Compositional Game Theory, Compositionally

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We present a new compositional approach to compositional game theory (CGT) based upon Arrows, a concept originally from functional programming, closely related to Tambara modules, and operators to build new Arrows from old. We model equilibria as a module over an Arrow and define an operator to build a new Arrow from such a module over an existing Arrow. We also model strategies as graded Arrows and define an operator which builds a new Arrow by taking the colimit of a graded Arrow. A final operator builds a graded Arrow from a graded bimodule. We use this compositional approach to CGT to show how known and previously unknown variants of open games can be proven to form symmetric monoidal categories.

1 Introduction

A new strain of game theory — *Compositional Game Theory (CGT)* — was introduced recently [9]. At its core, CGT involves a completely new representation of games — *open games* — with operators for constructing larger and more complex games from smaller, simpler (and hence easier to reason about) ones. Despite recent substantial interest and further development [13, 10, 7, 6, 4, 11, 14], open games remain complex structures, e.g. the simplest form of an open game is an 8-tuple consisting of 4 ports, a set of strategies, play and coplay functions and an equilibrium predicate. More advanced versions [4] require sophisticated structures such as coends. This causes problems: (i) complex definitions lead to complex and even error prone proofs; (ii) proofs are programs and overly complex proofs turn into overly complex programs; (iii) even worse, this complexity deters experimentation and innovation, for instance the authors of [10] were deterred from trying alternative definitions as the work became prohibitive; and (iv) worse still, this complexity suggests we do not fully understand the mathematical structure of open games.

Category theorists have an answer to such complexity: trade complex definitions in a simple category for simpler definitions in a more complex category. This simplified open games by factoring their definition via the category of lenses, and this more abstract perspective led directly to the discovery of Bayesian Open Games [4] where one naturally swaps out lenses in favour of optics when one starts with a monoidal category. Swapping out a subcomponent for another is a hallmark of compositionality, but, because the definition of open games is not compositional, one has to redevelop all the structure of open games built over optics from scratch. The same thing happens with open games with mixed strategies [10] where a small change — here to the type of the equilibrium predicate — leads to the need to redevelop the full theory of these open games. The same story is true if we want to (i) replace equilibria with best-response; or (ii) add type dependency; or (iii) allow the covariant play function and contravariant co-play function to live in different categories; or (iv) build open games over monadic effects; or (v) consider a host of other variations of open games. Every single time we must show from scratch that a given variant of open game forms a monoidal category — a highly non-trivial task.

We present a compositional approach to CGT which allows us exactly to make small modifications to the definition of an open game, check these modifications have suitable local properties, and conclude the resulting open games do indeed form a monoidal category. As with compositionality in general, we start with a semantic structure and define operators to build more complex examples from smaller ones. The semantic structure we use is a presentation of monoidal categories using profunctors and Arrows [17]. This simplifies the treatment of the monoidal structure to that of a strength, and also allows us to work in a purely functorial setting. We stress that arrows are just a convenient tool — it would also be possible to work with monoidal categories directly. This paper develops a three phase compositional axiomatisation of CGT: (i) the first phase consists of selecting an arrow to represent the play/coplay structure of open games; (ii) the second phase abstracts the equilibrium predicate to a bimodule over an arrow, and defines an operator for building new arrows from an old arrow and a bimodule over it; and (iii) the final phase abstracts the indexing of open games by strategies by defining an operator which maps a graded arrow to an arrow. This provides a simple, compositional axiomatisation of CGT in arrow theoretic terms. We also show we can add the strategies first and then later add the equilibrium predicate. This later approach also allows us to treat probabilistic games and open games based upon best response functions. The closest approach in the literature is Hedges [14, §8], and while there are notable similarities there are also significant differences, e.g. while Hedges [14] provides an axiomatisation of certain forms of open games, it does not do so compositionally, and is not general enough to cover probabilistic or dependently typed open games.

In summary, this more abstract picture clearly exposes the fundamental mathematical structure of open games in a way which has not been done before. The originality of the paper lies in its conceptual reimagining of open games as a compositional structure using canonical mathematical structures such as arrows, modules and grading. Its utility is demonstrated by the variety of examples — some known and others new to this paper — encompassed by our framework. As a result, we believe our presentation will both (i) lead to further new variants of open games; and (ii) become the default presentation of open games.

2   Profunctors and Arrows

Definition 1 (Profunctor)**.** Given two categories $\mathscr{C}$ and $\mathscr{D}$, a *profunctor* F from $\mathscr{C}$ to $\mathscr{D}$, denoted by $F : \mathscr{C} \nrightarrow \mathscr{D}$, is a functor $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$.

Example 2. Let $\mathscr{C}$ be a category. Then $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ is a profunctor.

Profunctors, strong profunctors and monoids therein form our bedrock and much of this paper is a toolbox for building new examples of arrows from old. The bicategory **Prof** has categories as objects, profunctors as morphisms, and natural transformations as 2-cells. Composition of profunctors is given by the the following coend formula, for $F : \mathscr{C} \nrightarrow \mathscr{D}$ and $G : \mathscr{D} \nrightarrