal{M}, w \Vdash G$ for any $\mathcal{M}$ and $w \in W$.

The following relates the semantic and axiomatic definitions of IK.

Theorem 2.5 (Soundness and Completeness [17, 34, 36]). $\vdash_{\text{IK}} G \iff \models G$.

Later in the paper, we make use of the following consequence of soundness.

Corollary 2.6. IK is consistent.

$$\begin{array}{ll}
jk_1 : [s](A \rightarrow B) \rightarrow ([t]A \rightarrow [s \cdot t]B) & j +_l : [s]A \rightarrow [s + t]A \\
jk_2 : [s](A \rightarrow B) \rightarrow (\langle m \rangle A \rightarrow \langle s \star m \rangle B) & j +_r : [t]A \rightarrow [s + t]A \\
jk_3 : \langle m \rangle (A \vee B) \rightarrow (\langle m \rangle A \vee \langle m \rangle B) & j \sqcup_l : \langle m \rangle A \rightarrow \langle m \sqcup n \rangle A \\
jk_4 : (\langle m \rangle A \rightarrow [t]B) \rightarrow [m \triangleright t](A \rightarrow B) & j \sqcup_r : \langle n \rangle A \rightarrow \langle m \sqcup n \rangle A \\
jk_5 : \langle m \rangle \perp \rightarrow \perp &
\end{array}$$

Figure 1: Intuitionistic justification axioms and rule

2.2 Intuitionistic Justification Logic

The sets of *proof terms* PrfTm and of *satisfier terms* SatTm are defined by mutual induction as follows

$$t ::= c \in \text{PrfConst} \mid x \in \text{PrfVar} \mid t \cdot t \mid t + t \mid !t \mid m \triangleright t \quad m ::= a \in \text{SatVar} \mid m \sqcup m \mid t \star m$$

where proof constants PrfConst , proof variables PrfVar and satisfier variables SatVar are countable sets. The *language* of justification logic $\mathcal{L}_J$ reuses the set Prop as shown in the following grammar

$$A ::= p \in \text{Prop} \mid \perp \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid [t]A \mid \langle m \rangle A$$

where $t \in \text{PrfTm}$ and $m \in \text{SatTm}$.

Before defining logics over $\mathcal{L}_J$, we provide some intuitions on the intended meaning of the operators over proof and satisfier terms.

In justification logic, a formula of the shape $[t]A$ can be read as t is a *proof* of A . Proof terms can hence be thought of as capturing *global reasoning*, i.e. they assert the *validity* of statements with respect to any model. Dually, satisfier terms relate to *local reasoning*: $\langle m \rangle A$ could be read as m is a *model* where A is satisfied, implying the *consistency* of A . The operations *proof sum* $+$, *application* $\cdot$ and *proof checker* $!$ are standard justification operations relating to common proof manipulations [1, 2]. The operation on satisfiers of *disjoint union* $\sqcup$ is quite naturally the local counterpart to the proof sum. The *propagation* operation $\star$ combines global and local reasoning by taking as arguments a proof term and a satisfier term [26]. For example, the locality of A in $\langle m \rangle A$ can be exploited via the global holding of $A \rightarrow (A \vee B)$ in $[t](A \rightarrow (A \vee B))$ to locally establish $(A \vee B)$ as $\langle t \star m \rangle (A \vee B)$. Finally, the *local update* operation $\triangleright$ expresses that if local reasoning implies global reasoning, global information can be updated using local information [29]. For example, if the satisfier m locally reasons about A and implies the proof term t globally reasons about B , then this connection is *updated* into global reasoning by making the proof term $m \triangleright t$ globally reason about $A \rightarrow B$. While the interpretations of operations on proof terms are formally backed up by the completeness proof with respect to Peano arithmetic in the context of classical Logic of Proofs [2], in the intuitionistic context, these merely constitute guidelines to help us reason about the operations at a high-level and would require to be studied within a meta-theory such as Heyting arithmetic or related to be given a proper formal understanding.

The interpretation we gave to our operators is reflected in the set of axioms Base , which extends Int with the additional *justification* axioms presented in the two left columns of Figure 1.

While traditionally a set of axioms like Base would generate a *unique* logic, justification logic allows for the definition of myriad logics via *constant specifications*.

Definition 2.7 (Constant specification). A constant specification for Base is any subset $\text{CS} \subseteq \{[c]A \mid c \in \text{PrfConst} \text{ and } A \in \text{Base}^I\}$. We call a constant specification CS *axiomatically appropriate* if for each axiom instance A of Base , there exists a proof constant c such that $[c]A \in \text{CS}$. We call an axiomatically appropriate constant specification CS *schematic* if for any instance A of an axiom in Base^I we have that $[c]A \in \text{CS}$ entails $[c]A' \in \text{CS}$ for any instance A' of the same axiom.

The parametricity in constant specifications of axiomatic systems for justification logics is illustrated by a characteristic rule of these calculi.

Definition 2.8. Given a constant specification CS for Base, we define the logic JIK_{CS} as the set of consecutions $\Gamma \vdash A$ provable by the system described by the rules from Definition 2.1 (over $\mathcal{L}_J$), where:

- Mod is replaced by Base and
- nec is replaced by the rule can presented below.

$$\frac{[c]A \in \text{CS} \quad k \geq 0}{\Gamma \vdash [!^k c] \dots [!c][c]A} \text{ can}$$

We adopt the conventions of Definition 2.1 with $\vdash_{\text{IK}}$ replaced by $\vdash_{\text{JIK}_{\text{CS}}}$.

Remark 2.9. The $!$ operator is not necessary to give a justification counterpart to non-transitive logics and we could opt for a formulation where $\text{CS} \subseteq \{[c_n][\dots][c_1]A \mid c_1, \dots, c_n \in \text{PrfConst and } A \in \text{Base}'\}$ [2]. We prefer to use the explicit $!$ operator for a more concise presentation of CS where proof constants can really be thought of as a *witness* to an axiom, and not to more complex formula of the form $[c]A$.

The following is an example of a proof in JIK_{CS} .

Example 2.10. We show that from $\vdash_{\text{JIK}_{\text{CS}}} [t](A \rightarrow (A \rightarrow B) \rightarrow B)$ (1) one can infer $\vdash_{\text{JIK}_{\text{CS}}} \langle m \rangle (A \rightarrow B) \rightarrow [s]A \rightarrow \langle (t \cdot s) \star m \rangle B$ for any proof term s and satisfier term m . First, we instantiate jk_1 and jk_2 as follows: $\vdash_{\text{JIK}_{\text{CS}}} [t](A \rightarrow (A \rightarrow B) \rightarrow B) \rightarrow [s]A \rightarrow [t \cdot s]((A \rightarrow B) \rightarrow B)$ (2) and $\vdash_{\text{JIK}_{\text{CS}}} [t \cdot s]((A \rightarrow B) \rightarrow B) \rightarrow \langle m \rangle (A \rightarrow B) \rightarrow \langle (t \cdot s) \star m \rangle B$ (3). Second, we apply mp to (1) and (2) to obtain $\vdash_{\text{JIK}_{\text{CS}}} [s]A \rightarrow [t \cdot s]((A \rightarrow B) \rightarrow B)$ (4). Via transitivity of $\rightarrow$ on (3) and (4), we get $\vdash_{\text{JIK}_{\text{CS}}} [s]A \rightarrow \langle m \rangle (A \rightarrow B) \rightarrow \langle (t \cdot s) \star m \rangle B$ (5). Then by propositional reasoning on (5), we finally infer $\vdash_{\text{JIK}_{\text{CS}}} \langle m \rangle (A \rightarrow B) \rightarrow [s]A \rightarrow \langle (t \cdot s) \star m \rangle B$. This is the justification version of the theorem $\Diamond(A \rightarrow B) \rightarrow \Box A \rightarrow \Diamond B$ in IK.

2.3 Properties of Justification Logic

In this section, we recall key definitions and properties of justification logic needed throughout the paper.

Theorem 2.11 (Deduction Theorem). *Let $\Gamma \cup \{A, B\} \subseteq \mathcal{L}_J$. Then: $\Gamma, A \vdash_{\text{JIK}_{\text{CS}}} B \iff \Gamma \vdash_{\text{JIK}_{\text{CS}}} A \rightarrow B$*

Proof. This is a standard argument [28]: from right to left we simply use mp and monotonicity on the left of provability, and in the other direction we proceed by induction on the given proof of $\Gamma, A \vdash_{\text{JIK}_{\text{CS}}} B$. $\square$

Next, we show that JIK_{CS} is closed under a notion of substitution modifying proof and satisfier terms.

Definition 2.12 (Justification substitution). A (*justification*) *substitution* is a pair of maps $\sigma_t : \text{PrfVar} \rightarrow \text{PrfTm}$ and $\sigma_s : \text{SatVar} \rightarrow \text{SatTm}$. For simplicity, we merge the two maps and write σ as a justification substitution. We define as expected the applications $t\sigma$, $m\sigma$ and $A\sigma$ of σ to, respectively, the proof term t , the satisfier term m and the formula A . We write $\Gamma\sigma$ to designate the set $\{A\sigma \mid A \in \Gamma\}$.

Lemma 2.13 (Substitution Lemma). *Let CS be a schematic constant specification, $\Gamma \cup \{A\} \subseteq \mathcal{L}_J$, and σ a substitution. Then: $\Gamma \vdash_{\text{JIK}_{\text{CS}}} A \implies \Gamma\sigma \vdash_{\text{JIK}_{\text{CS}}} A\sigma$*

Proof. The proof is similar to [28, Lemma 2.27], and goes by induction on the structure of the proof of $\Gamma \vdash_{\text{JIK}_{\text{CS}}} A$. The critical cases are when the last rule applied is ax or can. In the former case, it suffices to notice that if A is an instance of an axiom, then so is $A\sigma$. In the latter case, we make use of the fact that CS is schematic: with $[c]A \in \text{CS}$ and $[!^n c] \dots [c]A$ appearing in our consecution, we obtain a proof of $\Gamma\sigma \vdash ([!^n c] \dots [c]A)\sigma$ via can by noticing that $([!^n c] \dots [c]A)\sigma = [!^n c] \dots [c](A\sigma)$ and that $[c]A\sigma \in \text{CS}$ as CS is schematic. $\square$

The next lemma expresses an interesting feature of justification logic: its (object) language allows for the reflection of its own (meta) proofs within formulas. On top of its conceptual interest, we leverage this lemma in many places throughout the paper.

Lemma 2.14 (Lifting Lemma). *Let CS be axiomatically appropriate. Let $k \in \mathbb{N}$, $\{A_1, \dots, A_k, B, A\} \subseteq \mathcal{L}_J$, $\{t_1, \dots, t_k\} \subseteq \text{PrfTm}$ and $m \in \text{SatTm}$. Then*

1. *If $A_1, \dots, A_k \vdash_{\text{JIK}_{CS}} A$ then there exists a proof term t s.t. $[t_1]A_1, \dots, [t_k]A_k \vdash_{\text{JIK}_{CS}} [t]A$.*
2. *If $A_1, \dots, A_k, B \vdash_{\text{JIK}_{CS}} A$ then there exists a satisfier term n s.t. $[t_1]A_1, \dots, [t_k]A_k, \langle m \rangle B \vdash_{\text{JIK}_{CS}} \langle n \rangle A$.*

Proof. As in [29], we proceed by induction on the structure of the proof of $A_1, \dots, A_k \vdash_{\text{JIK}_{CS}} A$, thereby inspecting the last rule applied. Our requirement that CS is axiomatically appropriate shows in the base case when $A \in \text{Base}^I$: the existence of a constant c such that $[c]A \in \text{CS}$ is ensured, thereby giving us $[t_1]A_1, \dots, [t_k]A_k \vdash_{\text{JIK}_{CS}} [c]A$ via can . We expand on the case for can , as the remaining cases are standard. In this case $A = [!^I c][\dots][!^I c][c]B$ for some proof constant $c \in \text{PrfConst}$ and formula B with $[c]B \in \text{CS}$. Instantiating can one step further, we get $[t_1]A_1, \dots, [t_k]A_k \vdash_{\text{JIK}_{CS}} [!^{I+1} c][!^I c][\dots][!^I c][c]B$ and hence $[t_1]A_1, \dots, [t_k]A_k \vdash_{\text{JIK}_{CS}} [!^{I+1} c]A$. $\square$

The proof of the above lemma informs us that the terms t and n we build are parametric in $t_1, \dots, t_k$ and m . As a consequence, when $k = 0$ the first part of the lemma forces t to be a *ground* term, i.e. a proof term which contains no proof or satisfier variable.

The technical lemma below generalises the axioms governing the operators $+$ and $\sqcup$, a useful tool in the completeness proof we provide later on.

Lemma 2.15. *Let CS be axiomatically appropriate. Let $i, j \in \mathbb{N}$, $\{A_1, \dots, A_i, B_1, \dots, B_j\} \subseteq \mathcal{L}_J$, $\{t_1, \dots, t_i\} \subseteq \text{PrfTm}$ and $\{m_1, \dots, m_j\} \subseteq \text{SatTm}$. Then there exists a proof term t and a satisfier term m such that:*

1. $\vdash_{\text{JIK}_{CS}} ([t_1]A_1 \vee \dots \vee [t_i]A_i) \rightarrow [t](A_1 \vee \dots \vee A_i)$
2. $\vdash_{\text{JIK}_{CS}} (\langle m_1 \rangle A_1 \vee \dots \vee \langle m_j \rangle A_j) \rightarrow \langle m \rangle (A_1 \vee \dots \vee A_j)$

Proof. By the Lifting Lemma 2.14 and using the axioms $j +_l$ and $j +_r$ repetitively for the first statement, or $j \sqcup_l$ and $j \sqcup_r$ for the second statement. $\square$

2.4 From Modal to Justification – and back

We now proceed to formalise the connection between JIK_{CS} and IK : in one direction, through translating their respective languages; in the other, by using *realisation* – this is the overarching goal of our paper.

Definition 2.16 (Forgetful projection). The *forgetful projection* is a map $(\cdot)^f : \mathcal{L}_J \rightarrow \mathcal{L}_{\square}$ inductively defined below, where $*$ $\in \{\wedge, \vee, \rightarrow\}$.

$$\bullet \perp^f := \perp \quad \bullet p^f := p \quad \bullet (A * B)^f := (A^f * B^f) \quad \bullet ([t]A)^f := \square A^f \quad \bullet (\langle m \rangle A)^f := \diamond A^f$$

Theorem 2.17. *Let $A \in \mathcal{L}_J$. If $\Gamma \vdash_{\text{JIK}_{CS}} A$ then $\Gamma^f \vdash_{\text{IK}} A^f$.*

Proof. This follows from the fact that the forgetful projection on axioms of JIK_{CS} and conclusions of the can rule are theorems of IK . $\square$

Corollary 2.18. *JIK_{CS} is consistent.*

Proof. Suppose $\vdash_{\text{JIK}_{CS}} \perp$. Then by Theorem 2.17, $\vdash_{\text{IK}} \perp$ which is a contradiction. $\square$

The reverse direction is expressed via realisation maps.

Definition 2.19 (Realisation map). A *realisation map* is a function $r : \mathcal{L}_\square \rightarrow \mathcal{L}_J$ such that $r(G)^f = G$ for each $G \in \mathcal{L}_\square$.

In the remaining of the paper, we set ourselves to prove the next theorem.

Theorem 2.20 (Realisation Theorem). *For schematic CS, there is a realisation map r such that:*

$$\forall G \in \mathcal{L}_\square. \vdash_{IK} G \implies \vdash_{JK_{CS}} r(G).$$

This *realisation map* is precisely what embeds the modal logic IK into its corresponding justification logic JK_{CS} . The condition $r(G)^f = G$ ensures that each $\square$ (and each $\diamond$) occurring in G is replaced by exactly one proof term (and satisfier term, respectively). In other words, a realisation map takes a modal formula G into a justification formula with the same formula tree.

3 Basic Modular Models

Basic modular models were first introduced by Mkrtychev [32] in a classical setting. These models can be seen as an extension of valuations for classical propositional logic, which assign a truth value to each proposition, with an additional interpretation on proof terms.

In the intuitionistic setting, Marti and Studer [30] build basic modular models for an intuitionistic justification logic by instead enhancing the relational semantics for intuitionistic propositional logic. In this section, we extend these models with an interpretation on satisfier terms.

3.1 Definition and Soundness

For the definition of basic models for JK_{CS} , expanding upon the basic models given in [30], we need the following operation on sets of formulas $\Gamma \cdot \Delta := \{B \in \mathcal{L}_J \mid A \rightarrow B \in \Gamma \text{ and } A \in \Delta\}$.

Definition 3.1 (Basic model). A *basic (intuitionistic modular) model* is a tuple $\mathcal{B} = (W, \leq, *_{Tm}, *_{Prop})$, where W is a non-empty set of worlds, $\leq$ is a pre-order on W , $*_{Prop} : Prop \times W \rightarrow \{0, 1\}$, and $*_{Tm} : (PrfTm \cup SatTm) \times W \rightarrow \mathcal{P}(\mathcal{L}_J)$.

We abuse notation and combine $*_{Tm}$ and $*_{Prop}$ into a single function $*$, making basic modular models tuples of the shape $(W, \leq, *)$. We also write s_w^* for $*(s, w)$. In a basic modular model, the function $*$ satisfies the following conditions on proof and satisfier terms:

1. $s_w^* \cdot t_w^* \subseteq (s \cdot t)_w^*$.
2. $s_w^* \cdot m_w^* \subseteq (s * m)_w^*$.
3. $s_w^* \cup t_w^* \subseteq (s + t)_w^*$.
4. $m_w^* \cup n_w^* \subseteq (m \sqcup n)_w^*$.
5. $A \in t_w^*$ for any conclusion $[t]A$ of can.
6. If $A \vee B \in m_w^*$ then $A \in m_w^*$ or $B \in m_w^*$.
7. If for all $v \geq w$, $A \notin m_v^*$ or $B \in t_v^*$, then $A \rightarrow B \in (m \triangleright t)_w^*$.
8. $\perp \notin m_w^*$.

Definition 3.2 (Truth in a basic model). Given a basic model $\mathcal{B}$, a point $w \in W$ and a formula A , we define the *truth of A at w in $\mathcal{B}$* recursively on the structure of A :

$$\begin{aligned}
\mathcal{B}, w \Vdash p & \text{ iff } p_w^* = 1 \\
\mathcal{B}, w \Vdash \perp & \text{ never} \\
\mathcal{B}, w \Vdash A \wedge B & \text{ iff } \mathcal{B}, w \Vdash A \text{ and } \mathcal{B}, w \Vdash B \\
\mathcal{B}, w \Vdash A \vee B & \text{ iff } \mathcal{B}, w \Vdash A \text{ or } \mathcal{B}, w \Vdash B \\
\mathcal{B}, w \Vdash A \rightarrow B & \text{ iff } \forall v \geq w \text{ if } \mathcal{B}, v \Vdash A \text{ then } \mathcal{B}, v \Vdash B \\
\mathcal{B}, w \Vdash [t]A & \text{ iff } A \in t_w^* \\
\mathcal{B}, w \Vdash \langle m \rangle A & \text{ iff } A \in m_w^*
\end{aligned}$$

We write $\models_b A$ if for any basic model $\mathcal{B}$ and $w \in W$ we have $\mathcal{B}, w \Vdash A$.

The monotonicity conditions imposed on $*$ port to the truth of formulas.

Lemma 3.3 (Monotonicity Lemma). *Let $\mathcal{B} = (W, \leq, *)$ be a basic modular model, w and $v \in W$ such that $w \leq v$, and A be a formula. Then: $\mathcal{B}, w \Vdash A \implies \mathcal{B}, v \Vdash A$.*

Proof. We proceed by induction on A . The base case $A = p$ follows by the (M1) condition in Definition 3.1. The cases $A = B \wedge C, B \vee C, B \rightarrow C$ follow from a standard argument. When $A = [t]B$, we get $B \in \mathfrak{t}_w^*$ by Definition 3.2. Then, using (M2) of Definition 3.1 we obtain $B \in \mathfrak{t}_v^*$ hence $\mathcal{B}, v \Vdash [t]B$. The case where $A = \langle m \rangle B$ is similar. $\square$

The soundness of JIK_{CS} with respect to its semantics on basic models can now be established.

Theorem 3.4 (Soundness). *For any formula A : $\vdash_{\text{JIK}_{\text{CS}}} A \implies \models_b A$.*

Proof. By induction on the proof of A in JIK_{CS} . Fix a basic model $\mathcal{B}$ and a world $w \in W$. We show the validity of the new axioms and rules, and refer to [30] for the remaining ones.

- $\text{jk}_2 : [s](A \rightarrow B) \rightarrow \langle m \rangle A \rightarrow \langle s \star m \rangle B$. Let $v \geq w$ with $\mathcal{B}, v \Vdash [s](A \rightarrow B)$ i.e. $A \rightarrow B \in \mathfrak{s}_v^*$. Let $u \geq v$ with $\mathcal{B}, u \Vdash \langle m \rangle A$, i.e. $A \in \mathfrak{m}_u^*$. By the monotonicity property, $A \rightarrow B \in \mathfrak{s}_u^*$. So $B \in \mathfrak{s}_u^* \cdot \mathfrak{m}_u^* \subseteq (\mathfrak{s} \star \mathfrak{m})_u^*$ and hence $\mathcal{B}, u \Vdash \langle s \star m \rangle B$. Therefore by definition we have $\mathcal{B}, v \Vdash \langle m \rangle A \rightarrow [s \star m]B$ and hence $\mathcal{B}, w \Vdash [s](A \rightarrow B) \rightarrow \langle m \rangle A \rightarrow [s \star m]B$.
- $\text{jk}_3 : \langle m \rangle (A \vee B) \rightarrow (\langle m \rangle A \vee \langle m \rangle B)$. Let $v \geq w$ with $\mathcal{B}, v \Vdash \langle m \rangle (A \vee B)$ i.e. $A \vee B \in \mathfrak{m}_v^*$. By property 6 of Definition 3.1, $A \in \mathfrak{m}_v^*$ or $B \in \mathfrak{m}_v^*$. So $\mathcal{B}, v \Vdash \langle m \rangle A$ or $\mathcal{B}, v \Vdash \langle m \rangle B$, hence $\mathcal{B}, v \Vdash \langle m \rangle A \vee \langle m \rangle B$. This gives $\mathcal{B}, w \Vdash \langle m \rangle (A \vee B) \rightarrow (\langle m \rangle A \vee \langle m \rangle B)$.
- $\text{jk}_4 : (\langle m \rangle A \rightarrow [t]B) \rightarrow [m \triangleright t](A \rightarrow B)$. Let $v \geq w$ with $\mathcal{B}, v \Vdash \langle m \rangle A \rightarrow [t]B$ i.e. for all $u \geq v$, $A \notin \mathfrak{m}_u^*$ or $B \in \mathfrak{t}_u^*$. By property 7 of Definition 3.1, $A \rightarrow B \in (\mathfrak{m} \triangleright \mathfrak{t})_v^*$. So $\mathcal{B}, v \Vdash [m \triangleright t](A \rightarrow B)$. Hence $\mathcal{B}, w \Vdash (\langle m \rangle A \rightarrow [t]B) \rightarrow [m \triangleright t](A \rightarrow B)$.
- $\text{jk}_5 : \langle m \rangle \perp \rightarrow \perp$. Let $v \geq w$, by property 8 of Definition 3.1, we have $\perp \notin \mathfrak{m}_v^*$ and so $\mathcal{B}, v \nVdash \langle m \rangle \perp$. Therefore, $\mathcal{B}, w \Vdash \langle m \rangle \perp \rightarrow \perp$.
- $[!^n c][!^{n-1} c][\dots][!c][c]A$ is a conclusion of can. Then, as $[!^{n-1} c][\dots][!c][c]A \in (!^n c)_w^*$, we have that $\mathcal{B}, w \Vdash [!^n c][!^{n-1} c][\dots][!c][c]A$. $\square$

Remark 3.5. To show soundness, it is important that conditions 1-4 in Def. 3.1 are inclusions and not equalities. As for example, were they equalities, the formula $(\langle t \star m \rangle q \wedge \langle m \rangle p) \rightarrow [t](p \rightarrow q)$ would be valid. However, it is not a theorem of JIK_{CS} – otherwise, by Theorem 2.17 we would get $\vdash_{\text{IK}} (\Diamond q \wedge \Diamond p) \rightarrow \Box(p \rightarrow q)$ which is not true. We can build a countermodel $\mathcal{B}$ to this formula meeting the inclusion-based conditions: $W = \{w\}$, $\leq = \{(w, w)\}$, $(\mathfrak{t} \star \mathfrak{m})_w^* = \{q\}$, $\mathfrak{m}_w^* = \{p\}$, $\mathfrak{t}_w^* = \emptyset$. Note that $\mathfrak{t}_w^* \cdot \mathfrak{m}_w^* = \emptyset \subseteq (\mathfrak{t} \star \mathfrak{m})_w^*$. We have $\mathcal{B}, w \Vdash \langle \mathfrak{t} \star \mathfrak{m} \rangle q \wedge \langle \mathfrak{m} \rangle p$ but $\mathcal{B}, w \nVdash [t](p \rightarrow q)$, hence $\mathcal{B}, w \nVdash (\langle \mathfrak{t} \star \mathfrak{m} \rangle q \wedge \langle \mathfrak{m} \rangle p) \rightarrow [t](p \rightarrow q)$.

3.2 Completeness

In this section we prove the reverse direction of the last result, i.e. completeness.

Theorem 3.6 (Completeness). *For any formula A : $\models_b A \implies \vdash_{\text{JIK}_{\text{CS}}} A$.*

Our proof exploits the construction of a canonical model for intuitionistic justification logic. This syntactic structure uses prime sets as worlds.

Definition 3.7 (Prime Set). *A prime set $\Gamma \subseteq \mathcal{L}_J$ is a set of formulas satisfying the following:*

- *Deductive closure*: if $\Gamma \vdash_{\text{JIK}_{\text{CS}}} A$ then $A \in \Gamma$.
- *Primeness*: if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.
- *Consistency*: $\perp \notin \Gamma$.

Notation 3.8. Let $\Gamma, \Delta \subseteq \mathcal{L}_J$. We write $\Gamma \vdash_{\text{JIK}_{\text{CS}}} \Delta$ if there is a finite set of formulas $\{A_1, \dots, A_n\} \subseteq \Delta$ such that $\Gamma \vdash_{\text{JIK}_{\text{CS}}} A_1 \vee \dots \vee A_n$. We also write $\Gamma \not\vdash_{\text{JIK}_{\text{CS}}} \Delta$ when it does not hold that $\Gamma \vdash_{\text{JIK}_{\text{CS}}} \Delta$.

The completeness proof relies on projecting unprovable consecutions into the canonical model.

Lemma 3.9 (Prime Lemma). *Let $\Gamma, \Delta \subseteq \mathcal{L}_J$. If $\Gamma \not\vdash_{\text{JIK}_{\text{CS}}} \Delta$ then there exists a prime set $\Gamma' \supseteq \Gamma$ such that $\Gamma' \not\vdash_{\text{JIK}_{\text{CS}}} \Delta$. Furthermore, Γ' is a maximal set (w.r.t. inclusion) satisfying these conditions.*

Proof sketch. The proof goes via a standard argument: we exploit an enumeration of formulas to extend Γ step-by-step while ensuring that the extension does not entail Δ . The limit of this process can be shown to be a prime set Γ' not entailing Δ . The maximality of Γ' follows from the fact that if a formula could be added without entailing Δ , then it would indeed have been added when enumerated. $\square$

Next, we define a structure which we show to be a basic (modular) model. To define parts of this structure, we use the sets of formulas $\mathbf{t}^{-1}\Gamma := \{A \in \mathcal{L}_J \mid [\mathbf{t}]A \in \Gamma\}$ and $\mathbf{m}^{-1}\Gamma := \{A \in \mathcal{L}_J \mid \langle \mathbf{m} \rangle A \in \Gamma\}$.

Definition 3.10 (Canonical basic model). Let $\mathcal{B}^c$ be the tuple $(W^c, \leq^c, *^c)$ where:

- $W^c := \{\Gamma \mid \Gamma \text{ is a prime set}\};$
- $\leq^c := \subseteq;$
- $p_\Gamma^{*c} = 1$ iff $p \in \Gamma;$
- $\mathbf{t}_\Gamma^{*c} := \mathbf{t}^{-1}\Gamma;$
- $\mathbf{m}_\Gamma^{*c} := \mathbf{m}^{-1}\Gamma.$

Proposition 3.11. $\mathcal{B}^c$ is a basic model.

Proof. We are required to prove that this structure satisfies all the properties from Definition 3.1. As an example, we show that $\mathbf{s}_\Gamma^{*c} \cdot \mathbf{t}_\Gamma^{*c} \subseteq (\mathbf{s} \cdot \mathbf{t})_\Gamma^{*c}$. Let $B \in \mathbf{s}_\Gamma^{*c} \cdot \mathbf{t}_\Gamma^{*c}$. By definition, there is a formula A such that $A \rightarrow B \in \mathbf{s}_\Gamma^{*c}$ and $A \in \mathbf{t}_\Gamma^{*c}$. This means that $\Gamma \vdash_{\text{JIK}_{\text{CS}}} [\mathbf{s}](A \rightarrow B)$ and $\Gamma \vdash_{\text{JIK}_{\text{CS}}} [\mathbf{t}]A$. Using the jk_1 axiom we get $\Gamma \vdash_{\text{JIK}_{\text{CS}}} [\mathbf{s} \cdot \mathbf{t}]B$, hence $[\mathbf{s} \cdot \mathbf{t}]B \in \Gamma$ by deductive closure. Therefore $B \in (\mathbf{s} \cdot \mathbf{t})^{-1}\Gamma = (\mathbf{s} \cdot \mathbf{t})_\Gamma^{*c}$. $\square$

Lemma 3.12 (Truth Lemma). *Let A be a formula and Γ be a prime set. Then: $A \in \Gamma \iff \mathcal{B}^c, \Gamma \Vdash A$.*

Proof. By induction on A . All cases are covered in [30], but the case for $A = \langle \mathbf{m} \rangle B$ which we show:

$$\begin{aligned}
 \langle \mathbf{m} \rangle B \in \Gamma &\iff B \in \mathbf{m}^{-1}\Gamma \text{ by definition of } \mathbf{m}^{-1}\Gamma \\
 &\iff B \in \mathbf{m}_\Gamma^{*c} \text{ by definition of the model } \mathcal{B}^c \\
 &\iff \mathcal{B}^c, \Gamma \Vdash \langle \mathbf{m} \rangle B \text{ by definition of truth in a basic model.}
 \end{aligned}$$

$\square$

With these results in hand, we can prove completeness.

Proof of Theorem 3.6. We prove the contrapositive. Suppose $\not\vdash_{\text{JIK}_{\text{CS}}} A$. Then by the Prime Lemma 3.9, there exists a prime set Γ such that $\Gamma \not\vdash_{\text{JIK}_{\text{CS}}} A$. So $A \notin \Gamma$, and by the Truth Lemma 3.12 we have $\mathcal{B}^c, \Gamma \not\Vdash A$. Finally, Proposition 3.11 ensures that the $\mathcal{B}^c$ belongs to the adequate class of models. $\square$

4 Modular Models

In the classical setting, modular models [5] are epistemic models extending basic modular models with a modal accessibility relation to model the notion of *knowledge*. A key feature of these models is *justification yields belief*, which says that if a formula of the form $[t]A$ holds at a world, then A must hold in all accessible worlds. This behaviour imitates that of the $\Box$ operator in modal logic and hence the feature promotes “backwards compatibility” from the justification logic to its corresponding modal logic.

In our setting, we must make several adaptations. First, satisfaction of a modal formula $\Box G$ is not defined locally, but globally along the intuitionistic pre-order, so the justification yields belief principle must account for this. Second, to deal with satisfiers we introduce *satisfaction yield possibility*, which says that if a formula of the form $\langle m \rangle A$ holds at a world, then A must hold in some accessible world. Finally, to promote “backwards compatibility”, we have to ensure that the additional frame conditions of a birelational frame (Definition 2.3) [17] also hold.

4.1 Definition and Soundness

We define models for intuitionistic justification logic that bridge it with intuitionistic modal logic via the modal relation. To flesh out this connection, we make the interpretation of terms resonate with the modal relation by imposing conditions on the former using the latter and the notion of truth. To get there, we need to introduce preliminary structures on which we evaluate formulas.

Definition 4.1 (Intuitionistic quasi-model). An (intuitionistic) quasi-model is a tuple $\mathcal{M} = (W, \leq, R, *)$ where $\mathcal{B} = (W, \leq, *)$ is a basic model and R is a binary relation on W .

Definition 4.2 (Truth in a quasi-model). For a quasi-model $\mathcal{M} = (W, \leq, R, *)$, a point $w \in W$ and a formula A , we define the *truth of A at w in $\mathcal{M}$* as: $\mathcal{M}, w \Vdash A$ iff $\mathcal{B}, w \Vdash A$ where $\mathcal{B} = (W, \leq, *)$.

As the notion of truth is the same in basic and quasi-models, the following obviously follows.

Lemma 4.3 (Monotonicity Lemma). Let $\mathcal{M} = (W, \leq, R, *)$ be a quasi-model. Let $w, v \in W$ such that $w \leq v$. Let A be a formula. Then: $\mathcal{M}, w \Vdash A \implies \mathcal{M}, v \Vdash A$

Proof. By definition $\mathcal{M}, w \Vdash A$ implies $\mathcal{B}, w \Vdash A$ where $\mathcal{B} = (W, \leq, *)$. Using the Monotonicity Lemma 3.3 for basic modular models we get $\mathcal{B}, v \Vdash A$. By definition here again, we obtain $\mathcal{M}, v \Vdash A$. $\square$

Remark 4.4. Definition 4.2 exhibits the *locality of truth* [30, 28]. This notion in quasi-models helps us use these intermediate structures to bridge justification and modal logics: we obtain the adequate models by restricting the interpretation of terms by R and truth as shown in the next definition.

Definition 4.5 (Intuitionistic modular model). An (intuitionistic modular) model is a quasi-model $\mathcal{M} = (W, \leq, R, *)$ such that $\mathcal{M}_\Box := (W, \leq, R, *_{\text{Prop}})$ is a birelational intuitionistic model together with:

- *Justification yields belief* (JYB) principle: for all proof terms $t \in \text{PrfTm}$

$$t_w^* \subseteq \Box_w := \{A \mid \text{for all } w', v' \in W \text{ with } w \leq w' R v', \mathcal{M}, v' \Vdash A\}$$

- *Satisfaction yields possibility* (SYP) principle: for all satisfiers $m \in \text{SatTm}$

$$m_w^* \subseteq \Diamond_w := \{A \mid \text{there exists } v \in W \text{ such that } w R v \text{ and } \mathcal{M}, v \Vdash A\}$$

We write $\models_m A$ if for any model $\mathcal{M}$ and world $w \in W$ we have $\mathcal{M}, w \Vdash A$.

In contrast to the classical setting or other intuitionistic variants [5, 30, 33], the definition of (JYB) takes into account the global interpretation of $\Box$ via the pre-order $\leq$.

Exploiting soundness with respect to basic models, the result follows for models.

Theorem 4.6 (Soundness of modular models). *For any formula A : $\vdash_{\text{JIK}_{\text{CS}}} A \implies \models_m A$.*

Proof. Let $\mathcal{M} = (W, \leq, R, *)$ be a modular model and w a world. Since $\mathcal{B} = (W, \leq, *)$ is a basic modular model, we have $\mathcal{B}, w \Vdash A$ by Theorem 3.4. Hence, $\mathcal{M}, w \Vdash A$ by Definition 4.5. $\square$

Remark 4.7 (Differences between Modular models and Fitting models). The first epistemic models for classical justification logics were introduced by Fitting [18]. These differ from Artemov's modular models [5] on the definition of satisfaction for justification formulas of the form $[t]A$. Fitting models additionally require A to hold in all *modally accessible* worlds, i.e., $\mathcal{M}, w \Vdash [t]A$ iff $A \in \tau_w^*$ and for all $v \in W$ with wRv , $\mathcal{M}, v \Vdash A$. In contrast, modular models make an *ontological separation* with the modal accessibility relation which plays no role on the satisfaction of formulas. Yet, both classes of models ensure “backwards compatibility” with modal logic [28]; in fact, they are equivalent [27].

4.2 Completeness

We prove completeness w.r.t. the semantics on models, assuming that CS is axiomatically appropriate.

Theorem 4.8 (Completeness). *For any formula A : $\models_m A \implies \vdash_{\text{JIK}_{\text{CS}}} A$.*

We construct once more a canonical model using the basic canonical model from Definition 3.10 as a basis. In essence, we aim to show that an adequate modal relation can be grafted onto this basic model.

We first need to define, for a given $\Gamma \subseteq \mathcal{L}_J$, certain sets of formulas: $\Gamma^\sharp := \{A \mid \text{for some } t \in \text{PrfTm}, [t]A \in \Gamma\}$ and $\Gamma^\flat := \{A \mid \text{for all } m \in \text{SatTm}, \langle m \rangle A \notin \Gamma\}$.

Definition 4.9 (Canonical intuitionistic modular model). Let $\mathcal{M}^c$ be the tuple $(W^c, \leq^c, R^c, *^c)$ where $\mathcal{B}^c = (W^c, \leq^c, *^c)$ is the canonical basic model, and $\Gamma R^c \Delta$ is defined as: if $[t]A \in \Gamma$ then $A \in \Delta$ and for each $A \in \Delta$ there exists $\langle m \rangle A \in \Gamma$.

It is immediate that $\mathcal{M}^c$ is a quasi-model, as $\mathcal{B}^c$ is a basic model by Proposition 3.11. Utilising the locality of truth, this allows us to directly prove the Truth Lemma for $\mathcal{M}^c$.

Lemma 4.10 (Truth Lemma). *Let A be a formula and Γ be a prime set. Then: $A \in \Gamma \iff \mathcal{M}^c, \Gamma \Vdash A$.*

Proof. As $\mathcal{B}^c$ is a basic model, we have $\mathcal{M}^c, \Gamma \Vdash A \iff \mathcal{B}^c, \Gamma \Vdash A \iff A \in \Gamma$ by Lemma 3.12. $\square$

The canonical model also displays the following strong properties which ensure all collections of formulas made true by the same satisfier m (or false by the same proof term t) must be in fact true (or false, respectively) at the same world.

Lemma 4.11. *Let $\Pi \subseteq \mathcal{L}_J$ and $\Gamma \in W^c$.*

1. *If for all $t \in \text{PrfTm}$ and for all $A_1, \dots, A_n \in \Pi$, $\Gamma \not\vdash_{\text{JIK}_{\text{CS}}} [t](A_1 \vee \dots \vee A_n)$, then there are $\Gamma', \Delta' \in W^c$ such that $\Gamma \subseteq \Gamma' R^c \Delta'$ and $\Delta' \not\vdash_{\text{JIK}_{\text{CS}}} \Pi$.*
2. *If for some $m \in \text{SatTm}$ and for all $A_1, \dots, A_n \in \Pi$, $\Gamma \vdash_{\text{JIK}_{\text{CS}}} \langle m \rangle (A_1 \wedge \dots \wedge A_n)$, then there is $\Delta \in W^c$ such that $\Gamma R^c \Delta$ and $\Delta \vdash_{\text{JIK}_{\text{CS}}} A$ for all $A \in \Pi$.*

Proof. 1. Let $\tilde{\Pi} = \{[t](A_1 \vee \dots \vee A_n) \mid t \in \text{PrfTm and } A_1, \dots, A_n \in \Pi\}$. Since for all $t \in \text{PrfTm}$ and $A_1, \dots, A_n \in \Pi$ we have $\Gamma \not\vdash_{\text{JIK}_{\text{CS}}} [t](A_1 \vee \dots \vee A_n)$, we can establish that $\Gamma \not\vdash_{\text{JIK}_{\text{CS}}} \tilde{\Pi}$. Indeed, the primeness of Γ informs us that if it entailed a finite disjunction of elements of $\tilde{\Pi}$ then it would entail one of its disjuncts. Then, by the Prime Lemma 3.9 we get a maximal prime set Γ' with $\Gamma \subseteq \Gamma'$ and $\Gamma' \not\vdash_{\text{JIK}_{\text{CS}}} \tilde{\Pi}$. We can then show that $\Gamma'^{\#} \not\vdash_{\text{JIK}_{\text{CS}}} \Pi$. Otherwise, there would be $C_1, \dots, C_m \in \Gamma'^{\#}$ with each $[t_i]C_i \in \Gamma'$ and $A_1, \dots, A_n \in \Pi$ such that $C_1, \dots, C_m \vdash_{\text{JIK}_{\text{CS}}} A_1 \vee \dots \vee A_n$, which by the Lifting Lemma 2.14 would give $\Gamma' \vdash_{\text{JIK}_{\text{CS}}} [t](A_1 \vee \dots \vee A_n)$ for some $t \in \text{PrfTm}$. Then, we can apply the Prime Lemma 3.9 again on $\Gamma'^{\#} \not\vdash_{\text{JIK}_{\text{CS}}} \Pi$ to construct a prime set Δ' with $\Gamma'^{\#} \subseteq \Delta'$ and $\Delta' \not\vdash_{\text{JIK}_{\text{CS}}} \Pi$.

Finally we need to check that $\Gamma' R^c \Delta'$. Since $\Gamma'^{\#} \subseteq \Delta'$ by construction we have that if $[t]A \in \Gamma'$ then $A \in \Delta'$, hence it is enough to show that for each $A \in \Delta'$ there exists $\langle m \rangle A \in \Gamma'$. For such $A \in \Delta'$, suppose for a contradiction that for all $m \in \text{SatTm}$, $\langle m \rangle A \notin \Gamma'$. Then, for all $A_1, \dots, A_n \in \Pi$ we have $\Gamma', \langle m \rangle A \not\vdash_{\text{JIK}_{\text{CS}}} [t](A_1 \vee \dots \vee A_n)$ for each m and t , otherwise using the Deduction Theorem 2.11 and jk_4 we would have $\Gamma' \vdash_{\text{JIK}_{\text{CS}}} [m \triangleright t](A \rightarrow (A_1 \vee \dots \vee A_n))$ and hence $A \rightarrow (A_1 \vee \dots \vee A_n) \in \Gamma'^{\#} \subseteq \Delta'$, so we could deduce from $A \in \Delta'$ and primeness of Δ' that $A_i \in \Delta'$ for some $i \leq n$. However, we have $\Gamma' \subsetneq \Gamma' \cup \{\langle m \rangle A\}$ and so the Prime Lemma 3.9 would give a strictly larger prime set satisfying the same conditions as Γ' , contradicting its maximality – a similar use of maximality to e.g. [17, Lemma 3 and Theorem 7].

2. Let us fix $m \in \text{SatTm}$ such that for all $A_1, \dots, A_n \in \Pi$, $\Gamma \vdash_{\text{JIK}_{\text{CS}}} \langle m \rangle (A_1 \wedge \dots \wedge A_n)$, hence by deductive closure of Γ , $\langle m \rangle (A_1 \wedge \dots \wedge A_n) \in \Gamma$. We first show that $\Pi \cup \Gamma^{\#} \not\vdash_{\text{JIK}_{\text{CS}}} \Gamma^{\flat}$. Otherwise, there would be $A_1, \dots, A_l \in \Pi$, $B_1, \dots, B_n \in \Gamma^{\#}$ with $[t_i]B_i \in \Gamma$ for some t_i , and $C_1, \dots, C_p \in \Gamma^{\flat}$ such that $A_1, \dots, A_l, B_1, \dots, B_n \vdash_{\text{JIK}_{\text{CS}}} C_1 \vee \dots \vee C_p$, and therefore $A_1 \wedge \dots \wedge A_l, B_1, \dots, B_n \vdash_{\text{JIK}_{\text{CS}}} C_1 \vee \dots \vee C_p$. By the Lifting Lemma 2.14 this yields some $n \in \text{SatTm}$ such that $\langle m \rangle (A_1 \wedge \dots \wedge A_l), [t_1]B_1, \dots, [t_n]B_n \vdash_{\text{JIK}_{\text{CS}}} \langle n \rangle (C_1 \vee \dots \vee C_p)$, hence $\Gamma \vdash_{\text{JIK}_{\text{CS}}} \langle n \rangle (C_1 \vee \dots \vee C_p)$. Now, using the jk_3 axiom as well as the deductive closure and primeness of Γ , we get $\langle n \rangle C_j \in \Gamma$ for some $j \leq p$, which contradicts $C_j \in \Gamma^{\flat}$. Then, we can apply the Prime Lemma 3.9 on $\Pi \cup \Gamma^{\#} \not\vdash_{\text{JIK}_{\text{CS}}} \Gamma^{\flat}$ to get a prime set $\Delta \supseteq \Pi \cup \Gamma^{\#}$ with $\Delta \not\vdash_{\text{JIK}_{\text{CS}}} \Gamma^{\flat}$. Finally, $\Gamma R^c \Delta$ because as $\Gamma^{\#} \subseteq \Delta$ by construction we have that if $[t]A \in \Gamma$ then $A \in \Delta$, and as $\Delta \not\vdash_{\text{JIK}_{\text{CS}}} \Gamma^{\flat}$ we have that for each $A \in \Delta$ there must exist $\langle m \rangle A \in \Gamma$. $\square$

As our next step in showing that $\mathcal{M}^c$ is an intuitionistic modular model, we show that $\mathcal{M}_{\square}^c$ is an intuitionistic birelational model. We therefore proceed to establish the confluence properties.

Lemma 4.12 (Forwards and backwards confluence in $\mathcal{M}^c$). *Let $\Gamma, \Gamma', \Delta \in W^c$.*

(FC) *If $\Gamma \subseteq \Gamma'$ and $\Gamma R^c \Delta$, then there exists $\Delta' \in W^c$ with $\Delta \subseteq \Delta'$ and $\Gamma' R^c \Delta'$.*

(BC) *If $\Gamma R^c \Delta \subseteq \Delta'$, then there exists $\Gamma' \in W^c$ with $\Gamma \subseteq \Gamma' R^c \Delta'$.*

Proof. While our proof is similar to the one for modal birelational models for IK in [17], there are some subtleties in handling the satisfier terms. We expand on this point in the remark below. $\square$

Remark 4.13. The use of proof terms in classical justification logic is known to introduce some non-normal behaviours of the modality [4, 6].

Similarly, in the intuitionistic setting, the IK theorem $\Diamond(G_1 \wedge \dots \wedge G_n) \rightarrow (\Diamond G_1 \wedge \dots \wedge \Diamond G_n)$ does not distinguish the information the distinct diamonds are carrying. The justification version of this theorem, $\langle m \rangle (A_1 \wedge \dots \wedge A_n) \rightarrow (\langle t_1 \star m \rangle A_1 \wedge \dots \wedge \langle t_n \star m \rangle A_n)$ where m is a satisfier variable and each t_i is a ground term obtained by applying the Lifting Lemma 2.14 to $\vdash_{\text{IK}} (A_1 \wedge \dots \wedge A_n) \rightarrow A_i$, is more constrained due to the specific structure of the $\star$ -satisfier terms on the right-hand-side.

We finally establish that $\mathcal{M}^c$ is a modular model by proving that it obeys the JYB and SYP principles. *Justification yields belief* is already achieved from the definition of R^c , following the lines of [30].

Lemma 4.14 (JYB in $\mathcal{M}^c$). *Let $\mathfrak{t} \in \text{PrfTm}$ and $\Gamma \in W^c$. Then:*

$A \in \mathfrak{t}_\Gamma^{*c} \implies \text{for all } \Gamma', \Delta' \in W^c \text{ such that } \Gamma \subseteq \Gamma' R^c \Delta' \text{ we have } \mathcal{M}^c, \Delta' \Vdash A.$

Proof. Suppose $A \in \mathfrak{t}_\Gamma^{*c}$, then $[\mathfrak{t}]A \in \Gamma$ by Definition 3.10. So, for $\Gamma' \supseteq \Gamma$ we have $[\mathfrak{t}]A \in \Gamma'$ too. Hence, for Δ' with $\Gamma' R^c \Delta'$, we have $A \in \Delta'$ by Definition 4.9. By the Truth Lemma 4.10, we have $\mathcal{M}^c, \Delta' \Vdash A$. $\square$

The remaining principle of *satisfaction yields possibility* is slightly more involved, as shown below.

Lemma 4.15 (SYP in $\mathcal{M}^c$). *Let $\mathfrak{m} \in \text{SatTm}$ and $\Gamma \in W^c$. Then:*

$A \in \mathfrak{m}_\Gamma^{*c} \implies \text{there exists } \Delta \in W^c \text{ such that } \Gamma R^c \Delta \text{ and } \mathcal{M}^c, \Delta \Vdash A.$

Proof. Suppose $A \in \mathfrak{m}_\Gamma^{*c}$, then $\langle \mathfrak{m} \rangle A \in \Gamma$ by definition. So, $\Gamma \vdash_{\text{JIK}_{\text{CS}}} \langle \mathfrak{m} \rangle A$. Therefore, by Lemma 4.11 there is $\Delta \in W^c$ with $\Gamma R^c \Delta$ and $\Delta \vdash_{\text{JIK}_{\text{CS}}} A$. By deductive closure $A \in \Delta$ and apply the Truth Lemma 4.10 for $\mathcal{M}^c$ to get $\mathcal{M}^c, \Delta \Vdash A$. $\square$

The next proposition reveals the nature of $\mathcal{M}^c$ by gathering our results.

Proposition 4.16. *$\mathcal{M}^c$ is an intuitionistic modular model.*

Proof. First, $\mathcal{M}^c$ is a quasi-model as $\mathcal{B}^c$ is a basic model by Proposition 3.11. Then, Lemma 4.12 shows that forwards and backwards confluence hold in $\mathcal{M}^c$. Finally, Lemma 4.14 and Lemma 4.15 respectively prove that the properties JYB and SYP are satisfied in $\mathcal{M}^c$. $\square$

With this in hand, completeness follows.

Proof of Theorem 4.8. We reason contrapositively and assume $\not\vdash_{\text{JIK}_{\text{CS}}} A$. The Prime Lemma 3.9 informs us of the existence of a prime set Γ such that $\Gamma \not\vdash_{\text{JIK}_{\text{CS}}} A$. So $A \notin \Gamma$, which gives $\mathcal{M}^c, \Gamma \not\vdash A$ by the Truth Lemma 4.10. Finally, Proposition 4.16 ensures that the $\mathcal{M}^c$ belongs to the adequate class of models. $\square$

5 Realisation

We recall that our goal is to prove the following.

Theorem 2.20. For schematic CS, there is a realisation map r such that for all $G \in \mathcal{L}_\square$:

$$\vdash_{\text{IK}} G \implies \vdash_{\text{JIK}_{\text{CS}}} r(G).$$

This realisation function is constructed methodically in two steps:

1. Firstly, we reason semantically that a theorem G of IK can be *pre-realised* into a disjunction of justification formulas which have “roughly” the same structure as G ;
2. Secondly, we *condense* a pre-realisation into a realisation by syntactical reasoning and substitutions to remove the extraneous content.

Our goal is more specifically to define *normal* realisation map which differentiates modalities based on their polarity. Recall that a subformula of G is *positive* if its position in the formula tree of G is reached from the root by following the left branch of an implication an even number of times; and call it *negative* otherwise.

Definition 5.1 (Normal realisation). A realisation map $r : \mathcal{L}_\square \rightarrow \mathcal{L}_\text{J}$ is *normal* if, for any $G \in \mathcal{L}_\square$, its negative modal subformulas $\Box F$ (or $\Diamond F$) are realised as $[x]B$ (or $\langle a \rangle B$) with $x \in \text{PrfVar}$ (or $a \in \text{SatVar}$, respectively) occurring in $r(G)$ exactly once.

Remark 5.2. The choice of a normal realisation ensures that no independent subformulas are unintentionally identified. For example, the IK-theorem $(\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$ can be realised as

$$(\langle a \rangle p \vee \langle a \rangle q) \rightarrow \langle t_1 \star a \sqcup t_2 \star a \rangle (p \vee q) \quad \text{or} \quad (\langle a \rangle p \vee \langle b \rangle q) \rightarrow \langle t_1 \star a \sqcup t_2 \star b \rangle (p \vee q)$$

where t_1 and t_2 are ground terms obtained by applying the Lifting Lemma 2.14 to $\vdash_{\text{JIK}_{\text{CS}}} p \rightarrow (p \vee q)$ and $\vdash_{\text{JIK}_{\text{CS}}} q \rightarrow (p \vee q)$, respectively. But, the first instance cannot be the result of a normal realisation.

For this reason, we need to do some bookkeeping on $\Box$ s and $\Diamond$ s: depending on their (positive or negative) polarity, they will be replaced by a variable in $\text{PrfVar} \cup \text{SatVar}$ or a compound term in $\text{PrfTm} \cup \text{SatTm}$. So, we first annotate by natural numbers $\Box$ s and $\Diamond$ s in the modal logic formulas. Each annotation is unique. Formally:

Definition 5.3 (Annotated modal formulas). An *annotated modal formula* is built up like a standard modal formula but instead of single modalities $\Box$ and $\Diamond$, there is an infinite family of indexed modalities $\Box_k$ and $\Diamond_k$, for $k \in \mathbb{N}$. We assume that no index occurs more than once in any given annotated formula.

Example 5.4. Given the modal formula $\Box \Diamond G \rightarrow \Box(\Diamond J \wedge \Diamond \Box(H \vee I))$, a possible annotated version is $\Box_1 \Diamond_2 G \rightarrow \Box_3(\Diamond_4 J \wedge \Diamond_5 \Box_6(H \vee I))$.

Additionally, we need to track the signs of modal formulas.

Definition 5.5 (Signed modal formulas). *Signed modal formulas* are annotated modal formulas assigned with a polarity T or F. We denote this as TG or FG. The set of signed formulas is $\mathcal{L}_{\Box}^{\text{ann}}$.

We stress that these annotations do not add semantic information: they are bookkeeping devices. Where it is clear, we do not distinguish between a modal formula and its annotated version.

5.1 Pre-realisation

In this subsection, we adapt the methodology of [18, 28, 33] and reason semantically on both the canonical model $\mathcal{M}^c$ and its induced birelational intuitionistic model $\mathcal{M}_{\Box}^c$ (recall Definition 4.5) to *pre-realise* modal formulas.

Definition 5.6 (Pre-realiser). The *pre-realiser* is a function $\langle \cdot \rangle : \mathcal{L}_{\Box}^{\text{ann}} \rightarrow \mathcal{P}(\mathcal{L})$ defined as:

$$\begin{aligned} \langle \top p \rangle &:= \{p\} & \langle \text{F}p \rangle &:= \{p\} \\ \langle \text{T}G \wedge H \rangle &:= \{A \wedge B \mid A \in \langle \text{T}G \rangle, B \in \langle \text{T}H \rangle\} & \langle \text{F}G \wedge H \rangle &:= \{A \wedge B \mid A \in \langle \text{F}G \rangle, B \in \langle \text{F}H \rangle\} \\ \langle \text{T}G \vee H \rangle &:= \{A \vee B \mid A \in \langle \text{T}G \rangle, B \in \langle \text{T}H \rangle\} & \langle \text{F}G \vee H \rangle &:= \{A \vee B \mid A \in \langle \text{F}G \rangle, B \in \langle \text{F}H \rangle\} \\ \langle \text{T}G \rightarrow H \rangle &:= \{A \rightarrow B \mid A \in \langle \text{F}G \rangle, B \in \langle \text{T}H \rangle\} & \langle \text{F}G \rightarrow H \rangle &:= \{(A_1 \wedge \dots \wedge A_n) \rightarrow (B_1 \vee \dots \vee B_m) \mid \\ & & & A_i \in \langle \text{T}G \rangle, B_j \in \langle \text{F}H \rangle\} \\ \langle \text{T}\Box_n G \rangle &:= \{[x_n]A \mid A \in \langle \text{T}G \rangle\} & \langle \text{F}\Box_n G \rangle &:= \{[t](A_1 \vee \dots \vee A_m) \mid A_i \in \langle \text{F}G \rangle, \\ & & & t \in \text{PrfTm}\} \\ \langle \text{T}\Diamond_n G \rangle &:= \{\langle a_n \rangle(A_1 \wedge \dots \wedge A_m) \mid A_i \in \langle \text{T}G \rangle\} & \langle \text{F}\Diamond_n G \rangle &:= \{\langle m \rangle A \mid A \in \langle \text{F}G \rangle, m \in \text{SatTm}\} \end{aligned}$$

Definition 5.7. A modal formula G is *pre-realisable* in JIK_{CS} if $\vdash_{\text{JIK}_{\text{CS}}} \langle \text{F}G \rangle$, i.e., there exist some $A_1, \dots, A_n \in \langle \text{F}G \rangle$ such that $\vdash_{\text{JIK}_{\text{CS}}} A_1 \vee \dots \vee A_n$.

The following is the key result of this subsection and corresponds to the first step of the realisation proof described above and provides an initial but imperfect connection between theorems of IK and theorems of JIK_{CS} . It leverages the semantic machinery developed up to now.

Theorem 5.8 (Pre-realisation). *Let CS be axiomatically appropriate. For $G \in \mathcal{L}_{\Box}$, if $\vdash_{\text{IK}} G$, then G is pre-realisable in JIK_{CS} .*

This is shown through the semantic connection between a pre-realiser and its modal formula which is understood in the following:

Theorem 5.9. *For every annotated modal formula G . For a prime set Γ .*

1. *If for all $A \in \langle\langle TG \rangle\rangle$ we have $\mathcal{M}^c, \Gamma \Vdash A$, then $\mathcal{M}_{\Box}^c, \Gamma \Vdash G$.*
2. *If for all $A \in \langle\langle FG \rangle\rangle$ we have $\mathcal{M}^c, \Gamma \nVdash A$, then $\mathcal{M}_{\Box}^c, \Gamma \nVdash G$.*

Proof. We prove both simultaneously by induction on G . The cases when $G = p, H \wedge I, H \vee I, H \rightarrow I$ follow the argument in [33]. We extend the proof with an adapted case for $\Box$ and a new one for $\Diamond$.

If $G = \Box_n H$ then $A \in \langle\langle TG \rangle\rangle$ is of the form $[x_n]B$ for $B \in \langle\langle TH \rangle\rangle$ and $A \in \langle\langle FG \rangle\rangle$ is of the form $[t](B_1 \vee \dots \vee B_m)$ for $B_1, \dots, B_m \in \langle\langle FH \rangle\rangle$ and $t \in \text{PrfTm}$.

1. Suppose for all $A \in \langle\langle TG \rangle\rangle$, $\mathcal{M}^c, \Gamma \Vdash A$, that is, for all $B \in \langle\langle TH \rangle\rangle$, $\mathcal{M}^c, \Gamma \Vdash [x_n]B$. So, $B \in x_n^* \Gamma$ and by Lemma 4.14, for all Γ', Δ' with $\Gamma \subseteq \Gamma' R^c \Delta'$, $\mathcal{M}^c, \Delta' \Vdash B$. By (IH) on 1., $\mathcal{M}_{\Box}^c, \Delta' \Vdash H$ and therefore $\mathcal{M}_{\Box}^c, \Gamma \Vdash \Box_n H$.
2. Suppose for all $A \in \langle\langle FG \rangle\rangle$, $\mathcal{M}^c, \Gamma \nVdash A$, that is, for all $B_1, \dots, B_m \in \langle\langle FH \rangle\rangle$ and $t \in \text{PrfTm}$, $\mathcal{M}^c, \Gamma \nVdash [t](B_1 \vee \dots \vee B_m)$. So by the Truth Lemma 4.10, $[t](B_1 \vee \dots \vee B_m) \notin \Gamma$, hence $\Gamma \not\vdash_{\text{JKCS}} [t](B_1 \vee \dots \vee B_m)$. By Lemma 4.11, there exist $\Gamma', \Delta' \in W^c$ such that $\Gamma \subseteq \Gamma' R^c \Delta'$ and $\Delta' \not\vdash_{\text{JKCS}} \langle\langle FH \rangle\rangle$. So for all $B \in \langle\langle FH \rangle\rangle$, $B \notin \Delta'$ and by the Truth Lemma 4.10 again, $\mathcal{M}^c, \Delta' \nVdash B$. By (IH) on 2., $\mathcal{M}_{\Box}^c, \Delta' \nVdash H$, hence $\mathcal{M}_{\Box}^c, \Gamma \nVdash \Box_n H$.

If $G = \Diamond_n H$ then $A \in \langle\langle TG \rangle\rangle$ is of the form $\langle a_n \rangle (B_1 \wedge \dots \wedge B_m)$ for $B_1, \dots, B_m \in \langle\langle TH \rangle\rangle$, and $A \in \langle\langle FG \rangle\rangle$ is of the form $\langle m \rangle B$ for $B \in \langle\langle FH \rangle\rangle$ and $m \in \text{SatTm}$.

1. Suppose for all $A \in \langle\langle TG \rangle\rangle$, $\mathcal{M}^c, \Gamma \Vdash A$, that is, for all $B_1, \dots, B_m \in \langle\langle TH \rangle\rangle$, $\mathcal{M}^c, \Gamma \Vdash \langle a_n \rangle (B_1 \wedge \dots \wedge B_m)$. By Lemma 4.11, there exists Δ with $\Gamma R^c \Delta$ and $\Delta \vdash_{\text{JKCS}} B$ for all $B \in \langle\langle TH \rangle\rangle$. By deductive closure $B \in \Delta$ and by the Truth Lemma 4.10 we get $\mathcal{M}^c, \Delta \Vdash B$ for all $B \in \langle\langle TH \rangle\rangle$. Hence, by (IH) on 1., $\mathcal{M}_{\Box}^c, \Delta \Vdash H$ and so $\mathcal{M}_{\Box}^c, \Gamma \Vdash \Diamond_n H$.
2. Suppose for all $A \in \langle\langle FG \rangle\rangle$, $\mathcal{M}^c, \Gamma \nVdash A$, that is, for all $B \in \langle\langle FH \rangle\rangle$ and $m \in \text{SatTm}$, $\mathcal{M}^c, \Gamma \nVdash \langle m \rangle B$, and by the Truth Lemma 4.10, $\langle m \rangle B \notin \Gamma$. By definition of R^c (Definition 4.9), it means, for all $\Delta \in W^c$ with $\Gamma R^c \Delta$, that $B \notin \Delta$, and by the Truth Lemma 4.10 again, $\mathcal{M}^c, \Delta \nVdash B$ for all $B \in \langle\langle FH \rangle\rangle$. Hence, by (IH) on 2., $\mathcal{M}_{\Box}^c, \Delta \nVdash H$ and so $\mathcal{M}_{\Box}^c, \Gamma \nVdash \Diamond_n H$. $\square$

Now we can prove the Pre-realisation Theorem 5.8.

Proof of Theorem 5.8. Suppose G is not pre-realisable. Then $\not\vdash_{\text{JKCS}} \langle\langle FG \rangle\rangle$. By the Prime Lemma 3.9, there exists a prime set Γ such that $\Gamma \not\vdash_{\text{JKCS}} \langle\langle FG \rangle\rangle$. Hence, for all $A \in \langle\langle FG \rangle\rangle$, $A \notin \Gamma$, so by the Truth Lemma 4.10, $\mathcal{M}^c, \Gamma \nVdash A$. By Theorem 5.9, $\mathcal{M}_{\Box}^c, \Gamma \nVdash G$, so G is not a theorem of IK . $\square$

5.2 From Pre-realisation to Realisation

This subsection performs the second step of the realisation proof sketched above. From any finite subset of pre-realiseres for a given modal formula G , through a process called *condensing* [20] which syntactically removes the extra disjunctions and conjunctions, we construct a justification formula that *potentially* realises G , i.e. respects its syntax but not necessarily its semantics. Eventually, the realisation map, will choose a valid justification formula from the set of potential realiseres, guided by the Pre-realisation theorem 5.8 from the previous section.

Definition 5.10 (Potential realiser). The *potential realiser* is a map $\llbracket \cdot \rrbracket : \mathcal{L}_{\square}^{\text{ann}} \rightarrow \mathcal{P}(\mathcal{L}_J)$ defined as:

$$\begin{aligned} \llbracket \top p \rrbracket &:= \{p\} & \llbracket \text{F}p \rrbracket &:= \{p\} \\ \llbracket \text{T}G \wedge H \rrbracket &:= \{A \wedge B \mid A \in \llbracket \text{T}G \rrbracket, B \in \llbracket \text{T}H \rrbracket\} & \llbracket \text{F}G \wedge H \rrbracket &:= \{A \wedge B \mid A \in \llbracket \text{F}G \rrbracket, B \in \llbracket \text{F}H \rrbracket\} \\ \llbracket \text{T}G \vee H \rrbracket &:= \{A \vee B \mid A \in \llbracket \text{T}G \rrbracket, B \in \llbracket \text{T}H \rrbracket\} & \llbracket \text{F}G \vee H \rrbracket &:= \{A \vee B \mid A \in \llbracket \text{F}G \rrbracket, B \in \llbracket \text{F}H \rrbracket\} \\ \llbracket \text{T}G \rightarrow H \rrbracket &:= \{A \rightarrow B \mid A \in \llbracket \text{F}G \rrbracket, B \in \llbracket \text{T}H \rrbracket\} & \llbracket \text{F}G \rightarrow H \rrbracket &:= \{A \rightarrow B \mid A \in \llbracket \text{F}G \rrbracket, B \in \llbracket \text{T}H \rrbracket\} \\ \llbracket \text{T}\Box_n G \rrbracket &:= \{[x_n]A \mid A \in \llbracket \text{T}G \rrbracket\} & \llbracket \text{F}\Box_n G \rrbracket &:= \{[t]A \mid A \in \llbracket \text{F}G \rrbracket, t \in \text{PrfTm}\} \\ \llbracket \text{T}\Diamond_n G \rrbracket &:= \{[a_n]A \mid A \in \llbracket \text{T}G \rrbracket\} & \llbracket \text{F}\Diamond_n G \rrbracket &:= \{[m]A \mid A \in \llbracket \text{F}G \rrbracket, m \in \text{SatTm}\} \end{aligned}$$

The idea is that a realisation function on a modal formula G would choose a suitable justification formula $A \in \llbracket \text{F}G \rrbracket$. The choice of a formula from $\llbracket \text{F}G \rrbracket$, rather than $\llbracket \text{T}G \rrbracket$, ensures that for any negative subformula of $A \in \llbracket \text{F}G \rrbracket$ of the shape $[t]B$ (or $[m]B$), we have $t \in \text{PrfVar}$ (or $m \in \text{SatVar}$ respectively), which is required to reach a normal realisation. The difference in the definition of $\llbracket \text{F}G \rightarrow H \rrbracket$, $\llbracket \text{F}\Box_n G \rrbracket$ and $\llbracket \text{T}\Diamond_n G \rrbracket$ compared to the pre-realiser ensures the justification formula does not contain additional conjunctions and disjunctions and exactly matches the structure of the modal formula. More formally:

Proposition 5.11. *Let G be an annotated modal formula. Then for every $A \in \llbracket \text{T}G \rrbracket \cup \llbracket \text{F}G \rrbracket$, $A^f = G$, where $(\cdot)^f : \mathcal{L}_J \rightarrow \mathcal{L}_{\square}$ is the forgetful projection from Definition 2.16.*

Proof. This is a routine proof by induction on G . □

The pre-realiser from Definition 5.6 is converted into a potential realiser using substitutions and syntactic reasoning – this process is called *condensing* by Fitting [20, p. 640].

To construct adequate substitutions we use the following auxiliary definitions.

Definition 5.12 (No new variable condition). Let σ be a substitution. The *domain* of σ is the set $\text{dom}(\sigma) := \{x \in \text{PrfVar} \mid x\sigma \neq x\} \cup \{a \in \text{SatVar} \mid a\sigma \neq a\}$.

A substitution σ *meets the no new variable condition* if for each variable $x \in \text{dom}(\sigma)$ (or $a \in \text{dom}(\sigma)$), $x\sigma$ (or $a\sigma$) contains no other variables than x (or a).

Definition 5.13 (Lives on). Let G be an annotated modal formula and σ be a substitution. We say that σ *lives on G* if for every $x_k \in \text{dom}(\sigma)$, $\Box_k$ occurs in G and for every $a_k \in \text{dom}(\sigma)$, $\Diamond_k$ occurs in G .

With these definitions in hand, we tackle the condensing theorem.

Theorem 5.14 (Condensing Theorem). *Let $G \in \mathcal{L}_{\square}^{\text{ann}}$.*

1. *For each collection $A_1, \dots, A_n \in \llbracket \text{T}G \rrbracket$, there exist $A \in \llbracket \text{T}G \rrbracket$ and σ that lives on G and meets the no new variable condition such that $A \vdash_{\text{JIKCS}} (A_1 \wedge \dots \wedge A_n)\sigma$.*
2. *For each collection $A_1, \dots, A_n \in \llbracket \text{F}G \rrbracket$, there exist $A \in \llbracket \text{F}G \rrbracket$ and σ that lives on G and meets the no new variable condition such that $(A_1 \vee \dots \vee A_n)\sigma \vdash_{\text{JIKCS}} A$.*

Proof. We prove both simultaneously by induction on G . Except for diamonds, all cases are covered by Pischke [33]. So, we are left to consider the case when $G = \Diamond_k H$.

1. Let $\langle a_k \rangle (A_1^1 \wedge \dots \wedge A_{n_1}^1), \dots, \langle a_k \rangle (A_1^m \wedge \dots \wedge A_{n_m}^m) \in \llbracket \text{T}\Diamond_k H \rrbracket$ with $A_1^1, \dots, A_{n_1}^1, \dots, A_1^m, \dots, A_{n_m}^m \in \llbracket \text{T}H \rrbracket$. By the inductive hypothesis, there exist $A \in \llbracket \text{T}H \rrbracket$ and σ living on H which meets the no new variable condition such that $A \vdash_{\text{JIKCS}} (A_1^1 \wedge \dots \wedge A_{n_1}^1 \wedge \dots \wedge A_1^m \wedge \dots \wedge A_{n_m}^m)\sigma$. So, for each $i \leq m$, $\vdash_{\text{JIKCS}} A \rightarrow (A_1^i \wedge \dots \wedge A_{n_i}^i)\sigma$. By the Lifting Lemma 2.14, there exist ground terms $t_1, \dots, t_m \in \text{PrfTm}$ such that $\vdash_{\text{JIKCS}} [t_i](A \rightarrow (A_1^i \wedge \dots \wedge A_{n_i}^i)\sigma)$. Set $t := t_1 + \dots + t_m$ which is also a ground term, and by repetitive use of $j +_l$ and $j +_r$, we have $\vdash_{\text{JIKCS}} [t](A \rightarrow (A_1^1 \wedge \dots \wedge A_{n_1}^1)\sigma)$ for all $i \leq m$.

Now, define σ' as $\sigma'(a_k) = t \star a_k$ and as the identity otherwise – note as t is a ground term, σ' meets the no new variable condition. By the Substitution Lemma 2.13, Definition 2.12 and the fact that t is a

ground term, so $t\sigma' = t$, we have $\vdash_{\text{JIK}_{\text{CS}}} [t](A\sigma' \rightarrow (A_1^i \wedge \dots \wedge A_{n_i}^i)\sigma\sigma')$ for all $i = 1, \dots, m$. Using the jk_2 axiom and mp : $\vdash_{\text{JIK}_{\text{CS}}} \langle a_k \rangle A\sigma' \rightarrow \langle t \star a_k \rangle (A_1^i \wedge \dots \wedge A_{n_i}^i)\sigma\sigma'$. Which can be rewritten as $\langle a_k \rangle (A\sigma') \vdash_{\text{JIK}_{\text{CS}}} (\langle a_k \rangle (A_1^i \wedge \dots \wedge A_{n_i}^i))\sigma\sigma'$ since $a_k\sigma\sigma' = a_k\sigma' = t \star a_k$. Propositional reasoning gives $\langle a_k \rangle (A\sigma') \vdash_{\text{JIK}_{\text{CS}}} (\langle a_k \rangle (A_1^1 \wedge \dots \wedge A_{n_1}^1) \wedge \dots \wedge \langle a_k \rangle (A_1^m \wedge \dots \wedge A_{n_m}^m))\sigma\sigma'$ where indeed $\sigma\sigma'$ is a substitution which lives on $\Diamond_k H$, meets the no new variable condition and $\langle a_k \rangle (A\sigma') \in \llbracket T \Diamond_k H \rrbracket$.

2. Let $\langle m_1 \rangle A_1, \dots, \langle m_n \rangle A_n \in \llbracket F \Diamond_k H \rrbracket$ with $A_1, \dots, A_n \in \llbracket FH \rrbracket$ and $m_1, \dots, m_n \in \text{SatTm}$. By Proposition 2.15, there exists a satisfier $m \in \text{SatTm}$ such that $(*) \langle m_1 \rangle A_1 \vee \dots \vee \langle m_n \rangle A_n \vdash_{\text{JIK}_{\text{CS}}} \langle m \rangle (A_1 \vee \dots \vee A_n)$. By the inductive hypothesis, there exist $A \in \llbracket FH \rrbracket$ and σ that lives on H (and meets the no new variable condition) such that $(**) (A_1 \vee \dots \vee A_n)\sigma \vdash_{\text{JIK}_{\text{CS}}} A$. Note that by definition σ also lives on $\Diamond_k H$ trivially. By the Substitution Lemma 2.13 applied to $(*)$, we have $(\langle m_1 \rangle A_1 \vee \dots \vee \langle m_n \rangle A_n)\sigma \vdash_{\text{JIK}_{\text{CS}}} (\langle m \rangle (A_1 \vee \dots \vee A_n))\sigma$. By the Lifting Lemma 2.14 applied to $(**)$, there exists $n \in \text{SatTm}$ such that $\langle m\sigma \rangle (A_1 \vee \dots \vee A_n)\sigma \vdash_{\text{JIK}_{\text{CS}}} \langle n \rangle A$, equivalently (by Definition 2.12) $(\langle m \rangle (A_1 \vee \dots \vee A_n))\sigma \vdash_{\text{JIK}_{\text{CS}}} \langle n \rangle A$. Hence, $(\langle m_1 \rangle A_1 \vee \dots \vee \langle m_n \rangle A_n)\sigma \vdash_{\text{JIK}_{\text{CS}}} \langle n \rangle A$ where indeed $\langle n \rangle A \in \llbracket F \Diamond_k H \rrbracket$ and σ is a substitution living on $\Diamond_n H$ and meets the no new variable condition. $\square$

Finally, our central Realisation Theorem falls as a corollary of the results harvested this far: from pre-realiseres which are semantically justified by the Pre-realisation theorem 5.8 we extract potential realiseres which are also syntactically designed through the Condensing theorem 5.14 to follow precisely the original structure of the modal formula.

Proof of Theorem 2.20. Suppose $\vdash_{\text{IK}} G$. By the Pre-realisation Theorem 5.8, there exist $A_1, \dots, A_n \in \llbracket FG \rrbracket$ such that $\vdash_{\text{JIK}_{\text{CS}}} A_1 \vee \dots \vee A_n$. By the Condensing Theorem 5.14, there exists $A \in \llbracket FG \rrbracket$ and a substitution σ such that $\vdash_{\text{JIK}_{\text{CS}}} (A_1 \vee \dots \vee A_n)\sigma \rightarrow A$. By the Substitution Lemma 2.13, $\vdash_{\text{JIK}_{\text{CS}}} (A_1 \vee \dots \vee A_n)\sigma$ and so by modus ponens $\vdash_{\text{JIK}_{\text{CS}}} A$. Let $r(G) := A$ and we have $r(G)^f = G$ by Proposition 5.11. $\square$

6 Conclusion

We have presented basic modular and modular models for the justification counterparts to IK which let us derive an alternative semantic proof of the realisation theorem from IK to JIK. This method could be further adapted on the one hand to sublogics of IK such as the constructive modal logic CK and on the other to extensions of IK such as the other logics in the modal IS5 cube by following in the classical case [8, 35, 19, 27, 21, 10], or the family of Geach/Scott-Lemmon logics [20]. For example, the modal logic IS4 extends IK with the following axioms.

$$t_{\Box}: \Box G \rightarrow G \quad t_{\Diamond}: G \rightarrow \Diamond G \quad 4_{\Box}: \Box G \rightarrow \Box \Box G \quad 4_{\Diamond}: \Diamond \Diamond G \rightarrow \Diamond G$$

Semantically, this means imposing R is reflexive and transitive in any model $(W, \leq, R, *)$. In the corresponding justification logic [29], we add the following axioms to Base.

$$jt_{\Box}: [t]A \rightarrow A \quad jt_{\Diamond}: A \rightarrow \langle m \rangle A \quad j4_{\Box}: [t]A \rightarrow [!t][t]A \quad j4_{\Diamond}: \langle m \rangle \langle n \rangle A \rightarrow \langle n \rangle A$$

An adaptation to the basic modular models in Section 3 in line with models for the justification counterpart to S4 [32, 5], and imposing R is reflexive and transitive, should be sufficient. Similarly, we conjecture that one can find justification counterparts to the family of logics defined by extending IK with $\Diamond^k \Box^l G \rightarrow \Box^m \Diamond^n G$ where $k, l, m, n \in \mathbb{N}$, by using the frame conditions given in [34]. The challenge could be defining additional operations on proof terms and satisfier terms.

Regarding sublogics such as CK, the canonical model construction for modular models would need to be generalised to build over not only prime sets, but pairs of prime sets and *segments* following the methodology introduced by [39] and developed for all the logics between CK and IK by [22].

Some variants of intuitionistic justification logic can be seen as extending the typed λ -calculus with reflective capabilities [3]. Applications of intuitionistic justification logic could extend to λ -calculi and type theory [9], and lead to more expressive modal λ -calculi, notably for type constructors corresponding to the $\Diamond$ modality [38].

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Halo Semantics for Modal Logic

Yoàv Montacute

National Institute of Informatics

Tokyo, Japan

montacute@nii.ac.jp

In nonstandard analysis the halo of a point in a topological space is the intersection of the nonstandard extensions of all its open neighbourhoods. We define a parametric family of modal operators from the halo by varying which elements of the nonstandard extension are admitted as witnesses, and identify four canonical instances. Two recover well-known modalities: the topological closure and the Cantor derivative. A third reduces to Kripke semantics over the specialisation preorder.

The fourth, purely nonstandard instance admits only nonstandard witnesses. The Transfer Principle forces it to coincide with the ω -accumulation point operator, a classical topological notion not previously studied in modal logic. Unlike the Cantor derivative, the ω -accumulation operator maps arbitrary sets to closed sets without any separation axiom, yielding an ω -Cantor–Bendixson decomposition on all topological spaces. Axiom 4 holds universally, again without separation conditions. We prove that K4 is the complete logic over infinite spaces, and GL over infinite ω -scattered spaces.

1 Introduction

In Robinson’s nonstandard analysis, every point x in a topological space acquires an infinitesimal neighbourhood: the intersection of the nonstandard extensions of all its open sets, called the *halo*. It replaces the infinitary quantification over all open neighbourhoods that defines topological operations with a single membership test against one set. The halo encodes the local topology of the space in a single object. This invites the following inquiry: If a formula φ is true at various points of the space, then what is the relationship between the halo of a given point and the nonstandard extension $^{*}[\![\varphi]\!]$ of the truth set of such a formula?

We investigate this by defining a family of modal operators which specify for each point x , a set $E(x)$ of witnesses which are excluded from the nonstandard truth set $^{*}[\![\varphi]\!]$. The resulting family has four canonical members, on which we focus. The purely standard case collapses to Kripke semantics over the specialisation preorder. Two more turn out to coincide with well-known modalities: when nothing is excluded the operator equates to topological closure, and when the point itself is excluded it equates to the Cantor derivative. Hence, the classical semantics of McKinsey–Tarski [8] and Esakia [5] arise naturally from our framework.

The truly novel operator is the one obtained by excluding all standard points, so that witnesses must be nonstandard hyperpoints. The *Transfer Principle* of nonstandard analysis forces this condition to be equivalent to a well-known but modally unstudied topological notion: ω -accumulation. Unlike the Cantor derivative, the ω -accumulation operator maps arbitrary sets to closed sets without any separation axiom. This yields a Cantor–Bendixson-style decomposition on all topological spaces, paralleling the classical theorem. In addition, Axiom 4 holds on all topological spaces, again with no separation condition required. We show that the logic K4 is complete for the nonstandard semantics over all infinite topological spaces; and that GL is complete over all infinite ω -scattered spaces.

Outline. Section 2 recalls the relevant definitions from nonstandard analysis. Section 3 defines the parametric family and analyses all four exclusion choices with examples. Section 4 studies the purely standard operator. Section 5 discusses the closure and Cantor derivative operators. Section 6 develops the theory of the nonstandard modality, including the ω -accumulation characterisation, its coincidence with the Cantor derivative on T_1 spaces, the ω -Cantor-Bendixson decomposition, and the completeness of K4 and GL. We conclude with some remarks.

2 Nonstandard Framework

We recall the axiomatic framework for nonstandard analysis following Salbany and Todorov [10], who work within the superstructure of an infinite set. This framework specifies the conditions required for halos to have the appropriate topological properties.

2.1 Superstructures and the star map

Let S be an infinite set. In our applications $S = X \cup \mathbb{R}$, where (X, τ) is a topological space. The *superstructure* over S is the hierarchy

$$V(S) = \bigcup_{k \in \mathbb{N}_0} V_k(S),$$

built inductively as

$$V_0(S) = S, \quad V_{k+1}(S) = V_k(S) \cup \mathcal{P}(V_k(S)).$$

Each level adds all subsets of the previous level. The idea is that $V_0(S)$ contains the base objects, $V_1(S)$ adds all sets of base objects, $V_2(S)$ adds all sets of those, and so on. Concretely:

- $V_0(S) = S$: the points of X and the real numbers.
- $V_1(S) = S \cup \mathcal{P}(S)$: includes all subsets of S , in particular every open set $U \subseteq X$, since $U \subseteq X \subseteq S$ and hence $U \in \mathcal{P}(S) \subseteq V_1(S)$.
- $V_2(S)$: includes all subsets of $V_1(S)$, in particular the topology τ , since τ is a collection of open sets and each open set is in $V_1(S)$, so $\tau \subseteq V_1(S)$ and hence $\tau \in \mathcal{P}(V_1(S)) \subseteq V_2(S)$.

Functions are identified with their graphs (sets of ordered pairs), and ordered pairs are encoded as sets in the usual way ($\langle a, b \rangle = \{\{a\}, \{a, b\}\}$), so functions also appear at a finite level of $V(S)$. Every mathematical object used in this paper, such as points, open sets, the topology, valuations, and the neighbourhood filter, is an element of $V(S)$ at some finite level.

The formal language $\mathcal{L}(V(S))$ consists of *bounded quantifier formulas*: formulas built from $=$, $\in$, the logical connectives, and quantifiers restricted to the form $\forall x \in A$ or $\exists x \in A$ for some $A \in V(S)$. Restricting to bounded quantifiers ensures that the formula does not quantify over the entire (proper-class-sized) universe, only over specific sets already in $V(S)$. This language is expressive enough to state all standard topological properties of (X, τ) (see [10]).

Definition 2.1 (nonstandard model). A *nonstandard model* of $V(S)$ is a superstructure $V(*S)$ together with a *star map* $*$: $V(S) \rightarrow V(*S)$ satisfying the axioms NSA1, NSA2 and NSA3 below.

A set $B \in V(*S)$ is called *internal* if $B \in {}^*C$ for some $C \in V(S)$, i.e. if B is an element of the nonstandard extension of some standard set. A set that is not internal is called *external*.

(NSA1) **Extension Principle.** $*s = s$ for all $s \in S$; equivalently, $S \subseteq *S$.

- (NSA2) **Transfer Principle.** A bounded quantifier formula $\Phi(A_1, \dots, A_n)$ is true in $V(S)$ if and only if $\Phi(*A_1, \dots, *A_n)$ is true in $V(*S)$.
- (NSA3) **Saturation Principle.** Every family of internal sets of cardinality at most $\kappa \geq |V(S)|$ with the finite intersection property has non-empty intersection.

We call a model satisfying the axioms above *polysaturated*. Such models exist for any infinite S , as shown by the “ultrapower construction” [10]. We fix one from this point onward.

The distinction between internal and external sets is fundamental to nonstandard analysis. Any set explicitly constructed by Transfer from a standard formula, e.g. $*(U \cap A)$, $*U \setminus *A$, and so on is internal. The Transfer Principle applies only to internal sets as bounded quantifier formulas can be transferred between $V(S)$ and $V(*S)$ only when all sets quantified over are internal. Similarly, the Saturation Principle produces intersections only for families of internal sets. Important examples of external sets used in this paper are the standard universe X inside $*X$, the halo $\mu(x)$ and the set $*S \setminus X$ for infinite S .

For us this distinction is important when applying Saturation to a family such as $\{*(U \cap A) \setminus G : U \in \mathcal{N}(x), \text{ finite } G \subseteq X\}$, where $\mathcal{N}(x) = \{U \in \tau : x \in U\}$ is the open neighbourhood filter of x , because each individual member $*(U \cap A) \setminus G$ is internal; the intersection of this family, however, need not be internal, and indeed, it is a subset of $*A \setminus X$, which is external.

The Transfer Principle immediately yields the following structural properties of $*$, which we use throughout without further comment.

Lemma 2.2 (properties of the star map). *For all $A, B \subseteq X$ and $x \in X$:*

- (i) $*(A \cap B) = *A \cap *B$;
- (ii) $*(A \cup B) = *A \cup *B$;
- (iii) $*(X \setminus A) = *X \setminus *A$;
- (iv) $A \subseteq B \Rightarrow *A \subseteq *B$ (monotonicity);
- (v) $*\emptyset = \emptyset$ and $*X \supseteq X$;
- (vi) $x \in A$ if and only if $x \in *A$, for $x \in X$.

Proof. Each statement is a direct application of the Transfer Principle to the corresponding bounded quantifier formula expressing set operations and membership. For instance, (i) follows by transferring $\forall z \in X (z \in A \cap B \leftrightarrow z \in A \wedge z \in B)$, which yields $\forall z \in *X (z \in *(A \cap B) \leftrightarrow z \in *A \wedge z \in *B)$. In part (v), $*\emptyset = \emptyset$ follows by Transfer and $*X \supseteq X$ from Extension. Part (vi) holds because $x \in S \subseteq *S$ by Extension and the formula $x \in A$ is quantifier-free. It readily follows by transfer that $x \in A$ if and only if $x \in *A$. $\square$

2.2 The halo and nonstandard characterisations

Fix a topological space (X, τ) .

Definition 2.3 (halo). For $x \in X$, the *halo* of x is defined as

$$\mu(x) = \bigcap_{U \in \mathcal{N}(x)} *U.$$

This set is also called the *monad* of x in the literature [9, 10].

Lemma 2.4. *For all $x \in X$, $x \in \mu(x)$.*

Proof. For any $U \in \mathcal{N}(x)$ we have $x \in U$, hence $x \in {}^*U$ by Lemma 2.2(vi). Since this holds for every $U \in \mathcal{N}(x)$, $x \in \bigcap_{U \in \mathcal{N}(x)} {}^*U = \mu(x)$. $\square$

The following is immediate from the definition of a halo.

Lemma 2.5. *If $\tau' \subseteq \tau$, then $\mu_\tau(x) \subseteq \mu_{\tau'}(x)$.*

Definition 2.6 (beacon). For $B \subseteq {}^*X$, the *beacon* of B is defined as

$$\text{bcn}(B) = \{x \in X \mid \mu(x) \cap B \neq \emptyset\}.$$

Each standard point x casts a halo $\mu(x)$ into ${}^*X$. The beacon of B is the set of standard points whose halos reach B . As we show below, the beacon of ${}^*A$ is precisely the topological closure of A .

The following proposition collects the fundamental nonstandard equivalences. Parts (i) and (ii) are due to Robinson [9] (see also [7]); the remaining parts are standard consequences recorded, e.g. in [10].

Proposition 2.7 (nonstandard characterisations). *Let (X, τ) be a topological space with polysaturated nonstandard extension. For $A \subseteq X$ and $x \in X$:*

- (i) *Closure: $x \in \text{cl}(A)$ iff $\mu(x) \cap {}^*A \neq \emptyset$. Equivalently, $\text{cl}(A) = \text{bcn}({}^*A)$. So A is closed iff $A = \text{bcn}({}^*A)$.*
- (ii) *Open set: $A \in \tau$ iff $\mu(x) \subseteq {}^*A$ for every $x \in A$.*
- (iii) *Limit point: x is a limit point of A iff $\mu(x) \cap ({}^*A \setminus \{x\}) \neq \emptyset$.*
- (iv) *Isolated point: $x \in A$ is isolated in A iff $\mu(x) \cap {}^*A = \{x\}$.*
- (v) *T_1 space: (X, τ) is T_1 iff $\mu(x) \cap X = \{x\}$ for every $x \in X$.*
- (vi) *Hausdorff space: (X, τ) is Hausdorff iff $\mu(x) \cap \mu(y) = \emptyset$ for all distinct $x, y \in X$.*

3 The Halo Operator

3.1 Logic

Let $\mathbf{Prop} = \{p, q, r, \dots\}$ denote the set of propositional variables. The modal language has a single diamond primitive $\Diamond$. The language is defined inductively as follows:

$$\varphi ::= p \mid \perp \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi.$$

Boolean connectives $\top, \vee, \rightarrow, \leftrightarrow$ are standard abbreviations. The box is always the De Morgan dual: $\Box\varphi := \neg\Diamond\neg\varphi$. The axiomatic systems used in this paper are all normal modal logics, i.e. include the Axiom K. We recall the relevant axiom schemas:

- (K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- (T) $\Box\varphi \rightarrow \varphi$
- (4) $\Box\varphi \rightarrow \Box\Box\varphi$
- (w4) $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi \vee \varphi$
- (L) $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$

All of the axiomatic systems discussed in this paper include classical tautologies, Modus Ponens and Necessitation. The axiomatic systems we consider are:

$$S4 = K + T + 4; \quad K4 = K + 4; \quad wK4 = K + w4; \quad GL = K + L.$$

Definition 3.1 (topological model). A *topological model* is a pair $\mathcal{M} = ((X, \tau), \nu)$ where (X, τ) is a topological space and $\nu : \mathbf{Prop} \rightarrow \mathcal{P}(X)$ is a valuation. We work inside a fixed polysaturated nonstandard extension of X .

The *extension* $\llbracket \varphi \rrbracket^{\mathcal{M}} \subseteq X$ is defined inductively in the obvious way on Boolean connectives, with the modal clause determined by the choice of *exclusion function*.

Definition 3.2 (halo operator). Given a topological model $\mathcal{M}$ and an exclusion function $E : X \rightarrow \mathcal{P}(*X)$, the *halo operator* $\Diamond_E$ is defined by the semantic clause:

$$\llbracket \Diamond_E \varphi \rrbracket^{\mathcal{M}} = \{x \in X \mid \mu(x) \cap (*\llbracket \varphi \rrbracket^{\mathcal{M}} \setminus E(x)) \neq \emptyset\}.$$

We write $x \models \varphi$ for $x \in \llbracket \varphi \rrbracket^{\mathcal{M}}$ under the semantics determined by E . We omit the superscript $\mathcal{M}$ when clear from context.

We focus on four choices for E in this paper: the purely standard case, where $E(x) = *X \setminus X$; the closure case, where $E(x) = \emptyset$; the Cantor derivative case, where $E(x) = \{x\}$; and the purely nonstandard case, where $E(x) = X$. We denote the resulting modalities by $\Diamond_{\text{st}}$, $\Diamond_c$, $\Diamond_d$ and $\Diamond_{\text{ns}}$, respectively. The names will be justified in Section 5, where we show that the two operators $\Diamond_c$ and $\Diamond_d$ coincide with the closure and the Cantor derivative, respectively.

We establish the relationship between the four modalities.

Proposition 3.3. *For every topological model $\mathcal{M}$ and formula φ :*

$$\llbracket \Diamond_{\text{ns}} \varphi \rrbracket \subseteq \llbracket \Diamond_d \varphi \rrbracket \subseteq \llbracket \Diamond_c \varphi \rrbracket.$$

The purely standard operator satisfies $\llbracket \Diamond_{\text{st}} \varphi \rrbracket \subseteq \llbracket \Diamond_c \varphi \rrbracket$, but is incomparable with $\Diamond_d$.

Proof. The inclusions $\llbracket \Diamond_{\text{ns}} \varphi \rrbracket \subseteq \llbracket \Diamond_d \varphi \rrbracket \subseteq \llbracket \Diamond_c \varphi \rrbracket$ follow from $*A \setminus X \subseteq *A \setminus \{x\} \subseteq *A$. For $\llbracket \Diamond_{\text{st}} \varphi \rrbracket \subseteq \llbracket \Diamond_c \varphi \rrbracket$, every standard witness in $\mu(x) \cap \llbracket \varphi \rrbracket$ is also a witness for $\Diamond_c$, so $\llbracket \Diamond_{\text{st}} \varphi \rrbracket \subseteq \llbracket \Diamond_c \varphi \rrbracket$.

For incomparability, consider an arbitrary topological space (X, τ) , a point $x \in X$, and set $\nu(p) = \{x\}$. Since $x \in \mu(x)$ always, we have $\mu(x) \cap \llbracket p \rrbracket = \{x\} \neq \emptyset$, so $x \models \Diamond_{\text{st}} p$. But since $\mu(x) \cap (\{x\} \setminus \{x\}) = \emptyset$ then $x \not\models \Diamond_d p$. Hence $\llbracket \Diamond_{\text{st}} \varphi \rrbracket \not\subseteq \llbracket \Diamond_d \varphi \rrbracket$ in general.

For the other direction, consider $\mathbb{R}$ with the usual topology, let $x = 0$ and $\nu(p) = \{1/n : n \geq 1\}$. Then since every neighbourhood of 0 contains points of $\nu(p) \setminus \{0\}$, $0 \models \Diamond_d p$. But $\mathbb{R}$ is T_1 , so $\mu(0) \cap X = \{0\}$ by Proposition 2.7(v). Since $0 \notin \nu(p)$, it follows that $\mu(0) \cap \llbracket p \rrbracket = \emptyset$, and so $0 \not\models \Diamond_{\text{st}} p$. Hence $\llbracket \Diamond_d \varphi \rrbracket \not\subseteq \llbracket \Diamond_{\text{st}} \varphi \rrbracket$ in general. $\square$

Proposition 3.4. *For every topological model $\mathcal{M}$ and formula φ :*

$$\llbracket \Diamond_c \varphi \rrbracket = \llbracket \Diamond_{\text{st}} \varphi \rrbracket \cup \llbracket \Diamond_{\text{ns}} \varphi \rrbracket.$$

Proof. The inclusion $\supseteq$ follows from Proposition 3.3. For $\subseteq$, suppose $x \in \llbracket \Diamond_c \varphi \rrbracket$, so there exists $y \in \mu(x) \cap * \llbracket \varphi \rrbracket$. Either $y \in X$ or $y \notin X$. In the first case, $y \in \mu(x) \cap \llbracket \varphi \rrbracket$ by Lemma 2.2(vi), so $x \in \llbracket \Diamond_{\text{st}} \varphi \rrbracket$. In the second case $y \in \mu(x) \cap (* \llbracket \varphi \rrbracket \setminus X)$, so $x \in \llbracket \Diamond_{\text{ns}} \varphi \rrbracket$. $\square$

3.2 Examples

We illustrate all four operators on two examples. The first is a T_1 space ($\mathbb{R}$), which demonstrates the collapse $\Diamond_{\text{ns}} = \Diamond_d$ and the incomparability of $\Diamond_{\text{st}}$ with $\Diamond_d$. The second is a non- T_1 space (the Sierpiński space), chosen specifically to separate $\Diamond_d$ from $\Diamond_{\text{ns}}$.

Example 3.5 (the real line). Let $X = \mathbb{R}$ with the standard topology and $v(p) = (0, 1) \cup \{2\}$. Since $\mathbb{R}$ is T_1 , Proposition 2.7(v) implies $\mu(x) \cap \mathbb{R} = \{x\}$ for every x . The halo $\mu(x) = \{y \in {}^*\mathbb{R} : |y - x| < r \text{ for all standard } r > 0\}$ consists of all hyperreals infinitesimally close to x . By Transfer, ${}^*(0, 1) = \{y \in {}^*\mathbb{R} : 0 < y < 1\}$ and ${}^*\{c\} = \{c\}$, for each $c \in \mathbb{R}$.

At $x = 0$, each positive infinitesimal ε satisfies $\varepsilon \in \mu(0)$, $\varepsilon \in {}^*(0, 1) \subset {}^*v(p)$, and $\varepsilon \notin \mathbb{R}$. Hence $0 \in \llbracket \Diamond_c p \rrbracket$; since $\varepsilon \neq 0$, $0 \in \llbracket \Diamond_d p \rrbracket$; since $\varepsilon \notin \mathbb{R}$, $0 \in \llbracket \Diamond_{ns} p \rrbracket$. But $0 \notin v(p)$, so $0 \notin \llbracket \Diamond_{st} p \rrbracket$.

At $x = 2$, the point 2 is isolated in $v(p)$. For each $y \in \mu(2)$ with $y \neq 2$, $|y - 2|$ is a positive infinitesimal, so $y \notin {}^*(0, 1)$. Thus $\mu(2) \cap ({}^*v(p) \setminus \{2\}) = \emptyset$, so $2 \notin \llbracket \Diamond_d p \rrbracket$ and $2 \notin \llbracket \Diamond_{ns} p \rrbracket$. However $2 \in v(p)$, so $2 \in \llbracket \Diamond_{st} p \rrbracket$; and $2 \in \text{cl}(v(p))$, so $2 \in \llbracket \Diamond_c p \rrbracket$.

The example above does not separate $\Diamond_d$ from $\Diamond_{ns}$. We show that the two can be separated even on a two-point topological space.

Example 3.6 (the Sierpiński space). Let $X = \{a, b\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}\}$. The open sets containing a are $\{a\}$ and X ; the only open set containing b is X . This space is not T_1 : the singleton $\{a\}$ is open, hence $\{b\} = X \setminus \{a\}$ is closed, but $\{a\} = X \setminus \{b\}$ is not closed since $\{b\}$ is not open. Let $v(p) = \{a\}$. Since $\{a\}$ and X are the open sets containing a , we obtain $\mu(a) = {}^*\{a\} \cap {}^*X = {}^*\{a\}$. Since X is the only open set containing b , $\mu(b) = {}^*X$. Recall that by Transfer finite sets are fixed by * . In particular, ${}^*v(p) = {}^*\{a\} = \{a\}$. So a is in $\llbracket \Diamond_{st} p \rrbracket$ and $\llbracket \Diamond_c p \rrbracket$ but not in $\llbracket \Diamond_d p \rrbracket$ or $\llbracket \Diamond_{ns} p \rrbracket$.

At $x = b$, we have $\mu(b) \cap \{a\} = \{a\} \neq \emptyset$ and $\mu(b) \cap {}^*\{a\} = \{a\} \neq \emptyset$, so $b \models \Diamond_{st} p$ and $b \models \Diamond_c p$. Also $\mu(b) \cap (\{a\} \setminus \{b\}) = \{a\} \neq \emptyset$, so $b \models \Diamond_d p$. But $\mu(b) \cap (\{a\} \setminus X) = \{a\} \setminus \{a, b\} = \emptyset$, so $b \not\models \Diamond_{ns} p$.

4 The purely standard operator

The purely standard modality $\Diamond_{st}$ is obtained by setting $E(x) = {}^*X \setminus X$, so that only standard points are admitted as witnesses.

Proposition 4.1. *On arbitrary topological spaces (X, τ) , the operator $\Diamond_{st}$ computes exactly the Kripke diamond over the specialisation preorder $\leq_\tau$ defined by $x \leq_\tau y$ iff $x \in \text{cl}(\{y\})$, i.e.*

$$x \models \Diamond_{st} \varphi \iff \exists y \in X : x \leq_\tau y \text{ and } y \models \varphi.$$

Proof. Using ${}^*\llbracket \varphi \rrbracket \subseteq {}^*X$ and ${}^*A \cap X = A$ for all $A \subseteq X$ (Lemma 2.2(vi)):

$$x \models \Diamond_{st} \varphi \iff \mu(x) \cap ({}^*\llbracket \varphi \rrbracket \setminus ({}^*X \setminus X)) \neq \emptyset \iff \mu(x) \cap \llbracket \varphi \rrbracket \neq \emptyset.$$

For $y \in X$, $y \in \mu(x)$ iff $y \in {}^*U$ for every $U \in \mathcal{N}(x)$, and by Lemma 2.2(vi), $y \in {}^*U$ iff $y \in U$ for standard y and U . Hence $\mu(x) \cap X = \uparrow x := \{y \in X : x \leq_\tau y\}$, and so

$$x \models \Diamond_{st} \varphi \iff \mu(x) \cap \llbracket \varphi \rrbracket \neq \emptyset \iff \uparrow x \cap \llbracket \varphi \rrbracket \neq \emptyset \iff \exists y \geq_\tau x (y \models \varphi). \quad \square$$

Corollary 4.2. *On every T_1 topological space, $\Diamond_{st} \varphi \equiv \varphi$ for every formula φ .*

Proof. On a T_1 space, $\text{cl}(\{x\}) = \{x\}$ for every x , so $\leq_\tau$ is equality. Hence $x \models \Diamond_{st} \varphi$ iff $x \models \varphi$. $\square$

Recall that a topological space is *Alexandroff* if arbitrary intersections of open sets are open; equivalently, every point has a smallest open neighbourhood. Every finite space is Alexandroff.

Corollary 4.3. *On an Alexandroff topological space (X, τ) , the operator $\Diamond_{st}$ coincides with the standard Kripke semantics for the specialisation preorder $\leq_\tau$. The complete modal logic of $\Diamond_{st}$ over Alexandroff spaces is S4.*

Proof. In an Alexandroff space every point has a smallest open neighbourhood $\mathcal{U}(x)$, so $\mu(x) \cap X = \mathcal{U}(x)$, which is precisely the upset $\{y : x \leq_\tau y\}$. By Proposition 4.1, $\Diamond_{\text{st}}$ is the Kripke diamond over the preorder $(X, \leq_\tau)$, which is a reflexive, transitive frame. The complete logic of such frames is S4 [4]. $\square$

5 The closure operator and the Cantor derivative operator

It readily follows that the operator $\Diamond_c$ is exactly the topological closure.

Proposition 5.1. *For every model $\mathcal{M}$ and formula φ :*

$$\llbracket \Diamond_c \varphi \rrbracket^{\mathcal{M}} = \text{cl}(\llbracket \varphi \rrbracket^{\mathcal{M}}).$$

In particular, $\llbracket \Box_c \varphi \rrbracket^{\mathcal{M}} = \text{int}(\llbracket \varphi \rrbracket^{\mathcal{M}})$.

Proof. This is immediate from Proposition 2.7(i) with $A = \llbracket \varphi \rrbracket$. $\square$

Soundness and completeness for the closure semantics are due to McKinsey and Tarski [8].

Theorem 5.2 ([8]). *S4 is sound and complete for all (finite) spaces with respect to the closure semantics.*

Similarly, the operator $\Diamond_d$ is exactly the Cantor derivative operator.

Proposition 5.3. *For every model $\mathcal{M}$ and formula φ :*

$$\llbracket \Diamond_d \varphi \rrbracket^{\mathcal{M}} = d(\llbracket \varphi \rrbracket^{\mathcal{M}}),$$

where $d(A) = \{x \in X : x \text{ is a limit point of } A\}$ is the Cantor derivative operator.

Proof. This is immediate from Proposition 2.7(iii) with $A = \llbracket \varphi \rrbracket$. $\square$

Soundness and completeness for the Cantor derivative semantics originate with Esakia [5].

Theorem 5.4 ([5]).

- (i) *wK4 is sound and complete for all (finite) spaces with respect to the Cantor derivative semantics.*
- (ii) *K4 is sound and complete for all T_D spaces with respect to the Cantor derivative semantics.*

The following halo-based proof of Axiom w4 on all topological spaces demonstrates the internal mechanisms of the halo semantics.

Proposition 5.5. *For every topological space and each model $\mathcal{M}$:*

$$\llbracket \Diamond_d \Diamond_d \varphi \rrbracket \subseteq \llbracket \Diamond_d \varphi \rrbracket \cup \llbracket \varphi \rrbracket.$$

Proof. Let $A = \llbracket \varphi \rrbracket$ and suppose $x \in d(d(A))$. By Proposition 2.7(iii), there exists $y \in \mu(x) \cap (*d(A) \setminus \{x\})$. Since $y \in *d(A)$, from Transfer of the bounded formula defining $d(A)$, we obtain that for every $V \in *\tau$, if $y \in V$ then $V \cap (*A \setminus \{y\}) \neq \emptyset$. Since $y \in \mu(x)$, we have $y \in *U$ for every $U \in \mathcal{N}(x)$. Each $*U$ belongs to $*\tau$ by Transfer, so the transferred condition yields a point $z_U \in *U \cap (*A \setminus \{y\})$ for each $U \in \mathcal{N}(x)$.

If $x \in *A$ then $x \in A$ by Lemma 2.2(vi), so $x \in \llbracket \varphi \rrbracket$ and we are done. Otherwise $x \notin *A$. Then $z_U \neq x$ for every U , since $z_U \in *A$. Since also $z_U \neq y$, we obtain $z_U \in *(U \cap A) \setminus \{x\}$ for each $U \in \mathcal{N}(x)$. Note that each set $*(U \cap A) \setminus \{x\}$ is internal. For any $U_1, \dots, U_n \in \mathcal{N}(x)$, setting $U = U_1 \cap \dots \cap U_n \in \mathcal{N}(x)$ implies $z_U \in *(U \cap A) \setminus \{x\} \subseteq *(U_i \cap A) \setminus \{x\}$ for each i , so the family has the finite intersection property. By Saturation, there exists $w \in \bigcap_{U \in \mathcal{N}(x)} (*(U \cap A) \setminus \{x\})$. Then $w \in *U$ for every $U \in \mathcal{N}(x)$, so $w \in \mu(x)$; and $w \in *A$ with $w \neq x$, so $w \in \mu(x) \cap (*A \setminus \{x\})$ and $x \in d(A)$. $\square$

For $\Diamond_{\text{ns}}$, the stronger inclusion $\omega(\omega(A)) \subseteq \omega(A)$ holds outright (Lemma 6.15), giving the full Axiom 4 without the disjunct $\llbracket \varphi \rrbracket$. Strengthening w4 to the full Axiom 4 for $\Diamond_d$ requires T_D [1, 2]. In the next section, we show that this is not the case for the nonstandard semantics.

6 The nonstandard operator

Recall the definition of the nonstandard operator:

$$x \models \Diamond_{\text{ns}} \varphi \iff \mu(x) \cap (*\llbracket \varphi \rrbracket \setminus X) \neq \emptyset.$$

We show some basic validities of $\Diamond_{\text{ns}}$.

Proposition 6.1. *For every topological model $\mathcal{M}$ the following properties hold:*

- (i) $\models \neg \Diamond_{\text{ns}} \perp$;
- (ii) $\models \Diamond_{\text{ns}}(\varphi \vee \psi) \leftrightarrow (\Diamond_{\text{ns}} \varphi \vee \Diamond_{\text{ns}} \psi)$;
- (iii) *if $\models \varphi \rightarrow \psi$ then $\models \Diamond_{\text{ns}} \varphi \rightarrow \Diamond_{\text{ns}} \psi$;*
- (iv) $\models \Diamond_{\text{ns}} \varphi \rightarrow \Diamond_c \varphi$;
- (v) $\models \Diamond_{\text{ns}} \varphi \rightarrow \Diamond_d \varphi$.

Proof. (i) follows from $*\emptyset \setminus X = \emptyset$; (ii) follows from $(*(\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket)) \setminus X = (*\llbracket \varphi \rrbracket \cup *\llbracket \psi \rrbracket) \setminus X = (*\llbracket \varphi \rrbracket \setminus X) \cup (*\llbracket \psi \rrbracket \setminus X)$; (iii) follows since $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ entails $*\llbracket \varphi \rrbracket \subseteq *\llbracket \psi \rrbracket$ and so $*\llbracket \varphi \rrbracket \setminus X \subseteq *\llbracket \psi \rrbracket \setminus X$; (iv) follows from $*\llbracket \varphi \rrbracket \setminus X \subseteq *\llbracket \varphi \rrbracket$; and (v) follows from $*\llbracket \varphi \rrbracket \setminus X \subseteq *\llbracket \varphi \rrbracket \setminus \{x\}$. $\square$

Proposition 6.2. *Axiom T is not valid for $\Diamond_{\text{ns}}$, even on $\mathbb{R}$.*

Proof. $v(p) = \{0\}$. By Transfer $*\{0\} = \{0\}$, so $*\{0\} \setminus \mathbb{R} = \emptyset$. It follows that $0 \models p$ but $0 \not\models \Diamond_{\text{ns}} p$. $\square$

6.1 The ω -Accumulation Characterisation

We show that $\Diamond_{\text{ns}}$ has a classical topological characterisation analogous to those of $\Diamond_c$ and $\Diamond_d$. Recall that a point $x \in X$ is an ω -accumulation point of $S \subseteq X$ if $|U \cap S| \geq \aleph_0$ for every open set U containing x . We denote the set of ω -accumulation points of S by $\omega(S)$. This notion appears classically in connection with perfect sets and the Cantor–Bendixson theorem [6].

Before providing the full characterisation, we need to show the following.

Proposition 6.3. *Given $S \subseteq X$, the set $*S \setminus X$ is non-empty if, and only if, S is infinite.*

Proof. $(\Rightarrow)$ If S is finite, say $|S| = n$, then there exists a bijection $f: \{1, \dots, n\} \xrightarrow{\sim} S$. By Transfer, there exists a bijection $f: \{1, \dots, n\} \xrightarrow{\sim} *S$, so $|*S| = n$. Hence $*S$ consists exactly of the standard images of the elements of S by the Extension Principle, and therefore $*S = S \subseteq X$, and so $*S \setminus X = \emptyset$.

$(\Leftarrow)$ Suppose S is infinite. Consider the family

$$\mathcal{G} = \{*(S \setminus F) : \text{finite } F \subseteq S\}.$$

Each member of this family is internal and non-empty. For any finite F_1, F_2 : $*(S \setminus (F_1 \cup F_2)) \subseteq *(S \setminus F_1) \cap *(S \setminus F_2)$ and $S \setminus (F_1 \cup F_2)$ is non-empty, so $\mathcal{G}$ has the finite intersection property. By Saturation, there exists $y \in \bigcap \mathcal{G}$. Then $y \in *S$, and for every $s \in S$, taking $F = \{s\}$ implies $y \in *(S \setminus \{s\})$, so $y \neq s$. Since every standard element of $*S$ is of the form $s \in S$ (Lemma 2.2(vi)), y must be nonstandard, i.e. $y \notin X$. $\square$

Theorem 6.4 (ω -accumulation characterisation). *For every topological model $\mathcal{M}$ and formula φ :*

$$x \models \Diamond_{\text{ns}} \varphi \iff x \in \omega(\llbracket \varphi \rrbracket)$$

Proof. Set $A := \llbracket \varphi \rrbracket$.

($\Rightarrow$) Suppose $\mu(x) \cap (*A \setminus X) \neq \emptyset$. Consider an arbitrary open $U \in \mathcal{N}(x)$. Then $*U \supseteq \mu(x)$, so $*(U \cap A) \setminus X \neq \emptyset$. By Proposition 6.3, $U \cap A$ is infinite.

($\Leftarrow$) Suppose every $U \in \mathcal{N}(x)$ satisfies $|U \cap A| \geq \aleph_0$. Consider the following family indexed by all pairs (U, G) with $U \in \mathcal{N}(x)$ and finite $G \subseteq X$:

$$\mathcal{F} = \{*(U \cap A) \setminus G : U \in \mathcal{N}(x), \text{ finite } G \subseteq X\}.$$

Each member is internal and non-empty. Moreover, the finite intersection property holds: given finitely many pairs $(U_1, G_1), \dots, (U_n, G_n)$, set $U = U_1 \cap \dots \cap U_n \in \mathcal{N}(x)$ and $G = G_1 \cup \dots \cup G_n$; then $*(U \cap A) \setminus G \subseteq *(U_i \cap A) \setminus G_i$ for each i , and this set is non-empty. By Saturation there exists $y \in \bigcap \mathcal{F}$.

Let $G = \emptyset$; then $y \in *(U \cap A) \subseteq *U$ for each $U \in \mathcal{N}(x)$, so $y \in \mu(x)$ and also $y \in *A$. Take an arbitrary $a \in X$ and set $G = \{a\}$; then $y \neq a$. Since $a \in X$ was arbitrary, $y \notin X$. Thus $y \in \mu(x) \cap (*A \setminus X)$. $\square$

6.2 Some properties of the nonstandard semantics

We show some properties of the nonstandard topological semantics. We begin by completing the verification that $\Diamond_{\text{ns}}$ defines a normal modal logic.

Lemma 6.5. *Axiom K is valid for $\Diamond_{\text{ns}}$ on all topological spaces.*

Proof. This follows from Proposition 6.1(ii) and (iii). $\square$

We show that $\Diamond_{\text{ns}}$ coincides with $\Diamond_d$ on T_1 spaces; the proof illustrates how the halo formulation reduces to the classical characterisation.

Lemma 6.6. *If (X, τ) is a T_1 topological space then for every model $\mathcal{M}$ and formula φ :*

$$\llbracket \Diamond_{\text{ns}} \varphi \rrbracket = \llbracket \Diamond_d \varphi \rrbracket.$$

Proof. Since $\omega(A) \subseteq d(A)$, it suffices to show $d(A) \subseteq \omega(A)$ on T_1 spaces. Suppose $x \in d(A)$ and $U \in \mathcal{N}(x)$. Suppose for contradiction that $U \cap A$ is finite: $U \cap A = \{a_1, \dots, a_k\} \cup (\{x\} \cap A)$ with $a_1, \dots, a_k \in A \setminus \{x\}$. Since (X, τ) is T_1 , for each i there exists $W_i \ni x$ open with $a_i \notin W_i$. Let $W = U \cap W_1 \cap \dots \cap W_k$. Then $W \in \mathcal{N}(x)$ and $W \cap (A \setminus \{x\}) = \emptyset$, contradicting $x \in d(A)$. Hence $U \cap A$ is infinite.

Moreover, Proposition 2.7(v) implies $\mu(x) \cap X = \{x\}$ on T_1 spaces, so for $y \in \mu(x)$: $y \notin X \iff y \neq x$. Therefore $\mu(x) \cap (*A \setminus X) = \mu(x) \cap (*A \setminus \{x\})$, which is exactly the equality of the $\Diamond_{\text{ns}}$ and $\Diamond_d$ witness conditions. $\square$

Corollary 6.7. *On T_1 spaces, $\Diamond_{\text{ns}}$ satisfies Axiom 4.*

Proof. On T_1 spaces $\Diamond_{\text{ns}} = \Diamond_d$ by Lemma 6.6; since T_1 implies T_D , Axiom 4 holds by Theorem 5.4(ii). $\square$

While our study is centred around topological semantics for arbitrary topological spaces, it is worth noting the relationship of $\Diamond_{\text{ns}}$ to both finite topological spaces and Kripke frames.

Proposition 6.8. *Given a finite topological space (X, τ) , $\llbracket \Diamond_{\text{ns}} \varphi \rrbracket = \emptyset$ for every model $\mathcal{M}$ and formula φ .*

Proof. Since each $A \subseteq X$ is finite, by Transfer we obtain $|*A| = |A|$, so $*A = A \subseteq X$ and $*A \setminus X = \emptyset$. $\square$

Corollary 6.9. *The logic with $\Diamond_{\text{ns}}$ does not have the finite model property.*

Proposition 6.10. *The operator $\Diamond_{\text{ns}}$ cannot be represented as a Kripke modality on any topological space where ω is non-trivial. That is, if there exist $x \in X$ and $A \subseteq X$ with $x \in \omega(A)$, then there is no binary relation $R \subseteq X \times X$ satisfying*

$$y \in \omega(B) \iff R(y) \cap B \neq \emptyset \quad \text{for all } B \subseteq X, y \in X.$$

Proof. Suppose such R exists. Take any $y_0 \in R(x) \cap A$ (which exists since $x \in \omega(A)$ and the Kripke clause implies $R(x) \cap A \neq \emptyset$). Since $\{y_0\}$ is a finite set, $\omega(\{y_0\}) = \emptyset$ by Proposition 6.3: no point can have every neighbourhood meet a singleton in infinitely many points. The Kripke clause then implies $R(x) \cap \{y_0\} = \emptyset$, contradicting $y_0 \in R(x)$. $\square$

Note that Proposition 6.10 in particular holds for every infinite non-discrete T_1 space: Lemma 6.6 guarantees $\omega(X \setminus \{x\}) = d(X \setminus \{x\}) \ni x$ for any non-isolated x .

Proposition 6.11. *The bimodal property $\Diamond_c \phi \leftrightarrow \phi \vee \Diamond_{\text{ns}} \phi$ is valid in (X, τ) iff (X, τ) is T_1 .*

Proof. ($\Leftarrow$) Let (X, τ) be T_1 and $A = \llbracket \phi \rrbracket$. Since $\text{cl}(A) = A \cup d(A)$, by Proposition 5.1 we have $\llbracket \Diamond_c \phi \rrbracket = A \cup d(A)$. By Lemma 6.6, $\omega(A) = d(A)$ on T_1 spaces. Hence $\llbracket \phi \vee \Diamond_{\text{ns}} \phi \rrbracket = A \cup \omega(A) = A \cup d(A) = \llbracket \Diamond_c \phi \rrbracket$.

($\Rightarrow$) Suppose (X, τ) is not T_1 . Then some singleton $\{a\}$ is not closed, so there exists $b \neq a$ with $b \in \text{cl}(\{a\})$. Set $v(p) = \{a\}$, so $A = \{a\}$. Then $b \in \text{cl}(\{a\}) = \llbracket \Diamond_c p \rrbracket$. However, $A = \{a\}$ is finite, so $\ast\{a\} \setminus X = \emptyset$ by Proposition 6.3, giving $\omega(\{a\}) = \emptyset$. Hence $\llbracket p \vee \Diamond_{\text{ns}} p \rrbracket = \{a\} \cup \emptyset = \{a\}$ so $b \notin \llbracket p \vee \Diamond_{\text{ns}} p \rrbracket$. $\square$

The property $\Diamond_c \phi \leftrightarrow \phi \vee \Diamond_d \phi$ in contrast is valid on all topological spaces, since $\text{cl}(A) = A \cup d(A)$.

We now introduce the appropriate notions of perfect sets and scatteredness required for Löb Axiom and the decomposition of the ω -accumulation operator.

Definition 6.12 (ω -perfect set and ω -scattered space). A closed subset $P \subseteq X$ is ω -perfect if $\omega(P) \supseteq P$, that is, every point of P is an ω -accumulation point of P itself. A topological space (X, τ) is ω -scattered if it contains no non-empty ω -perfect subset.

On T_1 spaces, Lemma 6.6 implies $\omega(A) = d(A)$ for every $A \subseteq X$, so ω -perfect coincides with perfect and ω -scattered coincides with scattered in the sense of Cantor–Bendixson [6].

Lemma 6.13. *For every topological space (X, τ) and every $A \subseteq X$, the set $\omega(A)$ is closed.*

Proof. Let $x \in \text{cl}(\omega(A))$ and $U \in \mathcal{N}(x)$. Pick $y \in U \cap \omega(A)$. Since U is open and $y \in \omega(A)$, we have $|U \cap A| \geq \aleph_0$. Since U was arbitrary, $x \in \omega(A)$. $\square$

The Cantor derivative $d(A)$ is not closed in general; closedness requires T_D . In contrast, closedness of $\omega(A)$ holds without any separation assumption (Lemma 6.13), yielding the following decomposition, analogous to the classical Cantor–Bendixson theorem [6].

Theorem 6.14 (ω -Cantor–Bendixson). *Every topological space (X, τ) has a largest ω -perfect subset P_ω , which is closed. The complement $X \setminus P_\omega$ is open and ω -scattered. In particular, (X, τ) is ω -scattered if and only if $P_\omega = \emptyset$.*

Proof. First, note that ω is monotone: if $A \subseteq B$ then $U \cap A \subseteq U \cap B$ for every open U , so $\omega(A) \subseteq \omega(B)$.

Define $\omega^0 = X$, $\omega^{\alpha+1} = \omega(\omega^\alpha)$, and $\omega^\lambda = \bigcap_{\alpha < \lambda} \omega^\alpha$ for limit λ . Since $\omega(A)$ is closed (Lemma 6.13) and $\omega(A) \subseteq \text{cl}(A)$, if A is closed then $\omega(A) \subseteq A$. In particular $\omega^1 = \omega(X) \subseteq X = \omega^0$; if $\omega^\alpha \subseteq \omega^\beta$ for $\beta < \alpha$, then by monotonicity $\omega^{\alpha+1} = \omega(\omega^\alpha) \subseteq \omega(\omega^\beta) = \omega^{\beta+1}$, and at limits the inclusion is preserved by definition. Hence the sequence is decreasing. Since the ω^α are subsets of X and each successor

step either strictly shrinks the set or stabilises, the sequence must stabilise at some ordinal $\gamma \leq |X|$; set $P_\omega = \omega^\gamma$. By construction $\omega(P_\omega) = \omega^{\gamma+1} = \omega^\gamma = P_\omega$, so P_ω is ω -perfect. Since each ω^α is closed by Lemma 6.13, and intersections of closed sets are closed, P_ω is closed, and $X \setminus P_\omega$ is open.

It remains to show that P_ω is the largest ω -perfect subset. Let $Q \subseteq X$ be ω -perfect, so $Q \subseteq \omega(Q)$. We show $Q \subseteq \omega^\alpha$ for every α by transfinite induction. The base case $Q \subseteq \omega^0 = X$ is immediate. If $Q \subseteq \omega^\alpha$, then by monotonicity $\omega(Q) \subseteq \omega(\omega^\alpha) = \omega^{\alpha+1}$, and since $Q \subseteq \omega(Q)$ we obtain $Q \subseteq \omega^{\alpha+1}$. At limit λ , if $Q \subseteq \omega^\alpha$ for all $\alpha < \lambda$, then $Q \subseteq \bigcap_{\alpha < \lambda} \omega^\alpha = \omega^\lambda$. Hence $Q \subseteq P_\omega$.

Finally, if P is a non-empty ω -perfect subset of the subspace $X \setminus P_\omega$, then $P \subseteq \omega(P)$ since $X \setminus P_\omega$ is open. The set $\omega(P)$ is then a non-empty closed ω -perfect subset of X , so $\omega(P) \subseteq P_\omega$ by maximality, contradicting $P \subseteq X \setminus P_\omega$. Hence $X \setminus P_\omega$ is ω -scattered. $\square$

6.3 Topological completeness for K4 and GL

In this section we establish completeness with respect to the nonstandard semantics. Corollary 6.7 demonstrates the validity of Axiom 4 for $\Diamond_{\text{ns}}$ on T_1 spaces via the collapse to $\Diamond_d$. We now show that Axiom 4 holds on all topological spaces, with no separation condition required.

Lemma 6.15. *For every topological model $\mathcal{M}$ and every formula φ :*

$$\llbracket \Diamond_{\text{ns}} \Diamond_{\text{ns}} \varphi \rrbracket \subseteq \llbracket \Diamond_{\text{ns}} \varphi \rrbracket.$$

Proof. Let $A = \llbracket \varphi \rrbracket$ and suppose $x \in \omega(\omega(A)) = \llbracket \Diamond_{\text{ns}} \Diamond_{\text{ns}} \varphi \rrbracket$ by Theorem 6.4. Let $U \in \mathcal{N}(x)$ be arbitrary. By definition, $|U \cap \omega(A)| \geq \aleph_0$; pick a point $y \in U \cap \omega(A)$. Since U is an open neighbourhood of y and $y \in \omega(A)$, $|U \cap A| \geq \aleph_0$. Since U is arbitrary, $x \in \omega(A) = \llbracket \Diamond_{\text{ns}} \varphi \rrbracket$ by Theorem 6.4. $\square$

Theorem 6.16 (topological completeness for K4). *K4 is sound for $\Diamond_{\text{ns}}$ over all topological spaces. It is complete over all infinite topological spaces.*

Proof. Soundness follows from the results above. For completeness, suppose $\varphi \notin \text{K4}$. By the Kripke completeness of K4 [4], φ is falsified on some finite transitive Kripke frame. By Bezhanishvili, Esakia, and Gabelaia [2, 3], every such frame is the image of a d-map from a metrizable, hence infinite and T_1 , space (X, τ) , and φ is falsified under $\Diamond_d$ on (X, τ) . Since (X, τ) is T_1 , Lemma 6.6 implies $\Diamond_{\text{ns}} = \Diamond_d$, so φ is falsified under the nonstandard semantics on an infinite topological space. $\square$

We proceed to the next completeness result.

Lemma 6.17. *The L axiom*

$$\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$$

is valid for the nonstandard semantics over all ω -scattered topological spaces.

Proof. Fix a model $\mathcal{M}$ and set $A = \llbracket \neg\varphi \rrbracket^{\mathcal{M}}$. By the ω -accumulation characterisation, $\llbracket \Box_{\text{ns}} \varphi \rrbracket^{\mathcal{M}} = X \setminus \omega(A)$. From this we obtain $\llbracket \Box_{\text{ns}}(\Box_{\text{ns}} \varphi \rightarrow \varphi) \rrbracket = X \setminus \omega(A \setminus \omega(A))$. Hence L is false at $x \in X$ if and only if x belongs to the set $D := \omega(A) \setminus \omega(A \setminus \omega(A))$. In particular, L is valid in $\mathcal{M}$ if and only if $D = \emptyset$.

($\Leftarrow$) Suppose (X, τ) is ω -scattered. We show $D = \emptyset$ by proving that D is ω -perfect; ω -scatteredness then forces $D = \emptyset$. Let $x \in D$. By definition, since $x \in \omega(A)$, every open $U \ni x$ satisfies $|U \cap A| \geq \aleph_0$; since $x \notin \omega(A \setminus \omega(A))$, there is an open neighbourhood U_x of x with $|U_x \cap (A \setminus \omega(A))| < \aleph_0$. Let V be an

arbitrary open neighbourhood of x ; replacing it with $V \cap U_x$ if necessary, we may assume $V \subseteq U_x$. Since $x \in \omega(A)$, the set $V \cap A$ is infinite. Since $V \subseteq U_x$, the set $V \cap (A \setminus \omega(A))$ is finite. Therefore

$$V \cap A \cap \omega(A) = (V \cap A) \setminus (A \setminus \omega(A))$$

is infinite. We claim that every $y \in V \cap A \cap \omega(A)$ belongs to D , i.e. satisfies $y \notin \omega(A \setminus \omega(A))$. Suppose for contradiction that $y \in \omega(A \setminus \omega(A))$. Since $y \in V$ and V is open, $|V \cap (A \setminus \omega(A))| \geq \aleph_0$. But $V \subseteq U_x$ implies $V \cap (A \setminus \omega(A)) \subseteq U_x \cap (A \setminus \omega(A))$, which is finite, a contradiction.

Therefore $V \cap D \supseteq V \cap A \cap \omega(A)$, which is infinite. Since V was an arbitrary open neighbourhood of x , we have $x \in \omega(D)$. As $x \in D$ was arbitrary, $D \subseteq \omega(D)$, so $\omega(D)$ is a closed ω -perfect subset of X . Since (X, τ) is ω -scattered, $\omega(D) = \emptyset$, hence $D = \emptyset$, so L is valid in $\mathcal{M}$.

($\Rightarrow$) Suppose $P \subseteq X$ is a non-empty ω -perfect set, so $\omega(P) \supseteq P$. Define a model $\mathcal{M}$ such that $v(p) = X \setminus P$, and set $A := \llbracket \neg p \rrbracket = P$. Since $\omega(P) \supseteq P$,

$$\llbracket \Box_{\text{ns}} p \rrbracket = X \setminus \omega(P) \subseteq X \setminus P = \llbracket p \rrbracket,$$

so $\Box_{\text{ns}} p \rightarrow p$ is valid on $\mathcal{M}$. Hence $\llbracket \neg(\Box_{\text{ns}} p \rightarrow p) \rrbracket = \emptyset$ and so $\omega(\emptyset) = \emptyset$ implies $\llbracket \Box_{\text{ns}}(\Box_{\text{ns}} p \rightarrow p) \rrbracket = X \setminus \omega(\emptyset) = X$. On the other hand, for each $x \in P \subseteq \omega(P) = \omega(A)$, we have $x \notin X \setminus \omega(A) = \llbracket \Box_{\text{ns}} p \rrbracket$. Therefore, L is not valid on (X, τ) . $\square$

Theorem 6.18 (topological completeness for GL). *GL is sound for $\Diamond_{\text{ns}}$ over all ω -scattered topological spaces. It is complete over all infinite ω -scattered topological spaces.*

Proof. Soundness follows from the results above. For completeness, suppose $\varphi \notin \text{GL}$. By the Kripke completeness of GL [11], φ is falsified on some finite irreflexive transitive Kripke frame. By Bezhanishvili, Esakia, and Gabelaia [2, 3], every such frame is the image of a d-map from a scattered metrizable, hence infinite and T_1 , space (X, τ) , and φ is falsified under $\Diamond_d$ on (X, τ) .

Since (X, τ) is T_1 , Lemma 6.6 gives $\Diamond_{\text{ns}} = \Diamond_d$, so φ is falsified under the nonstandard semantics. It remains to show (X, τ) is ω -scattered. If P is a non-empty ω -perfect subset, then $P \subseteq \omega(P) = d(P)$ by Lemma 6.6, so P is a non-empty perfect set, contradicting scatteredness. Therefore (X, τ) is ω -scattered, and φ is falsified under $\Diamond_{\text{ns}}$ on an infinite ω -scattered space. $\square$

The argument above shows in particular that every scattered space is ω -scattered, but not vice versa.

7 Final remarks

We have introduced the halo operator, giving a uniform nonstandard-analytic framework from which the classical closure and derived-set semantics arise as instances. It yields a new operator which turns out to be the modal counterpart of the ω -accumulation point operator, with K4 and GL as its complete logics over infinite spaces and infinite ω -scattered spaces, respectively.

The four exclusion functions studied here are the canonical extremes, but intermediate choices are a natural next step. For instance, $E(x) = \{x\} \cup (\mu(x) \cap X)$ excludes the point itself together with all standard points in its halo. More generally, one could require witnesses to avoid any prescribed internal subset of $\mu(x)$, producing a rich family of operators interpolating between $\Diamond_d$ and $\Diamond_{\text{ns}}$.

Another interesting direction for future work is the logic of $\Diamond_{\text{ns}}$ over a single fixed infinite space such as $\mathbb{R}$. The natural candidate is K4 extended by the seriality axiom $\Diamond_{\text{ns}} \top$, since $\mathbb{R}$ is dense and every point is an ω -accumulation point of any infinite set, but a precise axiomatisation and proof remain to be established.

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Analytic Cut in Epistemic Logics with Distributed Knowledge

Ryo Murai

Independent Researcher
ryo.murai11@gmail.com

Sizhuo Liu

Graduate School of Humanities and Human Sciences
Hokkaido University
Hokkaido, Japan
liuliusizhuo@outlook.com

Katsuhiko Sano

Faculty of Humanities and Human Sciences
Hokkaido University
Hokkaido, Japan
v-sano@let.hokudai.ac.jp

Distributed knowledge is a notion of group knowledge studied in multi-agent epistemic logic. Semantically, the distributed knowledge of a group is interpreted via an accessibility relation given by the intersection of the epistemic accessibility relations of the agents in that group. This paper investigates sequent calculi for epistemic logics of distributed knowledge based on **K45**, **KD45**, and **S5**. While cut elimination holds in existing sequent calculi for modal logics **K45** and **KD45**, it fails in all the systems mentioned above. Instead, we establish the analytic cut property for all three systems by adapting Takano’s (2018) strategy, which restricts the cut formulas to the set of subformulas of the conclusion of the cut rule. As a corollary, the Craig interpolation theorem holds for all logics considered. We also show that all proof-theoretic results remain valid when the empty group is allowed for the distributed-knowledge operator, in which case the distributed knowledge for the empty group is interpreted as the global modality.

1 Introduction

Distributed knowledge is a notion of group knowledge in multi-agent epistemic logic [7] whose interpretation has varied in the literature. In this paper we follow the interpretation introduced in [2], i.e., a group G is said to have distributed knowledge of φ if an *external* observer with access to the epistemic states of all members of G can know that φ . In Kripke semantics, while individual knowledge (notation: $\Box_a \varphi$, read as “agent a knows that φ ”) is interpreted via an accessibility relation R_a for each agent a , distributed knowledge for a group G is interpreted via the intersection $\bigcap_{a \in G} R_a$ of the epistemic accessibility relations of the agents in G . Accordingly, $D_G \varphi$ holds at a state w if and only if φ holds at all states v such that $(w, v) \in \bigcap_{a \in G} R_a$. Reflexivity, transitivity, Euclideaness, and symmetry are preserved when passing from the individual level to the group level, i.e., under intersection of epistemic accessibility relations, whereas seriality is *not* preserved (see [3, Lemma 3.2]).

Sequent calculus, introduced by Gentzen in the 1930s, is a proof-theoretic framework where derivations are formulated as sequents $\Gamma \Rightarrow \Delta$ (read as “if all formulas in Γ hold, then some formula in Δ holds”), where Γ and Δ are finite multisets of formulas [9, 10]. A crucial aspect of sequent calculus is the cut rule

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} (Cut),$$

and the associated cut elimination theorem (Gentzen’s *Hauptsatz*), which states that every derivable sequent admits a cut-free derivation. As emphasized in [32], cut elimination is closely connected to proof normalization and the structural analysis of derivations.

For modal logics, Gentzen-style sequent calculi $G(\mathbf{K45})$ and $G(\mathbf{KD45})$ for $\mathbf{K45}$ and $\mathbf{KD45}$, respectively, were shown to be cut-free in [27, Theorem 2]. However, as shown in [21, p. 116], cut elimination fails for the sequent calculus $G(\mathbf{S5})$ corresponding to modal logic $\mathbf{S5}$. As an alternative, Takano [28] showed that every application of the cut rule in $G(\mathbf{S5})$ can be transformed into an *analytic* cut, i.e., a cut rule in which the cut formula is a subformula of the conclusion. Moreover, Takano later provided a semantic analysis of sequent calculi for modal logics in [30], where cut-free sequent calculi such as $G(\mathbf{K45})$ and $G(\mathbf{KD45})$ were semantically shown to enjoy cut elimination, while sequent calculi that lack cut elimination such as $G(\mathbf{S5})$ were shown to satisfy the extended subformula property. These results are established by proving the semantic completeness of sequent calculi where the cut rule is removed or restricted to an analytic cut. A key ingredient of the semantic proof is the notion of Ξ -*partial valuation*, i.e., a subformula-closed finite set of formulas, which originates from [29], that was later applied in [16, 15], [30], [23] and [33] to establish cut elimination or the analytic cut property for the sequent calculi for temporal epistemic logics, modal logics, bi-intuitionistic tense logic and awareness logic, respectively.

Model-theoretic investigations of distributed knowledge have been extensively documented in, for example, [11, 25, 2, 3], whereas proof-theoretic studies, namely sequent calculi for logics with distributed knowledge, remain comparatively limited. A labelled $G3$ -style sequent calculus, i.e., a calculus without structural rules in which each formula is equipped with a label, was introduced in [13]. A Gentzen-style sequent calculus based on $\mathbf{S4}$ was proposed in [24], in which the distributed knowledge operator is *not* parameterized by a group G . Gentzen-style and Kanger-style sequent calculi for $\mathbf{S5}$ were later studied in [12] in the same fashion. Label-free Gentzen-style sequent calculi for multi-agent epistemic propositional logics $\mathbf{K}_D$, $\mathbf{KD}_D$, $\mathbf{K4}_D$, $\mathbf{S4}_D$, and $\mathbf{S5}_D$ with group-parameterized distributed knowledge operators were introduced in [17]. Cut elimination was established for these systems except for $G(\mathbf{S5}_D)$, which is the sequent calculus for $\mathbf{S5}_D$. Moreover, the Craig interpolation theorem was obtained for the four systems via a method based on [26]. Furthermore, although groups are usually defined as *non-empty*, the operator $D_\emptyset$ corresponds naturally to the global modality when the empty set is allowed as a group.

A sequent calculus $G(\mathbf{S5}_D)$ for $\mathbf{S5}_D$ was introduced in [17, 19]. This paper extends that line of research by introducing sequent calculi $G(\mathbf{K45}_D)$ and $G(\mathbf{KD45}_D)$ for $\mathbf{K45}_D$ and $\mathbf{KD45}_D$, respectively, while also revisiting $G(\mathbf{S5}_D)$. Surprisingly, although cut elimination holds for $G(\mathbf{K45})$ and $G(\mathbf{KD45})$, it *fails* not only for $G(\mathbf{S5}_D)$, but *also* for $G(\mathbf{K45}_D)$ and $G(\mathbf{KD45}_D)$. Therefore, by adapting Takano's semantic approach [30], we establish the analytic cut property for all three systems. As a consequence, the Craig interpolation theorem is obtained via the method in [26]. We note that conditions on *both* propositional variables and *agents* are taken into account, which is a relatively new result for logics for distributed knowledge. To the best of the authors' knowledge, such conditions have recently been considered only in [17, 34] and [5]. Although the Craig interpolation theorem in this paper has been established in [34], the proof is model-theoretic rather than proof-theoretic as in [26]. In fact, a proof-theoretic result was presented earlier by the first and third authors in [19]. As emphasized in [17], this result suggests that the logics with distributed knowledge are "good" expansions of basic modal logic, since the Craig interpolation theorem *does not* hold for some expansions of basic modal logic [31]. Moreover, we show that when the empty group is admitted, the operator $D_\emptyset$ corresponds naturally to the global modality, and that the main proof-theoretic results are preserved under this extension.

The paper is organized as follows. Section 2 introduces the syntax and semantics of distributed epistemic logics based on $\mathbf{K45}$, $\mathbf{KD45}$, and $\mathbf{S5}$ together with their Hilbert systems and pseudo-model semantics. Section 3 presents the sequent calculi $G(\mathbf{K45}_D)$, $G(\mathbf{KD45}_D)$, and $G(\mathbf{S5}_D)$. Section 4 first demonstrates that cut elimination fails for these systems, then establishes the analytic cut property, while

the Craig interpolation theorem is proved in Section 5. In Section 6, we extend the framework by introducing the global modality corresponding to distributed knowledge of the empty group and show that the analytic cut property and interpolation results are preserved under this extension. Finally, Section 7 concludes the paper.

2 Preliminaries

Let Prop be a countably infinite set of propositional variables and Ag a non-empty finite set of agents. A *group* is a non-empty set $G \subseteq \text{Ag}$ of agents and the set of all groups is denoted by

$$\text{Grp} := \{G \in \mathcal{P}(\text{Ag}) \mid G \neq \emptyset\}.$$

The syntax L for distributed epistemic logic is defined inductively as follows:

$$L \ni \varphi ::= p \mid \perp \mid \neg\varphi \mid \varphi \rightarrow \psi \mid D_G\varphi,$$

where $p \in \text{Prop}$ and $G \in \text{Grp}$. Boolean connectives $\wedge, \vee, \leftrightarrow$ and the truth constant $\top$ are defined in the usual manner. We also define the operator $\Box_a\varphi$ for individual knowledge or belief as $\Box_a\varphi := D_{\{a\}}\varphi$. For a finite set Δ , $\bigwedge\Delta$ and $\bigvee\Delta$ are the conjunction or disjunction of all formulas in Δ (when Δ is empty, $\bigwedge\Delta := \top$ and $\bigvee\Delta := \perp$). As noted above, an expression of the form $D_\emptyset\varphi$ is not a well-formed formula, since we have excluded $\emptyset$ from our definition of groups.

We move on to the semantics. A *frame* F is a tuple $(W, (R_a)_{a \in \text{Ag}})$ where W is a non-empty set of states and $R_a \subseteq W \times W$ is a binary relation on W for each $a \in \text{Ag}$. A *model* M is a tuple $(W, (R_a)_{a \in \text{Ag}}, V)$ where $(W, (R_a)_{a \in \text{Ag}})$ is a frame and $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function. Given a model $M = (W, (R_a)_{a \in \text{Ag}}, V)$ and a state $w \in W$, the notion of φ being true at w in M (notation: $M, w \models \varphi$) is defined inductively as follows:

$$\begin{aligned} M, w \models p & \quad \text{iff} \quad w \in V(p), \\ M, w \models \perp & \quad \text{Never}, \\ M, w \models \neg\varphi & \quad \text{iff} \quad M, w \not\models \varphi, \\ M, w \models \varphi \rightarrow \psi & \quad \text{iff} \quad M, w \not\models \varphi \text{ or } M, w \models \psi, \\ M, w \models D_G\varphi & \quad \text{iff} \quad \text{for all } v \in W, (w, v) \in \bigcap_{a \in G} R_a \text{ implies } M, v \models \varphi. \end{aligned}$$

A pair (M, w) of a model M and a state w in M is called a *pointed model*. A formula φ is *valid* in a model M (notation: $M \models \varphi$) if $M, w \models \varphi$ for all $w \in W$. A formula φ is *valid* in a frame F (notation: $F \models \varphi$) if $(F, V) \models \varphi$ for all valuations V . Given a class $\mathbb{F}$ of frames, φ is *valid* in $\mathbb{F}$ if $F \models \varphi$ for every frame $F \in \mathbb{F}$.

To capture various aspects of individual knowledge operators, we consider a class of frames $\mathbb{F}$ in which all accessibility relations R_a for $a \in \text{Ag}$ satisfy the same set of frame properties, defined as follows.

Definition 2.1. We define $\mathbb{F}_{\mathbf{K45}}$ to be the class of all frames such that each R_a is transitive and Euclidean, $\mathbb{F}_{\mathbf{KD45}}$ to be the class of all frames such that each R_a is transitive, Euclidean and serial ($\forall w \in W \exists v (wR_av)$), and $\mathbb{F}_{\mathbf{S5}}$ to be the class of all frames such that each R_a is reflexive, transitive, and Euclidean (equivalently, symmetric). We define the distributed epistemic logics $\mathbf{K45}_D$, $\mathbf{KD45}_D$ and $\mathbf{S5}_D$ to be the sets of all L -formulas that are valid in the class of $\mathbb{F}_{\mathbf{K45}}$, $\mathbb{F}_{\mathbf{KD45}}$ and $\mathbb{F}_{\mathbf{S5}}$, respectively.

As proved in [3, Lemma 3.2], reflexivity, transitivity, Euclideaness and symmetry are preserved under taking the intersection $\bigcap_{a \in G} R_a$ of binary relations¹. It is well-known that all logics in Definition 2.1 can be axiomatized (see [7]).

Table 1: Hilbert system $H(\mathbf{K}_D)$, $H(\mathbf{K45}_D)$, $H(\mathbf{KD45}_D)$ and $H(\mathbf{S5}_D)$

All the Axioms and Rules of $H(\mathbf{K}_D)$		Additional Axiom Schemes	
(Taut)	all instances of propositional tautologies	(T)	$D_G \phi \rightarrow \phi$
(Incl)	$D_G \phi \rightarrow D_H \phi \ (G \subseteq H)$	(D)	$\neg D_{\{a\}} \perp$
(K)	$D_G(\phi \rightarrow \psi) \rightarrow (D_G \phi \rightarrow D_G \psi)$	(4)	$D_G \phi \rightarrow D_G D_G \phi$
(MP)	From $\phi \rightarrow \psi$ and ϕ infer ψ	(5)	$\neg D_G \phi \rightarrow D_G \neg D_G \phi$
(Nec)	From ϕ infer $D_G \phi$		

Definition 2.2. The Hilbert system $H(\mathbf{K}_D)$ is defined as in Table 1. The Hilbert system $H(\mathbf{K45}_D)$ is defined as an axiomatic expansion of $H(\mathbf{K}_D)$ with (4) and (5) of Table 1. Moreover, the Hilbert systems $H(\mathbf{KD45}_D)$ and $H(\mathbf{S5}_D)$ are defined as axiomatic expansions of $H(\mathbf{K45}_D)$ with (D) and (T) of Table 1, respectively.

The following notion of *pseudo-model* is used in, for instance, [7, 11], as an intermediate step for semantic completeness proofs of epistemic logics with distributed knowledge. However, in this paper, we use pseudo-models to establish semantically the analytic property of sequent calculi in Section 4.

Definition 2.3. A *pseudo frame* F is a tuple $(W, (R_G)_{G \in \text{Grp}})$ where W is a set of states, $R_G \subseteq W \times W$ is a binary relation on W such that for each $G, H \in \text{Grp}$ and $G \subseteq H$ implies $R_H \subseteq R_G$. A *pseudo model* M is a tuple $(W, (R_G)_{G \in \text{Grp}}, V)$ where $(W, (R_G)_{G \in \text{Grp}})$ is a pseudo frame and V is a function from Prop to $\mathcal{P}(W)$.

Given a frame $F = (W, (R_a)_{a \in \text{Ag}})$, we define $R_G := \bigcap_{a \in G} R_a$. Then $(W, (R_G)_{G \in \text{Grp}})$ is a pseudo frame. The notion of ϕ being true at w in M is defined in the same way as at the beginning of Section 2 except that

$$M, w \models D_G \phi \text{ iff for all } v \in W, w R_G v \text{ implies } M, v \models \phi.$$

Definition 2.4. We define $\mathbb{P}_{\mathbf{K45}_D}$ to be the class of all pseudo frames such that each R_G is transitive and Euclidean, $\mathbb{P}_{\mathbf{KD45}_D}$ to be the class of all pseudo frames $F = (W, (R_G)_{G \in \text{Grp}})$ such that $F \in \mathbb{P}_{\mathbf{K45}_D}$ and each $R_{\{a\}}$ is serial, $\mathbb{P}_{\mathbf{S5}_D}$ to be the class of all pseudo frames such that each R_G is reflexive, transitive, and Euclidean (equivalently, symmetric).

Fact 2.5. Let $\Lambda \in \{\mathbf{K45}_D, \mathbf{KD45}_D, \mathbf{S5}_D\}$. The following are all equivalent: 1. ϕ is a theorem of $H(\Lambda)$; 2. ϕ is valid in $\mathbb{P}_\Lambda$; 3. ϕ is valid in $\mathbb{F}_\Lambda$.

Proof. First, item 1 implies item 2 by the soundness of $H(\Lambda)$, which can be established straightforwardly. Moreover, item 2 implies item 3 since $\mathbb{F}_\Lambda$ can be regarded as a subclass of $\mathbb{P}_\Lambda$. So it remains to show that item 3 implies item 1, which is a result already established in the literature. A proof sketch for $H(\mathbf{K}_D)$ and $H(\mathbf{K45}_D)$ can be found in [7, Theorem 3.4.1] and [11, Theorem 3.30], respectively. Detailed proofs are provided in [3, Theorem 4.7] for $H(\mathbf{KD45}_D)$ and in [35, Theorem 10] for $H(\mathbf{S5}_D)$. $\square$

¹However, seriality is *not* preserved. As shown in [3, Lemma 3.2], we can construct a $\mathbf{KD45}$ model in which R_a and R_b are serial while $R_a \cap R_b$ is not. Thus, as discussed in [3], the axiom $\neg D_G \perp$ in [6] is not valid, hence the original axiomatization of $\mathbf{KD45}_D$ in [6] is unsound. This axiomatization was subsequently corrected in [7], where completeness is reported, although without a detailed proof.

3 Sequent Calculi for Distributed Knowledge Logics

In this section we introduce sequent calculi $G(\mathbf{K45}_D)$, $G(\mathbf{KD45}_D)$ and $G(\mathbf{S5}_D)$ built on $\mathbf{LK}_0$ (see [28]), which is the propositional fragment of a sequent calculus $\mathbf{LK}$ [9, 10] of the first-order classical logic. A *sequent* $\Gamma \Rightarrow \Delta$ is a pair of finite multisets Γ and Δ of formulas read as “if all formulas in Γ hold, then at least one formula in Δ holds”. Moreover, we use $D_{\{a\}}\Gamma$ to denote $\{D_{\{a\}}\varphi \mid \varphi \in \Gamma\}$ and Γ, Δ to denote the multiset union $\Gamma \cup \Delta$.

Definition 3.1. A sequent calculus $\mathbf{LK}_0$ consists of the following.

- Axioms:

$$\frac{}{\varphi \Rightarrow \varphi} (\text{id}) \quad \frac{}{\perp \Rightarrow} (\perp)$$

- Structural Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} (w \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} (\Rightarrow c) \quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} (c \Rightarrow)$$

- Logical Rules:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi} (\Rightarrow \neg) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta} (\neg \Rightarrow)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} (\Rightarrow \rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} (\rightarrow \Rightarrow)$$

- Cut:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} (\text{Cut})_\varphi$$

The sequent calculus $G(\mathbf{K45}_D)$ is $\mathbf{LK}_0$ expanded with the following rule:

$$\frac{\varphi_1, \dots, \varphi_m, D_{G_1}\varphi_1, \dots, D_{G_m}\varphi_m \Rightarrow D_{H_1}\psi_1, \dots, D_{H_n}\psi_n, \chi}{D_{G_1}\varphi_1, \dots, D_{G_m}\varphi_m \Rightarrow D_{H_1}\psi_1, \dots, D_{H_n}\psi_n, D_G\chi} (\Rightarrow D^{\mathbf{K45}_D}),$$

where $G_i, H_j \subseteq G$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$. The sequent calculus $G(\mathbf{KD45}_D)$ expands $G(\mathbf{K45}_D)$ with the following:

$$\frac{\Gamma, D_{\{a\}}\Gamma \Rightarrow D_{\{a\}}\Delta}{D_{\{a\}}\Gamma \Rightarrow D_{\{a\}}\Delta} (D^{\mathbf{KD45}_D}).$$

Finally, the sequent calculus $G(\mathbf{S5}_D)$ expands $\mathbf{LK}_0$ with the following rules:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta} (D \Rightarrow) \quad \frac{D_{G_1}\varphi_1, \dots, D_{G_m}\varphi_m \Rightarrow D_{H_1}\psi_1, \dots, D_{H_n}\psi_n, \chi}{D_{G_1}\varphi_1, \dots, D_{G_m}\varphi_m \Rightarrow D_{H_1}\psi_1, \dots, D_{H_n}\psi_n, D_G\chi} (\Rightarrow D^{\mathbf{S5}_D}),$$

where $G_i, H_j \subseteq G$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$ in $(\Rightarrow D^{\mathbf{S5}_D})$.

Note that m and n are possibly 0 in $(\Rightarrow D^{\mathbf{K45}_D})$ and $(\Rightarrow D^{\mathbf{S5}_D})$. For each calculus, we define the notion of derivability of a sequent as a finite tree generated from axioms (id) and $(\perp)$ by inference rules specific to the calculus. In addition, we use a double line $=$ as an abbreviation of finite applications of structural rules. Given a derivation $\mathcal{D}$, a sequent in the root node of $\mathcal{D}$ is called the *end-sequent* and we use $\text{root}(\mathcal{D}) \equiv \Gamma \Rightarrow \Delta$ to mean that the end-sequent of $\mathcal{D}$ is $\Gamma \Rightarrow \Delta$. Moreover we define the *height* of $\mathcal{D}$ to be the maximum length of branches in $\mathcal{D}$ from the end-sequent to an axiom.

Proposition 3.2. Let $\Lambda \in \{\mathbf{K45}_D, \mathbf{KD45}_D, \mathbf{S5}_D\}$. If $\Gamma \Rightarrow \Delta$ is derivable in $G(\Lambda)$ then $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid in $\mathbb{F}_\Lambda$.

Proof. By induction on the height of the derivation $\mathcal{D}$ to obtain $\Gamma \Rightarrow \Delta$. For base step, our goal is immediate. For inductive step, we divide the argument depending on the last rule applied. We only comment on the case of $(\Rightarrow D^{\mathbf{K45}_D})$ for $G(\mathbf{K45}_D)$:

$$\frac{\varphi_1, \dots, \varphi_m, D_{G_1} \varphi_1, \dots, D_{G_m} \varphi_m \Rightarrow D_{H_1} \psi_1, \dots, D_{H_n} \psi_n, \chi}{D_{G_1} \varphi_1, \dots, D_{G_m} \varphi_m \Rightarrow D_{H_1} \psi_1, \dots, D_{H_n} \psi_n, D_G \chi} (\Rightarrow D^{\mathbf{K45}_D}),$$

where $\bigcup_{i=1}^m G_i \cup \bigcup_{j=1}^n H_j \subseteq G$. Our goal is to show $\mathbb{F}_{\mathbf{K45}_D} \models \bigwedge_{i=1}^m D_{G_i} \varphi_i \rightarrow \bigvee_{j=1}^n D_{H_j} \psi_j \vee D_G \chi$. Fix any model M from $\mathbb{F}_{\mathbf{K45}_D}$ and any $w \in W$. Suppose $M, w \models \bigwedge_{i=1}^m D_{G_i} \varphi_i$. We show that $M, w \models \bigvee_{j=1}^n D_{H_j} \psi_j \vee D_G \chi$. Suppose $M, w \not\models \bigvee_{j=1}^n D_{H_j} \psi_j$ and we show that $M, w \models D_G \chi$. Fix any $v \in W$ such that $(w, v) \in \bigcap_{a \in G} R_a$. It suffices to show that $M, v \models \chi$. By induction hypothesis, we obtain $\mathbb{F}_{\mathbf{K45}_D} \models \bigwedge_{i=1}^m \varphi_i \wedge \bigwedge_{i=1}^m D_{G_i} \varphi_i \rightarrow \bigvee_{j=1}^n D_{H_j} \psi_j \vee \chi$. Since $G_i \subseteq G$, $(w, v) \in \bigcap_{a \in G} R_a \subseteq \bigcap_{b \in G_i} R_b$. Hence $M, v \models \bigwedge_{i=1}^m \varphi_i$ by our assumption $M, w \models \bigwedge_{i=1}^m D_{G_i} \varphi_i$. Moreover by transitivity we obtain $\mathbb{F}_{\mathbf{K45}_D} \models \bigwedge_{i=1}^m D_{G_i} \varphi_i$, since for every $u \in W$ such that $(v, u) \in \bigcap_{b \in G_i} R_b$, $(w, u) \in \bigcap_{b \in G_i} R_b$. Therefore $M, v \models \bigvee_{j=1}^n D_{H_j} \psi_j \vee \chi$. For our goal, it suffices to show $M, v \not\models \bigvee_{j=1}^n D_{H_j} \psi_j$. By $M, w \not\models \bigvee_{j=1}^n D_{H_j} \psi_j$, there exists a state $u_j \in W$ for every $1 \leq j \leq n$ such that $(w, u_j) \in \bigcap_{c \in H_j} R_c$ and $M, u_j \not\models \psi_j$. Then by Euclideaness, we obtain $(v, u_j) \in \bigcap_{c \in H_j} R_c$, which implies our goal $M, v \not\models \bigvee_{j=1}^n D_{H_j} \psi_j$ together with $M, u_j \not\models \psi_j$. $\square$

Proposition 3.3. Let $\Lambda \in \{\mathbf{K45}_D, \mathbf{KD45}_D, \mathbf{S5}_D\}$. If φ is a theorem of $H(\Lambda)$, then $\Rightarrow \varphi$ is derivable in $G(\Lambda)$.

Proof. We note that (Cut) is necessary to simulate (MP) . We only check the derivability of $(Incl)$ in each system. Suppose $G \subseteq H$. For $\Lambda \in \{\mathbf{K45}_D, \mathbf{KD45}_D, \mathbf{S5}_D\}$, the following derivations (left for $\mathbf{K45}_D, \mathbf{KD45}_D$ and right for $\mathbf{S5}_D$) witness $(Incl)$, where the side conditions are satisfied by the assumption $G \subseteq H$.

$$\frac{\frac{\varphi \Rightarrow \varphi}{\varphi, D_G \varphi \Rightarrow \varphi} (w \Rightarrow) \quad \frac{\varphi \Rightarrow \varphi}{D_G \varphi \Rightarrow \varphi} (D \Rightarrow)}{\frac{D_G \varphi \Rightarrow D_H \varphi}{\Rightarrow D_G \varphi \Rightarrow D_H \varphi} (\Rightarrow D^{\mathbf{K45}_D})} (\Rightarrow \rightarrow) \quad \frac{\frac{\varphi \Rightarrow \varphi}{D_G \varphi \Rightarrow \varphi} (D \Rightarrow) \quad \frac{D_G \varphi \Rightarrow D_H \varphi}{D_G \varphi \Rightarrow D_H \varphi} (\Rightarrow D^{\mathbf{S5}_D})}{\Rightarrow D_G \varphi \Rightarrow D_H \varphi} (\Rightarrow \rightarrow). \quad \square$$

As shown in [27], sequent calculi $G(\mathbf{K45})$ and $G(\mathbf{KD45})$ for modal logic $\mathbf{K45}$ and $\mathbf{KD45}$, respectively, are cut-free, while sequent calculus $G(\mathbf{S5})$ for modal logic $\mathbf{S5}$ is not [21, p. 116]. A sequent $\Rightarrow D_{\{a\}} \neg D_{\{a\}} p, D_{\{a,b\}} p$ is derivable in $G(\mathbf{K45}_D)$ and $G(\mathbf{KD45}_D)$ as follows:

$$\frac{\frac{\frac{D_{\{a\}} p \Rightarrow D_{\{a\}} p}{\Rightarrow \neg D_{\{a\}} p, D_{\{a\}} p} (\Rightarrow \neg) \quad \frac{p \Rightarrow p}{p, D_{\{a\}} p \Rightarrow p} (w \Rightarrow)}{\Rightarrow D_{\{a\}} \neg D_{\{a\}} p, D_{\{a\}} p} (\Rightarrow D^{\mathbf{K45}_D}) \quad \frac{p \Rightarrow p}{D_{\{a\}} p \Rightarrow D_{\{a,b\}} p} (\Rightarrow D^{\mathbf{K45}_D})}{\Rightarrow D_{\{a\}} \neg D_{\{a\}} p, D_{\{a,b\}} p} (Cut)_{D_{\{a\}} p},$$

Similarly, we can construct a derivation with (Cut) of the same sequent in $G(\mathbf{S5}_D)$:

$$\frac{\frac{\frac{D_{\{a\}} p \Rightarrow D_{\{a\}} p}{\Rightarrow \neg D_{\{a\}} p, D_{\{a\}} p} (\Rightarrow \neg) \quad \frac{p \Rightarrow p}{D_{\{a\}} p \Rightarrow p} (D \Rightarrow)}{\Rightarrow D_{\{a\}} \neg D_{\{a\}} p, D_{\{a\}} p} (\Rightarrow D^{\mathbf{S5}_D}) \quad \frac{p \Rightarrow p}{D_{\{a\}} p \Rightarrow D_{\{a,b\}} p} (\Rightarrow D^{\mathbf{S5}_D})}{\Rightarrow D_{\{a\}} \neg D_{\{a\}} p, D_{\{a,b\}} p} (Cut)_{D_{\{a\}} p}.$$

It is noted that the cut formula $D_{\{a\}} p$ is a subformula of the conclusion of the cut. Since $G(\mathbf{S5})$ is not cut-free, it is not surprising that $G(\mathbf{S5}_D)$ is also not cut-free. In what follows, however, we show that $\Rightarrow D_{\{a\}} \neg D_{\{a\}} p, D_{\{a,b\}} p$ is not derivable in $G(\mathbf{K45}_D)$, $G(\mathbf{KD45}_D)$ and $G(\mathbf{S5}_D)$ without the cut rule either.

Theorem 3.4. Let $\Lambda \in \{\mathbf{K45}_D, \mathbf{KD45}_D, \mathbf{S5}_D\}$. The sequent $\Rightarrow D_{\{a\}} \neg D_{\{a\}} p, D_{\{a,b\}} p$ is not derivable in $G^-(\Lambda)$, where $G^-(\Lambda)$ is the same calculus as $G(\Lambda)$ except that the cut rule is dropped.

Proof. Let φ^m denote m occurrences of φ . We comment on the case where $\Lambda = \mathbf{KD45}_D$. We can prove the following statements by simultaneous induction on $\mathcal{D}$ of $G^-(\mathbf{KD45}_D)$.

- (1) $\forall m, n, l \in \mathbb{N} (\text{root}(\mathcal{D}) \not\Rightarrow (D_{\{a\}} \neg D_{\{a\}} p)^m, (D_{\{a,b\}} p)^n, p^l),$
- (2) $\forall x_1, x_2, y, z \in \mathbb{N} (\text{root}(\mathcal{D}) \not\Rightarrow p^{x_1}, (D_{\{a\}} p)^{x_2} \Rightarrow (D_{\{a\}} \neg D_{\{a\}} p)^y, (\neg D_{\{a\}} p)^z).$

Then the statement of the lemma is a special case of (1) where $m = n = 1$ and $l = 0$. $\square$

4 Analytic Cut Property of $G(\mathbf{K45}_D)$, $G(\mathbf{KD45}_D)$ and $G(\mathbf{S5}_D)$

This section establishes the analytic cut property, i.e., every application of the cut rule in $G(\Lambda)$ can be replaced with an application of the cut rule such that the cut formula is in the set of *subformulas* of the conclusion of the cut. The set $\text{Sub}(\varphi)$ of subformulas of an L -formula φ is defined as usual. In particular, $\text{Sub}(D_G \varphi) = \text{Sub}(\varphi) \cup \{D_G \varphi\}$. For any multiset Γ of formulas, $\text{Sub}(\Gamma)$ is defined as $\bigcup_{\varphi \in \Gamma} \text{Sub}(\varphi)$. Moreover, we say that Ξ is *subformula-closed* if $\text{Sub}(\Xi) \subseteq \Xi$. Let $\Lambda \in \{\mathbf{K45}_D, \mathbf{KD45}_D, \mathbf{S5}_D\}$ in this section.

Definition 4.1. A cut

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} (Cut)_\varphi$$

is *analytic* if $\varphi \in \text{Sub}(\Gamma, \Pi, \Delta, \Sigma)$. The sequent calculus $G^a(\Lambda)$ is the same calculus as $G(\Lambda)$ except that (Cut) is always analytic. In what follows, when $\Gamma \Rightarrow \Delta$ is derivable in $G^a(\Lambda)$, we denote $G^a(\Lambda) \vdash \Gamma \Rightarrow \Delta$.

Let Ξ be a subformula-closed finite set of formulas in what follows. The following definition is taken from [30], whose notion originates from [29].

Definition 4.2. A pair (Γ, Δ) is a Ξ -*partial valuation* in $G^a(\Lambda)$ if all the following hold:

- (1) $G^a(\Lambda) \not\vdash \Gamma \Rightarrow \Delta;$
- (2) $\text{Sub}(\Gamma, \Delta) = \Gamma \cup \Delta;$
- (3) $\Gamma \cup \Delta \subseteq \Xi.$

Remark 4.3. It is worth noting that the condition for $\text{Sub}(\Gamma, \Delta)$ highlights the distinction between semantically eliminating only non-analytic cuts and eliminating all cuts. In fully cut-free sequent calculi (for example, $\mathbf{S4}$), the equality $\text{Sub}(\Gamma, \Delta) = \Gamma \cup \Delta$ may *not* hold; instead, the inclusion $\Gamma \cup \Delta \subseteq \text{Sub}(\Gamma, \Delta)$ is utilized. This is precisely the difference between the approach used in this paper to establish the analytic cut property and the one applied in [30] to prove semantic completeness for completely cut-free systems.

The following lemma can be established similarly to Lindenbaum's Lemma (cf. [33, Lemma 1]).

Lemma 4.4. Let $\Gamma \cup \Delta \subseteq \Xi$. If $G^a(\Lambda) \not\vdash \Gamma \Rightarrow \Delta$, then there exists a Ξ -*partial valuation* (Γ^+, Δ^+) such that $\Gamma \subseteq \Gamma^+$, $\Delta \subseteq \Delta^+$ and $\text{Sub}(\Gamma, \Delta) = \Gamma^+ \cup \Delta^+$.

Proof. Let $\varphi_1, \dots, \varphi_m$ be an enumeration of all formulas in $\text{Sub}(\Gamma, \Delta)$. In what follows we inductively construct a sequence $(\Gamma_k, \Delta_k) (1 \leq k \leq m)$ such that $G^a(\Lambda) \not\vdash \Gamma_k \Rightarrow \Delta_k$, $\Gamma_k \subseteq \Gamma_{k+1}$ and $\Delta_k \subseteq \Delta_{k+1}$ for all

$1 \leq k \leq m$. For base step, let $(\Gamma_0, \Delta_0) := (\Gamma, \Delta)$. It is clear that $G^a(\Lambda) \not\vdash \Gamma \Rightarrow \Delta$. For inductive step, we define $(\Gamma_{k+1}, \Delta_{k+1})$ as follows:

$$\begin{cases} \Gamma_{k+1} = \Gamma_k \text{ and } \Delta_{k+1} = \Delta_k \cup \{\varphi_k\} & \text{if } G^a(\Lambda) \not\vdash \Gamma_k \Rightarrow \Delta_k, \varphi_k \\ \Gamma_{k+1} = \Gamma_k \cup \{\varphi_k\} \text{ and } \Delta_{k+1} = \Delta_k & \text{otherwise} \end{cases}$$

Moreover, it is not the case that $G^a(\Lambda) \vdash \Gamma_k \Rightarrow \Delta_k, \varphi_k$ and $G^a(\Lambda) \vdash \varphi_k, \Gamma_k \Rightarrow \Delta_k$. Suppose it is the case for contradiction. Then we can derive $\Gamma_k \Rightarrow \Delta_k$ by

$$\frac{\frac{\Gamma_k \Rightarrow \Delta_k, \varphi_k \quad \varphi_k, \Gamma_k \Rightarrow \Delta_k}{\Gamma_k, \Gamma_k \Rightarrow \Delta_k, \Delta_k} (Cut)_{\varphi_k}^a}{\Gamma_k \Rightarrow \Delta_k} (w),$$

where $\varphi_k \in \text{Sub}(\Gamma_k, \Delta_k)$ holds clearly since $\varphi_k \in \text{Sub}(\Gamma, \Delta)$ and $\Gamma \subseteq \Gamma_k, \Delta \subseteq \Delta_k$. Hence we have $G^a(\Lambda) \vdash \Gamma_k \Rightarrow \Delta_k$, which contradicts the induction hypothesis. Therefore, either $\Gamma_k \Rightarrow \Delta_k, \varphi_k$ or $\varphi_k, \Gamma_k \Rightarrow \Delta_k$ is underivable in $G^a(\Lambda)$. Finally, let $(\Gamma^+, \Delta^+) := (\Gamma_{m+1}, \Delta_{m+1})$. It holds clearly that $\text{Sub}(\Gamma, \Delta) = \text{Sub}(\Gamma^+, \Delta^+) = \Gamma^+ \cup \Delta^+$. Moreover $\Gamma \subseteq \Gamma^+, \Delta \subseteq \Delta^+$ and $G^a(\Lambda) \not\vdash \Gamma^+ \Rightarrow \Delta^+$ by our definition. $\square$

The following definition of R_G^Λ is based on the definitions of accessibility relations for individual agents given in [16, Section 5.3] for individual agents, but is modified here to the group level to accommodate the semantics of distributed knowledge.

Definition 4.5. The pseudo model $M_\Xi^\Lambda = (W_\Xi^\Lambda, (R_G^\Lambda)_{G \in \text{Grp}}, V_\Xi^\Lambda)$ derived from Ξ is defined as follows:

- $W_\Xi^\Lambda = \{(\Gamma, \Delta) \mid (\Gamma, \Delta) \text{ is a } \Xi\text{-partial valuation in } G^a(\Lambda)\}$;
- R_G^Λ is defined depending on the choice of Λ :
 - $\Lambda = \mathbf{K45}_D$: $(\Gamma, \Delta) R_G^{\mathbf{K45}_D} (\Pi, \Sigma)$ iff $\forall H \subseteq G \{ \varphi, D_H \varphi \mid D_H \varphi \in \Gamma \} \subseteq \Pi$ and $\{ D_H \varphi \mid D_H \varphi \in \Pi \} \subseteq \Gamma$,
 - $\Lambda = \mathbf{KD45}_D$: $R_G^{\mathbf{KD45}_D} := R_G^{\mathbf{K45}_D}$;
 - $\Lambda = \mathbf{S5}_D$: $(\Gamma, \Delta) R_G^{\mathbf{S5}_D} (\Pi, \Sigma)$ iff $\forall H \subseteq G \{ \{ D_H \varphi \mid D_H \varphi \in \Gamma \} = \{ D_H \varphi \mid D_H \varphi \in \Pi \} \}$;
- $(\Gamma, \Delta) \in V_\Xi^\Lambda(p)$ iff $p \in \Gamma$.

Since Ξ is finite, we note that W_Ξ^Λ is finite. The following proposition shows that M_Ξ^Λ is indeed a pseudo model.

Proposition 4.6. *The following hold for the pseudo frame $F_\Xi^\Lambda = (W_\Xi^\Lambda, (R_G^\Lambda)_{G \in \text{Grp}})$ derived from Ξ :*
 1. $G \subseteq H$ implies $R_H^\Lambda \subseteq R_G^\Lambda$; 2. F_Ξ^Λ satisfies the corresponding properties of Λ .

Proof. 1. Fix any $G, H \in \text{Grp}$ such that $G \subseteq H$. Since $R_G^{\mathbf{KD45}_D} := R_G^{\mathbf{K45}_D}$, we only comment on the cases where $(\Lambda = \mathbf{K45}_D)$ and $(\Lambda = \mathbf{S5}_D)$.

- Let $\Lambda = \mathbf{K45}_D$. Fix any $(\Gamma, \Delta), (\Pi, \Sigma) \in W_\Xi^\Lambda$ such that $(\Gamma, \Delta) R_H^{\mathbf{K45}_D} (\Pi, \Sigma)$. We show that $(\Gamma, \Delta) R_G^{\mathbf{K45}_D} (\Pi, \Sigma)$. Fix any $I \subseteq G$. It suffices to show (i) $\{ \varphi, D_I \varphi \mid D_I \varphi \in \Gamma \} \subseteq \Pi$ and (ii) $\{ D_I \varphi \mid D_I \varphi \in \Pi \} \subseteq \Gamma$. By assumption, $I \subseteq G \subseteq H$, hence our goal follows from

$$(\Gamma, \Delta) R_H^{\mathbf{K45}_D} (\Pi, \Sigma).$$

- Let $\Lambda = \mathbf{S5}_D$. Fix any $(\Gamma, \Delta), (\Pi, \Sigma) \in W_\Xi^\Lambda$ such that $(\Gamma, \Delta) R_H^{\mathbf{S5}_D} (\Pi, \Sigma)$. We show that $(\Gamma, \Delta) R_G^{\mathbf{S5}_D} (\Pi, \Sigma)$. Fix any $I \subseteq G$. It suffices to show $\{ D_I \varphi \mid D_I \varphi \in \Gamma \} = \{ D_I \varphi \mid D_I \varphi \in \Pi \}$. By assumption, $I \subseteq G \subseteq H$, hence our goal follows from $(\Gamma, \Delta) R_H^{\mathbf{S5}_D} (\Pi, \Sigma)$.

2. • Let $\Lambda = \mathbf{K45}_D$. We show that $R_G^{\mathbf{K45}_D}$ is transitive and Euclidean.

Transitivity Fix any $(\Gamma, \Delta), (\Pi, \Sigma), (\Theta, \Omega) \in W_{\Xi}^{\mathbf{K45}_D}$. Suppose

$$(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Pi, \Sigma) \text{ and } (\Pi, \Sigma)R_G^{\mathbf{K45}_D}(\Theta, \Omega).$$

We show $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Theta, \Omega)$. Fix any $H \subseteq G$. It suffices to show (i) $\{\varphi, D_H\varphi \mid D_H\varphi \in \Gamma\} \subseteq \Theta$ and (ii) $\{D_H\varphi \mid D_H\varphi \in \Theta\} \subseteq \Gamma$. First, we show (i). Let us fix any $D_H\psi \in \Gamma$. We show $\psi, D_H\psi \in \Theta$. By assumption, $\{\varphi, D_H\varphi \mid D_H\varphi \in \Gamma\} \subseteq \Pi$. Then we have $D_H\psi \in \Pi$. Again by assumption, $\{\varphi, D_H\varphi \mid D_H\varphi \in \Pi\} \subseteq \Theta$. Therefore our goal of $\psi, D_H\psi \in \Theta$ follows from $D_H\psi \in \Pi$.

Second, we move to (ii). By assumption we have

$$\{D_H\varphi \mid D_H\varphi \in \Theta\} \subseteq \Pi \text{ and } \{D_H\varphi \mid D_H\varphi \in \Pi\} \subseteq \Gamma,$$

which implies our goal $\{D_H\varphi \mid D_H\varphi \in \Theta\} \subseteq \Gamma$.

Euclideaness Fix any $(\Gamma, \Delta), (\Pi, \Sigma), (\Theta, \Omega) \in W_{\Xi}^{\mathbf{K45}_D}$. Suppose $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Pi, \Sigma)$ and $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Theta, \Omega)$. We show $(\Pi, \Sigma)R_G^{\mathbf{K45}_D}(\Theta, \Omega)$. Fix any $H \subseteq G$. It suffices to show (i) $\{\varphi, D_H\varphi \mid D_H\varphi \in \Pi\} \subseteq \Theta$ and (ii) $\{D_H\varphi \mid D_H\varphi \in \Theta\} \subseteq \Pi$.

First, we establish (i). Fix any $D_H\psi \in \Pi$. We show that $\psi, D_H\psi \in \Theta$. By $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Pi, \Sigma)$ we have $D_H\psi \in \Gamma$. Then our goal holds by $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Theta, \Omega)$.

Second, we move to (ii). Fix any $D_H\psi$ with $D_H\psi \in \Theta$. We show that $D_H\psi \in \Pi$. By $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Theta, \Omega)$, we have $D_H\psi \in \Gamma$. Then our goal holds by $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Pi, \Sigma)$.

Seriality We prove the seriality of $R_{\{a\}}^{\mathbf{KD45}_D}$. Fix any $(\Gamma, \Delta) \in W_{\Xi}^{\mathbf{KD45}_D}$. We show that there exists $(\Pi, \Sigma) \in W_{\Xi}^{\mathbf{KD45}_D}$ such that $(\Gamma, \Delta)R_{\{a\}}^{\mathbf{KD45}_D}(\Pi, \Sigma)$. We define $\Pi := \{\varphi \mid D_{\{a\}}\varphi \in \Gamma\}$, $\Sigma := \{D_{\{a\}}\psi \mid D_{\{a\}}\psi \in \Delta\}$. Then $\Pi, D_{\{a\}}\Pi \Rightarrow \Sigma$ is underivable in $G^a(\mathbf{KD45}_D)$. Otherwise $\Gamma \Rightarrow \Delta$ is derivable from $\Pi, D_{\{a\}}\Pi \Rightarrow \Sigma$ via $(D^{\mathbf{KD45}_D})$ and weakening rules, which contradicts $G^a(\Lambda) \not\vdash \Gamma \Rightarrow \Delta$. By Lemma 4.4, there exists $(\Pi^+, \Sigma^+) \in W_{\Xi}^{\mathbf{KD45}_D}$ such that $\Pi \cup D_{\{a\}}\Pi \subseteq \Pi^+$, $\Sigma \subseteq \Sigma^+$ and $\text{Sub}(\Pi, D_{\{a\}}\Pi, \Sigma) = \Pi^+ \cup \Sigma^+$. Then it suffices to show $(\Gamma, \Delta)R_{\{a\}}^{\mathbf{KD45}_D}(\Pi^+, \Sigma^+)$. Fix any $D_{\{a\}}\chi \in \Xi$. First, we suppose $D_{\{a\}}\chi \in \Gamma$. We show that $\chi, D_{\{a\}}\chi \in \Pi^+$. Then $\chi \in \Pi \subseteq \Pi^+$. Moreover $D_{\{a\}}\chi \in D_{\{a\}}\Pi \subseteq \Pi^+$. Therefore, our goal $\chi \in \Pi^+$ and $D_{\{a\}}\chi \in \Pi^+$ holds. Second, we suppose $D_{\{a\}}\chi \in \Pi$. Our goal is to show $D_{\{a\}}\chi \in \Gamma$. Suppose $D_{\{a\}}\chi \notin \Gamma$ for contradiction. By $D_{\{a\}}\chi \in \Pi$, $D_{\{a\}}\chi \in \Pi^+ \cup \Sigma^+ = \text{Sub}(\Pi, D_{\{a\}}\Pi, \Sigma) \subseteq \text{Sub}(\Gamma, \Delta) = \Gamma \cup \Delta$. Since $D_{\{a\}}\chi \notin \Gamma$, we have $D_{\{a\}}\chi \in \Delta$, i.e., $D_{\{a\}}\chi \in \Sigma \subseteq \Sigma^+$. Then $\Pi^+ \Rightarrow \Sigma^+$ is derivable from $D_{\{a\}}\chi \Rightarrow D_{\{a\}}\chi$ by weakening rules, which is contradicted to (Π^+, Σ^+) being a Ξ -partial valuation. Therefore, our goal $D_{\{a\}}\chi \in \Gamma$ holds.

- Let $\Lambda = \mathbf{S5}_D$. We show that $R_G^{\mathbf{S5}_D}$ is reflexive, symmetric and transitive. Fix any $(\Gamma, \Delta), (\Pi, \Sigma), (\Theta, \Omega) \in W_{\Xi}^{\mathbf{S5}_D}$. First, we show that $(\Gamma, \Delta)R_G^{\mathbf{S5}_D}(\Gamma, \Delta)$, which holds clearly by

$$\{D_H\varphi \mid D_H\varphi \in \Gamma\} = \{D_H\varphi \mid D_H\varphi \in \Gamma\}.$$

Second, suppose $(\Gamma, \Delta)R_G^{\mathbf{S5}_D}(\Pi, \Sigma)$. We show that $(\Pi, \Sigma)R_G^{\mathbf{S5}_D}(\Gamma, \Delta)$. It suffices to show that $\{D_H\varphi \mid D_H\varphi \in \Pi\} = \{D_H\varphi \mid D_H\varphi \in \Gamma\}$, which follows from our assumption

$$(\Gamma, \Delta)R_G^{\mathbf{S5}_D}(\Pi, \Sigma),$$

i.e., $\{D_H\varphi \mid D_H\varphi \in \Gamma\} = \{D_H\varphi \mid D_H\varphi \in \Pi\}$. Finally, suppose

$$(\Gamma, \Delta)R_G^{\mathbf{S5}_D}(\Pi, \Sigma) \text{ and } (\Pi, \Sigma)R_G^{\mathbf{S5}_D}(\Theta, \Omega).$$

We show $(\Gamma, \Delta)R_G^{\mathbf{S5}_D}(\Theta, \Omega)$. Fix any $H \subseteq G$. It suffices to show $\{D_H\varphi \mid D_H\varphi \in \Gamma\} = \{D_H\varphi \mid D_H\varphi \in \Theta\}$. By assumption, we have $\{D_H\varphi \mid D_H\varphi \in \Gamma\} = \{D_H\varphi \mid D_H\varphi \in \Pi\}$ and $\{D_H\varphi \mid D_H\varphi \in \Pi\} = \{D_H\varphi \mid D_H\varphi \in \Theta\}$. Therefore, our goal holds. $\square$

Lemma 4.7. *Let (Γ, Δ) be a Ξ -partial valuation in $G^a(\Lambda)$. Then (Γ, Δ) satisfies the following:*

- $(\neg \Rightarrow) \neg\varphi \in \Gamma$ implies $\varphi \in \Delta$;
- $(\Rightarrow \neg) \neg\varphi \in \Delta$ implies $\varphi \in \Gamma$;
- $(\rightarrow \Rightarrow) \varphi \rightarrow \psi \in \Gamma$ implies $\varphi \in \Delta$ or $\psi \in \Gamma$;
- $(\Rightarrow \rightarrow) \varphi \rightarrow \psi \in \Delta$ implies $\varphi \in \Gamma$ and $\psi \in \Delta$;
- (D) $D_G\varphi \in \Delta$ implies $\varphi \in \Sigma$ for some $(\Pi, \Sigma) \in W_\Xi^\Lambda$ such that $(\Gamma, \Delta)R_G^\Lambda(\Pi, \Sigma)$.

Proof. We comment on item (D) for $\mathbf{K45}_D$ and $\mathbf{S5}_D$, because the other conditions are established easily (cf. [33, Lemma 2]).

- Let $\Lambda = \mathbf{K45}_D$. Suppose $D_G\varphi \in \Delta$. Define $\Pi := \{\psi, D_H\psi \mid D_H\psi \in \Gamma \text{ and } H \subseteq G\}$ and $\Sigma := \{D_I\gamma \mid D_I\gamma \in \Delta \text{ and } I \subseteq G\}$. It follows that $\Pi \Rightarrow \Sigma, \varphi$ is underivable in $G^a(\mathbf{K45}_D)$. Suppose not. Then we obtain the following:

$$\frac{\Pi \Rightarrow \Sigma, \varphi}{\Pi' \Rightarrow \Sigma, D_G\varphi} (\Rightarrow D^{\mathbf{K45}_D}),$$

where $\Pi' = \bigcup_{H \subseteq G} \{D_H\psi \mid D_H\psi \in \Gamma\}$, which is contradicted to (Γ, Δ) being a Ξ -partial valuation by weakening rules. So, $G^a(\mathbf{K45}_D) \not\vdash \Pi \Rightarrow \Sigma, \varphi$. By Lemma 4.4, there exists a Ξ -partial valuation (Π^+, Σ^+) such that $\Pi \subseteq \Pi^+$, $\Sigma \cup \{\varphi\} \subseteq \Sigma^+$ and $\text{Sub}(\Pi, \Sigma, \varphi) = \Pi^+ \cup \Sigma^+$. Then it suffices to show $(\Gamma, \Delta)R_G^{\mathbf{K45}_D}(\Pi^+, \Sigma^+)$, which can be proved similarly to the proof of Proposition 4.6 (2).

- Let $\Lambda = \mathbf{S5}_D$. Suppose $D_G\varphi \in \Delta$. Let Π and Σ be defined as:

$$\Pi := \{D_H\psi \mid D_H\psi \in \Gamma \text{ and } H \subseteq G\} \quad \Sigma := \{D_I\gamma \mid D_I\gamma \in \Delta \text{ and } I \subseteq G\}.$$

It follows that $\Pi \Rightarrow \Sigma, \varphi$ is underivable in $G^a(\mathbf{S5}_D)$, otherwise $\Gamma \Rightarrow \Delta$ is derivable by $(\Rightarrow D^{\mathbf{S5}_D})$ and weakening rules. Then by Lemma 4.4, there exists a Ξ -partial valuation (Π^+, Σ^+) such that $\Pi \subseteq \Pi^+$, $\Sigma \cup \{\varphi\} \subseteq \Sigma^+$ and $\text{Sub}(\Pi, \Sigma, \varphi) = \Pi^+ \cup \Sigma^+$. Then it suffices to show $(\Gamma, \Delta)R_G^{\mathbf{S5}_D}(\Pi^+, \Sigma^+)$. Fix any $C \subseteq G$ and $\chi \in \Xi$. We show that $D_C\chi \in \Gamma$ iff $D_C\chi \in \Pi^+$. $(\Rightarrow)$ Suppose $D_C\chi \in \Gamma$. By definition of Π , $D_C\chi \in \Pi$. Therefore our goal $D_C\chi \in \Pi^+$ holds by $\Pi \subseteq \Pi^+$. $(\Leftarrow)$ Suppose $D_C\chi \in \Pi^+$. Then $D_C\chi \in \Pi^+ \cup \Sigma^+ = \text{Sub}(\Pi, \Sigma, \varphi) \subseteq \text{Sub}(\Gamma, \Delta) = \Gamma \cup \Delta$. Our goal is to show that $D_C\chi \in \Gamma$. Suppose not. Then $D_C\chi \in \Delta$, i.e., $D_C\chi \in \Sigma \subseteq \Sigma^+$. Then $\Pi^+ \Rightarrow \Sigma^+$ is derivable from $D_C\varphi \Rightarrow D_C\varphi$. Therefore our goal $D_C\chi \in \Gamma$ holds. $\square$

By Lemma 4.7, we obtain the following.

Lemma 4.8. *The following holds for all formulas $\varphi \in \Xi$ and $(\Gamma, \Delta) \in W_\Xi^\Lambda$:*

1. $\varphi \in \Gamma$ implies $M_\Xi^\Lambda, (\Gamma, \Delta) \models \varphi$;
2. $\varphi \in \Delta$ implies $M_\Xi^\Lambda, (\Gamma, \Delta) \not\models \varphi$.

Proof. By induction on φ . We comment on the case where $\varphi \equiv D_G\psi$.

1. Suppose $D_G\psi \in \Gamma$. We show that $M_\Xi^\Lambda, (\Gamma, \Delta) \models D_G\psi$. Fix any $(\Pi, \Sigma) \in W_\Xi^\Lambda$ and suppose

$$(\Gamma, \Delta)R_G^\Lambda(\Pi, \Sigma).$$

We show that $M_\Xi^\Lambda, (\Pi, \Sigma) \models \psi$. We comment on the case for $\mathbf{K45}_D$ and $\mathbf{S5}_D$.

- (**K45_D**) We show $M_{\Xi}^{\mathbf{K45}_D}, (\Pi, \Sigma) \models \psi$. By induction hypothesis, it suffices to show $\psi \in \Pi$, which follows from Definition 4.5 and $D_G\psi \in \Gamma$.
- (**S5_D**) We show $M_{\Xi}^{\mathbf{S5}_D}, (\Gamma, \Delta) \models D_G\psi$. Fix any $(\Pi, \Sigma) \in W_{\Xi}^{\mathbf{S5}_D}$ and suppose $(\Gamma, \Delta) R_G^{\mathbf{S5}_D} (\Pi, \Sigma)$. Then $D_G\psi \in \Gamma$ implies $D_G\psi \in \Pi$. Then $\psi \in \text{Sub}(D_G\psi) \subseteq \text{Sub}(\Pi, \Sigma) = \Pi \cup \Sigma$. If $\psi \in \Pi$, our goal $M_{\Xi}^{\mathbf{K45}_D}, (\Pi, \Sigma) \models \psi$ follows from induction hypothesis. So suppose $\psi \notin \Pi$ for contradiction. Then $\psi \in \Sigma$, hence $\Pi \Rightarrow \Sigma$ is derivable by

$$\frac{\frac{\varphi \Rightarrow \varphi}{D_G\varphi \Rightarrow \varphi} (D \Rightarrow)}{\Pi \Rightarrow \Sigma} (w),$$

which contradicts (Π, Σ) being a Ξ -partial valuation. Therefore our goal $\psi \in \Pi$ holds.

2. Suppose $D_G\psi \in \Delta$. We show $M_{\Xi}^{\Lambda}, (\Gamma, \Delta) \not\models D_G\psi$. By Lemma 4.7, we can find some $(\Pi, \Sigma) \in W_{\Xi}^{\Lambda}$ such that $(\Gamma, \Delta) R_G^{\Lambda} (\Pi, \Sigma)$ and $\psi \in \Sigma$. By induction hypothesis, $\psi \in \Sigma$ implies $M_{\Xi}^{\Lambda}, (\Pi, \Sigma) \not\models \psi$. Therefore our goal $M_{\Xi}^{\Lambda}, (\Gamma, \Delta) \not\models D_G\psi$ holds. $\square$

Theorem 4.9 (Completeness of $G^a(\Lambda)$). *For any sequent $\Gamma \Rightarrow \Delta$, if $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid in all pseudo frames which satisfies the corresponding properties to Λ , then $G^a(\Lambda) \vdash \Gamma \Rightarrow \Delta$.*

Proof. We show the contraposition. Suppose $G^a(\Lambda) \not\vdash \Gamma \Rightarrow \Delta$. Let $\Xi := \text{Sub}(\Gamma, \Delta)$. By Lemma 4.4, there exists a Ξ -partial valuation (Γ^+, Δ^+) such that $\Gamma \subseteq \Gamma^+$, $\Delta \subseteq \Delta^+$ and $\text{Sub}(\Gamma, \Delta) = \Gamma^+ \cup \Delta^+$. By Lemma 4.8, $M_{\Xi}^{\Lambda}, (\Gamma^+, \Delta^+) \not\models \bigwedge \Gamma \rightarrow \bigvee \Delta$. Therefore our goal holds. $\square$

As a corollary of Theorem 4.9, we can show that $G(\Lambda)$ enjoys the analytic cut property.

Corollary 4.10. *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $G(\Lambda)$, then it is also derivable in $G^a(\Lambda)$.*

Proof. Suppose $G(\Lambda) \vdash \Gamma \Rightarrow \Delta$. By Proposition 3.2, $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid in $\mathbb{F}_{\Lambda}$. Therefore, our goal $G^a(\Lambda) \vdash \Gamma \Rightarrow \Delta$ holds by Theorem 4.9. $\square$

5 Craig Interpolation Theorem

This section establishes the Craig interpolation theorem for $G(\Lambda)$ using the analytic cut property via Maehara's method, originally introduced in [26]. An application of this method to basic modal logic can also be found in [22]. First, we introduce some necessary syntactic notions. As in the previous section, we assume in what follows that $\Lambda \in \{\mathbf{K45}_D, \mathbf{KD45}_D, \mathbf{S5}_D\}$.

Definition 5.1. A *partition* for a sequent $\Gamma \Rightarrow \Delta$ is defined as a tuple $\langle (\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2) \rangle$ such that $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$.

Definition 5.2. For a formula $\varphi \in L$, we define $\text{Prop}(\varphi)$ as the set of all propositional variables appearing in φ and $\text{Agt}(\varphi)$ as the set of all agents appearing in φ . For any multiset Γ of formulas, $\text{Prop}(\Gamma)$ and $\text{Agt}(\Gamma)$ are defined as $\bigcup_{\varphi \in \Gamma} \text{Prop}(\varphi)$ and $\bigcup_{\varphi \in \Gamma} \text{Agt}(\varphi)$, respectively.

In particular, it is noted that $\text{Agt}(D_G\varphi) = \text{Agt}(\varphi) \cup G$. The following lemma is immediate.

Lemma 5.3. *If $\varphi \in \text{Sub}(\psi)$, then $\text{Prop}(\varphi) \subseteq \text{Prop}(\psi)$ and $\text{Agt}(\varphi) \subseteq \text{Agt}(\psi)$.*

The following is the key lemma for the Craig interpolation theorem.

Lemma 5.4. *Suppose $G(\Lambda) \vdash \Gamma \Rightarrow \Delta$. Then for any partition $\langle (\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2) \rangle$ for the sequent $\Gamma \Rightarrow \Delta$, there exists a formula φ (called interpolant) that satisfies the following:*

- (1) $G(\Lambda) \vdash \Gamma_1 \Rightarrow \Delta_1, \varphi$ and $G(\Lambda) \vdash \varphi, \Gamma_2 \Rightarrow \Delta_2$;
- (2) $\text{Prop}(\varphi) \subseteq \text{Prop}(\Gamma_1, \Delta_1) \cap \text{Prop}(\Gamma_2, \Delta_2)$;
- (3) $\text{Agt}(\varphi) \subseteq \text{Agt}(\Gamma_1, \Delta_1) \cap \text{Agt}(\Gamma_2, \Delta_2)$.

Proof. In what follows, we let Λ be **K45**_D. This is because the cases where Λ is **KD45**_D or **S5**_D can be handled using an argument similar to that for **K45**_D. By Corollary 4.10, we can assume that $\Gamma \Rightarrow \Delta$ has a derivation $\mathcal{D}$ in $G^a(\Lambda)$. We prove Lemma 5.4 by induction on $\mathcal{D}$. Let the height of $\mathcal{D}$ be n . For the base case, our argument is standard (the reader is referred to [22, Lemma 36]). For the inductive step, let $n > 0$. First, we suppose $\Gamma_1 = \Delta_1 = \emptyset$ or $\Gamma_2 = \Delta_2 = \emptyset$. Then the interpolant is $\neg\perp$ or $\perp$. Hence, in what follows, we suppose it is not the case that $\Gamma_1 = \Delta_1 = \emptyset$ or $\Gamma_2 = \Delta_2 = \emptyset$, i.e., $\Gamma_1 \cup \Delta_1 \neq \emptyset$ and $\Gamma_2 \cup \Delta_2 \neq \emptyset$. We divide our argument depending on the last applied rule to obtain $\Gamma \Rightarrow \Delta$. We comment on the cases where the last applied rule is an analytic cut or $(\Rightarrow D^{\mathbf{K45}_D})$. Although an argument for the case of analytic cut is discussed already in [22, Section 6.4], we provide the details here in order to make the proof self-contained. For other cases, the reader is referred to [22, Lemma 34] for propositional connectives.

- Suppose that the last applied rule is an analytic cut:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \quad \varphi, \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}{\Gamma_1, \Gamma_2, \Pi_1, \Pi_2 \Rightarrow \Delta_1, \Delta_2, \Sigma_1, \Sigma_2} (Cut)_\varphi,$$

where $\varphi \in \text{Sub}(\Gamma_1, \Gamma_2, \Pi_1, \Pi_2, \Delta_1, \Delta_2, \Sigma_1, \Sigma_2)$. The partition is of the form

$$\langle (\Gamma_1, \Pi_1 : \Delta_1, \Sigma_1); (\Gamma_2, \Pi_2 : \Delta_2, \Sigma_2) \rangle.$$

We divide the argument into the following cases:

$$\varphi \in \text{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1) \text{ and } \varphi \in \text{Sub}(\Gamma_2, \Pi_2, \Delta_2, \Sigma_2).$$

We focus only on the former case. Suppose $\varphi \in \text{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$. By induction hypothesis, we can find formulas ψ_1 and ψ_2 such that

- both $\Gamma_1 \Rightarrow \Delta_1, \varphi, \psi_1$ and $\psi_1, \Gamma_2 \Rightarrow \Delta_2$ are derivable in $G(\Lambda)$ and $X(\psi_1) \subseteq X(\Gamma_1, \Delta_1, \varphi) \cap X(\Gamma_2, \Delta_2)$ for all $X \in \{\text{Prop}, \text{Agt}\}$;
- both $\varphi, \Pi_1 \Rightarrow \Sigma_1, \psi_2$ and $\psi_2, \Pi_2 \Rightarrow \Sigma_2$ are derivable in $G(\Lambda)$ and $X(\psi_2) \subseteq X(\varphi, \Pi_1, \Sigma_1) \cap X(\Pi_2, \Sigma_2)$ for all $X \in \{\text{Prop}, \text{Agt}\}$.

We show that $\psi_1 \vee \psi_2 := \neg\psi_1 \rightarrow \psi_2$ is a desired interpolant. The derivability condition is obtained as follows:

$$\frac{\frac{\Gamma_1 \Rightarrow \Delta_1, \varphi, \psi_1 \quad \varphi, \Pi_1 \Rightarrow \Sigma_1, \psi_2}{\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \psi_1, \psi_2} (Cut)_\varphi^a}{\frac{\neg\psi_1, \Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \psi_2}{\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \neg\psi_1 \rightarrow \psi_2} (\neg \Rightarrow)} (\Rightarrow \rightarrow), \quad \frac{\frac{\psi_1, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_2 \Rightarrow \Delta_2, \neg\psi_1} (\Rightarrow \neg) \quad \psi_2, \Pi_2 \Rightarrow \Sigma_2}{\neg\psi_1 \rightarrow \psi_2, \Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2} (\rightarrow \Rightarrow),$$

where the (Cut) is analytic by our assumption $\varphi \in \text{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$. Moreover the variable and agent conditions are verified as follows. Let $X \in \{\text{Prop}, \text{Agt}\}$. We proceed as follows:

$$\begin{aligned} X(\neg\psi_1 \rightarrow \psi_2) &= X(\psi_1) \cup X(\psi_2), \\ &\subseteq (X(\Gamma_1, \Delta_1, \varphi) \cap X(\Gamma_2, \Delta_2)) \cup (X(\varphi, \Pi_1, \Sigma_1) \cap X(\Pi_2, \Sigma_2)) \\ &\subseteq X(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1, \varphi) \cap X(\Gamma_2, \Pi_2, \Delta_2, \Sigma_2). \end{aligned}$$

By Lemma 5.3, $\varphi \in \text{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$ implies $X(\varphi) \subseteq X(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$. Therefore, we obtain $X(\neg\psi_1 \rightarrow \psi_2) \subseteq X(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1) \cap X(\Gamma_2, \Pi_2, \Delta_2, \Sigma_2)$.

- Suppose that the last applied rule is $(\Rightarrow D^{\mathbf{K45}_D})$:

$$\frac{\overrightarrow{\varphi_{1x}}, \overrightarrow{\varphi_{2y}}, \overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{G_{2y}} \varphi_{2y}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, \overrightarrow{D_{H_{2b}} \psi_{2b}}, \chi}{\overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{G_{2y}} \varphi_{2y}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, \overrightarrow{D_{H_{2b}} \psi_{2b}}, D_G \chi} (\Rightarrow D^{\mathbf{K45}_D}),$$

where $\overrightarrow{\gamma_{iz}}$ denotes a list $\gamma_{i1}, \dots, \gamma_{im_z}$ of formulas and $G_{1x}, H_{1a}, G_{2y}, H_{2b} \subseteq G$ hold for all indices. There are the following two possible forms of the partitions of the conclusion of the rule:

- (a) $\langle (\overrightarrow{D_{G_{1x}} \varphi_{1x}} : \overrightarrow{D_{H_{1a}} \psi_{1a}}, D_G \chi); (\overrightarrow{D_{G_{2y}} \varphi_{2y}} : \overrightarrow{D_{H_{2b}} \psi_{2b}}) \rangle$,
- (b) $\langle (\overrightarrow{D_{G_{1x}} \varphi_{1x}} : \overrightarrow{D_{H_{1a}} \psi_{1a}}); (\overrightarrow{D_{G_{2y}} \varphi_{2y}} : \overrightarrow{D_{H_{2b}} \psi_{2b}}, D_G \chi) \rangle$.

Due to space limitations, we focus on case (a) alone. By induction hypothesis, we can find a formula ψ such that both $\overrightarrow{\varphi_{1x}}, \overrightarrow{D_{G_{1x}} \varphi_{1x}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, \chi, \psi$ and $\psi, \overrightarrow{\varphi_{2y}}, \overrightarrow{D_{G_{2y}} \varphi_{2y}} \Rightarrow \overrightarrow{D_{H_{2b}} \psi_{2b}}$ are derivable in $G^a(\mathbf{K45}_D)$ and ψ satisfies $X(\psi) \subseteq X(\overrightarrow{\varphi_{1x}}, \overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{H_{1a}} \psi_{1a}}, \chi) \cap X(\overrightarrow{D_{G_{2y}} \varphi_{2y}}, \overrightarrow{D_{H_{2b}} \psi_{2b}})$, where $X \in \{\text{Prop}, \text{Agt}\}$. We show that $\neg D_H \neg \psi$ is a desired interpolant, where $H = \bigcup \overrightarrow{G_{2y}} \cup \bigcup \overrightarrow{H_{2b}}$. We note that $H \neq \emptyset$ since we have assumed $\Gamma_2 \cup \Delta_2 \neq \emptyset$. It follows that $H \subseteq G$.

$$\frac{\frac{\overrightarrow{\varphi_{1x}}, \overrightarrow{D_{G_{1x}} \varphi_{1x}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, \chi, \psi}{\neg \psi, \overrightarrow{\varphi_{1x}}, \overrightarrow{D_{G_{1x}} \varphi_{1x}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, \chi} (\neg \Rightarrow)}{\neg \psi, D_H \neg \psi, \overrightarrow{\varphi_{1x}}, \overrightarrow{D_{G_{1x}} \varphi_{1x}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, \chi} (w \Rightarrow)} \frac{\frac{\psi, \overrightarrow{\varphi_{2y}}, \overrightarrow{D_{G_{2y}} \varphi_{2y}} \Rightarrow \overrightarrow{D_{H_{2b}} \psi_{2b}}}{\overrightarrow{\varphi_{2y}}, \overrightarrow{D_{G_{2y}} \varphi_{2y}} \Rightarrow \overrightarrow{D_{H_{2b}} \psi_{2b}}, \neg \psi} (\Rightarrow \neg)}{\overrightarrow{D_{G_{2y}} \varphi_{2y}} \Rightarrow \overrightarrow{D_{H_{2b}} \psi_{2b}}, D_H \neg \psi} (\Rightarrow D^{\mathbf{K45}_D})} \frac{\overrightarrow{D_H \neg \psi}, \overrightarrow{D_{G_{1x}} \varphi_{1x}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, D_G \chi}{\overrightarrow{D_{G_{1x}} \varphi_{1x}} \Rightarrow \overrightarrow{D_{H_{1a}} \psi_{1a}}, D_G \chi, \neg D_H \neg \psi} (\Rightarrow \neg)}{\neg D_H \neg \psi, \overrightarrow{D_{G_{2y}} \varphi_{2y}} \Rightarrow \overrightarrow{D_{H_{2b}} \psi_{2b}}} (\neg \Rightarrow),$$

where both applications of $(\Rightarrow D^{\mathbf{K45}_D})$ are eligible. Since the condition for propositional variables is easily satisfied, we comment on the condition for agents. Our goal is to show that $\text{Agt}(\neg D_H \neg \psi) \subseteq \text{Agt}(\overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{H_{1a}} \psi_{1a}}, D_G \chi) \cap \text{Agt}(\overrightarrow{D_{G_{2y}} \varphi_{2y}}, \overrightarrow{D_{H_{2b}} \psi_{2b}})$. However, note that this argument is *not* possible when $\Gamma_2 = \Delta_2 = \emptyset$, in which case $\text{Agt}(\overrightarrow{D_{G_{2y}} \varphi_{2y}}, \overrightarrow{D_{H_{2b}} \psi_{2b}}) = \emptyset$. The same situation happens when $\Gamma_1 = \Delta_1 = \emptyset$ for case (b). This is precisely why we made the assumption $\Gamma_1 = \Delta_1 = \emptyset$ or $\Gamma_2 = \Delta_2 = \emptyset$ in the first place. Fix any $a \in \text{Agt}(\neg D_H \neg \psi) = H \cup \text{Agt}(\psi)$. We divide the argument into the following two cases: $a \in \text{Agt}(\psi)$ and $a \notin \text{Agt}(\psi)$.

- Suppose $a \in \text{Agt}(\psi)$. Then our goal

$$a \in \text{Agt}(\overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{H_{1a}} \psi_{1a}}, D_G \chi) \cap \text{Agt}(\overrightarrow{D_{G_{2y}} \varphi_{2y}}, \overrightarrow{D_{H_{2b}} \psi_{2b}})$$

follows from $\text{Agt}(\psi) \subseteq \text{Agt}(\overrightarrow{\varphi_{1x}}, \overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{H_{1a}} \psi_{1a}}, \chi) \cap \text{Agt}(\overrightarrow{D_{G_{2y}} \varphi_{2y}}, \overrightarrow{D_{H_{2b}} \psi_{2b}})$. Note that it is possible that $\{\overrightarrow{\varphi_{1x}}\} = \{\overrightarrow{D_{G_{1x}} \varphi_{1x}}\} = \emptyset$, in which case the argument remains valid.

- Suppose $a \notin \text{Agt}(\psi)$. Then $a \in H$. Our goal is to show

$$a \in \text{Agt}(\overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{H_{1a}} \psi_{1a}}, D_G \chi) \cap \text{Agt}(\overrightarrow{D_{G_{2y}} \varphi_{2y}}, \overrightarrow{D_{H_{2b}} \psi_{2b}}),$$

i.e., $a \in \text{Agt}(\overrightarrow{D_{G_{1x}} \varphi_{1x}}, \overrightarrow{D_{H_{1a}} \psi_{1a}}, D_G \chi)$ and $a \in \text{Agt}(\overrightarrow{D_{G_{2y}} \varphi_{2y}}, \overrightarrow{D_{H_{2b}} \psi_{2b}})$. The first condition is satisfied by $a \in H \subseteq G \subseteq \text{Agt}(D_G \chi)$, and the second condition is satisfied by $H = \bigcup \overrightarrow{G_{2y}} \cup \bigcup \overrightarrow{H_{2b}}$. $\square$

Theorem 5.5. Suppose $G(\Lambda) \vdash \Rightarrow \varphi \rightarrow \psi$. Then there exists a formula χ satisfying the following:

- (1) $G(\Lambda) \vdash \Rightarrow \varphi \rightarrow \chi$ and $G(\Lambda) \vdash \Rightarrow \chi \rightarrow \psi$;
- (2) $\text{Prop}(\chi) \subseteq \text{Prop}(\varphi) \cap \text{Prop}(\psi)$;
- (3) $\text{Agt}(\chi) \subseteq \text{Agt}(\varphi) \cap \text{Agt}(\psi)$.

Proof. Suppose $G(\Lambda) \vdash \Rightarrow \varphi \rightarrow \psi$. Then we can show $G(\Lambda) \vdash \varphi \Rightarrow \psi$ by

$$\frac{\Rightarrow \varphi \rightarrow \psi \quad \frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi, \varphi \rightarrow \psi \Rightarrow \psi} (\rightarrow \Rightarrow)}{\varphi \Rightarrow \psi} (\text{Cut})_{\varphi \rightarrow \psi}.$$

By Corollary 4.10, we have $G^a(\Lambda) \vdash \varphi \Rightarrow \psi$. Take a partition $\langle (\varphi : \emptyset); (\emptyset : \psi) \rangle$. Then by Lemma 5.4, we can find an interpolant χ that satisfies the required conditions. $\square$

6 Adding Global Modality

In this section, we introduce the framework with the global modality A . The syntax L^+ with the global modality is defined inductively as follows:

$$L^+ \ni \varphi ::= p \mid \perp \mid \neg \varphi \mid \varphi \rightarrow \psi \mid D_G \varphi \mid A \varphi$$

where $p \in \text{Prop}$ and $G \subseteq \text{Ag}$. The notion of $A \varphi$ being true at w in a model M or a pseudo model M is defined as:

$$M, w \models A \varphi \quad \text{iff} \quad \text{for all } v \in W, M, v \models \varphi.$$

As the readers may notice, though we have defined $G \neq \emptyset$, the motivation for the global modality is to obtain the distributed knowledge of an empty group of agents, namely $D_\emptyset \varphi$. This is because $\bigcap_{a \in \emptyset} = W \times W$. Under the semantics defined above, $D_\emptyset \varphi := A \varphi$.

Now we introduce sequent calculi $G(\mathbf{K45}_{DA})$, $G(\mathbf{KD45}_{DA})$ and $G(\mathbf{S5}_{DA})$, which are also expansions of $\mathbf{LK}_0$. For a finite multiset Γ of formulas, we use $A\Gamma$ to denote $\{A\varphi \mid \varphi \in \Gamma\}$. The sequent calculus $G(\mathbf{K45}_{DA})$ is $\mathbf{LK}_0$ expanded with the following rules:

$$\frac{\varphi_1, \dots, \varphi_m, D_{G_1} \varphi_1, \dots, D_{G_m} \varphi_m, A\Pi \Rightarrow A\Sigma, D_{H_1} \psi_1, \dots, D_{H_n} \psi_n, \chi}{D_{G_1} \varphi_1, \dots, D_{G_m} \varphi_m, A\Pi \Rightarrow A\Sigma, D_{H_1} \psi_1, \dots, D_{H_n} \psi_n, D_G \chi} (\Rightarrow D^{\mathbf{K45}_{DA}}),$$

$$\frac{A\Gamma \Rightarrow A\Delta, \varphi}{A\Gamma \Rightarrow A\Delta, A\varphi} (\Rightarrow A) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{A\varphi, \Gamma \Rightarrow \Delta} (A \Rightarrow)$$

where $\bigcup_{i=1}^m G_i \cup \bigcup_{j=1}^n H_j \subseteq G$ in $(\Rightarrow D^{\mathbf{K45}_{DA}})$. The sequent calculus $G(\mathbf{KD45}_{DA})$ expands $G(\mathbf{K45}_{DA})$ with the following rule:

$$\frac{\Gamma, D_{\{a\}} \Gamma, A\Pi \Rightarrow A\Sigma, D_{\{a\}} \Delta}{D_{\{a\}} \Gamma, A\Pi \Rightarrow A\Sigma, D_{\{a\}} \Delta} (D^{\mathbf{KD45}_{DA}}).$$

Finally, the sequent calculus $G(\mathbf{S5}_{DA})$ expands $\mathbf{LK}_0$ with $(\Rightarrow A)$, $(A \Rightarrow)$ and the following rules:

$$\frac{D_{G_1} \varphi_1, \dots, D_{G_m} \varphi_m, A\Pi \Rightarrow A\Sigma, D_{H_1} \psi_1, \dots, D_{H_n} \psi_n, \chi}{D_{G_1} \varphi_1, \dots, D_{G_m} \varphi_m, A\Pi \Rightarrow A\Sigma, D_{H_1} \psi_1, \dots, D_{H_n} \psi_n, D_G \chi} (\Rightarrow D^{\mathbf{S5}_{DA}}) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{D_G \varphi, \Gamma \Rightarrow \Delta} (D \Rightarrow).$$

where $\bigcup_{i=1}^m G_i \cup \bigcup_{j=1}^n H_j \subseteq G$ in $(\Rightarrow D^{\mathbf{S5}_{DA}})$. The following proposition can be established similarly as Proposition 3.2.

Proposition 6.1. *If $\Gamma \Rightarrow \Delta$ is derivable in $G(\Lambda)$ then $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid in the corresponding class $\mathbb{P}_\Lambda$ of pseudo frames to Λ .*

In what remains in this section, let $\Lambda \in \{\mathbf{K45}_{DA}, \mathbf{KD45}_{DA}, \mathbf{S5}_{DA}\}$, and explain that the proof-theoretic results, namely the analytic cut property and Craig interpolation theorem for $G(\Lambda)$ can be obtained in a similar way as in Theorems 4.9 and 5.5, respectively, except that a modification of the pseudo-model is required to establish the analytic cut property.

Definition 6.2. Define $M_\Xi^\Lambda = (W_\Xi^\Lambda, (R_G^\Lambda)_{G \in \text{Grp}}, V_\Xi^\Lambda)$ to be the pseudo model derived from Ξ in the same way as in Definition 4.5. Define $\sim_\Lambda$ on W_Ξ^Λ as follows:

$$(\Gamma, \Delta) \sim_\Lambda (\Pi, \Sigma) \text{ iff } \{A\phi \mid A\phi \in \Gamma\} = \{A\phi \mid A\phi \in \Pi\}.$$

It follows that $\sim_\Lambda$ is reflexive, transitive, symmetric and Euclidean.

Definition 6.3. Let (Φ, Ψ) be a Ξ -partial valuation. The pseudo model

$$M_{\Xi(\Phi, \Psi)}^\Lambda = (W_{\Xi(\Phi, \Psi)}^\Lambda, (R_G^\Lambda)_{G \in \text{Grp}}, V_{\Xi(\Phi, \Psi)}^\Lambda)$$

derived from Ξ and (Γ, Δ) is defined as:

- $W_{\Xi(\Phi, \Psi)}^\Lambda = \{(\Gamma, \Delta) \in W_\Xi^\Lambda \mid (\Phi, \Psi) \sim_\Lambda (\Gamma, \Delta)\}$;
- R_G^Λ is defined depending on the choice of Λ :
 - $\Lambda = \mathbf{K45}_{DA}$: $(\Gamma, \Delta)R_G^{\mathbf{K45}_{DA}}(\Pi, \Sigma)$ iff $\forall H \subseteq G(\{\phi, D_H\phi \mid D_H\phi \in \Gamma\} \subseteq \Pi \text{ and } \{D_H\phi \mid D_H\phi \in \Pi\} \subseteq \Gamma)$ and $\{A\phi \mid A\phi \in \Gamma\} = \{A\phi \mid A\phi \in \Pi\}$,
 - $\Lambda = \mathbf{KD45}_{DA}$: $R_G^{\mathbf{KD45}_{DA}} := R_G^{\mathbf{K45}_{DA}}$,
 - $\Lambda = \mathbf{S5}_{DA}$: $(\Gamma, \Delta)R_G^{\mathbf{S5}_{DA}}(\Pi, \Sigma)$ iff $\forall H \subseteq G(\{D_H\phi \mid D_H\phi \in \Gamma\} = \{D_H\phi \mid D_H\phi \in \Pi\})$ and $\{A\phi \mid A\phi \in \Gamma\} = \{A\phi \mid A\phi \in \Pi\}$;
- $(\Gamma, \Delta) \in V_{\Xi(\Phi, \Psi)}^\Lambda(p)$ iff $p \in \Gamma$.

The above definition of $W_{\Xi(\Phi, \Psi)}^\Lambda$ ensures that the global modality A means “for all states” by specifying the root (Φ, Ψ) . All the necessary lemmas can be proved in a manner similar to those for distributed knowledge logic without the global modality, with the exception of the following:

Lemma 6.4. *Let (Γ, Δ) be a Ξ -partial valuation in $G^a(\Lambda)$. Then (Γ, Δ) satisfies the following:*

- (A $\Rightarrow$) $A\phi \in \Gamma$ implies $\phi \in \Pi$ for all $(\Pi, \Sigma) \in W_{\Xi(\Phi, \Psi)}^\Lambda$;
- ($\Rightarrow$ A) $A\phi \in \Delta$ implies $\phi \in \Sigma$ for some $(\Pi, \Sigma) \in W_{\Xi(\Phi, \Psi)}^\Lambda$;
- (D) $D_G\phi \in \Delta$ implies $\phi \in \Sigma$ for some $(\Pi, \Sigma) \in W_{\Xi(\Phi, \Psi)}^\Lambda$ such that $(\Gamma, \Delta)R_G^\Lambda(\Pi, \Sigma)$;

For item (D), to find a desired witness $(\Pi, \Sigma) \in W_{\Xi(\Phi, \Psi)}^\Lambda$ reachable from (Φ, Ψ) , adding the contexts $A\Pi$ and $A\Sigma$ to the inference rules $(\Rightarrow D^{\mathbf{K45}_D})$, $(D^{\mathbf{KD45}_D})$ and $(\Rightarrow D^{\mathbf{S5}_D})$ of D_G plays a crucial role. By applying this lemma, we can prove the semantic completeness theorem, following the same line of argument as in the case without the global modality:

Theorem 6.5. *For any sequent $\Gamma \Rightarrow \Delta$, if $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid in the class $\mathbb{P}_\Lambda$ of all pseudo frames which satisfy the properties corresponding to Λ , then $G^a(\Lambda) \vdash \Gamma \Rightarrow \Delta$.*

Corollary 6.6. *For any sequent $\Gamma \Rightarrow \Delta$, if $G(\Lambda) \vdash \Gamma \Rightarrow \Delta$, then $G^a(\Lambda) \vdash \Gamma \Rightarrow \Delta$.*

For the rules $(\Rightarrow A)$ and $(A \Rightarrow)$, we can employ the same argument in terms of Maehara's method as used for the modal logic **S5** (see, e.g., [22]). Moreover, adding the contexts $A\Pi$ and $A\Sigma$ to the inference rules $(\Rightarrow D^{\mathbf{K45}_D})$, $(D^{\mathbf{KD45}_D})$ and $(\Rightarrow D^{\mathbf{S5}_D})$ of D_G does not change the outline of Maehara's argument or the information regarding the common vocabulary. Therefore, we obtain the following:

Theorem 6.7. *Let $\Lambda \in \{\mathbf{K45}_{DA}, \mathbf{KD45}_{DA}, \mathbf{S5}_{DA}\}$. Suppose $G(\Lambda) \vdash \Gamma \Rightarrow \Delta$. Then for any partition $\langle (\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2) \rangle$ for the sequent $\Gamma \Rightarrow \Delta$, there exists a formula φ that satisfies the following: $\Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\varphi, \Gamma_2 \Rightarrow \Delta_2$ are derivable in $G(\Lambda)$ and $X(\varphi) \subseteq X(\Gamma_1, \Delta_1) \cap X(\Gamma_2, \Delta_2)$ for all $X \in \{\text{Agt}, \text{Prop}\}$. Therefore, the Craig interpolation theorem holds for $G(\Lambda)$.*

The reader may wonder if all the sequent calculi equipped with the global modality are semantically complete in terms of models, i.e., pseudo models $M = (W, (R_G)_{G \in \text{Grp}}, V)$ such that $R_G = \bigcap_{a \in G} R_{\{a\}}$ holds for all $G \in \text{Grp}$. This can be done in terms of a technique called “tree unraveling”, which transforms a pseudo-model into one that satisfies $\bigcap_{a \in G} R_{\{a\}} = R_G$.

Theorem 6.8. *Let $\Lambda \in \{\mathbf{K45}_{DA}, \mathbf{KD45}_{DA}, \mathbf{S5}_{DA}\}$. If a formula φ is valid in the class $\mathbb{F}_\Lambda$ of all frames that satisfy the corresponding properties to Λ then φ is valid in the class $\mathbb{P}_\Lambda$ of all pseudo frames that satisfy the properties corresponding to Λ . Therefore, for any sequent $\Gamma \Rightarrow \Delta$, if $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid in the class $\mathbb{F}_\Lambda$, then $G^a(\Lambda) \vdash \Gamma \Rightarrow \Delta$.*

Proof. (Outline) The latter part follows immediately from the former part by Theorem 6.5. In what follows, we outline the proof of the former part. The following definitions and arguments are adapted from [8, 3, 18]. Suppose that a formula φ is valid in the class $\mathbb{F}_\Lambda$. To show that φ is valid in the pseudo-frame class $\mathbb{P}_\Lambda$, let us fix an arbitrary pseudo-frame $F = (W, (R_G)_{G \in \text{Grp}})$ in $\mathbb{P}_\Lambda$. Our goal is to establish the validity of φ on F . Let us fix an arbitrary valuation V and a state $w \in W$. We will show that $M, w \models \varphi$, where $M := (F, V)$. Without loss of generality, we assume that M is point-generated by w with respect to the binary relations $(R_G)_{G \in \text{Grp}}$. We define the tree unraveling $\text{Tree}^\Lambda(M, w)$ of M around w as follows:

- $\text{Finpath}(M, w)$ is defined as follows:

$$\text{Finpath}(M, w) := \{ \langle w_0, G_1, w_1, \dots, G_m, w_m \rangle \mid \\ m \geq 0, G_i \in \text{Grp}, w_0 = w, w_{i-1} R_{G_i} w_i \text{ for all } 1 \leq i \leq m \}.$$

We refer to an element of $\text{Finpath}(M, w)$ as a “path (from state w)” and denote it by $\vec{v}$, $\vec{u}$, etc.

- We define a binary relation $\mathcal{R}_G$ on $\text{Finpath}(M, w)$ as follows: $\vec{w} \mathcal{R}_G \vec{v}$ iff $\vec{v} = \vec{w} \frown (H, \text{tail}(\vec{w}))$ for some $H \supseteq G$, where $\text{tail}(\vec{w})$ denotes the last element of $\vec{w}$, and $\frown$ denotes concatenation. We use $\mathcal{R}_G^+$ (or $\mathcal{R}_G^*$) to mean the transitive closure (or the reflexive and transitive closure) of $\mathcal{R}_G$.
- Depending on our choice of Λ , we define $\mathcal{R}_G^\Lambda$ as follows:
 - For $\Lambda \in \{\mathbf{K45}_{DA}, \mathbf{KD45}_{DA}\}$, define $\vec{w} \mathcal{R}_G^\Lambda \vec{v}$ iff there exists $\vec{u} \in \text{Finpath}(M, w)$ such that $\vec{u} \mathcal{R}_G^* \vec{w}$ and $\vec{u} \mathcal{R}_G^+ \vec{v}$;
 - For $\Lambda = \mathbf{S5}_{DA}$, define $\vec{w} \mathcal{R}_G^\Lambda \vec{v}$ iff there exists $\vec{u} \in \text{Finpath}(M, w)$ such that $\vec{u} \mathcal{R}_G^* \vec{w}$ and $\vec{u} \mathcal{R}_G^* \vec{v}$.
- $\mathcal{V}(p) = \{ \vec{v} \in \text{Finpath}(M, w) \mid \text{tail}(\vec{v}) \in V(p) \}$ for every $p \in \text{Prop}$.
- Define $\text{Tree}^\Lambda(M, w) := (\text{Finpath}(M, w), (\mathcal{R}_G^\Lambda)_{G \in \text{Grp}}, \mathcal{V})$.

It is straightforward to verify that the frame part $(\text{Finpath}(M, w), (\mathcal{R}_G^\Lambda)_{G \in \text{Grp}})$ of $\text{Tree}^\Lambda(M, w)$ still belongs to $\mathbb{P}_\Lambda$. Moreover, we note that $\mathcal{R}_G^\Lambda = \bigcap_{a \in G} \mathcal{R}_{\{a\}}^\Lambda$ holds. We can also establish that the mapping $\text{tail} : \text{Finpath}(M, w) \rightarrow W$, which sends each $\vec{v}$ to its last element $\text{tail}(\vec{v})$, is a surjective bounded

morphism between pseudo-models. Here, surjectivity is necessary for preserving the truth of formulas of the form $A\psi$ (along the bounded morphism). Then we obtain $\text{Tree}^\Lambda(M, w), \langle w \rangle \models \varphi$ iff $M, \text{tail}(\langle w \rangle) \models \varphi$, where $\text{tail}(\langle w \rangle) = w$. Since $\mathcal{R}_G^\Lambda = \bigcap_{a \in G} \mathcal{R}_{\{a\}}^\Lambda$ holds, $(\text{Finpath}(M, w), (\mathcal{R}_{\{a\}}^\Lambda)_{a \in \text{Ag}}) \in \mathbb{F}_\Lambda$. Since φ is valid in $\mathbb{F}_\Lambda$, we obtain $(\text{Finpath}(M, w), (\mathcal{R}_{\{a\}}^\Lambda)_{a \in \text{Ag}}, \mathcal{V}), \langle w \rangle \models \varphi$, hence $\text{Tree}^\Lambda(M, w), \langle w \rangle \models \varphi$ in terms of pseudo models. Therefore, it follows from the above equivalence that $M, w \models \varphi$, as desired. $\square$

7 Conclusion

There are several directions for further research. First, there have been attempts to combine epistemic logic with coalition logic **CL**, which is a logic that concerns coalitional power (see [1]). It would be interesting to develop sequent calculi for **CL** with distributed knowledge and to investigate whether analytic cut properties can be established. Second, to the best of the authors' knowledge, a syntactic proof of the analytic cut property in the style of [28] is *not* available for our sequent calculi. The main obstacle appears to lie in the parameterization of the operator by a group G . However, a Gentzen-style sequent calculus for **S5** was introduced in [12], in which the operator is not parameterized by groups. It would therefore be interesting to investigate whether Takano's syntactic strategy can be adapted to systems combining this calculus with **CL**. Third, Hilbert systems and cut-free sequent calculi for intuitionistic **K**, **KT**, **KD**, **K4**, **K4D**, and **S4** with distributed knowledge were introduced in [18]. Building on this line of work, public announcement expansions of intuitionistic **K**, **KT**, **K4**, and **S4** with distributed knowledge were studied in [20]. It would therefore be worthwhile to investigate intuitionistic versions of **K45**, **KD45**, and **S5** with distributed knowledge, and to examine whether these intuitionistic systems can be expanded with public announcement logic via the approach used in [14], as well as whether the main proof-theoretic results are preserved under such extensions. Finally, an axiomatization for comparative knowledge was introduced in [4], which allows one to express that one group's (distributed) knowledge includes *all* of another group's (distributed) knowledge. It would therefore be interesting to develop sequent calculi for this expansion of distributed knowledge and examine whether the main proof-theoretic results established in this paper still hold.

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Uniform Interpolation of Basic Tense Logic

Katsuhiko Sano

Faculty of Humanities and Human Sciences, Hokkaido University
Sapporo, Japan
v-sano@let.hokudai.ac.jp

This paper establishes the uniform interpolation theorem for basic tense logic, which is also known as two-way modal logic or modal logic with converse. First introduced by Arthur Prior, basic tense logic is a syntactic expansion of basic modal logic with a converse modality. Its corresponding accessibility relation is defined as the converse of the standard accessibility relation in a given Kripke model. Although basic tense logic has been widely studied since its introduction, its uniform interpolation property has yet to be fully established. For basic modal logic **K**, Albert Visser (1996) provided a semantic argument formulated in terms of layered (or bounded) bisimulation, explicitly attributing the uniform interpolation property of **K** to Silvio Ghilardi. This paper extends Visser’s semantic argument to demonstrate that basic tense logic also enjoys the uniform interpolation property.

1 Introduction

Since its introduction by Arthur Prior in the 1950s [35, 36, 37, 38], basic tense logic has been a subject of intense study in logic, philosophy, formal semantics, and computer science, as evidenced by a vast body of literature including [27, 32, 10, 6, 3, 24, 21]. Tense logic arises from a temporal interpretation of modal logic, where the necessity operator $\Box$ is interpreted as a future necessity operator, read as “it will always be the case that” (originally denoted by the symbol “*G*”). To obtain tense logic syntactically, a converse modality $\blacksquare$ is added to basic modal logic. This modality is interpreted as a past necessity operator read as “it has always been the case that” (originally denoted by the symbol “*H*”). Key interaction axioms between $\Box$ and $\blacksquare$ are $p \rightarrow \Box \blacklozenge p$ and $p \rightarrow \blacksquare \lozenge p$, which jointly mean that the accessibility relation of $\blacksquare$ is the converse relation of the accessibility relation of $\Box$. In this sense, basic tense logic is also sometimes referred to as *modal logic with converse* or *two-way modal logic*.

The *Craig interpolation property* for a modal logic (possibly a tense logic or an intuitionistic logic) states that a provable implication $\varphi \rightarrow \psi$ of the logic has a formula θ , called an interpolant, such that $\varphi \rightarrow \theta$ and $\theta \rightarrow \psi$ are provable in the logic and the propositional variables in θ are contained in the common vocabulary of φ and ψ (for a survey of interpolation in non-classical logic, the reader is referred to [14]). This property is strengthened to the *uniform interpolation property* of a modal logic, which states that for a given formula φ and a propositional variable p , there exists a uniform interpolant $\exists p.\varphi$ such that its propositional variables are a subset of the propositional variables of φ excluding p and the following equivalence holds for all p -free formulas ψ : $\varphi \rightarrow \psi$ is provable if and only if $\exists p.\varphi \rightarrow \psi$ is provable. It is well known that the uniform interpolation property implies the Craig interpolation property.

The Craig interpolation property has been studied proof-theoretically and semantically for basic tense logic. For proof-theoretic studies, the reader is referred to Maruyama et al. [31], ten Cate et al. [12], Lyon et al. [29], and Sano and Yamasaki [40], all of which employ Maehara’s method [30] to compute an interpolant in a tableau or sequent calculus. For semantic studies, Wolter [44] provided a theorem on the failure of the Craig interpolation theorem for normal extensions of basic tense logic over unbounded

linear transitive frames. It is also worth noting that we can apply a semantic sufficient condition given in [11, Theorem 2.5.3] by ten Cate for the Craig interpolation of normal tense logics of an elementary (or first-order definable) frame class in terms of bisimulation products and generated subframes. This establishes that the Craig interpolation of the minimal basic tense logic **Kt** (i.e., the bimodal **K** with the two interaction axioms) holds, since the above two characteristic axioms relating the future and past necessity operators define frame properties that are closed under the specified frame constructions. However, as far as the author knows, the uniform interpolation property of the minimal basic tense logic is still unknown. This is rather surprising, given the importance of basic tense logic in the literature of philosophy, computer science, formal semantics, and mathematical logic.

As with the Craig interpolation property, there have been two main approaches to tackling the uniform interpolation property of the minimal modal logic **K**. In the proof-theoretic approach, Bílková [5] adapted Pitts's [34] argument for the uniform interpolation property of intuitionistic propositional logic to the minimal modal logic **K** using sequent calculus, where the termination of the calculus is crucial to the argument. This proof-theoretic approach has since been extended to various classical and non-classical (possibly non-normal) modal logics, e.g., in [26, 1, 2, 18].

Taking a semantic approach, Ghilardi and Zawadowski [22] combined the notion of layered bisimulation, also known as bounded bisimulation or n -bisimulation (cf. [7]), with algebraic and categorical considerations, to provide a semantic proof of Pitts's [34] result regarding the uniform interpolation property of intuitionistic propositional logic. Their semantic argument is an improvement upon Shavrukov's [41] semantic argument for the uniform interpolation property of the Gödel-Löb logic **GL**. As explicitly acknowledged by Visser [42], Ghilardi and Zawadowski's [22] argument can be adapted to prove the uniform interpolation property of **K**. In this sense, the uniform interpolation property of **K** is attributed to Ghilardi and Zawadowski [22] (cf. [42, footnote 2]). Visser [42] also provided the details of the model-theoretic argument in terms of layered bisimulations. It is worth noting that Ghilardi and Zawadowski [23] also established that the modal logic **S4** lacks the uniform interpolation property in terms of layered bisimulation (Visser [43] also provided a version of the argument). More recently, Visser's model-theoretic argument given in [42] was extended by Kurahashi [28] to prove the uniform Lyndon interpolation property, which generalizes both the uniform interpolation property and the Lyndon interpolation property by additionally accounting for the polarity of propositional variables in an interpolant. Kurahashi [28] proved that the modal logics **K**, **B**, **GL**, and **Grz** possess the uniform Lyndon interpolation property.

While the proof-theoretic approach provides an algorithm to compute a uniform interpolant, one of the merits of the semantic approach is that it reveals that the uniform interpolant in the basic modal logic **K** has a semantic clause as a bisimulation quantifier (cf. [20, 16]), as previously discussed by Visser [42]. This paper proves the uniform interpolation property of the basic tense logic **Kt** by employing Visser's [42] semantic argument in terms of *temporal* layered bisimulations. Consequently, we also prove that a uniform interpolant has a semantic clause as a temporal version of a bisimulation quantifier. Furthermore, we demonstrate that Ghilardi and Zawadowski's [23] semantic argument (and its alternative reformulation given by Visser [43]) for the failure of the uniform interpolation property in **S4** remains valid for the tense expansion of **S4**.

We proceed as follows. Section 2 reviews the syntax, Kripke semantics, and Hilbert system of basic tense logic, and defines the notion of a normal tense logic. We then formulate the uniform interpolation property for normal tense logics. Section 3 introduces bisimulations and layered bisimulations. In Section 4, we establish a sufficient condition for the uniform interpolation property of a normal tense logic in terms of the bisimulation expansion property of a frame class. Section 5 shows that the class of all frames possesses the bisimulation expansion property, and that the minimal basic tense logic has the

uniform interpolation property. Section 6 discusses the transfer of the uniform interpolation property via the mirror image, constant extension, and boxdot translation. Section 7 provides an example of a normal tense logic that lacks the uniform interpolation property; specifically, we examine a tense expansion of the modal logic **S4**. Finally, Section 8 concludes the paper with directions for future research.

2 Preliminaries

Given a countably infinite set Prop of propositional variables, we define the set Form of all formulas of basic tense logic as follows:

$$\text{Form} \ni \varphi ::= p \mid \perp \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid \blacksquare\varphi,$$

where $p \in \text{Prop}$. $\Box\varphi$ is read as “ φ holds for all future states” and $\blacksquare\varphi$ as “ φ holds for all past states”. In addition to the ordinary Boolean connectives such as $\wedge$, $\vee$ and $\neg$, we introduce $\Diamond\varphi$ and $\blacklozenge\varphi$ (they are also denoted by $F\varphi$ and $P\varphi$ in the literature) as abbreviations for $\neg\Box\neg\varphi$ and $\neg\blacksquare\neg\varphi$ whose readings are “ φ holds in some future state” and “ φ holds in some past state” respectively. Given a finite set Δ of formulas, we use $\bigwedge\Delta$ and $\bigvee\Delta$ to mean the conjunction and disjunction of all formulas in Δ respectively, where $\bigwedge\emptyset := \top$ and $\bigvee\emptyset := \perp$.

We define $P(\varphi)$ as the set of all propositional variables in φ . When $P(\varphi) = \emptyset$, we say that φ is a *constant formula*. Given a set $P \subseteq \text{Prop}$, we say that φ is *P-free* if $P(\varphi) \cap P = \emptyset$. When P is a singleton $\{p\}$, we simply say that φ is *p-free*. We define the *modal depth* $d(\varphi)$ of a formula φ inductively as follows:

$$\begin{aligned} d(p) &:= 0, & d(\perp) &:= 0, \\ d(\varphi_1 \rightarrow \varphi_2) &:= \max\{d(\varphi_1), d(\varphi_2)\}, & d(\bigcirc\varphi) &:= d(\varphi) + 1, \end{aligned}$$

where $\bigcirc \in \{\Box, \blacksquare\}$. For example, $d(p \rightarrow \Box\Diamond p) = 2$. Given a finite set $P \subseteq \text{Prop}$ of propositional variables and $n \in \omega$, we define $\text{Form}(P)$ and $\text{Form}(P; n)$ by

$$\text{Form}(P) := \{\varphi \in \text{Form} \mid P(\varphi) \subseteq P\}, \quad \text{Form}(P; n) := \{\varphi \in \text{Form} \mid P(\varphi) \subseteq P \text{ and } d(\varphi) \leq n\}.$$

We move to Kripke semantics of basic tense logic. We say that a pair $F = (W, R)$ is a *frame* if W is a non-empty set of *states* (or *moments*) and R is a binary relation on W , i.e., $R \subseteq W \times W$. We use $\mathbb{F}$, $\mathbb{G}$, etc. to denote a class of frames. Define $\mathbb{F}_{\text{all}}$ as the class of all frames. A *model* is a pair $M = (F, V)$ where $F = (W, R)$ is a frame and V is a valuation function $V : \text{Prop} \rightarrow \wp(W)$ where $\wp(W)$ is the powerset of W . A *pointed model* (M, w) is a pair of a model $M = (W, R, V)$ and a state $w \in W$ of M . We say that a pointed model (M, w) is *from* a frame class $\mathbb{F}$ if $(W, R) \in \mathbb{F}$ where $M = (W, R, V)$. Given a model $M = (W, R, V)$, a state $w \in W$ and a formula φ , we define the *satisfaction relation* $M, w \models \varphi$ (read: “ φ holds at w in M ”) inductively as follows:

$$\begin{aligned} M, w \models p & \quad \text{iff} \quad w \in V(p), \\ M, w \not\models \perp, \\ M, w \models \varphi_1 \rightarrow \varphi_2 & \quad \text{iff} \quad M, w \not\models \varphi_1 \text{ or } M, w \models \varphi_2, \\ M, w \models \Box\varphi & \quad \text{iff} \quad wRv \text{ implies } M, v \models \varphi, \text{ for all } v \in W, \\ M, w \models \blacksquare\varphi & \quad \text{iff} \quad vRw \text{ implies } M, v \models \varphi, \text{ for all } v \in W. \end{aligned}$$

For a set Γ of formulas, we write $M, w \models \Gamma$ to mean that $M, w \models \varphi$ for all $\varphi \in \Gamma$. Given a frame $F = (W, R)$, we say that φ is *valid* on F (notation: $F \models \varphi$) if $(F, V), w \models \varphi$ for all $w \in W$ and all valuation functions V on F . Given a class $\mathbb{F}$ of frames, we say that φ is *valid* on $\mathbb{F}$ (notation: $\mathbb{F} \models \varphi$) if $F \models \varphi$

(Taut)	All classical tautologies	(MP)	From φ and $\varphi \rightarrow \psi$, we may derive ψ .
(UP)	From φ , we may derive $\varphi\sigma$, where σ is a uniform substitution.		
(K $\Box$)	$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$	(K $\blacksquare$)	$\blacksquare(p \rightarrow q) \rightarrow (\blacksquare p \rightarrow \blacksquare q)$
($\Box\Diamond$)	$p \rightarrow \Box\Diamond p$	($\blacksquare\Diamond$)	$p \rightarrow \blacksquare\Diamond p$
(Nec $\Box$)	From φ , we may derive $\Box\varphi$.	(Nec $\blacksquare$)	From φ , we may derive $\blacksquare\varphi$.

Table 1 Hilbert System $H(\mathbf{Kt})$

for all frames $F \in \mathbb{F}$. Given a frame class $\mathbb{F}$, we define $\Lambda_{\mathbb{F}} := \{ \varphi \in \text{Form} \mid \mathbb{F} \models \varphi \}$, the set of all the valid formulas on the class $\mathbb{F}$. We say that a set Γ of formulas is *satisfiable* in a frame class $\mathbb{F}$ if there exist a frame $F \in \mathbb{F}$, a valuation V on F , and a state $w \in W$ such that $(F, V), w \models \Gamma$. When Γ is a singleton $\{ \varphi \}$, we simply say that φ is satisfiable in $\mathbb{F}$. Note that φ is satisfiable in $\mathbb{F}$ iff $\mathbb{F} \not\models \neg\varphi$.

Table 1 provides the axiomatization of the minimal basic tense logic. The underlying idea of the axiomatization is to extend the minimal bimodal logic with two axioms ($\Box\Diamond$) and ($\blacksquare\Diamond$), which state that the accessibility relation for $\blacksquare$ is the converse relation of the accessibility relation for $\Box$.

Definition 2.1. A set Λ of formulas is a *normal tense logic* if it contains all the axioms of Table 1 and it is closed under all the inference rules of Table 1. We denote by $\mathbf{Kt}$ the smallest normal tense logic. Given a normal tense logic Λ and a set Δ of formulas, we use $\Lambda \oplus \Delta$ to mean the smallest normal tense logic that contains $\Lambda \cup \Delta$.

Given a frame class $\mathbb{F}$, we note that $\Lambda_{\mathbb{F}} := \{ \varphi \in \text{Form} \mid \mathbb{F} \models \varphi \}$ is a normal tense logic. The following holds e.g., by [7, Corollary 4.36].

Fact 2.2. The axiomatization $H(\mathbf{Kt})$ is sound and complete for the class $\mathbb{F}_{\text{all}}$ of all frames, i.e., $\mathbf{Kt} = \Lambda_{\mathbb{F}_{\text{all}}} = \{ \varphi \in \text{Form} \mid \mathbb{F}_{\text{all}} \models \varphi \}$.

Definition 2.3. A normal tense logic Λ has the *Craig interpolation property* if every $\varphi \rightarrow \psi \in \Lambda$ has a *Craig interpolant* θ such that $\{ \varphi \rightarrow \theta, \theta \rightarrow \psi \} \subseteq \Lambda$ and $P(\theta) \subseteq P(\varphi) \cap P(\psi)$.

Both semantic and proof-theoretic proofs for the following are available; see, for example, [11, Theorem 2.5.3] and [29, 40], respectively.

Fact 2.4. The smallest normal tense logic $\mathbf{Kt}$ has the *Craig interpolation property*.

Definition 2.5. A normal tense logic Λ has the *uniform interpolation property* if every pair $(\varphi, p) \in \text{Form} \times \text{Prop}$ has a *(post-)uniform interpolant* $\exists p.\varphi$ in Λ such that the following three conditions are satisfied:

(variable) $P(\exists p.\varphi) \subseteq P(\varphi) \setminus \{ p \}$.

(derivability) $\varphi \rightarrow \exists p.\varphi \in \Lambda$.

(uniformity) if $\varphi \rightarrow \psi \in \Lambda$ then $\exists p.\varphi \rightarrow \psi \in \Lambda$, for all p -free formulas ψ .

It is easy to see that the converse implication of (uniformity) follows from (derivability). We note that when a normal tense logic Λ is inconsistent, i.e., $\perp \in \Lambda$, then it trivially has the uniform interpolation property, since $\Lambda = \text{Form}$.

Proposition 2.6. If a normal tense logic Λ has the uniform interpolation property, then it also has the *Craig interpolation property*.

Proof. Suppose that Λ has the uniform interpolation property. Fix any $\varphi \rightarrow \psi \in \Lambda$. If $P(\varphi) \setminus P(\psi) = \emptyset$, we obtain $P(\varphi) \subseteq P(\psi)$ and so φ is a Craig interpolant (both $\{ \varphi \rightarrow \varphi, \varphi \rightarrow \psi \} \subseteq \Lambda$ and $P(\varphi) \subseteq P(\varphi) \cap P(\psi)$).

$P(\varphi) \cap P(\psi)$ hold). Suppose that $P(\varphi) \setminus P(\psi)$ is non-empty. Put $P(\varphi) \setminus P(\psi) := \{p_1, \dots, p_n\}$. Our desired Craig interpolant is $\exists p_1 \dots \exists p_n. \varphi$. It is easy to see that $P(\exists p_1 \dots \exists p_n. \varphi) \subseteq P(\varphi) \cap P(\psi)$ by the condition of (variable). By (derivability), we obtain $\varphi \rightarrow \exists p_1 \dots \exists p_n. \varphi \in \Lambda$. Finally, the condition of (uniformity) and $\varphi \rightarrow \psi \in \Lambda$ jointly imply that $\exists p_1 \dots \exists p_n. \varphi \rightarrow \psi \in \Lambda$, since ψ is $\{p_1, \dots, p_n\}$ -free. $\square$

3 Bisimulation and Layered Bisimulation

This section reviews the notions of (temporal) bisimulation and layered bisimulation, along with their basic properties. First, we introduce the following notion of bisimulation (or *temporal* bisimulation, see, e.g., [7, Remark 2.25]). Hereafter, for basic tense logic, we shall refer to this simply as bisimulation throughout this paper.

Definition 3.1 ((Temporal) Bisimulation). Let $M = (W, R, V)$ and $M' = (W', R', V')$ be models and $P \subseteq \text{Prop}$. We say that a non-empty relation $Z \subseteq W \times W'$ is a *P-bisimulation* between M and M' if Z satisfies the following five conditions:

- (Atom) if wZw' then $(w \in V(p) \text{ iff } w' \in V'(p))$ for all $p \in P$;
- (Forth) if wZw' and wRv then there exists $v' \in W'$ such that $w'R'v'$ and vZv' .
- (Forth⁻¹) if wZw' and vRw then there exists $v' \in W'$ such that $v'R'w'$ and vZv' .
- (Back) if wZw' and $w'R'v'$ then there exists $v \in W$ such that wRv and vZv' .
- (Back⁻¹) if wZw' and $v'R'w'$ then there exists $v \in W$ such that vRw and vZv' .

When the underlying P is clear from the context, we simply say that Z is a *bisimulation*. If there exists a P -bisimulation between M and M' such that wZw' , we write $(M, w) \rightleftharpoons^P (M', w')$ and say that (M, w) and (M', w') are *bisimilar*.

Definition 3.2. We write $(M_1, w_1) \rightsquigarrow^P (M_2, w_2)$ to mean that $M_1, w_1 \models \varphi$ iff $M_2, w_2 \models \varphi$, for every formula $\varphi \in \text{Form}(P)$, i.e., the set of all formulas φ such that $P(\varphi) \subseteq P$.

It is well-known that bisimilarity implies formula equivalence (for basic modal logic, the reader is referred to [7, Theorem 2.20]), while the converse does not hold in general (cf. [7, Example 2.23]).

Proposition 3.3 (Formula Equivalence). *Let $M = (W, R, V)$ and $M' = (W', R', V')$ be models and $P \subseteq \text{Prop}$. If $(M, w) \rightleftharpoons^P (M', w')$ then $(M, w) \rightsquigarrow^P (M', w')$.*

Definition 3.4. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be models. We say that a function $f : W \rightarrow W'$ is a *P-bounded morphism* from M to M' if it satisfies the following four conditions:

- (Atom) $w \in V(p)$ iff $f(w) \in V'(p)$ for all $p \in P$.
- (Forth) wRv implies $f(w)R'f(v)$.
- (Back) if $f(w)R'v'$ then there exists $v \in W$ such that $f(v) = v'$ and wRv .
- (Back⁻¹) if $v'R'f(w)$ then there exists $v \in W$ such that $f(v) = v'$ and vRw .

A mapping $f : W \rightarrow W'$ is called a *bounded morphism* from (W, R) to (W', R') if it satisfies the two conditions other than (Atom).

We note that $f : W \rightarrow W'$ is a P -bounded morphism from M to M' iff $\text{Gr}(f) := \{(w, f(w)) \mid w \in W\}$ is a P -bisimulation. Therefore, for a P -bounded morphism f from M to M' , Proposition 3.3 enables us to obtain the equivalence: $M, w \models \varphi$ iff $M', f(w) \models \varphi$ for all formulas $\varphi \in \text{Form}(P)$.

The following definition is a temporal generalization of the notion of *layered bisimulation* (also known as *bounded bisimulation*) introduced in [23, Section 1], [42, Section 3], and [41].

Definition 3.5 (Layered (Temporal) Bisimulation). Let $M = (W, R, V)$ and $M' = (W', R', V')$ be models and $P \subseteq \text{Prop}$. We say that a non-empty relation $Z \subseteq W \times \omega \times W$ is a *layered P-bisimulation* between M and M' if Z satisfies the following five conditions, where we write $wZ_n w'$ to mean $(w, n, w') \in Z$ below:

- (Atom) if $wZ_n w'$ then $(w \in V(p) \text{ iff } w' \in V'(p))$ for all $p \in P$ ($n \in \omega$).
- (Forth) if $wZ_{n+1} w'$ and wRv then there exists $v' \in W'$ such that $w'R'v'$ and $vZ_n v'$ ($n \in \omega$).
- (Forth⁻¹) if $wZ_{n+1} w'$ and vRw then there exists $v' \in W'$ such that $v'R'w'$ and $vZ_n v'$ ($n \in \omega$).
- (Back) if $wZ_{n+1} w'$ and $w'R'v'$ then there exists $v \in W$ such that wRv and $vZ_n v'$ ($n \in \omega$).
- (Back⁻¹) if $wZ_{n+1} w'$ and $v'R'w'$ then there exists $v \in W$ such that vRw and $vZ_n v'$ ($n \in \omega$).

Let $n \in \omega$. If there exists a layered P-bisimulation between M and M' such that $wZ_n w'$, we write $(M, w) \rightleftharpoons_n^P (M', w')$. We say that a layered bisimulation Z between M and M' is *downward closed* if, for all $n \in \omega$, $wZ_n w'$ implies that $wZ_m w'$ for all $m \leq n$.

We can establish the following, similarly to [7, Proposition 2.29] for basic modal logic. The following can be also established in terms of *normal forms* [19, 33] with the *cover modality* or *nabla modality*. The reader can find an outline of the argument in Appendix A.

Proposition 3.6. *The quotient set $\text{Form}(P; n) / \equiv$ is finite, where $\equiv$ is logical equivalence.*

Definition 3.7. Let $P \subseteq \text{Prop}$. We define $\Phi(P; n)$ as a finite set of all fixed representatives of the equivalence classes in $\text{Form}(P; n) / \equiv$ by Proposition 3.6. Given a pointed model (M, w) , we define the *characteristic formula* $\chi_{(M, w)}^{P; n}$ with respect to (P, n) by $\bigwedge \{ \varphi \in \Phi(P; n) \mid M, w \models \varphi \}$. Let (M_i, w_i) be a pointed model ($i \in \{1, 2\}$). We write $(M_1, w_1) \rightsquigarrow_n^P (M_2, w_2)$ to mean that $M_1, w_1 \models \varphi$ iff $M_2, w_2 \models \varphi$, for every formula $\varphi \in \text{Form}(P; n)$.

Proposition 3.8. *Let $P \subseteq \text{Prop}$ be finite and $n \in \omega$. For every two pointed models (M_1, w_1) and (M_2, w_2) , the following are all equivalent: 1. $(M_1, w_1) \rightsquigarrow_n^P (M_2, w_2)$; 2. $M_1, w_1 \models \chi_{(M_2, w_2)}^{P; n}$; 3. There exists a downward layered P-bisimulation Z between M_1 and M_2 such that $(w_1, n, w_2) \in Z$; 4. $(M_1, w_1) \rightleftharpoons_n^P (M_2, w_2)$.*

Proof. The equivalence between items 1 and 2 is immediate by definition of $\chi_{(M_2, w_2)}^{P; n}$. The direction from item 1 to item 3 holds by defining $(w, n, w') \in Z$ iff $(M_1, w_1) \rightsquigarrow_n^P (M_2, w_2)$. The direction from item 3 to item 4 is trivial. Finally, the direction from item 4 to item 1 is proved by induction on $\varphi \in \text{Form}(P; n)$. $\square$

4 Bisimulation Expansion Property

This section formulates the bisimulation expansion property of a frame class $\mathbb{F}$, which provides a sufficient condition for the uniform interpolation property of the normal tense logic $\Lambda_{\mathbb{F}}$ associated with that class. Furthermore, we establish that the uniform interpolant in $\Lambda_{\mathbb{F}}$ coincides with a bisimulation quantifier, provided that the frame class $\mathbb{F}$ has the bisimulation expansion property.

While the following definition is based on [42, Lemma 6.1] and [28, Definition 11], we introduce four improvements: first, we refine the number of sets of propositional variables (whereas [42, Lemma 6.1] requires three pairwise disjoint sets); second, we shift the focus from model classes to frame classes; third, we prioritize bisimilarity over formula equivalences up to a given modal depth; and finally, we lift the finiteness restriction on P_2 below (cf. [28]).

Definition 4.1. A frame class $\mathbb{F}$ has the *bisimulation expansion property* if, for all $P_1, P_2 \subseteq \text{Prop}$ such that P_1 is finite and for all pointed models (M_1, w_1) and (M_2, w_2) from $\mathbb{F}$ and $n \in \omega$ such that $(M_1, w_1) \rightleftharpoons_n^{P_1 \cap P_2} (M_2, w_2)$, there exists a pointed model (N, x) from $\mathbb{F}$ such that $(M_1, w_1) \rightleftharpoons_n^{P_1} (N, x) \rightleftharpoons_n^{P_2} (M_2, w_2)$.

Lemma 4.2. *If a frame class $\mathbb{F}$ has the bisimulation expansion property, then $\Lambda_{\mathbb{F}} := \{\varphi \in \text{Form} \mid \mathbb{F} \models \varphi\}$ has the uniform interpolation property.*

Proof. Suppose that a frame class $\mathbb{F}$ has the bisimulation expansion property. Fix any pair $(\varphi, p) \in \text{Form} \times \text{Prop}$. Put $n := d(\varphi)$. Recall that $\Phi(P(\varphi) \setminus \{p\}; n)$ is a finite set of all fixed representatives of the equivalence classes in $\text{Form}(P(\varphi) \setminus \{p\}; n) / \equiv$. Define $\text{Cn}(\varphi) := \{\psi \in \Phi(P(\varphi) \setminus \{p\}; n) \mid \mathbb{F} \models \varphi \rightarrow \psi\}$. Then we define $\exists p.\varphi := \bigwedge \text{Cn}(\varphi)$. In what follows, we prove that $\exists p.\varphi (= \bigwedge \text{Cn}(\varphi))$ satisfies all the required conditions. The condition (variable) holds by definition. For (derivability), we show that $\mathbb{F} \models \varphi \rightarrow \bigwedge \text{Cn}(\varphi)$. Fix any $\psi \in \Phi(P(\varphi) \setminus \{p\}; n)$ such that $\mathbb{F} \models \varphi \rightarrow \psi$. It suffices to show that $\mathbb{F} \models \varphi \rightarrow \psi$, which is immediate.

So, in what follows, we focus on establishing the condition (uniformity). Fix any p -free formula ψ (i.e., $p \notin P(\psi)$) and suppose that $\mathbb{F} \models \varphi \rightarrow \psi$. We show that $\mathbb{F} \models \exists p.\varphi \rightarrow \psi$. Let us fix any pointed model (M_2, w_2) from $\mathbb{F}$ such that $M_2, w_2 \models \exists p.\varphi$. Our goal is to show that $M_2, w_2 \models \psi$. Define $\chi := \chi_{(M_2, w_2)}^{Q, n}$ where $Q := (P(\varphi) \setminus \{p\}) \cap P(\psi)$. Then $\varphi \wedge \chi$ is satisfiable in $\mathbb{F}$, i.e., $\mathbb{F} \not\models \neg(\varphi \wedge \chi)$. This is verified as follows: Suppose not, i.e., $\mathbb{F} \models \neg(\varphi \wedge \chi)$. It follows that $\mathbb{F} \models \varphi \rightarrow \neg\chi$. Since $\neg\chi$ is p -free, we obtain $\neg\chi \in \text{Cn}(\varphi)$ hence $\mathbb{F} \models \exists p.\varphi \rightarrow \neg\chi$ by $\exists p.\varphi = \bigwedge \text{Cn}(\varphi)$. Since $M_2, w_2 \models \exists p.\varphi$, we get $M_2, w_2 \models \neg\chi$, which is a contradiction with $M_2, w_2 \models \chi$. By the satisfiability of $\varphi \wedge \chi$ in $\mathbb{F}$, we can find a pointed model (M_1, w_1) from $\mathbb{F}$ such that $M_1, w_1 \models \varphi \wedge \chi$. We note that $Q = (P(\varphi) \setminus \{p\}) \cap P(\psi) = P(\varphi) \cap P(\psi)$, since ψ is p -free. It follows from $M_1, w_1 \models \chi$ that $(M_1, w_1) \equiv_n^{P(\varphi) \cap P(\psi)} (M_2, w_2)$. By the bisimulation expansion property of $\mathbb{F}$, we can find a pointed model (N, x) from $\mathbb{F}$ such that $(M_1, w_1) \equiv_n^{P(\varphi)} (N, x) \equiv_n^{P(\psi)} (M_2, w_2)$. It follows from $M_1, w_1 \models \varphi$ and $(M_1, w_1) \equiv_n^{P(\varphi)} (N, x)$ that $N, x \models \varphi$ (recall that $n = d(\varphi)$). By the initial supposition $\mathbb{F} \models \varphi \rightarrow \psi$, we obtain $N, x \models \psi$. Therefore, we derive from $(N, x) \equiv_n^{P(\psi)} (M_2, w_2)$ that $M_2, w_2 \models \psi$, as desired. $\square$

Remark 4.3. Even if we restrict our focus to formula equivalences up to a given modal depth and retain the finiteness restriction on P_2 in Definition 4.1, we can still provide a sufficient condition for the uniform interpolation property of $\Lambda_{\mathbb{F}}$. In the proof of Corollary 4.4, however, it is crucial to allow the set P_2 of propositional variables to be potentially infinite within the bisimulation expansion property. In this sense, Lemma 4.2 has broader applicability.

The following provides a generalization of Visser's argument for the definability of bisimulation quantifiers in the basic modal logic $\mathbf{K}$ (cf. [42, Theorem 6.2]) to a normal tense logic $\Lambda_{\mathbb{F}}$.

Corollary 4.4. *If a frame class $\mathbb{F}$ has the bisimulation expansion property, then the following equivalence holds for a uniform interpolant $\exists p.\varphi$ for $(\varphi, p) \in \text{Form} \times \text{Prop}$: for every (M, w) from $\mathbb{F}$,*

$$M, w \models \exists p.\varphi \quad \text{iff} \quad (M, w) \equiv^{\text{Prop} \setminus \{p\}} (N, v) \text{ and } N, v \models \varphi, \text{ for some } (N, v) \text{ from } \mathbb{F}.$$

Proof. Suppose that a frame class $\mathbb{F}$ has the bisimulation expansion property. By Lemma 4.2, $\Lambda_{\mathbb{F}}$ has the uniform interpolation property. Fix any pointed model (M, w) . For the right-to-left direction, let us suppose that we can find a pointed model (N, v) from $\mathbb{F}$ such that $(M, w) \equiv^{\text{Prop} \setminus \{p\}} (N, v)$ and $N, v \models \varphi$. Our goal is to show that $M, w \models \exists p.\varphi$. Since $\exists p.\varphi$ is a uniform interpolant for (φ, p) , we obtain $\mathbb{F} \models \varphi \rightarrow \exists p.\varphi$ by (derivability). It follows from $N, v \models \varphi$ that $N, v \models \exists p.\varphi$. By (variable) condition, $\exists p.\varphi$ is p -free. We derive from $(M, w) \equiv^{\text{Prop} \setminus \{p\}} (N, v)$ that $M, w \models \exists p.\varphi$, as desired.

For the left-to-right direction, assume that $M, w \models \exists p.\varphi$. Put $n := d(\varphi)$. We define

$$\chi := \chi_{(M, w)}^{P(\varphi) \setminus \{p\}; n} = \bigwedge \{\rho \in \Phi(P(\varphi) \setminus \{p\}; n) \mid M, w \models \rho\}.$$

Note that χ is p -free and $M, w \models \chi$ hence $(\exists p.\varphi) \wedge \chi$ is satisfiable in $\mathbb{F}$. We prove that $\varphi \wedge \chi$ is satisfiable in $\mathbb{F}$, i.e., $\mathbb{F} \models \neg(\varphi \wedge \chi)$. This is proved as follows: Suppose otherwise. Then $\mathbb{F} \models \varphi \rightarrow \neg\chi$. Since $\exists p.\varphi$ is a uniform interpolant for (φ, p) and $\neg\chi$ is p -free, we obtain $\mathbb{F} \models \exists p.\varphi \rightarrow \neg\chi$ by (uniformity). This means that $(\exists p.\varphi) \wedge \chi$ is not satisfiable in $\mathbb{F}$. A contradiction. Thus, $\varphi \wedge \chi$ is satisfiable in $\mathbb{F}$.

We can find a pointed model (M_0, w_0) from $\mathbb{F}$ such that $M_0, w_0 \models \varphi$ and $M_0, w_0 \models \chi$. By the latter, we obtain $(M_0, w_0) \stackrel{P(\varphi) \setminus \{p\}}{\simeq_n} (M, w)$. Since $P(\varphi) \setminus \{p\} = P(\varphi) \cap (\text{Prop} \setminus \{p\})$, the bisimulation expansion property of $\mathbb{F}$ enables us to find a pointed model (N, x) from $\mathbb{F}$ such that

$$(M_0, w_0) \stackrel{P(\varphi)}{\simeq_n} (N, x) \stackrel{\text{Prop} \setminus \{p\}}{\simeq} (M, w).$$

By $M_0, w_0 \models \varphi$, we obtain $(N, x) \models \varphi$. By $(N, x) \stackrel{\text{Prop} \setminus \{p\}}{\simeq} (M, w)$, we obtain our desired goal. $\square$

5 Uniform Interpolation Property of Basic Tense Logic

This section establishes that the class $\mathbb{F}_{\text{all}}$ of all frames has the bisimulation expansion property hence $\mathbf{Kt} (= \{\varphi \in \text{Form} \mid \mathbb{F}_{\text{all}} \models \varphi\})$ has the uniform interpolation property.

Lemma 5.1. *The class $\mathbb{F}_{\text{all}}$ of all frames has the bisimulation expansion property.*

Proof. Fix any $P_1, P_2 \subseteq \text{Prop}$ such that P_1 is finite and any $n \in \omega$. Let us also fix any pointed models (M_1, w_1) and (M_2, w_2) from $\mathbb{F}_{\text{all}}$ with $(M_1, w_1) \stackrel{P_1 \cap P_2}{\simeq_n} (M_2, w_2)$. Let us put $M_i := (W_i, R_i, V_i)$. Since $P_1 \cap P_2$ is finite, Proposition 3.8 enables us to find a downward-closed layered $P_1 \cap P_2$ -bisimulation $Z \subseteq W_1 \times \omega \times W_2$ such that $(w_1, n, w_2) \in Z$. We add a new state “T” to M_1 to define $M_1^T := (W_1^T, R_1^T, V_1^T)$ as follows:

$$W_1^T := W_1 \cup \{T\}; \quad R_1^T := R_1 \cup \{(T, x), (x, T) \mid x \in W_1 \cup \{T\}\}; \quad V_1^T(p) := V_1(p) \text{ for all } p \in \text{Prop}.$$

By noting that $\{-1\} \cup \omega (= 1 + \omega)$ and ω are order-isomorphic as well-ordered sets, we revise Z into Z^+ as follows: $Z^+ := Z \cup \{(T, -1, y) \mid y \in W_2\}$. We also define the *predecessor function* $\text{pd} : \{-1\} \cup \omega \rightarrow \{-1\} \cup \omega$ by $\text{pd}(\alpha) := \alpha - 1$ (if $\alpha \geq 1$); -1 (if $\alpha \in \{-1, 0\}$). Now we define our intended intermediate model $N = (Z^+, S, U)$ as follows:

- $(a_1, \alpha, a_2)S(b_1, \beta, b_2)$ iff $a_1 R_1^T b_1, a_2 R_2 b_2$ and $(\text{pd}(\alpha) = \beta \text{ or } \text{pd}(\beta) = \alpha)$.
- $U : P_1 \cup P_2 \rightarrow \wp(Z^+)$ is defined by

$$(a_1, \alpha, a_2) \in U(p) \quad \text{iff} \quad \begin{cases} a_1 \in V_1(p) & \text{if } p \in P_1 \setminus P_2; \\ a_2 \in V_2(p) & \text{if } p \in P_2. \end{cases}$$

In what follows, we establish $(M_1, w_1) \stackrel{P_1}{\simeq_n} (N, (w_1, n, w_2)) \stackrel{P_2}{\simeq} (M_2, w_2)$. Define the projections π_i ($i \in \{1, 2, 3\}$) from Z^+ to each component by the following: for any $(a_1, \alpha, a_2) \in Z^+$,

$$\pi_1 : (a_1, \alpha, a_2) \mapsto a_1; \quad \pi_2 : (a_1, \alpha, a_2) \mapsto \alpha; \quad \pi_3 : (a_1, \alpha, a_2) \mapsto a_2.$$

Given an element $\alpha \in \omega$, we define $Z(\alpha) := \pi_2^{-1}[\{\alpha\}]$, the inverse image of $\{\alpha\}$ under the projection π_2 . Note that we do not need to consider -1 as an index in the next claim. Define $Y_\alpha := \text{Gr}(\pi_1 \upharpoonright Z(\alpha))$ ($\alpha \in \omega$), where $\text{Gr}(f) = \{(x, f(x)) \mid x \in X\}$ is a graph of a mapping $f : X \rightarrow Y$ and $f \upharpoonright X'$ is the restriction of f to $X' \subseteq X$.

Claim 1. $(Y_\alpha)_{\alpha \in \omega}$ is a layered P_1 -bisimulation between N and M_1 .

(Proof of Claim) We establish (Atom), (Forth⁻¹) and (Back⁻¹) since (Forth) and (Back) are shown similarly to (Forth⁻¹) and (Back⁻¹), respectively.

- **[Atom]** Fix any $\alpha \in \omega$ and $p \in P_1$. Suppose that $\pi_1(a_1, \alpha, a_2) = a_1$. We show that $\pi_1(a_1, \alpha, a_2) \in V_1(p)$ iff $(a_1, \alpha, a_2) \in U(p)$. When $p \in P_1 \setminus P_2$, the equivalence is immediate by definition. So, we let $p \in P_1 \cap P_2$. Then $(a_1, \alpha, a_2) \in U(p)$ iff $a_2 \in V_2(p)$ by definition. Since $(a_1, \alpha, a_2) \in Z$, $a_2 \in V_2(p)$ is equivalent with $a_1 \in V_1(p)$, as desired.
- **[Forth⁻¹]** Assume $(a_1, \alpha + 1, a_2)Y_{\alpha+1}a_1$ and $Z^+ \ni (b_1, \beta, b_2)S(a_1, \alpha + 1, a_2)$. It follows from the definition of S that $b_1R_1^T a_1$ hence $b_1R_1 a_1$ by $\alpha + 1, \beta \in \omega$. In what follows, it suffices to establish $(b_1, \alpha, b_2)Y_\alpha b_1$. By assumption, we obtain i) $\text{pd}(\alpha + 1) = \beta$ or ii) $\text{pd}(\beta) = \alpha + 1$. When i) $\text{pd}(\alpha + 1) = \beta$ holds, we get $\beta = \alpha$. Since $(b_1, \alpha, b_2) \in Z(\alpha)$, we obtain $(b_1, \alpha, b_2)Y_\alpha b_1$. When ii) $\text{pd}(\beta) = \alpha + 1$ holds, we get $\beta = \alpha + 2$. Thus, $(b_1, \alpha + 2, b_2) \in Z$ by $\alpha \in \omega$. By the downward closure of Z , we obtain $(b_1, \alpha, b_2) \in Z$. Therefore, we get $(b_1, \alpha, b_2)Y_\alpha b_1$.
- **[Back⁻¹]** Assume that $(a_1, \alpha + 1, a_2)Y_{\alpha+1}a_1$ and $b_1R_1 a_1$. By $(a_1, \alpha + 1, a_2) \in Z$ and $b_1R_1 a_1$, we can find $b_2 \in W_2$ such that $b_2R_2 a_2$ and $(b_1, \alpha, b_2) \in Z$ by (Forth⁻¹) of Z . Then $(b_1, \alpha, b_2)Y_\alpha b_1$ and $(b_1, \alpha, b_2)S(a_1, \alpha + 1, a_2)$ by $\text{pd}(\alpha + 1) = \alpha$.

This finishes establishing $(Y_\alpha)_{\alpha \in \omega}$ is a layered P_1 -bisimulation between N and M_1 . $\dashv$

Claim 2. π_3 is a P_2 -bounded morphism from N to M_2 .

(Proof of Claim) By Definition 3.4, we check the following four conditions.

- **[Atom]** Fix any $p \in P_2$ and $(a_1, \alpha, a_2) \in Z^+$. We show that $\pi_3(a_1, \alpha, a_2) \in V_2(p)$ iff $(a_1, \alpha, a_2) \in U(p)$. This is immediate by definition of U .
- **[Forth]** Assume $(a_1, \alpha, a_2)S(b_1, \beta, b_2)$. It follows that $a_2R_2 b_2$ hence $\pi_3(a_1, \alpha, a_2)R_2 \pi_3(b_1, \beta, b_2)$.
- **[Back]** Fix any $(a_1, \alpha, a_2) \in Z^+$. Assume that $a_2 = \pi_3(a_1, \alpha, a_2)R_2 b_2$. We divide our argument into the following two cases: i) $\alpha \geq 1$; ii) $\alpha \in \{-1, 0\}$. For the case i), we proceed as follows: By $\alpha \geq 1$, we obtain $(a_1, \alpha, a_2) \in Z$. By $a_2R_2 b_2$ and (Back) of Z , we can find some $b_1 \in W_1$ such that $a_1R_1 b_1$ and $(b_1, \alpha - 1, b_2) \in Z$. So, $(a_1, \alpha, a_2)S(b_1, \alpha - 1, b_2)$ and $\pi_3(b_1, \alpha - 1, b_2) = b_2$. Finally, we consider the case ii): $\alpha \in \{-1, 0\}$. By definition of S , $(a_1, \alpha, a_2)S(T, -1, b_2)$ and $\pi_3(T, -1, b_2) = b_2$, as required.
- **[Back⁻¹]** Fix any $(a_1, \alpha, a_2) \in Z^+$. Assume that $b_2R_2 \pi_3(a_1, \alpha, a_2) = a_2$. We divide our argument into the following two cases: i) $\alpha \geq 1$; ii) $\alpha \in \{-1, 0\}$. For the case i), we proceed as follows: By $\alpha \geq 1$, we obtain $(a_1, \alpha, a_2) \in Z$. By $b_2R_2 a_2$ and (Back⁻¹) of Z , we can find some $b_1 \in W_1$ such that $b_1R_1 a_1$ and $(b_1, \alpha - 1, b_2) \in Z$. So, $(b_1, \alpha - 1, b_2)S(a_1, \alpha, a_2)$ and $\pi_3(b_1, \alpha - 1, b_2) = b_2$. Finally, we consider the case ii): $\alpha \in \{-1, 0\}$. By definition of S , $(T, -1, b_2)S(a_1, \alpha, a_2)$ and $\pi_3(T, -1, b_2) = b_2$, as required.

This finishes establishing that π_3 is a P_2 -bounded morphism from N to M_2 . $\dashv$

It is easy to see that $((w_1, n, w_2), w_1) \in \text{Gr}(\pi_1 \upharpoonright Z(n))$ and that $\pi_3 : (w_1, n, w_2) \mapsto w_2$. By Claims 1 and 2, we have established $(M_1, w_1) \rightleftharpoons_n^{P_1} (N, (w_1, n, w_2)) \rightleftharpoons^{P_2} (M_2, w_2)$. $\square$

By Lemmas 4.2 and 5.1, we obtain our main theorem of this paper as follows:

Theorem 5.2. *The smallest normal tense logic **Kt** has the uniform interpolation property.*

6 Uniform Interpolation Property of Normal Tense Extensions

This section begins with several examples of frame definability to introduce various extensions of **Kt**. We first discuss the notion of mirror images and prove that for a normal tense logic Λ invariant under the mirror image, the uniform interpolation property of $\Lambda \oplus \Delta$ implies that of Λ extended by the mirror image of Δ . Subsequently, we demonstrate that the uniform interpolation property is preserved under the addition of constant axioms. Finally, we establish that this preservation also holds for the reflexive counterpart of a normal tense logic under certain assumptions.

Let us say that a formula set Γ defines a frame class $\mathbb{F}$ if for all frames F , the following equivalence holds: $F \in \mathbb{F}$ iff $F \models \varphi$ for all $\varphi \in \Gamma$. Table 2 presents examples of frame class definability. The formulas LM and FM of Table 2 express the existence of a last and a first moment (cf. [10]), respectively. On the other hand, the formulas $D_{\Box}$ and $D_{\blacksquare}$ of Table 2 express the absence of a last and a first moment, respectively. We can say that LM and $D_{\Box}$ are *mirror images* of FM and $D_{\blacksquare}$, respectively, in the following sense.

Definition 6.1. Given a formula φ , the mirror image $\mathfrak{m}(\varphi)$ of φ is defined as the formula obtained by simultaneously replacing all occurrences of $\blacksquare$ with $\Box$ and vice versa. For a formula set Δ , define $\mathfrak{m}[\Delta] := \{\mathfrak{m}(\varphi) \mid \varphi \in \Delta\}$.

Given a normal modal logic Λ and a set Δ of formulas, recall that $\Lambda \oplus \Delta$ is the smallest normal modal logics containing $\Lambda \cup \Delta$. The following proposition is easy to obtain.

Proposition 6.2. 1. Let Λ be a normal tense logic. Then $\Box p \rightarrow p \in \Lambda$ iff $\blacksquare p \rightarrow p \in \Lambda$. Moreover, $\Box p \rightarrow \Box \Box p \in \Lambda$ iff $\blacksquare p \rightarrow \blacksquare \blacksquare p \in \Lambda$. 2. For any $\Delta \subseteq \{T_{\Box}, 4_{\Box}\}$, $\mathfrak{m}[\mathbf{Kt} \oplus \Delta] = \mathbf{Kt} \oplus \Delta$. 3. Let Λ be a normal tense logic such that $\mathfrak{m}[\Lambda] = \Lambda$ and Δ a formula set. Then $\varphi \in \Lambda \oplus \Delta$ iff $\mathfrak{m}(\varphi) \in \Lambda \oplus \mathfrak{m}[\Delta]$.

($D_{\Box}$)	$\Diamond \top$	$\forall x. \exists y. xRy$	($D_{\blacksquare}$)	$\Diamond \top$	$\forall x. \exists y. yRx$
($T_{\Box}$)	$\Box p \rightarrow p$	$\forall x. xRx$	($4_{\Box}$)	$\Box p \rightarrow \Box \Box p$	$\forall x, y, z. (xRy \wedge yRz \rightarrow xRz)$
(LM)	$\Box \perp \vee \Diamond \Box \perp$	$\forall x. (\exists y. (xRy) \rightarrow \exists z. (xRz \wedge \forall w. (\neg zRw)))$			
(FM)	$\blacksquare \perp \vee \Diamond \blacksquare \perp$	$\forall x. (\exists y. (yRx) \rightarrow \exists z. (zRx \wedge \forall w. (\neg wRz)))$			

Table 2 Examples of Frame Definability

We note that the mirror image $\mathfrak{m}$ does not change the set of propositional variables in a formula.

Proposition 6.3. Let Δ be a set of formulas and Λ be a normal tense logic such that $\mathfrak{m}[\Lambda] = \Lambda$. If $\Lambda \oplus \Delta$ has the uniform interpolation property, then $\Lambda \oplus \mathfrak{m}[\Delta]$ also has the uniform interpolation property.

Proof. Suppose that $\Lambda \oplus \Delta$ has the uniform interpolation property. For the uniform interpolation property of $\Lambda \oplus \mathfrak{m}[\Delta]$, let us fix any $(\varphi, p) \in \text{Form} \times \text{Prop}$. By the supposition, let $\exists p. \mathfrak{m}(\varphi)$ be a uniform interpolant for $(\mathfrak{m}(\varphi), p)$ in $\Lambda \oplus \Delta$. We show that $\mathfrak{m}(\exists p. \mathfrak{m}(\varphi))$ is our desired uniform interpolant for (φ, p) in $\Lambda \oplus \mathfrak{m}[\Delta]$. First, we show (variable): $P(\mathfrak{m}(\exists p. \mathfrak{m}(\varphi))) = P(\exists p. \mathfrak{m}(\varphi)) \subseteq P(\mathfrak{m}(\varphi)) \setminus \{p\} = P(\varphi) \setminus \{p\}$, as desired. Next, we move to (derivability). We prove that $\varphi \rightarrow \mathfrak{m}(\exists p. \mathfrak{m}(\varphi)) \in \Lambda \oplus \mathfrak{m}[\Delta]$. Since $\exists p. \mathfrak{m}(\varphi)$ is a uniform interpolant for $(\mathfrak{m}(\varphi), p)$ in $\Lambda \oplus \Delta$, we obtain $\mathfrak{m}(\varphi) \rightarrow \exists p. \mathfrak{m}(\varphi) \in \Lambda \oplus \Delta$ hence $\varphi \rightarrow \mathfrak{m}(\exists p. \mathfrak{m}(\varphi)) \in \Lambda \oplus \mathfrak{m}[\Delta]$ by item 3 of Proposition 6.2. Finally, we establish (uniformity). Fix any p -free formula ψ and suppose that $\varphi \rightarrow \psi \in \Lambda \oplus \mathfrak{m}[\Delta]$. Then $\mathfrak{m}(\varphi) \rightarrow \mathfrak{m}(\psi) \in \Lambda \oplus \Delta$ by item 3 of Proposition 6.2. Since $\exists p. \mathfrak{m}(\varphi)$ is a uniform interpolant for $(\mathfrak{m}(\varphi), p)$ in $\Lambda \oplus \Delta$ and $\mathfrak{m}(\psi)$ is still p -free, we obtain $\exists p. \mathfrak{m}(\varphi) \rightarrow \mathfrak{m}(\psi) \in \Lambda \oplus \Delta$. We conclude from item 3 of Proposition 6.2 that $\mathfrak{m}(\exists p. \mathfrak{m}(\varphi)) \rightarrow \psi \in \Lambda \oplus \mathfrak{m}[\Delta]$, as required. $\square$

By Theorem 5.2 and Propositions 6.2 and 6.3, we can establish the following.

Corollary 6.4. *If $\mathbf{Kt} \oplus \Delta$ has the uniform interpolation property, then $\mathbf{Kt} \oplus \mathfrak{m}[\Delta]$ has the uniform interpolation property.*

Recall that a formula φ is constant if $P(\varphi) = \emptyset$. The following extends Rautenberg's result on Craig interpolation [39, Theorem 3] and Kurahashi's results on uniform (Lyndon) interpolation [28, Proposition 3] for basic modal logic to the case of basic tense logic.

Proposition 6.5. *Let Δ be a set of constant formulas. If Λ has the uniform interpolation property, then $\Lambda \oplus \Delta$ also has the uniform interpolation property.*

Proof. Let Δ be a set of constant formulas and suppose that Λ has the uniform interpolation property. To show that $\Lambda \oplus \Delta$ has the uniform interpolation property, let us fix any formula φ and $p \in \text{Prop}$. By supposition, there exists a uniform interpolant $\exists p.\varphi$ in Λ such that it satisfies the specified three conditions. We show that the same $\exists p.\varphi$ is a (post-) uniform interpolant for (φ, p) in $\Lambda \oplus \Delta$. The conditions (variable) and (derivability) are easy to verify, so we focus on the final condition of (uniformity). Fix any formula ψ such that $p \notin P(\psi)$ (p -freeness) and $\varphi \rightarrow \psi \in \Lambda \oplus \Delta$. Our goal is to show that $\exists p.\varphi \rightarrow \psi \in \Lambda \oplus \Delta$. It follows from $\varphi \rightarrow \psi \in \Lambda \oplus \Delta$ that there exists a finite multiset Δ' constructed from a finite subset of Δ and a family $(L_\delta)_{\delta \in \Delta'} \subseteq \{\Box, \blacksquare\}^{<\omega}$ such that $\bigwedge \{L_\delta \delta \mid \delta \in \Delta'\} \rightarrow (\varphi \rightarrow \psi) \in \Lambda$ (since Δ is a set of constant formulas). Put $\bigwedge L \Delta' := \bigwedge \{L_\delta \delta \mid \delta \in \Delta'\}$. Then $\varphi \rightarrow (\bigwedge L \Delta' \rightarrow \psi) \in \Lambda$. Since Δ is a set of constant formulas, $\bigwedge L \Delta'$ is also constant. So $\bigwedge L \Delta' \rightarrow \psi$ is p -free. By supposition, we get $\exists p.\varphi \rightarrow (\bigwedge L \Delta' \rightarrow \psi) \in \Lambda \subseteq \Lambda \oplus \Delta$ hence $\bigwedge L \Delta' \rightarrow (\exists p.\varphi \rightarrow \psi) \in \Lambda \oplus \Delta$. Since $\bigwedge L \Delta' \in \Lambda \oplus \Delta$, this enables us to conclude that $\exists p.\varphi \rightarrow \psi \in \Lambda \oplus \Delta$, as desired. $\square$

By Theorem 5.2 and Proposition 6.5, we can establish the following.

Corollary 6.6. *Let Δ be a set of constant formulas. Then $\mathbf{Kt} \oplus \Delta$ has the uniform interpolation property. Therefore, for every $\Delta \subseteq \{\text{LM}, \text{FM}, \text{D}\Box, \text{D}\blacksquare\}$ of Table 2, $\mathbf{Kt} \oplus \Delta$ has the uniform interpolation property.*

Definition 6.7. We define the *boxdot translation* $(\cdot)^\Box : \text{Form} \rightarrow \text{Form}$ as follows:

$$\begin{aligned} p^\Box &:= p, & \perp^\Box &:= \perp, & (\varphi \rightarrow \psi)^\Box &:= \varphi^\Box \rightarrow \psi^\Box, \\ (\Box\varphi)^\Box &:= \varphi^\Box \wedge \Box(\varphi^\Box), & (\blacksquare\varphi)^\Box &:= \varphi^\Box \wedge \blacksquare(\varphi^\Box). \end{aligned}$$

Given any normal tense logic Λ , we define $\Box^{-1}(\Lambda) := \{\varphi \mid \varphi^\Box \in \Lambda\}$.

Proposition 6.8. *Let Λ be a normal tense logic. 1. If $\Box p \rightarrow p \in \Lambda$, then $\varphi \leftrightarrow \varphi^\Box \in \Lambda$. 2. $\Box^{-1}(\Lambda)$ is a normal tense logic. 3. $\Box p \rightarrow p \in \Box^{-1}(\Lambda)$. 4. $\Box^{-1}(\Lambda) \subseteq \Lambda \oplus \{\Box p \rightarrow p\}$. 5. If $\Lambda \subseteq \Box^{-1}(\Lambda)$ then $\Box^{-1}(\Lambda) = \Lambda \oplus \{\Box p \rightarrow p\}$.*

Proof. 1. We proceed by induction on φ . We only comment on the case where φ is of the form $\blacksquare\psi$. We establish $\blacksquare\psi \leftrightarrow (\psi^\Box \wedge \blacksquare(\psi^\Box))$. By the induction hypothesis, it suffices to establish that $\blacksquare\psi \rightarrow \psi \in \Lambda$, which is obtained from item 1 of Proposition 6.2.

2. We only check axiom $(\Box\Diamond)$, i.e., $p \rightarrow \Box\Diamond p \in \Box^{-1}(\Lambda)$. We show that $(p \rightarrow \Box\Diamond p)^\Box \in \Lambda$. We note that $(p \rightarrow \Box\Diamond p)^\Box \leftrightarrow (p \rightarrow (p \vee \Diamond p) \wedge \Box(p \vee \Diamond p)) \in \Lambda$. But $(p \rightarrow (p \vee \Diamond p) \wedge \Box(p \vee \Diamond p)) \in \Lambda$ by axiom $(\Box\Diamond)$.

3. It suffices to prove that $(\Box p \rightarrow p)^\Box \in \Lambda$. But $(\Box p \rightarrow p)^\Box := (\Box p \wedge p) \rightarrow p$ is an instance of tautologies, which is in Λ .

4. Fix any formula φ such that $\varphi^\Box \in \Lambda$. We show that $\varphi \in \Lambda \oplus \{\Box p \rightarrow p\}$. By item 1, we obtain $\varphi \leftrightarrow \varphi^\Box \in \Lambda \oplus \{\Box p \rightarrow p\}$. So it suffices to show that $\varphi^\Box \in \Lambda \oplus \{\Box p \rightarrow p\}$. But this is clear from $\varphi^\Box \in \Lambda$.

5. Suppose that $\Lambda \subseteq \Box^{-1}(\Lambda)$. By item 3 and the supposition, we obtain $\Lambda \oplus \{\Box p \rightarrow p\} \subseteq \Box^{-1}(\Lambda)$. It follows from item 4 that $\Box^{-1}(\Lambda) = \Lambda \oplus \{\Box p \rightarrow p\}$, as desired. $\square$

The following provides an extension of Kurahashi's result [28, Proposition 3] on the boxdot translation from normal modal logics to normal tense logics.

Proposition 6.9. *Let Λ be a normal tense logic such that $\Lambda \subseteq \Box^{-1}(\Lambda)$. If Λ has the uniform interpolation property, then $\Box^{-1}(\Lambda) (= \Lambda \oplus \{\Box p \rightarrow p\})$ also has the uniform interpolation property.*

Proof. Let Λ be a normal tense logic such that $\Lambda \subseteq \Box^{-1}(\Lambda)$ and that Λ has the uniform interpolation property. By item 5 of Proposition 6.8, we obtain $\Box^{-1}(\Lambda) = \Lambda \oplus \{\Box p \rightarrow p\}$. For the uniform interpolation property of $\Box^{-1}(\Lambda)$, let us fix any $(\varphi, p) \in \text{Form} \times \text{Prop}$. Let $\exists p.\varphi^\Box$ be a uniform interpolant for $(\varphi^\Box, p)$ in Λ . We show that $\exists p.\varphi^\Box$ is a uniform interpolant for (φ, p) in $\Box^{-1}(\Lambda)$. Since the boxdot translation does not change the set of propositional variables in a formula, $P(\exists p.\varphi^\Box) \subseteq P(\varphi^\Box) \setminus \{p\} = P(\varphi) \setminus \{p\}$. So (variable) condition holds. For (derivability), we prove that $\varphi \rightarrow \exists p.\varphi^\Box \in \Box^{-1}(\Lambda)$. Since $\varphi^\Box \rightarrow \exists p.\varphi^\Box \in \Lambda$ and $\varphi \leftrightarrow \varphi^\Box \in \Box^{-1}(\Lambda)$ (by items 1, 2 and 3 of Proposition 6.8), $\Lambda \subseteq \Box^{-1}(\Lambda)$ implies that $\varphi \rightarrow \exists p.\varphi^\Box \in \Box^{-1}(\Lambda)$, as required. For (uniformity), fix any p -free formula ψ with $\varphi \rightarrow \psi \in \Box^{-1}(\Lambda)$. We show that $\exists p.\varphi^\Box \rightarrow \psi \in \Box^{-1}(\Lambda)$. By $\varphi \rightarrow \psi \in \Box^{-1}(\Lambda)$, we obtain $\varphi^\Box \rightarrow \psi^\Box \in \Lambda$ by definition. Since $\psi^\Box$ is p -free, we obtain $\exists p.\varphi^\Box \rightarrow \psi^\Box \in \Lambda$. By $\Lambda \subseteq \Box^{-1}(\Lambda)$, it holds that $\exists p.\varphi^\Box \rightarrow \psi^\Box \in \Box^{-1}(\Lambda)$. Since $\rho \leftrightarrow \rho^\Box \in \Box^{-1}(\Lambda)$ for all formulas ρ , we obtain $\psi \leftrightarrow \psi^\Box \in \Box^{-1}(\Lambda)$. We conclude that $\exists p.\varphi^\Box \rightarrow \psi \in \Box^{-1}(\Lambda)$, as desired. $\square$

By Proposition 6.9, we obtain the following specific results on normal tense logics.

Corollary 6.10. 1. $\mathbf{Kt} \oplus \mathbf{T}_\Box$ has the uniform interpolation property.

2. Let $\Delta \subseteq \{\mathbf{D}_\Box, \mathbf{D}_\blacksquare\}$. If $\mathbf{Kt} \oplus \mathbf{4}_\Box \oplus \Delta$ has the uniform interpolation property, then $\mathbf{Kt} \oplus \{\mathbf{4}_\Box, \mathbf{T}_\Box\}$ also has the uniform interpolation property.

7 Failure of Uniform Interpolation Property of Tense S4

This section demonstrates that the tense expansion of modal logic **S4** lacks the uniform interpolation property. To establish this, we first characterize the uniform interpolation property for the normal tense logic of a frame class in terms of the bisimulation expansion property, under the assumption that the logic of the frame class is compact and the frame class satisfies the following Hennessy-Milner property.

Definition 7.1. A frame class $\mathbb{F}$ has the *Hennessy-Milner property* if, for every two pointed models (M_1, w_1) and (M_2, w_2) from $\mathbb{F}$, $(M_1, w_1) \rightsquigarrow^P (M_2, w_2)$ implies $(M_1, w_1) \rightleftharpoons^P (M_2, w_2)$ for every $P \subseteq \text{Prop}$.

It is easy to see that a class of finite frames always has the Hennessy-Milner property.

Definition 7.2. Given a class of frames $\mathbb{F}$, we say that the normal tense logic $\Lambda_{\mathbb{F}}$ is *compact* if, whenever every finite subset $\Gamma' \subseteq \Gamma$ is satisfiable in $\mathbb{F}$, the set Γ itself is also satisfiable in $\mathbb{F}$.

Proposition 7.3. Let $\mathbb{F}$ be a frame class satisfying the Hennessy-Milner property such that $\Lambda_{\mathbb{F}}$ is compact. The frame class $\Lambda_{\mathbb{F}}$ has the uniform interpolation property iff $\mathbb{F}$ has the bisimulation expansion property.

Proof. Let $\mathbb{F}$ be a frame class such that it has the Hennessy-Milner property. The right-to-left direction is already established in Lemma 4.2. So, we establish the left-to-right direction. Assume that $\Lambda_{\mathbb{F}}$ has the uniform interpolation property. To establish the bisimulation expansion property of $\mathbb{F}$, let us fix any subsets $P_1, P_2 \subseteq \text{Prop}$ such that P_1 is finite, any pointed models (M_1, w_1) and (M_2, w_2) from $\mathbb{F}$

and any $n \in \omega$. Suppose that $(M_1, w_1) \rightsquigarrow_n^{P_1 \cap P_2} (M_2, w_2)$. Put $P_1 \setminus P_2 := \{p_1, \dots, p_n\}$. Recall $\chi_{(M_1, w_1)}^{P_1; n} := \bigwedge \{ \chi \in \Phi(P_1; n) \mid M_1, w_1 \models \chi \}$. By our supposition, we can construct a uniform interpolant $\theta := \exists p_1 \dots \exists p_n \cdot \chi_{(M_1, w_1)}^{P_1; n}$. We have $P(\theta) = P(\chi_{(M_1, w_1)}^{P_1; n}) \setminus \{p_1, \dots, p_n\} = P_1 \setminus (P_1 \setminus P_2) = P_1 \cap P_2$. It follows from $\mathbb{F} \models \chi_{(M_1, w_1)}^{P_1; n} \rightarrow \theta$, $(M_1, w_1) \models \chi_{(M_1, w_1)}^{P_1; n}$ and $M_1, w_1 \rightsquigarrow_n^{P_1 \cap P_2} (M_2, w_2)$ that $M_2, w_2 \models \theta$. Define $\text{Th}(M_2, w_2) := \{ \rho \in \text{Form}(P_2) \mid M_2, w_2 \models \rho \}$. Then $\{ \theta \} \cup \text{Th}(M_2, w_2)$ is satisfiable in $\mathbb{F}$. We can also establish that $\{ \chi_{(M_1, w_1)}^{P_1; n} \} \cup \text{Th}(M_2, w_2)$ is satisfiable in $\mathbb{F}$. Suppose otherwise. Since $\text{Th}(M_2, w_2)$ is closed under conjunctions and $\Lambda_{\mathbb{F}}$ is compact, we can find a formula $\rho \in \text{Th}(M_2, w_2)$ such that $\mathbb{F} \models \chi_{(M_1, w_1)}^{P_1; n} \rightarrow \neg \rho$ holds. By $P(\rho) \cap \{p_1, \dots, p_n\} = \emptyset$, ρ is $\{p_1, \dots, p_n\}$ -free. So, we obtain $\mathbb{F} \models \theta \rightarrow \neg \rho$, which means that $\{ \theta \} \cup \text{Th}(M_2, w_2)$ is not satisfiable in $\mathbb{F}$. However, $\{ \theta \} \cup \text{Th}(M_2, w_2)$ is satisfiable in $\mathbb{F}$. We obtain a desired contradiction. By the satisfiability of $\{ \chi_{(M_1, w_1)}^{P_1; n} \} \cup \text{Th}(M_2, w_2)$ in $\mathbb{F}$, we can find a pointed model (N, x) from $\mathbb{F}$ such that $N, x \models \chi_{(M_1, w_1)}^{P_1; n}$ and $N, x \models \text{Th}(M_2, w_2)$. It follows that $(M_1, w_1) \rightleftharpoons_n^{P_1} (N, x)$ and $(N, x) \rightsquigarrow^{P_2} (M_2, w_2)$ hence $(N, x) \rightleftharpoons^{P_2} (M_2, w_2)$ by the Hennessy-Milner property of $\mathbb{F}$. This finishes showing the bisimulation expansion property of $\mathbb{F}$. $\square$

By Lemma 4.2, Corollary 4.4 and Proposition 7.3, we can also obtain the following.

Corollary 7.4. *Let $\mathbb{F}$ be a frame class satisfying the Hennessy-Milner property such that $\Lambda_{\mathbb{F}}$ is compact. Then if $\Lambda_{\mathbb{F}}$ has the uniform interpolation property, the uniform interpolant $\exists p. \varphi$ for $(\varphi, p) \in \text{Form} \times \text{Prop}$ in $\Lambda_{\mathbb{F}}$ is a bisimulation quantifier in the sense of Corollary 4.4.*

Definition 7.5. Let **S4t** be the smallest normal tense logic that contains $T_{\square}$ and $4_{\square}$ in Table 2. Let $\mathbb{F}_{\text{S4t}}^{\text{fin}}$ be the class of all reflexive and transitive finite frames (W, R) , i.e., all quasi-orders (W, R) .

The following can be readily shown via the canonical model construction and filtration technique. The finite model property of **S4t** was also established in [40, Theorem 3.6] using an analytic sequent calculus.

Proposition 7.6. *The normal tense logic **S4t** is sound and complete for the class $\mathbb{F}_{\text{S4t}}^{\text{fin}}$ of all finite reflexive and transitive frames, i.e., $\text{S4t} = \Lambda_{\mathbb{F}_{\text{S4t}}^{\text{fin}}}$. Moreover, $\text{S4t} = \Lambda_{\mathbb{F}_{\text{S4t}}^{\text{fin}}}$ is compact.*

The lemma below extends Visser's theorem for the modal logic **S4** [43, Theorem 16.3] to a tense-logic setting. The inclusion of the additional conditions (Forth^{-1}) and (Back^{-1}) does not alter the original line of reasoning. Given a model $M := (W, R, V)$, we use $V[s := X]$ to denote the same valuation as V except that it sends a propositional variable s to a set $X \subseteq W$, and we define $M[s := X] := (W, R, V[s := X])$.

Lemma 7.7. *Define $\varphi(p, q, r)$ as $\varphi(p, q, r) := p \wedge \square(p \rightarrow \diamond q) \wedge \square(q \rightarrow \diamond p) \wedge \square(p \rightarrow r) \wedge \square(q \rightarrow \neg r)$. Suppose that **S4t** has the uniform interpolation property. Let (M, w) be a pointed model from $\mathbb{F}_{\text{S4t}}^{\text{fin}}$, where $M = (W, R, V)$. Then $M, w \models \exists p. \exists q. \varphi(p, q, r)$ iff there exists an infinite R -sequence $(w_n)_{n \in \omega} \subseteq W$ such that $w_0 = w$, $w_k R w_{k+1}$, $M, w_{2k} \models r$ and $M, w_{2k+1} \not\models r$ hold, for all $k \in \omega$.*

Proof. Suppose that **S4t** has the uniform interpolation property. For the right-to-left direction, assume there exists an infinite R -sequence $(w_n)_{n \in \omega} \subseteq W$ satisfying the stated conditions. Define $E := \{w_{2k} \mid k \in \omega\}$ and $O := \{w_{2k+1} \mid k \in \omega\}$. By $r \notin \{p, q\}$, we have $M[p := E][q := O], w \models \varphi(p, q, r)$. Moreover, the following hold:

$$(M, w) \rightleftharpoons^{\text{Prop} \setminus \{p\}} (M[p := E], w) \rightleftharpoons^{\text{Prop} \setminus \{q\}} (M[p := E][q := O], w).$$

Therefore, we obtain $M, w \models \exists p. \exists q. \varphi(p, q, r)$ by Proposition 7.6 and Corollary 7.4.

For the left-to-right direction, suppose $M, w \models \exists p. \exists q. \varphi(p, q, r)$. By Proposition 7.6 and Corollary 7.4, there exist two pointed models (M_1, w_1) and (M_2, w_2) such that $(M, w) \rightleftharpoons^{\text{Prop} \setminus \{p\}} (M_1, w_1) \rightleftharpoons^{\text{Prop} \setminus \{q\}} (M_2, w_2)$.

(M_2, w_2) and $M_2, w_2 \models \varphi(p, q, r)$. The latter implies that there exists an infinite R_2 -path in M_2 along which the truth value of r alternates at every step. By the transitivity of bisimulation, it follows that $(M, w) \Leftrightarrow^{\text{Prop} \setminus \{p, q\}} (M_2, w_2)$. Since $r \notin \{p, q\}$, the valuation of r is preserved between M and M_2 . Therefore, by the (Back) condition, we can construct a corresponding infinite R -path in M along which r alternates between being true and false at every step, as desired. $\square$

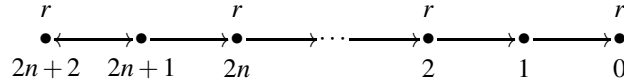


Figure 1: A model M in the proof of Lemma 7.8, with reflexive loops and transitive compositions omitted.

The following result extends Ghilardi and Zawadowski's [23] theorem on the failure of the uniform interpolation property in **S4** to a tense-logical setting. Our proof builds upon the one given in [43, Theorem 16.4], which provides an alternative proof of their theorem. The model used in our proof (see Figure 1) is identical to the one presented in [23, p. 269] and [43, Theorem 16.4]. Furthermore, the addition of the conditions (Forth^{-1}) and (Back^{-1}) to the notion of (non-temporal) layered bisimulations (e.g., from [42]) remains consistent with the model construction in [43].

Lemma 7.8. *There is no formula $\psi(r) \in \text{Form}(\{r\})$ such that, for all pointed models (M, w) from $\mathbb{F}_{\mathbf{S4}}^{\text{fin}}$ such that $M = (W, R, V)$, the following equivalence holds: $M, w \models \psi(r)$ iff there exists an infinite R -sequence $(w_n)_{n \in \omega} \subseteq W$ such that $w_0 = w$, $w_k R w_{k+1}$, $M, w_{2k} \models r$ and $M, w_{2k+1} \not\models r$ hold, for all $k \in \omega$.*

Proof. Suppose for contradiction that we can find a formula $\psi(r)$ such that the stated equivalence holds for all pointed models (M, w) from $\mathbb{F}_{\mathbf{S4}}^{\text{fin}}$. Fix some $n \in \omega$ such that $d(\psi(r)) \leq 2n$. Define a model $M = (W, R, V)$ as follows (see Figure 1): $W := \{0, 1, \dots, 2n, 2n+1, 2n+2\}$, xRy iff $x \geq y$ or $x \in \{2n+1, 2n+2\}$; $x \in V(r)$ iff x is even. The set $\{2n+1, 2n+2\}$ forms a *cluster* in terms of R . It is noted that $M, 2n+2 \models \psi(r)$ but $M, 2n \not\models \psi(r)$. Let us define $Z \subseteq W \times \omega \times W$ by

$$(x, n, x') \in Z \text{ iff } x = x' \text{ or } (x \equiv x' \pmod{2} \text{ and } n \leq \min(x, x')).$$

By definition, we obtain $(2n, 2n, 2n+2) \in Z$.

Claim 1. Z is a layered (temporal) $\{r\}$ -bisimulation between M and M .

(Proof of Claim) It is immediate to see that (**Atom**) holds by definition. In what follows, we check the remaining conditions. Suppose that $(x, k+1, x') \in Z$. If $x = x'$, then all four required conditions hold immediately. For example, (**Forth**) holds as follows: Assume xRy . Then $(y, k, y) \in Z$ holds by definition. So, we can find $y' \in W$ such that $x = x'Ry'$ and $(y, k, y') \in Z$, as desired.

Let us suppose that $x \neq x'$. Since $(x, k+1, x') \in Z$, it holds that $(x \equiv x' \pmod{2})$ and $k+1 \leq \min(x, x')$. Now we assume without loss of generality that $x < x'$. Since our definition of Z is symmetric with respect to x and x' , it suffices to prove the four remaining conditions under this assumption $x < x'$. First, we establish that $x \notin \{2n+1, 2n+2\}$. Suppose otherwise, i.e., $x \in \{2n+1, 2n+2\}$. If $x = 2n+1$, then $x' = 2n+2$, which is a contradiction with $x \equiv x' \pmod{2}$. If $x = 2n+2$, then it is impossible to have $x < x'$. So, we have shown that $x \notin \{2n+1, 2n+2\}$ hence $x \leq 2n$.

It is noted that arguments below for (**Forth**) and (**Back**) are the same as those in the proof of [43, Theorem 16.4].

- **[Forth]** Suppose that xRy , i.e., $y \leq x$ by $x \notin \{2n+1, 2n+2\}$. We show that there exists $y \in W$ such that $x'Ry'$ and $(y, k, y') \in Z$. We define $y' := y$. Since $y \leq x < x'$, we obtain $x'Ry$. We get $(y, k, y) \in Z$ immediately by definition.

- **Back** Suppose that $x'Ry'$. We show that there exists $y \in W$ such that xRy and $(y, k, y') \in Z$. We divide our argument into the following two cases: i) $y' \leq x$ and ii) $y' > x$.

- For case i) of $y' \leq x$, we proceed as follows: Define $y := y'$. It follows from $y' \leq x$ that xRy' by definition. Moreover, $(y', k, y') \in Z$ holds immediately.
- For case ii) of $y' > x$, we define our candidate y as follows:

$$y := \begin{cases} x & \text{if } y' \equiv x \pmod{2}; \\ x-1 & \text{if } y' \not\equiv x \pmod{2}. \end{cases}$$

It is noted that $k+1 \leq \min(x, x') = x$ implies $x \geq 1$ and so $x-1$ is well-defined. By definition, we obtain xRy . For both cases of y , it suffices to show that $(y, k, y') \in Z$. We note from $y' > x$ that $y' > y$ hence $y' \neq y$. It is easy to see that $y \equiv y' \pmod{2}$. So, it remains to establish $k \leq \min(y, y')$. This is verified as follows: Since $k+1 \leq \min(x, x') = x$ (by $x < x'$), we obtain $k \leq x-1 \leq y = \min(y, y')$ (by $y' > y$), as desired.

- **Forth⁻¹** Suppose that yRx . Whether $y \in \{2n+1, 2n+2\}$ holds or not, we obtain $x \leq y$ (by $x \notin \{2n+1, 2n+2\}$). We show that there exists $y' \in W$ such that $y'Rx'$ and $(y, k, y') \in Z$. We divide our argument into the following two cases: i) $x' \leq y$ and ii) $x' > y$.

- i) Assume $x' \leq y$. Then yRx' . Together with $(y, k, y) \in Z$, we have obtained our goal.
- ii) Assume $x' > y$. Let us define $\text{suc} : W \rightarrow W$ by $\text{suc}(z) := z+1$ (if $z \leq 2n+1$); $\text{suc}(z) := 2n+1$ (if $z = 2n+2$). We also define y' as follows:

$$y' := \begin{cases} x' & \text{if } y \equiv x' \pmod{2}; \\ \text{suc}(x') & \text{if } y \not\equiv x' \pmod{2}. \end{cases}$$

For both choices of y' , we obtain $y'Rx'$. By definition, it is clear that $y \equiv y' \pmod{2}$.

- * Let $y \equiv x' \pmod{2}$. To show that $(y, k, y') \in Z$, it suffices to establish $k \leq \min(y, y') = y$ as follows: $k \leq k+1 \leq \min(x, x') = x \leq y$, as desired.
- * Let $y \not\equiv x' \pmod{2}$. If $y \in \{2n+1, 2n+2\}$, then $(x', y) = (2n+2, 2n+1)$ by $x' > y$. It follows that $y' = \text{suc}(x') = 2n+1 = y$ hence $(y, k, y') \in Z$. In what follows, we let $y \notin \{2n+1, 2n+2\}$, i.e., $y \leq 2n$. By $x' > y$, we get $y' \neq y$ by definition of y' (this also holds when $x' = 2n+2$ and so $y' = 2n+1$). To show that $(y, k, y') \in Z$, it suffices to establish $k \leq \min(y, y') = y$ as follows: $k \leq k+1 \leq \min(x, x') = x \leq y$, as desired.

- **Back⁻¹** Suppose that $y'Rx'$. We show that there exists $y \in W$ such that yRx and $(y, k, y') \in Z$. Assume $y' \in \{2n+1, 2n+2\}$. Then we obtain $y'Rx$ and $(y', k, y') \in Z$. So it suffices to define $y := y'$. Assume otherwise, i.e., $y' \notin \{2n+1, 2n+2\}$. Then it follows from $y'Rx'$ that $x' \leq y'$ by definition of R . By $x < x'$, we obtain $x < y'$ hence yRx . Since $(y', k, y') \in Z$ holds immediately, it suffices to define $y := y'$.

This finishes establishing that Z is a layered $\{r\}$ -bisimulation between M and M . ⊢

By Claim 1 and Proposition 3.8, it follows from $M, 2n+2 \models \varphi(r)$ that $M, 2n \models \varphi(r)$, which is a contradiction with $M, 2n \not\models \varphi(r)$. □

Theorem 7.9. *The normal tense logic **S4t** does not have the uniform interpolation property.*

Proof. Suppose for contradiction that **S4t** has the uniform interpolation property. For the formula $\varphi(p, q, r)$ in the statement of Lemma 7.7 and $p, q \in \text{Prop}$, we can construct the formula $\exists p. \exists q. \varphi(p, q, r)$ in terms of uniform interpolants, whose semantic clause at (M, w) from $\mathbb{R}_{\mathbf{S4}}^{\text{fin}}$ means that there exists an infinite R -path along which the propositional variable r alternates between being true and false at every step. However, by noting that $P(\exists p. \exists q. \varphi(p, q, r)) \subseteq \{r\}$, this is impossible by Lemma 7.8. $\square$

By Corollary 6.10 and Theorem 7.9, we can also derive the following.

Corollary 7.10. *For every $\Delta \subseteq \{\mathbf{D}_{\square}, \mathbf{D}_{\blacksquare}\}$, $\mathbf{Kt} \oplus 4_{\square} \oplus \Delta$ lacks the uniform interpolation property.*

8 Further Directions

We have established that the basic tense logic **Kt** has the uniform interpolation property. It is straightforward to extend our argument to the multimodal (or indexed) version of **Kt** with modalities $[a]$ and $[a^{\sim}]$ ($a \in \text{Mod}$), as the addition of indices does not alter the outline of the argument. This extension also allows us to establish the uniform interpolation property for star-free propositional dynamic logic with converse [25], as there exists a translation from this logic into the indexed version of **Kt** that preserves the set of propositional variables occurring in a formula.¹

There are several directions for future research. First, an intriguing open question arising from this work concerns the extension of nabla (∇) normal forms to a tense-logical setting (see also Appendix A). In basic modal logic, ∇ -based normal forms (or disjunctive forms characterized by non-interacting conjunctive components) share deep structural links with both uniform interpolation and bisimulation quantifiers (cf. [4, Section 5]). A natural next step is to determine whether an analogous normal form can be formulated for **Kt** to prove its uniform interpolation property, which remains an open problem in [4, Section 5].

Second, we may investigate whether a Pitts–Bílková-style proof-theoretic approach (cf. [34, 5]) can be applied to the uniform interpolation property of **Kt** (even for modal logic **B**).

Third, we could provide further examples of normal tense logics with the uniform interpolation property. For example, we have not considered the axiom $5_{\square} (\diamond p \rightarrow \square \diamond p)$ and its mirror image $5_{\blacksquare} (\blacklozenge p \rightarrow \blacksquare \blacklozenge p)$ in this paper. In the context of basic modal logics, it is well known, e.g., in [45], that every normal modal logic containing $5_{\square}$ is *locally tabular* (i.e., the number of non-equivalent formulas generated from a finite set of propositional variables is finite, cf. [13, p.19]). Since local tabularity and the Craig interpolation property jointly imply the uniform interpolation property, it would be interesting to investigate the uniform interpolation property of the normal tense extension containing $\{5_{\square}, 5_{\blacksquare}\}$.

Fourth, we may extend our argument to establish the uniform Lyndon interpolation property of **Kt**, which further requires considering the polarities of propositional variables. As demonstrated by Kurahashi [28], this can be accomplished by introducing a polarized variant of layered bisimulation.

Finally, basic tense logic was pioneered by Arthur Prior [35, 36, 37, 38], who subsequently extended **Kt** into a *hybrid* logic equipped with nominals, the global modality, and a universal quantifier over nominals (cf. [9, 8]). In light of recent studies on propositionally quantified hybrid logic, exploring the philosophical implications of the uniform interpolation property for **Kt** becomes pertinent, given that it enables the definition of a bisimulation quantifier [16, 15], which is a distinct form of propositional quantification.

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A Proof of Proposition 3.6

This section provides an outline of a proof of Proposition 3.6 in terms of *canonical formulas* defined in [33] (see also [17]).

Definition A.1. Given a finite set $P \subseteq \text{Prop}$ and $Q \subseteq P$ of propositional variables, we define an abbreviation $\pi(Q, P) := \bigwedge Q \wedge \bigwedge \neg(P \setminus Q)$. Given a finite set Ψ of formulas, we also introduce the cover (or nabla) modalities $\nabla\Psi$ and $\blacktriangledown\Psi$ as follows:

$$\nabla\Psi := \bigwedge \Diamond\Psi \wedge \Box\Psi, \quad \blacktriangledown\Psi := \bigwedge \Diamond\Psi \wedge \blacksquare\Psi.$$

We note that $\Box\varphi$ and $\Diamond\varphi$ can be written as $\nabla\emptyset \vee \nabla\{\varphi\}$ and $\nabla\{\top, \varphi\}$, respectively.

Definition A.2. For a finite subset $P \subseteq \text{Prop}$ and $n \in \omega$, we define a *finite* set $C(P; n)$ of *canonical formulas* with respect to $(P; n)$ as follows:

$$\begin{aligned} C(P; 0) &:= \{ \pi(Q, P) \mid Q \subseteq P \}; \\ C(P; n+1) &:= \{ \pi \wedge \nabla\Psi \wedge \blacktriangledown\Psi' \mid \pi \in C(P; 0) \text{ and } \Psi, \Psi' \subseteq C(P; n) \}. \end{aligned}$$

It is noted that $C(P; n)$ is finite for each n and so $C(P; n)$ is well-defined.

Proposition A.3. Let $P \subseteq \text{Prop}$ be a finite subset and $n \in \omega$.

1. Given a pointed model (M, w) , there exists a unique $\chi \in C(P; n)$ such that $M, w \models \chi$.
2. Let $\chi \in C(P; n)$. For every $\varphi \in \text{Form}(P; n)$, it holds that $\mathbb{F}_{\text{all}} \models \chi \rightarrow \varphi$ or $\mathbb{F}_{\text{all}} \models \chi \rightarrow \neg\varphi$.
3. For all formulas $\varphi \in \text{Form}(P; n)$, there exists $\Phi \subseteq C(P; n)$ such that $\mathbb{F}_{\text{all}} \models \varphi \leftrightarrow \bigvee \Phi$.

Proof. (Outline) Items 1 and 2 can be established similarly to [33, Lemma 2.6] and [33, Lemma 2.7], respectively. Item 3 follows from item 2 (see also [17, Proposition 6]). $\square$

Proposition A.3 (3) implies Proposition 3.6, i.e., the quotient set $\text{Form}(P; n)/\equiv$ of $\text{Form}(P; n)$ in terms of logical equivalence $\equiv$ is finite.

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Logics Containing **wK4**: Selection à la Fine

Simon Santschi*

Mathematical Institute
University of Bern
Bern, Switzerland

simon.santschi@unibe.ch

Niels C. Vooijs*

Mathematical Institute
University of Bern
Bern, Switzerland

ncvooijs@gmail.com

We generalize Fine’s Iterative Selection Method to the weakly transitive setting. In particular, this provides a transparent frame-theoretic proof of the finite model property for (strongly) cofinal subframe logics extending **wK4**, which was previously established using algebraic methods. Using the same construction, we generalize Fine’s Finite Width Theorem to the weakly transitive setting, connecting these two celebrated theorems of Fine.

1 Introduction

A well-known theorem of Fine [10] states that every subframe logic extending **K4** has the finite model property (or FMP for short). Here, a normal modal logic Λ is called *subframe* if every definable submodel of a model of Λ is itself a model of Λ . Fine [10] gives two proofs for this fact. The first proof [10, Theorem 4.5] relies on subframe formulas and a relatively straightforward selection on the canonical model. The second proof [10, Theorem 6.6] is purely model-theoretic but relies on a more involved iterated selection-quotient procedure.¹

This second approach has several advantages. As Fine puts it: ‘the extra work involved in the direct proof is not wasted; for it enables us to extract further information about subreduct models’ [10, Section 6, p. 638]. Moreover, this approach generalizes the result from subframe logics to cofinal subframe logics, which would only later be introduced by Zakharyashev [15]. Finally, Zakharyashev’s theory of canonical formulas builds upon this construction [15, 16, 17].

We generalize Fine’s Iterative Selection Method to the weakly transitive setting, obtaining the following model-theoretic analogue to [4, Lemma 5.1].

Theorem A. *Let $\mathfrak{M}$ be a weakly transitive Kripke model and Σ a finite subformula-closed set of formulas. Then there exists a definable Σ -sub-p-morphism from $\mathfrak{M}$ onto a finite model.*

Here a Σ -sub-p-morphism is a generalization of Fine’s n -subreduction, see Section 3. As an immediate corollary, we obtain a model-theoretic proof of the following theorem, which was already proved algebraically, see [4, Theorem 5.9] and [2, Theorem 7.14].

Theorem B. *Every strongly cofinal subframe logic extending **wK4** has the FMP.*

Moreover, we generalize the construction beyond finite filtration sets, by extending the iteration to the transfinite. This allows us to generalize, using the selection construction, Fine’s Finite Width Theorem [9] to the weakly transitive setting.

Theorem C. *Every logic of finite width extending **wK4** is complete w.r.t. its conversely pre-well-founded frames, and is therefore, in particular, Kripke complete.*

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¹Fine [10] does not mention the definability of the selected subset, and therefore states the result for Kripke complete logics [10, Corollary 6.3]. The fact that the selected subset is definable was noted by Kracht [11].

That is, we obtain two of Fine’s celebrated theorems, generalized to the weakly transitive setting, from a single generalized construction, connecting the two results.

Compared to [4, Lemma 5.1], Theorem A is a very slight generalization in that it gives some more guarantees about the selected subset. The important corollary Theorem B follows equally from both constructions, where it should be noted that *strong* cofinality was introduced only in [2], and is thus not stated in [4, Theorem 5.9] but does easily follow from their construction, as noted in [2]. Thus, our construction can be understood as a purely model-theoretic alternative to [4], with the added benefit of uniting it with the Finite Width Theorem in a single construction.

Another strand in the literature, starting with Fine’s [10] first FMP proof, is to perform selection on canonical models. These, and more generally descriptive models, are easier to deal with for selection methods, since for every point x that satisfies a formula φ , there exists a *maximal* successor of x that satisfies φ . The selected subset can therefore be defined directly without the need for an iterated construction. A canonicity assumption on the logic then also obviates the need for definability of the selection. Examples of this approach in the weakly transitive setting are [1, 12].

The logic $\mathbf{wK4} = \mathbf{K} \oplus (p \wedge \Box p) \rightarrow \Box \Box p$ is the logic of weakly transitive frames [8]. This logic and its extensions have recently attracted attention [4, 2, 3, 7] for two reasons. First, $\mathbf{wK4}$ is the logic of topological spaces when $\Diamond$ is interpreted as the Cantor derivative. This so-called *topological d -semantics* was introduced by McKinsey and Tarski [13] along with the classical *closure semantics*, but completeness of $\mathbf{wK4}$ with respect to this semantics was only proved in [8]. Second, many results for extensions of $\mathbf{K4}$ generalize to extensions of $\mathbf{wK4}$, but the proofs often require non-trivial modification.

Outline. The paper is organized as follows. In Section 2, we recall the basic notions and notations needed in the paper. In Section 3, we generalize Fine’s n -subreductions to the weakly transitive setting and introduce auxiliary notions used in our selection construction, which is described in Section 4. Sections 5 and 6 are concerned with the termination of the construction and the definability of the selected subset, respectively. Both results rely on suitably defined bisimulations. In Section 7, we combine the results from the earlier sections to derive Theorems A to C. Finally, in Section 8, we discuss possible directions for further research.

2 Preliminaries

We assume familiarity with basic Kripke semantics for normal unimodal logic, see, e.g., [6, 5]. By a *logic* we always mean a normal unimodal logic. We write $\Box\varphi := \varphi \wedge \Box\varphi$.

Let $\mathfrak{F} = \langle X, R \rangle$ be a (Kripke) frame. For $x \in X$, we write $R(x) := \{y \in X \mid \langle x, y \rangle \in R\}$. For $y \in R(x)$ we say x *sees* y and y is a *successor* of x . If, moreover, $x \notin R(y)$, then we call y a *strict successor* of x . We write $R^{\text{op}} := \{\langle y, x \rangle \mid \langle x, y \rangle \in R\}$ for the converse of R , and R^* for the reflexive transitive closure of R . Given a set $Y \subseteq X$ and $x \in X$, we write $\text{sp}_Y(x) := R(x) \cap Y$ for the Y -span of x , $\text{sp}_Y^*(x) := R^*(x) \cap Y$, and $\equiv_{Y\text{-sp}}$ for the equivalence relation on X induced by sp_Y . For any equivalence relation $\equiv_\bullet$, we write $[x]_\bullet$ for the equivalence class of x . For a binary relation S , we denote its *domain* by $\text{dom}(S) := \{x \mid \langle x, y \rangle \in S\}$. We apply logical connectives to subsets of a frame, e.g., $\Diamond Y = R^{\text{op}}(Y)$ and $Y \rightarrow Z = \neg Y \cup Z$.

Recall that $\mathfrak{F}$ is *weakly transitive* if whenever $y \in (R \circ R)(x)$, then either $y \in R(x)$ or $y = x$. In this case, R^* coincides with the reflexive closure of R . For the rest of the paper, we will only be concerned with weakly transitive frames.

A *cluster* is a maximal set C such that $C \times C \subseteq R^*$. Every point is a member of exactly one cluster. A set $U \subseteq X$ is called an *upset* or *upwards closed* if $R(U) \subseteq U$. We say that $Y \subseteq X$ *covers* $Y' \subseteq X$ if

$Y' \subseteq (R^*)^{\text{op}}(Y)$. It is *cofinal* if it covers X and *strongly cofinal* if, moreover, $(R \circ R)^{\text{op}}(Y) \subseteq R^{\text{op}}(Y)$.

For a model $\mathfrak{M} = \langle X, R, V \rangle$ on the frame $\mathfrak{F}$ and a formula φ , we write $\llbracket \varphi \rrbracket_{\mathfrak{M}}$ for the set of points x satisfying φ , i.e. $\langle \mathfrak{M}, x \rangle \models \varphi$. Given a set of formulas Σ , we write $\text{ty}_{\Sigma}(\mathfrak{M}, x) := \{\varphi \in \Sigma \mid x \in \llbracket \varphi \rrbracket_{\mathfrak{M}}\}$ for the Σ -type of x . In both cases we omit $\mathfrak{M}$ from the notation when it is clear from the context. We write $\equiv_{\Sigma\text{-ty}}$ for the equivalence relation on X induced by ty_{Σ} . We say $\mathfrak{M}$ *validates* φ if $\llbracket \varphi \rrbracket_{\mathfrak{M}} = X$, and $\mathfrak{M}$ is a *model* of a logic Λ or is a Λ -*model* if it validates every $\varphi \in \Lambda$. A logic Λ is said to have the *finite model property* (or *FMP* for short) if it is complete with respect to its finite models.

A set $Y \subseteq X$ is called *definable* in $\mathfrak{M}$ if there exists a formula φ such that $Y = \llbracket \varphi \rrbracket_{\mathfrak{M}}$, and $x \in X$ is *definable* if $\{x\}$ is definable. A *definable variant* of $\mathfrak{M}$ is a model $\mathfrak{M}'$ on $\mathfrak{F}$ such that $\llbracket p \rrbracket_{\mathfrak{M}'}$ is definable in $\mathfrak{M}$ for every atomic proposition p . In this case, if $\mathfrak{M}$ is a Λ -model for some logic Λ , then $\mathfrak{M}'$ is a Λ -model as well. $\mathfrak{M}$ is called *differentiated* if for all $x, y \in X$, if $x \neq y$, then there exists a definable set Y with $x \in Y$ and $y \notin Y$, and *atomic* if every point is definable.

Recall that a *p-morphism* between models is a surjection which is also a bisimulation. Any bisimulation equivalence $\equiv$ on $\mathfrak{M}$ induces a *p-morphism* from $\mathfrak{M}$ to the quotient $\mathfrak{M}/\equiv$. A submodel $\mathfrak{M}'$ of $\mathfrak{M}$ is the restriction of $\mathfrak{M}$ to some set $Y \subseteq X$, and a *p-morphism* from $\mathfrak{M}'$ to $\mathfrak{N}$ is called a *sub-p-morphism* from $\mathfrak{M}$ to $\mathfrak{N}$. Both the submodel and the sub-p-morphism are called *cofinal*, *definable*, etc. if Y has the property in question in $\mathfrak{M}$. If $\mathfrak{M}'$ is a definable submodel of $\mathfrak{M}$, then every set definable in $\mathfrak{M}'$ is definable in $\mathfrak{M}$.

A logic Λ is called *subframe*, *cofinal subframe*, or *strongly cofinal subframe*, respectively, if for every Λ -model $\mathfrak{M}$, every definable, definable cofinal, or definable strongly cofinal submodel of $\mathfrak{M}$, respectively, is itself a Λ -model.

We say that a frame $\mathfrak{F}$ or model $\mathfrak{M}$ has *finite width* if every anti-chain in it is finite, and a logic (extending **wK4**) is of *finite width* if it is complete w.r.t. its models of finite width. Note that we do not require a fixed finite bound on the size of anti-chains. We call $\mathfrak{F}$ *conversely pre-well-founded* if there is no infinite sequence $f: \mathbb{N} \rightarrow X$ in it such that $f(n+1)$ is a strict successor of $f(n)$ for every $n \in \mathbb{N}$. The frame $\mathfrak{F}$ has the *finite cover property* if every set $Y \subseteq X$ is covered by a finite subset $Y' \subseteq Y$. The finite cover property is equivalent to the combination of the finite width and converse pre-well-foundedness properties, and is also equivalent to the statement that for every infinite decreasing sequence of upsets $(U_n)_{n \in \mathbb{N}}$, there exists $i \in \mathbb{N}$ such that $U_i = U_{i+1}$.

3 Towards the Selection Construction

In this section, we generalize Fine's n -subreductions and Zakhariashev's Σ -subreductions to the weakly transitive setting, and set up some further definitions we will need for the selection construction in the next section. Let us fix a subformula-closed set of formulas Σ , and denote by $\mathbb{P}$ the set of atomic propositions in Σ . Moreover, fix, for the rest of this section, a model $\mathfrak{M}$ on a weakly transitive frame $\mathfrak{F} = \langle X, R \rangle$.

Definition 3.1 (Eliminability). Let $Y \subseteq X$. Then $x \in X \setminus Y$ is called Σ -*eliminable* w.r.t. Y iff for every $\varphi \in \Sigma$, there exists $y \in \text{sp}_Y(x)$ with $x \in \llbracket \varphi \rrbracket \iff y \in \llbracket \varphi \rrbracket$. We write $\mathcal{E}_{\Sigma}(\mathfrak{M}, Y)$ for the set of Σ -eliminable points w.r.t. Y , and generally omit Σ from the notation. A point $x \in X$ is called Σ -*eliminable* if it is Σ -eliminable w.r.t. $\{y \in R(x) \mid x \notin R(y)\}$.

Lemma 3.2. If Σ is finite and Y is definable in $\mathfrak{M}$, then so is $\mathcal{E}_{\Sigma}(\mathfrak{M}, Y)$.

Proof. Let $\mathcal{A}$ be the set containing $\llbracket \varphi \rrbracket$ and $\llbracket \neg \varphi \rrbracket$ for each $\varphi \in \Sigma$. Then

$$\mathcal{E}_{\Sigma}(\mathfrak{M}, Y) = \neg Y \cap \bigcap \{A \rightarrow \Diamond(Y \cap A) \mid A \in \mathcal{A}\}.$$

Since Σ is finite, the intersection is finite. □

Next, we generalize Σ -sub-p-morphisms to the weakly transitive case.

Definition 3.3 (Σ -sub-p-morphism). We call a set $X' \subseteq X$ a Σ -subset of $\mathfrak{M}$ if $\mathcal{E}_\Sigma(\mathfrak{M}, X') = X \setminus X'$ and for all $\varphi \in \Sigma$, $x \in X'$ and $y \in R(x) \cap R^{\text{op}}(x)$ such that $x \in \llbracket \varphi \rrbracket \iff y \in \llbracket \varphi \rrbracket$, there is $y' \in \text{sp}_{X'}(x)$ with $x \in \llbracket \varphi \rrbracket \iff y' \in \llbracket \varphi \rrbracket$. A sub-p-morphism f from $\mathfrak{M}$ to $\mathfrak{N}$ is called a Σ -sub-p-morphism if $\text{dom}(f)$ is a Σ -subset of $\mathfrak{M}$.

If $\mathfrak{M}$ is transitive, then the latter requirement is void. In this case, if Σ is finite and, up to logical equivalence, closed under Boolean connectives, then we obtain Zakharyashev's definition of Σ -subreduction. If Σ is, up to logical equivalence, the set of formulas of modal depth up to n , we obtain Fine's definition of n -subreduction. As in Fine's and Zakharyashev's definitions, it is immediate that every Σ -subset is cofinal. In addition, we obtain the following key property of Σ -subsets.

Lemma 3.4. *Let X' be a Σ -subset of $\mathfrak{M}$, $\varphi \in \Sigma$, $x \in X$ and $y \in R(x)$. Then there exists $y' \in \text{sp}_{X'}(x)$ with $y \in \llbracket \varphi \rrbracket \iff y' \in \llbracket \varphi \rrbracket$.*

Proof. If $y \in X'$ then take $y' := y$, and we are done. So suppose $y \in \mathcal{E}(\mathfrak{M}, X')$. Then there is $z \in \text{sp}_{X'}(y)$ such that $y \in \llbracket \varphi \rrbracket \iff z \in \llbracket \varphi \rrbracket$. If $z \in R(x)$, then take $y' := z$. Otherwise, $z \notin R(x)$ and, by weak transitivity, $x = z$, so $y \in R(x) \cap R^{\text{op}}(x)$ and $y \in \llbracket \varphi \rrbracket \iff x \in \llbracket \varphi \rrbracket$. By the second property of Σ -subsets, there is $y' \in \text{sp}_{X'}(x)$ with $x \in \llbracket \varphi \rrbracket \iff y' \in \llbracket \varphi \rrbracket$. $\square$

From this it immediately follows that every Σ -subset of $\mathfrak{M}$ is strongly cofinal. Moreover, a routine induction on formulas shows that they preserve truth of formulas in Σ .

Proposition 3.5. *Let f be a Σ -sub-p-morphism from $\mathfrak{M}$ to $\mathfrak{N}$ and let $\mathfrak{M}'$ be the restriction of $\mathfrak{M}$ to $\text{dom}(f)$. Then, for every $x \in \text{dom}(f)$,*

$$\text{ty}_\Sigma(\mathfrak{M}, x) = \text{ty}_\Sigma(\mathfrak{M}', x) = \text{ty}_\Sigma(\mathfrak{N}, f(x)).$$

From Lemma 3.4 and Proposition 3.5 it also follows that, as in the transitive setting of Zakharyashev, Σ -sub-p-morphisms compose. However, we will not use this property.

In the selection construction we will be iteratively removing points that are eliminable w.r.t. some upset U , and extending this U . For the latter step, we introduce $\text{Top}(U)$, the set of points in the 'highest' clusters below U .

Definition 3.6. For an upset U in $\mathfrak{M}$ and $D := X \setminus U$, we define

$$\text{Top}(U) := \{x \in D \mid \text{sp}_D(x) \subseteq R^{\text{op}}(x)\}.$$

The next lemma follows straightforwardly from the above definition.

Lemma 3.7. *If U is an upset in $\mathfrak{M}$, then also $\text{Top}(U) \cup U$ is an upset.*

In the selection construction we will take quotients that are maximal w.r.t. these upsets.

Definition 3.8. For an upset U in $\mathfrak{M}$, we call $x, y \in X$ *bisimilar relative to U* if x and y are bisimilar in the extension of $\mathfrak{M}$ with a fresh propositional variable that evaluates to U . Equivalently, bisimilarity relative to U is the largest bisimulation equivalence on $\mathfrak{M}$ that does not identify points in U and $X \setminus U$. We say that $\mathfrak{M}$ is *collapsed relative to U* if bisimilarity relative to U is the diagonal $\Delta_X := \{\langle x, x \rangle \mid x \in X\}$.

4 Generalized Selection Construction

In this section, we define the selection construction. Compared to Fine's original construction, the definition of the top T_α is modified to accommodate weakly transitive frames, quotienting out bisimilarity is postponed (and not even part of the construction), and the construction is generalized to infinitely many stages and infinite Σ , which we will use to re-derive and generalize Fine's Finite Width Theorem.

Let us fix a weakly transitive Kripke model $\mathfrak{M} = \langle X, R, V \rangle$.

Construction 4.1. We define a decreasing sequence of sets $X_\alpha \subseteq X$ and an increasing sequence of upsets U_α in the restriction of $\mathfrak{M}$ to X_α by transfinite induction as follows.

For any ordinal α we write

- $\mathfrak{M}_\alpha = \langle X_\alpha, R_\alpha, V_\alpha \rangle$ for the restriction of $\mathfrak{M}$ to X_α ,
- $\equiv_\alpha$ for bisimilarity relative to U_α on $\mathfrak{M}_\alpha$,
- $\tilde{\mathfrak{M}}_\alpha = \langle \tilde{X}_\alpha, \tilde{R}_\alpha, \tilde{V}_\alpha \rangle$ for the quotient of $\mathfrak{M}_\alpha$ through $\equiv_\alpha$, and
- q_α for the natural quotient map from $\mathfrak{M}_\alpha$ to $\tilde{\mathfrak{M}}_\alpha$.

Define $X_0 := X$ and $U_0 := \emptyset$. For the successor step, suppose X_α and U_α are already defined. Let $T_\alpha := q_\alpha^{-1}(\text{Top}(q_\alpha(U_\alpha)))$, where $\text{Top}(q_\alpha(U_\alpha))$ is taken in the model $\tilde{\mathfrak{M}}_\alpha$. Define

$$U_{\alpha+1} := T_\alpha \cup U_\alpha \quad \text{and} \quad X_{\alpha+1} := X_\alpha \setminus \mathcal{E}(\mathfrak{M}_\alpha, U_{\alpha+1}).$$

Notice that $U_\alpha \subseteq U_{\alpha+1} \subseteq X_{\alpha+1} \subseteq X_\alpha$ as required. Moreover, $U_{\alpha+1}$ is an upset in $\mathfrak{M}_\alpha$ by Lemma 3.7.

For the limit step, define

$$U_\lambda := \bigcup_{\alpha < \lambda} U_\alpha \quad \text{and} \quad X_\lambda := X'_\lambda \setminus \mathcal{E}(\mathfrak{M}|_{X'_\lambda}, U_\lambda) \quad \text{where} \quad X'_\lambda := \bigcap_{\alpha < \lambda} X_\alpha,$$

and notice that $U_\lambda \subseteq X_\lambda$ is an upset in $\mathfrak{M}_\lambda$ as required.

If $X_\alpha = U_\alpha$ then we say that the construction *has terminated* at stage α . We say that the construction *terminates* if it has terminated at stage α for some ordinal α . If $X_\alpha \neq U_\alpha$ but there exists $\beta < \alpha$ such that $U_\alpha = U_\beta$ and $X_\alpha = X_\beta$, we say that the construction is *stuck* at stage α . Finally, if neither is the case, we say the construction *progresses* at stage α .

Clearly, the construction either eventually terminates or eventually gets stuck.

Lemma 4.2. *Let α and β be ordinals with $\alpha \leq \beta$. Then*

- (i) *if the construction has terminated at stage α , then it has terminated at stage β ,*
- (ii) *if the construction is stuck at stage α , then it is stuck at stage β ,*
- (iii) *if the construction progresses at stage β , then it progresses at stage α , and*
- (iv) *if α is a cardinal strictly larger than the cardinality of X , then the construction has either terminated or is stuck at stage α .*

Therefore, we want a sufficient criterion for the construction not to get stuck. We will give such a criterion in the next section. First, we state some simple properties about the construction.

Lemma 4.3. *Let α be an ordinal.*

- (i) $U_\alpha = \bigcup \{T_\beta \mid \beta < \alpha\}$ and $\tilde{\mathfrak{M}}_\alpha$ is collapsed relative to $q_\alpha(U_\alpha)$.
- (ii) If β is an ordinal with $\alpha \leq \beta$, and $x, y \in U_\alpha$, then $x \equiv_\alpha y$ iff $x \equiv_\beta y$.

(iii) If $x, y \in U_{\alpha+1}$ and $x \rightleftharpoons_{\alpha} y$, then $x \rightleftharpoons_{\alpha+1} y$.

(iv) If $x, y \in U_{\alpha}$ such that $y \in R(x)$ and $x \equiv_{\Sigma \cdot \gamma} y$, then there exists $\beta < \alpha$ such that $x, y \in T_{\beta}$.

Proof. (i) is clear. For (ii) note that both $\rightleftharpoons_{\alpha}$ and $\rightleftharpoons_{\beta}$ restricted to U_{α} are equal and for (iii) additionally that $\rightleftharpoons_{\alpha}$ does not identify points in U_{α} with points in T_{α} .

For (iv), note that by (i) there exist $\beta, \gamma < \alpha$ such that $x \in T_{\beta}$ and $y \in T_{\gamma}$. Since $U_{\beta+1}$ is upward closed in $\mathfrak{M}_{\beta}$ and $x \in T_{\beta} \subseteq U_{\beta+1}$, we get $y \in U_{\beta+1}$ and hence $\gamma \leq \beta$. If the inequality were strict, then $x \in \mathcal{E}(\mathfrak{M}_{\gamma}, U_{\gamma+1})$, contradicting that $x \in T_{\beta} \subseteq X_{\gamma+1}$. $\square$

Lemma 4.4. *For any ordinal α , the set X_{α} is a Σ -subset of $\mathfrak{M}$.*

Proof. For a successor ordinal α , define $X'_{\alpha} := X_{\alpha-1}$, and for a limit ordinal λ , let X'_{λ} be as in the construction. Write $\mathfrak{M}'_{\beta}$ for the restriction of $\mathfrak{M}$ to X'_{β} . We prove by induction on ordinals α that X'_{α} is a Σ -subset of $\mathfrak{M}$. Suppose, as induction hypothesis, that X'_{β} is a Σ -subset of $\mathfrak{M}$ for every $\beta < \alpha$. First, let $x \in X \setminus X'_{\alpha}$, say $x \in \mathcal{E}(\mathfrak{M}'_{\beta}, U_{\beta})$ for some $\beta < \alpha$. By the induction hypothesis and Proposition 3.5, $\mathcal{E}(\mathfrak{M}'_{\beta}, U_{\beta}) \subseteq \mathcal{E}(\mathfrak{M}, U_{\beta})$. Since $U_{\beta} \subseteq X'_{\alpha}$, we conclude that $x \in \mathcal{E}(\mathfrak{M}, X'_{\alpha})$.

Second, let $\varphi \in \Sigma$, $x \in X'_{\alpha}$ and $y \in R(x) \cap R^{\text{op}}(x)$ such that $x \in \llbracket \varphi \rrbracket_{\mathfrak{M}} \iff y \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$. If $y \in X'_{\alpha}$ then we are done. Otherwise, $y \in \mathcal{E}(\mathfrak{M}'_{\beta}, U_{\beta})$ for some $\beta < \alpha$. Using the induction hypothesis and Proposition 3.5, there exists $y' \in U_{\beta} \cap R(y)$ with $y \in \llbracket \varphi \rrbracket_{\mathfrak{M}} \iff y' \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$. Suppose $y' \notin R(x)$. Then, by weak transitivity, $y' = x$, yielding $x \in U_{\beta}$. Since U_{β} is an upset in $\mathfrak{M}'_{\beta}$, we get $y \in U_{\beta}$, but this contradicts $y \in \mathcal{E}(\mathfrak{M}'_{\beta}, U_{\beta})$. Therefore $y' \in R(x)$. $\square$

The next two lemmata will be used in the proof of the Finite Width Theorem.

Lemma 4.5. *For any ordinal α , the restriction of $\tilde{\mathfrak{M}}_{\alpha}$ to $q_{\alpha}(U_{\alpha})$ is conversely pre-well-founded.*

Proof. By induction on α . For $\alpha = 0$ this is trivial. The limit step is routine.

For the successor step, suppose that the restriction of $\tilde{\mathfrak{M}}_{\alpha}$ to $U := q_{\alpha}(U_{\alpha})$ is conversely pre-well-founded. Then clearly also the restriction to $U \cup \text{Top}(U)$ is conversely pre-well-founded. Note that, by construction, $q_{\alpha}(U_{\alpha+1}) = U \cup \text{Top}(U)$. However, by Lemma 4.3 (iii), $\tilde{\mathfrak{M}}_{\alpha+1}$ restricted to $q_{\alpha+1}(U_{\alpha+1})$ is a p-morphic image of $\tilde{\mathfrak{M}}_{\alpha}$ restricted to $q_{\alpha}(U_{\alpha+1})$. But the p-morphic image of a conversely pre-well-founded frame is conversely pre-well-founded. $\square$

Lemma 4.6. *For any ordinal α , no $\tilde{x} \in q_{\alpha}(U_{\alpha})$ is Σ -eliminable in $\tilde{\mathfrak{M}}_{\alpha}$.*

Proof. Let $x \in U_{\alpha}$. Then there exists $\beta < \alpha$ such that $x \in T_{\beta}$. As $U_{\beta} \cup T_{\beta}$ is an upset in $\mathfrak{M}_{\beta}$, we get $R_{\beta}(x) \subseteq U_{\beta} \cup T_{\beta}$. We show that $q_{\alpha}(x)$ is not Σ -eliminable.

If for every $\varphi \in \Sigma$, there exists a $y \in R(x) \cap U_{\beta}$ with $x \in \llbracket \varphi \rrbracket \iff y \in \llbracket \varphi \rrbracket$, then x is Σ -eliminable w.r.t. U_{β} , a contradiction. So there is a $\varphi \in \Sigma$ such that for every $y \in R_{\beta}(x)$ with $x \in \llbracket \varphi \rrbracket \iff y \in \llbracket \varphi \rrbracket$, we have $y \in T_{\beta}$. Then $q_{\beta}(y) \in q_{\beta}(T_{\beta}) = \text{Top}(q_{\beta}(U_{\beta}))$, yielding $q_{\beta}(x) \in \tilde{R}_{\beta}(q_{\beta}(y))$. It follows from Lemma 4.3 (ii) and (iii) that $q_{\alpha}(x) \in \tilde{R}_{\alpha}(q_{\alpha}(y))$. We conclude that, since q_{α} is a p-morphism, for all $\tilde{y} \in \tilde{R}_{\alpha}(q_{\alpha}(x))$ with $q_{\alpha}(x) \in \llbracket \varphi \rrbracket \iff \tilde{y} \in \llbracket \varphi \rrbracket$, we have $q_{\alpha}(x) \in \tilde{R}_{\alpha}(\tilde{y})$. Hence, $q_{\alpha}(x)$ is not Σ -eliminable. $\square$

5 Covering via Stability and Bisimulation

In this section we prove that, in Construction 4.1, if the restriction of $\tilde{\mathfrak{M}}_\alpha$ to $q_\alpha(U_\alpha)$ does not contain an infinite anti-chain, then T_α covers $X_\alpha \setminus U_\alpha$. In particular, it then follows that the construction is not stuck at stage α . To prove this result we first need to develop a theory of stable points, and construct a bisimulation equivalence on them.

We first introduce the general notion of stability w.r.t. an equivalence relation or function.

Definition 5.1 (Stable). Let $\mathfrak{F} = \langle X, R \rangle$ be a frame, $\equiv$ an equivalence relation on X , and $Y \subseteq X$. Then a point $y \in Y$ is called $\equiv$ -stable in Y iff

$$\forall y' \in R^*(y) \cap Y. y \equiv y'.$$

If $f: X \rightarrow Z$ is a function, then a point $y \in Y$ is called f -stable in Y if it is $\equiv_f$ -stable in Y , where $\equiv_f$ is the binary equivalence relation on $\text{dom}(f)$ induced by f , i.e. $y \equiv_f y'$ iff $f(y) = f(y')$.

Now we define the set of stable points we are interested in, and an equivalence relation on it that will turn out to be a bisimulation equivalence. This set and bisimulation lie at the heart of both the original version of Fine's selective filtration theorem and Fine's finite width theorem.

Fix a weakly transitive Kripke model $\mathfrak{M} = \langle X, R, V \rangle$ with an upset $U \subseteq X$ and $D := X \setminus U$.

Construction 5.2. We define

$$\text{Stab} := \{x \in D \mid x \text{ is } \text{sp}_U\text{- and } (\text{ty}_{\mathbb{P}} \circ \text{sp}_D)\text{-stable in } D\}.$$

Let $\equiv_{\text{ty}_{\mathbb{P}}}$ and $\equiv_U$ be the equivalence relations induced by $\text{ty}_{\mathbb{P}} \circ \text{sp}_D$ and the characteristic function of U , respectively, and define $\equiv_\sigma$ to be the intersection of these two equivalence relations with $\equiv_{\mathbb{P}\text{-ty}}$ and $\equiv_{U\text{-sp}}$, restricted to $\text{Stab} \cup U$. Note that $\text{Stab} \cup U$ is an upset.

The set Stab consists of points in D such that while going up in the accessibility relation and staying in D neither the U -span nor the set of $\mathbb{P}$ -types seen in D change.

Lemma 5.3. *The equivalence relation $\equiv_\sigma$ is a bisimulation equivalence on $\text{Stab} \cup U$.*

Proof. Since $\equiv_\sigma \subseteq \equiv_{\mathbb{P}\text{-ty}}$, it preserves atomic propositions. For the back- and forth-conditions, let $x \equiv_\sigma y$ and $x' \in R(x)$. Suppose $x' \in U$. Then $x' \in \text{sp}_U(x) = \text{sp}_U(y)$, hence $x' \in R(y)$. Note that $x' \equiv_\sigma x'$, by reflexivity, since $x' \in U \subseteq \text{Stab} \cup U$.

Otherwise, $x' \in D$ and, since D is a downset, also $x \in D$. Hence $x \in \text{Stab}$ and therefore $y \in \text{Stab}$ as well. Then $\text{ty}_{\mathbb{P}}(x') \in (\text{ty}_{\mathbb{P}} \circ \text{sp}_D)(x) = (\text{ty}_{\mathbb{P}} \circ \text{sp}_D)(y)$, so there exists $y' \in R(y) \cap D$ with $x' \equiv_{\mathbb{P}\text{-ty}} y'$. By the sp_U -stability of x , we get $x \equiv_{U\text{-sp}} x'$ and analogously $y \equiv_{U\text{-sp}} y'$, so we conclude that $x' \equiv_{U\text{-sp}} x \equiv_{U\text{-sp}} y \equiv_{U\text{-sp}} y'$ as desired. The argument for $\equiv_{\text{ty}_{\mathbb{P}}}$ is analogous. Finally, since $x', y' \in D$ we obtain $x' \equiv_U y'$. $\square$

For technical reasons we will work with $\text{sp}_D(\text{Stab})$ instead of Stab . This effectively removes from Stab the irreflexive points 'at the bottom'. The following lemma, which easily follows from the fact that $\text{Stab} \cup U$ is an upset, shows that this is not a problem.

Lemma 5.4. $\text{sp}_D(\text{Stab}) \subseteq \text{Stab}$, and $\text{sp}_D(\text{Stab}) \cup U$ is an upset.

We use Construction 5.2, in particular the set $\text{sp}_D(\text{Stab})$ and the restriction of $\equiv_\sigma$ to it, as a tool for reasoning about $\text{Top}(U)$.

Lemma 5.5. *If $\mathbb{P}$ is finite, and the restriction of $\mathfrak{M}$ to U has the finite cover property, then $\text{sp}_D(\text{Stab}) \cup \text{Top}(U)$ covers D .*

Proof. Let $T := \text{sp}_D(\text{Stab}) \cup \text{Top}(U)$ and $x \in D \setminus T$. Suppose that $R(x) \cap \text{Top}(U) = \emptyset$. Then each $y \in \text{sp}_D(x)$ has a strict successor in D . We will first show that x has a sp_U -stable successor in D .

Suppose for a contradiction that x has no sp_U -stable successor in D . Then, since, by weak-transitivity, sp_U is monotone decreasing with respect to R , using the axiom of dependent choice, we can construct an injective monotone sequence $f: \langle \mathbb{N}, \leq \rangle \rightarrow \langle D, R \cap D^2 \rangle$ such that $\text{sp}_U \circ f$ is injective and for each $i \in \mathbb{N}$, $\text{sp}_U(f(i+1)) \subseteq \text{sp}_U(f(i))$. Hence, since $\mathfrak{M}$ restricted to U has the finite cover property, there exist $i, j \in \mathbb{N}$ with $i < j$ such that $\text{sp}_U(f(i)) = \text{sp}_U(f(j))$, a contradiction.

Therefore, by weak transitivity, we may assume that x is sp_U -stable. It remains to show that x has a $(\text{ty}_{\mathbb{P}} \circ \text{sp}_D)$ -stable successor which will, in particular, be sp_U -stable. Since $\mathbb{P}$ is finite, $\text{ty}_{\mathbb{P}}$ attains only finitely many values. Moreover, by weak transitivity, the function sp_D is strictly decreasing on D with respect to the strict successor relation. Hence, there exists a $(\text{ty}_{\mathbb{P}} \circ \text{sp}_D)$ -stable $y' \in \text{sp}_D(x)$. Now there exists a strict successor $y \in D$ of y' which is also $(\text{ty}_{\mathbb{P}} \circ \text{sp}_D)$ -stable, and, by weak transitivity, $y \in \text{sp}_D(x)$. $\square$

Lemma 5.6. *If $\mathfrak{M}$ is collapsed relative to U , then $\text{sp}_D(\text{Stab}) \subseteq \text{Top}(U)$.*

Proof. Let $x' \in \text{sp}_D(\text{Stab})$ and $y \in \text{sp}_D(x')$. We prove that $y \in R^{\text{op}}(x')$, i.e. $R(y, x')$. Pick $x \in \text{Stab}$ such that $x' \in \text{sp}_D(x)$. Then $y \in R^*(x)$, so by stability $x \equiv_{\text{ty}_{\text{sp}}} y$. Therefore, as $x' \in \text{sp}_D(x)$, there exists $y' \in \text{sp}_D(y)$ with $x' \equiv_{\mathbb{P}\text{-ty}} y'$. Now $x' \equiv_{\mathbb{P}\text{-ty}} y'$, and by stability $x' \equiv_{U\text{-sp}} y'$ and $x' \equiv_{\text{ty}_{\text{sp}}} y'$. Since x' and y' are both in D , $x' \equiv_U y'$. We conclude that $x' \equiv_{\sigma} y'$.

By Lemma 5.3, $\equiv_{\sigma}$ is a bisimulation equivalence on the upset $\text{Stab} \cup U$. So it extends to a bisimulation equivalence on X that, in particular, does not identify points in U and D . Hence x' and y' are bisimilar relative to U and, since $\mathfrak{M}$ is collapsed relative to U , it follows that $x' = y'$. But we know that $R(y, y')$, hence $R(y, x')$, and we are done. $\square$

These two lemmata combine into the sought criterion for Construction 4.1 to not get stuck.

Proposition 5.7. *Suppose that $\mathbb{P}$ is finite and let α be an ordinal. Moreover, suppose that $\tilde{\mathfrak{M}}_{\alpha}$ restricted to $q_{\alpha}(U_{\alpha})$ in Construction 4.1 is of finite width. Then T_{α} covers $X_{\alpha} \setminus U_{\alpha}$, and, in particular, the construction does not get stuck at stage α .*

Proof. We apply Construction 5.2 to $\tilde{\mathfrak{M}}_{\alpha}$ with upset $U := q_{\alpha}(U_{\alpha})$, defining Stab .

By assumption $\tilde{\mathfrak{M}}_{\alpha}$ restricted to U is of finite width, and by Lemma 4.5 it is conversely pre-well-founded. Therefore, it has the finite cover property and, by Lemma 5.5, $\text{sp}_D(\text{Stab}) \cup \text{Top}(U)$ covers $D := \tilde{X}_{\alpha} \setminus U$. Moreover, $\tilde{\mathfrak{M}}_{\alpha}$ is collapsed relative to U . Hence, by Lemma 5.6, $\text{sp}_D(\text{Stab}) \subseteq \text{Top}(U)$, so $\text{Top}(U)$ covers D . Therefore, $T_{\alpha} = q_{\alpha}^{-1}(\text{Top}(U))$ covers $q_{\alpha}^{-1}(D) = q_{\alpha}^{-1}(\tilde{X}_{\alpha} \setminus q_{\alpha}(U_{\alpha})) = X_{\alpha} \setminus U_{\alpha}$, where we use that $\Rightarrow_{\alpha}$ by definition does not identify points in U_{α} with points outside of it. $\square$

Together with Lemmata 4.2 and 4.5, we obtain the following result.

Corollary 5.8. *If, in Construction 4.1, $\mathbb{P}$ is finite and $\mathfrak{M}$ is of finite width, then the construction has terminated at some stage α and $\tilde{\mathfrak{M}}_{\alpha}$ has the finite cover property.*

6 Definability via the Diagram Bisimulation

For the selection theorem, we need one more technical result: for finite Σ , the sets U_n and X_n from Construction 4.1 are definable for each $n \in \mathbb{N}$. To obtain this result, we establish a criterion which ensures that Top is definable and introduce another bisimulation equivalence on Top , which we will call the

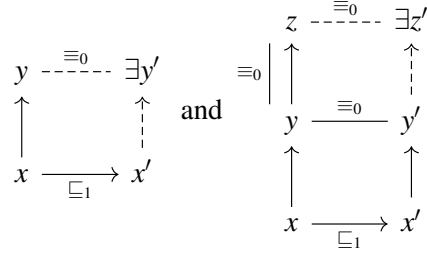


Figure 1: Diagram describing the definition of $\sqsubseteq_1$ in Construction 6.1. All variables live in D .

diagram bisimulation. This relation is defined in three stages, where each next stage is defined in terms of the previous stages by conditions resembling the back- and forth-condition for bisimulations.

We fix a weakly transitive Kripke model $\mathfrak{M} = \langle X, R, V \rangle$ with upset $U \subseteq X$ and $D := X \setminus U$.

Construction 6.1. Let $\equiv_0$ be the intersection $\equiv_{\mathbb{P}\text{-ty}} \cap \equiv_{U\text{-sp}}$ restricted to D and define the relation $\sqsubseteq_1$ on D by $x \sqsubseteq_1 x'$ if and only if the following two conditions hold:

- (C1) for every $y \in \text{sp}_D(x)$, there exists $y' \in \text{sp}_D(x')$ such that $y \equiv_0 y'$, and
- (C2) for all $y \in \text{sp}_D(x)$, $z \in \text{sp}_D(y)$, $y' \in \text{sp}_D(x')$ with $y' \equiv_0 y \equiv_0 z$, there exists $z' \in \text{sp}_D(y')$ with $z \equiv_0 z'$.

The conditions are depicted in Figure 1. We define $\equiv_1$ to be the intersection of $\sqsubseteq_1$ and its converse, and write $\equiv_{01}$ for the intersection $\equiv_0 \cap \equiv_1$. Note that, in general, only $\text{dom}(\equiv_1) \subseteq \text{dom}(\sqsubseteq_1)$.

The next lemma easily follows from the transitivity of $\equiv_0$ and the definition of $\sqsubseteq_1$.

Lemma 6.2. *The relation $\sqsubseteq_1$ is transitive, and $\equiv_1$ is an equivalence relation on its domain.*

The next lemma shows that the domain of $\equiv_1$ contains $\text{Top}(U)$.

Lemma 6.3. $\text{Top}(U) \subseteq \text{dom}(\equiv_1)$.

Proof. Let $x \in \text{Top}(U)$. Note that it suffices to show $x \sqsubseteq_1 x$. We check (C1) and (C2) for $\sqsubseteq_1$. For (C1) note that if $y \in \text{sp}_D(x)$, then, since $\equiv_0$ is reflexive, we have $y \equiv_0 y$.

For (C2) assume that $y, y' \in \text{sp}_D(x)$ and $z \in \text{sp}_D(y)$ with $y' \equiv_0 y \equiv_0 z$. There are two cases. If $y = y'$, then we are done. Otherwise, $y \neq y'$ and, since $x \in \text{Top}(U)$, $x \in R(y')$. By weak transitivity, $y \in R(y')$, and by assumption $z \equiv_0 y$. $\square$

As the aim of this section is to prove a definability result for $\text{Top}(U)$, we are interested in the definability of the equivalence classes of the introduced relations.

Lemma 6.4. *Let $\mathfrak{M} = \langle X, R, V \rangle$ be a model, $x \in X$, and let $Y \subseteq X$ be such that Y is finite and every point in Y is definable. Then $[x]_{Y\text{-sp}}$ is definable in $\mathfrak{M}$.*

Proof. Notice that $[x]_{Y\text{-sp}} = \Box(Y \rightarrow \text{sp}_Y(x)) \cap \bigcap \{\Diamond\{y\} \mid y \in \text{sp}_Y(x)\}$, which is definable, since the intersection is finite and Y , $\text{sp}_Y(x)$, and each $\{y\}$ are definable. $\square$

The next lemma is the first ingredient to our definability result.

Lemma 6.5. *Suppose that U and $\mathbb{P}$ are finite and $I := \text{dom}(\equiv_1)$.*

- (i) $D/\equiv_0$ is finite, and if every point in U is definable, then each set in $D/\equiv_0$ is definable.
- (ii) $I/\equiv_1$ is finite, and if every point in U is definable, then each set in $I/\equiv_1$ is definable.

Proof. (i) Note that both $\text{ty}_{\mathbb{P}}$ and sp_U attain only finitely many values on D , so $D/\equiv_0$ is finite. For the second part suppose that every point in U is definable. The set $\mathbb{P}$ is finite, so the $\equiv_{\mathbb{P}\text{-ty}}$ -equivalence class of x is definable, since we have

$$[x]_{\mathbb{P}\text{-ty}} = \bigcap (\{ \llbracket p \rrbracket \mid p \in \text{ty}_{\mathbb{P}}(x) \} \cup \{ \llbracket \neg p \rrbracket \mid p \in \mathbb{P} \setminus \text{ty}_{\mathbb{P}}(x) \}).$$

Moreover, since U is finite and each point in it is definable, by Lemma 6.4, $[x]_{Y\text{-sp}}$ is definable. Hence, $[x]_0 = [x]_{\mathbb{P}\text{-ty}} \cap [x]_{Y\text{-sp}} \cap D$ is definable.

(ii) We will show that there are only finitely many $\sqsubseteq_1$ -upsets which under the additional assumption are all definable. Then, the same holds for $\sqsubseteq_1$ -downsets, since every downset is the complement of an upset, and it follows that there are only finitely many $\equiv_1$ -equivalence classes which under the additional assumption are all definable, since each such equivalence class is the intersection of a $\sqsubseteq_1$ -upset with a $\sqsubseteq_1$ -downset. For $x \in D$ we define the two sets

$$\begin{aligned} Y_x &:= D \cap \bigcap \{ \diamond[y]_0 \mid y \in \text{sp}_D(x) \}, \\ Y'_x &:= D \cap \bigcap \{ \square([y]_0 \rightarrow \diamond[z]_0) \mid y \in \text{sp}_D(x), z \in \text{sp}_D(y), y \equiv_0 z \}. \end{aligned}$$

It is straightforward to check that Y_x is the set of points x' such that condition (C1) of $x \sqsubseteq_1 x'$ is satisfied, and Y'_x is the set of points x' such that condition (C2) of $x \sqsubseteq_1 x'$ is satisfied. Hence $Y_x \cap Y'_x$ is the $\sqsubseteq_1$ -upset generated by x . It follows, by (i), that there are only finitely many $\equiv_0$ -equivalence classes, so Y_x and Y'_x attain only finitely many values for $x \in D$. Therefore, there are finitely many point-generated $\sqsubseteq_1$ -upsets and as $\sqsubseteq_1$ -upset is a union of point-generated upsets, there are only finitely many $\sqsubseteq_1$ -upset.

Finally, if each point in U is definable, then, by (i), it follows straightforwardly that for each $x \in D$ the sets Y_x and Y'_x are definable, yielding that every $\sqsubseteq_1$ -upset is definable. $\square$

Using the relations $\equiv_0$ and $\equiv_{01}$ we can construct a full bisimulation.

Construction 6.6 (extending Construction 6.1). Let $\equiv_{\delta}$ be the largest relation contained in $\equiv_{01}$ such that for all $x \equiv_{\delta} x'$, $y \in \text{sp}_D^*(x)$ and $y' \in \text{sp}_D^*(x')$, if $y \equiv_0 y'$ then $y \equiv_1 y'$. This definition is diagrammatically depicted on the right, where arrows with $*$ at the tail denote edges in R^* .

$$\begin{array}{ccc} y & \xrightarrow{\equiv_1} & y' \\ \uparrow & \equiv_0 & \uparrow \\ x & \xrightarrow{\equiv_{01}} & x' \end{array}$$

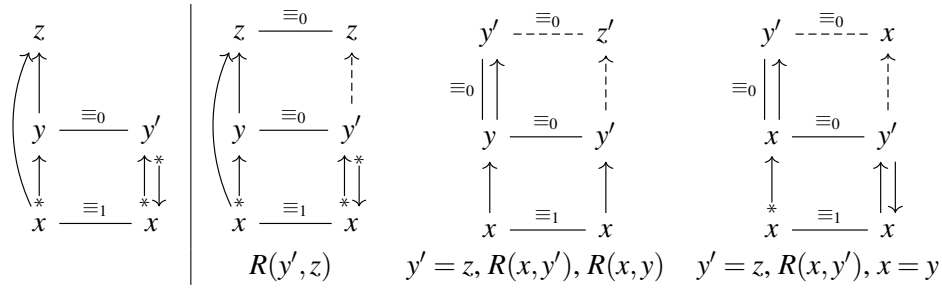
Lemma 6.7. (i) Let $x, x' \in D$, $y \in \text{sp}_D(x)$ and $y' \in \text{sp}_D(x')$ with $x \equiv_{\delta} x'$ and $y \equiv_0 y'$. Then $y \equiv_{\delta} y'$. In particular, $U \cup \text{dom}(\equiv_{\delta})$ is an upset.

(ii) The relation $\equiv_{\delta} \cup \Delta_U$ is a bisimulation equivalence on $U \cup \text{dom}(\equiv_{\delta})$.

Proof. (i) By the defining property of $\equiv_{\delta}$, we see that $y \equiv_1 y'$, hence $y \equiv_{01} y'$. Suppose $z \in \text{sp}_D^*(y)$ and $z' \in \text{sp}_D^*(y')$ with $z \equiv_0 z'$. By weak transitivity, $z \in \text{sp}_D^*(x)$ and $z' \in \text{sp}_D^*(x')$. Then, since $x \equiv_{\delta} x'$, it follows that $z \equiv_1 z'$. We conclude that $y \equiv_{\delta} y'$.

(ii) Clearly $\equiv_{\delta} \cup \Delta_U$ is an equivalence relation. For the preservation of atomic propositions, notice that $\equiv_{\delta} \subseteq \equiv_0 \subseteq \equiv_{\mathbb{P}\text{-ty}}$. For the back- and forth-conditions, suppose $x \equiv_{\delta} x'$ and $y \in R(x)$. If $x \in U$ or $x' \in U$ then $x = x'$, and hence $y \in R(x')$. Otherwise, $x, x' \in \text{dom}(\equiv_{\delta})$. If $y \in U$, then $y \equiv_{\delta} y$ and $y \in \text{sp}_U(x) = \text{sp}_U(x')$, since $x \equiv_{U\text{-sp}} x'$. So suppose $y \in D$. Since $x \equiv_1 x'$, by condition (C1) of $\equiv_1$, there exists $y' \in \text{sp}_D(x')$ such that $y \equiv_0 y'$ and (i) yields $y \equiv_{\delta} y'$. $\square$

The usefulness of this bisimulation equivalence $\equiv_{\delta}$ lies in the fact that its equivalence classes are, under mild conditions on U , definable and finitely many, like those of $\equiv_0$ and $\equiv_1$.

Figure 2: Case 1 ($R(x, z)$) in the proof of (C1) in Lemma 6.9.

Lemma 6.8. Suppose that U and $\mathbb{P}$ are finite. Then there are only finitely many $\equiv_\delta$ -equivalence classes. If, in addition, each point in U is definable, then each $\equiv_\delta$ -equivalence class is definable and, in particular, $\text{dom}(\equiv_\delta)$ is definable.

Proof. Notice that for $x \in \text{dom}(\equiv_\delta)$, we have $[x]_\delta = [x]_{01} \cap \bigcap \{ \Box([y]_0 \rightarrow [y]_1) \mid y \in \{x\} \cup \text{sp}_D^*(x) \}$. By Lemma 6.5, all the involved sets admit only finitely many values, and under the additional assumption they are definable. $\square$

The next lemma shows that $\text{Top}(U)$ is contained in $\text{dom}(\equiv_\delta)$.

Lemma 6.9. $\text{Top}(U) \subseteq \text{dom}(\equiv_\delta)$.

Proof. Let $x \in \text{Top}(U)$. By Lemma 6.3, we have $\text{Top}(U) \subseteq \text{dom}(\equiv_1)$.

Hence, for proving that $x \in \text{dom}(\equiv_\delta)$, we need to show that for all $y, y' \in \text{sp}_D^*(x)$, if $y \equiv_0 y'$ then $y \equiv_1 y'$. So let $y, y' \in \text{sp}_D^*(x)$ with $y \equiv_0 y'$. It suffices to show $y \sqsubseteq_1 y'$, since $\equiv_0$ is symmetric. So we need to check the two defining properties (C1) and (C2).

For (C1), note first that, by definition of $\text{Top}(U)$, we have $x \in \text{sp}_D^*(y')$ and $y, y' \in \text{Top}(U)$. Let $z \in R(y)$. We show that there exists a $z' \in \text{sp}_D(y')$ with $z \equiv_0 z'$. We distinguish two cases.

Case 1. Suppose $R(x, z)$. This case is depicted diagrammatically in Figure 2.

If $R(y', z)$, then we are done, since by reflexivity of $\equiv_0$, we get $z \equiv_0 z$. So suppose $\neg R(y', z)$. Since $R^*(y', x)$ and $R(x, z)$, by weak transitivity, we get $y' = z$ and $y' \neq x$. Hence, $R(x, y')$, $R(y', x)$, and $y \equiv_0 y' = z$. We distinguish two subcases.

- If $R(x, y)$, then we can apply (C2) to find $z' \in \text{sp}_D(y')$ with $z \equiv_0 z'$.
- If $\neg R(x, y)$, then, since $R^*(x, y)$, we get $x = y$; so for $z' := x$, we get $z \equiv_0 z'$ and $R(y', z')$.

Case 2. Suppose $\neg R(x, z)$. By weak transitivity $x = z$. Since $y \in \text{Top}(U)$ and $R(y, z)$, we get $R(x, y)$. This is depicted in Figure 3. We distinguish two subcases.

- Suppose $R(x, y')$. Since $x \in \text{Top}(U)$, we get $R(y', x)$. So $z' := x$ closes the diagram.
- Suppose $\neg R(x, y')$. Since $R^*(x, y')$, we obtain $x = y'$, so $z' := y$ closes the diagram.

For (C2) let $z \in \text{sp}_D(y)$, $w \in \text{sp}_D(z)$ and $z' \in \text{sp}_D(y')$ with $z' \equiv_0 z \equiv_0 w$. We want to show that there exists $w' \in \text{sp}_D(z')$ such that $w \equiv_0 w'$, i.e., we want to complete the diagram on the left in Figure 4. We distinguish four cases which are depicted diagrammatically in Figure 4.

Case 1. If $R(x, z)$ and $R(x, z')$, then we can apply (C2) directly to find $w' \in \text{sp}_D(z')$ with $w \equiv_0 w'$.

Case 2. If $\neg R(x, z)$ and $R(x, z')$, then, by weak transitivity, $x = z$, so $x \equiv_0 w$. Moreover, since $x \in \text{Top}(U)$, we also get $R(z', x)$, so $w' := x$ closes the diagram.

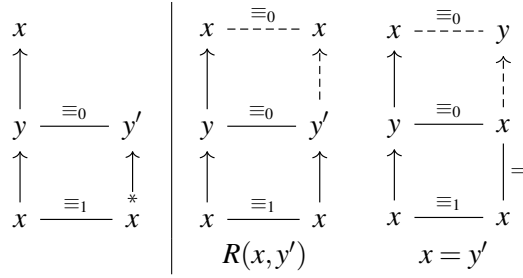
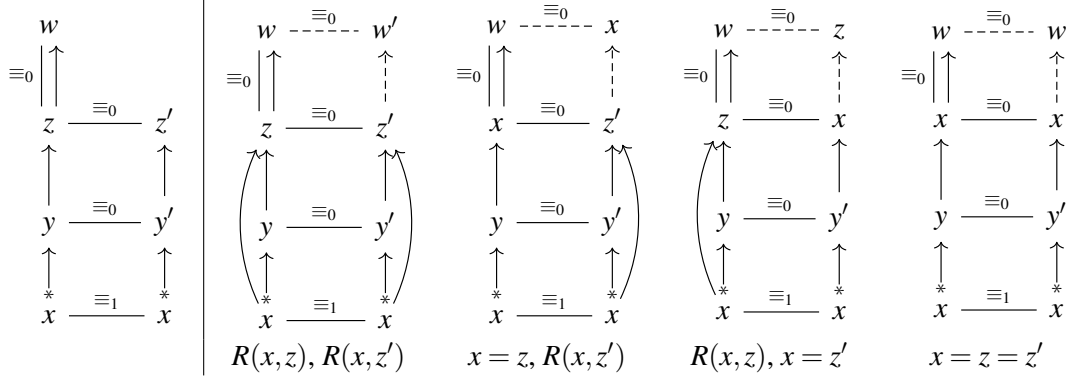
Figure 3: Case 2 ($x = z$) in the proof of (C1) in Lemma 6.9.

Figure 4: The four cases in the proof of (C2) in Lemma 6.9.

Case 3. If $R(x, z)$ and $\neg R(x, z')$, then, by weak transitivity, $x = z'$, so $w' := z$ closes the diagram.

Case 4. If $\neg R(x, z)$ and $\neg R(x, z')$, then, by weak transitivity, $x = z = z'$, so $w' := w$ closes the diagram. $\square$

In conclusion, we find the following criterion for the definability of Top .

Lemma 6.10. *Suppose $\mathbb{P}$ is finite, U is finite, and $\mathfrak{M}$ is collapsed relative to U . Then $\text{Top}(U)$ is finite. If, additionally, every point in U is definable, then each point in $\text{Top}(U)$ is definable.*

Proof. By Lemma 6.7, $\equiv_\delta \cup \Delta_U$ is a bisimulation equivalence on $\mathfrak{M}$ restricted to the upset $U \cup \text{dom}(\equiv_\delta)$. Since $\mathfrak{M}$ is collapsed relative to U , it follows that $\equiv_\delta$ is the diagonal on $\text{dom}(\equiv_\delta)$. Therefore, by Lemma 6.8, $\text{dom}(\equiv_\delta)$ is finite and under the extra assumption each point in it is definable. Moreover, by Lemma 6.9, $\text{Top}(U) \subseteq \text{dom}(\equiv_\delta)$, so the same is true for $\text{Top}(U)$. $\square$

Applied to the selection construction, we obtain the following result for the finite stages.

Proposition 6.11. *If the set of atomic propositions $\mathbb{P}$ is finite, then, in Construction 4.1, $q_n(U_n)$ is finite and X_n is definable in $\mathfrak{M}$ for each $n \in \mathbb{N}$.*

Proof. We prove by induction on $n \in \mathbb{N}$ that $q_n(U_n)$ is finite, and that U_n , X_n , and the elements of $U_{n+1}/\equiv_{n+1}$ are definable in $\mathfrak{M}$. For $n = 0$ this holds vacuously.

Suppose that $\tilde{U}_n = q_n(U_n)$ is finite, and that U_n , X_n , and the elements of $U_n/\equiv_n$ are definable in $\mathfrak{M}$. Let $\mathfrak{M}_n^+$ be the extension of $\mathfrak{M}_n$ with, for every $A \in U_n/\equiv_n$, a fresh atomic proposition p_A which evaluates to A . Clearly, $\mathfrak{M}_n^+$ is a definable submodel of a definable variant of $\mathfrak{M}$. Moreover, since $\equiv_n$ is a bisimulation equivalence on $\mathfrak{M}_n$ and each $A \in U_n/\equiv_n$ is by definition closed under it, $\equiv_n$ is a bisimulation equivalence on $\mathfrak{M}_n^+$. Write $\tilde{\mathfrak{M}}_n^+$ for the quotient $\mathfrak{M}_n^+/\equiv_n$.

By definition, every point in $q_n(U_n)$ is definable in $\mathfrak{M}_n^+$, $\mathfrak{M}_n^+$ is collapsed relative to $q_n(U_n)$, and, by the induction hypothesis, $q_n(U_n)$ is finite. Therefore, by Lemma 6.10, $\text{Top}(q_n(U_n))$ is finite and every point in it is definable in $\mathfrak{M}_n^+$. Hence, T_n and the $\equiv_n$ equivalence classes in it are definable in $\mathfrak{M}_n^+$, and hence in $\mathfrak{M}$. By Lemma 4.3 (iii), each $\equiv_{n+1}$ equivalence class in U_{n+1} is the union of $\equiv_n$ equivalence classes, hence there are only finitely many such equivalence classes which all are definable. In particular, $q_{n+1}(U_{n+1})$ is finite. Finally, by Lemma 3.2, it follows that $\mathcal{E}(\mathfrak{M}_n, U_{n+1})$ is definable in $\mathfrak{M}$, and the definability of X_{n+1} follows. $\square$

7 Fine's Theorems

With the necessary theory developed, we now prove the main results of the paper. We start with the Selection Theorem.

Theorem A. *Let $\mathfrak{M}$ be a weakly transitive Kripke model and Σ a finite subformula-closed set of formulas. Then there exists a definable Σ -sub-p-morphism from $\mathfrak{M}$ onto a finite model.*

Proof. We apply Construction 4.1 to $\mathfrak{M}$. It suffices to show that the construction terminates at stage $m := 2^{|\Sigma|}$. Then, by Lemma 4.4 together with Proposition 6.11, it follows that q_m is a Σ -sub-p-morphism onto the finite model $\mathfrak{M}_m$.

So let $n \in \mathbb{N}$. Then, by Proposition 6.11, $q_n(U_n)$ is finite, and hence, by Proposition 5.7, T_n covers $X_n \setminus U_n$. In particular the construction does not get stuck at any finite stage. Suppose for a contradiction that the construction has not terminated at stage m . Then $T_m \neq \emptyset$, so there exists $x_m \in T_m$, and inductively, we find $x_m, \dots, x_0$ such that $x_i \in T_i$ and $x_i \in R_m(x_{i+1})$ for each $i \leq m$. By Lemma 4.3 (iv), the points $x_m, \dots, x_0$ have pairwise distinct Σ -types. But there are only $m = 2^{|\Sigma|}$ distinct Σ -types, a contradiction. $\square$

Now suppose Λ is a strongly cofinal subframe logic extending **wK4**, and $\varphi \notin \Lambda$. Then there exists a weakly transitive Λ -model $\mathfrak{M}$ refuting φ . Let Σ be the set of subformulas of φ . By the previous theorem, we obtain a definable Σ -sub-p-morphism from $\mathfrak{M}$ onto a finite model $\mathfrak{N}$. But every Σ -subset of $\mathfrak{M}$ is strongly cofinal, so $\mathfrak{N}$ is a Λ -model. Moreover, by Proposition 3.5, $\mathfrak{N}$ still refutes φ . Thus, we get the claimed FMP result:

Theorem B. *Every strongly cofinal subframe logic extending **wK4** has the FMP.*

Next, we prove Fine's Finite Width Theorem for **wK4**, using our selection construction with an infinite filtration set. We will show that the resulting model is atomic, and finally that the underlying Kripke frame still has the same logic.

Let Fm be the set of all modal formulas over finitely many atomic propositions $\mathbb{P}$. We apply our generalized selection construction to a given weakly transitive model $\mathfrak{M}$ of finite width with $\Sigma := \text{Fm}$. By Corollary 5.8, the construction terminates at some stage α and the resulting model $\mathfrak{M}_\alpha$ has the finite cover property. Moreover, by Lemma 4.6, it is free of Fm -eliminable points. For models with these properties, we derive a Hennessy-Milner-like theorem.

Theorem 7.1. *Suppose $\mathbb{P}$ is finite. Let $\mathfrak{M} = \langle X, R, V \rangle$ be a weakly transitive model free of Fm -eliminable points, such that every point generated subframe has the finite cover property. Then $\equiv_{\text{Fm-ty}}$ and bisimilarity on $\mathfrak{M}$ coincide.*

In the weakly transitive case, image-finiteness is strictly stronger than point generated subframes having the finite cover property. Relaxing this hypothesis compared to the Hennessy-Milner Theorem comes at the cost of requiring the frame to be free of Fm -eliminable points.

Proof. It suffices to prove that $\equiv_{\text{Fm-ty}}$ is a bisimulation equivalence. Let $x, x', y \in X$ be such that $y \in R(x)$ and $x \equiv_{\text{Fm-ty}} x'$. For $n \in \mathbb{N}$, let φ_n be a formula of modal depth at most n such that

$$\llbracket \varphi_n \rrbracket = \bigcap \{ \llbracket \varphi \rrbracket \mid y \in \llbracket \varphi \rrbracket, \varphi \text{ is of modal depth at most } n \}.$$

Then $y \in \llbracket \varphi_n \rrbracket$ and therefore $x \in \llbracket \Diamond \varphi_n \rrbracket$. Since $x' \equiv_{\text{Fm-ty}} x$, there exists $y'_n \in R(x')$ with $y'_n \in \llbracket \varphi_n \rrbracket$. For each $n \in \mathbb{N}$, pick such a y'_n , and define $Y' := \{y'_n \mid n \in \mathbb{N}\}$. Suppose for a contradiction that there does not exist $y' \in Y'$ with $y' \equiv_{\text{Fm-ty}} y$. Then for each $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $y'_n \notin \llbracket \varphi_m \rrbracket$. Since $\mathbb{P}$ is finite, every cluster is finite up to bisimilarity, so Y' contains points in infinitely many clusters. Hence, by the finite cover property for the subframe generated by the point x' , Y' contains an infinite strictly decreasing chain, so there is an infinite set $I \subseteq \mathbb{N}$ such that for $n, m \in I$, if $n < m$ then $R(y'_m, y'_n)$ and $y'_n \notin \llbracket \varphi_m \rrbracket$.

For $n \in I$, let $s(n) := \min\{m \in I \mid n < m\}$. Then for each $n \in I$, $y'_{s(n)} \in \llbracket \Diamond(\varphi_n \wedge \neg \varphi_{s(n)}) \rrbracket$ and, since $\Diamond(\varphi_n \wedge \neg \varphi_{s(n)})$ has modal depth $s(n) + 1 \leq s(s(n))$, we get $y \in \llbracket \Diamond(\varphi_n \wedge \neg \varphi_{s(n)}) \rrbracket$. Therefore, there exists $y_n \in R(y)$ with $y_n \in \llbracket \varphi_n \wedge \neg \varphi_{s(n)} \rrbracket$. Since the cluster of y is finite up to bisimilarity, the set $J = \{n \in I \mid y \notin R(y_n)\}$ is infinite. Finally, let $\varphi \in \text{Fm}$ and suppose $y \in \llbracket \varphi \rrbracket$. Let n be the modal depth of φ and $m \in J$ with $n < m$. The point y_m is a strict successor of y in $\llbracket \varphi_m \rrbracket \subseteq \llbracket \varphi_n \rrbracket \subseteq \llbracket \varphi \rrbracket$, i.e., $y_m \in \llbracket \varphi \rrbracket$. Hence, since φ was arbitrary, y is Fm-eliminable, contradicting the hypotheses. $\square$

Since the model $\tilde{\mathfrak{M}}_\alpha$ resulting from the selection is collapsed relative to its domain, by the previous Theorem, the relation $\equiv_{\text{Fm-ty}}$ on $\tilde{\mathfrak{M}}_\alpha$ is the diagonal, i.e. $\tilde{\mathfrak{M}}_\alpha$ is differentiated. The following lemma strengthens this to atomicity.

Lemma 7.2. *Suppose that $\mathbb{P}$ is finite. Let $\mathfrak{M} = \langle X, R, V \rangle$ be a weakly transitive differentiated model that has the finite cover property and is free of Fm-eliminable points. Then $\mathfrak{M}$ is atomic.*

Proof. Let $x \in X$ and define $\text{clu}(x) := R(x) \cap R^{\text{op}}(x) \cup \{x\}$. We prove that $\{x\}$ is definable. Since x is not eliminable there exists φ_0 such that x is maximal in $\llbracket \varphi_0 \rrbracket$. By assumption, there exists a finite cover C of $\llbracket \varphi_0 \rrbracket \setminus \text{clu}(x)$. Since $\mathbb{P}$ is finite and $\mathfrak{M}$ is a weakly transitive differentiated model, $\text{clu}(x)$ is finite. Thus, by differentiation, there exists φ_x such that $\text{clu}(x) \cap \llbracket \varphi_x \rrbracket = \{x\}$ and for each $z \in C$, there exists φ_z such that $\text{clu}(x) \subseteq \llbracket \varphi_z \rrbracket$ but $x \notin \llbracket \varphi_z \rrbracket$. Consider the definable set

$$Y := \llbracket \varphi_0 \rrbracket \cap \llbracket \varphi_x \rrbracket \cap \bigcap \{ \llbracket \Box(\varphi_0 \rightarrow \varphi_z) \rrbracket \mid z \in C \}.$$

We claim that $\{x\} = Y$.

For the left-to-right inclusion, note that $x \in \llbracket \varphi_0 \rrbracket \cap \llbracket \varphi_x \rrbracket$. Moreover, since x is maximal in $\llbracket \varphi_0 \rrbracket$, for $z \in C$ and $x' \in R^*(x)$ with $x' \in \llbracket \varphi_0 \rrbracket$, we get $x' \in \text{clu}(x) \subseteq \llbracket \varphi_z \rrbracket$. Hence $x \in Y$.

For the converse, suppose that $x' \in Y \setminus \{x\}$. Then $x' \in \llbracket \varphi_0 \rrbracket \setminus \text{clu}(x)$ and x' is covered by some $z \in C$. By weak transitivity, $z \in R^*(x')$. We know by assumption that $x' \in \llbracket \Box(\varphi_0 \rightarrow \varphi_z) \rrbracket$. Hence, $z \in \llbracket \varphi_0 \rightarrow \varphi_z \rrbracket$ and $z \in C \subseteq \llbracket \varphi_0 \rrbracket$, but, by definition of φ_z , $z \notin \llbracket \varphi_z \rrbracket$, contradiction. $\square$

We conclude that the model $\tilde{\mathfrak{M}}_\alpha$ resulting from the selection construction is atomic.

Proposition 7.3. *Suppose that $\mathbb{P}$ is finite, $\Sigma = \text{Fm}$, and in Construction 4.1 $\mathfrak{M}$ is of finite width. Then the construction terminates, say at stage α , and $\tilde{\mathfrak{M}}_\alpha$ is atomic, has the finite cover property, and validates exactly the same formulas as $\mathfrak{M}$.*

In particular, any logic of finite width, is complete w.r.t. such conversely pre-well-founded and atomic models. It now suffices to prove persistence for these models. We first derive an auxiliary lemma.

Lemma 7.4. *Let Σ be finite and let $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ be models such that $\mathfrak{M}$ has the finite cover property and $\tilde{\mathfrak{M}}$ is finite, and let f be Σ -sub- p -morphism from $\mathfrak{M}$ to $\tilde{\mathfrak{M}}$. Then there exists a restriction g of f with finite domain which is still a Σ -sub- p -morphism from $\mathfrak{M}$ to $\tilde{\mathfrak{M}}$.*

Proof. Let $\tilde{x}$ be a point of $\tilde{\mathfrak{M}}$. By the finite cover property, there exists a finite cover $C_{\tilde{x}}$ of $f^{-1}(\tilde{x})$. Without loss of generality, we may assume that for each $x \in C_{\tilde{x}}$, if $(R(x) \cap R^{\text{op}}(x)) \cap f^{-1}(\tilde{x}) \neq \emptyset$, then $(R(x) \cap R^{\text{op}}(x)) \cap C_{\tilde{x}} \neq \emptyset$, that is, if for $x \in C_{\tilde{x}}$ the set $f^{-1}(\tilde{x})$ contains a point in the cluster of x , then also $C_{\tilde{x}}$ contains a point in the cluster of x .

For every $\varphi \in \Sigma$ and $x \in C_{\tilde{x}}$, if there exists $y \in R(x) \cap R^{\text{op}}(x)$ with $x \in \llbracket \varphi \rrbracket \iff y \in \llbracket \varphi \rrbracket$, then there exists $y' \in \text{sp}_{\text{dom}(f)}(x)$ with $x \in \llbracket \varphi \rrbracket \iff y' \in \llbracket \varphi \rrbracket$. Whenever this is the case, pick some such y' and add it to $C_{\tilde{x}}$ to obtain $C'_{\tilde{x}}$ which is finite, since Σ is finite. Define g to be the restriction of f to the union of the finitely many $C'_{\tilde{x}}$.

It is easy to check that $\text{dom}(g)$ is a finite Σ -subset of $\mathfrak{M}$, and monotonicity of g is inherited from f . We check the back-condition for g . Let $x \in \text{dom}(g)$, $\tilde{x} := g(x)$ and $\tilde{y}$ a successor of $\tilde{x}$. By the back-condition of f , there exists $y \in R(x) \cap f^{-1}(\tilde{y})$. Since $C_{\tilde{y}}$ covers $f^{-1}(\tilde{y})$, there is $z \in R^*(y) \cap C_{\tilde{y}}$. If $z \in R(x)$, then we are done. Otherwise $z \notin R(x)$, and, by weak transitivity, $x = z$, so $\tilde{x} = \tilde{y}$ and $y \in (R(x) \cap R^{\text{op}}(x)) \cap f^{-1}(\tilde{x})$. Hence, there exists $y' \in (R(x) \cap R^{\text{op}}(x)) \cap C_{\tilde{x}}$. $\square$

Proposition 7.5. *Let Λ be an extension of **wK4** and $\mathfrak{M}$ an atomic Λ -model on finitely many propositional variables $\mathbb{P}$ with the finite cover property. Then the frame $\mathfrak{F} = \langle X, R \rangle$ underlying $\mathfrak{M}$ is a Λ -frame.*

Proof. Let $\mathfrak{N}$ be a model on $\mathfrak{F}$ and let Σ_n be the set of formulas of modal depth at most n up to logical equivalence. It is enough to show that for each $n \in \mathbb{N}$, $\mathfrak{N}$ satisfies all the formulas in $\Sigma_n \cap \Lambda$. So let $n \in \mathbb{N}$, and write $\Sigma := \Sigma_n$. By Theorem A and Lemma 7.4, there exists a finite Σ -subset Y of $\mathfrak{N}$. Write $\mathfrak{N}|Y$ for the restriction of $\mathfrak{N}$ to Y .

Let $x \in X$. Since Y is a Σ -subset of $\mathfrak{N}$ and Σ is Boolean closed (up to equivalence), there exists $y \in R^*(x) \cap Y$ with $\text{ty}_{\Sigma}(\mathfrak{N}, x) = \text{ty}_{\Sigma}(\mathfrak{N}, y)$. Then by Lemma 3.4

$$\forall \diamond \psi \in \Sigma. x \in R^{\text{op}}(Y \cap \llbracket \psi \rrbracket_{\mathfrak{N}}) \quad \text{iff} \quad y \in R^{\text{op}}(Y \cap \llbracket \psi \rrbracket_{\mathfrak{N}}). \quad (1)$$

Fix a total order $\leq$ on Y . Define the map $h: X \rightarrow Y$ such that $h(x)$ is the $\leq$ -minimal $y \in R(x) \cap Y$ for which eq. (1) holds, if $x \notin Y$, and $h(x) := x$ otherwise.

Since Y is finite and $\mathfrak{M}$ is atomic, by Lemma 6.4, $\equiv_{Y\text{-sp}}$ is definable in $\mathfrak{M}$, and moreover has finitely many equivalence classes. Note that $h^{-1}(y)$ is a union of equivalence classes of $\equiv_{Y\text{-sp}}$ restricted to $X \setminus Y$ and $\{y\}$, hence definable in $\mathfrak{M}$.

Let V be the valuation of $\mathfrak{N}$, and define a new valuation V' on $\mathfrak{F}$ by $V'(p) := h^{-1}(V(p) \cap Y)$. Write $\mathfrak{M}'$ for the resulting model. Since Y is finite, $V'(p)$ is definable in $\mathfrak{M}$, so $\mathfrak{M}'$ is a definable variant of $\mathfrak{M}$, hence a Λ -model. We show that $\mathfrak{M}'$ satisfies the same Σ -formulas as $\mathfrak{N}$.

Note that by definition $\mathfrak{M}'|Y = \mathfrak{N}|Y$. By Proposition 3.5, $Y \cap \llbracket \psi \rrbracket_{\mathfrak{N}} = \llbracket \psi \rrbracket_{\mathfrak{N}|Y}$. Hence, by definition of h , we get for all $x \in X$, $h(x) \in R^*(x)$ and

$$\forall \diamond \psi \in \Sigma. x \in R^{\text{op}}(\llbracket \psi \rrbracket_{\mathfrak{M}'|Y}) \quad \text{iff} \quad h(x) \in R^{\text{op}}(\llbracket \psi \rrbracket_{\mathfrak{M}'|Y}). \quad (2)$$

We prove by induction on formulas $\psi \in \Sigma$, that $x \in \llbracket \psi \rrbracket_{\mathfrak{M}'}$ iff $h(x) \in \llbracket \psi \rrbracket_{\mathfrak{M}'|Y}$, for all $x \in X$. For atomic propositions this is trivial, and the steps for meet and negation are standard. For the diamond step, let $\diamond \psi \in \Sigma$ and $x \in X$. Suppose $x \in \llbracket \diamond \psi \rrbracket_{\mathfrak{M}'}$ and find a successor $x' \in \llbracket \psi \rrbracket_{\mathfrak{M}'}$. By the induction hypothesis $h(x') \in \llbracket \psi \rrbracket_{\mathfrak{M}'|Y}$. If $h(x') \in R(x)$ then, by eq. (2), we get $h(x) \in \llbracket \diamond \psi \rrbracket_{\mathfrak{M}'|Y}$ and we are done. So suppose $h(x') \notin R(x)$. Then, since $x' \in R(x)$ and $h(x') \in R^*(x')$, by weak transitivity, it follows that

$x = h(x')$, yielding $x \neq x'$ and $x \in R(x')$. Moreover, $x \in Y$ and hence $h(x) = x$. Now $x' \in R^{\text{op}}(\llbracket \psi \rrbracket_{\mathfrak{M}'|Y})$, so, by eq. (2), we get $x = h(x') \in R^{\text{op}}(\llbracket \psi \rrbracket_{\mathfrak{M}'|Y})$, yielding $x \in \llbracket \Diamond \psi \rrbracket_{\mathfrak{M}'|Y}$.

Conversely, suppose $h(x) \in \llbracket \Diamond \psi \rrbracket_{\mathfrak{M}'|Y}$. By eq. (2), there exists $y \in R(x) \cap \llbracket \psi \rrbracket_{\mathfrak{M}'|Y}$. But, since $y \in Y$ we have $h(y) = y$. Hence, by the induction hypothesis, $y \in \llbracket \psi \rrbracket_{\mathfrak{M}'}$, yielding $x \in \llbracket \Diamond \psi \rrbracket_{\mathfrak{M}'}$.

Now suppose $\mathfrak{N}$ refutes $\phi \in \Sigma_n$. Since Y is a Σ_n -subset in $\mathfrak{N}$, there is $y \in Y$ such that $y \in \llbracket \neg \phi \rrbracket_{\mathfrak{N}}$. By Proposition 3.5, we have $y \in \llbracket \neg \phi \rrbracket_{\mathfrak{N}|Y} = \llbracket \neg \phi \rrbracket_{\mathfrak{M}'|Y}$, and since $h(y) = y$ it follows, by the inductive proof above, that $y \in \llbracket \neg \phi \rrbracket_{\mathfrak{M}'}$. Since $\mathfrak{M}'$ is a Λ -model, we conclude $\phi \notin \Lambda$. $\square$

Combining Propositions 7.3 and 7.5, we obtain the Finite Width Theorem for **wK4**:

Theorem C. *Every logic of finite width extending **wK4** is complete w.r.t. its conversely pre-well-founded frames, and is therefore, in particular, Kripke complete.*

8 Concluding Remarks

We have generalized Fine's Iterative Selection Method to the weakly transitive setting and to infinite filtration sets Σ , and have shown that using the set of all formulas on finitely many atomic propositions as the filtration set provides an alternative proof of Fine's Finite Width Theorem. A natural next step is to look for further applications of our construction. Our construction might help to drop canonicity assumptions in existing selection constructions on canonical models of subframe logics, e.g., [1, 12]. On the other hand, infinite filtration sets might be useful for non-subframe logics. Another direction is to try to generalize it beyond weakly transitive logics, for example, to n -transitive logics. A limitative result for the latter was given by Wolter [14, Section 4.3] who constructed examples of subframe logics extending **K** that lack the FMP.

Zakharyashev's theory of canonical formulas forms a major application of Fine's Selection Construction, and could, using our result, be generalized to **wK4**. This theory has already been generalized using algebraic methods [2], but the frame theoretic perspective might provide an easier path forward for handling the quasi-normal case, cf. [16, Section 7].

Even though there exists a computable bound on the size of the finite models produced by the construction, it was shown in [16, Theorem 3.3] that there are continuum-many subframe logics extending **K4**, so in particular there are undecidable ones. This is related to the partial order defined by $\mathfrak{M} \preceq \mathfrak{N}$ if there exists a definable sub-p-morphism from $\mathfrak{M}$ to $\mathfrak{N}$ [11, Section 7.1]. Similarly, we can define $\mathfrak{M} \preceq_{\Sigma} \mathfrak{N}$ if there exists a definable Σ -sub-p-morphism from $\mathfrak{M}$ to $\mathfrak{N}$. It is straightforward to show that starting with a model $\mathfrak{M}$ and applying our selection construction, we obtain a $\preceq_{\Sigma}$ -minimal model. However, it is not clear whether it is the unique such model.

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Non-classical Topological Evidence Logic

Igor Sedlár

Institute of Computer Science, Czech Academy of Sciences
Prague, The Czech Republic

sedlar@cs.cas.cz

Topological Evidence Logic (TEL) is a recent approach to epistemic logic that uses topological tools to model coherent epistemic justification. Specifically, a hypothesis is coherently justified if and only if it is entailed by a dense open set. In its simplest form, TEL can be formulated as an extension of S4 with a global modality. All currently studied forms of TEL are based on classical propositional logic, which has been heavily criticised for misrepresenting the way in which ordinary agents reason. In this article, we show that the TEL approach is robust under modifications to the propositional base. We show that an extension of the intuitionistic modal framework recently introduced by de Groot and Shillito, incorporating a global modality, enables coherent justification to be expressed in an intuitionistic setting. Furthermore, we adapt the recent work of Standefer et al., which extends relevant logic with a global modality, to show that coherent justification can be expressed in a relevant setting if an interior-of-complement operator is added to the language. Our main technical result is a soundness and completeness theorem for relevant TEL based on the weak relevant modal logic BS4.

1 Introduction

Topology has proven to be a useful tool in logic and computer science. McKinsey and Tarski's famous results [19, 20, 38] establish a *topological semantics* for modal logic. Smyth's topological perspective on the basic concepts of programming language semantics [31] led to the modelling of *observable properties* of computational systems by open sets [1, 15, 32, 39]. This angle was further elaborated by Kelly [18] who provided a topological foundation of formal learning theory.

In topological semantics of the modal logic S4, the set of possible worlds X is seen as a topological space and the proposition $\Box P$, for arbitrary $P \subseteq X$, is the *interior* of P . Hence, $x \in \Box P$ iff there is an open set U such that $x \in U$ and $U \subseteq P$. Assuming that open sets are interpreted as observable properties or *evidence*, then $\Box P$ means that there is truthful evidence that supports P .

Topological Evidence Logic (TEL) is a recent extension of the topological approach to modal logic that explicitly uses topology to model epistemic notions; see [4, 5, 6, 7, 8, 9, 11, 16, 23, 40] for example. The key concept in TEL is the topological representation of *coherent justification*. Given a space X , a hypothesis $P \subseteq X$ is coherently justified if $U \subseteq P$ for a dense open set U . Intuitively, this means that P is supported by evidence that is *consistent with all (consistent) evidence*. Accordingly, P is *known* in a state $x \in X$ if it is coherently justified by an open set U such that $x \in U$.

A simple form of TEL is an extension of S4 with the global modality A , representing truth in all states. Since $\Box P$ is the interior of $P \subseteq X$, the proposition $\Diamond P$, defined as $(\Box(P^c))^c$, is the *closure* of P . It follows that $A\Diamond P = X$ iff P is dense. Consequently, $A\Diamond\Box P = X$ iff $P \subseteq U$ for some dense open set U . In other words, $A\Diamond\Box P$ says that P is supported by some dense open set, i.e. that P is coherently justified.

All currently studied forms of TEL are based on classical propositional logic and the use of Boolean negation appears to be essential for TEL's ability to express coherent justification. However, classical propositional logic has been heavily criticised for misrepresenting how agents reason. This has been a major theme in 20th century logic, giving rise to a plethora of non-classical logics such as intuitionistic,

many-valued and relevant logics. In recent years, many non-classical epistemic logics have emerged; see [3, 10, 28, 29, 30, 36, 37]. By using a non-classical propositional base, these epistemic logics avoid certain problematic closure principles such as ‘if an agent believes the consequent of an implication, then they believe the implication’ or ‘if an agent has inconsistent beliefs, then they believe everything’. However, these logics lack the attractive evidence-based approach of TEL and are not able to represent coherent justification in the way TEL does.

In this article, we bridge this gap by *studying intuitionistic and relevant versions of TEL*. Recent work by de Groot and Shillito [13] marks the first step towards a non-classical version of TEL. They extend the Kripke-style relational semantics for propositional intuitionistic logic with the interior semantics for the modal operator $\Box$. However, their logic iS4 lacks the resources to express the density-based notion of coherent justification that is central to TEL. Our first contribution is to show that coherent justification can be expressed in an extension of iS4 with A.

In the context of relevant logic, systems with A have recently been considered by Standefer and co-authors [33, 35, 36]. However, completeness results for these logics have remained elusive. Our second contribution is to show that density-based justification can be expressed within an extension of the a semantic framework for the weak relevant modal logic BS4, in which the $\Box$ operator is interpreted as interior and in which an interior-of-complement operator $\Box^c$ and the global modality A are introduced. We also prove soundness and completeness of the resulting logic. Due to the modular nature of our axiomatisation, our result solves the problem of axiomatising relevant logic with A, left open in [35].

The article is structured as follows. Section 2 discusses intuitionistic TEL and shows that coherent justification can be expressed in an extension of de Groot and Shillito’s iS4 with a global modality. In Section 3, we provide a sound and complete axiomatisation for intuitionistic TEL. Section 4 formulates a first naive attempt at relevant TEL and shows that it fails. Relevant TEL is introduced in Section 5 as an extension of the relevant modal logic BS4 with the global modality A and the interior-of-complement operator $\Box^c$. A soundness and completeness result is established in Section 6. Section 7 offers a summary of the paper and an outline of future work. To ensure the paper remains self-contained for the non-expert reader, full proofs of two standard results in relevant logic are provided in the appendix.

2 Intuitionistic TEL

In this section, we present an intuitionistic version of TEL, extending the work of de Groot and Shillito [13]. We show that incorporating the global modality A into their logic iS4 enables coherent justification to be expressed in a manner analogous to that in classical TEL. We assume that the reader is familiar with the basics of modal logic [12] and general topology [21].

Definition 2.1. Let $Prop$ be a countably infinite set of propositional variables. The language $\mathcal{L}_A^\Box$ is defined using the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid A\varphi$$

where $p \in Prop$. We define $\Diamond\varphi := \neg\Box\neg\varphi$ and $E\varphi := \neg A\neg\varphi$.

Definition 2.2. Let $\langle X, \leq \rangle$ be a pre-ordered set. An *up-set topology* on $\langle X, \leq \rangle$ is a topology $\mathcal{T}$ on X such that all $U \in \mathcal{T}$ are closed upwards under $\leq$ (up-sets). An *up-space* is $\langle X, \leq, \mathcal{T} \rangle$ such that $\mathcal{T}$ is an up-set topology on $\langle X, \leq \rangle$. Given an up-space $\mathcal{S} = \langle X, \leq, \mathcal{T} \rangle$, the *$\mathcal{S}$ -interior* of $P \subseteq X$ is $Int_{\mathcal{S}}(P) := \bigcup \{U \mid U \in \mathcal{T} \text{ \& } U \subseteq P\}$ and the *$\mathcal{S}$ -closure* of P is $Cl_{\mathcal{S}}(P) := \bigcap \{K \mid K^c \in \mathcal{T} \text{ \& } P \subseteq K\}$, where K^c is the complement of K with respect to X .

We will write *Int* and *Cl* if $\mathcal{S}$ is clear from the context or immaterial.

Example 2.3. Let $\langle X, \leq \rangle$ be a pre-ordered set. The *Alexandroff topology* on $\langle X, \leq \rangle$ is $\mathcal{T}_{\leq}$ comprising all up-sets on $\langle X, \leq \rangle$. Then $\langle X, \leq, \mathcal{T}_{\leq} \rangle$ is an up-space. In fact, $\mathcal{T}_{\leq}$ is the *finest* up-set topology on $\langle X, \leq \rangle$.

Example 2.4. Let $\langle \mathbb{R}, \leq \rangle$ be the set of real numbers with their standard ordering and choose a set $O \subseteq \mathbb{R}$ that intuitively represents the ‘observable’ real numbers. We require that for each $a \in \mathbb{R}$ there is $b \in O$ such that $b \leq a$. Take $\mathcal{T}_O$ to be the right-order topology determined by the basis $\mathcal{B}_O = \{[x, \infty) \mid x \in O\}$. The structure $\langle \mathbb{R}, \leq, \mathcal{T}_O \rangle$ is an up-space. One example of this construction is $\mathcal{T}_{\mathbb{Z}}$, representing possible measurements of a real quantity using a device with integer outputs.

Example 2.5 ([32, 39]). Let 2^ω be the set of all finite and countably infinite binary words with the prefix ordering $\sqsubseteq$, where $w \sqsubseteq u$ if w is a prefix of u . Let W be a set of *finite* binary words containing the empty word, representing possible observations. This set may contain all finite binary words, but this is not a requirement. We assume that the observer can observe some finite words but not infinite ones. Let $\uparrow w = \{u \in 2^\omega \mid w \sqsubseteq u\}$, comprising words that are ‘consistent’ with prefix w . Take the topology $\mathcal{T}_W$ generated by the basis $\mathcal{B}_W = \{\uparrow w \mid w \text{ finite \& } w \in W\}$. Then $\langle 2^\omega, \sqsubseteq, \mathcal{T}_W \rangle$ is an up-space. Open sets in this topology are the observable (or *semidecidable*) properties of binary words, i.e. properties that can be confirmed by observing a finite prefix.

Example 2.6 ([13, 22]). Let $\langle X, \leq, R \rangle$ be a bi-relational Kripke model for intuitionistic modal logic [22], where R is a preorder on X such that $x \leq y$ implies Rxy . Let $\mathcal{T}_R$ be the topology comprising all subsets of X that are upwards closed under the preorder R (the Alexandroff topology determined by R). Then $\langle X, \leq, \mathcal{T}_R \rangle$ is an up-space. In fact, all up-spaces $\langle X, \leq, \mathcal{T} \rangle$ give rise to a bi-relational Kripke model $\langle X, \leq, \leq_{\mathcal{T}} \rangle$ where $\leq_{\mathcal{T}}$ is the *specialisation order* of $\mathcal{T}$, in which $x \leq_{\mathcal{T}} y$ iff $\forall U \in \mathcal{T}$: if $x \in U$, then $y \in U$. As discussed below, however, not all $\mathcal{T}$ are identical to $\mathcal{T}_{\leq_{\mathcal{T}}}$.

Relational structures based on pre-ordered sets $\langle X, \leq \rangle$ are often used in the semantics of non-classical logics. The usual informal interpretation of these structures is in terms of information support by partial situations: $x \leq y$ iff all information supported by situation x is supported by situation y as well [14, 17, 25]. The Alexandroff topology $\mathcal{T}_{\leq}$ makes this informal interpretation explicit: its open sets are all the pieces of information that $\leq$ cares about since $x \leq y$ iff $\forall U \in \mathcal{T}_{\leq}$: if $x \in U$, then $y \in U$. Conversely, any topological space $\langle X, \mathcal{T} \rangle$ can be transformed into a pre-ordered set $\langle X, \leq_{\mathcal{T}} \rangle$ where $\leq_{\mathcal{T}}$ is the specialisation order of $\mathcal{T}$ (see Example 2.6). However, a perfect match between the pre-order (implicit) and topological (explicit) model of available information cannot be expected in general. The Alexandroff topology determined by the specialisation order $\leq_{\mathcal{T}}$ of an arbitrary topology $\mathcal{T}$ is the closure of $\mathcal{T}$ under arbitrary intersections, not necessarily $\mathcal{T}$ itself. However, if $\mathcal{T}$ is already closed under arbitrary intersections ($\langle X, \mathcal{T} \rangle$ is an Alexandroff space), then $\mathcal{T}$ is the Alexandroff topology determined by $\leq_{\mathcal{T}}$.

In a general up-space $\langle X, \leq, \mathcal{T} \rangle$ where $\mathcal{T}$ is typically coarser than $\mathcal{T}_{\leq}$, the pre-order $\leq$ is relative to *all information* while $\mathcal{T}$ comprises the *observable information*. The latter is a subset of the former and so open sets of $\mathcal{T}$ are up-sets with respect to $\leq$. In general, not all information is observable. For example, while $\mathcal{T}_{\leq}$ is closed under arbitrary intersections, $\mathcal{T}$ does not have to be – an observer cannot usually make an infinite number of observations and combine them. However, $\mathcal{T}$ may differ from $\mathcal{T}_{\leq}$ even if it is closed under arbitrary intersections. In Example 2.4, $a \in \mathbb{R}$ represent possible values of some quantity (e.g. temperature) and intervals $[x, \infty)$ for $x \in \mathbb{Z}$ represent temperature thresholds verifiable by a sensor with integer outputs. Collections of such intervals are closed under arbitrary intersections.

Definition 2.7. Let $\mathcal{U}(X, \leq)$ be the collection of up-sets on $\langle X, \leq \rangle$. An *up-model* is $\langle X, \leq, \mathcal{T}, \mathcal{V} \rangle$ where $\mathcal{V} : Prop \rightarrow \mathcal{U}(X, \leq)$. A *pointed up-model* is a pair $\langle \mathbf{M}, x \rangle$ where $\mathbf{M}$ is an up-model on X and $x \in X$. The *satisfaction relation* $\models$ between pointed up-models and formulas $\chi \in \mathcal{L}_A^\square$ is defined by structural induction on χ as follows (where $\llbracket \varphi \rrbracket_{\mathbf{M}} := \{x \in X \mid \langle \mathbf{M}, x \rangle \models \varphi\}$):

- $\mathbf{M}, x \models p$ iff $x \in \mathcal{V}(p)$;
- $\mathbf{M}, x \models \neg\phi$ iff $y \notin \llbracket \phi \rrbracket_{\mathbf{M}}$ for all $y \geq x$;
- $\mathbf{M}, x \models \phi \wedge \psi$ iff $\mathbf{M}, x \models \phi$ and $\mathbf{M}, x \models \psi$;
- $\mathbf{M}, x \models \phi \vee \psi$ iff $\mathbf{M}, x \models \phi$ or $\mathbf{M}, x \models \psi$;
- $\mathbf{M}, x \models \phi \rightarrow \psi$ iff, for all $y \geq x$, if $\mathbf{M}, y \models \phi$, then $\mathbf{M}, y \models \psi$;
- $\mathbf{M}, x \models \Box\phi$ iff $x \in \text{Int}_{\mathbf{M}}(\llbracket \phi \rrbracket_{\mathbf{M}})$;
- $\mathbf{M}, x \models A\phi$ iff $\llbracket \phi \rrbracket_{\mathbf{M}} = X$.

A formula ϕ is valid in an up-model $\mathbf{M} = \langle X, \leq, \mathcal{T}, \mathcal{V} \rangle$ iff $\llbracket \phi \rrbracket_{\mathbf{M}} = X$. Validity in up-spaces and classes thereof is defined as usual.

We will often write $x \models \phi$ and $x \in \llbracket \phi \rrbracket$ if $\mathbf{M}$ is clear from the context or immaterial. Recall that a set $P \subseteq X$ is *dense* in $\langle X, \mathcal{T} \rangle$ iff $\text{Cl}(P) = X$. Equivalently, P is dense iff $P \cap U \neq \emptyset$ for all $U \in \mathcal{T}$ such that $U \neq \emptyset$. Note that if P is dense and $P \subseteq Q$, then Q is dense.

Thanks to the fact that the *interior of complement* can be expressed using the interior operator and intuitionistic negation, support by a dense open set can be expressed in the same way as in classical TEL.

Lemma 2.8. $\text{Int}(\llbracket \neg\phi \rrbracket) = \text{Int}(\llbracket \phi \rrbracket^c)$.

Proof. We have $\text{Int}(\llbracket \neg\phi \rrbracket) \subseteq \text{Int}(\llbracket \phi \rrbracket^c)$ since $\llbracket \neg\phi \rrbracket \subseteq \llbracket \phi \rrbracket^c$. Conversely, $\text{Int}(\llbracket \phi \rrbracket^c) \subseteq \llbracket \neg\phi \rrbracket$ since all open sets are upwards closed under $\leq$. Hence, $\text{Int}(\llbracket \phi \rrbracket^c) \subseteq \text{Int}(\llbracket \neg\phi \rrbracket)$. $\square$

Proposition 2.9. *The following hold for all pointed up-models $\mathbf{M}, x$ and all formulas ϕ :*

1. $\mathbf{M}, x \models A\Diamond\phi$ iff $\llbracket \phi \rrbracket_{\mathbf{M}}$ is a dense set;
2. $\mathbf{M}, x \models A\Diamond\Box\phi$ iff $U \subseteq \llbracket \phi \rrbracket_{\mathbf{M}}$ for a dense open set U ;
3. $\mathbf{M}, x \models \Box\phi \wedge A\Diamond\Box\phi$ iff $U \subseteq \llbracket \phi \rrbracket_{\mathbf{M}}$ for a dense open set U such that $x \in U$.

Proof. 1. $\llbracket \Diamond\phi \rrbracket = X$ iff $\forall x: x \notin \text{Int}(\llbracket \neg\phi \rrbracket)$ iff $\forall x: x \notin \text{Int}(\llbracket \phi \rrbracket^c)$ (Lemma 2.8) iff $\text{Cl}(\llbracket \phi \rrbracket) = X$ iff $\llbracket \phi \rrbracket$ is dense.

2. By the first claim, $\llbracket \Diamond\Box\phi \rrbracket = X$ iff $\text{Int}(\llbracket \phi \rrbracket)$ is dense. But clearly $\text{Int}(\llbracket \phi \rrbracket)$ is dense iff there is a dense $U \in \mathcal{T}$ such that $U \subseteq \llbracket \phi \rrbracket$ (since the collection of dense sets is closed under supersets).

3. By the second claim, we have $x \models \Box\phi \wedge A\Diamond\Box\phi$ iff $x \in U$ for some open $U \subseteq \llbracket \phi \rrbracket$ and there is a dense open V such that $V \subseteq \llbracket \phi \rrbracket$. It follows that $U \cup V$ is a dense open such that $x \in U \cup V \subseteq \llbracket \phi \rrbracket$. Conversely, if $x \in U$ for some dense open $U \subseteq \llbracket \phi \rrbracket$, then obviously $x \models \Box\phi$ and $\text{Int}(\llbracket \phi \rrbracket)$ is non-empty and dense, whence $x \models A\Diamond\Box\phi$. $\square$

The upshot of this section is that TEL is easily extended to the intuitionistic setting. Take the usual Kripke-style semantics for intuitionistic propositional logic, add a topological interpretation of $\Box$ as interior and extend the framework with the global modality A . Even though negation is non-classical, we can still express interior-of-complement and therefore density. In Section 4, we consider TEL based on relevant logic. We will see that the situation is more interesting there. However, before proceeding to relevant TEL, we will provide an axiomatisation of intuitionistic TEL.

$$\begin{array}{c}
\heartsuit(\varphi \rightarrow \psi) \rightarrow (\heartsuit\varphi \rightarrow \heartsuit\psi) \quad \heartsuit\varphi \rightarrow \varphi \quad \heartsuit\varphi \rightarrow \heartsuit\heartsuit\varphi \quad \frac{\varphi}{\heartsuit\varphi} \\
A\varphi \vee A\neg A\varphi \quad A\gamma \rightarrow \Box A\gamma \quad \neg A\gamma \rightarrow \Box \neg A\gamma
\end{array}$$

Figure 1: Modal axioms and rules of the system $iS4_A$, where $\heartsuit \in \{\Box, A\}$.

3 Axiomatisation of Intuitionistic TEL

To the best of our knowledge, there is no explicit axiomatisation of intuitionistic modal logic with A in the literature. As discussed in [35, p. 1304], for example, intuitionistic propositional logic with A can be axiomatised using Ono's system L_4 from [22]. Here, we extend this axiomatisation to fit the language with both A and $\Box$. While the completeness proof may be relatively unexciting for an expert reader, we include it for the sake of exposition.

Definition 3.1. Let $iS4_A$ be the Hilbert-style system that extends a fixed Hilbert-style system for intuitionistic propositional logic with the axioms and inference rules shown in Figure 1. We write $iS4_A \vdash \varphi$ to express that φ is a theorem of $iS4_A$.

The proof of the following lemma is straightforward, so we will omit it.

Lemma 3.2. For all $\varphi \in \mathcal{L}_A^\Box$, if $iS4_A \vdash \varphi$, then φ is valid in all up-spaces.

Both modal operators clearly distribute over conjunctions, $iS4_A \vdash \heartsuit(\varphi \wedge \psi) \leftrightarrow (\heartsuit\varphi \wedge \heartsuit\psi)$, and so they are monotonic: if $iS4_A \vdash \varphi \rightarrow \psi$, then $iS4_A \vdash \heartsuit\varphi \rightarrow \heartsuit\psi$. We note that $\neg A\varphi \rightarrow A\neg A\varphi$ is provable using the axiom $A\varphi \vee A\neg A\varphi$, distributivity of $\wedge$ over $\vee$ and the fact that $\perp \vee \psi \rightarrow \psi$ is provable for each contradiction $\perp$.

Definition 3.3. An $iS4_A$ -theory is $\Gamma \subseteq \mathcal{L}_A^\Box$ such that (a) if $iS4_A \vdash \varphi \rightarrow \psi$ and $\varphi \in \Gamma$, then $\psi \in \Gamma$ and (b) if $iS4_A \vdash \varphi$, then $\varphi \in \Gamma$. An $iS4_A$ -theory Γ is *prime* if $\varphi \vee \psi \in \Gamma$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$. An $iS4_A$ -theory Γ is *non-trivial* iff $\Gamma \neq \mathcal{L}_A^\Box$.

We denote as $|\varphi|$ the set of non-trivial prime $iS4_A$ -theories Γ such that $\varphi \in \Gamma$.

Lemma 3.4. The collection of sets of the form $|\Box\varphi|$ for $\varphi \in \mathcal{L}_A^\Box$ is a basis for an up-set topology on the set of all non-trivial prime $iS4_A$ -theories ordered by set inclusion.

Proof. A modification of the standard argument [2] for classical $S4$. Let Γ be any non-trivial prime $iS4_A$ -theory. Firstly, by Necessitation, $\Box(p \rightarrow p) \in \Gamma$. Hence, every Γ belongs to a basic set. Secondly, if Γ belongs to an intersection of two basic sets, i.e. $\Gamma \in |\Box\varphi| \cap |\Box\psi|$ for some φ, ψ , then $\Gamma \in |\Box(\varphi \wedge \psi)|$ since $iS4_A \vdash \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$. Hence, it follows that Γ belongs to a basic subset of the intersection of the two basic sets. Each basic set $|\Box\varphi|$ is clearly closed under $\subseteq$. $\square$

Definition 3.5. The $iS4_A$ -space is $\langle X, \subseteq, \mathcal{T} \rangle$, where X is the set of all non-empty non-trivial prime $iS4_A$ -theories and $\mathcal{T}$ is the topology generated by the basis comprising sets $|\Box\varphi| = \{x \in X \mid \Box\varphi \in x\}$.

It follows from Lemma 3.4 that the $iS4_A$ -space is a well-defined up-space. It can be extended to an up-model $\mathcal{C}$ by defining $\mathcal{V}$ as usual, namely, $\mathcal{V}(p) := |p|$. We can prove the Truth Lemma, stating that $\mathcal{C}, x \models \varphi$ iff $\varphi \in x$, for all formulas without A . However, as usual in modal logic, it is not the case that $A\varphi \in x$ for some $x \in X$ iff $\varphi \in y$ for all $y \in X$. What we need to do is to *anchor* $\mathcal{C}$ in a fixed $w \in X$, that is restrict $\mathcal{C}$ to prime theories that agree with w on all formulas of the form $A\varphi$.

Definition 3.6. Let w be any non-trivial prime $iS4_A$ -theory. The *canonical up-model for $iS4_A$ anchored in w* is $\mathbf{C}^w = \langle X^w, \subseteq^w, \mathcal{T}^w, \mathcal{V}^w \rangle$ where

- $X^w := \bigcap_{A\phi \in w} |A\phi| \cap \bigcap_{A\psi \notin w} |\neg A\psi|$ and $x \subseteq^w y$ iff $x \subseteq y$ and $x, y \in X^w$;
- $\mathcal{T}^w$ is the subspace topology on X^w derived from $\mathcal{T}$ (i.e. $U \in \mathcal{T}^w$ iff $\exists V \in \mathcal{T}: U = X^w \cap V$);
- $\mathcal{V}^w(p) = |p| \cap X^w$.

Lemma 3.7. $x \in X^w$ iff $A\phi \in x \iff A\phi \in w$ for all ϕ .

Proof. If $x \in X^w$ and $A\phi \in w$, then $A\phi \in x$ by definition. If $A\phi \notin w$, then $A\neg A\phi \in w$ using the axiom $A\phi \vee A\neg A\phi$ of $iS4_A$. Then $A\neg A\phi \in x$ by definition, whence $\neg A\phi \in x$ using the axiom $A\psi \rightarrow \psi$. Since x is non-trivial and $iS4_A \vdash \psi \wedge \neg \psi \rightarrow \chi$, we have $A\phi \notin x$.

Conversely, assume that $x \notin X^w$. Then either there is $A\phi \in w$ such that $A\phi \notin x$ and we are done, or there is $A\phi \notin w$ such that $\neg A\phi \notin x$. In the latter case, axiom $A\phi \vee A\neg A\phi$ makes sure that $A\neg A\phi \in w$. However, $A\neg A\phi \notin x$ since that would entail $\neg A\phi \in x$ using axiom $A\psi \rightarrow \psi$. $\square$

Definition 3.8. A pair $\langle \Delta, \nabla \rangle$ of subsets of $\mathcal{L}_A^\square$ is *$iS4_A$ -independent* if there are no $\phi_1, \dots, \phi_n \in \Delta$ and $\psi_1, \dots, \psi_m \in \nabla$ such that $iS4_A \vdash \bigwedge_{i=1}^n \phi_i \rightarrow \bigvee_{j=1}^m \psi_j$.

Lemma 3.9 (Pair extension for $iS4_A$). *If $\langle \Delta, \nabla \rangle$ is an $iS4_A$ -independent pair, then there is a prime $iS4_A$ -theory Δ' such that $\Delta \subseteq \Delta'$ and $\nabla \cap \Delta' = \emptyset$. If both Δ and ∇ are non-empty, then Δ' is non-empty and non-trivial.*

Proof. See Appendix A.1. $\square$

Lemma 3.10. *Let Γ be a non-empty set of formulas and ψ an arbitrary formula. If $(\bigcap_{\phi \in \Gamma} |\phi|) \subseteq |\psi|$, then there is a finite $\Delta \subseteq \Gamma$ such that $\vdash (\bigwedge_{\delta \in \Delta} \delta) \rightarrow \psi$.*

Proof. Standard argument using Zorn's lemma, the fact that $iS4_A$ is finitary and Lemma 3.9. $\square$

We define $|\chi|^w := |\chi| \cap X^w$.

Lemma 3.11 (Truth Lemma). *For all $\chi \in \mathcal{L}_A^\square$, $|\chi|^w = \llbracket \chi \rrbracket_{\mathbf{C}^w}$.*

Proof. Structural induction on χ . The interesting cases are (i) $\chi = \phi \rightarrow \psi$, (ii) $\chi = \Box\phi$ and (iii) $\chi = A\phi$.

(i) $\chi = \phi \rightarrow \psi$. Assume that $\phi \rightarrow \psi \in x$, $x \subseteq^w y$ and, using the induction hypothesis, $\phi \in y$. Then $\phi \rightarrow \psi \in y$ and $\psi \in y$ since y is closed under modus ponens. Conversely, if $\phi \rightarrow \psi \notin x$, then there is y such that $x \subseteq y$, $\phi \in y$ and $\psi \notin y$ by Lemma 3.9 since the pair $\langle x \cup \{\phi\}, \{\psi\} \rangle$ is independent. We prove that $y \in X^w$ as follows. If $A\gamma \in w$, then $A\gamma \in x$. Since $A\gamma \rightarrow (\phi \rightarrow A\gamma)$ is a theorem of $iS4_A$, we obtain $\phi \rightarrow A\gamma \in y$. Hence, $A\gamma \in y$. If $A\gamma \notin w$, then $\neg A\gamma \in x$. Since $\neg A\gamma \rightarrow (\phi \rightarrow \neg A\gamma)$ is a theorem of $iS4_A$, we obtain $\phi \rightarrow \neg A\gamma \in y$. Hence, $\neg A\gamma \in y$.

(ii) $\chi = \Box\phi$. If $\Box\phi \in x$, then $x \in |\Box\phi|^w \in \mathcal{T}^w$. Using the axiom $\Box\phi \rightarrow \phi$, we obtain $|\Box\phi| \subseteq |\phi|$ and so $|\Box\phi|^w \subseteq |\phi|^w$. Conversely, assume that there is $U \in \mathcal{T}^w$ such that $x \in U \subseteq |\phi|^w$. We know that $U = V \cap X^w$ for some $V \in \mathcal{T}$. Without loss of generality, we may assume that $x \in |\Box\psi| \cap X^w \subseteq |\phi|$ for some specific ψ . By Lemma 3.10, there are $\{A\chi_i\}_{i=1}^n \subseteq w \cap x$ and $\{A\chi'_j\}_{j=1}^m \subseteq w^c$ such that

$$\vdash \Box\psi \wedge \bigwedge_{i=1}^n A\chi_i \wedge \bigwedge_{j=1}^m \neg A\chi'_j \rightarrow \phi$$

and $\{\neg A\chi'_j\}_{j=1}^m \subseteq x$. We can infer the following using axioms $A\gamma \rightarrow \Box A\gamma$, $\neg A\gamma \rightarrow \Box \neg A\gamma$ and the fact that $\Box$ is monotonic and distributes over conjunctions in $iS4_A$:

$$\vdash \Box \psi \wedge \bigwedge_{i=1}^n A\chi_i \wedge \bigwedge_{j=1}^m \neg A\chi'_j \rightarrow \Box \varphi.$$

It follows that $\Box \varphi \in x$.

(iii) $\chi = A\varphi$. If $A\varphi \in x$, then $A\varphi \in w$ by Lemma 3.7. Hence, $A\varphi \in y$ for all $y \in X^w$. Using the axiom $A\varphi \rightarrow \varphi$, we infer that $\varphi \in y$ for all $y \in X^w$. Conversely, assume that $A\varphi \notin x$. We show that the pair $\langle \{A\psi \mid A\psi \in w\} \cup \{\neg A\psi' \mid A\psi' \notin w\}, \{\varphi\} \rangle$ is independent. If not, then we have

$$\vdash \bigwedge_{i=1}^n A\psi_i \wedge \bigwedge_{j=1}^m \neg A\psi'_j \rightarrow \varphi$$

for some $\{A\psi_i\}_{i=1}^n \subseteq w$ and $\{A\psi'_j\}_{j=1}^m \subseteq w^c$. We infer the following using axiom $A\psi \rightarrow AA\psi$, the theorem $\neg A\psi \rightarrow A\neg A\psi$ and the fact that A is monotonic and distributes over conjunctions in $iS4_A$:

$$\vdash \bigwedge_{i=1}^n A\psi_i \wedge \bigwedge_{j=1}^m \neg A\psi'_j \rightarrow A\varphi.$$

Since the antecedent of this implication is in x (as $x \in X^w$), we conclude that $A\varphi \in x$, which contradicts our assumption. Hence, the pair is independent and we have $y \in X^w$ such that $\varphi \notin y$ by Lemma 3.9. $\square$

Theorem 3.12. *A formula φ is valid in all up-spaces iff it is provable in $iS4_A$.*

Proof. Soundness is established by Lemma 3.2. Completeness: if $\not\vdash \varphi$, then $\langle T, \{\varphi\} \rangle$ is independent, where T is the set of all theorems of $iS4_A$. Then there is a non-trivial prime $iS4_A$ -theory w such that $\varphi \notin w$. By Lemma 3.11, φ is not satisfied in w in $\mathbf{C}^w$. Hence, φ is not valid in all up-spaces. $\square$

4 Towards relevant TEL

The results of Section 2 suggest a strategy for obtaining relevant TEL: first, take ordered frames for relevant logics [24, 26, 27] and endow them with an up-set topology; add a modal $\Box$ to the language and interpret it as interior; add the global modality A . In effect, this is the framework of [35], extended with a topologically interpreted $\Box$.

Firstly, we will set up this framework in more detail. Secondly, we will show that it cannot express interior-of-complement and that it cannot express the fact that a given formula is supported by a dense open set. In the next section, we extend the framework so that these facts become expressible.

Definition 4.1. A Routley–Meyer frame [27, 34] is $\mathbf{F} = \langle X, \leq, N, R, * \rangle$ where $\langle X, \leq \rangle$ is a pre-ordered set, $N \subseteq X$ is closed upwards under $\leq$, R is a ternary relation on X and $*$ is a unary operation on X such that

- (RM1) if $x \in N$ and $Rxyz$, then $y \leq z$;
- (RM2) for all y , there is $x \in N$ such that $Rxyy$;
- (RM3) if $Rxyz$ and $x' \leq x$, $y' \leq y$, $z \leq z'$, then $Rx'y'z'$;
- (RM4) if $x \leq y$, then $y^* \leq x^*$;
- (RM5) $x = x^{**}$.

A *Routley–Meyer up-space* is a pair $\langle \mathbf{F}, \mathcal{T} \rangle$ where $\mathcal{T}$ is an up-set topology on $\mathbf{F}$.

The motivation for adopting the frame conditions (RM1–5) is explained after Definition 4.4. Before we get to the definition, we give two examples of RM up-models.

Example 4.2. Expanding Example 2.4, take $\mathbf{F}_{\mathbb{R}} = \langle \mathbb{R}, \leq, N, R, - \rangle$ where $\leq$ is the standard ordering of real numbers, $N = [0, \infty)$, and $Rxyz$ iff $x + y \leq z$. $\mathbf{F}_{\mathbb{R}}$ can be endowed by the right order topology $\mathcal{T}_{\mathbb{Z}} = \{\emptyset, \mathbb{R}\} \cup \{[x, \infty) \mid x \in \mathbb{Z}\}$.

Example 4.3. Let Σ^* be the set of all finite words over an alphabet Σ , including the empty word ε . Take $\mathbf{F}_{\Sigma^*} = \langle \mathcal{P}(\Sigma^*), \subseteq, N, R, ^c \rangle$, where $A \in N$ iff $\varepsilon \in A$ and $R(A, B, C)$ iff $AB = \{wu \mid w \in A \text{ \& } u \in B\} \subseteq C$. $\mathbf{F}_{\Sigma^*}$ can be endowed with the topology of ‘observable properties’, generated by the sub-basis comprising sets $U_w = \{A \subseteq \Sigma^* \mid w \in A\}$ for all $w \in \Sigma^*$. Intuitively, the open sets in this topology are properties of languages that can be verified by observing finite words.

Definition 4.4. A *RM up-model* is $\mathbf{M} = \langle \mathbf{F}, \mathcal{T}, \mathcal{V} \rangle$ where $\mathcal{V} : Prop \rightarrow \mathcal{U}(X, \leq)$. Satisfaction is defined as follows (as before, $\llbracket \varphi \rrbracket_{\mathbf{M}} = \{x \mid \mathbf{M}, x \models \varphi\}$):

- $\mathbf{M}, x \models p$ iff $x \in \mathcal{V}(p)$;
- $\mathbf{M}, x \models \varphi \wedge \psi$ iff $\mathbf{M}, x \models \varphi$ and $\mathbf{M}, x \models \psi$;
- $\mathbf{M}, x \models \varphi \vee \psi$ iff $\mathbf{M}, x \models \varphi$ or $\mathbf{M}, x \models \psi$;
- $\mathbf{M}, x \models \varphi \rightarrow \psi$ iff, for all y, z , if $Rxyz$ and $\mathbf{M}, y \models \varphi$, then $\mathbf{M}, z \models \psi$;
- $\mathbf{M}, x \models \neg \varphi$ iff $\mathbf{M}, x^* \not\models \varphi$;
- $\mathbf{M}, x \models \Box \varphi$ iff $x \in \text{Int}_{\mathbf{M}}(\llbracket \varphi \rrbracket_{\mathbf{M}})$;
- $\mathbf{M}, x \models A\varphi$ iff $\mathbf{M}, y \models \varphi$ for all $y \in X$.

A formula φ is *valid* in an RM up-model $\mathbf{M}$ iff $N \subseteq \llbracket \varphi \rrbracket_{\mathbf{M}}$. Let BS4_A be the logic of all RM up-spaces, defined as the set of all $\mathcal{L}_A^{\Box}$ -formulas valid in all RM up-models.

The next lemma, useful in proving validity of implicational formulas, provides a motivation for the frame conditions (RM1–4). The frame condition (RM5) is needed for the double negation law to hold.

Lemma 4.5. $\varphi \rightarrow \psi$ is valid in $\mathbf{M}$ iff $\llbracket \varphi \rrbracket_{\mathbf{M}} \subseteq \llbracket \psi \rrbracket_{\mathbf{M}}$.

Proof. This holds thanks to (RM1), (RM2) and the fact that $\llbracket \varphi \rrbracket_{\mathbf{M}}$ is an up-set for all φ . The latter fact holds for $\varphi = \psi \rightarrow \chi$ by (RM3) and for $\varphi = \neg \psi$ by (RM4). $\square$

We illustrate the satisfaction clauses of Definition 4.4 in the spaces discussed in previous examples.

Example 4.2, continued. In any model $\mathbf{M}$ on $\mathbf{F}_{\mathbb{R}}$, we have $x \models \varphi \rightarrow \psi$ iff, for all $y, z \in \mathbb{R}$, if $y \models \varphi$, then $x + y \models \psi$ (‘adding’ φ gives ψ). On the other hand, $x \models \neg \varphi$ iff $-x \not\models \varphi$. Not all ‘properties’ are observable directly if our measurement device has only integer outputs. For example, if $\llbracket p \rrbracket = [0.5, \infty)$, then $\llbracket \Box p \rrbracket = [1, \infty)$, meaning that measuring 1 is the closest we can get to verifying p .

Example 4.3, continued. Formulas express properties of languages. In particular, $A \models \varphi \rightarrow \psi$ if $AB \models \psi$ for all B such that $B \models \varphi$ (‘concatenating with a φ -language gives a ψ -language’). Not every property of languages can be verified by observing finite words. E.g. since every non-empty open set contains both finite and infinite languages, $\llbracket \Box p \rrbracket = \emptyset$ if $\llbracket p \rrbracket = \{A \subseteq \Sigma^* \mid A \text{ is infinite}\}$.

Now we show that, over RM up-spaces, $\mathcal{L}_A^{\Box}$ lacks the expressivity for the salient TEL concepts.

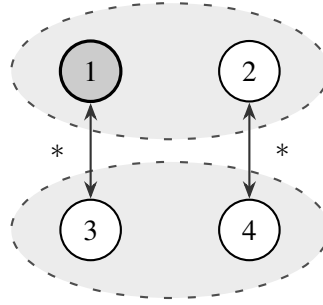


Figure 2: A model showing that interior of complement is not expressible in $\mathcal{L}_A^\square$.

Proposition 4.6. *For any given $p \in Prop$:*

1. *there is no $\varphi(p) \in \mathcal{L}_A^\square$ such that, for all $\mathbf{M}$ we have $\mathbf{M}, x \models \varphi(p)$ iff $x \in Int_{\mathbf{M}}(\llbracket p \rrbracket_{\mathbf{M}}^c)$;*
2. *there is no $\psi(p) \in \mathcal{L}_A^\square$ such that, for all $\mathbf{M}$ we have $\mathbf{M}, x \models \psi(p)$ iff there is a dense open U such that $U \subseteq \llbracket p \rrbracket_{\mathbf{M}}$.*

Proof. 1. We construct an RM up-model $\mathbf{M}$ such that $\llbracket \varphi \rrbracket_{\mathbf{M}} \neq Int(\llbracket p \rrbracket_{\mathbf{M}}^c)$ for all φ . See Figure 2, where $N = X = \{1, \dots, 4\}$, $\leq$ is the identity relation and $Rxyz$ iff $y = z$. The Routley star operation is indicated by the solid arrows. The topology is $\mathcal{T} = \{\emptyset, \{1, 2\}, \{3, 4\}, X\}$. Moreover, $\mathcal{V}(p) = \{1\}$ and $\mathcal{V}(q) = \emptyset$ for all other $q \in Prop$. Let $\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 4\}, X\}$. We prove by structural induction that $\llbracket \varphi \rrbracket \in \mathcal{F}$ for all $\varphi \in \mathcal{L}_A^\square$. The claim is obvious for elements of $Prop$. The cases of the induction step corresponding to $\wedge$ and $\vee$ are also clear. For $\neg$, note that $\neg\emptyset := \{x \mid x^* \notin \emptyset\} = X$, $\neg\{1\} = \{1, 2, 4\}$, $\neg\{1, 2\} = \{1, 2\}$, $\neg\{1, 2, 4\} = \{1\}$ and $\neg X = \emptyset$. For $\rightarrow$, note that $\llbracket \varphi \rightarrow \psi \rrbracket = X$ if $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ and $= \emptyset$ otherwise. For $\Box$, note that $\mathcal{F}$ is closed under Int . Specifically, $Int(\{1\}) = \emptyset$ and $Int(\{1, 2, 4\}) = \{1, 2\}$. The final case of the induction step corresponding to $\Box$ is trivial. Note, however, that $Int(\llbracket p \rrbracket^c) = \{3, 4\}$, but $\{3, 4\} \notin \mathcal{F}$.

2. We present a pair of RM up-models $\mathbf{M}_1$ and $\mathbf{M}_2$ such that: (i) $\llbracket \varphi \rrbracket_1 = \llbracket \varphi \rrbracket_2$ for all $\varphi \in \mathcal{L}_A^\square$, and (ii) there is a dense open subset of $\llbracket p \rrbracket_1$ while there is no dense open subset of $\llbracket p \rrbracket_2$. The two models are shown in Figure 3, with $\mathbf{M}_1$ on the left and $\mathbf{M}_2$ on the right. The models are defined in the same way as the model in Figure 2; they differ from it only in the valuation of p ($\mathcal{V}_1(p) = \mathcal{V}_2(p) = \{1, 2\}$) and their respective topologies. In $\mathbf{M}_1$, we have $\mathcal{T}_1 = \{\emptyset, \{1, 2\}, X\}$ and in $\mathbf{M}_2$ we have $\mathcal{T}_2 = \{\emptyset, \{1, 2\}, \{3\}, \{1, 2, 3\}, X\}$. Let $\mathcal{F} = \{\emptyset, \{1, 2\}, X\}$. Similarly as before, we can prove that $\llbracket \varphi \rrbracket_i \in \mathcal{F}$ for all $\varphi \in \mathcal{L}_A^\square$ and $i \in \{1, 2\}$. (Note that $\mathcal{F}$ is closed under both Int_1 and Int_2 .) Using this fact, we can easily show that $\llbracket \varphi \rrbracket_1 = \llbracket \varphi \rrbracket_2$ for all $\varphi \in \mathcal{L}_A^\square$. Note that $\{1, 2\}$ is dense in $\mathcal{T}_1$. Therefore, in $\langle X, \mathcal{T}_1 \rangle$, there is a dense open subset of $\llbracket p \rrbracket_1$. However, $\{1, 2\}$ is not dense in $\mathcal{T}_2$ and so there is no dense open subset of $\llbracket p \rrbracket_2$ in $\langle X, \mathcal{T}_2 \rangle$. Hence, there is no $\varphi \in \mathcal{L}_A^\square$ that can express the existence of a dense open subset $\llbracket p \rrbracket$. $\square$

5 Relevant TEL

The previous section shows that $\mathcal{L}_A^\square$ needs to be extended in order to express the salient TEL concepts in the context of relevant logic. In this section, we introduce one such extension. It is perhaps the most direct one – we add a modal operator $\Box^c$, interpreted as *interior-of-complement*.

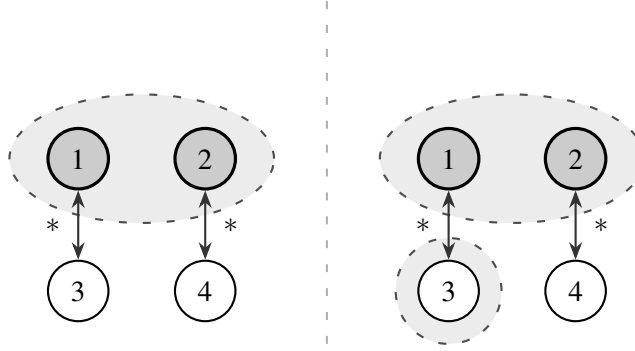


Figure 3: A pair of models showing that support by a dense open set is not expressible in $\mathcal{L}_A^\square$.

Definition 5.1. The language $\mathcal{L}_{A,c}^\square$ is defined using the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid A\varphi \mid \Box^c\varphi$$

where $p \in Prop$. We define $\Diamond^c\varphi := \neg\Box^c\varphi$ and $E\varphi := \neg A\neg\varphi$. We define $\perp := Ap \wedge \neg Ap$ and $\top := Ap \vee \neg Ap$ for some fixed $p \in Prop$.

Definition 5.2. In RM up-models, the satisfaction relation $\models$ is defined for $\mathcal{L}_{A,c}^\square$ as before, with the following clause added:

- $\mathbf{M}, x \models \Box^c\varphi$ iff $x \in Int(\llbracket\varphi\rrbracket^c)$.

Let $BS4_{A,c}$ be the logic of all RM up-models, defined as the set of all $\mathcal{L}_{A,c}^\square$ -formulas valid in all RM up-models.

Note that $\mathbf{M}, x \models \Diamond^c\varphi$ iff $x^* \in Cl(\llbracket\varphi\rrbracket_{\mathbf{M}})$. Moreover, $\mathbf{M}, x \models E\varphi$ iff $\llbracket\varphi\rrbracket_{\mathbf{M}} \neq \emptyset$ (for the right-to-left implication, we need to assume that every $y = z^*$ for some z), $\mathbf{M}, x \not\models \perp$ and $\mathbf{M}, x \models \top$ for all $x \in X$.

With $\Box^c$ at hand we can express density and coherent justification.

Proposition 5.3. The following hold for all pointed up-models $\mathbf{M}, x$:

1. $\mathbf{M}, x \models A\Diamond^c\varphi$ iff $\llbracket\varphi\rrbracket_{\mathbf{M}}$ is a dense set;
2. $\mathbf{M}, x \models A\Diamond^c\Box\varphi$ iff $U \subseteq \llbracket\varphi\rrbracket_{\mathbf{M}}$ for a dense open set U ;
3. $\mathbf{M}, x \models \Box\varphi \wedge A\Diamond^c\Box\varphi$ iff $U \subseteq \llbracket\varphi\rrbracket_{\mathbf{M}}$ for a dense open set U such that $x \in U$.

Proof. 1. $\llbracket\Diamond^c\varphi\rrbracket = X$ iff $\forall x: x^* \in Cl(\llbracket\varphi\rrbracket)$ iff $\forall y: y \in Cl(\llbracket\varphi\rrbracket)$ (since $\forall y\exists x: y = x^*$ by RM5) iff $\llbracket\varphi\rrbracket$ is dense. Claims 2. and 3. follow from Claim 1. exactly as in the proof of Proposition 2.9. $\square$

This shows that $BS4_{A,c}$ is a version of relevant TEL that is able to express support by truthful evidence as $\Box\varphi$ and coherent justification by truthful evidence as $\Box\varphi \wedge A\Diamond^c\Box\varphi$.

We conclude this section with the following observation. Note that $\Box^c$ resembles intuitionistic negation under its topological interpretation, i.e. $\llbracket\neg\varphi\rrbracket = Int(\llbracket\varphi\rrbracket^c)$. One might wonder whether our language also allows us to express intuitionistic implication. The following proposition shows that it does not.

Proposition 5.4. For any given $p, q \in Prop$, there is no $\varphi(p, q) \in \mathcal{L}_{A,c}^\square$ such that, for all $\mathbf{M}$ we have $\llbracket\varphi(p, q)\rrbracket_{\mathbf{M}} = Int(\llbracket p \rrbracket_{\mathbf{M}}^c \cup \llbracket q \rrbracket_{\mathbf{M}})$.

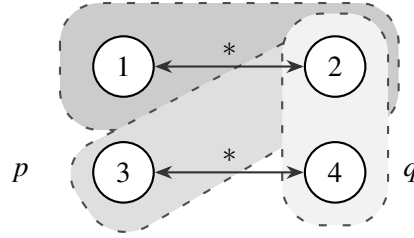


Figure 4: A model showing that intuitionistic implication is not expressible in $\mathcal{L}_{A,c}^{\square}$.

Proof. We fix $p, q \in Prop$ and we construct an RM up-model $\mathbf{M}$ such that $\llbracket \varphi \rrbracket_{\mathbf{M}} \neq Int(\llbracket p \rrbracket_{\mathbf{M}}^c \cup \llbracket q \rrbracket_{\mathbf{M}})$ for all $\varphi \in \mathcal{L}_{A,c}^{\square}$. The countermodel is shown in Figure 4. As before, $\leq$ is the identity relation, $N = X$, the $*$ operation is indicated by the arrows, and $Rxyz$ iff $y = z$. The topology $\mathcal{T}$ is generated by the sub-basis $\{B = \{1, 2\}, P = \{2, 3\}, Q = \{2, 4\}\}$. We set $\mathcal{V}(p) = P$ and $\mathcal{V}(q) = Q$. Observe that $Int(P^c \cup Q) = Int(\{1, 2, 4\}) = \{1, 2, 4\}$. However, we can show by structural induction that $\llbracket \varphi \rrbracket \in \{\emptyset, X, \{2\}, \{2, 3\}, \{2, 4\}, \{2, 3, 4\}\}$ for all $\varphi \in \mathcal{L}_{A,c}^{\square}$. $\square$

6 Axiomatisation of Relevant TEL

In this section we provide a sound and complete axiomatisation of $BS4_{A,c}$. This solves a problem left open in [35], namely, the problem of axiomatising B_A , the basic relevant logic B with A . The axiomatisation of B_A is obtained from our axiomatisation of $BS4_{A,c}$ by omitting all axioms and rules that feature the modal operators $\square$ and $\square^c$.

Recall that B is the logic of all RM frames in the language $\mathcal{L}$, the purely propositional fragment of $\mathcal{L}_{A,c}^{\square}$. A standard Hilbert-style system for B , denoted as B , is shown in Figure 5.

We note that the De Morgan laws $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$ and $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$ are provable in B , as is the double negation law $\neg\neg\varphi \leftrightarrow \varphi$. The transitivity rule is derivable in B : if $B \vdash \varphi \rightarrow \psi$ and $B \vdash \psi \rightarrow \chi$, then $B \vdash \varphi \rightarrow \chi$. The contraposition rule is also derivable: if $B \vdash \varphi \rightarrow \psi$, then $B \vdash \neg\psi \rightarrow \neg\varphi$.

Definition 6.1. Let $BS4_{A,c}$ be the Hilbert-style system that extends B with the axioms and rules shown in Figure 6. Provability in $BS4_{A,c}$ is defined in the standard way. We write $\vdash \varphi$ if φ is provable in $BS4_{A,c}$.

Axioms (13–18) for A say that A is a relevant S5-style modality; see [33]. (19–21) say that $\square^c\varphi$ expresses the largest open proposition disjoint from the proposition expressed by φ . Axioms (22–29)

$\varphi \rightarrow \varphi$	(1)	$\neg\neg\varphi \rightarrow \varphi$	(7)
$(\varphi \wedge \psi) \rightarrow \varphi, (\varphi \wedge \psi) \rightarrow \psi$	(2)	$\varphi, \varphi \rightarrow \psi \implies \psi$	(8)
$\varphi \rightarrow (\varphi \vee \psi), \psi \rightarrow (\varphi \vee \psi)$	(3)	$\varphi, \psi \implies \varphi \wedge \psi$	(9)
$((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \wedge \chi))$	(4)	$\varphi \rightarrow \neg\psi \implies \psi \rightarrow \neg\varphi$	(10)
$((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)$	(5)	$\varphi \rightarrow \psi \implies (\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)$	(11)
$(\varphi \wedge (\psi \vee \chi)) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$	(6)	$\varphi \rightarrow \psi \implies (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$	(12)

Figure 5: A Hilbert-style system B for the basic relevant logic B .

$$\begin{array}{ll}
(\heartsuit\varphi \wedge \heartsuit\psi) \rightarrow \heartsuit(\varphi \wedge \psi) & (13) \\
\heartsuit\varphi \rightarrow \varphi & (14) \\
\heartsuit\varphi \rightarrow \heartsuit\heartsuit\psi & (15) \\
\varphi \rightarrow \psi \implies \heartsuit\varphi \rightarrow \heartsuit\psi & (16) \\
\neg A\varphi \rightarrow A\neg A\varphi & (17) \\
A(\varphi \rightarrow \psi) \rightarrow (A\varphi \rightarrow A\psi) & (18) \\
\Box^c\varphi \rightarrow \Box\Box^c\varphi & (19) \\
\Box^c\varphi \wedge \varphi \rightarrow \perp & (20) \\
\Box\psi \wedge \varphi \rightarrow \perp \implies \Box\psi \rightarrow \Box^c\varphi & (21) \\
(A\varphi \wedge \neg A\varphi) \rightarrow \psi & (22) \\
\varphi \rightarrow (A\psi \vee \neg A\psi) & (23) \\
A\varphi \rightarrow (\neg A\varphi \rightarrow \psi) & (24) \\
\neg A\varphi \rightarrow (A\varphi \rightarrow \psi) & (25) \\
A\varphi \rightarrow (\psi \rightarrow A\varphi) & (26) \\
\neg A\varphi \rightarrow (\psi \rightarrow \neg A\varphi) & (27) \\
A\varphi \rightarrow \Box A\varphi & (28) \\
\neg A\varphi \rightarrow \Box\neg A\varphi & (29)
\end{array}$$

Figure 6: The extra axioms and rules of $BS4_{A,c}$, to be added to B , where $\heartsuit \in \{\Box, A\}$.

impose *uniformity* of formulas with the global modality under $*$ and R and within open sets; note that, in the words of Standefer and French [35], a certain amount of *classicality* is thereby imposed. These axioms are crucial for showing that the anchored canonical model is an RM up-model, as well as for proving the Truth Lemma for the model. We have not attempted to remove any redundancies from the axiom system.

As a first step towards our main result, we show that $BS4_{A,c}$ is sound with respect to the class of all RM up-spaces.

Lemma 6.2. *If φ is provable in $BS4_{A,c}$, then it is valid in all RM up-spaces.*

Proof. Standard induction on the length of proofs. We show some cases.

(17) Suppose $x \models \neg A\varphi$; we have to prove that $x \models A\neg A\varphi$. That is, $y^* \not\models A\varphi$ for an arbitrary y . But since $x^* \not\models A\varphi$, there is $z \not\models \varphi$, which implies that $z' \not\models A\varphi$ for all z' . (18) Suppose $x \models A(\varphi \rightarrow \psi)$. To prove that $x \models A\varphi \rightarrow A\psi$, assume that $Rxyz$ and $y \models A\varphi$. Pick an arbitrary u . We have to prove $u \models \psi$. By (RM2), there is $v \in N$ such that $Rvuu$. By the assumptions, $v \models \varphi \rightarrow \psi$ and $u \models \varphi$. Hence, $u \models \psi$, as desired. (19–20) hold since $\llbracket \Box^c\varphi \rrbracket = \text{Int}(\llbracket \varphi \rrbracket^c)$ is an open set U such that $U \cap \llbracket \varphi \rrbracket = \emptyset$. (21) holds since $\llbracket \Box^c\varphi \rrbracket$ is the largest such set. (22–23) hold since $x \models \neg A\varphi$ iff $x^* \not\models A\varphi$ iff $\exists y: y \not\models \varphi$. Hence, $\llbracket \neg A\varphi \rrbracket = \llbracket A\varphi \rrbracket^c$. (24–27) all hold because $\llbracket A\varphi \rrbracket, \llbracket \neg A\varphi \rrbracket \in \{\emptyset, X\}$ and $(\emptyset \rightarrow P) = (Q \rightarrow X)$ for all $P, Q \subseteq X$ (where $Q \rightarrow P = \{x \mid \forall y, z: \text{ if } y \in Q, \text{ then } z \in P\}$). (28–29) hold for a similar reason: since $\llbracket A\varphi \rrbracket, \llbracket \neg A\varphi \rrbracket \in \{\emptyset, X\}$, they are open sets. $\square$

Comparing $BS4_{A,c}$ and $iS4_A$, we note that the K-axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ is not valid and is thus replaced by the conjunctive regularity axiom $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ and the monotonicity rule $\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$. Because of a different definition of validity, we do not have the necessitation rule for $\Box$ or A . Ono's axiom $A\varphi \vee A\neg A\varphi$ is not sufficient for deriving $\neg A\varphi \rightarrow A\neg A\varphi$, which is used in our completeness proof. Therefore, we replace it with $\neg A\varphi \rightarrow A\neg A\varphi$ directly.

Definition 6.3. A $BS4_{A,c}$ -theory is $\Gamma \subseteq \mathcal{L}_{A,c}^\Box$ such that (a) if $\vdash \varphi \rightarrow \psi$ and $\varphi \in \Gamma$, then $\psi \in \Gamma$ and (b) if $\varphi \in \Gamma$ and $\psi \in \Gamma$, then $\varphi \wedge \psi \in \Gamma$. A $BS4_{A,c}$ -theory Γ is *prime* iff $\varphi \vee \psi \in \Gamma$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$; *regular* iff $\vdash \varphi$ only if $\varphi \in \Gamma$; and *non-trivial* if $\Gamma \neq \mathcal{L}_{A,c}^\Box$.

We denote as $|\varphi|$ the set of all non-empty non-trivial prime $BS4_{A,c}$ -theories that contain φ .

Lemma 6.4. *The collection of sets of the form $|\Box\varphi|$ for $\varphi \in \mathfrak{L}_{A,c}^\Box$ is a basis for an up-set topology on the set of all non-empty and non-trivial prime $BS4_{A,c}$ -theories, ordered by set inclusion.*

Proof. Fix a theory x . We know that $\varphi \in x$ for some φ . It follows that we have $A\psi \in x$ or $\neg A\psi \in x$ by (23). In either case, we have $x \in |\Box\chi|$ for some χ by (28) or (29). Hence, every x is an element of some basic set. The rest of the proof is similar to the proof of Lemma 3.4. $\square$

Definition 6.5. The $BS4_{A,c}$ -space is $\langle X, \subseteq, \mathcal{T} \rangle$ where X is the set of all non-empty non-trivial prime $BS4_{A,c}$ -theories and $\mathcal{T}$ is the topology generated by the basis comprising sets $|\Box\varphi|$ for all $\varphi \in \mathfrak{L}_{A,c}^\Box$.

The $BS4_{A,c}$ -space can be extended to an RM up-model $\mathbf{C}$ where $N, R, *$ and $\mathcal{V}$ are defined as usual in canonical models for relevant logics, namely by taking N to be the set of regular prime theories, defining $Rxyz$ iff $\forall \varphi, \psi$: if $\varphi \rightarrow \psi \in x$ and $\varphi \in y$, then $\psi \in z$; $x^* := \{\varphi \mid \neg\varphi \notin x\}$; and $\mathcal{V}(p) := \{x \in X \mid p \in x\}$. However, the Truth Lemma does not hold in $\mathbf{C}$. Similarly as in the intuitionistic setting, it is not the case that $A\varphi \in x$ iff $\varphi \in y$ for all $y \in X$. As in Section 3, we will solve this problem by anchoring $\mathbf{C}$ in a fixed $w \in X$ and restricting all the canonical relations to theories that agree with w on all global modal formulas. We can assume that w is a regular theory.

Definition 6.6. Let w be any regular non-trivial prime $BS4_{A,c}$ -theory. The *canonical up-model for $BS4_{A,c}$ anchored in w* is $\mathbf{C}^w = \langle X^w, \subseteq^w, \mathcal{T}^w, N^w, R^w, *, \mathcal{V}^w \rangle$ where:

- $X^w := \bigcap_{A\varphi \in w} |\Box\varphi| \cap \bigcap_{A\psi \notin w} |\Box\neg\psi|$ and $x \subseteq^w y$ iff $x \subseteq y$ and $x, y \in X^w$;
- $\mathcal{T}^w$ is the subspace topology on X^w derived from $\mathcal{T}$ (i.e. $U \in \mathcal{T}^w$ iff $\exists V \in \mathcal{T}: U = X^w \cap V$);
- $N^w := \{x \in X^w \mid x \text{ is regular}\}$
- R^wxyz iff $x, y, z \in X^w$ and $\forall \varphi, \psi$, if $\varphi \rightarrow \psi \in x$ and $\varphi \in y$, then $\psi \in z$;
- $x^{*^w} := \{\varphi \mid \neg\varphi \notin x\}$;
- $\mathcal{V}^w(p) := |p| \cap X^w$.

We define $|\varphi|^w := |\varphi| \cap X^w$.

Lemma 6.7. $\mathbf{C}^w$ is well-defined.

Proof. We have to show that $x \in X^w$ implies $x^{*^w} \in X^w$. We have $x^* \in X$ by De Morgan and double negation laws. Now suppose that $x^{*^w} \notin X^w$. If $A\varphi \in w$ and $A\varphi \notin x^*$, then $A\varphi \wedge \neg A\varphi \in x$ by (23). By (22), this contradicts the assumption that x is non-trivial. If $\neg A\psi \notin x^*$ for some $A\psi \notin w$, then $\neg\neg A\psi \in x$, which entails $A\psi \in x$ by (7). However, $\neg A\psi \in x$ as well, contradicting non-triviality of x by (22). $\square$

The following lemma establishes that $x \in X^w$ iff x agrees with w on all $A\varphi$.

Lemma 6.8. $x \in X^w$ iff $A\varphi \in x \iff A\varphi \in w$ for all φ .

Proof. Firstly, assume that $x \in X^w$. If $A\varphi \in w$, then $A\varphi \in x$ by definition. If $A\varphi \notin w$, then $\neg A\varphi \in x$ by definition and so $A\varphi \notin x$ by (22). Secondly, assume that $x \notin X^w$. Either there is $A\varphi \in w$ such that $A\varphi \notin x$ and we are done, or there is $A\psi \notin w$ such that $\neg A\psi \notin x$. In the latter case, $A\psi \in x$ by (23). $\square$

Definition 6.9. A pair $\langle \Delta, \nabla \rangle$ of subsets of $\mathfrak{L}_{A,c}^\Box$ is *$BS4_{A,c}$ -independent* if there are no $\varphi_1, \dots, \varphi_n \in \Delta$ and $\psi_1, \dots, \psi_m \in \nabla$ such that $\bigwedge_{i=1}^n \varphi_i \rightarrow \bigvee_{j=1}^m \psi_j$.

Lemma 6.10 (Pair extension for $BS4_{A,c}$). *If $\langle \Delta, \nabla \rangle$ is an $BS4_{A,c}$ -independent pair, then there is a prime $BS4_{A,c}$ -theory Δ' such that $\Delta \subseteq \Delta'$ and $\nabla \cap \Delta' = \emptyset$. If both Δ and ∇ are non-empty, then Δ' is non-empty and non-trivial.*

Proof. See Appendix A.1. $\square$

Lemma 6.11. $\mathcal{C}^w$ is an RM up-model.

Proof. (RM1) If $x \in N^w$, R^wxyz and $\phi \in y$, then $\phi \in z$ since $\vdash \phi \rightarrow \phi$. Hence, $y \subseteq z$.

(RM2) Consider any $y \in X^w$. We have to show that there is $x \in N^w$ such that R^wxy . Firstly, consider the pair $\langle T, \{\phi \rightarrow \psi \mid \phi \in y \ \& \ \psi \notin y\} \rangle$, where T is the set of theorems of $BS4_{A,c}$. This pair is independent. If it were not, then by adjunction (9) there would be a theorem χ such that $\vdash \chi \rightarrow \bigvee_{i=1}^n (\phi_i \rightarrow \psi_i)$ for $\phi_i \in y$ and $\psi_i \notin y$. This entails that $\vdash \chi \rightarrow (\bigwedge_{i=1}^n \phi_i \rightarrow \bigvee_{i=1}^n \psi_i)$, and so we obtain $\vdash \bigwedge_{i=1}^n \phi_i \rightarrow \bigvee_{i=1}^n \psi_i$. Since $\bigwedge_{i=1}^n \phi_i \in y$, we have $\psi_i \in y$ for some i , contradicting the assumption. Hence, the pair must be independent. Since y is non-empty and non-trivial, it follows by Lemma 6.10 that there is a regular $x \in X$ such that if $\phi \rightarrow \psi \in x$ and $\phi \in y$, then $\psi \in y$.

Now we prove that $x \in N^w$. Firstly, suppose that $A\phi \in w$ and $A\phi \notin x$. We have $\neg A\phi \in x$ by (23). Since $\psi \notin y$ for some ψ , we obtain $A\phi \rightarrow \psi \in x$ by (25). Since $y \in X^w$, we have $A\phi \in y$. It follows that $\psi \in y$, contradicting our assumption. Secondly, suppose that $A\psi \in x$ and $A\psi \notin w$. We have $\phi \in y$ for some ϕ and so $\phi \rightarrow A\psi \in x$ by (26). Consequently, $A\psi \in y$, contradicting the assumption that $y \in X^w$.

(RM3) is obviously satisfied and the rest is standard: (RM4) is established using contraposition and (RM5) holds thanks to double negation. $\square$

Lemma 6.12. Let Γ be a non-empty set of formulas and ψ an arbitrary formula. If $(\bigcap_{\phi \in \Gamma} |\phi|) \subseteq |\psi|$, then there is a finite $\Delta \subseteq \Gamma$ such that $\vdash (\bigwedge_{\delta \in \Delta} \delta) \rightarrow \psi$.

Proof. Standard argument using Zorn's lemma, the fact that $BS4_{A,c}$ is finitary and Lemma 6.10. $\square$

Lemma 6.13. For all χ , $|\chi|^w = \llbracket \chi \rrbracket_{\mathcal{C}^w}$.

Proof. Structural induction on χ . The only interesting cases are (i) $\chi = \phi \rightarrow \psi$, (ii) $\chi = \Box\phi$, (iii) $\chi = \Box^c\phi$ and (iv) $\chi = A\phi$.

(i) $\chi = \phi \rightarrow \psi$. If $\phi \rightarrow \psi \in x$ and R^wxyz where $\phi \in y$, then $\psi \in z$ by the definition of R^w . Conversely, if $\phi \rightarrow \psi \notin x$, then we can show that there are $y, z \in X$ such that $Rxyz$, $\phi \in y$ and $\psi \notin z$; see Appendix A.2. We prove that $y, z \in X^w$ as follows. Assume that $y \notin X^w$. Firstly, if there is $A\gamma \in w$ such that $A\gamma \notin y$, then we have $A\gamma \in x$ by definition and $\neg A\gamma \in y$ by (23). Since $\psi \notin z$ for some ψ , we obtain $\neg A\gamma \rightarrow \psi \in x$ by (24), and so $\psi \in z$. This is a contradiction. Secondly, assume that there is $A\gamma \notin w$ such that $A\gamma \in y$. Since $\psi \notin z$ for some ψ , we obtain $A\gamma \rightarrow \psi \in x$ by (25), and so $\psi \in z$. Contradiction. The fact that $z \in X^w$ is established similarly, using (26) and (27) and the fact that y is non-empty.

(ii) $\chi = \Box\phi$. This case is established similarly as in the proof of Lemma 3.11, using Lemma 6.12 and (13), (14), (15), (16), (28), (29).

(iii) $\chi = \Box^c\phi$. If $\Box^c\phi \in x$, then $x \in U = |\Box^c\phi|^w$ by (19). Obviously $U \in \mathcal{T}^w$. Moreover, $U \subseteq |\Box^c\phi|^w \subseteq (|\phi|^w)^c$ by (14) and (20). Hence, $x \in \text{Int}((|\phi|^w)^c)$, as desired. Conversely, assume that $x \in U$ such that $U \subseteq (|\phi|^w)^c$. Without loss of generality, we may assume that $x \in |\Box\psi|^w$ for some ψ such that $|\Box\psi|^w \subseteq (|\phi|^w)^c$. In other words, $|\Box\psi| \cap |\phi| \cap X^w = \emptyset$. By Lemma 6.12, there are $\{A\chi_i\}_{i=1}^n \subseteq w \cap x$ and $\{A\chi'_j\}_{j=1}^m \subseteq w^c$ such that

$$\vdash \Box\psi \wedge \phi \wedge \bigwedge_{i=1}^n A\chi_i \wedge \bigwedge_{j=1}^m \neg A\chi'_j \rightarrow \perp.$$

By (14) and (16), we obtain

$$\vdash \Box \left(\psi \wedge \bigwedge_{i=1}^n A\chi_i \wedge \bigwedge_{j=1}^m \neg A\chi'_j \right) \wedge \varphi \rightarrow \perp,$$

which we denote by $\vdash \Box \alpha \wedge \varphi \rightarrow \perp$. Using (21), $\vdash \Box \alpha \rightarrow \Box^c \varphi$. We know that $\Box \alpha \in x$, and so we conclude that $\Box^c \varphi \in x$, as desired.

(iv) $\chi = A\varphi$. This case is established similarly as in the proof of Lemma 3.11, using Lemma 6.10 and (13), (14), (15), (16) and (17). $\square$

Theorem 6.14. *A formula φ is valid in all RM up-spaces iff it is provable in $BS4_{A,c}$.*

Proof. Soundness is established in Lemma 6.2. Completeness: if $\not\vdash \varphi$, then $\langle T, \{\varphi\} \rangle$ is independent. Then there is a regular prime theory w such that $\varphi \notin w$. By Lemmas 6.7, 6.11 and 6.13, φ is not valid in all RM up-models. $\square$

7 Conclusion

This paper made the first steps in studying versions of Topological Evidence Logic (TEL) based on non-classical propositional logics. Specifically, we studied intuitionistic and relevant versions of TEL. We have shown that an extension of de Groot and Shillito's [13] modal logic iS4 with a global modality A can express TEL-style coherent justification (existence of a dense open subset). We have argued that a language with $\Box$ and A is not sufficient for a relevant version of TEL since coherent justification is not expressible. Accordingly, for relevant TEL we added an interior-of-complement operator $\Box^c$ and we showed that coherent justification is expressible in the language even in a weak relevant background. As our main technical result, we provided a sound and complete axiomatisation of a relevant TEL based on the weak modal relevant logic BS4. We have also provided a sound and complete axiomatisation of intuitionistic TEL based on iS4 with A .

The paper barely scratches the surface, leaving many interesting problems to be investigated in the future. Firstly, both iS4 and BS4 are decidable and we plan to extend the decidability result to iS4_A and BS4_{A,c}. It should not be too difficult to combine proofs of the finite model property with the anchoring technique. Secondly, it will be interesting to look at relevant TEL based on stronger propositional logics such as R or E. It is not clear how much the extra frame conditions needed for these logics interfere with the anchoring technique. Thirdly, it is intriguing in the setting of RM frames to replace topological spaces seen as locales of up-sets, where evidence combination is represented by set intersection, by *quantales* of up-sets where evidence combination is represented by the ternary relation R , and investigate the corresponding modal logic.

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A Technical appendix

A.1 Proof of the Pair Extension Lemma

We prove the lemma in a slightly more general setting that entails both Lemma 3.9 and Lemma 6.10; see [24, pp. 92–95]. Let L be any extension of B with extra axiom schemata and rules, over any language $\mathcal{L}'$ extending $\mathcal{L}$ such that $Fm_{\mathcal{L}'}$ is denumerable. We express the fact that φ is provable in L by writing $\vdash \varphi$. As expected, a pair $\langle \Delta, \nabla \rangle$ of subsets of $\mathcal{L}'$ is L -independent iff there are no $\varphi_1, \dots, \varphi_n \in \Delta$ and $\psi_1, \dots, \psi_m \in \nabla$ such that $\vdash \bigwedge_{i=1}^n \varphi_i \rightarrow \bigvee_{j=1}^m \psi_j$. (Prime, regular) L -theories are defined as expected.

Note that transitivity is a derived rule in L : if $\vdash_L \varphi \rightarrow \psi$ and $\vdash_L \psi \rightarrow \chi$, then applying the suffixing rule (12) to $\vdash_L \varphi \rightarrow \psi$ yields $\vdash (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi)$. Because we have $\vdash \psi \rightarrow \chi$, applying modus ponens (8) yields $\vdash \varphi \rightarrow \chi$.

Lemma A.1 (Pair Extension). *If $\langle \Delta, \nabla \rangle$ is an L -independent pair of subsets of $\mathcal{L}'$, then there is a prime L -theory Δ_∞ such that $\Delta \subseteq \Delta_\infty$ and $\Delta_\infty \cap \nabla = \emptyset$. Δ_∞ is non-empty and non-trivial if Δ and ∇ are both non-empty.*

To prove Lemma A.1, we first require a step lemma demonstrating that L -independence can be preserved when adding a new formula to either the left or right set.

Lemma A.2. *For all $\chi \in Fm_{\mathcal{L}'}$, if $\langle \Delta, \nabla \rangle$ is L -independent, then $\langle \Delta \cup \{\chi\}, \nabla \rangle$ is L -independent or $\langle \Delta, \nabla \cup \{\chi\} \rangle$ is L -independent.*

Proof. Suppose that both $\langle \Delta \cup \{\chi\}, \nabla \rangle$ and $\langle \Delta, \nabla \cup \{\chi\} \rangle$ are not L -independent. It follows that there are conjunctions φ, φ' of elements of Δ and disjunctions ψ, ψ' of elements of ∇ such that

$$\vdash_L (\varphi \wedge \chi) \rightarrow \psi \quad \text{and} \quad \vdash \varphi' \rightarrow (\chi \vee \psi').$$

(We use $\wedge$ -elimination (2) and $\vee$ -introduction (3) if needed.) By $\wedge$ -elimination (2) and transitivity, our second assumption entails $\vdash_L (\varphi \wedge \varphi') \rightarrow (\chi \vee \psi')$. Since $\vdash (\varphi \wedge \varphi') \rightarrow \varphi$ (2), we can use adjunction (9), $\wedge$ -introduction (4) and modus ponens (8) to obtain

$$\vdash_L (\varphi \wedge \varphi') \rightarrow (\varphi \wedge (\chi \vee \psi')).$$

By the distributivity axiom (6) and transitivity:

$$\vdash_L (\varphi \wedge \varphi') \rightarrow ((\varphi \wedge \chi) \vee (\varphi \wedge \psi')). \quad (30)$$

By $\wedge$ -elimination (2), $\vdash (\varphi \wedge \psi') \rightarrow \psi'$. By $\vee$ -introduction (3) and transitivity,

$$\vdash (\varphi \wedge \psi') \rightarrow ((\varphi \wedge \chi) \vee \psi').$$

By $\vee$ -introduction (3), $\vdash (\varphi \wedge \chi) \rightarrow ((\varphi \wedge \chi) \vee \psi')$. Hence, adjunction (9), $\vee$ -introduction (3) and modus ponens (8) yield $\vdash ((\varphi \wedge \chi) \vee (\varphi \wedge \psi')) \rightarrow ((\varphi \wedge \chi) \vee \psi')$. By transitivity (30), we obtain:

$$\vdash_L (\varphi \wedge \varphi') \rightarrow ((\varphi \wedge \chi) \vee \psi'). \quad (31)$$

Our first assumption $\vdash_L (\varphi \wedge \chi) \rightarrow \psi$ entails $\vdash (\varphi \wedge \chi) \rightarrow (\psi \vee \psi')$ by $\vee$ -introduction (3) and transitivity. By $\vee$ -introduction (3), adjunction, $\vee$ -elimination (5) and modus ponens (8), we obtain

$$\vdash ((\varphi \wedge \chi) \vee \psi') \rightarrow (\psi \vee \psi'). \quad (32)$$

Applying transitivity to (31) and (32) yields:

$$\vdash_L (\varphi \wedge \varphi') \rightarrow (\psi \vee \psi').$$

Because $\varphi \wedge \varphi'$ is a conjunction of elements of Δ , and $\psi \vee \psi'$ is a disjunction of elements of ∇ , this contradicts the assumption that $\langle \Delta, \nabla \rangle$ is L -independent. $\square$

Proof of Lemma A.1. Assume that the pair $\langle \Delta, \nabla \rangle$ is L -independent. Since $Fm_{\mathcal{L}'}$ is denumerable, we can assume its formulas are enumerated as $\langle \chi_i \rangle_{i=1}^\infty$. We define a sequence of pairs $\langle \Delta_i, \nabla_i \rangle$ from $i = 1$ to ∞ :

- $\Delta_1 = \Delta$ and $\nabla_1 = \nabla$
- $\langle \Delta_{i+1}, \nabla_{i+1} \rangle = \begin{cases} \langle \Delta_i \cup \{\chi_i\}, \nabla_i \rangle & \text{if } \langle \Delta_i \cup \{\chi_i\}, \nabla_i \rangle \text{ is } L\text{-independent;} \\ \langle \Delta_i, \nabla_i \cup \{\chi_i\} \rangle & \text{otherwise.} \end{cases}$

We define $\Delta_\infty = \bigcup_{i=1}^\infty \Delta_i$ and $\nabla_\infty = \bigcup_{i=1}^\infty \nabla_i$.

Claim 1: Pairs $\langle \Delta_i, \nabla_i \rangle$ are L -independent for all i . We proceed by induction on i . The base case $i = 1$ holds by assumption, and the induction step is established directly using Lemma A.2.

Claim 2: $\Delta \subseteq \Delta_\infty$ and $\nabla \cap \Delta_\infty = \emptyset$. The first part holds by definition. For the second part, suppose for a contradiction that there is a formula $\chi \in \nabla \cap \Delta_\infty$. Because $\chi \in \Delta_\infty$, we have $\chi \in \Delta_i$ for some i . Since $\nabla \subseteq \nabla_i$, we also have $\chi \in \nabla_i$. By the identity axiom (1), the presence of χ in both sets means the pair $\langle \Delta_i, \nabla_i \rangle$ is not L -independent. This contradicts Claim 1.

Claim 3: $\Delta_\infty \cup \nabla_\infty = Fm_{\mathcal{L}'}$. This follows from the construction. The sequence $\langle \chi_i \rangle_{i=1}^\infty$ enumerates all formulas in $Fm_{\mathcal{L}'}$, and at each step i , the formula χ_i is added to either Δ_{i+1} or ∇_{i+1} .

Claim 4: Δ_∞ is a prime L -theory. Firstly, we show that Δ_∞ is closed under L -provable implications. Assume $\vdash_L \varphi \rightarrow \psi$ and $\varphi \in \Delta_\infty$. If $\psi \notin \Delta_\infty$, then $\psi \in \nabla_\infty$ by Claim 3. Thus, there exists an index k large enough such that both $\varphi \in \Delta_k$ and $\psi \in \nabla_k$. However, since $\vdash_L \varphi \rightarrow \psi$, $\langle \Delta_k, \nabla_k \rangle$ is not L -independent, contradicting Claim 1. Therefore, $\psi \in \Delta_\infty$.

Secondly, we show that Δ_∞ is closed under conjunction. Assume $\varphi, \psi \in \Delta_\infty$. This implies there exist indices i, j such that $\varphi \in \Delta_i$ and $\psi \in \Delta_j$. Let $k = \max(i, j)$, which ensures $\varphi, \psi \in \Delta_k$. If $\varphi \wedge \psi \notin \Delta_\infty$, then $\varphi \wedge \psi \in \nabla_\infty$, meaning there is some index m such that $\varphi \wedge \psi \in \nabla_m$. By taking $n = \max(k, m)$, we have $\varphi, \psi \in \Delta_n$ and $\varphi \wedge \psi \in \nabla_n$. Since $\vdash (\varphi \wedge \psi) \rightarrow (\varphi \wedge \psi)$ by (1), this contradicts the L -independence of $\langle \Delta_n, \nabla_n \rangle$. Thus, $\varphi \wedge \psi \in \Delta_\infty$.

Finally, we show that Δ_∞ is prime. Let $\varphi \vee \psi \in \Delta_\infty$. Suppose for a contradiction that $\varphi \notin \Delta_\infty$ and $\psi \notin \Delta_\infty$. By Claim 3, this implies $\varphi, \psi \in \nabla_\infty$. Consequently, there exists an index k large enough such that $\varphi \vee \psi \in \Delta_k$ and $\varphi, \psi \in \nabla_k$. Since $\vdash (\varphi \vee \psi) \rightarrow (\varphi \vee \psi)$ by (1), $\langle \Delta_k, \nabla_k \rangle$ is not L -independent, which once again contradicts Claim 1. Therefore, either $\varphi \in \Delta_\infty$ or $\psi \in \Delta_\infty$. $\square$

Since $BS4_{A,c}$ is an extension of B in a denumerable language, Lemma A.1 entails Lemma 6.10. Since intuitionistic propositional logic can be axiomatised by a Hilbert-style system that extends B , the system $iS4_A$ is also an extension of B in a denumerable language. Hence, Lemma A.1 entails Lemma 3.9.

A.2 Truth Lemma for relevant implication

Let L be any extension of B with extra axiom schemata and rules, over any language $\mathcal{L}'$ extending $\mathcal{L}$ such that $Fm_{\mathcal{L}'}$ is denumerable. Hence, Lemma A.1 applies. We write $\vdash \varphi$ to indicate that φ is provable in L . Assume that for each $\varphi, \psi \in Fm_{\mathcal{L}'}$ there is a formula $\omega(\varphi, \psi) \in Fm_{\mathcal{L}'}$ such that $\vdash \varphi \rightarrow (\psi \rightarrow \omega(\varphi, \psi))$ and a formula $\omega'(\varphi, \psi)$ such that $\vdash \varphi \rightarrow (\omega'(\varphi, \psi) \rightarrow \psi)$.

We note that such formulas are available in B over an extended language containing *fusion* $\circ$ and *converse implication* $\leftarrow$ since they come with axioms $\varphi \rightarrow (\psi \rightarrow (\varphi \circ \psi))$ and $\varphi \rightarrow ((\psi \leftarrow \varphi) \rightarrow \psi)$. However, any extension of B containing the axioms for A is also sufficient since there we have $\vdash \varphi \rightarrow (\psi \rightarrow E\varphi)$ and $\vdash \varphi \rightarrow (\neg E\varphi \rightarrow \psi)$ by (14), (25) and (27).

Let X be the set of all non-empty non-trivial prime L -theories. We define a ternary relation R on X by stipulating that $Rxyz$ iff $\forall \varphi, \psi \in Fm_{\mathcal{L}}$, if $\varphi \rightarrow \psi \in x$ and $\varphi \in y$, then $\psi \in z$.

Lemma A.3. $\varphi \rightarrow \psi \in x$ iff $\forall y, z \in X$: if $Rxyz$ and $\varphi \in y$, then $\psi \in z$.

Proof. The left-to-right implication follows from the definition of R . The converse implication is established as follows. Assume that $\varphi \rightarrow \psi \notin x$. We define $y_0 = \{\alpha \mid \vdash \varphi \rightarrow \alpha\}$. Take the following pair:

$$\Delta_z = \{\beta \mid \exists \alpha \in y_0: \alpha \rightarrow \beta \in x\} \quad \nabla_z = \{\psi\}$$

Note that $\varphi \in y_0$ and consequently Δ_z is non-empty: since $\psi \in x$ for some ψ and $\vdash \psi \rightarrow (\varphi \rightarrow \omega(\psi, \varphi))$, we have $\varphi \rightarrow \omega(\psi, \varphi) \in x$, meaning that $\omega(\psi, \varphi) \in \Delta_z$. We prove that $\langle \Delta_z, \nabla_z \rangle$ is L -independent. If not, there is a conjunction $\beta = \beta_1 \wedge \dots \wedge \beta_n$ of elements of Δ_z such that $\vdash \beta \rightarrow \psi$. For each β_i there is $\alpha_i \in y_0$ such that $\alpha_i \rightarrow \beta_i \in x$. Let $\alpha = \alpha_1 \wedge \dots \wedge \alpha_n$. Since $\vdash \alpha \rightarrow \alpha_i$ for all i , we have $\vdash (\alpha_i \rightarrow \beta_i) \rightarrow (\alpha \rightarrow \beta_i)$ using suffixing (12). Hence, $\alpha \rightarrow \beta_i \in x$ for all i , which means that $\alpha \rightarrow \beta \in x$ by repeated application of $\wedge$ -introduction (4). Since $\vdash \beta \rightarrow \psi$, we obtain $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \psi)$ by suffixing (12). Hence, $\alpha \rightarrow \psi \in x$. Since $\alpha_i \in y_0$ for all i , we have $\vdash \varphi \rightarrow \alpha_i$, which implies $\vdash \varphi \rightarrow \alpha$ using adjunction (9) and $\wedge$ -introduction (4). By suffixing (12), $\vdash (\alpha \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$. Because $\alpha \rightarrow \psi \in x$, we obtain $\varphi \rightarrow \psi \in x$, contradicting our assumption. Hence, $\langle \Delta_z, \nabla_z \rangle$ must be L -independent. Since both Δ_z and ∇_z are non-empty, Lemma A.1 implies that there is a non-empty non-trivial prime L -theory z extending Δ_z and disjoint from ∇_z .

Now consider the following pair:

$$\Delta_y = \{\varphi\} \quad \nabla_y = \{\gamma \mid \exists \delta \notin z: \gamma \rightarrow \delta \in x\}$$

Note that $\chi \in x$ and $\delta \notin z$ for some χ, δ . Since $\vdash \chi \rightarrow (\omega'(\chi, \delta) \rightarrow \delta)$, we have $\omega'(\chi, \delta) \rightarrow \delta \in x$. Hence, $\omega'(\chi, \delta) \in \nabla_y$. We prove that $\langle \Delta_y, \nabla_y \rangle$ is L -independent. If not, then there is a disjunction $\gamma = \gamma_1 \vee \dots \vee \gamma_m$ such that $\vdash \varphi \rightarrow \gamma$. For each i , there is $\delta_i \notin z$ such that $\gamma_i \rightarrow \delta_i \in x$. We have $\vdash \delta_i \rightarrow \delta$ for all i and so, by prefixing (11), $\vdash (\gamma_i \rightarrow \delta_i) \rightarrow (\gamma_i \rightarrow \delta)$. Since $\gamma_i \rightarrow \delta_i \in x$ for all i , we obtain $\gamma_i \rightarrow \delta$ for all i . By closure under conjunction and $\vee$ -introduction (3), $\gamma \rightarrow \delta \in x$. Since we assume that $\vdash \varphi \rightarrow \gamma$, we obtain $\vdash (\gamma \rightarrow \delta) \rightarrow (\varphi \rightarrow \delta)$ by suffixing (12). Hence, $\varphi \rightarrow \delta \in x$. We know that $\varphi \in y_0$ and so $\delta \in \Delta_z$. Since $\Delta_z \subseteq z$, we have $\delta \in z$. This means that $\delta_i \in z$ for some i , contradicting our assumption. Hence, $\langle \Delta_y, \nabla_y \rangle$ is L -independent. Since both Δ_y and ∇_y are non-empty, Lemma A.1 implies that there is a non-empty non-trivial prime L -theory y extending Δ_y and disjoint from ∇_y .

By construction of y and z , we know that $\varphi \in y$ and $\psi \notin z$. To show that $Rxyz$, assume $\alpha \rightarrow \beta \in x$ and $\beta \notin z$. Then $\alpha \notin y$ by the definition of ∇_y , as desired. $\square$

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Uniform Lyndon Interpolation via Non-wellfounded Proofs*

Borja Sierra Miranda

University of Bern
Bern, Switzerland

borja.sierra@unibe.ch

Thomas Studer

University of Bern
Bern, Switzerland

thomas.studer@unibe.ch

Non-wellfounded proof theory has been applied to establish uniform interpolation and Lyndon interpolation (separately) for multiple logics. However, it has not yet been used to prove uniform Lyndon interpolation. We close this gap by showing uniform Lyndon interpolation for the provability logic GLS. This logic was known to have uniform interpolation, but it was open whether it has uniform Lyndon interpolation (or at least non-uniform Lyndon interpolation). The methodology we provide is easy to adapt to other provability logics if a non-wellfounded sequent calculus is available for them. In addition, we offer an alternative proof of cut elimination for GLS via non-wellfounded proofs.

1 Introduction

Non-wellfounded proof theory provides a powerful machinery to study of logics with explicit fixpoints, see e.g. [10, 11, 16, 5, 2, 3, 12, 23]. However, its use is not restricted to these logics. In particular, provability logics are closely connected to a simple class of non-wellfounded proof systems [22, 21, 6, 24, 13, 20].

Non-wellfounded sequent calculi have been successfully applied to obtain many interpolation results. In particular, they have been used to establish Craig interpolation [9, 15], Lyndon interpolation [22] and uniform interpolation [1, 14]. Still, the combination of uniform and Lyndon interpolation (the so-called uniform Lyndon interpolation property [4, 17]) has not been studied with these methods. This paper is the first investigation in that direction, using non-wellfounded proofs to establish uniform Lyndon interpolation of the logic GLS.

Solovay's logic GLS (also known simply as S) is the provability logic corresponding to the provability predicate of PA according to true arithmetic [26], and it is known to have Craig interpolation [8, 7, 18].¹ No proof that it has Lyndon or uniform Lyndon interpolation is known. Usually for such a proof the use of non-wellfounded proof theory is beneficial, as non-wellfounded sequent calculi get rid of what is usually called the *diagonal formula*, which, in wellfounded proofs, does not preserve the polarity. In this paper we will show that, indeed, GLS has uniform Lyndon interpolation using non-wellfounded proofs.

Contributions. In this paper the contributions are three-fold.

1. An alternative proof of cut elimination for GLS is provided using non-wellfounded proofs (the original proof using ideas from the wellfounded cut elimination of GL can be found in [7] and more detailed in [18]).

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¹Uniform interpolation for GLS follows easily from the fact that GL has uniform interpolation, by the same proof as in [7]. Note though that the translation between GL and GLS described in that paper does not yield Lyndon interpolation, as it does not preserve the polarity of variables.

2. The introduction of the concepts of Lyndon fixpoints and Lyndon equational systems, which are necessary to show that the uniform interpolant created via non-wellfounded proofs respects polarities.
3. The first proof that GLS has uniform Lyndon interpolation.

Organization. In Section 2 we will introduce the concepts needed from non-wellfounded proof theory. In Section 3 we will give the definition of GLS and its sequent calculi, with an alternative proof of cut elimination using non-wellfounded proofs. In Section 4 we will introduce the notions of Lyndon fixpoint and Lyndon equational system, which are at the heart of our proof of uniform Lyndon interpolation. We conclude by proving the promised result, uniform Lyndon interpolation for GLS.

2 Preliminaries: local progress proof theory

Let us start by fixing the notions concerning trees. Given a set X we will write X^* (X^+) to mean the set of (non-empty) finite sequences of elements in X , ε will denote the empty sequence. As usual, we will write $w \leq v$ to mean that w is an initial prefix of v and $(w, v]$ to mean the set $\{u \in X^* \mid w < u \leq v\}$. A (finitely branching) tree on A is a function T whose image is contained in A and whose domain is a prefix-closed non-empty subset of $\mathbb{N}^*$ such that for every $w \in \text{Dom}(T)$ there is a unique natural number k (called the *arity of w in T*) such that $wi \in \text{Dom}(T)$ iff $i < k$. The $wi \in \text{Dom}(T)$ are also called the *immediate successors of w* . Note that the domain of any tree always contains the word ε , which is called the *root of T* .

The elements of $\text{Dom}(T)$ are also called the *nodes of T* , the 0-ary nodes are called *leaves* and the rest of the nodes are called *interior nodes*. Finally, an *infinite branch in a tree T* is sequence of nodes $(w_i)_{i \in \mathbb{N}}$ such that $w_0 = \varepsilon$ and for each i there is a j such that $w_{i+1} = w_i j$.

Fix a set Seq whose elements we will call *sequents*, a sequent rule is a subset of Seq^+ . Given a rule R , we say that $(S_0, \dots, S_{n-1}, S)$ is an *instance of R with premises $S_0, \dots, S_{n-1}$ and conclusion S* if $(S_0, \dots, S_{n-1}, S) \in R$. A rule R is said to be *n -ary* if $R \subseteq \text{Seq}^{n+1}$. We introduce the kind of sequent calculi we are going to use, called *local progress sequent calculi*.

Definition 2.1. A *local progress sequent calculus* is a pair $\mathcal{G} = (\mathcal{R}, (L_R)_{R \in \mathcal{R}})$ where $\mathcal{R}$ is a set of sequent rules and each L_R is a function that takes an instance $r = (S_0, \dots, S_{n-1}, S)$ of R and returns a subset of $\{0, \dots, n-1\}$. $\mathcal{G}$ is said to be *wellfounded* if each L_R is the constant function returning $\emptyset$.

We are prepared to define the notion of proof in a local progress calculus. From the definition of proof we can infer that a wellfounded sequent calculus is just a sequent calculus in the usual (wellfounded) proof theory.

Definition 2.2. Given a local progress sequent calculus $\mathcal{G} = (\mathcal{R}, (L_R)_{R \in \mathcal{R}})$, a *preproof in $\mathcal{G}$* is a tree with labels in $\text{Seq} \times \mathcal{R}$ such that for each node w with immediate successors $w0, \dots, w(n-1)$ we have that $(S_0, \dots, S_{n-1}, S) \in R$ where S is the sequent at w , R is the rule at w and S_i is the sequent at wi .

Let w be a node in π with immediate successors $w0, \dots, w(n-1)$. We say that wi is a *progressing node* if $i \in L_R(r)$ where R is the rule at w and $r = (S_0, \dots, S_{n-1}, S)$ where S is the sequent at w and S_i the sequent at wi . A proof is a preproof in which any infinite branch has infinitely many progressing nodes.

We will write $\mathcal{G} \vdash S$ to mean that S is provable in $\mathcal{G}$ and $\pi \vdash_{\mathcal{G}} S$ to mean that π is a proof of S in $\mathcal{G}$, omitting the subscript $\mathcal{G}$ when it is clear from context. It will be common to write an instance $(S_0, \dots, S_{n-1}, S)$ of a rule R as

$$\frac{S_0 \quad \dots \quad S_{n-1}}{S} R$$

Given a proof π in $\mathcal{G}$ we will define its main local fragment as the finite tree obtained from cutting the tree at the first progressing nodes from the root (in particular removing the progressing nodes). The *local height* of π , denoted $\text{lhg}(\pi)$, is the height of its main local fragment. The *local rules* of π , denoted $\text{IRul}(\pi)$, is the set of rules occurring in the main local fragment. We will say that π is *locally R-free* if $R \notin \text{IRul}(\pi)$. We note that if $\mathcal{G}$ is a wellfounded calculus the notion of local height agrees with the usual notion of height and the local rules is the set of rules occurring in the proof.

Given a local progress calculus $\mathcal{G} = (\mathcal{R}, (L_R)_{R \in \mathcal{R}})$ and a rule R' we define $\mathcal{G} + R'$ as the local progress calculus with rules $\mathcal{R} \cup \{R'\}$ and $L_{R'}$ the constant function returning $\emptyset$. Rules can interact with local progress calculi in different ways, the following definition introduce some of these ways and the theorem below shows that some of them are equivalent.

Definition 2.3. Let $\mathcal{G}$ be a sequent calculus and R be a rule. We say that:

1. R is admissible if for any instance $(S_0, \dots, S_{n-1}, S) \in R$ we have that $\pi_i \vdash_{\mathcal{G}} S_i$ for $i < n$ implies the existence of $\pi \vdash_{\mathcal{G}} S$. In addition we say that R is *admissible preserving local height* if $\text{lhg}(\pi) \leq \max_{i < n} \text{lhg}(\pi_i)$ and *admissible preserving local rules* if $\text{IRul}(\pi) \subseteq \bigcup_{i < n} \text{IRul}(\pi_i)$.
2. R is eliminable if $\mathcal{G} + R \vdash S$ implies $\mathcal{G} \vdash S$.²
3. R is locally admissible if for any instance $(S_0, \dots, S_{n-1}, S) \in R$ we have that if there are locally R -free $\pi_i \vdash_{\mathcal{G}} S_i$ for $i < n$ then there is a locally R -free $\pi \vdash_{\mathcal{G}} S$.
4. An n -ary rule R is i -invertible in $\mathcal{G}$ (where $i < n$) if the rule $\{(S, S_i) \mid (S_0, \dots, S_{n-1}, S) \in R\}$ is admissible. R is *invertible* if it is i -invertible for each $i < n$.

We will talk about invertibility preserving local height and/or local rules with the obvious meaning.

Given an admissible rule R in $\mathcal{G}$ an instance $(S_0, \dots, S_{n-1}, S) \in R$ and proofs $\pi_i \vdash_{\mathcal{G}} S_i$ for $i < n$ we will write $R(\pi_0, \dots, \pi_{n-1})$ to mean a proof of S in $\mathcal{G}$ that exists by admissibility. For the invertibility of a rule R we will write instead $\text{inv}R(\pi)$.

The following result allows an easy development of local progress proof theory (for the details see [13, 24]).

Theorem 2.4. Let $\mathcal{G}$ be a local progress sequent calculus, then R is eliminable in $\mathcal{G}$ iff R is locally admissible in $\mathcal{G}$. If $\mathcal{G}$ is wellfounded, then both are equivalent to R is admissible in $\mathcal{G}$.

Finally, sometimes we will need to work with non-wellfounded proofs that have a particular finite representation. To define this notion we introduce the notion of trees with backedges. A *tree with backedges* is an ordered pair $\tau = (T, (\cdot)^\circ)$ such that T is a finite tree and $(\cdot)^\circ$ is a function from a subset of the leafs of τ to the nodes of τ such that if w is in the domain then $w^\circ < w$, the nodes in the domain of $(\cdot)^\circ$ are called *repeat nodes*.

Definition 2.5. Given a local progress calculus $\mathcal{G} = (\mathcal{R}, (L_R)_{R \in \mathcal{R}})$, a *cyclic preproof* in $\mathcal{G}$ is a tree with backedges $\pi = (T, (\cdot)^\circ)$ on $\text{Seq} \times \mathcal{R}$ such that for any non-repeat node w with immediate successors $w_0, \dots, w_{n-1}$, we have that $(S_0, \dots, S_{n-1}, S) \in R$ where S is the sequent at w , R is the rule at w and S_i is the sequent at w_i . and for any repeat node w , w and w° have the same sequent and rule. A *cyclic proof* in $\mathcal{G}$ is a cyclic preproof such that for any repeat node w there is a progressing node in $(w^\circ, w]$.

In cyclic preproofs we will annotate repeat nodes with the word Rep at the right, instead of the rule at the node.

²In the proof-theoretic tradition, it is common to study the concept of effective cut elimination, i.e. to study who to get rid of the cut rule in proofs using computable (effective) means. Often, cut-elimination is understood as effective cut-elimination and non-effective cut-elimination is called cut admissibility. This is justified since eliminability and admissibility, as given in Definition 2.3, are equivalent for wellfounded proofs. However, this equivalence does not hold for non-wellfounded proofs. The best approximation we can get is Theorem 2.4. For this reason, we will not follow the tradition. For us, a rule being eliminable means exactly what it says: that the rule can be eliminated from the calculus without affecting the provability of sequents. In case we want to talk about the effectiveness of our methodology, we will say effective cut elimination explicitly.

$$\begin{array}{c}
\frac{}{p, \Gamma \gg p, \Delta} \text{ax} \quad \frac{}{\perp, \Gamma \gg \Delta} \perp\text{L} \quad \frac{\Gamma \gg \Delta}{\Gamma \gg \Delta, \perp} \perp\text{R} \\
\\
\frac{\Gamma \gg \phi, \Delta \quad \psi, \Gamma \gg \Delta}{\phi \rightarrow \psi, \Gamma \gg \Delta} \rightarrow\text{L} \quad \frac{\phi, \Gamma \gg \psi, \Delta}{\Gamma \gg \phi \rightarrow \psi, \Delta} \rightarrow\text{R} \\
\\
\frac{\Box \Sigma, \Box \phi \Rightarrow \phi}{\Box \Sigma, \Gamma \gg \Box \phi, \Delta} \Box^{\text{GL}} \quad \frac{\Box \Sigma \Rightarrow \phi}{\Box \Sigma, \Gamma \gg \Box \phi, \Delta} \Box^{\text{K}^4} \quad \frac{\Box \phi, \Gamma \Rightarrow \Delta}{\Box \phi, \Gamma \Rightarrow \Delta} \text{T}
\end{array}$$

Figure 1: Sequent Rules

3 The logic GLS and its sequent calculi

We fix an infinite countable set Var whose elements are called *propositional variables*.

Definition 3.1. We define the language $\mathcal{L}_{\Box}$ given by the following Backus-Naur form:

$$\phi ::= p \mid \perp \mid \phi \rightarrow \phi \mid \Box \phi,$$

where $p \in \text{Var}$. The expressions of $\mathcal{L}_{\Box}$ are called (*modal*) *formulas*. The complexity of a formula ϕ is the number of logical connectives of ϕ , and will be denoted as $|\phi|$.

Given a multisets of formulas Γ , we will write $\Box \Gamma$ to mean the multiset $\{\Box \phi \mid \phi \in \Gamma\}$ and $\Box \Gamma$ to mean the multiset $\Gamma \cup \Box \Gamma$. Then, for a formula ϕ , $\Box \phi$ will be the multiset with two elements $\{\phi, \Box \phi\}$.

Definition 3.2. We define the logic GL as the smallest set of formulas such that

1. every classical propositional tautology in $\mathcal{L}_{\Box}$ is in GL,
2. (K) $\Box(\phi \rightarrow \psi) \rightarrow \Box \phi \rightarrow \Box \psi$ is in GL,
3. (L) $\Box(\Box \phi \rightarrow \phi) \rightarrow \Box \phi$ is in GL,

and that is closed under the rules

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \text{MP} \quad \frac{\phi}{\Box \phi} \text{NEC}$$

The elements of GL are also called the *theorems of GL*. We will write $\text{GL} \vdash \phi$ to mean $\phi \in \text{GL}$.

We define the logic GLS as the smallest set of formulas such that

1. every theorem of GL is in GLS,
2. (T) $\Box \phi \rightarrow \phi$ is in GLS,

and that is closed under the rule (MP). The elements of GLS are also called the *theorems of GLS*. We will write $\text{GLS} \vdash \phi$ to mean $\phi \in \text{GLS}$.

Now we will define sequent calculi for GLS. We start by fixing the notion of sequent we will work with. A *sequent* is a triple (Γ, Δ, i) where Γ, Δ are multisets of formulas and $i \in \{0, 1\}$. We write $(\Gamma, \Delta, 0)$ as $\Gamma \Rightarrow \Delta$ and $(\Gamma, \Delta, 1)$ as $\Gamma \Rightarrow \Delta$. We use $\gg$, possibly with subindexes, to mean either $\Rightarrow$ or $\Rightarrow$. Given a multiset Σ , we let Σ^s be the *set* with the same elements as Σ (but without repetitions).

We introduce some terminology related to the rules of Figure 1. In $(\perp\text{R})$, $(\rightarrow\text{L})$, $(\rightarrow\text{R})$, (T), $(\Box^{\text{GL}})$ and $(\Box^{\text{K}^4})$ the displayed formula at the conclusion is called *principal formula*. In $(\Box^{\text{GL}})$ and $(\Box^{\text{K}^4})$ the formulas at the conclusion in $\Box \Sigma$ will be called *auxiliary*. In $(\Box^{\text{GL}})$ the formula $\Box \phi$ at the premise is called *diagonal formula*. In (ax), $(\perp\text{L})$, $(\Box^{\text{GL}})$ and $(\Box^{\text{K}^4})$ the formulas in Γ and Δ will be said to belong to the *weakening part*. Note that the weakneing part of any rule instance can be changed arbitrarily and still we will have a rule instance of the same rule.

$$\frac{\Gamma \gg \Delta}{\Gamma, \Gamma' \gg \Delta, \Delta'} \text{Wk} \quad \frac{\Gamma, \Gamma, \Gamma' \gg \Delta, \Delta, \Delta'}{\Gamma, \Gamma' \gg \Delta, \Delta'} \text{Ctr} \quad \frac{\Gamma \gg \Delta, \chi \quad \chi, \Gamma \gg \Delta}{\Gamma \gg \Delta} \text{Cut}$$

Figure 2: Structural Rules

Definition 3.3. We make the following definitions.

1. $\mathcal{G}\text{GLS}$ is the wellfounded calculus with rules (ax), ($\perp\text{L}$), ($\perp\text{R}$), ($\rightarrow\text{L}$), ($\rightarrow\text{R}$), ($\Box^{\text{GL}}$) and (T).³
2. $\mathcal{G}^\infty\text{GLS}$ is the local progress calculus with rules (ax), ($\perp\text{L}$), ($\perp\text{R}$), ($\rightarrow\text{L}$), ($\rightarrow\text{R}$), ($\Box^{\text{K}4}$) and (T). Progress only occurs at the premise of ($\Box^{\text{K}4}$).

Lemma 3.4. For any formula ϕ and multisets Γ, Δ we have that

1. $\mathcal{G}\text{GLS} \vdash \phi, \Gamma \gg \phi, \Delta$ and $\mathcal{G}^\infty\text{GLS} \vdash \phi, \Gamma \gg \phi, \Delta$. We denote the use of this inside proofs as (Ax).
2. The following rule is admissible in $\mathcal{G}\text{GLS}$ (+Cut) and in $\mathcal{G}^\infty\text{GLS}$ (+Cut):

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{Prom}$$

Proof. The proof of 1. is by induction on ϕ while the proof of 2. is by induction on the local height of the proof. $\square$

In Figure 2 we can see some common structural rules. As usual, these rules behave well with the sequent calculi, as we record in the following lemma. The proof, which is omitted due to constraints of space, is just using induction on the local height and using Theorem 2.4 when needed.

Lemma 3.5. In $\mathcal{G}\text{GLS}$ (+Cut) and in $\mathcal{G}^\infty\text{GLS}$ (+Cut) we have that

1. Wk is eliminable and admissible preserving local height and local rules,
2. ($\perp\text{R}$), ($\rightarrow\text{L}$), ($\rightarrow\text{R}$) are invertible preserving local height and local rules,
3. Ctr is admissible preserving local height and local rules.

Using the admissibility of structural rules is easy to show the following (see [18]).

Theorem 3.6. For any multisets Γ, Δ we have that

$$\mathcal{G}\text{GLS} + \text{Cut} \vdash \Gamma \Rightarrow \Delta \text{ iff } \text{GL} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta \quad \text{and} \quad \mathcal{G}\text{GLS} + \text{Cut} \vdash \Gamma \Rightarrow \Delta \text{ iff } \text{GLS} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta.$$

3.1 Translations between wellfounded and non-wellfounded proofs

To establish cut elimination for $\mathcal{G}\text{GLS}$ via cut elimination for $\mathcal{G}^\infty\text{GLS}$, we will need to be able to translate proofs between both calculi. In this section we define the necessary translations. The following lemma follows from using (Cut) and ($\Box^{\text{GL}}$).

Lemma 3.7 (Löb's rule). The following rule is admissible in $\mathcal{G}\text{GLS} + \text{Cut}$:

$$\frac{\Box \Sigma, \Box \phi \Rightarrow \phi}{\Box \Sigma \Rightarrow \phi} \text{Löb}$$

Lemma 3.8. If $\mathcal{G}\text{GLS} + \text{Cut} \vdash \Gamma \Rightarrow \Delta$, then $\mathcal{G}^\infty\text{GLS} + \text{Cut} \vdash \Gamma \Rightarrow \Delta$.

³This calculus is an minor variation of the ones appearing in [7, 18].

Proof. We define corecursively a function α from proofs in $\mathcal{G}\text{GLS} + \text{Cut}$ to proofs in $\mathcal{G}^\infty\text{GLS} + \text{Cut}$ by cases on the last rule of the input proof. It commutes with all rules different from $(\Box^{\text{GL}})$, and for $(\Box^{\text{GL}})$ it is defined as

$$\frac{\pi_0}{\frac{\Box \Sigma, \Box \phi \Rightarrow \phi}{\Box \Sigma, \Gamma \Rightarrow \Box \phi, \Delta} \Box^{\text{GL}}} \xrightarrow{\alpha} \frac{\alpha(\text{Löb}(\pi_0))}{\frac{\Box \Sigma \Rightarrow \phi}{\Box \Sigma, \Gamma \Rightarrow \Box \phi, \Delta} \Box^{\text{K4}}}$$

We notice that $\alpha(\pi)$ is indeed a proof and not a preproof, the corecursive call only increase in height if progress is made in the path from the root to the corecursive call. $\square$

Definition 3.9. Let ϕ be a formula. We define its *set of subformulas*, denoted $\text{Sub}(\phi)$, recursively on ϕ as

$$\begin{aligned} \text{Sub}(p) &= \{p\}, & \text{Sub}(\perp) &= \{\perp\}, \\ \text{Sub}(\phi_0 \rightarrow \phi_1) &= \{\phi_0 \rightarrow \phi_1\} \cup \text{Sub}(\phi_0) \cup \text{Sub}(\phi_1), & \text{Sub}(\Box \phi_0) &= \{\Box \phi_0\} \cup \text{Sub}(\phi_0). \end{aligned}$$

Given a multiset Γ we will write $\text{Sub}(\Gamma)$ to denote the set $\bigcup_{\phi \in \Gamma} \text{Sub}(\phi)$, and given a sequent $\Gamma \gg \Delta$ we will write $\text{Sub}(\Gamma \gg \Delta)$ to denote $\text{Sub}(\Gamma) \cup \text{Sub}(\Delta)$.

The following lemma follows by inspection of Figure 1.

Lemma 3.10 (Local subformula property). *Let*

$$\frac{S_0 \quad \cdots \quad S_{n-1}}{S} R$$

be an instance of a rule of $\mathcal{G}\text{GLS}$ or $\mathcal{G}^\infty\text{GLS}$. Then $\text{Sub}(S_i) \subseteq \text{Sub}(S)$ for $i < n$.

Lemma 3.11. *For any finite set Λ , we have that $\mathcal{G}^\infty\text{GLS} \vdash \Gamma \gg \Delta$ implies $\mathcal{G}\text{GLS} \vdash \Box \Lambda, \Gamma \gg \Delta$.*

Proof. Let $\pi \vdash \Gamma \gg \Delta$ in $\mathcal{G}^\infty\text{GLS}$, we proceed by induction on the measure $\omega \cdot |\text{Sub}(\Gamma \gg \Delta) \setminus \Lambda| + \text{lh}(\pi)$ and cases on the last rule of π . If the last rule of π is not $\Box^{\text{K4}}$ then it suffices to apply the induction hypothesis and reapply the rule. Now assume that the last rule of π is $\Box^{\text{K4}}$, i.e., π is

$$\frac{\pi_0}{\frac{\Box \Sigma \Rightarrow \phi}{\Box \Sigma, \Gamma' \gg \Box \phi, \Delta'} \Box^{\text{K4}}}$$

where $\Gamma = \Box \Sigma, \Gamma'$ and $\Delta = \Box \phi, \Delta'$. Let us denote the conclusion of π_0 as S_0 and the conclusion of π as S . We proceed by cases. If $\phi \in \Lambda$ the desired proof is obtained by using (Ax) , as $\Box \phi$ will be in both sides of the sequent. Now assume that $\phi \notin \Lambda$, then $|\text{Sub}(S_0) \setminus (\Lambda \cup \{\phi\})| < |\text{Sub}(S_0) \setminus \Lambda| \leq |\text{Sub}(S) \setminus \Lambda|$, so by the induction hypothesis we have a proof $\tau \vdash \Box \phi, \Box \Lambda, \Box \Sigma \Rightarrow \phi$. The desired proof is then

$$\frac{\text{Wk}(\tau)}{\frac{\Box \phi, \Box \Lambda, \Box \Sigma \Rightarrow \phi}{\Box \Lambda, \Box \Sigma, \Gamma' \gg \Box \phi, \Delta'} \Box^{\text{GL}}}$$

$\square$

3.2 Cut elimination

Finally, we show local admissibility of Cut in $\mathcal{G}^\infty\text{GLS}$ which implies its eliminability. Thanks to the translations of the previous subsection we also obtain cut elimination for $\mathcal{G}\text{GLS}$.

Theorem 3.12. *Cut is locally admissible in $\mathcal{G}^\infty\text{GLS}$. As a corollary, Cut is eliminable in $\mathcal{G}^\infty\text{GLS}$.*

Proof. Let $\pi \vdash \Gamma \gg \Delta, \chi$ and $\tau \vdash \chi, \Gamma \gg \Delta$ be locally cut free proofs in $\mathcal{G}^\infty\text{GLS} + \text{Cut}$. We proceed by induction on the measure $\omega \cdot |\chi| + (\text{lh}(\pi) + \text{lh}(\tau))$. Most cases follow the usual cut reductions in cut elimination (for the details see Appendix A), so we will consider only two special cases. Assume that π and τ are of the following shape

$$\frac{\pi_0}{\frac{\Box \Sigma \Rightarrow \chi}{\Box \Sigma, \Gamma' \gg \Box \chi, \Box \phi, \Delta'} \Box^{K4}} \quad \frac{\tau_0}{\frac{\Box \chi, \Box \Sigma \Rightarrow \phi}{\Box \chi, \Box \Sigma, \Gamma' \gg \Box \phi, \Delta'} \Box^{K4}}$$

where $\Gamma = \Box \Sigma, \Gamma'$, $\Delta = \Box \phi, \Delta'$. We can assume that the auxiliary $\Box$ -formulas on the left hand side are the same, since otherwise we can use Wk on π_0 and τ_0 preserving height and local cut freeness. Then, the desired proof is

$$\frac{\text{Wk}(\pi_0) \quad \frac{\frac{\pi_0}{\Box \Sigma \Rightarrow \chi} \Box^{K4} \quad \frac{\tau_0}{\Box \chi, \Box \Sigma \Rightarrow \phi} \text{Cut}}{\Box \Sigma \Rightarrow \phi, \chi} \text{Cut}}{\frac{\Box \Sigma \Rightarrow \phi}{\Box \Sigma, \Gamma' \Rightarrow \Box \phi, \Delta'} \Box^{K4}}$$

Notice that the proof is trivially locally cut free.

The other case is as follows. Assume π and τ are of the following shape.

$$\frac{\pi_0}{\frac{\Box \Sigma \Rightarrow \chi}{\Box \Sigma, \Gamma' \Rightarrow \Box \chi, \Delta} \Box^{K4}} \quad \frac{\tau_0}{\frac{\Box \chi, \Box \Sigma, \Gamma' \Rightarrow \Delta}{\Box \chi, \Box \Sigma, \Gamma' \Rightarrow \Delta} \text{T}}$$

where $\Gamma = \Box \Sigma, \Gamma'$. Then the desired proof is

$$\frac{\text{Wk}(\text{Prom}(\pi_0)) \quad \frac{\frac{\pi_0}{\Box \Sigma \Rightarrow \chi} \Box^{K4} \quad \frac{\text{Wk}(\tau_0)}{\Box \chi, \Box \Sigma, \Gamma' \Rightarrow \Delta} \text{Cut (I.H.)}}{\Box \Sigma, \Gamma' \Rightarrow \chi, \Delta} \text{Cut (I.H.)}}{\frac{\Box \Sigma, \Gamma' \Rightarrow \Delta}{\Box \Sigma, \Gamma' \Rightarrow \Delta} \text{T}}$$

where the double line denotes multiple applications of (T). The first cut is justified by a smaller sum of local heights and the second cut by a smaller cut formula. $\square$

Thanks to the translations of the previous subsection, we obtain the following.

Corollary 3.13. *Cut is eliminable in $\mathcal{G}\text{GLS}$.⁴*

Corollary 3.14. *Let Γ, Δ be multisets. We have the following:*

1. $\text{GL} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$ iff $\mathcal{G}\text{GLS} (+\text{Cut}) \vdash \Gamma \Rightarrow \Delta$ iff $\mathcal{G}^\infty\text{GLS} (+\text{Cut}) \vdash \Gamma \Rightarrow \Delta$.
2. $\text{GLS} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$ iff $\mathcal{G}\text{GLS} (+\text{Cut}) \vdash \Gamma \Rightarrow \Delta$ iff $\mathcal{G}^\infty\text{GLS} (+\text{Cut}) \vdash \Gamma \Rightarrow \Delta$.

⁴In fact, this cut elimination procedure can be made effective (see the footnote on page 713) since a finite approximation of the non-wellfounded proof will suffice to compute each of the steps.

4 Uniform Lyndon Interpolation for GLS

Having established the connection between the sequent and Hilbert calculi for GLS, we are ready to show uniform Lyndon interpolation for GLS. In order to respect the polarity of variables, it is necessary to avoid introducing the diagonal formula in the modal rule. Hence the use of $\mathcal{G}^\infty\text{GLS}$ instead of $\mathcal{G}\text{GLS}$ is essential.

To properly talk about uniform Lyndon interpolation, first we need to define the positive and negative vocabulary of a formula. From now on, with *vocabulary* we simply mean a set of variables.

Definition 4.1. Let ϕ be a formula. We define its *positive* and *negative vocabulary*, denoted $\text{Voc}_+(\phi)$ and $\text{Voc}_-(\phi)$ respectively, recursively by

$$\begin{aligned} \text{Voc}_+(p) &= \{p\}, & \text{Voc}_-(p) &= \emptyset, \\ \text{Voc}_+(\perp) &= \emptyset, & \text{Voc}_-(\perp) &= \emptyset, \\ \text{Voc}_+(\phi_0 \rightarrow \phi_1) &= \text{Voc}_-(\phi_0) \cup \text{Voc}_+(\phi_1), & \text{Voc}_-(\phi_0 \rightarrow \phi_1) &= \text{Voc}_+(\phi_0) \cup \text{Voc}_-(\phi_1), \\ \text{Voc}_+(\Box \phi_0) &= \text{Voc}_+(\phi_0), & \text{Voc}_-(\Box \phi_0) &= \text{Voc}_-(\phi_0). \end{aligned}$$

Given a multiset Γ we will write $\text{Voc}_+(\Gamma)$ to mean the set $\bigcup_{\phi \in \Gamma} \text{Voc}_+(\phi)$, and given a sequent $\Gamma \gg \Delta$ we will write $\text{Voc}_+(\Gamma \gg \Delta)$ to mean $\text{Voc}_+(\Gamma) \cup \text{Voc}_+(\Delta)$. We will use the same notation for Voc_- .

We will write $\overline{+}$ to mean $-$ and $\overline{-}$ to mean $+$. The following lemma can be proved by induction on the complexity of ϕ (although we omit the proof as it is highly technical and not particularly illuminating).

Lemma 4.2. Let p be a propositional variable and $\phi(p_0, \dots, p_{n-1}, q_0, \dots, q_{m-1})$, $\psi_0, \dots, \psi_{n-1}$, and $\chi_0, \dots, \chi_{m-1}$ be formulas. If $\text{Voc}_-(\phi) \cap \{p_0, \dots, p_{n-1}\} = \emptyset$ and $\text{Voc}_+(\phi) \cap \{q_0, \dots, q_{m-1}\} = \emptyset$, then for $b \in \{+, -\}$

$$\begin{aligned} \text{Voc}_b(\phi(\psi_0, \dots, \psi_{n-1}, \chi_0, \dots, \chi_{m-1})) \subseteq \\ \text{Voc}_b(\phi) \setminus \{p_0, \dots, p_{n-1}, q_0, \dots, q_{m-1}\} \cup \bigcup_{i < n} \text{Voc}_b(\psi_i) \cup \bigcup_{j < m} \text{Voc}_{\overline{b}}(\chi_j). \end{aligned}$$

We are ready to recall the definition of uniform Lyndon interpolation.

Definition 4.3. A logic L has the *Uniform Lyndon Interpolation Property* (or ULIP in short) if for any formula ϕ and vocabularies V_+, V_- , there is a formula ι called *uniform Lyndon interpolant* such that

1. $\text{Voc}_+(\iota) \subseteq V_+$ and $\text{Voc}_-(\iota) \subseteq V_-$,
2. $L \vdash \phi \rightarrow \iota$, and
3. for any ψ such that $\text{Voc}_+(\psi) \subseteq V_+$ and $\text{Voc}_-(\psi) \subseteq V_-$ we have that $L \vdash \phi \rightarrow \psi$ implies $L \vdash \iota \rightarrow \psi$.

The rest of the paper will be dedicated to show that GLS has the ULIP.

4.1 Lyndon equational systems

The first step towards calculating uniform Lyndon interpolants will be to solve equational systems of formulas. In addition, as the polarities of variables matters, we will need to keep track of them. We will do so by introducing the notions of *Lyndon fixpoint* and *Lyndon equational system*.

Definition 4.4 (Lyndon Fixpoints). Let $\phi(p)$ and ψ be formulas. We say that ψ is a *Lyndon fixpoint* of ϕ (with respect to p) in GLS if $\text{Voc}_b(\psi) \subseteq \text{Voc}_b(\phi) \setminus \{p\}$ for $b \in \{+, -\}$ and $\text{GLS} \vdash \psi \leftrightarrow \phi(\psi)$.

A formula ϕ is modalized in a variable p if all the occurrences of p in ϕ are under the scope of a $\Box$.

Definition 4.5 (Lyndon equational systems). Let V_+, V_- be vocabularies and $\bar{p} = (p_0, \dots, p_{n-1})$ be a finite sequence of pairwise different variables not occurring in $V_+ \cup V_-$. A *Lyndon equational system* over $(\bar{p}, V_+, V_-)$ is a collection of triples $\mathcal{E} = \{(p_i, b_i, \phi_i) \mid i < n\}$ such that $b_i \in \{+, -\}$, ϕ_i is a formula, $\text{Voc}_{b_i}(\phi_i) \subseteq V_+ \cup B_+$ and $\text{Voc}_{\bar{b}_i}(\phi_i) \subseteq V_- \cup B_-$, where $B_+ = \{p_i \mid b_i = +\}$ and $B_- = \{p_i \mid b_i = -\}$. The elements of $\bar{p}$ are called *unknowns* and those of $\mathcal{E}$ are called *equations*.

A *solution in L* to $\mathcal{E}$ is a sequence $(\psi_0, \dots, \psi_{n-1})$ such that for each $i \in n$, we have $\text{Voc}_{b_i}(\psi_i) \subseteq V_+$, $\text{Voc}_{\bar{b}_i}(\psi_i) \subseteq V_-$ and $L \vdash \psi_i \leftrightarrow \phi_i[\psi_0/p_0, \dots, \psi_{n-1}/p_{n-1}]$. $\mathcal{E}$ is said to be

1. *Solvable in L* if it has a solution in L .
2. *Simple* if ϕ_i is a $\Box$ -formula for $i < n$.
3. *Modalized* if ϕ_i is modalized in $p_0, \dots, p_i$.
4. *Positive* if $b_i = +$ for $i < n$.

Given an equational system $\mathcal{E}$ over $(\bar{p}, V_+, V_-)$ with solution $(\psi_0, \dots, \psi_{n-1})$ we will also call the substitution $(\cdot)^*$ where $p_i^* = \psi_i$ and $q^* = q$ for q not in $\bar{p}$ a solution of $\mathcal{E}$.

Lemma 4.6. *We have that*

1. *For any formula $\phi(p)$, $\Box \phi(\top)$ is a Lyndon fixpoint of $\Box \phi(p)$ in GLS.*
2. *Simple Lyndon equational systems have a solution in GLS.*

Proof. Proof 1. It is wellknown (e.g. see [25, p. 78]) that $\text{GL} \vdash \Box \phi(\top) \leftrightarrow \Box \phi(\Box \phi(\top))$, so it is also a theorem of GLS. By the properties of substitution, we get $\text{Voc}_b(\phi(\top)) \subseteq \text{Voc}_b(\phi) \setminus \{p\}$ for $b \in \{+, -\}$.

Proof of 2. Let $\mathcal{E} = \{(p_i, b_i, \Box \phi_i) \mid i < n\}$ be a simple Lyndon equational system over $(\bar{p}, V_+, V_-)$. We proceed by induction on n , the number of unknowns. Take $\Box \phi_0(p_0, \dots, p_{n-1})$, we know that it has a Lyndon fixpoint ψ_0 in GLS. Note that $\mathcal{E}' = \{(p_i, b_i, \Box \phi_i[\psi_0/p_0]) \mid 1 \leq i < n\}$ is a simple Lyndon equational system solvable in GLS by the induction hypothesis, let $(\chi_1, \dots, \chi_n)$ be a solution in GLS of it. Let us define $\chi_0 = \psi_0[\chi_1/p_1, \dots, \chi_n/p_n]$, then we claim that $(\chi_0, \dots, \chi_n)$ is a solution of $\mathcal{E}$.

First, note that we already have that $\text{Voc}_{b_i}(\chi_i) \subseteq V_+$ and $\text{Voc}_{\bar{b}_i}(\chi_i) \subseteq V_-$ for $1 \leq i < n$. We notice that for $1 \leq i < n$ we have that if $b_i = b_0$ then $p_i \notin \text{Voc}_-(\psi_0)$ and if $b_i \neq b_0$ then $p_i \notin \text{Voc}_+(\psi_0)$ (it suffices to do cases on b_i and b_0). Using Lemma 4.2 we have that

$$\begin{aligned} \text{Voc}_{b_0}(\chi_0) &\subseteq \text{Voc}_{b_0}(\psi_0) \setminus \{p_1, \dots, p_n\} \cup \bigcup_{\substack{1 \leq i < n \\ b_i = b_0}} \text{Voc}_{b_0}(\chi_i) \cup \bigcup_{\substack{1 \leq i < n \\ b_i \neq b_0}} \text{Voc}_{\bar{b}_0}(\chi_i) \\ &\subseteq \text{Voc}_{b_0}(\Box \phi_0) \setminus \{p_0, \dots, p_n\} \cup \bigcup_{\substack{1 \leq i < n \\ b_i = b_0}} \text{Voc}_{b_i}(\chi_i) \cup \bigcup_{\substack{1 \leq i < n \\ b_i \neq b_0}} \text{Voc}_{b_i}(\chi_i) \subseteq V_+. \end{aligned}$$

where we used that $\text{Voc}_{b_0}(\Box \phi_0) \subseteq V_+ \cup B_+$. We have an analogous reasoning for $\text{Voc}_{\bar{b}_0}(\chi_0)$, so the desired polarity conditions hold.

Finally, we show the desired equivalences. We have $\text{GLS} \vdash \chi_i \leftrightarrow (\Box \phi_i[\psi_0/p_0])[\chi_1/p_1, \dots, \chi_n/p_n]$ for $1 \leq i < n$. Using that $p_0 \neq p_i$ for $1 \leq i < n$ we obtain that

$$\begin{aligned} (\Box \phi_i[\psi_0/p_0])[\chi_1/p_1, \dots, \chi_n/p_n] &= \Box \phi_i[\psi_0[\chi_1/p_1, \dots, \chi_n/p_n]/p_0, \chi_1/p_1, \dots, \chi_n/p_n] \\ &= \Box \phi_i[\chi_0/p_0, \chi_1/p_1, \dots, \chi_n/p_n], \end{aligned}$$

as desired. All left to show that $\text{GLS} \vdash \chi_0 \leftrightarrow \Box \phi_0[\chi_0/p_0, \dots, \chi_n/p_n]$. We have that, as ψ_0 is a fix-point, $\text{GLS} \vdash \Box \psi_0 \leftrightarrow \Box \phi_0[\psi_0/p_0]$. As GLS is closed under substitutions we obtain that $\text{GLS} \vdash \Box \chi_0 \leftrightarrow (\Box \phi_0[\psi_0/p_0])[\chi_1/p_1, \dots, \chi_n/p_n]$, and we can use the same equalities as before since $p_0 \neq p_i$ for $1 \leq i < n$. $\square$

We say that ϕ is positive in p if $p \notin \text{Voc}_-(\phi)$.

Theorem 4.7. *We have that*

1. *Every formula positive and modalized in p has a Lyndon fixpoint in GLS.*
2. *Positive modalized Lyndon equational system are solvable in GLS.*

Proof. Proof of 1.⁵ Let $\phi(p)$ be a formula that is modalized in p and positive in p . Then $\phi(p)$ can be written as $\phi'(\Box \psi_0(p), \dots, \Box \psi_{n-1}(p), \Box \chi_0(p), \dots, \Box \chi_{m-1}(p))$, where $\phi'(q_0, \dots, q_{n-1}, r_0, \dots, r_{m-1})$ does not contain p , and $\{q_0, \dots, q_{n-1}\} \subseteq \text{Voc}_+(\phi') \setminus \text{Voc}_-(\phi')$, $\{r_0, \dots, r_{m-1}\} \subseteq \text{Voc}_-(\phi') \setminus \text{Voc}_+(\phi')$. From this is easy to infer that for $i < n$, $j < m$ and $b \in \{+, -\}$:

$$\text{Voc}_b(\psi_i) \subseteq \text{Voc}_b(\phi), \quad \text{Voc}_b(\chi_j) \subseteq \text{Voc}_{\bar{b}}(\phi), \quad p \notin \text{Voc}_-(\psi_i) \quad \text{and} \quad p \notin \text{Voc}_+(\chi_j).$$

Consider the following equational system $\{(q_i, +, \Box \psi_i(\phi')) \mid i < n\} \cup \{(r_j, -, \Box \chi_j(\phi')) \mid j < m\}$. This is a simple Lyndon $(\bar{q}\bar{r}, \text{Voc}_+(\phi) \setminus \{p\}, \text{Voc}_-(\phi) \setminus \{p\})$ -equational system. Thus, it has a solution $(\cdot)^* : \bar{q}\bar{r} \rightarrow \mathcal{L}_\Box$ in GLS. Define $\eta = \phi'(q_0^*, \dots, q_{n-1}^*, r_0^*, \dots, r_{m-1}^*)$, by Lemma 4.2 we know that $\text{Voc}_+(\eta) \subseteq \text{Voc}_+(\phi) \setminus \{p\}$ and $\text{Voc}_-(\eta) \subseteq \text{Voc}_-(\phi)$. Also, since $(\cdot)^*$ is a solution of the equational system, we have that $\text{GLS} \vdash q_i^* \leftrightarrow \Box \psi_i(\eta)$ and $\text{GLS} \vdash r_j^* \leftrightarrow \Box \chi_j(\eta)$. So we obtain

$$\text{GLS} \vdash \eta \leftrightarrow \phi'(\Box \psi_0(\eta), \dots, \Box \psi_{n-1}(\eta), \Box \chi_0(\eta), \dots, \Box \chi_{m-1}(\eta)).$$

In other words, $L \vdash \eta \leftrightarrow \phi(\eta)$, so η is a Lyndon fixpoint of ϕ with respect to p .

Proof of 2. The proof is similar to the second point of Lemma 4.6 using that if ϕ_i is modalised in $p_0, \dots, p_i$ then $\phi_i[\psi/p_0]$ is also modalised in $p_0, \dots, p_i$. $\square$

The fundamental result will be the second part of Theorem 4.7. When constructing an interpolant we will see that a positive modalized Lyndon equational system will arise. Solving it and using the solution on a particular formula will give us the uniform Lyndon interpolant.

4.2 Interpolation templates

It is common to relate uniform interpolation to proof search. We will make this connection explicit via defining interpolation templates, which is a proof search where we assume that we only have a part of the desired sequent. Since there is a rule in the system which increase the complexity of sequents (namely rule (T)), we will need to base our proof search on a notion of saturation.

Definition 4.8. Let $\Gamma \gg \Delta$ be a sequent. We say that a formula ϕ is

1. <i>Left saturated in $\Gamma \gg \Delta$ if</i> (a) ϕ is atomic, (b) $\phi = \phi_0 \rightarrow \phi_1$ and $\phi_0 \in \Delta$ or $\phi_1 \in \Gamma$, (c) $\phi = \Box \phi_0$ and $\phi_0 \in \Gamma$ or $\gg \Rightarrow$.	2. <i>Right saturated in $\Gamma \gg \Delta$ if</i> (a) ϕ is atomic or a $\Box$-formula, (b) $\phi = \phi_0 \rightarrow \phi_1$ and $\phi_0 \in \Gamma$, $\phi_1 \in \Delta$.

⁵This proof follows the proof in [19] for GL, with the addition of the condition on the polarity of variables.

$$\frac{}{p, \Gamma \gg p, \Delta} \text{ax} \quad \frac{}{\perp, \Gamma \gg \Delta} \perp\text{L} \quad \frac{}{\top \gg} \text{Emp} \quad \frac{\Box \phi, \Gamma \Rightarrow \Delta}{\Box \phi, \Gamma \Rightarrow \Delta} \text{T}_{\text{sat}}$$

where in (T_{sat}) the formula $\Box \phi$ is not left saturated in the conclusion.

$$\frac{\phi \rightarrow \psi, \Gamma \gg \phi, \Delta \quad \psi, \phi \rightarrow \psi, \Gamma \gg \Delta}{\phi \rightarrow \psi, \Gamma \gg \Delta} \rightarrow\text{L}_{\text{sat}} \quad \frac{\phi, \Gamma \gg \psi, \phi \rightarrow \psi, \Delta}{\Gamma \gg \phi \rightarrow \psi, \Delta} \rightarrow\text{R}_{\text{sat}}$$

where in $(\rightarrow\text{L}_{\text{sat}})$ the formula $\phi \rightarrow \psi$ is not left saturated in the conclusion and in $(\rightarrow\text{R}_{\text{sat}})$ the formula $\phi \rightarrow \psi$ is not right saturated in the conclusion.

$$\frac{\Box \Sigma^s \Rightarrow \quad [\Box \Sigma^s \Rightarrow \phi]_{\phi \in \Theta}}{\Box \Sigma, \Gamma \gg \Box \Theta, \Delta} \Box_{\text{sat}}^{\text{K4}}$$

where $\Box \Sigma, \Gamma \gg \Box \Theta, \Delta$ is saturated, Γ, Δ contain no $\Box$ -formulas, $(\Gamma \cap \Delta) \cap \text{Var} = \emptyset$ and $\perp \notin \Gamma$.

Figure 3: Interpolation Template Rules

Given a sequent $S = \Gamma \gg \Delta$ we define its *saturation complexity*, denoted $|S|_{\text{sat}}$, as the multiset

$$\{|\phi| \mid \phi \in \Gamma, \phi \text{ not left saturated in } S\} \cup \{|\phi| \mid \phi \in \Delta, \phi \text{ not right saturated in } S\}.$$

We assume the saturation complexity comes with the multiset ordering attached to it, so $|S|_{\text{sat}} < |S'|_{\text{sat}}$ means that $|S|_{\text{sat}}$ is obtained from $|S'|_{\text{sat}}$ by the process of taking out at least one element and replace it with a finite number of strictly smaller elements. We remember that the multiset ordering is wellfounded.

Definition 4.9. An interpolation template is a cyclic proof constructed in the local progress calculus whose rules are displayed in Figure 3 and progress is only made at the premises of $\Box_{\text{sat}}^{\text{K4}}$.

It is easy to show the following lemma, the details of the proof can be found in the appendix.

Lemma 4.10. Every sequent has an interpolation template.

We will consider formulas on an expanded set of propositional variables. For each $w \in \mathbb{N}^*$ we pick a new variable x_w different from the variables in Var and such that if $w, v \in \mathbb{N}^*$ and $w \neq v$ then $x_w \neq x_v$. We define $B = \{x_w \mid w \in \mathbb{N}^*\}$ and for an interpolation template T we define $B_T = \{x_w \mid w \text{ repeat node of } T\}$. The elements of B will be called *bound variables* and the formulas on this expanded set of variables will be called *pseudoformulas*. For the rest of the subsection we fix two vocabularies V_+ and V_- for which we want to calculate the uniform Lyndon interpolant.

Definition 4.11. Let T be an interpolation template and w a node of T , let us denote the sequent at w in T as $\Gamma_w \gg_w \Delta_w$. We will annotate each of the sequents of T with a pseudoformula κ called the *preinterpolant at w* , denoted as $\kappa : \Gamma_w \gg_w \Delta_w$. We proceed by recursion on the tree structure of T as follows.

$$\frac{}{\perp : p, \Gamma \gg p, \Delta} \text{ax} \quad \frac{}{\perp : \perp, \Gamma \gg \Delta} \perp\text{L} \quad \frac{}{\top \gg} \text{Emp} \quad \frac{}{x_w : \Gamma_w \gg_w \Delta_w} \text{Rep} \quad \frac{\kappa : \Box \phi, \Gamma \Rightarrow \Delta}{\kappa : \Box \phi, \Gamma \Rightarrow \Delta} \text{T}_{\text{sat}}$$

$$\frac{\kappa_0 : \phi \rightarrow \psi, \Gamma \gg \phi, \Delta \quad \kappa_1 : \psi, \phi \rightarrow \psi, \Gamma \gg \Delta}{\kappa_0 \vee \kappa_1 : \phi \rightarrow \psi, \Gamma \gg \Delta} \rightarrow\text{L}_{\text{sat}} \quad \frac{\kappa : \phi, \Gamma \gg \psi, \phi \rightarrow \psi, \Delta}{\kappa : \Gamma \gg \phi \rightarrow \psi, \Delta} \rightarrow\text{R}_{\text{sat}}$$

$$\frac{\kappa^\Box : \Box \Sigma \Rightarrow \quad [\kappa_\phi^\Diamond : \Box \Sigma \Rightarrow \phi]_{\phi \in \Theta}}{\Box \kappa^\Box \wedge \bigwedge_{\phi \in \Theta} \kappa_\phi^\Diamond \wedge \bigwedge (\Gamma \cap V_+) \wedge \neg(\Delta \cap V_-) : \Box \Sigma, \Gamma \gg \Box \Theta, \Delta} \Box_{\text{sat}}^{\text{K4}}$$

As promised, every interpolation template has an associated positive modalized equational system. The proof of this fact can be found in the appendix.

Lemma 4.12. *For any interpolation template T and for any $x_w \in B_T$ let us write κ_{x_w} to denote the preinterpolant of T at w° . There is an enumeration $\bar{x} = (x_0, \dots, x_{n-1})$ of B_T such that the equational system $\mathcal{E}_T = \{(x_i, +, \kappa_{x_i}) \mid i < n\}$ is a positive modalized equational system over $(\bar{x}, V_+, V_-)$. As a corollary, $\mathcal{E}_T$ is solvable in GLS.*

Definition 4.13 (Interpolant). Let T be an interpolation template, $(\cdot)^*$ a solution of $\mathcal{E}_T$ and for each node w of T let us denote the preinterpolant at w as κ_w . The interpolant given by T is defined as $\iota_T = \kappa_\varepsilon^*$.

The interpolant given by an interpolation template T depends on the chosen solution of $\mathcal{E}_T$. This does not matter to us, as in the end one can show that any two uniform interpolants are logically equivalent. Also note that, by definition of solution of a Lyndon equational system over $(\bar{x}, V_+, V_-)$ we automatically have that each ι_T is a formula (and not a pseudoformula) and $\text{Voc}_b(\iota_T) \subseteq V_b$ for $b \in \{+, -\}$. Finally, to show that the interpolant has the necessary properties, we will need two additional lemmas.

Lemma 4.14. *Let T be an interpolation template of $\Gamma \gg \Delta$. Then $\mathcal{G}^\infty\text{GLS} \vdash \Gamma \gg \Delta, \iota_T$.*

Proof. Given a node w of T let $\Gamma_w \gg_w \Delta_w$ denote the sequent at w and κ_w be the preinterpolant of T at w . The trick is to build a function α such that given a node w of T , $\alpha(w)$ is a proof in $\mathcal{G}^\infty\text{GLS} + \text{Cut} + \text{Wk} + \text{Ctr}$ of $\Gamma_w \gg_w \Delta_w, \kappa_w^*$. Since Ctr is derivable with Cut, Wk is eliminable in $\mathcal{G}^\infty\text{GLS} + \text{Cut}$ and Cut is eliminable in $\mathcal{G}^\infty\text{GLS}$, we can obtain the desired proof in $\mathcal{G}^\infty\text{GLS}$. α is defined in Appendix B. $\square$

Lemma 4.15. *Let T be an interpolation template of $\Gamma \gg \Delta$ and let $\Phi \gg \Psi$ be a sequent such that $\text{Voc}_b(\Phi \gg \Psi) \subseteq V_b$ for $b \in \{+, -\}$. Then $\mathcal{G}^\infty\text{GLS} \vdash \Gamma, \Gamma' \gg \Delta, \Delta'$ implies $\mathcal{G}^\infty\text{GLS} \vdash \iota_T, \Gamma' \gg \Delta'$.*

Proof. Given a node w of T let $\Gamma_w \gg_w \Delta_w$ denote the sequent at w and κ_w be the preinterpolant of T at w . The trick is to build a function β such that given a node w of T and a proof $\pi \vdash \Gamma_w, \Phi \gg_w \Delta_w, \Psi$ in $\mathcal{G}^\infty\text{GLS}$ such that $\text{Voc}_b(\Phi \gg_w \Psi) \subseteq V_b$ for $b \in \{+, -\}$, $\beta(w, \pi)$ is a proof in $\mathcal{G}^\infty\text{GLS} + \text{Cut} + \text{Wk} + \text{Ctr}$ of $\kappa_w^*, \Phi \gg_w \Psi$. Since Ctr is derivable with Cut, Wk is eliminable in $\mathcal{G}^\infty\text{GLS} + \text{Cut}$ and Cut is eliminable in $\mathcal{G}^\infty\text{GLS}$, we can obtain the desired proof in $\mathcal{G}^\infty\text{GLS}$. The definition of β can be found in Appendix B. $\square$

We finish the paper with the desired result.

Theorem 4.16. *GLS has uniform Lyndon interpolation.*

Proof. Let T be an interpolation template for $\phi \Rightarrow$. It is easy to show using Lemmas 4.14 and 4.15 that ι_T is the uniform Lyndon interpolant of ϕ for given vocabularies (V^+, V_-) . $\square$

Future work

The method to establish uniform Lyndon interpolation via non-wellfounded proofs can be applied to a plethora of provability logics, once sequent calculi for these logics have been developed. We leave it as future work to apply the methodology to other provability logics, such as unary interpretability or bimodal provability logics. In addition, there are two possible extensions of the method which need further exploration. One direction is to investigate logics without simple Lyndon fixpoints, such as interpretability logic IL, and see if it is still possible to solve positive modalized Lyndon equational systems. The other direction is to study whether the method is applicable to intuitionistic provability logics, where the resulting Lyndon equational system does not need to be positive.

Acknowledgements

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A Cut reductions

We display some possible cut reductions which were not consider at the proof of Theorem 3.12.

Weakening part. If χ belongs to the weakening part of the last rule instance of π or τ we can delete directly, as weakening parts can be modified arbitrarily. From now on we assume that χ does not occur in the weakening part of the rule instances.

Axiomatic. Assume π ends in (ax), the case for τ is analogous. Then $\chi = p$ for some variable p and the desired reduction is

$$\frac{}{p, \Gamma' \gg \Delta, p} \text{ax} \quad \frac{}{p, p, \Gamma' \gg \Delta} \tau \mapsto \frac{\text{Ctr}(\tau)}{p, \Gamma' \gg \Delta}$$

where $\Gamma = p, \Gamma'$.

If π ends in $\perp L$ then χ would belong to the weakening part, which is already covered. Assume τ ends in $\perp L$, so $\chi = \perp$.

$$\frac{}{\Gamma \gg \Delta, \perp} \pi \quad \frac{}{\perp, \Gamma \gg \Delta} \tau \mapsto \frac{\text{inv}_{\perp R}(\pi)}{\Gamma \gg \Delta}$$

From now own we assume that neither π nor τ end in (ax) or ($\perp L$).

$\perp R$ case. Assume π ends in an application of ($\perp R$), the case for τ is analogous. If $\perp$ is the cut formula, the desired cut reduction is obtained taking the immediate subproof of π . If $\perp$ is not the cut formula, the desired cut reduction is

$$\frac{\frac{}{\Gamma \gg \Delta', \chi} \pi_0}{\Gamma \gg \perp, \Delta', \chi} \perp R \quad \frac{}{\chi, \Gamma \gg \perp, \Delta'} \tau \mapsto \frac{\frac{}{\Gamma \gg \Delta', \chi} \pi_0 \quad \frac{}{\chi, \Gamma \gg \Delta'} \text{inv}_{\perp R}(\tau)}{\Gamma \gg \Delta'} \text{Cut(I.H.)}$$

where $\Delta = \perp, \Delta'$.

From now own we assume that neither π nor τ end in ($\perp R$).

Principal cut reduction. Assume χ is principal in π and τ . Then either χ is an implication or a $\Box$ -formula. The second case is covered at the proof of Theorem 3.12, so we just show the cut reduction for the first.

$$\begin{aligned} & \frac{\frac{}{\chi_0, \Gamma \gg \Delta, \chi_1} \pi_0}{\Gamma \gg \Delta, \chi_0 \rightarrow \chi_1} \rightarrow R \quad \frac{\frac{}{\Gamma \gg \Delta, \chi_0} \tau_0 \quad \frac{}{\chi_1, \Gamma \gg \Delta} \tau_1}{\chi_0 \rightarrow \chi_1, \Gamma \gg \Delta} \rightarrow L \\ & \mapsto \frac{\frac{\text{Wk}(\tau_0)}{\Gamma \gg \Delta, \chi_1, \chi_0} \quad \frac{\pi_0}{\chi_0, \Gamma \gg \Delta, \chi_1} \text{Cut(I.H.)}}{\Gamma \gg \Delta, \chi_1} \quad \frac{\tau_1}{\chi_1, \Gamma \gg \Delta} \text{Cut(I.H.)} \\ & \quad \frac{}{\Gamma \gg \Delta} \end{aligned}$$

Commutative cut reduction. Finally, assume that the cut formula is not principal in either π or τ . Assume that the cut formula is not principal in π and the last rule of π is ($\rightarrow R$), the cases where the rule is instead ($\rightarrow R$) or (T), or where any of these occurs at τ are analogous. The desired cut reduction is

$$\frac{\frac{}{\phi, \Gamma \Rightarrow \psi, \Delta', \chi} \pi_0}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta', \chi} \rightarrow R \quad \frac{}{\chi, \Gamma \Rightarrow \phi \rightarrow \psi, \Delta'} \tau \mapsto \frac{\frac{}{\phi, \Gamma \Rightarrow \psi, \Delta', \chi} \pi_0 \quad \frac{}{\chi, \phi, \Gamma \Rightarrow \psi, \Delta'} \text{inv}_{\rightarrow R}(\tau)}{\frac{\phi, \Gamma \Rightarrow \psi, \Delta'}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta'} \rightarrow R} \text{Cut (I.H.)}$$

where $\Delta = \phi \rightarrow \psi, \Delta'$

Finally, we have that the cut formula is not principal in either π or τ with last rule $(\Box^{K4})$. We notice that this cannot occur in π , as then the cut formula would belong to the weakening part, so it must occur in τ . To make the cut formula not belong to the weakening part it must be the case that $\chi = \Box\chi_0$. If χ where not principal in π , then we would be in one of the cases which we already covered, so we can assume it is principal in π , so π must end in $(\Box^{K4})$. This case was covered at the proof of Theorem 3.12.

B Proofs about interpolation templates

Proof of Lemma 4.10. We will show that we can obtain a cyclic preproof, this preproof will be a proof since all the rules strictly lower the saturation complexity of sequents from conclusion to premises except for $(\Box_{\text{sat}}^{K4})$ (so to repeat a sequent we must at least apply $(\Box_{\text{sat}}^{K4})$ once, as otherwise the repeated sequent would have strictly saturation complexity than itself). Take the sequent $\Gamma \gg \Delta$ and start applying the rules of Figure 3 as possible (note that for any sequent it is always possible to apply at least one), and whenever a repetition in the same branch is found make the leaf a repetition node. If we show that in this process no infinite branch is generated we would finish.

Assume otherwise, so there is a branch without repetitions. As the only rule which not strictly lowers the saturation complexity is $(\Box_{\text{sat}}^{K4})$, it must be the case that $(\Box_{\text{sat}}^{K4})$ is applied infinitely often. We notice that the rules of Figure 3 fulfill the subformula property, so all the formulas occurring on the branch must be in the set $\text{Sub}(\Gamma \gg \Delta)$. Also, the premises of $(\Box_{\text{sat}}^{K4})$ are determined by choosing a formula (or nothing in the case of the left-most premise) and a set of formulas, so in fact the possible number of such premises is not bigger than $(k+1)2^k$, where k is the cardinality of $\text{Sub}(\Gamma \gg \Delta)$. Since there is infinitely many nodes in the branch which are premises of $(\Box_{\text{sat}}^{K4})$, and only finitely many possibilities, there must be a repetition, absurd.

Proof of Lemma 4.12. Given a node w of T let $\text{hg}(w)$ denote the height of w , i.e., the height of the subtree generated by w (then leafs have height 0 and the root has the same height as the tree). If $\text{hg}(T) = H$, let $\bar{x}_i$ for $i \leq H$ be an arbitrary linear ordering of $\{x_w \mid w \text{ repeat node and } \text{hg}(w^\circ) = i\}$. Consider the enumeration of B_T defined as $\bar{x}_T = \bar{x}_0 \cdots \bar{x}_H$ and let us show that $\mathcal{E}_T$ is an positive modalized equational system over $(\bar{x}_T, V_+, V_-)$.

The positivity of $\mathcal{E}_T$ is straightforward by definition. In addition, one can check that for any node w , $\text{Voc}_+(\kappa_w) \subseteq V_+ \cup B_+$ and $\text{Voc}_-(\kappa_w) \subseteq V_-$ by induction on the length of w . So all left to show is that it is modalized, let $\bar{x}_T = (x_0, \dots, x_{n-1})$ and for each $i < n$ let us write w_i to mean the repeat node such that $x_i = x_{w_i}$. First, we notice three facts.

1. If w is a repeat node and $v \not\leq w$ then x_w does not occur in κ_w .
2. If w is the conclusion of a $\Box_{\text{sat}}^{K4}$ rule, then κ_w is modalized in all the variables of B_T .
3. For any $x \in B_T$, if w is a non-repeat node with immediate successors $w0, \dots, w(n-1)$ and $\kappa_{w0}, \dots, \kappa_{w(n-1)}$ are modalized in x then κ_w is modalized in κ_w .

Additionally, thanks to the definition of $\bar{x}_T$ we have that if $i \leq j$ then either $w_j^\circ \leq w_i^\circ$ or w_j° and w_i° are incomparable. Using this with the previous facts gives that for any $i \leq j$ we have that $\kappa_{w_j^\circ}$ is modalised in x_i , as desired.

Definition of α at Lemma 4.14. We will define α corecursively, that $\alpha(w)$ is a preproof will be trivial from construction. When the definition is finished we will show that it is indeed a proof. We proceed by cases on the shape of w .

Case w is (ax) or ($\perp$ L). $\alpha(w)$ is obtained by applying rule (ax) or ($\perp$ L), respectively.

Case w is (Emp). $\alpha(w)$ is a proof of $\gg_w \top$, which is straightforward to obtain.

Case w is (T_{sat}). Then the function is defined as

$$\frac{\begin{array}{c} w0 \\ \kappa : \Box \phi, \Gamma'_w \Rightarrow \Delta_w \\ \kappa : \Box \phi, \Gamma'_w \Rightarrow \Delta_w \end{array} T_{\text{sat}}}{\kappa : \Box \phi, \Gamma'_w \Rightarrow \Delta_w} \xrightarrow{\alpha} \frac{\begin{array}{c} \alpha(w0) \\ \Box \phi, \Gamma'_w \Rightarrow \Delta_w, (\kappa)^* \\ \Box \phi, \Gamma'_w \Rightarrow \Delta_w, (\kappa)^* \end{array} T}{\Box \phi, \Gamma'_w \Rightarrow \Delta_w, (\kappa)^*}$$

where $\kappa_w = \kappa$, $\gg_w = \Rightarrow$ and $\Gamma_w = \Box \phi, \Gamma'_w$.

Case w is ($\rightarrow L_{\text{sat}}$) or ($\rightarrow R_{\text{sat}}$). Analogous to the (T_{sat}) case.

Case w is (Rep). The function is defined as

$$\frac{\overline{x_w : \Gamma_w \gg_w \Delta_w} \text{ Rep}}{\Gamma_w \gg_w \Delta_w} \xrightarrow{\alpha} \frac{\begin{array}{c} \alpha(w^\circ) \\ \Gamma_w \gg_w \Delta_w, \kappa_w^* \\ \Gamma_w \gg_w \Delta_w, \kappa_w^*, \kappa_w^* \end{array} \text{ Wk} \quad \frac{\tau}{\kappa_w^*, \Gamma_w \gg_w \Delta_w, \kappa_w^*} \text{ Cut}}{\Gamma_w \gg_w \Delta_w, \kappa_w^*}$$

where τ is a proof of $\kappa_w^*, \Gamma_w \gg_w \Delta_w, \kappa_w^*$ in $\mathcal{G}^\infty\text{GLS}$ which exists since $\text{GLS} \vdash x_w^* \leftrightarrow \kappa_w^*$.

Case w is ($\Box_{\text{sat}}^{K4}$). Then w is of shape

$$\frac{\begin{array}{c} w^\Box \\ \kappa^\Box : \Box \Sigma^s \Rightarrow \end{array} \quad \left[\begin{array}{c} w^\Diamond_\phi \\ \kappa^\Diamond_\phi : \Box \Sigma^s \Rightarrow \phi \end{array} \right]_{\phi \in \Theta}}{\Box \kappa^\Box \wedge \bigwedge_{\phi \in \Theta} \Diamond \kappa^\Diamond_\phi \wedge \bigwedge (\Gamma'_w \cap V_+) \wedge \neg(\Delta'_w \cap V_-) : \Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w} \Box_{\text{sat}}^{K4}$$

where $\kappa_w = \Box \kappa^\Box \wedge \bigwedge_{\phi \in \Theta} \Diamond \kappa^\Diamond_\phi \wedge \bigwedge (\Gamma'_w \cap V_+) \wedge \neg(\Delta'_w \cap V_-)$, $\gg_w = \gg$, $\Gamma_w = \Box \Sigma, \Gamma'_w$ and $\Delta_w = \Box \Theta, \Delta'_w$. Since $(\cdot)^* : B_T \rightarrow \mathcal{L}_\Box$ and $B_T \cap \text{Var} = \emptyset$ we have that

$$\kappa_w^* = \Box (\kappa^\Box)^* \wedge \bigwedge_{\phi \in \Theta} \Diamond (\kappa^\Diamond_\phi)^* \wedge \bigwedge (\Gamma'_w \cap V_+) \wedge \neg(\Delta'_w \cap V_-).$$

So to build a preproof of $\Gamma_w \gg_w \Delta, \kappa_w^*$ we build a preproof for each of the conjuncts and join them using ($\wedge R$).

- Proof for $\Box \kappa^\Box$ and $\Diamond \kappa^\Diamond_\phi$ for $\phi \in \Theta$. $\alpha(w)$ is respectively:

$$\frac{\begin{array}{c} \alpha(w^\Box) \\ \Box \Sigma^s \Rightarrow (\kappa^\Box)^* \\ \Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w, \Box (\kappa^\Box)^* \end{array} \Box^{K4}}{\Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w, \Box (\kappa^\Box)^*} \quad \frac{\begin{array}{c} \alpha(w^\Diamond_\phi) \\ \Box \Sigma^s \Rightarrow \phi, (\kappa^\Diamond_\phi)^* \\ \Box \neg(\kappa^\Diamond_\phi)^*, \Box \Sigma^s \Rightarrow \phi \\ \Box \neg(\kappa^\Diamond_\phi)^*, \Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w \\ \Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w, \Diamond (\kappa^\Diamond_\phi)^* \end{array} \neg L + Wk \quad \Box^{K4} \quad \neg R}{\Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w, \Diamond (\kappa^\Diamond_\phi)^*}$$

- Proof for $p \in \Gamma'_w \cap V_+$ and $\neg q$ where $q \in \Delta'_w \cap V_-$. In each case the preproof is, respectively:

$$\frac{\overline{\Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w, p} \text{ ax}}{\Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w, p} \quad \frac{\overline{q, \Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w} \text{ ax}}{\Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w, \neg q} \neg R$$

To each node w assign the measure $(|\Gamma_w \gg_w \Delta_w|_{\text{sat}}, \text{lg}h(w))$ with the lexicographic order. We notice that in each case the measure decreases from w to its corecursive calls, except when w is $(\Box_{\text{sat}}^{\text{K4}})$. However, in this case there is an application of $\Box^{\text{K4}}$ from the root of $\alpha(w)$ to the corecursive calls, guaranteeing progress.

Definition of β at Lemma 4.15. We will define β corecursively. That $\beta(w, \pi)$ is a preproof will be trivial from construction, when the definition is finished we will show that it is indeed a proof. We proceed by cases on the shape of w .

Case w is (ax) or $(\perp\text{L})$. $\beta(w, \pi)$ is obtained by applying rule $(\perp\text{L})$.

Case w is (Emp). $\beta(w, \pi)$ is obtained from π by applying the weakening rule.

Case w is (T_{sat}) . Then the function is defined as

$$\left(\frac{w0}{\kappa : \Box \phi, \Gamma'_w \Rightarrow \Delta_w} \text{T}_{\text{sat}}, \quad \Box \phi, \Gamma'_w, \Phi \Rightarrow \Delta_w, \Psi \right) \xrightarrow{\beta} \frac{\beta(w0, \text{Wk}(\pi))}{\frac{\kappa^*, \Phi \Rightarrow \Psi}{\kappa^*, \Phi \Rightarrow \Psi} \text{Wk}}$$

where $\kappa_w = \kappa, \gg_w = \Rightarrow$ and $\Gamma_w = \Box \phi, \Gamma'_w$.

Case w is $(\rightarrow\text{L}_{\text{sat}})$ or $(\rightarrow\text{R}_{\text{sat}})$. Analogous to the (T_{sat}) case.

Case w is (Rep). The function is defined as

$$\frac{\frac{\frac{x_w : \Gamma_w \gg_w \Delta_w}{\text{Rep}}, \quad \Gamma_w, \Phi \gg_w \Delta_w, \Psi}{\xrightarrow{\beta} \frac{\tau}{x_w^*, \Phi \gg_w \Psi, \kappa_{w^\circ}^*} \frac{\frac{\beta(w^\circ, \pi)}{\kappa_{w^\circ}^*, \Phi \gg_w \Psi} \text{Wk}}{x_w^*, \Phi \gg_w \Psi} \text{Cut}}}$$

where τ is a proof of $x_w^*, \Phi \gg_w \Psi, \kappa_{w^\circ}^*$ in $\mathcal{G}^\infty\text{GLS}$ which exists since $\text{GLS} \vdash x_w^* \leftrightarrow \kappa_{w^\circ}^*$.

Case w is $(\Box_{\text{sat}}^{\text{K4}})$. Then w is of shape

$$\frac{\kappa^\Box : \Box \Sigma^s \Rightarrow \left[\kappa_\phi^\Diamond : \Box \Sigma^s \Rightarrow \phi \right]_{\phi \in \Theta}}{\Box \kappa^\Box \wedge \bigwedge_{\phi \in \Theta} \Diamond \kappa_\phi^\Diamond \wedge \bigwedge (\Gamma'_w \cap V_+) \wedge \neg (\Delta'_w \cap V_-) : \Box \Sigma, \Gamma'_w \gg \Box \Theta, \Delta'_w} \Box_{\text{sat}}^{\text{K4}}$$

where $\kappa_w = \Box \kappa^\Box \wedge \bigwedge_{\phi \in \Theta} \Diamond \kappa_\phi^\Diamond \wedge \bigwedge (\Gamma'_w \cap V_+) \wedge \neg (\Delta'_w \cap V_-)$, $\gg_w = \gg$, $\Gamma_w = \Box \Sigma, \Gamma'_w$ and $\Delta_w = \Box \Theta, \Delta'_w$. We proceed by further case analysis in the last rule applied to π .

- Last rule is ax. We have that a propositional variable p must occur at the left and right side of the sequent. Since $(\Gamma'_w \cap \Delta'_w) \cap \text{Var} = \emptyset$ we have three options. If $p \in \Phi \cap \Psi$ then the desired proof is just ax. If $p \in \Gamma'_w \cap \Psi$, then $p \in \Psi$ implies that $p \in V_+$ and the desired proof is using that p is a conjunct of κ_w^* . If $p \in \Phi \cap \Delta'_w$, then $p \in \Phi$ implies that $p \in V_-$ and the desired proof is using that $\neg p$ is a conjunct of κ_w^* .
- Last rule is $\perp\text{L}$. Since $\perp \notin \Gamma'_w$ (by definition of $\Box_{\text{sat}}^{\text{K4}}$), it must be the case that $\perp \in \Phi$. Then the desired preproof is obtained using $\perp\text{L}$.
- Last rule is $\perp\text{R}$. If $\perp \in \Delta'_w$ then $\Delta'_w = \perp, \Delta''_w$. We define β as

$$\left(w, \quad \frac{\pi_0}{\frac{\Box \Sigma, \Gamma'_w, \Phi \gg \Box \Theta, \Delta''_w, \Psi}{\Box \Sigma, \Gamma'_w, \Phi \gg \Box \Theta, \perp, \Delta''_w, \Psi} \perp\text{R}} \right) \xrightarrow{\beta} \frac{\beta(w, \text{Wk}(\pi_0))}{\frac{\kappa_w^*, \Phi \gg \Psi}{\kappa_w^*, \Phi \gg \Psi} \text{Wk}}$$

If $\perp \notin \Delta'_w$ then $\perp \in \Psi$ so $\Psi = \perp, \Psi'$. We define β as

$$\left(w, \frac{\pi_0}{\frac{\Box \Sigma, \Gamma'_w, \Phi \gg \Box \Theta, \Delta'_w, \Psi'}{\Box \Sigma, \Gamma'_w, \Phi \gg \Box \Theta, \Delta'_w, \perp, \Psi'} \perp R} \right) \xrightarrow{\beta} \frac{\beta(w, \pi_0)}{\frac{\Phi \gg \Psi'}{\Phi \gg \perp, \Psi'} \perp R}$$

- Last rule is (T). If the principal formula $\Box \phi$ of (T) is in $\Box \Sigma$, then we know by saturation that $\phi \in \Box \Sigma, \Gamma'_w$. Let $\Sigma' = \phi, \Sigma$, we define β as

$$\left(w, \frac{\pi_0}{\frac{\Box \phi, \Box \Sigma', \Gamma'_w, \Phi \Rightarrow \Box \Theta, \Delta'_w, \Psi}{\Box \phi, \Box \Sigma', \Gamma'_w, \Phi \Rightarrow \Box \Theta, \Delta'_w, \Psi} T} \right) \xrightarrow{\beta} \frac{\beta(w, \text{Ctr}(\pi_0))}{\frac{\kappa_w^*, \Phi \Rightarrow \Psi}{\kappa_w^*, \Phi \Rightarrow \Psi} \text{Wk}}$$

where we contract π_0 to get rid of one instance of ϕ at the left. If the principal formula $\Box \phi$ of (T) is not in $\Box \Sigma$, it must be in Φ , so let $\Phi = \Box \phi, \Phi'$. We define β as

$$\left(w, \frac{\pi_0}{\frac{\Box \Sigma, \Gamma'_w, \Box \phi, \Phi' \Rightarrow \Box \Theta, \Delta'_w, \Psi}{\Box \Sigma, \Gamma'_w, \Box \phi, \Phi' \Rightarrow \Box \Theta, \Delta'_w, \Psi} T} \right) \xrightarrow{\beta} \frac{\beta(w, \pi_0)}{\frac{\kappa_w^*, \Box \phi, \Phi' \Rightarrow \Psi}{\kappa_w^*, \Box \phi, \Phi' \Rightarrow \Psi} T}$$

- Last rule is $(\rightarrow L)$ or $(\rightarrow R)$. Analogous to (T) case (with cases on where the principal formula occurs and possibly needed to use admissibility of contraction and weakening at proof π).
- Last rule is $\Box^{K4}$. If the principal formula $\Box \phi$ of $\Box^{K4}$ is in $\Box \Theta$, then let $\Box \Theta = \Box \phi, \Box \Theta'$. We define β as

$$\left(w, \frac{\pi_0}{\frac{\Box \Sigma', \Box \Phi_{\Box} \Rightarrow \phi}{\Box \Sigma, \Gamma'_w, \Phi \gg \Box \phi, \Box \Theta', \Delta'_w, \Psi} \Box^{K4}} \right) \xrightarrow{\beta} \frac{\beta(w_{\phi}^{\Diamond}, \text{Wk}(\pi_0))}{\frac{(\kappa_{\phi}^{\Diamond})^*, \Box \Phi_{\Box} \Rightarrow}{\Box \Phi_{\Box} \Rightarrow \neg(\kappa_{\phi}^{\Diamond})^*} \neg R} \frac{\neg R}{\frac{\Phi \gg \Box \neg(\kappa_{\phi}^{\Diamond})^*, \Psi}{\Diamond(\kappa_{\phi}^{\Diamond})^*, \Phi \gg \Psi} \Box^{K4}} \frac{\neg L}{\frac{\kappa_w^*, \Phi \gg \Psi}{\kappa_w^*, \Phi \gg \Psi} \text{Wk} + \wedge L}$$

where $\Sigma' \subseteq \Sigma$, $\Box \Phi_{\Box} \subseteq \Phi$ and we applied weakening on π_0 to obtain the full $\Box \Sigma$ instead of $\Box \Sigma'$. If the principal formula $\Box \phi$ of $\Box^{K4}$ is in Ψ , then let $\Psi = \Box \phi, \Psi'$. We define β as

$$\left(w, \frac{\pi_0}{\frac{\Box \Sigma', \Box \Phi_{\Box} \Rightarrow \phi}{\Box \Sigma, \Gamma'_w, \Phi \gg \Box \Theta, \Delta'_w, \Box \phi, \Psi'} \Box^{K4}} \right) \xrightarrow{\beta} \frac{\beta(w^{\Box}, \text{Wk}(\pi_0))}{\frac{(\kappa^{\Box})^*, \Box \Phi_{\Box} \Rightarrow \phi}{\Box(\kappa^{\Box})^*, \Box \Phi_{\Box} \Rightarrow \phi} \text{Wk}} \frac{\Box^{K4}}{\frac{\Box(\kappa^{\Box})^*, \Phi \gg \Box \phi, \Psi'}{\kappa_w^*, \Phi \gg \Box \phi, \Psi'} \text{Wk} + \wedge L}$$

where $\Sigma' \subseteq \Sigma$, $\Box \Phi_{\Box} \subseteq \Phi$ and we applied weakening on π_0 to obtain the full $\Box \Sigma$ instead of $\Box \Sigma'$.

To each pair (w, π) assign the measure $(|\Gamma_w \gg_w \Delta_w|_{\text{sat}}, \text{lgh}(w), \text{lgh}(\pi))$ with the lexicographic order. We notice that in each case the measure decreases from (w, π) to its corecursive calls, except when w is $(\Box_{\text{sat}}^{K4})$ and π ends in the $(\Box^{K4})$ rule. However, in this case there is an application of $(\Box^{K4})$ from the root of $\alpha(w)$ to the corecursive calls, guaranteeing progress.

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Knowing-Value Logic with Successor Arithmetic

Hongyi Wang

Department of Philosophy
Peking University
Beijing, China

wanghongyi@stu.pku.edu.cn

In their prior work [7], Wang and Fan proposed conditional knowing-value logic and provided a complete axiomatization. However, in natural language scenarios and logic puzzles, knowing-value reasoning often appears together with arithmetic operations, which motivates us to enrich knowing-value logic with arithmetic function symbols. In this paper, we extend the language of conditional knowing-value logic with equality and the successor function. Due to the failure of compactness over the class of standard models, we additionally introduce non-standard models to facilitate the technical analysis. Our main results establish the finite model property and provide an axiomatization that is strongly complete with respect to the class of non-standard models and weakly complete with respect to the class of standard models. Furthermore, we extend our logic with public announcement operators and use the resulting system to formalize and solve the “Consecutive Numbers” puzzle. This work provides a novel framework for integrating epistemic logic with arithmetic.

1 Introduction

Epistemic logic is a branch of philosophical logic that studies reasoning about knowledge, belief, and related concepts. In this field, propositional knowledge—usually expressed as “knowing-that”—has been widely studied in classical epistemic logic. However, as [5] points out, natural language also contains many “knowing-wh” expressions, such as knowing-what, knowing-why, knowing-how, and knowing-who, which require systematic logical analysis. Therefore, logics of “knowing-wh” have attracted much attention in recent years.

This paper focuses on “knowing-value” logic, a specific subfield of “knowing-wh” research. It formalizes expressions such as “S knows the value of the password” or “S knows what the password is”. In [4], Plaza first proposed the knowing-value operator Kv . Intuitively, $Kv_i c$ means that agent i knows the value of the non-rigid constant c . Plaza also provided an axiomatization for **ELKv** (epistemic logic with the knowing-value operator). Building on this work, Wang and Fan introduced a relativized knowing-value operator in [6]. Intuitively, $Kv_i(\phi, c)$ means that agent i knows the value of c given the condition ϕ . This operator can formalize statements such as “S knows the value of the password, given that the password is a 4-digit number”. In [6, 7], they provided a complete axiomatization for the corresponding logics **ELKv** and **PALKv** (public announcement logic with the relativized knowing-value operator). There are also other studies on knowing-value logic, such as [3], which introduced the KvP operator to express knowing the value of a predicate.

However, the logics **ELKv** and **PALKv** still have some limitations in both syntax and semantics. From a syntactic perspective, because they lack equality and numerical terms, their formalization can be coarse-grained. For instance, **ELKv** expresses “S knows the value of the password given that it is a 4-digit number” as $Kv(p, c)$, using a propositional letter p for the condition. Similarly, “S knows that if the password is 4-digit, then it is 1111” is formalized as $K(p \rightarrow q)$, using q for “the password is 1111”. This approach hides the internal structure of these propositions. As a result, **ELKv** fails to

capture some reasoning in natural language. In natural language, the second statement clearly implies the first. But in $\mathbf{ELKv}^r$, $K(p \rightarrow q)$ does not imply $Kv(p, c)$ because p and q are just independent letters. Generally, reasoning that involves the interaction between equality, knowing-value, and arithmetic is common in natural language scenarios, but $\mathbf{ELKv}^r$ cannot fully express it. From a semantic perspective, a model for $\mathbf{ELKv}^r$ assigns values to constants over a bare, unstructured set of objects O . To study specific mathematical structures, such as natural numbers with the successor function $(\mathbb{N}, 0^{\mathbb{N}}, S^{\mathbb{N}})$ or with addition and multiplication $(\mathbb{N}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, +^{\mathbb{N}}, \times^{\mathbb{N}})$, we must enrich the language with arithmetic function symbols and add arithmetic axioms to the proof system. All these reasons motivate us to enrich knowing-value logic with equality and arithmetic function symbols, starting with the successor function. To further demonstrate the need for this extension, we present the following puzzle.

Example 1.1 ([1]). Anne and Bill get to hear the following: “Given are two natural numbers. They are consecutive numbers. I am going to whisper one of these numbers to Anne and the other number to Bill.” This happens. Anne and Bill now have the following conversation.

- Anne: “I don’t know your number.”
- Bill: “I don’t know your number.”
- Anne: “I know your number.”
- Bill: “I know your number.”

How is that possible? What surely is one of the two numbers?

In this puzzle, introducing the successor function allows us to formalize the statement “Anne knows that a and b are consecutive numbers, but Anne does not know the value of b ” as $K_A(a \approx Sb \vee b \approx Sa) \wedge \neg Kv_A b$. Instead of using a simple propositional letter p to represent “ a and b are consecutive”, this formalization preserves the internal structure and relevant details of the puzzle. Furthermore, the reasoning in the puzzle crucially relies on the premise that Anne and Bill both know that 0 is not a successor in natural numbers. In first-order logic, this can be derived by an axiom of successor arithmetic: $\forall x \neg(Sx \approx 0)$. Although our logic does not have quantifiers, we can capture this by introducing an axiom schema $\neg Sc \approx 0$ for any constant c . We formalize and solve the puzzle in Section 6.

Our main contributions are as follows. First, we introduce $\mathbf{PALKvSA}^r$, an extension of $\mathbf{PALKv}^r$ with equality and the successor function; its announcement-free fragment is $\mathbf{ELKvSA}^r$. We define semantics for these languages over both standard and non-standard models. Second, for $\mathbf{ELKvSA}^r$, we propose the axiom system $\mathbf{ELKVSA}^r$ and prove that it is strongly complete with respect to non-standard models and weakly complete with respect to standard models. Third, we establish that $\mathbf{ELKvSA}^r$ has the finite model property and is decidable. Finally, we show how the “Consecutive Numbers” puzzle can be formalized and solved within this setting. Overall, our work develops a novel framework that integrates epistemic logic with arithmetic, which is of independent theoretical interest and has many potential applications.

The rest of the paper is organized as follows. Section 2 briefly reviews the preliminaries. Section 3 introduces the language and semantics of our logic. Section 4 presents the axiom system $\mathbf{ELKVSA}^r$ and proves its strong completeness over non-standard models. Section 5 establishes the finite model property, weak completeness over standard models, and decidability. Section 6 formalizes the puzzle using public announcement operators. Finally, Section 7 concludes the paper.

2 Preliminaries

This section briefly recalls the background needed in our work: the conditional knowing-value logic and first-order successor arithmetic.

2.1 Conditional Kv Logic

Definition 2.1 (Language **PALKv^r**). Given a countably infinite set of proposition letters **P**, a countably infinite set of agent names **I**, and a countably infinite set of (non-rigid) constant symbols **C**, the language of **PALKv^r** is defined as follows:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid Kv_i(\phi, c) \mid [\phi]\phi$$

where $p \in \mathbf{P}, i \in \mathbf{I}, c \in \mathbf{C}$.

We use the standard abbreviations $\phi \rightarrow \psi := \neg(\phi \wedge \neg\psi)$, $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, $\perp := \neg\top$, and $\hat{K}_i\phi := \neg K_i\neg\phi$. A model $\mathcal{M}$ is defined as $\langle W, \{\sim_i \mid i \in I\}, O, V, V_C \rangle$, where W is a non-empty set of possible worlds, $\sim_i$ is an equivalence relation on W , O is a non-empty set of objects, V is a valuation function assigning a set of worlds to each $p \in \mathbf{P}$, and $V_C : \mathbf{C} \times W \rightarrow O$ assigns an object for each constant-world pair. For a formula ψ , write $\mathcal{M}|_\psi$ for the restriction of $\mathcal{M}$ to $W_\psi = \{w \in W \mid \mathcal{M}, w \models \psi\}$. The semantics is defined as follows:

$\mathcal{M}, w \models \top$	always holds
$\mathcal{M}, w \models p$	$\Leftrightarrow w \in V(p)$
$\mathcal{M}, w \models \neg\phi$	$\Leftrightarrow \mathcal{M}, w \not\models \phi$
$\mathcal{M}, w \models \phi \wedge \psi$	$\Leftrightarrow \mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models K_i\psi$	$\Leftrightarrow$ for all u such that $w \sim_i u : \mathcal{M}, u \models \psi$
$\mathcal{M}, w \models Kv_i(\phi, c)$	$\Leftrightarrow$ for any $u, v \in W$ such that $w \sim_i u$ and $w \sim_i v$: if $\mathcal{M}, u \models \phi$ and $\mathcal{M}, v \models \phi$, then $V_C(c, u) = V_C(c, v)$
$\mathcal{M}, w \models [\psi]\phi$	$\Leftrightarrow \mathcal{M}, w \models \psi$ implies $\mathcal{M} _\psi, w \models \phi$

We denote the announcement-free part of **PALKv^r** as **ELKv^r**. It is shown in [7] that the following system **ELKV^r** completely axiomatizes **ELKv^r**.

System **ELKV^r**

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$K_i(\phi \rightarrow \psi) \rightarrow (K_i\phi \rightarrow K_i\psi)$
T	$K_i\phi \rightarrow \phi$
4	$K_i\phi \rightarrow K_iK_i\phi$
5	$\neg K_i\phi \rightarrow K_i\neg K_i\phi$
DISTKv ^r	$K_i(\phi \rightarrow \psi) \rightarrow (Kv_i(\psi, c) \rightarrow Kv_i(\phi, c))$
Kv ^r 4	$Kv_i(\phi, c) \rightarrow K_iKv_i(\phi, c)$
Kv ^r $\perp$	$Kv_i(\perp, c)$
Kv ^r $\vee$	$\hat{K}_i(\phi \wedge \psi) \wedge Kv_i(\phi, c) \wedge Kv_i(\psi, c) \rightarrow Kv_i(\phi \vee \psi, c)$

Rules

MP	$\frac{\phi, \phi \rightarrow \psi}{\psi}$
NECK	$\frac{\psi}{\phi}$
RE	$\frac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$

By reduction, **ELKV^r** together with the following reduction axioms completely axiomatize **PALKv^r**. The resulting system is called **PALKV^r**.

!ATOM	$[\psi]p \leftrightarrow (\psi \rightarrow p)$
!NEG	$[\psi]\neg\phi \leftrightarrow (\psi \rightarrow \neg[\psi]\phi)$
!CON	$[\psi](\phi \wedge \chi) \leftrightarrow ([\psi]\phi \wedge [\psi]\chi)$
!K	$[\psi]K_i\phi \leftrightarrow (\psi \rightarrow K_i[\psi]\phi)$
!Kv ^r	$[\phi]Kv_i(\psi, c) \leftrightarrow (\phi \rightarrow Kv_i(\phi \wedge [\phi]\psi, c))$

The following reduction axiom !COM is derivable from **PALKV^r**.

$$\text{!COM} \quad [\psi][\chi]\phi \leftrightarrow [\psi \wedge [\psi]\chi]\phi$$

2.2 First-Order Successor Arithmetic

Theorem 2.2. *Given the symbol set $\{0, S\}$ and the structure $\mathfrak{N} = (\mathbb{N}, \mathbf{S})$ where $\mathbb{N}$ is the set of natural numbers and $\mathbf{S}$ is the successor function on it, the first-order theory of this structure $Th(\mathfrak{N})$ is axiomatized by the following three axioms and an additional axiom schema.*

$$\begin{aligned} & \forall x (\neg Sx \approx 0) \\ & \forall x \forall y (Sx \approx Sy \rightarrow x \approx y) \\ & \forall x (\neg x \approx 0 \rightarrow \exists y x \approx Sy) \\ & \forall x (\neg S^n(x) \approx x) \text{ for every natural number } n \geq 1 \end{aligned}$$

where $S^n(x)$ is an abbreviation of $\underbrace{SS \cdots S}_{n \text{ times}}(x)$.

Proof. Its proof can be found in many first-order logic textbooks, for example, [2]. □

3 Language and Semantics

In this section, we formally define the language and semantics of our logic. We define both standard and non-standard models. Note that the classes of standard and non-standard models induce two distinct semantics. When a property or result is stated w.r.t. standard (or non-standard) models, it is to be understood as holding under the semantics induced by the respective class of models. We now introduce the language **PALKvSA^r** and write **ELKvSA^r** for its announcement-free fragment.

Definition 3.1 (Language **PALKvSA^r(I, C)**). Given a countable constant symbol set **C** and a countable set of agents **I**, the set of terms **Term** and formulas are defined as follows.

$$\begin{aligned} t &::= 0 \mid c \mid St \\ \varphi &::= \top \mid t \approx t \mid \neg \varphi \mid \varphi \wedge \varphi \mid K_i \varphi \mid K_{v_i}(\varphi, t) \mid [\varphi] \varphi \end{aligned}$$

where $c \in \mathbf{C}$, $i \in \mathbf{I}$, 0 is a rigid constant symbol, S is the successor function symbol. Formulas of the form $t \approx t'$ are called atomic formulas (in **PALKv^r**, atomic formulas are propositional letters). We write $K_{v_i}t$ as an abbreviation of $K_{v_i}(\top, t)$.

Definition 3.2 (Semantics). A standard model is a tuple $\mathcal{M} = \langle W, \mathbb{N}, \mathbf{S}, \{\sim_i\}_{i \in \mathbf{I}}, V_{\mathbf{C}} \rangle$, where $\sim_i$ is an equivalence relation on the set of possible worlds W for each agent i . Here $\mathbb{N}$ is the set of natural numbers and $\mathbf{S}$ is the successor function on $\mathbb{N}$, and $V_{\mathbf{C}} : \mathbf{C} \times W \rightarrow \mathbb{N}$ assigns a natural number for each constant-world pair.

Define the assignment for terms $V_{\mathbf{T}} : \mathbf{Term} \times W \rightarrow \mathbb{N}$ as follows:

$$\begin{aligned} V_{\mathbf{T}}(0, w) &= \mathbf{0} \\ V_{\mathbf{T}}(c, w) &= V_{\mathbf{C}}(c, w) \\ V_{\mathbf{T}}(St, w) &= \mathbf{S}(V_{\mathbf{T}}(t, w)) \end{aligned}$$

For a formula ψ , write $\mathcal{M}|_{\psi}$ for the restriction of $\mathcal{M}$ to $W_{\psi} = \{w \in W \mid \mathcal{M}, w \models \psi\}$. Then the satisfaction relation is defined as follows:

$\mathcal{M}, w \models t_1 \approx t_2$	$\Leftrightarrow$	$V_T(t_1, w) = V_T(t_2, w)$
$\mathcal{M}, w \models Kv_i(\varphi, t)$	$\Leftrightarrow$	for any $u, v \in W$ such that $w \sim_i u, w \sim_i v$, if $\mathcal{M}, u \models \varphi$ and $\mathcal{M}, v \models \varphi$, then $V_T(t, u) = V_T(t, v)$
$\mathcal{M}, w \models [\varphi]\psi$	$\Leftrightarrow$	$\mathcal{M}, w \models \varphi$ implies $\mathcal{M} _\varphi, w \models \psi$

Similar to **PALKv**^r, if an axiom system $\mathbb{S}$ completely axiomatizes **ELKvSA**^r, then $\mathbb{S}$ combined with the reduction axioms !ATOM, !NEG, !CON, !K, !Kv^r and RE completely axiomatizes **PALKvSA**^r, where !ATOM is $[\varphi]\alpha \leftrightarrow (\varphi \rightarrow \alpha)$ for any atomic formula α . Therefore, we will focus exclusively on axiomatizing **ELKvSA**^r in the remainder of this paper. However, as the following proposition demonstrates, **ELKvSA**^r is not compact with respect to the class of standard models. Consequently, there is no sound and strongly complete axiomatization¹ for **ELKvSA**^r over standard models.

Proposition 3.3. ***ELKvSA**^r is not compact w.r.t. standard models.*

Proof. Consider the formula set $\{\neg c \approx S^n 0 \mid n \in \mathbb{N}\}$. It is finitely satisfiable, but not satisfiable. Also consider the formula set $\{Kv_i c\} \cup \{\neg K_i(c \approx S^n 0) \mid n \in \mathbb{N}\}$. It is also finitely satisfiable, but not satisfiable. $\square$

Therefore, for technical reasons, we introduce non-standard models to facilitate our analysis. $(\mathbb{N}^*, \mathbf{S}^*)$ is a non-standard model of first-order successor arithmetic, where $\mathbb{N}^* = \mathbb{N} \cup \{(n, z) \mid n \in \mathbb{N}, z \in \mathbb{Z}\}$, and $\mathbf{S}^*(n) = n + 1$, $\mathbf{S}^*(n, z) = (n, z + 1)$. A non-standard model for our logic is then a tuple based on this arithmetic structure:

$$\mathcal{N} = \langle W, \mathbb{N}^*, \mathbf{S}^*, \{\sim_i\}_{i \in I}, V_{\mathbf{C}}^* \rangle$$

The definition of $V_{\mathbf{C}}^*$, V_T^* and the satisfaction relation are as before, with $(\mathbb{N}^*, \mathbf{S}^*)$ in place of $(\mathbb{N}, \mathbf{S})$. For convenience, let $Z_n := \{(n, z) \mid z \in \mathbb{Z}\}$ be the n th $\mathbb{Z}$ -chain.

Remark 3.4. Note that our non-standard models are defined to have countably infinitely many $\mathbb{Z}$ -chains (indexed by $n \in \mathbb{N}$). This specific cardinality is intentionally chosen. On one hand, if we restrict the class to non-standard models with only finitely many $\mathbb{Z}$ -chains, compactness would still fail. On the other hand, introducing uncountably many $\mathbb{Z}$ -chains is unnecessary, as constant symbols are countable. In the rest of the paper, “non-standard models” means models based on the fixed arithmetic structure $(\mathbb{N}^*, \mathbf{S}^*)$ above. However, once completeness is proved for this class, it also holds for any larger class of models that contains it, such as the class obtained by allowing arbitrary non-standard models of successor arithmetic.

4 Axiomatization of **ELKvSA**^r

In this section, we present the axiom system **ELKVSA**^r for the logic **ELKvSA**^r. We first establish the soundness of the system with respect to both standard and non-standard models. Subsequently, we prove the strong completeness of **ELKVSA**^r over the class of non-standard models.

¹Strong completeness means that, for every set of formulas Γ and formula φ , if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$; Weak completeness means that, for every formula φ , if $\models \varphi$, then $\vdash \varphi$.

4.1 System $\text{ELKVS}A^r$

System $\text{ELKVS}A^r$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$
T	$K_i\varphi \rightarrow \varphi$
4	$K_i\varphi \rightarrow K_iK_i\varphi$
5	$\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$
ID	$t \approx t$
SI	$t \approx t' \rightarrow (\alpha \rightarrow \alpha(t'/t))$ where α is an atomic formula
SA1	$\neg 0 \approx St$
SA2	$St_1 \approx St_2 \rightarrow t_1 \approx t_2$
SA3	$\neg S^n t \approx t$, where $n \geq 1$
$Kv^r 0$	$Kv_i 0$
$Kv^r S$	$Kv_i(\varphi, t) \leftrightarrow Kv_i(\varphi, St)$
$Kv^r 4$	$Kv_i(\varphi, t) \rightarrow K_iKv_i(\varphi, t)$
KVE1	$Kv_it_1 \wedge K_i(\varphi \rightarrow t_1 \approx t_2) \rightarrow Kv_i(\varphi, t_2)$
KVE2	$Kv_it_1 \wedge Kv_it_2 \rightarrow (\widehat{K}_i(t_1 \approx t_2) \rightarrow K_i(t_1 \approx t_2))$

Rules

MP	$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$
NECK	$\frac{\psi}{K_i\psi}$
RE	$\frac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
Kv^r -Elim	$\frac{(Kv_ic \wedge K_i(\varphi \rightarrow c \approx t)) \rightarrow \psi}{Kv_i(\varphi, t) \rightarrow \psi}$ where c does not occur in φ, ψ, t

Proposition 4.1. *The Kv^r -Elim rule preserves validity w.r.t. both standard and non-standard models.*

Proof. Assume for contradiction that the premise $\models (Kv_ic \wedge K_i(\varphi \rightarrow c \approx t)) \rightarrow \psi$ is valid, but the conclusion $\models Kv_i(\varphi, t) \rightarrow \psi$ is not. Since the conclusion is not valid, there exists a model $\mathcal{M} = \langle W, \mathbb{O}, \mathbf{S}^\mathbb{O}, \{\sim_i \mid i \in I\}, V_C \rangle$ and a world $w \in W$ such that $\mathcal{M}, w \models Kv_i(\varphi, t)$ but $\mathcal{M}, w \not\models \psi$, where $(\mathbb{O}, \mathbf{S}^\mathbb{O})$ is either $(\mathbb{N}, \mathbf{S})$ or $(\mathbb{N}^*, \mathbf{S}^*)$. From $\mathcal{M}, w \models Kv_i(\varphi, t)$, we know there exists an element $o \in \mathbb{O}$ such that for all $u \in W$ with $w \sim_i u$, if $\mathcal{M}, u \models \varphi$, then $V_T(t, u) = o$. Now, we construct a new model $\mathcal{M}' = \langle W, \mathbb{O}, \mathbf{S}^\mathbb{O}, \{\sim_i \mid i \in I\}, V'_C \rangle$ which is exactly the same as $\mathcal{M}$, except that we force the interpretation of the constant c to be o in all worlds (i.e., $V'_C(c, v) = o$ for all $v \in W$). Because c does not occur in φ, ψ , or t , the truth values of these formulas and the valuation of t remain completely unchanged in $\mathcal{M}'$. In particular, we still have $\mathcal{M}', w \not\models \psi$. However, let us evaluate the premise in $\mathcal{M}'$ at w :

- Since c is assigned the same value o globally, $\mathcal{M}', w \models Kv_ic$ is true.
- For any $u \in W$ such that $w \sim_i u$, if $\mathcal{M}', u \models \varphi$, then $\mathcal{M}, u \models \varphi$, which implies $V'_T(t, u) = V_T(t, u) = o$. Since $V'_C(c, u) = o$, we have $V'_C(c, u) = V'_T(t, u)$, meaning $\mathcal{M}', u \models c \approx t$. Thus, $\mathcal{M}', w \models K_i(\varphi \rightarrow c \approx t)$ is true.

Combining these, we get $\mathcal{M}', w \models Kv_ic \wedge K_i(\varphi \rightarrow c \approx t)$. Since we assumed the premise $(Kv_ic \wedge K_i(\varphi \rightarrow c \approx t)) \rightarrow \psi$ is globally valid, it must hold in $\mathcal{M}'$ at w . This forces $\mathcal{M}', w \models \psi$. This directly contradicts our earlier deduction that $\mathcal{M}', w \not\models \psi$. Therefore, our initial assumption must be false, and the rule preserves validity. $\square$

Theorem 4.2. *$\text{ELKVS}A^r$ is sound w.r.t. both standard and non-standard models.*

We now explain the intuition behind the axioms and rules of $\text{ELKVS}A^r$. The axioms ID and SI govern equality; from them, we can derive the reflexivity, symmetry, and transitivity of equality. SA1-SA3 come from first-order successor arithmetic. Note that we cannot express the property “every non-zero number is a successor” because our language lacks quantifiers. The axioms KVE1 and KVE2 bridge knowing-value and equality. KVE1 states that if agent i knows the value of t_1 and knows that φ implies

$t_1 \approx t_2$, then agent i also knows the value of t_2 given condition φ . KVE2 states that if agent i knows the values of both t_1 and t_2 , then the agent knows whether they are equal.

A notable feature of the system is the Kv^r -Elim rule. Semantically, the formula $Kv_i(\varphi, c)$ can be understood as $\exists x K_i(\varphi \rightarrow c \approx x)$. Conceptually, the Kv^r -Elim rule plays a role similar to $\exists$ -Elimination in first-order logic, allowing us to safely eliminate the relativized knowing-value operator in derivations. This rule is strong enough to derive standard axioms such as $\text{DISTK}v^r$ and $Kv^r\vee$ from $\text{ELK}V^r$. Furthermore, we will show that $\text{GK}v^r\vee$ and GKVE —the generalized versions of $Kv^r\vee$ and KVE—are also derivable in $\text{ELK}VSA^r$ via this rule.

Before deriving theorems in the axiom system, we provide a general proof method using the Kv^r -Elim rule to derive formulas of the following form: $(\bigwedge_{k=0}^n Kv_i(\varphi_k, t_k)) \rightarrow \psi$. To prove this using Kv^r -Elim, we only need to show the following implication for $n+1$ fresh, pairwise distinct constants $c_0, \dots, c_n$ that do not occur anywhere in the original theorem (φ_k, t_k , or ψ):

$$\left(\bigwedge_{k=0}^n (Kv_i c_k \wedge K_i(\varphi_k \rightarrow c_k \approx t_k)) \right) \rightarrow \psi$$

Once this is established, we can repeatedly apply Kv^r -Elim to eliminate the freshly introduced constants c_k one by one, thereby deriving the original theorem.

The following propositions present several theorems derivable in $\text{ELK}VSA^r$ that will be used in our later completeness proof.

Proposition 4.3. *The symmetry and transitivity of equality are derivable in $\text{ELK}VSA^r$. Moreover, $t_1 \approx t_2 \rightarrow St_1 \approx St_2$ is also derivable.*

Proposition 4.4. *Both $\text{GK}v^r\vee$ and GKVE are derivable in the system $\text{ELK}VSA^r$.*

1. $\text{GK}v^r\vee$: $(\bigwedge_{k=0}^n Kv_i(\varphi_k, t_k)) \wedge \left(\bigwedge_{k=0}^{n-1} \widehat{K}_i(\varphi_k \wedge \varphi_{k+1} \wedge t_k \approx t_{k+1}) \right) \rightarrow Kv_i(\bigvee_{k=0}^n (\varphi_k \wedge t \approx t_k), t)$
2. GKVE : $(\bigwedge_{k=0}^n Kv_i(\varphi_k, t_k)) \wedge \left(\bigwedge_{k=0}^{n-1} \widehat{K}_i(\varphi_k \wedge \varphi_{k+1} \wedge t_k \approx t_{k+1}) \right) \rightarrow K_i(\varphi_0 \wedge \varphi_n \rightarrow t_0 \approx t_n)$

Proof. By the general proof method above, we introduce $n+1$ fresh, pairwise distinct constants $c_0, \dots, c_n$ which do not occur in φ_k, t_k , or t . Let $\Gamma := \bigwedge_{k=0}^n (Kv_i c_k \wedge K_i(\varphi_k \rightarrow c_k \approx t_k))$ and $\Delta := \bigwedge_{k=0}^{n-1} \widehat{K}_i(\varphi_k \wedge \varphi_{k+1} \wedge t_k \approx t_{k+1})$. To derive the two theorems, we only need to prove that $\Gamma \rightarrow (\Delta \rightarrow \psi)$ holds in $\text{ELK}VSA^r$ for each respective consequent ψ . Assume Γ and Δ .

First, for each $0 \leq k \leq n-1$, from $K_i(\varphi_k \rightarrow c_k \approx t_k)$ and $K_i(\varphi_{k+1} \rightarrow c_{k+1} \approx t_{k+1})$, we can derive $K_i((\varphi_k \wedge \varphi_{k+1} \wedge t_k \approx t_{k+1}) \rightarrow c_k \approx c_{k+1})$. Combining this with $\widehat{K}_i(\varphi_k \wedge \varphi_{k+1} \wedge t_k \approx t_{k+1})$ from Δ yields $\widehat{K}_i(c_k \approx c_{k+1})$. By KVE2 and $Kv_i c_k$ and $Kv_i c_{k+1}$, we derive $K_i(c_k \approx c_{k+1})$ for each $0 \leq k \leq n-1$. By the transitivity of equality inside the K_i modality, this naturally yields $K_i(c_0 \approx c_k)$ for all $0 \leq k \leq n$.

1. For $\text{GK}v^r\vee$: From $K_i(\varphi_k \rightarrow c_k \approx t_k)$ in Γ and $K_i(c_0 \approx c_k)$ derived above, we deduce $K_i(\varphi_k \rightarrow c_0 \approx t_k)$, which implies $K_i((\varphi_k \wedge t \approx t_k) \rightarrow c_0 \approx t)$. Taking the disjunction of the antecedents gives: $K_i(\bigvee_{k=0}^n (\varphi_k \wedge t \approx t_k) \rightarrow c_0 \approx t)$. Since we have $Kv_i c_0$ from Γ , by KVE1, we can deduce $Kv_i(\bigvee_{k=0}^n (\varphi_k \wedge t \approx t_k), t)$.
2. For GKVE : With $K_i(c_0 \approx c_n)$ derived, along with $K_i(\varphi_0 \rightarrow c_0 \approx t_0)$ and $K_i(\varphi_n \rightarrow c_n \approx t_n)$ from Γ , standard equality reasoning yields $K_i(\varphi_0 \wedge \varphi_n \rightarrow t_0 \approx t_n)$.

□

Proposition 4.5. *The following theorems can be derived in $\text{ELK}VSA^r$:*

1. $DISTKv^r : K_i(\varphi \rightarrow \psi) \rightarrow (Kv_i(\psi, t) \rightarrow Kv_i(\varphi, t))$
2. $Kv^r \perp : Kv_i(\perp, t)$
3. $Kv^r \vee : \widehat{K}_i(\phi \wedge \psi) \wedge Kv_i(\phi, c) \wedge Kv_i(\psi, c) \rightarrow Kv_i(\phi \vee \psi, c)$
4. $Kv^r 5 : \neg Kv_i(\varphi, t) \rightarrow K_i \neg Kv_i(\varphi, t)$

Proof. We only show the proofs for the first two items. The proof of 3 is similar to the above proposition, and the proof of 4 can be found in [7].

1. We rewrite the axiom as $Kv_i(\psi, t) \rightarrow (K_i(\varphi \rightarrow \psi) \rightarrow Kv_i(\varphi, t))$. Using the Kv^r -Elim method, we introduce a fresh constant c and aim to prove: $(Kv_i c \wedge K_i(\psi \rightarrow c \approx t)) \rightarrow (K_i(\varphi \rightarrow \psi) \rightarrow Kv_i(\varphi, t))$. Assume $Kv_i c$, $K_i(\psi \rightarrow c \approx t)$, and $K_i(\varphi \rightarrow \psi)$. From the latter two, we derive $K_i(\varphi \rightarrow c \approx t)$. Then, according to KVE1, $Kv_i c \wedge K_i(\varphi \rightarrow c \approx t)$ immediately implies $Kv_i(\varphi, t)$. Applying Kv^r -Elim concludes the proof.
2. By TAUT and NECK we get $K_i(\perp \rightarrow 0 \approx t)$. From $Kv^r 0$, we have $Kv_i 0$. Using KVE1, we derive $Kv_i(\perp, t)$.

□

4.2 Completeness w.r.t. Non-Standard Models

In this subsection, we prove the strong completeness w.r.t. non-standard models. We first consider the classical canonical frame $\mathcal{F}^c = \langle W^c, \{\sim_i^c\}_{i \in I} \rangle$ similar to [6]:

- $W^c = MCS \times \{1, 2\}$, where MCS is the set of all maximal consistent sets.
- For any $w, v \in W^c$, $w \sim_i^c v$ iff for any formula φ , $K_i \varphi \in w \Rightarrow \varphi \in v$.²

Obviously, $\sim_i^c$ is an equivalence relation on W^c . The following lemma can be proved by an argument similar to the proof of Proposition 3.10 in [7]:

Lemma 4.6. *Suppose $\Psi_c^i(w) = \{\psi \mid Kv_i(\psi, c) \in w\}$. If $\neg Kv_i(\varphi, c) \in w$, then we can construct $u, v \in MCS$ such that $w \sim_i^c (u, 1)$, $w \sim_i^c (v, 2)$, $\varphi \in u \cap v$, but $u \cap v \cap \Psi_c^i(w) = \emptyset$.*

However, as shown in [6], the canonical frame $\mathcal{F}^c$ is not sufficient for the multi-agent completeness proof. Wang and Fan give an insightful but rather involved solution in [7]. Here we use a different and more direct method: we construct a tree-like frame to facilitate the valuation. The proof is divided into five steps.

Step 1: Construction of the Tree-like Frame. Given a consistent set of formulas Γ_0 , by Lindenbaum's lemma, we can extend it to a maximal consistent set Γ . Let $w_0 = (\Gamma, 1)$. Construct $\langle W', \{\rightarrow_j\}_{j \in I} \rangle$ level by level. The elements of W' are histories. A history is a sequence $\vec{h} = \langle w_0, i_1, w_1, \dots, i_k, w_k \rangle$ where $w_{m-1} \sim_{i_m}^c w_m$ in $\mathcal{F}^c$. Define $\text{Last}(\vec{h}) = w_k$, $\text{height}(\vec{h}) = k$, and $\text{Ag}(\langle w_0 \rangle) = \star \notin I$; for $k > 0$, $\text{Ag}(\vec{h}) = i_k$. In the following, for $d = 1, 2$, $\vec{h} \cdot (j, (w, d))$ denotes the history obtained by appending the agent label j and the canonical world (w, d) to $\vec{h}$. When $\vec{h}'$ is added as a direct j -witness for $\vec{h}$, we put $\vec{h} \rightarrow_j \vec{h}'$; the S5 closure is taken in Step 2. Let $H_0 = \{\langle w_0 \rangle\}$ be the root of the tree. Suppose H_k has been constructed. We construct H_{k+1} as follows. For each $\vec{h} \in H_k$ and $j \in I$:

1. If $j = \text{Ag}(\vec{h})$: Skip. The parent of $\vec{h}$ has already constructed the required witnesses for this case, and these will remain accessible to $\vec{h}$ under the S5 closure.

²Strictly speaking, $\phi \in w$ means $\phi \in \pi_1(w)$, where π_1 is the projection function from W^c to MCS . We omit π_1 for simplicity.

2. If $j \neq \text{Ag}(\vec{h})$:

- For $\neg K_j \psi \in \text{Last}(\vec{h})$, by Existence Lemma, we can construct a MCS w such that $\neg \psi \in w$ and $\text{Last}(\vec{h}) \sim_j^c w$. Let $\vec{h}' = \vec{h} \cdot (j, (w, 1))$ be the new history. Add $\vec{h}'$ to H_{k+1} and let $\vec{h} \rightarrow_j \vec{h}'$.
- For $\neg K_j v_j(\phi, t) \in \text{Last}(\vec{h})$: by Kv'S, we need only consider the case in which t is a constant symbol c . By Lemma 4.6, we can construct $u, v \in \text{MCS}$ such that $\text{Last}(\vec{h}) \sim_j^c (u, 1)$, $\text{Last}(\vec{h}) \sim_j^c (v, 2)$, $\phi \in u \cap v$ but $u \cap v \cap \Psi_c^j(\text{Last}(\vec{h})) = \emptyset$. Let $\vec{h}' = \vec{h} \cdot (j, (u, 1))$ and $\vec{h}'' = \vec{h} \cdot (j, (v, 2))$ be the new histories. Add them to H_{k+1} and let $\vec{h} \rightarrow_j \vec{h}'$ and $\vec{h} \rightarrow_j \vec{h}''$.

Let $W' = \bigcup_{k \in \mathbb{N}} H_k$. We show that W' is countable. Fix Gödel codes for formulas and hence for finite sequences of formulas together with labels from $\{1, 2\}$. In the construction above, whenever witnesses are required, they are constructed by a fixed deterministic Lindenbaum-style procedure. Each history has at least one construction record, which is a finite sequence: the root is recorded by 1, and a history of height $k > 0$ is recorded as

$$(1, (\theta_1, d_1), \dots, (\theta_k, d_k)),$$

where each θ_m is the formula whose requirement caused the m th successor world to be constructed, and $d_m \in \{1, 2\}$ is the corresponding label. A history may have more than one such record, since the same world may witness several formulas. For each history $\vec{h}$, let $m(\vec{h})$ be the least Gödel code among its construction records. Since a construction record determines the whole history, if $m(\vec{h}) = m(\vec{r})$, then $\vec{h} = \vec{r}$. Hence m is injective, and therefore W' is countable.

Step 2: S5 Closure. Define $\mathcal{F}' = \langle W', \{\sim'_j\}_{j \in I} \rangle$, where $\sim'_j$ is the reflexive, symmetric, transitive closure of $\rightarrow_j$. Due to the construction in Step 1, for each $j \in I$, there are no paths of the form $\vec{h}_1 \rightarrow_j \vec{h}_2 \rightarrow_j \vec{h}_3$. Consequently, the structure of $\sim'_j$ equivalence classes is restricted. Each class contains exactly one node $\vec{r}$ with $\text{Ag}(\vec{r}) \neq j$, and all other nodes in the class are immediate j -successors of $\vec{r}$. Moreover, $\sim'_j$ has the following property.

Proposition 4.7. *If $\vec{h} \sim'_j \vec{r}$, then $\text{Last}(\vec{h}) \sim_j^c \text{Last}(\vec{r})$ in the canonical frame $\mathcal{F}^c$, and consequently $\text{Last}(\vec{h})$ and $\text{Last}(\vec{r})$ contain the same K_j -formulas and the same Kv_j -formulas.*

Proof. By the construction, if $\vec{h} \rightarrow_j \vec{r}$, then $\text{Last}(\vec{h}) \sim_j^c \text{Last}(\vec{r})$. Let R be a binary relation on W' defined by $\vec{x} R \vec{y}$ iff $\text{Last}(\vec{x}) \sim_j^c \text{Last}(\vec{y})$. We have $\rightarrow_j \subseteq R$, and R is an equivalence relation on W' . Then $\sim'_j \subseteq R$. The sharing of K_j -formulas follows from axiom 4 and the symmetry of $\sim_j^c$; similarly, the sharing of Kv_j -formulas follows from Kv'4 and the symmetry of $\sim_j^c$. $\square$

Step 3: Preparation for Valuation. Before defining the valuation function, we first need to determine which constant-history pairs must be placed in the same $\mathbb{Z}$ -chain (or in the natural-number part). To this end, we define relations R_i on $(\mathbf{C} \cup \{0\}) \times W'$. $(c, \vec{h}) R_i (d, \vec{r})$ ³ iff one of the following conditions holds:

- Type 1: $\vec{h} = \vec{r}$ and there exist $a, b \in \mathbb{N}$ such that $S^a c \approx S^b d \in \text{Last}(\vec{h})$. In this case, we write $(c, \vec{h}) \xrightarrow{a-b} (d, \vec{h})$ and also $(d, \vec{h}) \xrightarrow{b-a} (c, \vec{h})$.
- Type 2: c and d are the same symbol, $\vec{h} \sim'_i \vec{r}$ and there exists a formula ϕ such that $Kv_i(\phi, c) \wedge \phi \in \text{Last}(\vec{h}) \cap \text{Last}(\vec{r})$. In this case, $(c, \vec{h})$ and $(d, \vec{r})$ must receive the same value. We write $(c, \vec{h}) \xrightarrow{\phi} (c, \vec{r})$, and also $(c, \vec{r}) \xrightarrow{0} (c, \vec{h})$ when the distance is important.

³Note that c and d may also represent 0.

Let $\equiv_i$ be the transitive closure of R_i . It is an equivalence relation on $(\mathbf{C} \cup \{0\}) \times W'$. $(c, \vec{h}) \equiv_i (d, \vec{r})$ implies that $\vec{h} \sim'_i \vec{r}$ and thus $\text{Last}(\vec{h}) \sim_i^c \text{Last}(\vec{r})$. By arithmetic axioms, $(c, \vec{h}) \xRightarrow{a} (d, \vec{h}) \xRightarrow{b} (e, \vec{h})$ implies $(c, \vec{h}) \xRightarrow{a+b} (e, \vec{h})$. If $(c, \vec{h}) \xRightarrow{\varphi} (c, \vec{r}) \xRightarrow{\psi} (c, \vec{u})$, then $\widehat{K}_i(\varphi \wedge \psi) \wedge K v_i(\varphi, c) \wedge K v_i(\psi, c) \in \text{Last}(\vec{h}) \cap \text{Last}(\vec{r}) \cap \text{Last}(\vec{u})$; by $K v^r \vee$, this implies $K v_i(\varphi \vee \psi, c) \wedge (\varphi \vee \psi) \in \text{Last}(\vec{h}) \cap \text{Last}(\vec{u})$. Thus $(c, \vec{h}) \xRightarrow{\varphi \vee \psi} (c, \vec{u})$. Now it is time to show the power of $\text{GKv}^r \vee$ and GKVE .

Lemma 4.8. *For any $(c, \vec{h}), (d, \vec{h}) \in (\mathbf{C} \cup \{0\}) \times W'$, if $(c, \vec{h}) \equiv_i (d, \vec{h})$, i.e. there exist $(c_0, \vec{h}_0), \dots, (c_n, \vec{h}_n)$ such that $(c, \vec{h}) = (c_0, \vec{h}_0) \xRightarrow{a_1} (c_1, \vec{h}_1) \xRightarrow{a_2} \dots \xRightarrow{a_n} (c_n, \vec{h}_n) = (d, \vec{h})$, where $a_1, \dots, a_n \in \mathbb{Z}$ and $\sum_k a_k = a$, then $S^a c \approx d \in \text{Last}(\vec{h})$ if $a \geq 0$, and $c \approx S^{-a} d \in \text{Last}(\vec{h})$ if $a < 0$.*

Proof. We first normalize the witnessing chain. Each step is either: (i) a type 1 step in a single world node, written $(c, \vec{r}) \xRightarrow{a} (d, \vec{r})$, or (ii) a type 2 step between i -equivalent world nodes, written $(c, \vec{r}) \xRightarrow{\varphi} (c, \vec{u})$. Two consecutive type 1 steps can be merged into one by arithmetic axioms, and two consecutive type 2 steps can be merged into one by $K v^r \vee$. If there are only type 1 steps, then the conclusion follows immediately from arithmetic axioms. Now we present the case where the chain starts and ends with a type 2 step; if it starts (or ends) with a type 1 step, simply remove that initial (or terminal) type 1 step, apply the argument below, and then reattach it using arithmetic axioms. Thus we may assume an alternating chain of the form

$$(c_0, \vec{h}_0) \xRightarrow{\phi_0} (c_0, \vec{h}_1) \xRightarrow{a_0} (c_1, \vec{h}_1) \xRightarrow{\phi_1} (c_1, \vec{h}_2) \xRightarrow{a_1} \dots \xRightarrow{a_{n-1}} (c_n, \vec{h}_n) \xRightarrow{\phi_n} (c_n, \vec{h}_0),$$

where $a_0, \dots, a_{n-1} \in \mathbb{Z}$, $\phi_0, \dots, \phi_n$ are formulas, and $\sum_{i=0}^{n-1} a_i = a$. Note that $\vec{h}_0 = \vec{h}$ and all $\vec{h}_s$ are $\sim'_i$ -equivalent. Then all the $\text{Last}(\vec{h}_s)$ are $\sim_i^c$ -equivalent, and thus share the same K_i -formulas and $K v_i$ -formulas.

Let $s_k := \sum_{x=k}^{n-1} a_x$ for $k = 0, \dots, n$, and let $K := \min\{s_k \mid 0 \leq k \leq n\}$. Define $t_k := S^{s_k - K} c_k$. Then all exponents are non-negative. Since $s_n = 0$, we have $t_n = S^{-K} c_n$, and $t_0 = S^{a-K} c_0$.

For each $k < n$, the type 1 step $(c_k, \vec{h}_{k+1}) \xRightarrow{a_k} (c_{k+1}, \vec{h}_{k+1})$ yields $t_k \approx t_{k+1} \in \text{Last}(\vec{h}_{k+1})$ by applying successor on both sides the appropriate number of times. Also, the type 2 step $(c_k, \vec{h}_k) \xRightarrow{\phi_k} (c_k, \vec{h}_{k+1})$ gives $\phi_k \in \text{Last}(\vec{h}_k) \cap \text{Last}(\vec{h}_{k+1})$ and $K v_i(\phi_k, c_k) \in \text{Last}(\vec{h}_k) \cap \text{Last}(\vec{h}_{k+1})$. By $K v^r S$, $K v_i(\phi_k, t_k) \in \text{Last}(\vec{h}_k) \cap \text{Last}(\vec{h}_{k+1})$. Additionally, $\phi_n \wedge K v_i(\phi_n, t_n) \in \text{Last}(\vec{h}_n) \cap \text{Last}(\vec{h}_0)$. Since all the $\text{Last}(\vec{h}_s)$ share the same $K v_i$ -formulas, $K v_i(\phi_k, t_k) \in \text{Last}(\vec{h}_0)$ for each $k = 0, \dots, n$. Moreover, each $\text{Last}(\vec{h}_{k+1})$ contains $\phi_k \wedge \phi_{k+1} \wedge t_k \approx t_{k+1}$ for $k < n$, so $\widehat{K}_i(\phi_k \wedge \phi_{k+1} \wedge t_k \approx t_{k+1}) \in \text{Last}(\vec{h}_0)$ for all $k < n$. Therefore, the premises of GKVE hold in $\text{Last}(\vec{h}_0)$, and we obtain $K_i(\phi_0 \wedge \phi_n \rightarrow t_0 \approx t_n) \in \text{Last}(\vec{h}_0)$. Since $\phi_0 \wedge \phi_n \in \text{Last}(\vec{h}_0)$, we get $t_0 \approx t_n \in \text{Last}(\vec{h}_0)$. Finally, $t_0 \approx t_n$ is $S^{a-K} c \approx S^{-K} d$. Cancelling S^{-K} on both sides using arithmetic axioms repeatedly yields $S^a c \approx d$ when $a \geq 0$, and $c \approx S^{-a} d$ when $a < 0$. Hence the lemma holds. $\square$

Corollary 4.9. *For any loop of the form $(c, \vec{h}) = (c_0, \vec{h}_0) \xRightarrow{a_1} (c_1, \vec{h}_1) \xRightarrow{a_2} \dots \xRightarrow{a_n} (c_n, \vec{h}_n) = (c, \vec{h})$, where $a_1, \dots, a_n \in \mathbb{Z}$, we have $\sum_k a_k = 0$. Consequently, if there are two paths from $(c, \vec{h})$ to $(d, \vec{r})$, one with total weight a and the other with total weight b , then $a = b$.*

Proof. By the above lemma, we have $S^{\sum_k a_k} c \approx c \in \text{Last}(\vec{h})$ if $\sum_k a_k \geq 0$ and $c \approx S^{-\sum_k a_k} c \in \text{Last}(\vec{h})$ if $\sum_k a_k < 0$. But we also have $\neg S^n c \approx c \in \text{Last}(\vec{h})$ for $n > 0$. Since $\text{Last}(\vec{h})$ is consistent, we have $\sum_k a_k = 0$. $\square$

Lemma 4.10. *For any $(c, \vec{h}), (c, \vec{r}) \in (\mathbf{C} \cup \{0\}) \times W'$, if $(c, \vec{h}) \equiv_i (c, \vec{r})$, i.e. there exist $(c_0, \vec{h}_0), \dots, (c_n, \vec{h}_n)$ such that $(c, \vec{h}) = (c_0, \vec{h}_0) \xRightarrow{a_1} (c_1, \vec{h}_1) \xRightarrow{a_2} \dots \xRightarrow{a_n} (c_n, \vec{h}_n) = (c, \vec{r})$, where $a_1, \dots, a_n \in \mathbb{Z}$ and $\sum_k a_k = 0$, then $(c, \vec{h}) \xRightarrow{0} (c, \vec{r})$, i.e. if $\vec{h} \neq \vec{r}$, then there exists a formula φ such that $K v_i(\varphi, c) \wedge \varphi \in \text{Last}(\vec{h}) \cap \text{Last}(\vec{r})$.*

Proof. We first normalize the path as before. We present the case where the chain starts and ends with a type 1 step. If the chain starts or ends with a type 2 step, insert the trivial type 1 step of weight 0 (using $c \approx c$) so that the chain starts and ends with type 1 steps. Thus we may assume the chain has the alternating form

$$(c_0, \vec{h}_0) \xRightarrow{a_0} (c_1, \vec{h}_0) \xRightarrow{\phi_1} (c_1, \vec{h}_1) \xRightarrow{a_1} (c_2, \vec{h}_1) \xRightarrow{\phi_2} \cdots \xRightarrow{\phi_n} (c_n, \vec{h}_n) \xRightarrow{a_n} (c_0, \vec{h}_n),$$

where $a_0, \dots, a_n \in \mathbb{Z}$, $\phi_1, \dots, \phi_n$ are formulas, and $\sum_{k=0}^n a_k = 0$. Note that all $\vec{h}_k$ are $\sim'_i$ -equivalent.

Let $s_k := \sum_{x=k}^n a_x$ for $k = 0, \dots, n$, and let $K := \min\{s_k \mid 0 \leq k \leq n\}$. Define $t_k := S^{s_k - K} c_k$. Then all exponents are non-negative. Since $s_0 = 0$, we have $t_0 = S^{-K} c_0$. $(c_n, \vec{h}_n) \xRightarrow{a_n} (c_0, \vec{h}_n)$ implies $t_n \approx t_0 \in \text{Last}(\vec{h}_n)$. For each $k < n$, the type 1 step $(c_k, \vec{h}_k) \xRightarrow{a_k} (c_{k+1}, \vec{h}_k)$ implies $t_k \approx t_{k+1} \in \text{Last}(\vec{h}_k)$. Also, for each $k = 1, \dots, n$, since $(c_k, \vec{h}_{k-1}) \xRightarrow{\phi_k} (c_k, \vec{h}_k)$, we have $\phi_k \in \text{Last}(\vec{h}_{k-1}) \cap \text{Last}(\vec{h}_k)$ and $Kv_i(\phi_k, c_k) \in \text{Last}(\vec{h}_{k-1}) \cap \text{Last}(\vec{h}_k)$. By $Kv^r S$, $Kv_i(\phi_k, t_k) \in \text{Last}(\vec{h}_{k-1}) \cap \text{Last}(\vec{h}_k)$. Since all the $\text{Last}(\vec{h}_s)$ share the same Kv_i -formulas, $Kv_i(\phi_k, t_k) \in \text{Last}(\vec{h}_0) \cap \text{Last}(\vec{h}_n)$ for every k .

Moreover, $\text{Last}(\vec{h}_k)$ contains $\phi_k \wedge \phi_{k+1} \wedge t_k \approx t_{k+1}$ for $1 \leq k < n$, so since $\vec{h}_0 \sim'_i \vec{h}_k$ and $\vec{h}_n \sim'_i \vec{h}_k$, we have $\widehat{K}_i(\phi_k \wedge \phi_{k+1} \wedge t_k \approx t_{k+1}) \in \text{Last}(\vec{h}_0) \cap \text{Last}(\vec{h}_n)$. Therefore, the premises of $GKv^r \vee$ are satisfied in both $\text{Last}(\vec{h}_0)$ and $\text{Last}(\vec{h}_n)$ for the sequence $(\phi_1, t_1), \dots, (\phi_n, t_n)$. Hence $Kv_i\left(\bigvee_{k=1}^n (\phi_k \wedge t_0 \approx t_k), t_0\right) \in \text{Last}(\vec{h}_0) \cap \text{Last}(\vec{h}_n)$. Let $\psi := \bigvee_{k=1}^n (\phi_k \wedge t_0 \approx t_k)$. We have $\phi_1 \wedge t_0 \approx t_1 \in \text{Last}(\vec{h}_0)$ and $\phi_n \wedge t_0 \approx t_n \in \text{Last}(\vec{h}_n)$, so $\psi \in \text{Last}(\vec{h}_0) \cap \text{Last}(\vec{h}_n)$. By $Kv^r S$, $Kv_i(\psi, c_0) \in \text{Last}(\vec{h}_0) \cap \text{Last}(\vec{h}_n)$. Thus $(c_0, \vec{h}_0) \xRightarrow{\psi} (c_0, \vec{h}_n)$. $\square$

Step 4: Valuation Function. Our goal now is to define the valuation function $V_{\mathcal{C}}^*$ on W' . Note that W' is countable, and for each agent i , every $\sim'_i$ -equivalence class in W' forms a cluster consisting of exactly one parent node and all its immediate i -successors. This allows us to define the valuation level by level. Fix an explicit partition of the $\mathbb{Z}$ -chains into pairwise disjoint countable sets $\{E_n\}_{n \in \mathbb{N}}$ as follows. Enumerate all $\mathbb{Z}$ -chains as $\{Z_k \mid k \in \mathbb{N}\}$, and fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. Then define

$$E_n := \{Z_{\langle n, m \rangle} \mid m \in \mathbb{N}\}.$$

Each E_n is countably infinite and the E_n are pairwise disjoint, and $\bigcup_n E_n$ is the set of all $\mathbb{Z}$ -chains. Intuitively, E_n is the pool of fresh $\mathbb{Z}$ -chains reserved for equivalence classes that first appear when we define the valuation on histories of height n .

Now define $V_{\mathcal{C}}^*$ on W' as follows: For $\langle w_0 \rangle \in H_0$, intuitively, we only respect the atomic formulas in w_0 . Technically, we can consider $\equiv_i$ classes in $(\mathbf{C} \cup \{0\}) \times \{\langle w_0 \rangle\}$ for arbitrary i . For each equivalence class, we can find a representative. If $(0, \langle w_0 \rangle)$ is in this class, $(0, \langle w_0 \rangle)$ is the representative; otherwise, $(c_n, \langle w_0 \rangle)$ is the representative, where c_n is the constant symbol with the smallest index in this class. The equivalence class $[0, \langle w_0 \rangle]$ is placed in the natural-number part. $[(c_n, \langle w_0 \rangle)]$ is placed in $Z_{\langle 0, 2^{n+1} \rangle}$ (the representative receives value 0 in this $\mathbb{Z}$ -chain). For any other $(d, \langle w_0 \rangle)$ in this class, there are paths from the representative to it. However, by Corollary 4.9, all these paths have the same sum of weights. Then we define $V_{\mathcal{C}}^*(d, \langle w_0 \rangle)$ as the value of the representative plus the sum of weights on the path within the same $\mathbb{Z}$ -chain.

First note that for any two pairs $(c, \langle w_0 \rangle)$ and $(d, \langle w_0 \rangle)$, if there is a z -weight path from one to the other, then $V_{\mathcal{C}}^*(c, \langle w_0 \rangle) + z = V_{\mathcal{C}}^*(d, \langle w_0 \rangle)$. This is because if there is an x -weight path from the representative to $(c, \langle w_0 \rangle)$ and a y -weight path from the representative to $(d, \langle w_0 \rangle)$, then there is a path from $(c, \langle w_0 \rangle)$ to $(d, \langle w_0 \rangle)$ with weight $y - x$. By Corollary 4.9, we have $y - x = z$, and thus $V_{\mathcal{C}}^*(c, \langle w_0 \rangle) + z = V_{\mathcal{C}}^*(d, \langle w_0 \rangle)$.

Now we prove that $S^a c \approx S^b d \in w_0$ iff $V_C^*(c, \langle w_0 \rangle) + a = V_C^*(d, \langle w_0 \rangle) + b$ (where c, d may represent 0). First if $S^a c \approx S^b d \in w_0$, then $(c, \langle w_0 \rangle) \xrightarrow{a-b} (d, \langle w_0 \rangle)$, by the above observation, $V_C^*(c, \langle w_0 \rangle) + a = V_C^*(d, \langle w_0 \rangle) + b$. If $\neg S^a c \approx S^b d \in w_0$, then if there is a x -weight path from $(c, \langle w_0 \rangle)$ to $(d, \langle w_0 \rangle)$, then by the observation above, $V_C^*(c, \langle w_0 \rangle) + x = V_C^*(d, \langle w_0 \rangle)$. But by Lemma 4.8, $x \neq a - b$. Thus $V_C^*(c, \langle w_0 \rangle) + a \neq V_C^*(d, \langle w_0 \rangle) + b$. If there is no path between them, they will be placed in different $\mathbb{Z}$ -chains, and thus $V_C^*(c, \langle w_0 \rangle) + a \neq V_C^*(d, \langle w_0 \rangle) + b$ as well.

Assume we have defined the values of constants for all histories with height $< n$, and we have used only chains from $E_0, \dots, E_{n-1}$ so far. Suppose also that we have verified that for any history $\vec{h}$ with height $< n$, $S^a c \approx S^b d \in \text{Last}(\vec{h})$ iff $V_C^*(c, \vec{h}) + a = V_C^*(d, \vec{h}) + b$ (where c, d may represent 0).

Now consider any $\sim'_j$ equivalence class in $H_{n-1} \cup H_n$. It contains one parent $\vec{p} \in H_{n-1}$ where the values of constants have already been defined, and countably many children in H_n which are immediate j -successors of $\vec{p}$. Then we consider the equivalence classes by $\equiv_j$ over this $\sim'_j$ equivalence class. The valuation of $\vec{p}$ is compatible: by the assumption, if $(c, \vec{p})$ and $(d, \vec{p})$ (where c, d may represent 0) have been placed in the same component (both in $\mathbb{N}$ or both in the same $\mathbb{Z}$ -chain) with relative distance z , then they are connected in $\equiv_j$ by a path of weight z . If they are placed in different components, they must be $\equiv_j$ inequivalent. Therefore, for any $\equiv_j$ equivalence class containing some pair from $\vec{p}$, if it contains $(0, \vec{p})$, we take $(0, \vec{p})$ as the representative, place the whole class in the natural-number part, and give the representative value 0. By $\text{Kv}^r 0$ with $\top$ as the witnessing formula, every $(0, \vec{u})$ in this $\sim'_j$ -class belongs to this same $\equiv_j$ -class. Otherwise, we choose the representative to be $(c_k, \vec{p})$ where k is the smallest index such that $(c_k, \vec{p})$ is in this class, and we place the elements in this class in the same component as $(c_k, \vec{p})$, keeping the already defined value of $(c_k, \vec{p})$.

For each $\equiv_j$ equivalence class that does not contain any pair of the form $(c, \vec{p})$, by the previous paragraph, such a class contains no pair of the form $(0, \vec{u})$. We can also find a representative because histories were encoded as natural numbers above. Specifically, choose the pair $(c_k, \vec{v})$ in the class with lexicographically least $(k, m(\vec{v}))$, where $m(\vec{v})$ is the code of $\vec{v}$. $[(c_k, \vec{v})]$, the $\equiv_j$ -class of this representative, is placed in $\mathbb{Z}_{(n, 2^{k+1}, 3m(\vec{v}))}$. Also, $(c_k, \vec{v})$ receives value 0 in this $\mathbb{Z}$ -chain.

In both kinds of classes, we have fixed where the representative is placed in $\mathbb{N}^*$, either in the natural-number part or in a specified $\mathbb{Z}$ -chain, and we have also fixed its value there. As before, for any other $(d, \vec{r})$ in the class, we define its value as the value of the representative plus the sum of weights on any path from the representative to it. Then we can similarly verify that for any two pairs whose values have been defined, if there is a z -weight path from one to the other, then their values differ by z . Moreover, for any history $\vec{h}$ with height $\leq n$, $S^a c \approx S^b d \in \text{Last}(\vec{h})$ iff $V_C^*(c, \vec{h}) + a = V_C^*(d, \vec{h}) + b$ (where c, d may represent 0). Finally, throughout the construction, each pair $(c, \vec{h})$ receives exactly one value. This is because when $\vec{h}$ is a child in the equivalence class, we define its value; when $\vec{h}$ is a parent, we respect the value already defined for it.

Step 5: Truth Lemma. We can prove the truth lemma by induction on formulas.

Lemma 4.11 (Truth Lemma). *Let $\mathcal{M} := \langle W', \mathbb{N}^*, \mathbf{S}^*, \{\sim'_i\}_{i \in I}, V_C^* \rangle$ be the model constructed above. For any formula φ and any history $\vec{h} \in W'$, $\mathcal{M}, \vec{h} \models \varphi$ iff $\varphi \in \text{Last}(\vec{h})$.*

Proof. By induction on φ , we prove that for any $\vec{h} \in W'$, $\mathcal{M}, \vec{h} \models \varphi$ iff $\varphi \in \text{Last}(\vec{h})$. The atomic case was proved when we defined V_C^* . The boolean cases are trivial.

For the case $\varphi := K_i \psi$: if $K_i \psi \in \text{Last}(\vec{h})$, then for any $\vec{u} \in W'$ with $\vec{h} \sim'_i \vec{u}$, we have $\text{Last}(\vec{h}) \sim_i^c \text{Last}(\vec{u})$ by Proposition 4.7. Thus $\psi \in \text{Last}(\vec{u})$. The induction hypothesis gives $\mathcal{M}, \vec{u} \models \psi$, so $\mathcal{M}, \vec{h} \models K_i \psi$. On the other hand, if $\neg K_i \psi \in \text{Last}(\vec{h})$, we consider two cases. If $i \neq \text{Ag}(\vec{h})$, there exists a direct witness $\vec{u} \in W'$ such that $\vec{h} \rightarrow_i \vec{u}$ and $\neg \psi \in \text{Last}(\vec{u})$. If $i = \text{Ag}(\vec{h})$, the witness construction for i was skipped for $\vec{h}$.

However, consider its i -parent $\vec{p}$, since $\text{Last}(\vec{h}) \sim_i^c \text{Last}(\vec{p})$, $\neg K_i \psi \in \text{Last}(\vec{p})$. By the construction of W' , we have selected a witness $\vec{u}$ such that $\vec{p} \rightarrow_i \vec{u}$ and $\neg \psi \in \text{Last}(\vec{u})$. In both cases, $\vec{h} \sim_i^c \vec{u}$ and $\neg \psi \in \text{Last}(\vec{u})$. By the induction hypothesis, $\mathcal{M}, \vec{u} \not\models \psi$, establishing $\mathcal{M}, \vec{h} \not\models K_i \psi$.

For the case $\varphi := K_{v_i}(\psi, t)$, by applying axiom $\text{Kv}'S$, it suffices to consider the case in which t is a constant c . If $K_{v_i}(\psi, c) \in \text{Last}(\vec{h})$, let $\vec{u}, \vec{v} \in W'$ be such that $\vec{h} \sim_i^c \vec{u}$, $\vec{h} \sim_i^c \vec{v}$ and both $\mathcal{M}, \vec{u} \models \psi$ and $\mathcal{M}, \vec{v} \models \psi$. By the induction hypothesis, $\psi \in \text{Last}(\vec{u}) \cap \text{Last}(\vec{v})$. Since $\text{Last}(\vec{h}) \sim_i^c \text{Last}(\vec{u}) \sim_i^c \text{Last}(\vec{v})$, $K_{v_i}(\psi, c) \in \text{Last}(\vec{u}) \cap \text{Last}(\vec{v})$. We also have $\vec{u} \sim_i^c \vec{v}$. The definition of R_i then guarantees $(c, \vec{u}) \xrightarrow{\psi} (c, \vec{v})$. Thus, $(c, \vec{u}) \equiv_i (c, \vec{v})$ with weight 0. By the construction of $V_{\mathbf{C}}^*$, $V_{\mathbf{C}}^*(c, \vec{u}) = V_{\mathbf{C}}^*(c, \vec{v})$, yielding $\mathcal{M}, \vec{h} \models K_{v_i}(\psi, c)$. Conversely, if $\neg K_{v_i}(\psi, c) \in \text{Last}(\vec{h})$, we again consider two cases. If $i \neq \text{Ag}(\vec{h})$, the construction directly provides witnesses $\vec{u}, \vec{v} \in W'$ such that $\vec{h} \rightarrow_i \vec{u}$ and $\vec{h} \rightarrow_i \vec{v}$, with $\psi \in \text{Last}(\vec{u}) \cap \text{Last}(\vec{v})$ and $\text{Last}(\vec{u}) \cap \text{Last}(\vec{v}) \cap \Psi_c^i(\text{Last}(\vec{h})) = \emptyset$. If $i = \text{Ag}(\vec{h})$, the witnesses were selected by its i -parent $\vec{p}$; hence there are $\vec{u}$ and $\vec{v}$ such that $\vec{p} \rightarrow_i \vec{u}$ and $\vec{p} \rightarrow_i \vec{v}$, with $\psi \in \text{Last}(\vec{u}) \cap \text{Last}(\vec{v})$ and $\text{Last}(\vec{u}) \cap \text{Last}(\vec{v}) \cap \Psi_c^i(\text{Last}(\vec{h})) = \text{Last}(\vec{u}) \cap \text{Last}(\vec{v}) \cap \Psi_c^i(\text{Last}(\vec{p})) = \emptyset$. In either case, we have $\vec{h} \sim_i^c \vec{u}$, $\vec{h} \sim_i^c \vec{v}$, $\psi \in \text{Last}(\vec{u}) \cap \text{Last}(\vec{v})$ and $\text{Last}(\vec{u}) \cap \text{Last}(\vec{v}) \cap \Psi_c^i(\text{Last}(\vec{h})) = \emptyset$. By the induction hypothesis, $\mathcal{M}, \vec{u} \models \psi$ and $\mathcal{M}, \vec{v} \models \psi$. If $(c, \vec{u}) \equiv_i (c, \vec{v})$ with weight 0, then by Lemma 4.10, there exists a formula χ such that $K_{v_i}(\chi, c) \wedge \chi \in \text{Last}(\vec{u}) \cap \text{Last}(\vec{v})$. Then $\chi \in \Psi_c^i(\text{Last}(\vec{h}))$, contradicting the choice of $\vec{u}$ and $\vec{v}$. If it is not the case that $(c, \vec{u}) \equiv_i (c, \vec{v})$ with weight 0, then by the construction of $V_{\mathbf{C}}^*$, $V_{\mathbf{C}}^*(c, \vec{u}) \neq V_{\mathbf{C}}^*(c, \vec{v})$, yielding $\mathcal{M}, \vec{h} \not\models K_{v_i}(\psi, c)$. $\square$

This yields the desired completeness result.

Theorem 4.12. *ELKVSAR is strongly complete w.r.t. non-standard models.*

Proof. Given a consistent formula set Γ_0 , $\mathcal{M}, \langle (\Gamma, 1) \rangle \models \Gamma_0$. $\square$

5 From Non-standard Models to Standard Models

In this section, we bridge the gap between non-standard models and standard models. This allows us to establish the weak completeness of ELKVSAR with respect to standard models. In the process, we also obtain the finite model property w.r.t. both standard and non-standard models, which will be used to establish the decidability of ELKvSA' . We first show that satisfiability in a finite non-standard model implies satisfiability in a finite standard model.

Lemma 5.1. *For any formula ϕ satisfiable in a finite non-standard model $\mathcal{N}$, it is also satisfiable in a finite standard model.*

Proof. Let $\mathcal{N} = \langle W, \mathbb{N}^*, \mathbf{S}^*, \{\sim_i\}_{i \in \mathbf{I}}, V_{\mathbf{C}}^* \rangle$ be a non-standard model such that $\mathcal{N}, w_0 \models \phi$, where W is finite. Let

$$U = \{V_{\mathbf{C}}^*(c, w) \mid c \text{ occurs in } \phi, w \in W\} \cup \{0\}.$$

Intuitively, U is the set of all the numbers that are relevant to the truth of ϕ in $\mathcal{N}$. Since W is finite and ϕ contains only finitely many constants, U is finite and thus $U \cap Z_n$ and $U \cap \mathbb{N}$ are finite for each n .

Let $B = \max(\{a + b \mid \exists x, y \in \mathbf{C} \cup \{0\}, S^a x \approx S^b y \text{ is a subformula of } \phi\} \cup \{0\})$. We build an injective map $f : U \rightarrow \mathbb{N}$ such that:

1. For $x, y \in U$ in the same component (both in $\mathbb{N}$ or both in the same Z_n), $f(x) - f(y) = x - y$.
2. For $x, y \in U$ in different components, $|f(x) - f(y)| > B$.

3. $f(x) = x$ for all $x \in U \cap \mathbb{N}$.

Such an f exists because $U \cap \mathbb{N}$ and each $U \cap Z_n$ is finite, so we can preserve all local distances inside a component and separate different components by gaps greater than B . Define a standard model $\mathcal{M} = \langle W, \mathbb{N}, \mathbf{S}, \{\sim_i\}_{i \in \mathbf{I}}, V_C \rangle$ by defining $V_C(c, u) = f(V_C^*(c, u))$ for each constant c occurring in ϕ and $u \in W$. For other constants, assign arbitrary values. We show by induction on subformulas ψ of ϕ that for all $u \in W$,

$$\mathcal{N}, u \models \psi \iff \mathcal{M}, u \models \psi.$$

- Atomic case: $\psi := t_1 \approx t_2$. Suppose t_1 and t_2 are $S^a c_1$ and $S^b c_2$ respectively (the case of $S^k 0$ can be proved similarly). If $\mathcal{N}, u \models t_1 \approx t_2$, then $V_T(t_1, u) = V_T(t_2, u)$, which means $V_C^*(c_1, u) + a = V_C^*(c_2, u) + b$. By the definition of f , we have $f(V_C^*(c_1, u)) + a = f(V_C^*(c_2, u)) + b$, which means $\mathcal{M}, u \models t_1 \approx t_2$. Suppose $\mathcal{N}, u \models \neg S^a c_1 \approx S^b c_2$, then if $V_C^*(c_1, u)$ and $V_C^*(c_2, u)$ are in the same component but $V_C^*(c_1, u) + a \neq V_C^*(c_2, u) + b$, then by property 1, $f(V_C^*(c_1, u)) + a \neq f(V_C^*(c_2, u)) + b$ and $\mathcal{M}, u \models \neg t_1 \approx t_2$. If $V_C^*(c_1, u)$ and $V_C^*(c_2, u)$ are in different components, then by property 2, $|f(V_C^*(c_1, u)) - f(V_C^*(c_2, u))| > B \geq |a - b|$, so $f(V_C^*(c_1, u)) + a \neq f(V_C^*(c_2, u)) + b$ and $\mathcal{M}, u \models \neg t_1 \approx t_2$.
- Boolean connectives and $\psi := \top$: Trivial.
- $\psi := K_i \chi$. The IH is: $\forall v \in W, \mathcal{N}, v \models \chi \iff \mathcal{M}, v \models \chi$.

$$\begin{aligned} \mathcal{N}, u \models K_i \chi &\iff \forall v \in W (u \sim_i v \Rightarrow \mathcal{N}, v \models \chi) \\ &\iff \forall v \in W (u \sim_i v \Rightarrow \mathcal{M}, v \models \chi) \quad (\text{by IH}) \\ &\iff \mathcal{M}, u \models K_i \chi \end{aligned}$$

- $\psi := K_{v_i}(\chi, t)$. By the semantics, $K_{v_i}(\chi, S^k c) \leftrightarrow K_{v_i}(\chi, c)$ and $K_{v_i}(\chi, S^k 0) \leftrightarrow \top$ are valid. So we only need to consider the case in which t has the form c . Note that the IH is: $\forall v \in W, \mathcal{N}, v \models \chi \iff \mathcal{M}, v \models \chi$.

$$\begin{aligned} \mathcal{N}, u \models K_{v_i}(\chi, c) &\iff \forall v_1, v_2 \sim_i u, \text{ if } \mathcal{N}, v_1 \models \chi \text{ and } \mathcal{N}, v_2 \models \chi, \\ &\quad \text{then } V_C^*(c, v_1) = V_C^*(c, v_2) \\ &\iff \forall v_1, v_2 \sim_i u, \text{ if } \mathcal{M}, v_1 \models \chi \text{ and } \mathcal{M}, v_2 \models \chi, \\ &\quad \text{then } f(V_C^*(c, v_1)) = f(V_C^*(c, v_2)) \quad (\text{IH and } f \text{ is injective}) \\ &\iff \forall v_1, v_2 \sim_i u, \text{ if } \mathcal{M}, v_1 \models \chi \text{ and } \mathcal{M}, v_2 \models \chi, \\ &\quad \text{then } V_C(c, v_1) = V_C(c, v_2) \\ &\iff \mathcal{M}, u \models K_{v_i}(\chi, c) \end{aligned}$$

Hence $\mathcal{M}, w_0 \models \phi$, so ϕ is satisfiable in a finite standard model. $\square$

To apply the above lemma, we prove the finite model property for non-standard models via a selection argument from the canonical tree-like model.

Theorem 5.2. *ELKvSA^r has finite model property w.r.t. non-standard models. Moreover, if a formula ϕ is satisfiable in a non-standard model, it is satisfiable in a finite non-standard model whose size is bounded by $(2|\Sigma|)^{\text{depth}(\phi)+1}$, where Σ is the set of subformulas of ϕ closed under single negation, and $\text{depth}(\phi)$ is the modal depth of ϕ .*

Proof. For any formula ϕ satisfiable in a non-standard model, it is satisfiable in the tree-like model constructed in the completeness proof in Subsection 4.2. This tree-like model is $\mathcal{M} = \langle W', \mathbb{N}^*, \mathbf{S}^*, \{\sim'_i\}_{i \in I}, V_{\mathbf{C}}^* \rangle$. Suppose the root is $\langle w_0 \rangle$, where $\phi \in w_0$. Our goal is to select a finite submodel from $\mathcal{M}$ that still satisfies ϕ . Let Σ be the set of subformulas of ϕ closed under single negation. We define the **modal depth** $\text{depth}(\phi)$ as:

- $\text{depth}(\alpha) = 0$ for atomic formulas and Boolean combinations of atomics.
- $\text{depth}(\neg\phi) = \text{depth}(\phi)$, $\text{depth}(\phi \wedge \psi) = \max(\text{depth}(\phi), \text{depth}(\psi))$.
- $\text{depth}(K_i\phi) = \text{depth}(\phi) + 1$.
- $\text{depth}(Kv_i(\phi, t)) = \text{depth}(\phi) + 1$.

Let $D = \text{depth}(\phi)$.

Selection. Construct $W_s \subset W'$ level by level. Recall that $W' = \bigcup_{k=0}^{\infty} H_k$ where H_k is the set of nodes with height k . Let $H'_0 = H_0 = \{\langle w_0 \rangle\}$. Assume H'_k is constructed for $k < D$. We construct H'_{k+1} as follows. For each $\vec{h} \in H'_k$ and $j \in I$:

1. If $j = \text{Ag}(\vec{h})$: Skip.
2. If $j \neq \text{Ag}(\vec{h})$:
 - For $\neg K_j\psi \in \Sigma$: If $\mathcal{M}, \vec{h} \models \neg K_j\psi$ and $\text{depth}(\neg K_j\psi) \leq D - k$, select one witness $\vec{h}' \in H_{k+1}$ with $\mathcal{M}, \vec{h}' \not\models \psi$ and add to H'_{k+1} .
 - For $\neg Kv_j(\chi, S^k c) \in \Sigma$: If $\mathcal{M}, \vec{h} \models \neg Kv_j(\chi, S^k c)$ and $\text{depth}(\neg Kv_j(\chi, S^k c)) \leq D - k$, select two witnesses $\vec{h}_1, \vec{h}_2 \in H_{k+1}$ with $\mathcal{M}, \vec{h}_1 \models \chi$ and $\mathcal{M}, \vec{h}_2 \models \chi$ such that $V_{\mathbf{C}}^*(c, \vec{h}_1) \neq V_{\mathbf{C}}^*(c, \vec{h}_2)$. Add them to H'_{k+1} . (This can be done because of the construction of W' .)

Let $W_s = \bigcup_{k=0}^D H'_k$. W_s is finite. Let $\sim_i^s$ be the restriction of $\sim_i'$ to W_s , and $V_{\mathbf{C}}^s$ be the restriction of $V_{\mathbf{C}}^*$ to W_s . Let $\mathcal{M}_s = \langle W_s, \mathbb{N}^*, \mathbf{S}^*, \{\sim_i^s\}_{i \in I}, V_{\mathbf{C}}^s \rangle$. We next prove the following lemma.

Lemma 5.3. *For each d such that $0 \leq d \leq D$ and for each formula $\psi \in \Sigma$ with $\text{depth}(\psi) = d$, and for any node $\vec{h} \in W_s$, if the height condition $\text{height}(\vec{h}) \leq D - d$ is satisfied, then: $\mathcal{M}_s, \vec{h} \models \psi \iff \mathcal{M}, \vec{h} \models \psi$.*

Proof of the lemma. By induction on the modal depth d ($0 \leq d \leq D$).

Base Case ($d = 0$): ψ is atomic or Boolean combination of atomics. The condition $\text{height}(\vec{h}) \leq D$ always holds for $\vec{h} \in W_s$. Since valuations are inherited directly from $\mathcal{M}$, the equivalence holds trivially.

Inductive Hypothesis: Assume the Claim holds for all formulas with $\text{depth} < d$, i.e., for any formula $\chi \in \Sigma$ with $\text{depth}(\chi) < d$, and for any node $\vec{h} \in W_s$ satisfying $\text{height}(\vec{h}) \leq D - \text{depth}(\chi)$, we have: $\mathcal{M}_s, \vec{h} \models \chi \iff \mathcal{M}, \vec{h} \models \chi$.

Inductive Step: We consider formulas of depth d .

Subcase 1: $\psi := K_j\chi$. Here $\text{depth}(\psi) = d$, so $\text{depth}(\chi) = d - 1$. Assume the condition holds for $\vec{h}$: $\text{height}(\vec{h}) \leq D - d$.

($\Leftarrow$) Suppose $\mathcal{M}, \vec{h} \models \neg K_j\chi$.

- If $j \neq \text{Ag}(\vec{h})$: Since $\text{height}(\vec{h}) \leq D - d$, the selection condition for $\vec{h}$ is satisfied. Thus, during the construction of $H'_{\text{height}(\vec{h})+1}$, a witness $\vec{h}' \in H'_{\text{height}(\vec{h})+1}$ was selected such that $\vec{h} \sim_j' \vec{h}'$ and $\mathcal{M}, \vec{h}' \not\models \chi$.

As $\text{height}(\vec{h}') = \text{height}(\vec{h}) + 1 \leq D - d + 1 = D - \text{depth}(\chi)$, the condition for IH on $\vec{h}'$ holds. By IH, $\mathcal{M}_s, \vec{h}' \not\models \chi$, and since $\vec{h} \sim_j^s \vec{h}'$, we have $\mathcal{M}_s, \vec{h} \models \neg K_j\chi$.

- If $j = \text{Ag}(\vec{h})$: By the property of W' , $\vec{h}$ is a j -successor of its parent $\vec{p}$. Since $\vec{h} \sim'_j \vec{p}$, $\mathcal{M}, \vec{p} \models \neg K_j \chi$. Since $\text{height}(\vec{p}) = \text{height}(\vec{h}) - 1 < D - d$, the selection condition for $\vec{p}$ strictly holds. Thus, when constructing successors for $\vec{p}$, a witness $\vec{w} \in H'_{\text{height}(\vec{p})+1}$ was selected such that $\vec{p} \sim'_j \vec{w}$ and $\mathcal{M}, \vec{w} \not\models \chi$. In $\mathcal{M}_s$, $\vec{h}, \vec{p}, \vec{w}$ are in the same j -cluster, so $\vec{h} \sim_j^s \vec{w}$. Moreover, $\text{height}(\vec{w}) = \text{height}(\vec{p}) + 1 = \text{height}(\vec{h}) \leq D - d < D - \text{depth}(\chi)$. The condition for IH holds, entailing $\mathcal{M}_s, \vec{w} \not\models \chi$. Thus $\mathcal{M}_s, \vec{h} \models \neg K_j \chi$.

($\Rightarrow$) Suppose $\mathcal{M}_s, \vec{h} \models \neg K_j \chi$. There exists $\vec{z} \in W_s$ with $\vec{h} \sim_j^s \vec{z}$ and $\mathcal{M}_s, \vec{z} \not\models \chi$. Since $\vec{z}$ is in the same j -cluster as $\vec{h}$ in W' , $\vec{z}$ is either $\vec{h}$ itself, its parent, its child, or its sibling. In all cases, $\text{height}(\vec{z}) \leq \text{height}(\vec{h}) + 1 \leq D - \text{depth}(\chi)$. The condition for IH on $\vec{z}$ holds, so $\mathcal{M}_s, \vec{z} \not\models \chi \Rightarrow \mathcal{M}, \vec{z} \not\models \chi$. Since $\vec{h} \sim'_j \vec{z}$ in $\mathcal{M}$, we get $\mathcal{M}, \vec{h} \models \neg K_j \chi$.

Subcase 2: $\psi := \neg K_{v_j}(\chi, S^k c)$. Here $\text{depth}(\psi) = d$, so $\text{depth}(\chi) = d - 1$. Assume $\text{height}(\vec{h}) \leq D - d$. ($\Leftarrow$) Suppose $\mathcal{M}, \vec{h} \models \neg K_{v_j}(\chi, S^k c)$.

- If $j \neq \text{Ag}(\vec{h})$: Since $\text{height}(\vec{h}) \leq D - d$, the selection condition is satisfied, so there exist two selected witnesses $\vec{h}_1, \vec{h}_2 \in H'_{\text{height}(\vec{h})+1}$ with $\vec{h} \sim'_j \vec{h}_1$, $\vec{h} \sim'_j \vec{h}_2$, $\mathcal{M}, \vec{h}_1 \models \chi$, $\mathcal{M}, \vec{h}_2 \models \chi$, and $V_{\mathbf{C}}^*(c, \vec{h}_1) \neq V_{\mathbf{C}}^*(c, \vec{h}_2)$. By the IH (applicable since $\text{height}(\vec{h}_\ell) \leq D - \text{depth}(\chi)$, $\ell = 1, 2$), $\mathcal{M}_s, \vec{h}_1 \models \chi$ and $\mathcal{M}_s, \vec{h}_2 \models \chi$. Since $\vec{h} \sim_j^s \vec{h}_1 \sim_j^s \vec{h}_2$, we get $\mathcal{M}_s, \vec{h} \models \neg K_{v_j}(\chi, S^k c)$.
- If $j = \text{Ag}(\vec{h})$: Let $\vec{p}$ be the parent of $\vec{h}$. Then $\mathcal{M}, \vec{p} \models \neg K_{v_j}(\chi, S^k c)$ and $\text{height}(\vec{p}) = \text{height}(\vec{h}) - 1 < D - d$, so the selection condition for $\vec{p}$ is satisfied. Hence there exist two selected witnesses $\vec{w}_1, \vec{w}_2 \in H'_{\text{height}(\vec{p})+1}$ in the same j -cluster with $\mathcal{M}, \vec{w}_1 \models \chi$, $\mathcal{M}, \vec{w}_2 \models \chi$, and $V_{\mathbf{C}}^*(c, \vec{w}_1) \neq V_{\mathbf{C}}^*(c, \vec{w}_2)$. By IH, $\mathcal{M}_s, \vec{w}_1 \models \chi$ and $\mathcal{M}_s, \vec{w}_2 \models \chi$. Since $\vec{h}$ is in the same cluster as $\vec{p}, \vec{w}_1, \vec{w}_2$, $\mathcal{M}_s, \vec{h} \models \neg K_{v_j}(\chi, S^k c)$.

($\Rightarrow$) Assume $\mathcal{M}_s, \vec{h} \models \neg K_{v_j}(\chi, S^k c)$. By semantics, there exist $\vec{z}_1, \vec{z}_2 \in W_s$ such that $\vec{h} \sim_j^s \vec{z}_1$, $\vec{h} \sim_j^s \vec{z}_2$, $\mathcal{M}_s, \vec{z}_1 \models \chi$, $\mathcal{M}_s, \vec{z}_2 \models \chi$, and $V_{\mathbf{C}}^s(c, \vec{z}_1) \neq V_{\mathbf{C}}^s(c, \vec{z}_2)$. Since $\vec{z}_1, \vec{z}_2$ are in the same j -cluster as $\vec{h}$, $\text{height}(\vec{z}_\ell) \leq \text{height}(\vec{h}) + 1 \leq D - \text{depth}(\chi)$, the IH gives $\mathcal{M}, \vec{z}_\ell \models \chi$ ($\ell = 1, 2$). Moreover, $V_{\mathbf{C}}^*(c, \vec{z}_1) = V_{\mathbf{C}}^s(c, \vec{z}_1) \neq V_{\mathbf{C}}^s(c, \vec{z}_2) = V_{\mathbf{C}}^*(c, \vec{z}_2)$. Since $\vec{h} \sim'_j \vec{z}_1$ and $\vec{h} \sim'_j \vec{z}_2$ in $\mathcal{M}$, we get $\mathcal{M}, \vec{h} \models \neg K_{v_j}(\chi, S^k c)$.

Subcase 3: boolean connectives. Trivial. $\square$

Conclusion: Since $\text{depth}(\phi) = D$, and $\text{height}(\langle w_0 \rangle) = 0$, the condition $0 \leq D - D$ holds. Applying the Claim with $d = D$, we get $\mathcal{M}_s, \langle w_0 \rangle \models \phi$. Moreover, $|W_s| \leq (2|\Sigma|)^{D+1}$. $\square$

This yields the following theorem and corollary.

Theorem 5.4. ELKVSA^r is sound and weakly complete w.r.t. standard models.

Proof. Given a consistent formula ϕ , it is satisfiable in a non-standard model. Then by Theorem 5.2, it is satisfiable in a finite non-standard model. By Lemma 5.1, it is satisfiable in a finite standard model. $\square$

Corollary 5.5. ELKVSA^r has finite model property w.r.t. standard models.

We now establish the decidability of ELKVSA^r using the results above.

Theorem 5.6. ELKVSA^r is decidable.

Proof. Given a satisfiable formula ϕ , it is satisfiable in a finite standard model $\mathcal{M} = \langle W, \mathbb{N}, \mathbf{S}, \{\sim_i\}_{i \in I}, V_C \rangle$ of size $f(|\phi|)$, where f is a computable function. Let C_ϕ be the set of constants occurring in ϕ . Let $B = \max(\{a + b \mid \exists x, y \in C_\phi \cup \{0\}, S^a x \approx S^b y \text{ is a subformula of } \phi\} \cup \{0\})$. Let $U = \{V_C(c, w) \mid c \in C_\phi, w \in W\} \cup \{0\}$. We have $|U| \leq |C_\phi| \cdot f(|\phi|) + 1$. Construct an injective map $g : U \rightarrow \mathbb{N}$ as follows: Start from $g(0) = 0$, consider the least number $x \in U$ such that $g(x)$ is not defined. Suppose the greatest number $y \in U$ with $g(y)$ defined is $g(y) = m$. If $x - y \leq B$, let $g(x) = m + (x - y)$; otherwise, let $g(x) = m + B + 1$. Let $\mathcal{M}' = \langle W, \mathbb{N}, \mathbf{S}, \{\sim_i\}_{i \in I}, V'_C \rangle$ be the new model defined by $V'_C(c, w) = g(V_C(c, w))$ for each $c \in C_\phi$ and $w \in W$. By a similar argument as in the proof of Lemma 5.1, we can show that $\mathcal{M}', w \models \phi$. Hence if ϕ is satisfiable, it is satisfiable in a finite standard model, where the size of the model is bounded by $f(|\phi|)$ and $V_C(c, w) \in \{0, 1, \dots, (|C_\phi| \cdot f(|\phi|) + 1) \cdot (B + 1)\}$ for each $c \in C_\phi$ and $w \in W$. We can then enumerate all such models (the size of the model is bounded, the codomain of V_C is bounded, and we only need to consider agents occurring in ϕ and constants in C_ϕ) and check if ϕ is satisfiable in any of them. There are only finitely many such models, and checking satisfiability in each model is decidable, so we can decide the satisfiability of ϕ . Hence **ELKvSA**^r is decidable. $\square$

6 Consecutive Numbers

In this section, we return to the original motivation of this paper, the “consecutive numbers” puzzle in Example 1.1. Before presenting the formalization, let us first spell out the intuitive solution of the puzzle. If Anne’s number were 0, Bill’s number would have to be 1, so Anne would know it. Thus Anne’s first announcement says, in effect, that her number is not 0. Similarly, after this announcement, Bill’s ignorance says that his number is neither 0 nor 1: if it were 0, Anne’s number would be 1, and if it were 1, Anne’s number could only be 2. Anne can now use Bill’s announcement. She can know Bill’s number only if her own number is 1 or 2; for any larger number, Bill could have either the predecessor or the successor. Hence everyone can know the remaining possibilities are precisely $(a, b) = (2, 3)$ and $(a, b) = (1, 2)$. In either case Bill can determine Anne’s number.

The key point is that each public announcement about an agent’s epistemic state corresponds to the public announcement of an arithmetic fact about the two numbers. We can capture this process within our system. Let **PALKVSA**^r be **ELKVSA**^r plus **!ATOM**, **!NEG**, **!CON**, **!K**, **!Kv^r**, where **!ATOM** is $[\psi]\alpha \leftrightarrow (\psi \rightarrow \alpha)$ for atomic α . **!COM** is derivable in **PALKVSA**^r. The following definition formalizes the puzzle and its conclusion.

Definition 6.1. Let the set of premises be $\Gamma_0 = \{Kv_A a, Kv_B b, \neg a \approx 0 \rightarrow \neg Kv_A b, \neg b \approx 0 \rightarrow \neg Kv_B a\} \cup \{K_i(a \approx Sb \vee b \approx Sa) \mid i \in \{A, B, C\}\}$. Define $\Gamma_{n+1} := \Gamma_n \cup \{K_A \psi, K_B \psi, K_C \psi \mid \psi \in \Gamma_n\}$. The conclusion is:

$$[\neg Kv_A b][\neg Kv_B a][Kv_A b](Kv_B a \wedge K_C((a \approx SS0 \wedge b \approx SSS0) \vee (b \approx SS0 \wedge a \approx S0)))$$

The premise $\neg a \approx 0 \rightarrow \neg Kv_A b$ warrants some explanation. In the puzzle, when Anne’s number is 0, she can deduce that Bill’s number is 1. Otherwise, she does not know Bill’s number. So this premise matches the setup of the puzzle. Although this premise might not be strictly necessary, $\Gamma_0 \setminus \{\neg a \approx 0 \rightarrow \neg Kv_A b, \neg b \approx 0 \rightarrow \neg Kv_B a\}$ is provably insufficient to deduce the conclusion. Moreover, this premise can simplify the proof of the conclusion, so we include it in the premises.

Notation 6.2. Set $\chi_m^n := (\bigwedge_{k=0}^m \neg a \approx S^k 0) \wedge (\bigwedge_{k=0}^n \neg b \approx S^k 0)$, $\varphi_1 := \neg Kv_A b$, and

$$\varphi_n := \begin{cases} \varphi_{n-1} \wedge [\varphi_{n-1}] \neg Kv_B a & \text{if } n \text{ is even} \\ \varphi_{n-1} \wedge [\varphi_{n-1}] \neg Kv_A b & \text{if } n \text{ is odd} \end{cases} \quad \psi_n := \begin{cases} [\varphi_{n-1}] Kv_B a & \text{if } n \text{ is even} \\ [\varphi_{n-1}] Kv_A b & \text{if } n \text{ is odd} \end{cases}$$

If n or m is less than 0, the corresponding part is simply ignored.

We can solve a generalized version of the puzzle. The following proposition makes precise the intuition that each announcement in the puzzle conveys arithmetic information about the two numbers. The proofs are omitted, but they can be established by induction on n .

Proposition 6.3. 1. If n is even, $\Gamma_{n-1} \vdash (\varphi_n \leftrightarrow \chi_{n-2}^{n-1}) \wedge (\psi_n \leftrightarrow \neg \chi_{n-2}^{n-1})$
 2. If n is odd, $\Gamma_{n-1} \vdash (\varphi_n \leftrightarrow \chi_{n-1}^{n-2}) \wedge (\psi_n \leftrightarrow \neg \chi_{n-1}^{n-2})$.

With !COM, this yields the following solution in $\text{PALKVSA}'$.

Proposition 6.4. 1. If n is even, $\Gamma_n \vdash \underbrace{[\neg K_{v_A} b][\neg K_{v_B} a] \cdots [\neg K_{v_B} a]}_n [K_{v_A} b] (K_{v_B} a \wedge K_C((a \approx S^n 0 \wedge b \approx S^{n+1} 0) \vee (b \approx S^n 0 \wedge a \approx S^{n-1} 0)))$.
 2. If n is odd, $\Gamma_n \vdash \underbrace{[\neg K_{v_A} b][\neg K_{v_B} a] \cdots [\neg K_{v_A} b]}_n [K_{v_B} a] (K_{v_A} b \wedge K_C((b \approx S^n 0 \wedge a \approx S^{n+1} 0) \vee (a \approx S^n 0 \wedge b \approx S^{n-1} 0)))$.

7 Conclusion and Future Work

In this paper, we introduced equality and the successor function symbol into knowing-value logic, resulting in the logic ELKVSA' . We provided an axiom system ELKVSA' , which we proved to be strongly complete with respect to non-standard models and weakly complete with respect to standard models. By formalizing and solving the ‘‘Consecutive Numbers’’ puzzle, we demonstrated the enhanced expressiveness of our logic. Furthermore, we established the decidability of ELKVSA' , illustrating that our framework achieves a balance between expressiveness and computational complexity.

For future work, the language can be further enriched with addition and multiplication symbols, which would enable the formalization and resolution of more complex problems, such as the well-known ‘‘Sum and Product’’ puzzle. In our current logic, $K_{v_i}(\phi, S^k c)$ is equivalent to $K_{v_i}(\phi, c)$, which significantly simplifies the completeness proof. However, when extending the language with addition and multiplication, an agent may know the sum or product of two numbers without knowing the exact value of the numbers themselves. Furthermore, addition and multiplication give rise to more complex arithmetic structures. Consequently, establishing completeness and decidability for such an extended logic will be more challenging. Overall, given the rich interplay between mathematical structures and epistemic reasoning, the integration of epistemic logic and arithmetic holds great potential for future research.

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Better Understanding, Understanding Better

Yu Wei

Department of Philosophy
East China Normal University
Shanghai, China
ywei@philo.ecnu.edu.cn

“Any fool can know; the point is to understand.” A well-known remark often attributed to Einstein captures a widely shared intuition: understanding is more than merely knowing. Yet epistemic logic has paid relatively little attention to understanding, despite its central role in contemporary epistemology, philosophy of science, and recent debates about AI. A recurring theme in the philosophical literature is that, unlike knowledge, understanding comes in degrees: one may understand something more or less well, and one’s understanding may be better than another’s. We introduce a comparative epistemic logic of understanding with graded modalities U_i^τ and a comparative connective $(i \succ j)\varphi$ for “ i understands why φ better than j ”. Semantically, we enrich multi-agent epistemic models with agent-indexed graded explanation structures and a justification-style term algebra. This yields a unified framework for representing minimal, ordinary, more demanding, and ideal understanding, together with comparisons between agents with respect to the same formula at issue. We distinguish a finitary bounded-level calculus from an infinitary full-language companion system. We establish soundness and strong completeness, and show that each fixed finite-level fragment is decidable.

1 Introduction

Why does the Earth orbit the Sun? This question can be simple enough for a classroom. A child may be credited with understanding why by giving a basic gravity story (e.g., “the Sun’s gravity keeps the Earth going around”); in another context, say in an undergraduate astrophysics seminar, that attribution may be withdrawn once stricter explanatory standards are imposed.¹ The withdrawal does not mean “no understanding at all”: it means the required explanatory standard has shifted. The same pattern repeats at higher levels. An undergraduate physics student may count as understanding in a seminar, yet not in a panel of astrophysics experts discussing nearby phenomena. Again, the point is not that the student’s understanding evaporates. The point is that what it takes to qualify as understanding in that context is more epistemically demanding. This is exactly the phenomenon of understanding coming in degrees. As noted in [26], one prominent reason for distinguishing understanding from knowledge is that understanding is widely taken to admit degrees, whereas knowledge is typically not treated in this way.

It already reveals the core logical issue of the present work. Besides, the same example also displays a *comparative* dimension. It is natural to say that some people understand why something is the case better than others. Two agents may both explain why the Earth orbits the Sun, yet one explanation can be deeper, more integrated, or more counterfactually robust. We therefore need to express not only whether an agent understands why a proposition holds, but also whether one agent understands it better than another. Modern epistemic logic has rich tools for knowledge and belief, but much less machinery for these two features taken together: degree-sensitive understanding and comparative understanding.

This logical gap mirrors a deeper epistemological distinction: standard philosophical cases separate knowing from understanding. A child may know via testimony that faulty wiring caused a fire while

¹We borrow this example from [11].

lacking the ability to explain the relevant mechanism. A scientist may identify oxygen as a decisive factor in a reaction yet still lack understanding of why that dependence holds [24, 21]. These stories support, in particular, the contrast between knowing why and understanding why. They suggest not only a non-identity claim: understanding is often taken to require explanatory grasp beyond merely possessing true information, and is therefore a richer epistemic achievement than bare knowing. Given this distinction and its epistemic significance, understanding is now a central topic in contemporary epistemology and philosophy of science rather than a marginal one [19, 4, 15].

The issue is equally salient in AI-facing debates. In the wake of large language models, disputes about whether, and in what sense, AI genuinely understands have moved to the center of discussion [23, 5]. Yet the pressure is not new: well before the recent LLM wave, authors had already noted that “understanding” is often invoked operationally without a stable theoretical account [29]. If formal epistemology is to contribute here, it must represent not only knowledge but also comparative understanding.

There is also a historical reason to revisit the topic. Understanding was not alien to earlier logical traditions: medieval epistemic logic treated it as an epistemic mode in its own right [6, 7]. As Boh emphasizes, “the epistemic modality of understanding seems to be treated as even more basic than knowing or believing” [6, p. 100]. Our aim is to recover understanding as a formally disciplined notion within contemporary epistemic logic, while keeping close contact with current philosophical discussions. Formally, we build on [31, 35] by moving to a graded and comparative setting that integrates an epistemic base, a justification-style explanation algebra, and an explicit comparative connective within one framework.

1.1 Background and Related Work

Philosophers distinguish multiple uses of “understanding”: understanding-that, understanding-wh, and objectual understanding [14, 3]. Among these, understanding-wh (especially understanding-why) is often treated as the central case [20]. We follow this line and focus on understanding-why.

A central philosophical thesis is that understanding is explanation-involving. Understanding why is widely characterized as “*explanatory understanding*”,² which already signals this connection. Wilkenfeld argues that explanations are the kinds of things that bring about understanding [32], and Strevens puts the point succinctly: “No understanding without explanation” [27]. At the same time, theories of explanation differ sharply over what counts as explanans and explanatory support [17, 16, 25, 34]. For a logic with broad applicability, this suggests a methodological stance: explanatory structure should be represented abstractly enough to avoid commitment to any one substantive theory, but concretely enough for logical analysis. A nearby line treats understanding as involving compression: understanding is not a matter of storing a long list of disconnected facts, but of having a compact representation of relevant structure that can be used to recover relevant information about the target phenomenon [33, 8]. This point fits the present project well: useful compression must still retain enough information to support explanation, and the formal framework below will distinguish more specific explanations from coarser explanatory resources that may provide weaker support than their more specific counterparts.

Recent epistemic logic has extensively studied non-standard knowledge operators (what/how/why, etc.); see [30]. A key predecessor for our setting is the logic of knowing why [35], which combines epistemic accessibility with explanation terms. Intuitively, agent i knows why p when i knows that p and has a uniform explanation witness that works across all i -accessible worlds. In this sense, knowing-why has an existential explanation profile, which can be rendered schematically as $\exists t K_i(t : p)$ in a justification-style

²For example, see [4, 20] and the bibliographies therein. We do not treat the “why” in “understanding why” as restrictive: some cases are more naturally phrased as *understanding how* [20]. For example, one may say “understanding how the dinosaurs went extinct” rather than “understanding why they went extinct,” without intending any substantial difference.

metalanguage. Philosophically inspired by [21] and technically by [35], Wei [31] models understanding-why via higher-order explanation, schematically $\exists t_1 \exists t_2 K_i(t_2 : (t_1 : \varphi))$, where $t_1 : \varphi$ means that t_1 is an explanation for φ , and $t_2 : (t_1 : \varphi)$ means that t_2 is a *higher-order* explanation of “ t_1 explains φ ”. This logic thereby embodies the philosophical idea that understanding why requires at least two explanations at different levels, beyond what standard knowing-why requires.

The present paper builds directly on this line. The predecessor captures the key insight that understanding requires higher-order explanatory support, but its formal architecture remains too coarse for a full hierarchy of explanatory depth and lacks an explicit comparative connective with a unified meta-theory. We therefore replace the packed operator with a level-indexed family, add a comparative connective, and use graded explanations in the semantics. Intuitively, higher grades mark explanations as eligible to support higher-level, and hence more demanding, understanding claims.

This work is best read as a synthesis of epistemic, comparative, and justification-style ideas. At its base, the framework remains an epistemic logic in the strict technical sense: its semantics is based on multi-agent epistemic models, and modal interaction principles are developed inside that setting. In this respect, the project continues the “beyond knowing that” line in contemporary epistemic logic, but shifts the focus to understanding-why and comparative understanding.

The language is also genuinely comparative, because it contains an explicit connective $(i \succ j)\varphi$ stating that agent i understands why φ better than j . Comparative modal work such as [9] already includes a comparative operator $i \succeq j$ to express that, locally at a given world, all formulas known by agent i are also known by agent j . In that framework, the global axiom scheme $K_i\psi \rightarrow K_j\psi$ is replaced by a local inference-rule treatment. Our approach follows this local-comparative spirit, but with a different semantic source. The comparative clause is not a primitive relation-inclusion postulate; it is induced by explanation-sensitive understanding levels and evaluated relative to the formula at issue, with an epistemic side condition. This design has substantial technical consequences, developed in detail later.

The semantic machinery also borrows core ingredients from justification logic: explanation terms, algebraic term operations, and admissibility constraints [13, 1, 2]. Earlier logics [35, 31] omit the sum operator $+$, understandably: a disjunctive or choice term can be too indeterminate to serve as a determinate “reason-why” witness in ordinary knowledge-why ascriptions.³ The present framework nevertheless includes the full term algebra $(\cdot, +, !, c)$. For present purposes, $+$ has a natural explanation-theoretic reading: $t + s$ is available as an explanation of φ whenever either t or s is available. The compression perspective mentioned above explains why this operation should be treated with care. It does not require us to exclude $+$; rather, it shows that a sum term preserves only the fact that some explanation is available, while losing the information about which explanation supplies the support. For this reason, $+$ is included, but it is strictly grade-downgrading. Thus the system is justification-style, but not a standard justification logic: the object language has no formulas of the form $t : \varphi$. Explanation terms enter only semantically, as graded witnesses governed by the agent-indexed functions $\mathcal{E}_i$ and the epistemic accessibility relations used for knowledge and comparison.

1.2 Framework and Contributions

We propose a comparative epistemic logic of understanding with two operator families: $U_i^r\varphi$ for level-indexed understanding-why and $(i \succ j)\varphi$ for comparative understanding with respect to a formula. At

³For example, in a simple finite case, the agent may consider several possible causes of a forest fire: lightning, arson, an unattended campfire, or a power-line fault. If each alternative has its own explanation term, repeated use of $+$ can combine these terms into one disjunctive witness. This may provide a uniform witness across these alternatives, but the combined term no longer specifies which cause explains the fire at the actual world. See Xu et al. [35].

the conceptual level, we take finite indices τ to range over $\mathbb{N}^+ = \{1, 2, 3, \dots\}$. This already lets the language capture the spectrum emphasized in [20]: minimal understanding $<$ everyday understanding $<$ typical scientist's understanding $<$ ideal understanding. Schematically, these may be represented by $U_i^1\varphi, U_i^2\varphi, U_i^m\varphi, U_i^\omega\varphi$. Level 1 is intended to model a minimal explanatory foothold (a necessary but not yet typical notion of understanding), while level 2 is intended to capture ordinary or everyday explanatory understanding in a higher-order sense [21, 31]. Some finite level $m > 2$ can then be used to represent typical scientific understanding. Higher levels correspond to explanations that are more systematically organized and more deeply integrated into scientific understanding. This level-sensitive hierarchy fits Khalifa's spectrum picture, in which ideal understanding is a limit notion [20]. Formally, we capture that limit by requiring every finite level: $\bigwedge_{n \in \mathbb{N}^+} U_i^n\varphi$.

The numerical levels are a formal idealization. Finite indices on the understanding operators represent an ordered family of explanatory standards fixed in a model; a larger index marks a more demanding standard for the same formula. Depending on the application, such standards may concern specificity, systematic integration, or counterfactual robustness. The logic does not decide which substantive features make one standard stronger than another in every application; it only provides a framework for representing such standards once fixed. Meeting a stronger standard entails meeting weaker standards for the same formula, not possessing every particular explanation that might satisfy a weaker standard.

Our main technical contributions are fourfold. First, we give a graded semantics for level-indexed and comparative understanding, with comparisons always relative to the formula at issue. Second, for each fixed finite understanding-level domain, we introduce a finitary calculus and establish soundness, strong completeness, and decidability. Third, for the full language, we move to an infinitary extension of the finitary calculus, reflecting the ideal-understanding level, and prove soundness and strong completeness. Fourth, we isolate the boundary between the bounded and full languages: in each fixed finite-level fragment the comparative connective is eliminable, while in the full language it is not; moreover, the full language is non-compact, so no sound finitary proof system can be strongly complete for it.

The rest follows this architecture. Section 2 introduces syntax and semantics. Section 3 presents the calculi and soundness. Section 4 proves bounded and full completeness and gives a bounded decidability result. Section 5 concludes.

2 Syntax and Semantics

Definition 2.1. Given nonempty countable sets P of proposition letters and I of agents, and a nonempty understanding-level domain $L \subseteq \mathbb{N}^+ \cup \{\omega\}$, where $\mathbb{N}^+ := \{1, 2, 3, \dots\}$, the comparative epistemic language of understanding $\mathbf{CELU}(L)$ is defined by (where $p \in P, i, j \in I, \tau \in L$):

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid U_i^\tau\varphi \mid (i \succ j)\varphi.$$

We use only these two instances in what follows:

$$\mathbf{CELU} := \mathbf{CELU}(\mathbb{N}^+ \cup \{\omega\}), \quad \mathbf{CELU}^\kappa := \mathbf{CELU}(\{1, \dots, \kappa\}) \quad (\kappa \geq 2).$$

Here τ is an *understanding-level index*. As explained above, finite indices on U_i^τ are intended to distinguish explanatory standards of different strength. Thus $U_i^n\varphi$ says that agent i meets the n th such standard for understanding why φ . Note that we set $Ky_i\varphi$ in [35, 31] as $U_i^1\varphi$ now, which can be read as minimal understanding.⁴ The formula $U_i^\omega\varphi$ expresses ideal understanding and is interpreted as “for all

⁴It may be tempting to set $K_i\varphi := U_i^0\varphi$, i.e., knowledge is just level-0 understanding. However, our philosophical point is that understanding is more than knowledge, which does not require that $K_i\varphi$ itself is already a very minimal form of understanding.

finite levels n , $U_i^n \varphi$ ". By abuse of notation, we write U for the family of modalities $U_i^?$. The comparative understanding formula $(i \succ j)\varphi$ indicates that agent i understands why φ better than agent j does.

Informally, the bounded language $\mathbf{CELU}^\kappa$ keeps comparatives but restricts all understanding-level indices to $\{1, \dots, \kappa\}$, so ideal and unbounded distinctions are not expressible. Technically, $\mathbf{CELU}^\kappa$ is the base language for both the finitary calculus and the bounded decidability analysis.

We accept the view in [35] that although something is a tautology, one may still lack minimal understanding (knowledge why) of that tautology. A special set of "self-evident" tautologies Λ is introduced, which the agent is assumed to minimally understand. For example, we can let all the instances of $\varphi \wedge \psi \rightarrow \varphi$ and $\varphi \wedge \psi \rightarrow \psi$ be Λ . Such simple choices will behave well in the bounded decidability argument below. At present, we do not suppose any necessitation rule for U in general.

Definition 2.2. Fix a level domain L and the associated language $\mathbf{CELU}(L)$. A graded explanatory epistemic $\mathbf{CELU}(L)$ -model $\mathcal{M}$ is a tuple $(W, \{R_i \mid i \in I\}, V, E, \text{gr}, \{\mathcal{E}_i \mid i \in I\})$ where $(W, \{R_i \mid i \in I\}, V)$ is a standard multi-agent epistemic model, i.e. for each $i \in I$, R_i is an equivalence relation on W , and:

- E is a nonempty set of explanations, closed under $\cdot$, $+$, and $!$, and containing the constant c .
- $\text{gr} : E \rightarrow \mathbb{N}$ is a grade map. For each $n \in \mathbb{N}$, let $E_n := \{t \in E \mid \text{gr}(t) \geq n\}$. The grade map satisfies:
 - $\text{gr}(t \cdot s) \geq \min\{\text{gr}(t), \text{gr}(s)\}$;
 - if $\min\{\text{gr}(t), \text{gr}(s)\} = 0$ then $\text{gr}(t + s) = 0$;
 - if $\min\{\text{gr}(t), \text{gr}(s)\} \geq 1$ then $\text{gr}(t + s) < \min\{\text{gr}(t), \text{gr}(s)\}$.
 - $\text{gr}(!t) = 0$ if $\text{gr}(t) = 0$, and $\text{gr}(!t) = 2$ if $\text{gr}(t) \geq 1$;
 - $\text{gr}(c) = 1$.
- $\{\mathcal{E}_i \mid i \in I\}$ is a family of admissible explanation functions, each $\mathcal{E}_i : E \times \mathbf{CELU}(L) \rightarrow 2^W$ satisfies:

Explanation Application $\mathcal{E}_i(t, \varphi \rightarrow \psi) \cap \mathcal{E}_i(s, \varphi) \subseteq \mathcal{E}_i(t \cdot s, \psi)$.

Explanation Sum $\mathcal{E}_i(t, \varphi) \cup \mathcal{E}_i(s, \varphi) \subseteq \mathcal{E}_i(t + s, \varphi)$.

Constant Specification If $\varphi \in \Lambda$, then $\mathcal{E}_i(c, \varphi) = W$.

Epistemic Introspection For all $t \in E$, $\mathcal{E}_i(t, K_i \varphi) \subseteq \mathcal{E}_i(!t, K_i \varphi)$.

The term operations $\cdot$, $+$, $!$, and the constant c are standard in justification logic. Application $\cdot$ combines explanations, $+$ has the usual disjunctive or choice reading, $!$ represents positive introspection, and the constant c serves as a self-evident explanation for formulas in Λ . Unlike the models in [35, 31], every explanation term here also carries a grade.

There is another generalization. In [35, 31], a single admissible explanation function, not indexed by agents, records for each term t and formula φ the worlds at which t explains φ . Here we replace that single function with an agent-indexed family $\{\mathcal{E}_i\}_{i \in I}$. Thus $w \in \mathcal{E}_i(t, \varphi)$ means that, at w , t is available as an explanation of φ for agent i . In this respect, the present model is more general: explanatory support can vary with the subject whose understanding is at issue.

The grade constraint on $+$ works together with the Sum condition on $\mathcal{E}_i$. The Sum condition preserves availability: if either t or s explains φ for agent i at a world, then $t + s$ does too. But the sum term $t + s$ is less specific than either input, since it does not by itself specify whether t or s is the available explanation at that world. For this reason, $+$ is strictly grade-lowering whenever the input minimum is positive. Grade-0 terms are allowed as closure-generated explanation terms. They belong to E_0 , but not to any positive-grade class $E_n = \{t \in E \mid \text{gr}(t) \geq n\}$ with $n \geq 1$. As the truth conditions below make explicit, grade-0 terms cannot support positive-level understanding claims.

The first three conditions on $\mathcal{E}_i$ are standard. The fourth condition, $\mathcal{E}_i(t, K_i \varphi) \subseteq \mathcal{E}_i(!t, K_i \varphi)$, requires a separate reading. Its source is the positive-introspection principle from justification logic. In standard

justification logic, $t : \varphi \rightarrow !t : (t : \varphi)$ says that if t is a justification for φ , then $!t$ is a justification for the claim that t justifies φ [12]. Intuitively, $!t$ records a reflective confirmation of the justificatory status of t .

The introspection condition uses this idea only for knowledge claims about the same agent. Suppose p_{orb} says that the Earth orbits the Sun. A request to explain why agent i knows p_{orb} is normally a request for i 's reasons or evidence for believing p_{orb} , not a request to explain why the Earth orbits the Sun. This is in line with the view that requests to justify knowledge claims commonly ask for the agent's reasons or evidence [22]. In such cases, these reasons can play a justification-like explanatory role: an explanation of $K_i\varphi$ gives the agent's reasons or evidence for believing φ .

The condition above says that whenever, at a world w , t is available to agent i as an explanation of $K_i\varphi$, the reflected term $!t$ is also available to i at w as an explanation of the same knowledge claim. The point is not that $!t$ adds another explanation of φ itself. Rather, $!t$ makes explicit that i can cite t as the support for knowing φ .

This should be read together with the grading rule for $!$: if $\text{gr}(t) = 0$ then $\text{gr}(!t) = 0$, while if $\text{gr}(t) \geq 1$ then $\text{gr}(!t) = 2$. Level 1 is minimal understanding, corresponding to knowing why; applied to $K_i\varphi$, it means that agent i has an explanation of why she knows φ . The term $!t$ represents reflecting on that reason as her reason, so for own-knowledge claims level 2 marks this reflective grasp. Thus the semantic condition validates only the restricted introspection principle $U_i^1 K_i\varphi \rightarrow U_i^2 K_i\varphi$: it is not a general mechanism for producing ever higher levels, does not validate $U_i^n K_i\varphi \rightarrow U_i^{n+1} K_i\varphi$, and does not apply to $U_i^1 K_j\varphi \rightarrow U_i^2 K_j\varphi$ for $j \neq i$ or to non-epistemic formulas.

Definition 2.3 (Truth conditions). Fix a level domain L , a **CELU**(L)-model $\mathcal{M}$, and let $L_{\text{fin}} := L \cap \mathbb{N}^+$. The satisfaction relation for **CELU**(L) is defined inductively as follows (for $p \in P$, $i, j \in I$, and $n \in L_{\text{fin}}$):

$\mathcal{M}, w \models p$	$\Leftrightarrow$	$w \in V(p)$
$\mathcal{M}, w \models \neg\varphi$	$\Leftrightarrow$	$\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \varphi \wedge \psi$	$\Leftrightarrow$	$\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models K_i\varphi$	$\Leftrightarrow$	$\mathcal{M}, v \models \varphi$ for all v such that wR_iv
$\mathcal{M}, w \models U_i^n\varphi$	$\Leftrightarrow$	(1) there exists $t \in E_n$ such that for all $v \in W$ with wR_iv , $v \in \mathcal{E}_i(t, \varphi)$, (2) for all $v \in W$ with wR_iv , $\mathcal{M}, v \models \varphi$
$\mathcal{M}, w \models U_i^\omega\varphi$	$\Leftrightarrow$	(if $\omega \in L$) for all $n \in L_{\text{fin}}$, $\mathcal{M}, w \models U_i^n\varphi$
$\mathcal{M}, w \models (i \succ j)\varphi$	$\Leftrightarrow$	(1) $\text{deg}_i(w, \varphi) > \text{deg}_j(w, \varphi)$, and (2) $\mathcal{M}, w \models K_j\varphi$

where the *understanding degree function* is $\text{deg}_i(w, \varphi) := \sup(\{n \in L_{\text{fin}} \mid \mathcal{M}, w \models U_i^n\varphi\} \cup \{0\})$.

Thus $\text{deg}_i(w, \varphi)$ is not a primitive measure, but the supremum of the finite levels at which i understands why φ at w . The existential quantifier in $U_i^n\varphi$ is a witness condition: a level- n claim requires one and the same explanation term of grade at least n to be available at every i -accessible world.

If $\omega \notin L$, as in the bounded languages **CELU**^K, then $U_i^\omega\varphi$ is not a well-formed formula, so the U^ω clause is absent. In the full language **CELU**, where $L = \mathbb{N}^+ \cup \{\omega\}$, $U_i^\omega\varphi$ expresses ideal understanding as the limit case requiring $U_i^n\varphi$ for every $n \in \mathbb{N}^+$. Equivalently, $\mathcal{M}, w \models U_i^\omega\varphi$ iff $\text{deg}_i(w, \varphi) = \omega$.

Although $(i \succ j)\varphi$ is primitive in the syntax, its semantic value is fixed by two independently defined notions: the induced degrees $\text{deg}_i, \text{deg}_j$ and the epistemic condition $K_j\varphi$. The comparison is formula-relative: it compares agents only with respect to the same φ , not globally. The truth condition states $K_j\varphi$ explicitly because $\text{deg}_i(w, \varphi) > \text{deg}_j(w, \varphi) \geq 0$ already implies $\mathcal{M}, w \models U_i^1\varphi$, hence $\mathcal{M}, w \models K_i\varphi$.

$(i \succ j)\varphi$ covers both cases where both agents understand φ but at different levels, and cases where i has minimal understanding while j merely knows φ . The latter remains a comparison within a shared epistemic issue: having an explanation already supports the ordinary judgment “I understand it better”.

Accordingly, for finite m with $m, m+1 \in L_{\text{fin}}$, the conjunction $\bigvee_i^m \varphi \wedge \neg \bigvee_i^{m+1} \varphi$ isolates the exact finite understanding degree m of agent i with respect to φ .

Returning to the opening example, let $p_{\text{orb}} \in P$ say that the Earth orbits the Sun, and let $i_C, i_S, i_R \in I$ denote the child, the student, and the researcher. In one natural formalization, a world w may satisfy $1 \leq \deg_{i_C}(w, p_{\text{orb}}) < \deg_{i_S}(w, p_{\text{orb}}) < \deg_{i_R}(w, p_{\text{orb}})$. Then $\mathcal{M}, w \models (i_S \succ i_C)p_{\text{orb}}$ and $\mathcal{M}, w \models (i_R \succ i_S)p_{\text{orb}}$. If a bystander i_B merely knows that p_{orb} while $\deg_{i_B}(w, p_{\text{orb}}) = 0$, then $\mathcal{M}, w \models (i_C \succ i_B)p_{\text{orb}}$ holds.

Remark 2.4. $(i \succ j)\varphi$ becomes eliminable in the comparative-free fragment of **CELU**^K. That is, for every model $\langle \mathcal{M}, w \rangle$ over level domain $L = \{1, \dots, \kappa\}$,

$$\mathcal{M}, w \models (i \succ j)\varphi \iff \mathcal{M}, w \models K_j\varphi \wedge \bigvee_{m=1}^{\kappa} (\bigvee_i^m \varphi \wedge \neg \bigvee_i^{m+1} \varphi).$$

Proposition 2.5. In **CELU**, the $(i \succ j)\varphi$ is in general not eliminable by any comparative-free formula.

Proof. It suffices to show non-eliminability for one instance, say $(i \succ j)p$. Suppose χ is a comparative-free formula. Since χ contains only finitely many finite understanding-level indices, let n be greater than all those indices. Take two pointed models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$, each with just one world, with the same valuation, in particular with p true. All accessibility relations are reflexive, so $K_j p$ holds at both points. Specify the explanation functions by putting $w \in \mathcal{E}_i(t_i, p)$, $w \in \mathcal{E}_j(t_j, p)$, $w' \in \mathcal{E}_i'(t_i, p)$, and $w' \in \mathcal{E}_j'(t_j, p)$, and by adding only what is required by the admissibility clauses. Write gr and gr' for the grade maps of $\mathcal{M}$ and $\mathcal{M}'$, respectively, and let the only relevant difference be the grades: $\text{gr}(t_i) = n$, $\text{gr}'(t_i) = n+1$, and $\text{gr}(t_j) = \text{gr}'(t_j) = n$. Choose the remaining grades so that all generated terms have bounded grade; hence every $\bigvee_i^\omega$ -formula is false at both points. Then $\deg_i(w, p) = n$, $\deg_i(w', p) = n+1$, and $\deg_j(w, p) = \deg_j(w', p) = n$. The two points satisfy the same comparative-free formulas whose finite indices are below n : this is proved by induction on formulas, with the $\bigvee_k^m$ case using the same available witnesses, if any, for every $m < n$, and with the $\bigvee_i^\omega$ case handled by the preceding boundedness observation. Hence $\mathcal{M}, w \models \chi \iff \mathcal{M}', w' \models \chi$. But $\mathcal{M}, w \not\models (i \succ j)p$, and $\mathcal{M}', w' \models (i \succ j)p$. $\square$

Below we show that the full language **CELU** is non-compact.

Proposition 2.6. Fix distinct agents $i \neq j$ and an atom p . Let $\Sigma_{\succ} := \{(i \succ j)p\} \cup \{\bigvee_j^n p \mid n \in \mathbb{N}^+\}$. Then every finite subset of $\Sigma_{\succ}$ is satisfiable over **CELU**-models, but $\Sigma_{\succ}$ itself is unsatisfiable.

Proof sketch. For any finite $\Delta \subseteq \Sigma_{\succ}$, let m be the largest n such that $\bigvee_j^n p \in \Delta$. Then a one-state model with p true, reflexive accessibility, and $\deg_j(w, p) = m < m+1 = \deg_i(w, p)$ satisfies Δ and $(i \succ j)p$.

If $\mathcal{M}, w \models \bigvee_j^n p$ for all $n \geq 1$, then $\deg_j(w, p) = \sup(\{n \mid \mathcal{M}, w \models \bigvee_j^n p\} \cup \{0\}) = \omega$. So $\mathcal{M}, w \models (i \succ j)p$ is impossible, since it requires $\deg_i(w, p) > \deg_j(w, p) = \omega$. Hence $\Sigma_{\succ}$ is unsatisfiable. $\square$

Therefore, no finitary proof system over **CELU** can be both sound and strongly complete.

As in [35, 31], explanation factivity, defined below, is not built into the model definition.

Definition 2.7. Fix a level domain L and a **CELU**(L)-model $\mathcal{M}$. We say that $\mathcal{M}$ has *explanation factivity* if whenever $w \in \mathcal{E}_i(t, \varphi)$ for some $i \in I$ and $t \in E$, then $\mathcal{M}, w \models \varphi$.

Given a **CELU**(L)-model $\mathcal{M} = (W, \{R_i \mid i \in I\}, V, E, \text{gr}, \{\mathcal{E}_i \mid i \in I\})$, define its factive companion $\mathcal{M}^F = (W, \{R_i \mid i \in I\}, V, E, \text{gr}, \{\mathcal{E}_i^F \mid i \in I\})$ by $\mathcal{E}_i^F(t, \varphi) = \mathcal{E}_i(t, \varphi) \setminus \{w \mid \mathcal{M}, w \not\models \varphi\}$. By direct checking, $\mathcal{M}^F$ is again a **CELU**(L)-model. The proposition below asserts that **CELU**(L)-truth is invariant under this factive transformation. The proof is routine and omitted for reasons of space.

Proposition 2.8. For any **CELU**(L)-formula φ and any $w \in W$, $\mathcal{M}, w \models \varphi \iff \mathcal{M}^F, w \models \varphi$.

3 Axiomatization

We present two related systems over two aligned languages: the bounded finitary calculus SCU_κ for CELU^κ , and the infinitary system SCU for CELU .

Definition 3.1 (Finitary calculus). Fix $\kappa \geq 2$. Let SCU_κ be the finitary Hilbert system over CELU^κ whose axiom schemas are (where $i, j, k \in I$, and understanding-level indices range over $L = \{1, \dots, \kappa\}$):

- (TAUT) Propositional tautologies
- (DISTK) $K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$
- (T) $K_i\varphi \rightarrow \varphi$
- (4) $K_i\varphi \rightarrow K_iK_i\varphi$
- (5) $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$
- (DISTU) $U_i^n(\varphi \rightarrow \psi) \rightarrow (U_i^n\varphi \rightarrow U_i^n\psi) \quad (n \in L)$
- (DEG) $U_i^n\varphi \rightarrow U_i^m\varphi \quad (n, m \in L, n > m)$
- (UYK) $U_i^n\varphi \rightarrow K_i\varphi \quad (n \in L)$
- (4*) $U_i^n\varphi \rightarrow K_iU_i^n\varphi \quad (n \in L)$
- (KYU) $U_i^1K_i\varphi \rightarrow U_i^2K_i\varphi$
- (UYC0) $U_i^1\varphi \wedge K_j\varphi \wedge \neg U_j^1\varphi \rightarrow (i \succ j)\varphi$
- (UYC) $U_i^n\varphi \wedge U_j^m\varphi \wedge \neg U_j^{m+1}\varphi \rightarrow (i \succ j)\varphi \quad (n, m, m+1 \in L, n > m)$
- (CMPn) $(i \succ j)\varphi \wedge U_j^n\varphi \rightarrow U_i^{n+1}\varphi \quad (n, n+1 \in L)$
- (CMP1) $(i \succ j)\varphi \rightarrow U_i^1\varphi \wedge K_j\varphi$
- (CMP κ) $(i \succ j)\varphi \rightarrow \neg U_j^\kappa\varphi$.

The inference rules are:

- (MP) Modus Ponens (N) $\vdash \varphi / \vdash K_i\varphi$ (NE) $\varphi \in \Lambda / \vdash U_i^1\varphi$

The axiom (KYU) captures a restricted introspective step: from level-1 understanding of why agent i knows φ , agent i can move to level-2 understanding of that same knowledge claim by reflecting on her own reason for knowing φ . It is the proof-theoretic counterpart of the semantic *epistemic introspection* condition on $\mathcal{E}_i$. Axiom (CMP κ) records the upper bound of the finite language CELU^κ : if i understands φ better than j , then j cannot already satisfy the top available level κ . In the full system below, it is replaced by (CMP ω), which plays the same role for ideal understanding.

Definition 3.2 (Full calculus). The system SCU over CELU is the ω -companion of SCU_κ : it has the same finitary schemas/rules as Definition 3.1 with all finite level indices ranging over $\mathbb{N}^+$, except that the finite-language upper-bound axiom (CMP κ) is replaced by

$$(\text{CMP}\omega) \quad (i \succ j)\varphi \rightarrow \neg U_j^\omega\varphi,$$

and it additionally includes the following axiom schema:

$$(\text{DEG}_\omega^n) \quad U_i^\omega\varphi \rightarrow U_i^n\varphi \quad (n \in \mathbb{N}^+),$$

and the following infinitary rule schema:

$$(\omega\text{I}) \quad \frac{\{\chi \rightarrow K_{j_1}(\theta_1 \rightarrow \dots K_{j_m}(\theta_m \rightarrow U_i^n\varphi) \dots) \mid n \in \mathbb{N}^+\}}{\chi \rightarrow K_{j_1}(\theta_1 \rightarrow \dots K_{j_m}(\theta_m \rightarrow U_i^\omega\varphi) \dots)}.$$

Here $m \geq 0$, $j_1, \dots, j_m \in I$, and $\theta_1, \dots, \theta_m \in \mathbf{CELU}$. If $m = 0$, the rule is read without any outer K -operators, i.e., $\{\chi \rightarrow U_i^n \varphi \mid n \in \mathbb{N}^+\} / (\chi \rightarrow U_i^\omega \varphi)$. If $m = 1$ and $\theta_1 = \top$, the rule includes the instance $\{\chi \rightarrow K_j U_i^n \varphi \mid n \in \mathbb{N}^+\} / (\chi \rightarrow K_j U_i^\omega \varphi)$. That is, under the same assumption χ , if j knows that i has understanding of φ at every finite level, then j also knows that i has ideal understanding of φ . Thus $m > 0$ makes the same limit step available inside epistemic implication contexts.

Since ω -introduction is an infinitary rule with countably many premises, we represent derivability of system SCU by well-founded proof trees, following the treatment of infinitary formal proofs in [18].

Definition 3.3 (SCU-derivability). A *proof tree* from Σ is a well-founded tree whose nodes are labeled by formulas. Each node is either an assumption leaf (label in Σ), an axiom leaf (instance of a SCU-axiom), or is obtained by one of the following rules:

- (MP): from children labeled $\psi \rightarrow \chi$ and ψ , conclude χ ;
- (NE): conclude $U_i^1 \psi$ for $\psi \in \Lambda$;
- (N): from one *closed* child labeled ψ , conclude $K_i \psi$;
- (ω I): for $m \in \mathbb{N}$, from children labeled $\chi \rightarrow K_{j_1}(\theta_1 \rightarrow \dots K_{j_m}(\theta_m \rightarrow U_i^n \psi) \dots)$, one for each $n \in \mathbb{N}^+$, conclude the corresponding formula with $U_i^\omega \psi$ in place of $U_i^n \psi$;

where “closed” means that the corresponding subtree has no assumption leaves. We write $\Sigma \vdash_{\text{SCU}} \varphi$ iff there exists such a proof tree with root label φ . A set Σ is *SCU-consistent* iff $\Sigma \not\vdash_{\text{SCU}} \perp$.

Lemma 3.4. *If $\Sigma \vdash_{\text{SCU}} \psi$ and $\Sigma \cup \{\psi\} \vdash_{\text{SCU}} \chi$, then $\Sigma \vdash_{\text{SCU}} \chi$.*

Proof. Graft a fresh copy of a proof tree of ψ from Σ onto each assumption leaf labeled ψ in a proof tree of χ from $\Sigma \cup \{\psi\}$, leaving all other nodes unchanged and preserving the original rule applications. The result is a proof tree from Σ . The only point requiring checking is the requirement in (N) that its premise child be closed: this is preserved because a closed child subtree contains no assumption leaves, so no grafting takes place inside it. Well-foundedness is also preserved: any descending chain either stays in the original tree or eventually enters a single grafted copy, both of which are well-founded. $\square$

Lemma 3.5. *If $\Sigma \not\vdash_{\text{SCU}} \varphi$, then $\Sigma \cup \{\neg \varphi\}$ is SCU-consistent.*

Proof. Suppose that $\Sigma \cup \{\neg \varphi\} \vdash_{\text{SCU}} \perp$. By induction on proof trees, we first obtain: if $\Sigma \cup \{\alpha\} \vdash_{\text{SCU}} \beta$, then $\Sigma \vdash_{\text{SCU}} \alpha \rightarrow \beta$. The finitary cases are standard. For an ω -introduction instance, write its premises as $\chi \rightarrow B_n$ ($n \in \mathbb{N}^+$) and its conclusion as $\chi \rightarrow B_\omega$, where B_ω is obtained from B_n by replacing the displayed $U_i^n \psi$ with $U_i^\omega \psi$. By the induction hypothesis, $\Sigma \vdash_{\text{SCU}} \alpha \rightarrow (\chi \rightarrow B_n)$ for all n . Hence $\Sigma \vdash_{\text{SCU}} (\alpha \wedge \chi) \rightarrow B_n$ for all n , so (ω I) gives $\Sigma \vdash_{\text{SCU}} (\alpha \wedge \chi) \rightarrow B_\omega$, and therefore $\Sigma \vdash_{\text{SCU}} \alpha \rightarrow (\chi \rightarrow B_\omega)$. Then we get $\Sigma \vdash_{\text{SCU}} \neg \varphi \rightarrow \perp$, therefore $\Sigma \vdash_{\text{SCU}} \varphi$, contradicting the hypothesis. So $\Sigma \cup \{\neg \varphi\}$ is consistent. $\square$

The next derived principles are useful later in the completeness proofs. They also show that the formula $(i \succ j)\varphi$ behaves like a strict comparison: irreflexivity, transitivity, and asymmetry follow from the axioms connecting comparison with level-indexed understanding. For the full system, we also derive the ideal-level analogues of the epistemic principles for understanding formulas.

Proposition 3.6. *The following principles are provable in the indicated systems:*

$$\begin{array}{ll}
 (5^*) & \neg U_i^n \varphi \rightarrow K_i \neg U_i^n \varphi & (4_\omega^*) & U_i^\omega \varphi \rightarrow K_i U_i^\omega \varphi \\
 (5_\omega^*) & \neg U_i^\omega \varphi \rightarrow K_i \neg U_i^\omega \varphi & (\text{IRREF}) & \neg(i \succ i)\varphi \\
 (\text{TRANS}) & (i \succ j)\varphi \wedge (j \succ k)\varphi \rightarrow (i \succ k)\varphi & (\text{ASYM}) & (i \succ j)\varphi \rightarrow \neg(j \succ i)\varphi
 \end{array}$$

Here (5^*) , (IRREF), (TRANS), and (ASYM) are provable in both SCU_κ and SCU; for (5^*) , $n \in \{1, \dots, \kappa\}$ in SCU_κ and $n \in \mathbb{N}^+$ in SCU. (4_ω^*) and (5_ω^*) are provable in SCU.

Proof. The derivation of (5^*) is routine from $(T), (5), (4^*)$ together with classical reasoning. In SCU, (DEG_ω^n) and (4^*) give $U_i^\omega \varphi \rightarrow K_i U_i^n \varphi$ for every $n \in \mathbb{N}^+$. By modal reasoning, these yield the premises $U_i^\omega \varphi \rightarrow K_i(\top \rightarrow U_i^n \varphi)$ of the $m = 1$ instance of (ωI) , which gives $U_i^\omega \varphi \rightarrow K_i(\top \rightarrow U_i^\omega \varphi)$, and hence (4_ω^*) . Then (5_ω^*) follows by the same argument as (5^*) . Given (IRREF) and (TRANS) , the derivation of (ASYM) is routine. For (IRREF) in SCU_κ :

- (1) $(i \succ i)\varphi \rightarrow U_i^1 \varphi$ (CMP1)
- (2_m) $(i \succ i)\varphi \wedge U_i^m \varphi \rightarrow U_i^{m+1} \varphi$ (CMPn) $(1 \leq m < \kappa)$
- (3) $(i \succ i)\varphi \rightarrow U_i^\kappa \varphi$ (1), (2_m), induction on m $(1 \leq m < \kappa)$
- (4) $(i \succ i)\varphi \rightarrow \neg U_i^\kappa \varphi$ (CMP κ)
- (5) $\neg(i \succ i)\varphi$ (3), (4), classical reasoning

For (TRANS) in SCU_κ , write $A := (i \succ j)\varphi \wedge (j \succ k)\varphi$ and $B := (i \succ k)\varphi$. Then:

- (1) $A \rightarrow \neg U_k^\kappa \varphi$ (CMP κ), propositional reasoning
- (2) $A \rightarrow U_i^1 \varphi \wedge K_k \varphi$ (CMP1), propositional reasoning
- (3) $A \wedge \neg U_k^1 \varphi \rightarrow B$ (2), (UYC0)
- (4_m) $(j \succ k)\varphi \wedge U_k^m \varphi \rightarrow U_j^{m+1} \varphi$ (CMPn) $(1 \leq m < \kappa)$
- (5_m) $U_j^{m+1} \varphi \rightarrow U_j^m \varphi$ (DEG) $(1 \leq m < \kappa)$
- (6_m) $(i \succ j)\varphi \wedge U_j^m \varphi \rightarrow U_i^{m+1} \varphi$ (CMPn) $(1 \leq m < \kappa)$
- (7_m) $A \wedge U_k^m \varphi \rightarrow U_i^{m+1} \varphi$ (4_m), (5_m), (6_m)
- (8_m) $A \wedge U_k^m \varphi \wedge \neg U_k^{m+1} \varphi \rightarrow B$ (7_m), (UYC) $(1 \leq m < \kappa)$
- (9_m) $A \wedge \neg B \wedge U_k^m \varphi \rightarrow U_k^{m+1} \varphi$ (8_m), classical reasoning
- (10) $A \wedge \neg B \rightarrow U_k^\kappa \varphi$ (3), (9_m), induction on m
- (11) $A \rightarrow B$ (1), (10), classical reasoning

For (IRREF) in SCU:

- (1) $(i \succ i)\varphi \rightarrow U_i^1 \varphi$ (CMP1)
- (2_m) $(i \succ i)\varphi \wedge U_i^m \varphi \rightarrow U_i^{m+1} \varphi$ (CMPn) $(m \in \mathbb{N}^+)$
- (3_n) $(i \succ i)\varphi \rightarrow U_i^n \varphi$ (1), (2_m), induction on n
- (4) $(i \succ i)\varphi \rightarrow U_i^\omega \varphi$ (ωI) applied to $(3_n)_{n \in \mathbb{N}^+}$
- (5) $(i \succ i)\varphi \rightarrow \neg U_i^\omega \varphi$ (CMP ω)
- (6) $\neg(i \succ i)\varphi$ (4), (5), classical reasoning

For (TRANS) in SCU, use the same abbreviations $A := (i \succ j)\varphi \wedge (j \succ k)\varphi$ and $B := (i \succ k)\varphi$. Then:

- (1) $A \rightarrow \neg U_k^\omega \varphi$ (CMP ω), propositional reasoning
- (2) $A \rightarrow U_i^1 \varphi \wedge K_k \varphi$ (CMP1), propositional reasoning
- (3) $A \wedge \neg U_k^1 \varphi \rightarrow B$ (2), (UYC0)
- (4_m) $(j \succ k)\varphi \wedge U_k^m \varphi \rightarrow U_j^{m+1} \varphi$ (CMPn) $(m \in \mathbb{N}^+)$
- (5_m) $U_j^{m+1} \varphi \rightarrow U_j^m \varphi$ (DEG) $(m \in \mathbb{N}^+)$
- (6_m) $(i \succ j)\varphi \wedge U_j^m \varphi \rightarrow U_i^{m+1} \varphi$ (CMPn) $(m \in \mathbb{N}^+)$
- (7_m) $A \wedge U_k^m \varphi \rightarrow U_i^{m+1} \varphi$ (4_m), (5_m), (6_m)
- (8_m) $A \wedge U_k^m \varphi \wedge \neg U_k^{m+1} \varphi \rightarrow B$ (7_m), (UYC) $(m \in \mathbb{N}^+)$
- (9_m) $A \wedge \neg B \wedge U_k^m \varphi \rightarrow U_k^{m+1} \varphi$ (8_m), classical reasoning
- (10_n) $A \wedge \neg B \rightarrow U_k^n \varphi$ (3), (9_m), induction on n
- (11) $A \wedge \neg B \rightarrow U_k^\omega \varphi$ (ωI) applied to $(10_n)_{n \in \mathbb{N}^+}$
- (12) $A \rightarrow B$ (1), (11), classical reasoning

□

Theorem 3.7 (Soundness for SCU_κ). *Fix $\kappa \geq 2$. SCU_κ is sound over CELU^κ -models.*

Proof sketch. Below we record only two representative nontrivial cases.

For (KYU), assume $\mathcal{M}, w \models U_i^1 K_i \varphi$, witnessed by $t \in E_1$. Then $\text{gr}(t) \geq 1$, so $\text{gr}(!t) = 2$ and hence $!t \in E_2$. For each v with $w R_i v$, we have $v \in \mathcal{E}_i(t, K_i \varphi)$ and $\mathcal{M}, v \models K_i \varphi$; by epistemic introspection of $\mathcal{E}_i$, also $v \in \mathcal{E}_i(!t, K_i \varphi)$. Thus $!t$ witnesses $\mathcal{M}, w \models U_i^2 K_i \varphi$.

For (CMP κ), if $\mathcal{M}, w \models (i \succ j) \varphi \wedge U_j^\kappa \varphi$, then $\deg_j(w, \varphi) = \kappa$ while over CELU^κ always $\deg_i(w, \varphi) \leq \kappa$, contradicting $\deg_i(w, \varphi) > \deg_j(w, \varphi)$; hence $\mathcal{M}, w \models \neg U_j^\kappa \varphi$. □

Theorem 3.8 (Soundness for SCU). *SCU is sound over CELU models.*

Proof sketch. By induction on SCU proof trees from Definition 3.3. The shared finitary axioms/rules are sound as in Theorem 3.7. For (DEG $_\omega^n$), assume $\mathcal{M}, w \models U_i^\omega \varphi$. By the semantic clause of U^ω , this means $\mathcal{M}, w \models U_i^n \varphi$ for every $n \geq 1$. (CMP ω) is valid because $(i \succ j) \varphi$ requires $\deg_i(w, \varphi) > \deg_j(w, \varphi)$, impossible if $U_j^\omega \varphi$ holds (then $\deg_j(w, \varphi) = \omega$). For (ω I), assume all its premises are true at w . If $\mathcal{M}, w \not\models \chi$, the conclusion is immediate. Suppose $\mathcal{M}, w \models \chi$. If $m = 0$, the premises give $\mathcal{M}, w \models U_i^n \varphi$ for every $n \geq 1$, hence $\mathcal{M}, w \models U_i^\omega \varphi$ by the semantics. If $m > 0$, take arbitrary worlds $w_1, \dots, w_m$ with $w R_{j_1} w_1, \dots, w R_{j_m} w_m$. If there is a first r such that θ_r fails at w_r , the corresponding implication is true. If all θ_r hold, then the premises give $\mathcal{M}, w_m \models U_i^n \varphi$ for every $n \geq 1$, so $\mathcal{M}, w_m \models U_i^\omega \varphi$. Thus the conclusion is true at w . □

4 Completeness and Bounded Decidability

This section proves strong completeness for the bounded finitary and full infinitary calculi, and also establishes bounded decidability. We therefore use two canonical-model constructions: a bounded one, following the witness-based strategy of [35, 31] but adapted to graded witnesses, agent-indexed explanation stores, and comparative formulas, and a full-language construction based on maximal consistent sets closed under the infinitary rule.

We first construct the bounded canonical model for SCU_κ . Let Ω be the set of all maximal SCU_κ -consistent sets of CELU^κ -formulas.

Definition 4.1. The canonical model for SCU_κ is $\mathcal{M}^c = (W^c, \{R_i^c \mid i \in I\}, V^c, E^c, \text{gr}^c, \{\mathcal{E}_i^c \mid i \in I\})$, where:

- E^c is generated by the BNF: $t ::= c \mid \varphi^n \mid (t \cdot t) \mid (t + t) \mid !t$, where $\varphi \in \text{CELU}^\kappa$, $n \in \{1, \dots, \kappa\}$.
- The canonical grade map $\text{gr}^c : E^c \rightarrow \mathbb{N}$ is defined inductively by:

$$\begin{aligned} \text{gr}^c(c) &= 1, \\ \text{gr}^c(\varphi^n) &= n, \\ \text{gr}^c(t \cdot s) &= \min\{\text{gr}^c(t), \text{gr}^c(s)\}, \\ \text{gr}^c(!t) &= \begin{cases} 0, & \text{if } \text{gr}^c(t) = 0, \\ 2, & \text{if } \text{gr}^c(t) > 0, \end{cases} \\ \text{gr}^c(t + s) &= \begin{cases} 0, & \text{if } \min\{\text{gr}^c(t), \text{gr}^c(s)\} = 0, \\ \min\{\text{gr}^c(t), \text{gr}^c(s)\} - 1, & \text{otherwise.} \end{cases} \end{aligned}$$

For each $n \in \mathbb{N}$, let $E_n^c := \{t \in E^c \mid \text{gr}^c(t) \geq n\}$.

- W^c is the set of all triples $\langle \Gamma, \mathcal{F}, \vec{f} \rangle$ such that $\Gamma \in \Omega$ and:
 - $\vec{f} = \{f_i^n \mid i \in I, n \in \{1, \dots, \kappa\}\}$ with $f_i^n : \{\varphi \mid \bigcup_i^n \varphi \in \Gamma\} \rightarrow \{t \in E^c \mid \text{gr}^c(t) = n\}$;
 - $\mathcal{F} = \{F_i \mid i \in I\}$, where each $F_i \subseteq E^c \times \mathbf{CELU}^\kappa$ is the *agent- i explanation store* satisfying the following base and admissibility conditions relative to Γ and $\vec{f}$:

$$\begin{aligned}
 (\text{Base}_i) \quad & \{\langle c, \varphi \rangle \mid \varphi \in \Lambda\} \subseteq F_i \text{ and } \{\langle f_i^n(\varphi), \varphi \rangle \mid \bigcup_i^n \varphi \in \Gamma\} \subseteq F_i; \\
 (\text{App}_i) \quad & \text{if } \langle t, \varphi \rightarrow \psi \rangle, \langle s, \varphi \rangle \in F_i \text{ then } \langle t \cdot s, \psi \rangle \in F_i; \\
 (\text{Sum}_i) \quad & \text{if } \langle t, \varphi \rangle \in F_i \text{ or } \langle s, \varphi \rangle \in F_i \text{ then } \langle t + s, \varphi \rangle \in F_i; \\
 (\text{Int}_i) \quad & \text{if } \langle t, K_i \varphi \rangle \in F_i, \text{ then } \langle !t, K_i \varphi \rangle \in F_i.
 \end{aligned}$$

- $\langle \Gamma, \mathcal{F}, \vec{f} \rangle R_i^c \langle \Delta, \mathcal{G}, \vec{g} \rangle$ iff $\{\varphi \mid K_i \varphi \in \Gamma\} \subseteq \Delta$ and $\forall n \in \{1, \dots, \kappa\} (f_i^n = g_i^n)$.
- For each $i \in I$, $\mathcal{E}_i^c : E^c \times \mathbf{CELU}^\kappa \rightarrow 2^{W^c}$ is defined by $\mathcal{E}_i^c(t, \varphi) := \{\langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c \mid \langle t, \varphi \rangle \in F_i\}$.
- $V^c(p) := \{\langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c \mid p \in \Gamma\}$.

In this construction, each canonical world $\langle \Gamma, F, \vec{f} \rangle \in W^c$ packages exact-grade witnesses for the $\bigcup$ -formulas in Γ . For each $\bigcup_i^n \varphi \in \Gamma$, the map f_i^n selects a term $f_i^n(\varphi) \in E^c$ with $\text{gr}^c(f_i^n(\varphi)) = n$, and the pair $\langle f_i^n(\varphi), \varphi \rangle$ is placed into the base of F_i . Each F_i is then closed under (App_i) , (Sum_i) , and (Int_i) . The specific clauses in the canonical grading gr^c are chosen so that the canonical model still satisfies the original semantic constraints and, at the same time, supports the later technical arguments.

We next define the closure operation used to build explanation stores from their base pairs. It adds exactly the pairs forced by the admissibility conditions for application, sum, and epistemic introspection.

Definition 4.2. For $X \subseteq E^c \times \mathbf{CELU}^\kappa$ and $i \in I$, define the one-step closure operator $\text{cl}^i(X) \subseteq E^c \times \mathbf{CELU}^\kappa$ by $\text{cl}^i(X) := X \cup \text{cl}_{\text{app}}(X) \cup \text{cl}_{\text{sum}}(X) \cup \text{cl}_{\text{int}}^i(X)$, where

$$\begin{aligned}
 \text{cl}_{\text{app}}(X) &:= \{\langle t \cdot s, \psi \rangle \mid \exists \chi (\langle t, \chi \rightarrow \psi \rangle \in X \wedge \langle s, \chi \rangle \in X)\}, \\
 \text{cl}_{\text{sum}}(X) &:= \{\langle t + s, \varphi \rangle \mid \langle t, \varphi \rangle \in X \text{ or } \langle s, \varphi \rangle \in X\}, \\
 \text{cl}_{\text{int}}^i(X) &:= \{\langle !t, K_i \varphi \rangle \mid \langle t, K_i \varphi \rangle \in X\}.
 \end{aligned}$$

Given $S \subseteq E^c \times \mathbf{CELU}^\kappa$, let $S_i^0 := S$ and $S_i^{k+1} := \text{cl}^i(S_i^k)$ for $k \in \mathbb{N}$. Define $S_i^\infty := \bigcup_{k \in \mathbb{N}} S_i^k$.

Lemma 4.3. For every maximal SCU_κ -consistent set $\Gamma \in \Omega$, there exist $\mathcal{F} = \{F_i \mid i \in I\}$ and a witness family $\vec{f}$ such that $\langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$. In particular, $W^c \neq \emptyset$.

Proof. For each $i \in I$ and $n \in \{1, \dots, \kappa\}$, define $f_i^n(\varphi) := \varphi^n$ for φ with $\bigcup_i^n \varphi \in \Gamma$. This is well-typed since $\text{gr}^c(\varphi^n) = n$. For each fixed $i \in I$, let

$$S_{\Gamma, \vec{f}, i} := \{\langle c, \varphi \rangle \mid \varphi \in \Lambda\} \cup \{\langle f_i^n(\varphi), \varphi \rangle \mid n \in \{1, \dots, \kappa\}, \bigcup_i^n \varphi \in \Gamma\}.$$

Using the notation of Definition 4.2, set $S_{\Gamma, \vec{f}, i}^0 := S_{\Gamma, \vec{f}, i}$, $S_{\Gamma, \vec{f}, i}^{k+1} := \text{cl}^i(S_{\Gamma, \vec{f}, i}^k)$, $F_i := S_{\Gamma, \vec{f}, i}^\infty$. Let $\mathcal{F} := \{F_i \mid i \in I\}$. Then each F_i is base-extending and closed under (App_i) , (Sum_i) , and (Int_i) by construction. Hence $\langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$.

For nonemptiness, take any theorem $\top$ of SCU_κ . Then $\{\top\}$ is SCU_κ -consistent, so by Lindenbaum there exists $\Gamma \in \Omega$. Applying the first part to this Γ gives $\langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$. Hence $W^c \neq \emptyset$. $\square$

Lemma 4.4. For each $i \in I$, R_i^c is an equivalence relation on W^c .

We omit the proof. Regarding $\mathcal{E}_i^c$ in canonical models, it is obvious to check the following:

Lemma 4.5. *For each $i \in I$, the function $\mathcal{E}_i^c$ satisfies all conditions in Definition 2.2 for $L = \{1, \dots, \kappa\}$.*

We conclude that the canonical model is well-defined, based on the above lemmas.

Proposition 4.6. $\mathcal{M}^c$ is a $\mathbf{CELU}^\kappa$ -model (equivalently, a $\mathbf{CELU}(L)$ -model with $L = \{1, \dots, \kappa\}$).

Proof. E^c is nonempty and closed under $\cdot, +, !$ by its BNF generation. The clauses defining gr^c satisfy all grade constraints in Definition 2.2. By Lemma 4.3, $W^c \neq \emptyset$. By Lemma 4.4, each R_i^c is an equivalence relation on W^c . By Lemma 4.5, each $\mathcal{E}_i^c$ satisfies the required admissibility clauses. $\square$

We now establish the existence ingredients for the Truth Lemma.

Lemma 4.7 (K_i -Existence Lemma). *For any canonical world $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$, if $\neg K_i \phi \in \Gamma$, then there exists a canonical world $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle \in W^c$ such that $w R_i^c w'$ and $\neg \phi \in \Delta$.*

Proof. Fix $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$ with $\neg K_i \phi \in \Gamma$. Let $\Delta^- := \{\psi \mid K_i \psi \in \Gamma\} \cup \{\neg \phi\}$. It is routine to show that Δ^- is SCU_κ -consistent, by (N) and (DISTK). Extend Δ^- to a maximal SCU_κ -consistent set $\Delta \in \Omega$. Then $\{\psi \mid K_i \psi \in \Gamma\} \subseteq \Delta$ and $\neg \phi \in \Delta$.

For each $n \in \{1, \dots, \kappa\}$ and formula χ , we have $U_i^n \chi \in \Gamma \Leftrightarrow U_i^n \chi \in \Delta$. Indeed, if $U_i^n \chi \in \Gamma$, then (4*) gives $K_i U_i^n \chi \in \Gamma$, so $U_i^n \chi \in \Delta$. Conversely, if $U_i^n \chi \notin \Gamma$, then $\neg U_i^n \chi \in \Gamma$ by maximality; by theorem (5*), $K_i \neg U_i^n \chi \in \Gamma$, hence $\neg U_i^n \chi \in \Delta$, so $U_i^n \chi \notin \Delta$. Thus setting $g_i^n := f_i^n$ is type-correct for the Δ -domain requirement in W^c .

Define $\vec{g} = \{g_j^n \mid j \in I, n \in \{1, \dots, \kappa\}\}$ by: for $j = i$, let $g_i^n := f_i^n$; for $j \neq i$, let $g_j^n(\psi) := \psi^n$ whenever $U_j^n \psi \in \Delta$. This is well-defined because $\psi^n \in E^c$ and $\text{gr}^c(\psi^n) = n$.

To construct $\mathcal{G} = \{G_j \mid j \in I\}$, for each $j \in I$ define

$$S_{\Delta, \vec{g}, j} := \{\langle c, \phi \rangle \mid \phi \in \Delta\} \cup \{\langle g_j^n(\psi), \psi \rangle \mid n \in \{1, \dots, \kappa\}, U_j^n \psi \in \Delta\},$$

and then let $S_{\Delta, \vec{g}, j}^0 := S_{\Delta, \vec{g}, j}$, $S_{\Delta, \vec{g}, j}^{m+1} := \text{cl}^j(S_{\Delta, \vec{g}, j}^m)$, $S_{\Delta, \vec{g}, j}^\infty := \bigcup_{m \in \mathbb{N}} S_{\Delta, \vec{g}, j}^m$, and $G_j := S_{\Delta, \vec{g}, j}^\infty$. Each G_j is base-extending and closed under (App_j), (Sum_j), and (Int_j). Set $\mathcal{G} := \{G_j \mid j \in I\}$ and let $w' := \langle \Delta, \mathcal{G}, \vec{g} \rangle$. Then $w' \in W^c$.

By the definition of R_i^c , since $\{\psi \mid K_i \psi \in \Gamma\} \subseteq \Delta$ and $f_i^n = g_i^n$ for all $n \in \{1, \dots, \kappa\}$, we have $w R_i^c w'$. Finally, $\neg \phi \in \Delta$ by construction. $\square$

Lemma 4.8 (U_i^n -Existence Lemma). *Let $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$, fix $i \in I$ and $n \in \{1, \dots, \kappa\}$, and assume $U_i^n \phi \notin \Gamma$ and $\langle t, \phi \rangle \in F_i$ for some $t \in E_n^c$. There exists $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle \in W^c$ such that $w R_i^c w'$ and $\langle t, \phi \rangle \notin G_i$.*

Proof. Fix $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle$. Define, for this proof:

$$C^0 := \{\langle c, \phi \rangle \mid \phi \in \Delta\} \cup \{\langle f_i^m(\psi), \psi \rangle \mid m \in \{1, \dots, \kappa\}, U_i^m \psi \in \Gamma\},$$

and $C^{k+1} := \text{cl}^i(C^k)$, $C^\infty := \bigcup_{k \in \mathbb{N}} C^k$. Thus C^∞ is the least agent- i explanation store generated from Γ and the witness maps $\vec{f}$ by the admissibility conditions.

Claim 1. If $\langle t, \phi \rangle \in C^\infty$ and $\text{gr}^c(t) \geq n \geq 1$, then $U_i^n \phi \in \Gamma$.

Claim 2. Given the fixed $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$, $i \in I$ and $t \in E^c$, if $\langle t, \phi \rangle \notin C^\infty$, then there exists $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle \in W^c$ such that $w R_i^c w'$ and $\langle t, \phi \rangle \notin G_i$.

Proof of Claim 1. We prove by induction on $k \in \mathbb{N}$ that each pair in C^k has, in Γ , all positive understanding formulas up to the grade of its term:

$$P(k) : \forall \langle u, \psi \rangle \in C^k \forall \ell (1 \leq \ell \leq \text{gr}^c(u) \Rightarrow U_i^\ell \psi \in \Gamma).$$

For $k = 0$, there are two kinds of pairs in C^0 . If $\langle u, \psi \rangle = \langle c, \lambda \rangle$ with $\lambda \in \Lambda$, then $\text{gr}^c(c) = 1$ and $U_i^1 \lambda$ is derivable by (NE), hence belongs to Γ . If $\langle u, \psi \rangle = \langle f_i^m(\psi), \psi \rangle$, then $U_i^m \psi \in \Gamma$ by definition of the domain of f_i^m ; for any $1 \leq \ell \leq m$, (DEG) yields $U_i^\ell \psi \in \Gamma$.

For the induction step, take $\langle u, \psi \rangle \in C^{k+1} = \text{cl}^i(C^k)$ and $1 \leq \ell \leq \text{gr}^c(u)$. If $\langle u, \psi \rangle \in C^k$, apply IH. Otherwise:

(App) $u = r \cdot s$ with $\langle r, \chi \rightarrow \psi \rangle, \langle s, \chi \rangle \in C^k$. Let $a := \text{gr}^c(r)$, $b := \text{gr}^c(s)$, and $h := \min\{a, b\}$; then $\text{gr}^c(u) = h$. By IH and (DEG), $U_i^h(\chi \rightarrow \psi), U_i^h \chi \in \Gamma$. By (DISTU), $U_i^h \psi \in \Gamma$, and then (DEG) yields $U_i^\ell \psi \in \Gamma$.

(Sum) $u = r + s$ and either $\langle r, \psi \rangle \in C^k$ or $\langle s, \psi \rangle \in C^k$. Assume $\langle r, \psi \rangle \in C^k$. Set $a := \text{gr}^c(r)$ and $h := \text{gr}^c(u)$. Since $1 \leq \ell \leq h$, we have $h \geq 1$. By the grade definition for $+$, this yields $h < a$, hence $\ell < a$. By IH, $U_i^a \psi \in \Gamma$, and then (DEG) gives $U_i^\ell \psi \in \Gamma$.

(Int) $u = !r$ with $\langle r, K_i \psi \rangle \in C^k$. If $\text{gr}^c(r) = 0$, then $\text{gr}^c(u) = 0$, impossible since $1 \leq \ell \leq \text{gr}^c(u)$. If $\text{gr}^c(r) \geq 1$, then $\text{gr}^c(u) = 2$. By IH, $U_i^1 K_i \psi \in \Gamma$. By (KYU), $U_i^2 K_i \psi \in \Gamma$, and then (DEG) yields $U_i^\ell K_i \psi \in \Gamma$.

So $P(k+1)$ holds. Hence $P(k)$ holds for all k . If $\langle t, \phi \rangle \in C^\infty$, then $\langle t, \phi \rangle \in C^k$ for some k , and applying $P(k)$ with $\ell := n$ (since $1 \leq n \leq \text{gr}^c(t)$) yields $U_i^n \phi \in \Gamma$.

Proof of Claim 2. Set $\Delta := \Gamma$, and for each $m \in \{1, \dots, \kappa\}$ let $g_i^m := f_i^m$. For $j \neq i$, define $g_j^m(\psi) := \psi^m$ for $U_j^m \psi \in \Delta$. Then $\vec{g}$ satisfies the typing condition in the definition of W^c , and $g_i^m = f_i^m$ for all m .

Let $G_i := C^\infty$. By hypothesis, $\langle t, \phi \rangle \notin G_i$. For each $j \neq i$, let

$$S_{\Delta, \vec{g}, j} := \{\langle c, \phi \rangle \mid \phi \in \Lambda\} \cup \{\langle g_j^m(\psi), \psi \rangle \mid m \in \{1, \dots, \kappa\}, U_j^m \psi \in \Delta\},$$

and define $S_{\Delta, \vec{g}, j}^0 := S_{\Delta, \vec{g}, j}$, $S_{\Delta, \vec{g}, j}^{k+1} := \text{cl}^j(S_{\Delta, \vec{g}, j}^k)$, $S_{\Delta, \vec{g}, j}^\infty := \bigcup_{k \in \mathbb{N}} S_{\Delta, \vec{g}, j}^k$, and $G_j := S_{\Delta, \vec{g}, j}^\infty$. Let $\mathcal{G} := \{G_j \mid j \in I\}$ and define $w' := \langle \Delta, \mathcal{G}, \vec{g} \rangle$. For $j = i$, $G_i = C^\infty$ is base-extending and closed under (App_i), (Sum_i), (Int_i) by construction. For $j \neq i$, each G_j has the same closure properties by the iterative construction above. Hence $w' \in W^c$. Also, $\{\psi \mid K_i \psi \in \Gamma\} \subseteq \Delta$ and $f_i^m = g_i^m$ for all m , so $wR_i^c w'$. Finally, $\langle t, \phi \rangle \notin G_i$.

Now we conclude the lemma. If $\langle t, \phi \rangle \in C^\infty$, then Claim 1 gives $U_i^n \phi \in \Gamma$, contradiction. Hence $\langle t, \phi \rangle \notin C^\infty$. By Claim 2 there exists an i -successor $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle \in W^c$ with $\langle t, \phi \rangle \notin G_i$. $\square$

With these existence lemmas in place, we prove the bounded Truth Lemma.

Lemma 4.9 (Truth Lemma for $\mathbf{CELU}^K$). *For $\phi \in \mathbf{CELU}^K$ and $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$, $\mathcal{M}^c, w \models \phi$ iff $\phi \in \Gamma$.*

Proof. We use induction on formula structure. The Boolean cases are standard. For $\phi = K_i \psi$:

- Assume $K_i \psi \in \Gamma$, and let $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle$ satisfy $wR_i^c w'$. By definition of R_i^c , $\{\chi \mid K_i \chi \in \Gamma\} \subseteq \Delta$, hence $\psi \in \Delta$. By IH, $\mathcal{M}^c, w' \models \psi$. So $\mathcal{M}^c, w \models K_i \psi$.
- Assume $\mathcal{M}^c, w \models K_i \psi$. If $K_i \psi \notin \Gamma$, then $\neg K_i \psi \in \Gamma$ by maximality. By Lemma 4.7, there is $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle$ with $wR_i^c w'$ and $\neg \psi \in \Delta$. By IH, $\mathcal{M}^c, w' \not\models \psi$, contradiction. Hence $K_i \psi \in \Gamma$.

For $\phi = U_i^n \psi$:

- Assume $U_i^n \psi \in \Gamma$. Let $t := f_i^n(\psi)$; then $\text{gr}^c(t) = n$, so $t \in E_n^c$. We show that t witnesses $U_i^n \psi$ at w . Let $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle$ satisfy $wR_i^c w'$. By the definition of R_i^c , $g_i^n = f_i^n$. Since $\psi \in \text{dom}(f_i^n)$, we have $\psi \in \text{dom}(g_i^n)$, hence $U_i^n \psi \in \Delta$. The base condition for G_i gives $\langle g_i^n(\psi), \psi \rangle \in G_i$, and therefore $\langle t, \psi \rangle \in G_i$. Thus $w' \in \mathcal{E}_i^c(t, \psi)$. From (UYK) and $U_i^n \psi \in \Gamma$, we get $K_i \psi \in \Gamma$. By the K_i -case already proved, $\mathcal{M}^c, w \models K_i \psi$. Therefore $\mathcal{M}^c, w \models U_i^n \psi$.

- Assume $\mathcal{M}^c, w \models U_i^n \psi$. Then there exists $t \in E_n^c$ such that $\forall v \in W^c (wR_i^c v \Rightarrow v \in \mathcal{E}_i^c(t, \psi))$. Since R_i^c is reflexive, $w \in \mathcal{E}_i^c(t, \psi)$, i.e. $\langle t, \psi \rangle \in F_i$. If $U_i^n \psi \notin \Gamma$, then Lemma 4.8 gives $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle$ with $wR_i^c w'$ and $\langle t, \psi \rangle \notin G_i$. Hence $w' \notin \mathcal{E}_i^c(t, \psi)$, contradiction. Therefore $U_i^n \psi \in \Gamma$.

For $\varphi = (i \succ j)\psi$. First handle the special case $i = j$. By theorem (IRREF), no Γ contains $(i \succ i)\psi$. Semantically, $\mathcal{M}^c, w \models (i \succ i)\psi$ is impossible since it would require $\deg_i(w, \psi) > \deg_i(w, \psi)$. Hence, $\mathcal{M}^c, w \models (i \succ i)\psi \Leftrightarrow (i \succ i)\psi \in \Gamma$. Now assume $i \neq j$.

- Assume $(i \succ j)\psi \in \Gamma$. From (CMP1), $K_j \psi \in \Gamma$ and $U_i^1 \psi \in \Gamma$. From (CMP κ), $\neg U_j^\kappa \psi \in \Gamma$. By IH, $\mathcal{M}^c, w \models K_j \psi$. Let $S_j := \{n \leq \kappa \mid U_j^n \psi \in \Gamma\}$. By (DEG), S_j is downward closed. Since $\neg U_j^\kappa \psi \in \Gamma$, we have $S_j \neq \{1, \dots, \kappa\}$. If $S_j = \emptyset$, then $\deg_j(w, \psi) = 0$, while $U_i^1 \psi \in \Gamma$ gives $\deg_i(w, \psi) \geq 1$ by IH. If $S_j \neq \emptyset$, let $m+1 \leq \kappa$ be the least not in S_j . Then $m \leq \kappa-1$ and $S_j = \{1, \dots, m\}$. Hence $U_j^m \psi \in \Gamma$ and $U_j^{m+1} \psi \notin \Gamma$; then $\neg U_j^{m+1} \psi \in \Gamma$, and by (CMP n), $U_i^{m+1} \psi \in \Gamma$. By IH, $\deg_j(w, \psi) = m$ and $\deg_i(w, \psi) \geq m+1$. In either case, $\deg_i(w, \psi) > \deg_j(w, \psi)$. So $\mathcal{M}^c, w \models (i \succ j)\psi$.
- Assume $\mathcal{M}^c, w \models (i \succ j)\psi$. Then $\mathcal{M}^c, w \models K_j \psi$, so $K_j \psi \in \Gamma$ by IH. If $\deg_j(w, \psi) = 0$, then $\deg_i(w, \psi) \geq 1$, hence $\mathcal{M}^c, w \models U_i^1 \psi$, and $\mathcal{M}^c, w \models \neg U_j^1 \psi$; by IH, both belong to Γ , hence $(i \succ j)\psi \in \Gamma$ by (UYC0). If $\deg_j(w, \psi) = m \geq 1$, then necessarily $m \leq \kappa-1$. Then $\mathcal{M}^c, w \models U_j^m \psi \wedge \neg U_j^{m+1} \psi$ and $\mathcal{M}^c, w \models U_i^{m+1} \psi$. By IH these formulas are in Γ , and (UYC) yields $(i \succ j)\psi \in \Gamma$.

□

Theorem 4.10 (Completeness for SCU_κ). *Fix $\kappa \geq 2$. For $\Sigma \cup \{\varphi\} \subseteq \text{CELU}^\kappa$, $\Sigma \models \varphi$ implies $\Sigma \vdash_{\text{SCU}_\kappa} \varphi$.*

Proof. Assume $\Sigma \not\vdash_{\text{SCU}_\kappa} \varphi$. Then $\Sigma \cup \{\neg\varphi\}$ is SCU_κ -consistent. Extend it to a maximal SCU_κ -consistent set $\Gamma \in \Omega$. By Lemma 4.3, choose $\mathcal{F}, \vec{f}$ with $w := \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W^c$. By Lemma 4.9, $\mathcal{M}^c, w \models \Sigma$ and $\mathcal{M}^c, w \models \neg\varphi$. So $\Sigma \not\models \varphi$. □

Before stating decidability, we isolate the effective assumption on Λ used in the finite search below. As in the finitary-model strategy of justification logic, what is needed is effective control of generated evidence, not merely decidable membership in the constant specification [28]. Say that Λ is κ -effective if the following can be done effectively. Given finite $C \subseteq \text{CELU}^\kappa$, finite $J \subseteq I$, and finite sets X_i ($i \in J$) of pairs $\langle s, \psi \rangle$, with s a graded explanation term and $\psi \in C$, one can decide, for each $i \in J$, each $\varphi \in C$, and each $1 \leq m \leq \kappa$, whether there exists a term t such that $\text{gr}(t) \geq m$ and $\langle t, \varphi \rangle \in (X_i \cup \{\langle c, \lambda \rangle \mid \lambda \in \Lambda\})_i^\infty$, where the iterated closure is as in Definition 4.2. Simple effective finite-schema choices of Λ satisfy this condition; for example, Λ may be generated by the schemata $\varphi \wedge \psi \rightarrow \varphi$ and $\varphi \wedge \psi \rightarrow \psi$.

Theorem 4.11 (Decidability of fixed-level fragments). *Fix $\kappa \geq 2$ and assume that Λ is κ -effective. Then satisfiability and validity over CELU^κ are decidable.*

Proof sketch. We prove an effective finite-model property. Given $\varphi \in \text{CELU}^\kappa$, form the finite closure C of φ under subformulas, negation, downward closure for understanding levels ($U_i^{m+1} \psi$ brings in $U_i^m \psi$), the $K_i \psi$ associated with $U_i^m \psi$, and, for each comparative subformula $(i \succ j)\psi$, the formulas $K_j \psi, U_i^m \psi, U_j^m \psi$ for $1 \leq m \leq \kappa$. Only proposition letters and agents occurring in C are relevant.

Call $X \subseteq C$ a complete C -type if it decides every formula in C and respects the Boolean connectives. Enumerate finite structures whose states are complete C -types, with equivalence relations R_i matching the K_i -formulas. For U -formulas, require class-uniformity: if XR_iY , then X and Y contain the same formulas $U_i^m \psi$ from C . Let $\delta_i(X, \psi)$ be the largest such m , or 0 if none exists. For each $\delta_i(X, \psi) > 0$, add one fresh witness $e_{i, [X]_i, \psi}$ of that grade, shared across the R_i -class $[X]_i$. Using κ -effectiveness, decide whether the least store generated from Λ and these shared witness pairs contains a pair $\langle t, \psi \rangle$ with $\text{gr}(t) \geq m$. Accept

exactly those structures in which the $U_i^m \psi$ -labels match this generated-witness condition and the truth of ψ throughout the relevant R_i -class, and in which $(i \succ j) \psi \in X$ iff $\delta_i(X, \psi) > \delta_j(X, \psi)$ and $K_j \psi \in X$.

Filtration of any model satisfying φ through C yields an accepted finite structure, and any accepted structure is realized as a finite model with its states, relations, atom valuation, and least generated explanation stores. Induction on formulas in C gives agreement between truth and membership in the corresponding type. Hence φ is satisfiable iff some accepted finite structure contains φ . Since finitely many such structures are checked and all checks are effective, satisfiability is decidable. Validity follows. $\square$

For the system SCU, the remaining key step is to build maximal consistent sets closed under all ω -introduction instances. We use a countable Lindenbaum–Henkin construction with finite failure witnesses, along the lines of [10].

Lemma 4.12 (ω -Lindenbaum extension). *Every SCU-consistent set extends to a maximal SCU-consistent set that is closed under all ω -introduction instances.*

Proof. Assume Σ is SCU-consistent. Note that **CELU** and the set of ω -introduction instances are countable. Enumerate the formulas of **CELU** as $(\theta_s)_{s \in \mathbb{N}}$. Let $\mathcal{R}$ be the set of all ω -introduction instances, and fix a listing $r : \mathbb{N} \rightarrow \mathcal{R}$ in which every instance occurs infinitely often. At stage s , write $r(s)$ as the rule with premises $\chi_s \rightarrow B_s^n$ ($n \in \mathbb{N}^+$) and conclusion $\chi_s \rightarrow B_s^\omega$, where B_s^ω is obtained from B_s^n by replacing the displayed U^n with U^ω . We define the sequence $(\Gamma_s)_{s \in \mathbb{N}}$ recursively by $\Gamma_0 := \Sigma$, and, given Γ_s , first set

$$H_s := \begin{cases} \Gamma_s \cup \{\theta_s\}, & \Gamma_s \cup \{\theta_s\} \text{ is SCU-consistent,} \\ \Gamma_s \cup \{\neg\theta_s\}, & \text{otherwise.} \end{cases}$$

If $\neg(\chi_s \rightarrow B_s^\omega) \in H_s$, let

$$\Gamma_{s+1} := H_s \cup \{\neg(\chi_s \rightarrow B_s^{n_s})\},$$

where n_s is the least positive integer such that $H_s \cup \{\neg(\chi_s \rightarrow B_s^{n_s})\}$ is SCU-consistent. Otherwise let $\Gamma_{s+1} := H_s$.

We first show that this recursion is well-defined and that every Γ_s is SCU-consistent. The basis is immediate. For the step, suppose Γ_s is consistent. At least one of $\Gamma_s \cup \{\theta_s\}$ and $\Gamma_s \cup \{\neg\theta_s\}$ is consistent; otherwise, Lemma 3.5 gives $\Gamma_s \vdash_{\text{SCU}} \theta_s$, while inconsistency of $\Gamma_s \cup \{\theta_s\}$ implies inconsistency of $\Gamma_s \cup \{\neg\neg\theta_s\}$ by tautology and Lemma 3.4, so another application of Lemma 3.5 gives $\Gamma_s \vdash_{\text{SCU}} \neg\theta_s$, contradiction. Thus H_s is consistent. If no such n_s existed in this finite-failure-witness step, then for every $n \in \mathbb{N}^+$, $H_s \cup \{\neg(\chi_s \rightarrow B_s^n)\}$ would be inconsistent; by Lemma 3.5, $H_s \vdash_{\text{SCU}} \chi_s \rightarrow B_s^n$ for all n , and the ω -introduction instance $r(s)$ would yield $H_s \vdash_{\text{SCU}} \chi_s \rightarrow B_s^\omega$, contradicting consistency of H_s .

Let $\Gamma^* := \bigcup_{s \in \mathbb{N}} \Gamma_s$. The construction has the following finite-failure property: if the conclusion $\chi \rightarrow B^\omega$ of an ω -introduction instance has its negation in Γ^* , then $\neg(\chi \rightarrow B^n) \in \Gamma^*$ for some $n \in \mathbb{N}^+$. Indeed, once $\neg(\chi \rightarrow B^\omega)$ has entered the construction, choose a later stage s at which $r(s)$ is that same instance. Such a stage exists because every instance occurs infinitely often. The witness step at stage s then adds $\neg(\chi \rightarrow B^n)$ for some n .

It remains to check that Γ^* is a maximal SCU-consistent set and is closed under all ω -introduction instances. First, Γ^* decides every formula: if $\theta = \theta_s$, then stage $s + 1$ puts either θ or $\neg\theta$ into Γ^* .

Every SCU-theorem belongs to Γ^* . Otherwise, since Γ^* decides formulas, $\neg\eta \in \Gamma^*$ for some theorem η . Then $\neg\eta \in \Gamma_s$ for some s , and the theorem proof of η together with the tautology $\eta \rightarrow (\neg\eta \rightarrow \perp)$ makes Γ_s inconsistent.

The set Γ^* is closed under (MP) in the usual way. It is also closed under ω -introduction: if all premises $\chi \rightarrow B^n$ of an instance are in Γ^* but the conclusion $\chi \rightarrow B^\omega$ is not, then $\neg(\chi \rightarrow B^\omega) \in \Gamma^*$. By

the finite-failure property, $\neg(\chi \rightarrow B^n) \in \Gamma^*$ for some n . Since the construction is increasing, some Γ_s contains both $\chi \rightarrow B^n$ and its negation, contradicting consistency of Γ_s .

Finally, Γ^* is SCU-consistent. Otherwise take a proof tree of $\perp$ from Γ^* . By induction on the tree, every node label belongs to Γ^* : assumptions by definition; axioms and (NE)-nodes because theorems belong to Γ^* ; (MP)- and ω -introduction nodes by the closure just proved; and (N)-nodes because their premise subtrees are closed and hence prove theorems. Thus $\perp \in \Gamma^*$, so $\perp \in \Gamma_s$ for some s , contradicting consistency of Γ_s . $\square$

Now let Ω_∞ be the set of maximal SCU-consistent sets closed under all ω -introduction instances.

Lemma 4.13. *Let W_∞^c be the world set obtained by rebuilding Definition 4.1 with Ω replaced by Ω_∞ , CELU^κ replaced by CELU , and every bounded index range $\{1, \dots, \kappa\}$ replaced by $\mathbb{N}^+$. For every $\Gamma \in \Omega_\infty$, there exist $\mathcal{F} = \{F_i \mid i \in I\}$ and a witness family $\vec{f}$ such that $\langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W_\infty^c$.*

Proof. Exactly as in Lemma 4.3: define $f_i^n(\psi) := \psi^n$ on $\{\psi \mid U_i^n \psi \in \Gamma\}$, and let each F_i be the least closure of the corresponding base under (App_i) , (Sum_i) , and (Int_i) . The construction is purely syntactic and independent of whether $\Gamma \in \Omega$ or $\Gamma \in \Omega_\infty$. $\square$

Let $\mathcal{M}_\infty^c = (W_\infty^c, \{R_i^c\}_{i \in I}, V^c, E^c, \text{gr}^c, \{\mathcal{G}_i^c\}_{i \in I})$ be the canonical model over Ω_∞ obtained in this way.

Lemma 4.14 (Full K_i -Existence Lemma). *If $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W_\infty^c$ and $\neg K_i \varphi \in \Gamma$, then there exists $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle \in W_\infty^c$ such that $w R_i^c w'$ and $\neg \varphi \in \Delta$.*

Proof. Let $\Delta^- := \{\psi \mid K_i \psi \in \Gamma\} \cup \{\neg \varphi\}$. We first show that Δ^- is SCU-consistent. Suppose otherwise. Let a proof tree of $\perp$ from Δ^- be given. We claim, by induction on this tree, that for every node label α , $K_i(\neg \varphi \rightarrow \alpha) \in \Gamma$.

For assumption leaves, either $\alpha = \psi$ with $K_i \psi \in \Gamma$, or $\alpha = \neg \varphi$. In the first case, normal modal reasoning gives $K_i(\neg \varphi \rightarrow \psi) \in \Gamma$ from $K_i \psi \in \Gamma$; in the second, $K_i(\neg \varphi \rightarrow \neg \varphi) \in \Gamma$ follows by necessitation. Axiom leaves, (NE)-nodes, and closed (N)-nodes are theorems, so the desired formula $K_i(\neg \varphi \rightarrow \alpha)$ also follows by necessitation. The (MP) case is standard.

For an ω -introduction node, write its premises as $\chi \rightarrow B_n$ ($n \in \mathbb{N}^+$) and its conclusion as $\chi \rightarrow B_\omega$. By the induction hypothesis, $K_i(\neg \varphi \rightarrow (\chi \rightarrow B_n)) \in \Gamma$ for every n . Equivalently, by normal modal reasoning, $K_i((\neg \varphi \wedge \chi) \rightarrow B_n) \in \Gamma$ for every n . By propositional reasoning, these give the premises $\top \rightarrow K_i((\neg \varphi \wedge \chi) \rightarrow B_n)$ of the corresponding (ωI) -instance, whose conclusion is $\top \rightarrow K_i((\neg \varphi \wedge \chi) \rightarrow B_\omega)$. Since Γ is closed under all ω -introduction instances, we obtain $\top \rightarrow K_i((\neg \varphi \wedge \chi) \rightarrow B_\omega) \in \Gamma$, hence $K_i((\neg \varphi \wedge \chi) \rightarrow B_\omega) \in \Gamma$, and therefore $K_i(\neg \varphi \rightarrow (\chi \rightarrow B_\omega)) \in \Gamma$.

At the root we obtain $K_i(\neg \varphi \rightarrow \perp) \in \Gamma$, and hence $K_i \varphi \in \Gamma$, contradicting $\neg K_i \varphi \in \Gamma$. Thus Δ^- is consistent. By Lemma 4.12, extend Δ^- to some $\Delta \in \Omega_\infty$.

For every $n \in \mathbb{N}^+$ and formula χ , $U_i^n \chi \in \Gamma$ iff $U_i^n \chi \in \Delta$: the forward direction uses (4^*) , and the backward direction uses (5^*) , exactly as in the bounded K_i -Existence Lemma. Hence we may set $g_i^n := f_i^n$ for all n . For $j \neq i$, let $g_j^n(\psi) := \psi^n$ whenever $U_j^n \psi \in \Delta$. Build $\mathcal{G} = \{G_j \mid j \in I\}$ from Δ and $\vec{g}$ by the same base-and-closure construction used in Lemma 4.13. Then $w' := \langle \Delta, \mathcal{G}, \vec{g} \rangle$ belongs to W_∞^c . Since $\{\psi \mid K_i \psi \in \Gamma\} \subseteq \Delta$ and $g_i^n = f_i^n$ for all n , we have $w R_i^c w'$. Finally, $\neg \varphi \in \Delta$ by construction. $\square$

Lemma 4.15. *Let $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W_\infty^c$, fix $i \in I$ and $n \in \mathbb{N}^+$, and assume $U_i^n \varphi \notin \Gamma$ and $\langle t, \varphi \rangle \in F_i$ for some $t \in E_n$. There exists $w' = \langle \Delta, \mathcal{G}, \vec{g} \rangle \in W_\infty^c$ such that $w R_i^c w'$ and $\langle t, \varphi \rangle \notin G_i$.*

Proof. It is the same as Lemma 4.8, with $\mathbb{N}^+$ in place of $\{1, \dots, \kappa\}$. ω -introduction is not used. $\square$

Lemma 4.16 (Truth Lemma for CELU). *For $\varphi \in \text{CELU}$ and $w = \langle \Gamma, \mathcal{F}, \vec{f} \rangle \in W_\infty^c$: $\mathcal{M}_{\infty, w}^c \models \varphi$ iff $\varphi \in \Gamma$.*

Proof. Induction on φ . The Boolean cases are standard. The K_i case uses Lemma 4.14. The finite-level U_i^n case is the same as in Lemma 4.9, using Lemma 4.15 for the refutation direction.

For $\varphi = U_i^\omega \psi$.

- If $U_i^\omega \psi \in \Gamma$, then by (DEG_ω^n) , $U_i^n \psi \in \Gamma$ for every $n \geq 1$. By IH, $\mathcal{M}_{\infty, w}^c \models U_i^n \psi$ for every $n \geq 1$, hence $\mathcal{M}_{\infty, w}^c \models U_i^\omega \psi$.
- If $\mathcal{M}_{\infty, w}^c \models U_i^\omega \psi$, then by semantics $\mathcal{M}_{\infty, w}^c \models U_i^n \psi$ for all $n \geq 1$, so by the finite-level U-case $U_i^n \psi \in \Gamma$ for all $n \geq 1$. By closure under the basic ω -introduction instance, $U_i^\omega \psi \in \Gamma$.

For $\varphi = (i \succ j)\psi$. The case $i = j$ is the same as in Lemma 4.9. Assume $i \neq j$.

- Assume $(i \succ j)\psi \in \Gamma$. By (CMP1) and $(\text{CMP}\omega)$, $K_j \psi, U_i^1 \psi, \neg U_j^\omega \psi \in \Gamma$; hence $\mathcal{M}_{\infty, w}^c \models K_j \psi$ by the preceding K-case. Let $S_j := \{n \geq 1 \mid U_j^n \psi \in \Gamma\}$. By (DEG) , S_j is downward closed and $S_j \neq \mathbb{N}^+$; otherwise the basic ω -introduction instance would contradict $\neg U_j^\omega \psi \in \Gamma$. If $S_j = \emptyset$, then $\deg_j(w, \psi) = 0$, while $U_i^1 \psi \in \Gamma$ gives $\deg_i(w, \psi) \geq 1$ by the finite-level U-case. If $S_j \neq \emptyset$, let $m+1$ be the least not in S_j . Then $m \geq 1$ and $S_j = \{1, \dots, m\}$. Hence $U_j^m \psi, \neg U_j^{m+1} \psi \in \Gamma$ and, by $(\text{CMP}n)$, $U_i^{m+1} \psi \in \Gamma$. By the finite-level U-case, $\deg_j(w, \psi) = m$ and $\deg_i(w, \psi) \geq m+1$. In either case, $\deg_i(w, \psi) > \deg_j(w, \psi)$, so $\mathcal{M}_{\infty, w}^c \models (i \succ j)\psi$.
- Assume $\mathcal{M}_{\infty, w}^c \models (i \succ j)\psi$. Then $K_j \psi \in \Gamma$ by the K-case. The degree inequality also gives $\deg_j(w, \psi) \neq \omega$. If $\deg_j(w, \psi) = 0$, then $\deg_i(w, \psi) \geq 1$ and $U_i^1 \psi$ is true. By the finite-level U-case, $U_i^1 \psi, \neg U_j^1 \psi \in \Gamma$; hence $(i \succ j)\psi \in \Gamma$ by (UYC0) . If $\deg_j(w, \psi) = m \geq 1$, then $\deg_i(w, \psi) \geq m+1$. By the finite-level U-case, $U_j^m \psi, \neg U_j^{m+1} \psi, U_i^{m+1} \psi \in \Gamma$; then (UYC) yields $(i \succ j)\psi \in \Gamma$.

□

Theorem 4.17 (Strong completeness for SCU). *For all $\Sigma \cup \{\varphi\} \subseteq \text{CELU}$, $\Sigma \models \varphi$ implies $\Sigma \vdash_{\text{SCU}} \varphi$.*

Proof. Assume $\Sigma \not\vdash_{\text{SCU}} \varphi$. By Lemmas 3.5 and 4.12, choose $\Gamma^\infty \in \Omega_\infty$ with $\Sigma \cup \{\neg\varphi\} \subseteq \Gamma^\infty$. By Lemma 4.13, choose $\mathcal{F}, \vec{f}$ such that $w := \langle \Gamma^\infty, \mathcal{F}, \vec{f} \rangle \in W_\infty^c$. Let $\mathcal{M}_\infty^c$ be the corresponding canonical model over Ω_∞ . By Lemma 4.16, $\mathcal{M}_{\infty, w}^c \models \Sigma$ and $\mathcal{M}_{\infty, w}^c \models \neg\varphi$. Thus $\Sigma \not\models \varphi$. □

5 Conclusion

We have developed a comparative epistemic logic of understanding. On the semantic side, the framework is based on agent-indexed graded explanations; on the proof-theoretic side, it separates a bounded finitary layer from a full infinitary layer. In this way, the logic captures understanding in degrees and comparative understanding with respect to the same formula at issue, while also supporting strong completeness for the intended calculi and decidability for each fixed finite-level fragment.

The framework also suggests several natural directions for further work. On the technical side, it would be worth studying richer proof-theoretic presentations for the full language. On the dynamic side, one may ask how graded and comparative understanding behaves under information change, communication, or learning. More broadly, the present logic invites closer comparison both with other comparative epistemic frameworks and with neighboring formal accounts of explanation and understanding.

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Non-finite Axiomatizability of Generalized Medvedev Logics

Han Xiao

The Tsinghua-UvA JRC for Logic
Department of Philosophy
Tsinghua University
Beijing, China
han_xiao@mail.tsinghua.edu.cn

We introduce a generalized form of Medvedev logics obtained by removing the greatest element from finite products of rooted Kripke frames with a top. We show that, before removing the top, the intermediate logic characterized by such finite products is exactly KC. Classical Medvedev logic is characterized by topless products of 2-chains, and a theorem of Maksimova, Skvortsov and Shehtman establishes that it is not finitely axiomatizable. Motivated by this result, Nick Bezhanishvili conjectured that non-finite axiomatizability extends to topless products of arbitrary finite chains and, more generally, to topless products of finite rooted frames with a top. We prove that every such generalized Medvedev logic is not finitely axiomatizable, thereby settling both conjectures in the affirmative. In 2003, van Benthem, Guram Bezhanishvili, and Gehrke introduced Cheq, the logic of chequered sets, and we show that whenever Cheq is a sublogic of a generalized Medvedev logic, the latter is not finitely axiomatizable over Cheq. Finally, we investigate the order structure of generalized Medvedev logics. We prove that there are at least countably many distinct generalized Medvedev logics and that no least such logic exists. These results extend the classical theory of Medvedev logic and clarify the behaviour of intermediate logics generated by topless product constructions.

1 Introduction

1.1 General motivation

In 1962, Medvedev introduced the logic of finite problems Med [12] as a formalization of Kolmogorov's [8] interpretation of intuitionistic logic as a calculus of problems. One of the most important results about *Medvedev logic* Med was obtained by Maksimova, Skvortsov, and Shehtman, who proved in 1979 that Med is not finitely axiomatizable [11, Corollary 5].

Beyond its role in the theory of intermediate logics, Med also arises in the study of *modal logics of forcing*. This line of research, initiated by Hamkins and Löwe [7], investigates the modal logic of the class of models of set theory with the forcing relation. In particular, determining the *modal logics of c.c.c. forcing* has become a central problem in this area. A notable connection with Medvedev logic was established by Hamkins, Leibman, and Löwe, who proved that the modal logic of ω_1 -preserving forcing is contained in S4.tBA, the least modal companion of Med [6, Theorem 36]. They further conjectured that S4.tBA also provides an upper bound for the modal logic of c.c.c. forcing.

A useful structural perspective on Medvedev logic is that it is characterized by *topless products* of the 2-chain. More precisely, the Medvedev frame $\mathbf{P}_0(n)$ is obtained from the n -fold product of the 2-chain by removing its greatest element. From this perspective, it is natural to ask whether these constructions lead to new intermediate logics with similar structural properties. In particular, one may replace the 2-chain by more general finite frames and consider the topless products obtained by removing the greatest element from their finite products. This idea was proposed by Nick Bezhanishvili, who asked whether the classical non-finite axiomatizability result for Med extends to such generalized constructions. More

specifically, he conjectured that non-finite axiomatizability should already hold for topless products of arbitrary non-singleton finite chains and, more generally, for topless products of arbitrary non-singleton finite rooted frames with a top. These considerations lead naturally to a broader class of intermediate logics that we call *generalized Medvedev logics*. Studying these logics provides a systematic way to understand how structural features of Kripke frames—such as branching and product constructions—affect the axiomatizability.

1.2 Outline of the paper

We first study the logics characterized by finite products of a non-singleton finite rooted frame with a top. We prove that the intermediate logic characterized by these products is always the weak excluded middle logic KC (Theorem 3.3).

In contrast, once the top element is removed, the resulting generalized Medvedev logics exhibit non-finite axiomatizability. As a first structural observation, we show that the classical Medvedev logic is the largest generalized Medvedev logic (Lemma 3.4). The main result of the paper proves that every generalized Medvedev logic is not finitely axiomatizable, thereby confirming conjectures of Bezhanishvili for non-trivial finite distributive lattices (Theorem 4.1) and, more generally, for arbitrary non-singleton finite rooted frames with a top (Theorem 4.8).

Beyond this central result, we further analyze the relationship between generalized Medvedev logics and other well-studied intermediate logics. In particular, we extend Fontaine’s theorem by proving that whenever Cheq is a sublogic of a generalized Medvedev logic, the latter is not finitely axiomatizable over Cheq (Theorem 5.3).

Finally, we investigate the order structure of generalized Medvedev logics, showing that there are at least countably many pairwise distinct such logics (Theorem 5.4) and that no least generalized Medvedev logic exists (Theorem 5.5).

2 Definitions

Throughout this paper, we assume familiarity with the basic theory of intermediate logics. We write IPC for *intuitionistic propositional calculus*. By an (intuitionistic Kripke) frame we mean a poset. We use the terms ‘frame’ and ‘poset’ interchangeably and all posets considered are finite.

Given a frame $\mathbf{F}$, we write $\text{Log}(\mathbf{F})$ for the propositional logic consisting of all formulas valid on the frame $\mathbf{F}$; if $L \subseteq \text{Log}(\mathbf{F})$, we call $\mathbf{F}$ an *L-frame*. If $\mathcal{C}$ is a class of frames, we write $\text{Log}(\mathcal{C})$ for the intermediate logic consisting of all formulas valid on every frame of $\mathcal{C}$; in this case, we say that $\mathcal{C}$ *characterizes* the logic.

2.1 Posets as Kripke frames

Let $\mathbf{F} = \langle F, \leq \rangle$ be a finite poset; we write $x < y$ if $x \leq y$ and $x \neq y$. An element $x \in F$ is called *maximal* if there is no $y \in F$ such that $x < y$; it is called a *top* (or *greatest element*) if everything is below it. A top is necessarily unique. We say that $\mathbf{F}$ is a frame *with a top* if it contains a top. If $x < y$, we call y an *immediate successor* of x if there is no z such that $x < z < y$. The *branching degree* of x , denoted by $b(x)$, is defined as the number of immediate successors of x .

The notation $x \uparrow$ (resp. $x \downarrow$) denotes the *upset* $\{y \in F : x \leq y\}$ (resp. the *downset* $\{y \in F : x \geq y\}$). A subframe $\mathbf{S} = \langle S, \leq \rangle$ of $\mathbf{F} = \langle F, \leq \rangle$ is called a *generated subframe* if S is upward closed in F ; we say that $\mathbf{S}$ is *generated by* x if $S = x \uparrow$. If $\mathbf{F}$ is generated by x , then $\mathbf{F}$ is called *rooted* and x the *root* of $\mathbf{F}$. For $n \geq 1$,

let $[n] := \{1, 2, \dots, n\}$. The n -chain is the frame $\mathbf{C}(n) := \langle [n], \leq \rangle$, where $\leq$ is the usual order on $[n]$. The *depth* of a frame $\mathbf{F}$ is the length of the longest chain in $\mathbf{F}$. For $x \in F$, the *depth of x* , denoted by $d(x)$, is the depth of the subframe generated by x .

Let $\mathbf{F} = \langle F, \leq_0 \rangle$ and $\mathbf{G} = \langle G, \leq_1 \rangle$ be posets. Their *product* is the poset $\mathbf{F} \times \mathbf{G} := \langle F \times G, \leq \rangle$, where the order $\leq$ is defined coordinatewise by $(x, y) \leq (x', y')$ if and only if $x \leq_0 x'$ and $y \leq_1 y'$. For a poset $\mathbf{F}$, we write $\mathbf{F}^2 := \mathbf{F} \times \mathbf{F}$ and define inductively $\mathbf{F}^n := \mathbf{F}^{n-1} \times \mathbf{F}$. Now let $\mathbf{F}$ be a finite frame with a top. Its *topless frame* $\mathbf{F}^-$ is the frame obtained from $\mathbf{F}$ by removing its top element. For $n \geq 1$, the n -fold *topless product* of $\mathbf{F}$ is $\mathbf{F}^{n-} := (\mathbf{F}^n)^-$, that is, the topless frame of the product $\mathbf{F}^n$.

We write $\mathbf{F} \oplus \mathbf{G}$ for their *linear sum*, obtained by placing all elements of $\mathbf{G}$ strictly above all elements of $\mathbf{F}$. We define the iterated finite sum by recursion via $\mathbf{F} \times 1 := \mathbf{F}$ and $\mathbf{F} \times (n+1) := (\mathbf{F} \times n) \oplus \mathbf{F}$. The *vertical sum* of $\mathbf{F}$ and $\mathbf{G}$ is obtained from the linear sum of $\mathbf{F}$ and $\mathbf{G}$ by identifying the top of $\mathbf{F}$ with the root of $\mathbf{G}$ (provided they exist).

2.2 Morphisms

If $\mathbf{F}$ and $\mathbf{G}$ are posets and $f : F \rightarrow G$ is a map, we call f a *p-morphism* if for all $x, y \in F$, if $x \leq y$, then $f(x) \leq f(y)$ and if $f(x) \leq u$, then there is some $y \geq x$ such that $f(y) = u$. If f is onto, then we call $\mathbf{G}$ a *p-morphic image* of $\mathbf{F}$.

Theorem 2.1 (Folklore). *If $\mathbf{G}$ is a p-morphic image of $\mathbf{F}$, then $\mathbf{F} \models \varphi$ implies $\mathbf{G} \models \varphi$, for any formula φ .*

Theorem 2.2 (Jankov-de Jongh Theorem). *Given a finite rooted frame $\mathbf{F}$, there exists a formula $\chi(\mathbf{F})$ such that for every frame $\mathbf{G}$,*

$$\mathbf{G} \models \chi(\mathbf{F}) \quad \text{iff} \quad \mathbf{F} \text{ is a p-morphic image of a generated subframe of } \mathbf{G}.$$

The formula $\chi(\mathbf{F})$ is called the Jankov-de Jongh formula of $\mathbf{F}$.

2.3 Medvedev logic

In 1932, Kolmogorov [8] suggested a constructive interpretation of intuitionistic logic as a calculus of problems. To make precise the description proposed by Kolmogorov, Medvedev established the foundational framework for the logic of finite problems in [12]. We recall the definition of the *Medvedev frame* and the corresponding *Medvedev logic*.

Definition 2.3. For $n \geq 1$, the *Medvedev frame on n atoms* $\mathbf{P}_0(n)$ is defined by $\mathbf{P}_0(n) := \langle \{X \subseteq [n] : X \neq \emptyset\}, \supseteq \rangle$. The *Medvedev logic* Med is the intermediate logic of all Medvedev frames, that is, $\text{Med} := \text{Log}(\{\mathbf{P}_0(n) : n \in \omega\})$.

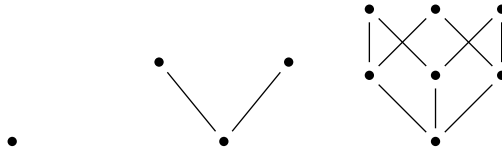


Figure 1: Medvedev frames.

Lemma 2.4 (Maksimova, Skvortsov, & Shehtman; cf. [11, Lemma 4]). *If $\mathbf{F}$ is a finite rooted frame with a top, then $\mathbf{F}$ is a Med-frame.*

Inheriting such a tradition on the frame for logic of finite problems, the order in our *finite Boolean algebra* here is reverse inclusion $\supseteq$.

Definition 2.5. For $n \geq 1$, the *Boolean algebra on n generators* $\mathbf{P}(n)$ is defined by $\mathbf{P}(n) := (\mathcal{P}([n]), \supseteq)$, where $\mathcal{P}([n])$ denotes the powerset of $[n]$.

2.4 The logic Cheq

The modal logic of chequered subsets was introduced by van Benthem, Guram Bezhanishvili, and Gehrke in 2003, who asked whether it is finitely axiomatizable [1, in particular, p. 343]. This logic plays an important role in the study of modal logics of various topological spaces. Litak [9] later denoted its intermediate companion by Cheq. Let $\mathbf{V}$ be the *two-fork*, that is, the poset consisting of three elements $\{o, p, q\}$ such that o is the root and p and q are incomparable maximal elements. Define recursively $\mathbf{V}_1 := \mathbf{V}$ and $\mathbf{V}_{n+1} := \mathbf{V}_n \times \mathbf{V}$, where $\times$ denotes the usual product of posets. The logic Cheq is the intermediate logic characterized by the class $\{\mathbf{V}_n : n \in \omega\}$.

3 Products and generalized Medvedev logics

Observe that $\mathbf{P}(n)$ is isomorphic to $\mathbf{C}(2)^n$, the n -fold product of the 2-chain, and that the Medvedev frame $\mathbf{P}_0(n)$ is obtained from $\mathbf{C}(2)^n$ by removing its greatest element. Thus, Medvedev logic can be viewed as the logic characterized by the family of frames obtained as topless products of the 2-chain.

This structural description suggests a natural generalization. Instead of taking products of the 2-chain, one may consider finite products of more general frames and remove their top elements. In particular, one may replace the 2-chain by an arbitrary finite chain, or more generally by an arbitrary non-singleton finite rooted frame with a top. Motivated by this observation, Bezhanishvili proposed two conjectures concerning the logics of such topless products. These conjectures predict that the non-finite axiomatizability known for Medvedev logic should persist for these generalized constructions.

In the remainder of this section we introduce the corresponding class of logics, which we call generalized Medvedev logics, and investigate their properties.

Proposition 3.1. *The Medvedev frame $\mathbf{P}_0(n)$ is isomorphic to $\mathbf{C}(2)^{n-}$.*

Proof. Define a map $f : \mathbf{P}_0(n) \rightarrow \mathbf{C}(2)^{n-}$ by sending each non-empty subset $X \subseteq [n]$ to the tuple $f(X) = (x_1, \dots, x_n)$, where $x_i = 2$ if $i \notin X$ and $x_i = 1$ otherwise. This map is clearly bijective. Moreover, $X \supseteq Y$ if and only if $f(X) \leq f(Y)$, so f is an isomorphism of posets. $\square$

Definition 3.2. Let $\mathbf{F}$ be a non-singleton finite rooted frame with a top. We define $\mathbf{P}_{\mathbf{F}} := \text{Log}(\{\mathbf{F}^n : n \in \omega\})$ and $\text{TLP}_{\mathbf{F}} := \text{Log}(\{\mathbf{F}^{n-} : n \in \omega\})$. The logic $\text{TLP}_{\mathbf{F}}$ is called the *generalized Medvedev logic over $\mathbf{F}$* .

Throughout the rest of this paper, unless otherwise stated, $\mathbf{F}$ is assumed to be a non-singleton finite rooted frame with a top. The following two results, Theorem 3.3 and Lemma 3.4, provide an initial description of the logics $\mathbf{P}_{\mathbf{F}}$ and $\text{TLP}_{\mathbf{F}}$. We begin with the case of products. The logic characterized by the finite products of $\mathbf{F}$ turns out to be the weak excluded middle logic $\text{KC} = \text{IPC} + (\neg p \vee \neg \neg p)$, also known as *Jankov logic*. Recall that KC is the logic of all finite rooted posets that have a top [2, Theorem 2.13] and, in particular, is characterized by the products of the 2-chain. The following theorem shows that this result persists for finite products of an arbitrary finite rooted frame with a top. Theorem 3.3 was conjectured by Bezhanishvili and the proof presented below is due to the author of the present paper.

Theorem 3.3. $\mathbf{P}_{\mathbf{F}} = \text{KC}$.

Proof. Since KC is the intermediate logic of all finite rooted frames with a top and each $\mathbf{F}^n$ has a top, we immediately obtain $P_{\mathbf{F}} \supseteq KC$. Let t denote the top of $\mathbf{F}$. Define a map $f_1 : \mathbf{F} \rightarrow \mathbf{C}(2)$ by

$$f_1(x) = \begin{cases} 2, & \text{if } x = t, \\ 1, & \text{otherwise.} \end{cases}$$

A straightforward verification shows that f_1 is a p -morphism. For each $n \geq 1$, define $f_n : \mathbf{F}^n \rightarrow \mathbf{C}(2)^n$ by $f_n((x_1, x_2, \dots, x_n)) = (f_1(x_1), f_1(x_2), \dots, f_1(x_n))$. It is easy to check that f_n is a p -morphism from $\mathbf{F}^n$ onto $\mathbf{C}(2)^n$. Consequently, $\text{Log}(\mathbf{F}^n) \subseteq \text{Log}(\mathbf{C}(2)^n)$ and thus $P_{\mathbf{F}} = \bigcap_{n \in \omega} \text{Log}(\mathbf{F}^n) \subseteq \bigcap_{n \in \omega} \text{Log}(\mathbf{C}(2)^n)$. Since $KC = \bigcap_{n \in \omega} \text{Log}(\mathbf{C}(2)^n)$ (see, e.g., [11, Corollary 1]), we obtain $P_{\mathbf{F}} \subseteq KC$. Therefore, $P_{\mathbf{F}} = KC$. $\square$

From the perspective of topless products $\mathbf{F}^{n-}$, the classical Medvedev logic is the largest generalized Medvedev logic with respect to inclusion.

Lemma 3.4. $\text{TLP}_{\mathbf{F}} \subseteq \text{Med}$.

Proof. We use the maps f_n constructed in the proof of Theorem 3.3. Recall that f_n is a p -morphism from $\mathbf{F}^n$ onto $\mathbf{C}(2)^n$ and the Medvedev frame $\mathbf{P}_0(n)$ is the topless frame of $\mathbf{C}(2)^n$. Define $f_n^0 : \mathbf{F}^{n-} \rightarrow \mathbf{P}_0(n)$ by restriction $f_n^0 := f_n \upharpoonright \mathbf{F}^{n-}$. The only element of $\mathbf{F}^n$ mapped to the top of $\mathbf{C}(2)^n$ by f_n is the top of $\mathbf{F}^n$. Hence, the restriction f_n^0 remains a p -morphism. Thus, $\text{Log}(\mathbf{F}^{n-}) \subseteq \text{Log}(\mathbf{P}_0(n))$. Taking intersections over all n , we obtain $\text{TLP}_{\mathbf{F}} = \bigcap_{n \in \omega} \text{Log}(\mathbf{F}^{n-}) \subseteq \bigcap_{n \in \omega} \text{Log}(\mathbf{P}_0(n)) = \text{Med}$. $\square$

This paper resolves two conjectures concerning the finite axiomatizability of generalized Medvedev logics, which were suggested by Nick Bezhanishvili in personal communication.

Conjecture 3.5 (Bezhanishvili's First Conjecture). *If $\mathbf{F} = \mathbf{C}(n)$ is a finite chain with $n \geq 2$, then $\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable.*

Conjecture 3.6 (Bezhanishvili's Second Conjecture). *$\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable.*

These conjectures extend the classical result of Maksimova, Skvortsov, and Shehtman [11, Corollary 5], which shows that Med is not finitely axiomatizable.

4 Non-finite axiomatization of $\text{TLP}_{\mathbf{F}}$

This section contains the main technical contribution of the paper, namely the proof that the generalized Medvedev logic $\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable. In the general case, the proof relies on a structural analysis of p -morphisms from generated subframes of the topless products $\mathbf{F}^{m-}$. Lemma 4.6 provides an upper bound on branching degrees, which allows us to construct, for each frame $\mathbf{F}$, appropriate Chinese lantern frames. Combining this construction with classical results of Maksimova, Skvortsov, and Shehtman on Chinese lantern frames, we obtain the non-finite axiomatizability of $\text{TLP}_{\mathbf{F}}$.

4.1 The distributive lattice case

We begin with the case of (non-trivial) finite distributive lattices, the following theorem shows that, the distributive lattice case is essentially the Medvedev logic. This follows from a recent characterization of Medvedev logic due to Grilletti [5].

Theorem 4.1. *Let $\mathbf{F}$ be a non-trivial finite distributive lattice. Then $\text{TLP}_{\mathbf{F}} = \text{Med}$.*

Proof. By Lemma 3.4, since Med is the largest generalized Medvedev logic, we already have $\text{TLP}_{\mathbf{F}} \subseteq \text{Med}$. Conversely, since $\mathbf{F}$ is a finite distributive lattice, each finite product $\mathbf{F}^n$ is again a finite distributive lattice. Hence $\mathbf{F}^{n-}$ is a frame obtained from a finite distributive lattice by removing its top element. By Grilletti's characterization of Medvedev logic as the logic of all such frames [5], we have $\text{Med} \subseteq \text{Log}(\mathbf{F}^{n-})$ for every n . Taking the intersection over all n , we obtain $\text{Med} \subseteq \bigcap_{n \in \omega} \text{Log}(\mathbf{F}^{n-}) = \text{TLP}_{\mathbf{F}}$. Therefore $\text{TLP}_{\mathbf{F}} = \text{Med}$. $\square$

Corollary 4.2. *For every finite chain $C(n)$ with $n \geq 2$, $\text{TLP}_{C(n)} = \text{Med}$. Consequently, $\text{TLP}_{C(n)}$ is not finitely axiomatizable.*

Proof. Every finite chain is a finite distributive lattice, so the first claim follows from the Theorem 4.1. The non-finite axiomatizability follows from the classical result of Maksimova, Skvortsov and Shehtman that Med is not finitely axiomatizable. $\square$

4.2 The general case

Given integers s and n , the *Chinese lantern* $\mathbf{L}_{s,n}$ is the frame shown on the left in Figure 2. Let $\mathbf{T}_n$ be the poset consisting of a root and exactly n immediate successors, and let $\mathbf{D}$ be the poset consisting of two incomparable points. Then $\mathbf{L}_{s,n} := \mathbf{T}_n \oplus (\mathbf{D} \times (s+1))$. By removing the point $(m, 1)$ from $\mathbf{L}_{s,n}$, we obtain the *Chinese lantern* $\mathbf{L}'_{s,n,m}$ shown on the right in Figure 2.

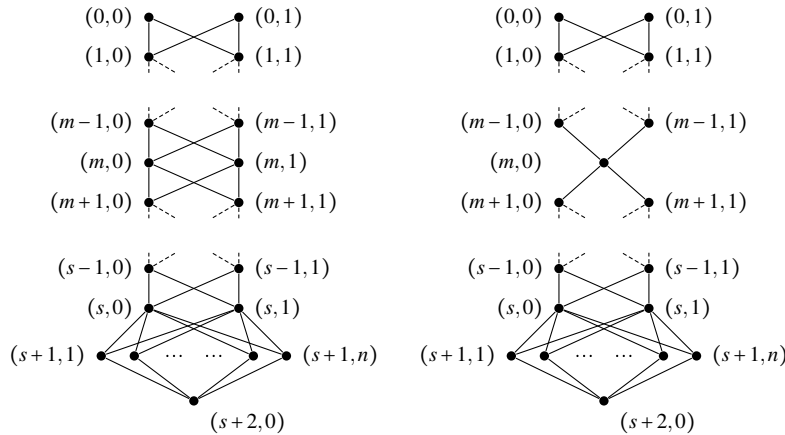


Figure 2: The Chinese lanterns $\mathbf{L}_{s,n}$ (left) and $\mathbf{L}'_{s,n,m}$ (right).

We define the n -th *suspension* of a poset $\mathbf{T}$ inductively by $\mathbf{T}^{(0)} := \mathbf{T}$ and $\mathbf{T}^{(n)} := \mathbf{T} \oplus (\mathbf{D} \times n)$. The following lemma appears as [11, Corollary 4]; a proof can also be found in [4, Claim 9].

Lemma 4.3 (Maksimova, Skvortsov, & Shehtman). *Let $\mathbf{F}$ be a finite rooted frame with a top. Then for every $n \geq 0$, the n -th suspension $\mathbf{F}^{(n)}$ is a p -morphic image of some Medvedev frame.*

Lemma 4.3 can be used to establish a connection between Chinese lantern frames and $\text{TLP}_{\mathbf{F}}$ -frames.

Lemma 4.4. *Each $\mathbf{L}'_{s,n,m}$ is a $\text{TLP}_{\mathbf{F}}$ -frame.*

Proof. The downset $(m,0) \downarrow$ in $\mathbf{L}'_{s,n,m}$ is a finite rooted frame with a top $(m,0)$. Moreover, $\mathbf{L}'_{s,n,m}$ is obtained as the m th suspension of $(m,0) \downarrow$. In other words, $\mathbf{L}'_{s,n,m} = ((m,0) \downarrow)^{(m)}$. By Lemma 4.3, $\mathbf{L}'_{s,n,m}$

is a p -morphic image of some Medvedev frame. Hence $\mathbf{L}'_{s,n,m}$ is a Med-frame. Since, by Lemma 3.4, every generalized Medvedev logic $\text{TLP}_{\mathbf{F}}$ is contained in Med, it follows that $\mathbf{L}'_{s,n,m}$ is a $\text{TLP}_{\mathbf{F}}$ -frame. $\square$

The following propositions correspond to [11, Lemma 9]; for a proof, see [4, Proposition 11].

Proposition 4.5 (Maksimova, Skvortsov, & Shehtman). *For any formula φ with s variables, there is an $m \leq s$ such that $\mathbf{L}_{s,n} \models \varphi$ if and only if $\mathbf{L}'_{s,n,m} \models \varphi$.*

Therefore, Chinese lantern frames provide an effective tool for detecting gaps between logics, in particular for determining whether they differ with respect to finite axiomatizability. We return to this idea in §5.1, where it is developed further in the context of Cheq.

To prove Conjecture 3.6, we construct, for each finite rooted frame $\mathbf{F}$ with a top, a Chinese lantern frame that is not a $\text{TLP}_{\mathbf{F}}$ -frame. The lemma below provides the crucial construction. Beyond its role in the proof of the conjecture, it will later be used in our structural analysis of generalized Medvedev logics.

Lemma 4.6. *Let $\mathbf{A}$ be a finite rooted frame such that no point of $\mathbf{A}$ has exactly one immediate successor. Let $\mathbf{F}$ be a non-singleton finite rooted frame with a top, and let $b := \max\{b(u) : u \in \mathbf{F}\}$. If $\mathbf{A}$ is a p -morphic image of a generated subframe of some $\mathbf{F}^{m-}$, then $b(x) < b \cdot 2^{d(x)}$ for every $x \in \mathbf{A}$.*

Proof. Let $f : \mathbf{S} \rightarrow \mathbf{A}$ be a p -morphism, where $\mathbf{S}$ is a generated subframe of $\mathbf{F}^{m-}$. For $u = (u_1, \dots, u_m) \in \mathbf{S}$, let t be the top of $\mathbf{F}$ and define $\#(u) := |\{i : u_i \neq t\}|$. We prove by induction on $d(x)$ that for every $x \in \mathbf{A}$ there exists $u_x \in \mathbf{S}$ such that $f(u_x) = x$ and $\#(u_x) < 2^{d(x)}$.

If $d(x) = 1$, then x is maximal. Choose u with $f(u) = x$ and extend it to a maximal element $u_x \geq u$ in $\mathbf{S}$. Then $f(u_x) = x$ and $\#(u_x) = 1 < 2$. Assume the claim holds for depth d . Let $d(x) = d + 1$. Choose u with $f(u) = x$. The frame $u\uparrow$ in $\mathbf{S}$ is also a generated subframe of $\mathbf{F}^{m-}$. The restriction of f is a p -morphism from $u\uparrow$ to the finite rooted frame $x\uparrow$. By induction, for any proper successor y of x , there is a $u_y \in u\uparrow$ such that $f(u_y) = y$ and $\#(u_y) < 2^{d(y)}$.

Since no point of $\mathbf{A}$ has exactly one immediate successor, x has at least two distinct immediate successors y, z . Thus there exist $u_y, u_z \in u\uparrow$ with $f(u_y) = y$, $f(u_z) = z$ and $\#(u_y) < 2^{d(y)}$, $\#(u_z) < 2^{d(z)}$. Write $u = (u_1, \dots, u_m)$, $u_y = (y_1, \dots, y_m)$, and $u_z = (z_1, \dots, z_m)$. For each i , let $x_i = t$ if $y_i = z_i = t$, and let $x_i = u_i$ otherwise. Put $u_x = (x_1, \dots, x_m)$. Hence $u_x \in u\uparrow$ and $u_x < u_y$, $u_x < u_z$. Because f is a p -morphism, $f(u_x) \in x\uparrow$ and $f(u_x) \leq f(u_y) = y$, $f(u_x) \leq f(u_z) = z$. But y and z are distinct immediate successors of x , thus $f(u_x) = x$.

Observe that for each $1 \leq i \leq m$, $x_i \neq t$ if and only if $y_i \neq t$ or $z_i \neq t$. Hence, $\{i : x_i \neq t\} = \{i : y_i \neq t\} \cup \{i : z_i \neq t\}$. Therefore, $\#(u_x) = |\{i : x_i \neq t\}| \leq |\{i : y_i \neq t\}| + |\{i : z_i \neq t\}| = \#(u_y) + \#(u_z)$. By the induction hypothesis, $\#(u_y) < 2^{d(y)}$ and $\#(u_z) < 2^{d(z)}$. Since $d(y) \leq d$ and $d(z) \leq d$, we obtain $\#(u_x) < 2^{d(y)} + 2^{d(z)} \leq 2^{d+1} = 2^{d(x)}$. This completes the induction.

Since $b = \max\{b(u) : u \in \mathbf{F}\}$, we have $b(x_i) \leq b$ whenever $x_i \neq t$, and $b(x_i) = 0$ if $x_i = t$. The immediate successors of $u_x = (x_1, \dots, x_m)$ in $\mathbf{F}^{m-}$ are obtained by replacing exactly one coordinate x_i by an immediate successor of x_i in $\mathbf{F}$, except possibly when the resulting tuple is the deleted top element $(t, \dots, t)$. Therefore, $b(u_x) \leq \sum_{i=1}^m b(x_i) \leq b \cdot \#(u_x) < b \cdot 2^{d(x)}$.

Fix $x \in \mathbf{A}$ and choose u_x maximal among preimages of x satisfying $\#(u_x) < 2^{d(x)}$. Such a choice is possible since $\mathbf{S}$ is finite. Let y be an immediate successor of x . Because $f(u_x) = x \leq y$ and f is a p -morphism, there exists $u_y \in \mathbf{S}$ with $u_x \leq u_y$ and $f(u_y) = y$. Since $u_x < u_y$, there exists an immediate successor u' of u_x with $u_x \leq u' \leq u_y$ and $x = f(u_x) \leq f(u') \leq f(u_y) = y$. If $f(u') = x$, then u' would satisfy $f(u') = x$ and $\#(u') \leq \#(u_x) < 2^{d(x)}$, contradicting the maximality of u_x . Hence $f(u') = y$. Thus every immediate successor y of x lifts to an immediate successor u' of u_x . Consequently, $b(x) \leq b(u_x)$. Together with the bound $b(u_x) < b \cdot 2^{d(x)}$, this yields $b(x) < b \cdot 2^{d(x)}$ as required. $\square$

Corollary 4.7. $\mathbf{L}_{s,b \cdot 2^{s+3}}$ is not a $\text{TLP}_{\mathbf{F}}$ -frame, where $b = \max\{b(u) : u \in \mathbf{F}\}$.

Proof. Let x be the root of $\mathbf{L}_{s,b \cdot 2^{s+3}}$. Then $d(x) = s + 3$ and $b(x) = b \cdot 2^{s+3}$. By Lemma 4.6, $\mathbf{L}_{s,b \cdot 2^{s+3}}$ is not a p -morphic image of any generated subframe of $\mathbf{F}^{m-}$.

Let χ be the Jankov-de Jongh formula of $\mathbf{L}_{s,b \cdot 2^{s+3}}$. For every m , $\mathbf{F}^{m-} \not\models \chi$ if and only if $\mathbf{L}_{s,b \cdot 2^{s+3}}$ is a p -morphic image of a generated subframe of $\mathbf{F}^{m-}$. Since the latter is impossible, we conclude that $\mathbf{F}^{m-} \models \chi$ for all m . Thus $\chi \in \bigcap_{m \in \omega} \text{Log}(\mathbf{F}^{m-}) = \text{TLP}_{\mathbf{F}}$. On the other hand, $\mathbf{L}_{s,b \cdot 2^{s+3}} \not\models \chi$. Therefore, $\mathbf{L}_{s,b \cdot 2^{s+3}}$ is not a $\text{TLP}_{\mathbf{F}}$ -frame. $\square$

Theorem 4.8. $\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable.

Proof. Suppose, towards a contradiction, that $\text{TLP}_{\mathbf{F}}$ is finitely axiomatizable. Since a finite set of formulas $\varphi_1, \dots, \varphi_n$ axiomatizes the same extension as the single formula $\varphi_1 \wedge \dots \wedge \varphi_n$, we may assume that $\text{TLP}_{\mathbf{F}}$ is axiomatized by a single formula $\varphi(p_1, \dots, p_s)$, where s is the number of propositional variables occurring in it. By Proposition 4.5, there exists $m \leq s$ such that $\mathbf{L}_{s,b \cdot 2^{s+3}} \models \varphi$ if and only if $\mathbf{L}'_{s,b \cdot 2^{s+3},m} \models \varphi$.

By Corollary 4.7, $\mathbf{L}_{s,b \cdot 2^{s+3}} \not\models \varphi$, whereas by Lemma 4.4, $\mathbf{L}'_{s,b \cdot 2^{s+3},m} \models \varphi$. This contradiction shows that $\text{TLP}_{\mathbf{F}}$ cannot be finitely axiomatized with s variables. Since s was arbitrary, the generalized Medvedev logics $\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable. $\square$

5 Further results on generalized Medvedev logics

This section collects two further consequences of the main results. First, we strengthen Fontaine's theorem on Med by showing that, whenever $\text{Cheq} \not\subseteq \text{TLP}_{\mathbf{F}}$, the logic $\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable even over Cheq. Second, we turn to the global picture of generalized Medvedev logics under inclusion. We show that there are countably many pairwise distinct generalized Medvedev logics and no least generalized Medvedev logic exists.

5.1 $\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable over Cheq

In 2006, Fontaine showed that Med is not finitely axiomatizable over Cheq [3, Theorem 9]. We extend this result to generalized Medvedev logics. The key observation is that all Chinese lantern frames are Cheq-frames. Let $\mathbf{W}$ denote the *bowtie*, the poset consisting of four elements: two incomparable minimal elements and two incomparable maximal elements, each maximal element lying above each minimal one; see Figure 3. Our result relies on the following Bowtie Lemma, whose proof can be found in [13, Lemma 6.2.3].

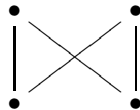


Figure 3: The bowtie $\mathbf{W}$.

Lemma 5.1 (Bowtie Lemma). *If $\mathbf{F}$ is any finite rooted frame, then $\mathbf{F} \oplus \mathbf{W}$ is a p -morphic image of some $\mathbf{V}_n$, that is, the n -fold product of the two-fork.*

The Bowtie Lemma clarifies structural properties of Cheq-frames and therefore provides a useful tool for the analysis of Cheq-frame structures.

Corollary 5.2. *The Chinese lantern frames $\mathbf{L}_{s,n}$ and $\mathbf{L}'_{s,n,m}$ are Cheq-frames.*

Proof. Every Chinese lantern $\mathbf{L}_{s,n}$ is a linear sum of a finite rooted frame and a bowtie. Hence, by the Bowtie Lemma 5.1, $\mathbf{L}_{s,n}$ is a Cheq-frame. For $m \geq 0$, the structure of $(m,0)\uparrow$ in $\mathbf{L}'_{s,n,m}$ is determined by m . If $m \geq 2$, it is a linear sum of a finite rooted frame and a bowtie. If $m = 1$, it is the frame $\mathbf{V}$. If $m = 0$, it is a singleton. In each case, $(m,0)\uparrow$ is a p -morphic image of some V_{n_1} by the Bowtie Lemma 5.1. Because $(m,0)\downarrow$ is a finite rooted frame with a top, it is also a p -morphic image of some Medvedev frame and thus a p -morphic image of some V_{n_2} . By Claim 20 in [4], the vertical sum of the p -morphic images of V_{n_1} and V_{n_2} is a Cheq-frame. As $\mathbf{L}'_{s,n,m}$ is precisely the vertical sum of $(m,0)\downarrow$ and $(m,0)\uparrow$, it follows that $\mathbf{L}'_{s,n,m}$ is a Cheq-frame. $\square$

Theorem 5.3. *If $\text{Cheq} \not\subseteq \text{TLP}_{\mathbf{F}}$, then $\text{TLP}_{\mathbf{F}}$ is not finitely axiomatizable over Cheq.*

Proof. Assume towards a contradiction that $\text{TLP}_{\mathbf{F}}$ is finitely axiomatizable over Cheq. We may then assume that there is a single formula φ with s propositional variables such that $\text{TLP}_{\mathbf{F}} = \text{Cheq} + \varphi$. By Proposition 4.5, we find $m \leq s$ such that $\mathbf{L}_{s,b \cdot 2^{s+3}} \models \varphi$ if and only if $\mathbf{L}'_{s,b \cdot 2^{s+3},m} \models \varphi$. By Corollary 5.2, both $\mathbf{L}_{s,b \cdot 2^{s+3}}$ and $\mathbf{L}'_{s,b \cdot 2^{s+3},m}$ are Cheq-frames. However, as in the proof of Theorem 4.8, these two Chinese lanterns disagree on the validity of φ . Contradiction! $\square$

5.2 The order structure of generalized Medvedev logics

Since no generalized Medvedev logic is finitely axiomatizable, it follows that many well-known logics cannot be generalized Medvedev logics. Together with the facts that Med is the greatest generalized Medvedev logic and that $\text{Med} \subseteq \text{KC}$, this yields an approximate description of the “geographic coordinates” of generalized Medvedev logics in the lattice of intermediate logics.

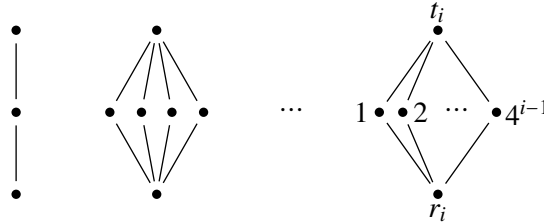


Figure 4: The members of the family $\{\mathbf{N}(i) : i \in \omega\}$.

Theorem 5.4. *There are at least countably many pairwise distinct generalized Medvedev logics.*

Proof. We construct a family $\{\mathbf{N}(i) : i \in \omega\}$ of finite rooted frames with its top. We begin by setting $\mathbf{N}(1) = \mathbf{C}(3)$. By Theorem 4.1, we have $\text{TLP}_{\mathbf{N}(1)} = \text{Med}$. Next, define $\mathbf{N}(2) = \langle N_2, R_2 \rangle$ by $N_2 = \{r_2, 1, 2, 3, 4, t_2\}$ and $R_2 = \{(r_2, x) : x \in N_2\} \cup \{(x, t_2) : x \in N_2\} \cup \{(x, x) : x \in N_2\}$. Thus r_2 is the root of $\mathbf{N}(2)^-$, its depth is 2, and its branching degree is 4. Since $b(r_2) = 2^{d(r_2)}$, by Lemma 4.6, $\mathbf{N}(2)^-$ is not a p -morphic image of any generated subframe of $\mathbf{N}(1)^{m-}$ for any m . By Jankov-de Jongh Theorem, $\chi(\mathbf{N}(2)^-) \in \text{TLP}_{\mathbf{N}(1)}$ and $\chi(\mathbf{N}(2)^-) \notin \text{Log}(\mathbf{N}(2)^-)$. Since $\text{TLP}_{\mathbf{N}(1)} = \text{Med}$ is the greatest generalized Medvedev logic, $\text{TLP}_{\mathbf{N}(2)}$ is strictly contained in $\text{TLP}_{\mathbf{N}(1)}$.

We now define, for $i \geq 1$, $\mathbf{N}(i) = \langle N_i, R_i \rangle$ by $N_i = \{r_i, 1, 2, \dots, 4^{i-1}, t_i\}$ and $R_i = \{(r_i, x) : x \in N_i\} \cup \{(x, t_i) : x \in N_i\} \cup \{(x, x) : x \in N_i\}$. For $1 \leq i < j$, there is a natural p -morphism $f_{ji} : \mathbf{N}(j) \rightarrow \mathbf{N}(i)$ given by

$$f_{ji}(x) = \begin{cases} 4^{i-1}, & \text{if } x \in \{4^{i-1} + 1, \dots, 4^{j-1}\}, \\ x, & \text{otherwise.} \end{cases}$$

In addition, only t_i of $\mathbf{N}(i)$ can be sent to t_j of $\mathbf{N}(j)$ via f_{ji} , thus for $1 \leq i < j$, $\text{TLP}_{\mathbf{N}(j)} \subseteq \text{TLP}_{\mathbf{N}(i)}$. Moreover, the maximal branching degree of $\mathbf{N}(i)$ is 4^{i-1} . By Lemma 4.6, $\mathbf{N}(j)^-$ is not a $\text{TLP}_{\mathbf{N}(i)}$ -frame when $i < j$, since in $\mathbf{N}(j)^-$ the root has depth 2 and branching degree $4^{j-1} \geq 2^{d(r_j)} \cdot 4^{i-1}$. Therefore, $\text{TLP}_{\mathbf{N}(j)} \not\subseteq \text{TLP}_{\mathbf{N}(i)}$ whenever $1 \leq i < j$. We obtain a strictly descending chain $\text{Med} = \text{TLP}_{\mathbf{N}(1)} \not\subseteq \text{TLP}_{\mathbf{N}(2)} \not\subseteq \dots \not\subseteq \text{TLP}_{\mathbf{N}(i)} \not\subseteq \dots$, which completes the proof. $\square$

Theorem 5.5. *There is no least generalized Medvedev logic.*

Proof. Assume towards a contradiction that there exists a least generalized Medvedev logic B . Then there is a finite rooted frame $\mathbf{F}_0$ with a top such that $B = \text{TLP}_{\mathbf{F}_0}$ and $B \subseteq \text{TLP}_{\mathbf{F}}$ for every finite rooted frame $\mathbf{F}$ with a top. Let $b_0 = \max\{b(u) : u \in \mathbf{F}_0\}$. We construct a finite frame $\mathbf{M} = \langle M, R \rangle$ by $M = \{r_0, 1, 2, \dots, 4b_0, t_0\}$ and $R = \{(r_0, x) : x \in M\} \cup \{(x, t_0) : x \in M\} \cup \{(x, x) : x \in M\}$. In $\mathbf{M}^-$, the root r_0 has depth $d(r_0) = 2$ and branching degree $4b_0$. Since $b(r_0) = 4b_0 = b_0 \cdot 2^{d(r_0)}$, Lemma 4.6 implies that $\mathbf{M}^-$ is not a p -morphic image of any generated subframe of any $\mathbf{F}_0^{m-}$. Thus $\chi(\mathbf{M}^-) \in \text{TLP}_{\mathbf{F}_0} = B$. On the other hand, $B \subseteq \text{TLP}_{\mathbf{M}} = \bigcap_{m \in \omega} \text{Log}(\mathbf{M}^{m-}) \subseteq \text{Log}(\mathbf{M}^-)$, so $\chi(\mathbf{M}^-) \in \text{Log}(\mathbf{M}^-)$, a contradiction. $\square$

Corollary 5.6. *The intersection of all generalized Medvedev logics is no longer a generalized Medvedev logic. IPC is not a generalized Medvedev logic.*

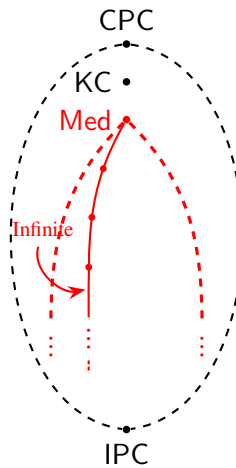


Figure 5: A schematic “geography” of generalized Medvedev logics. The red region indicates generalized Medvedev logics, all of which are not finitely axiomatizable.

6 Open problems and further directions

Several directions for further research naturally arise from the present work. We already show that the class of generalized Medvedev logics has a non-trivial order structure: it contains a countable strictly descending chain, but has no least element. A related question is whether one can construct a countable antichain of generalized Medvedev logics under inclusion. It would also be interesting to understand when two finite rooted frames with a top give rise to the same generalized Medvedev logic, or to comparable ones. Given finite rooted frames with a top, $\mathbf{F}$ and $\mathbf{G}$, what features of $\mathbf{F}$ and $\mathbf{G}$ determine whether the corresponding logics are equal, comparable, or incomparable? In particular, the present paper only gives some sufficient conditions for separating such logics, based on branching degree arguments. Another direction concerns the possible algebraic structure of this class. Although we have obtained a coarse “geography” of generalized Medvedev logics within the lattice of intermediate logics, a more refined structural description remains open. In particular, what algebraic or order-theoretic structure is formed by the class of all generalized Medvedev logics under inclusion? One may also consider a more general version of the construction, where instead of fixing a single finite rooted frame $\mathbf{F}$ with a top, one takes topless products whose frames are chosen from a class of finite rooted frames with a top. The present paper focuses on the single frame case because this is the direct analogue of the classical Medvedev construction, where one fixes the 2-chain and considers the frames $\mathbf{C}(2)^{n-}$. The general class-based construction is therefore a natural further direction, but we leave this generalization for future work.

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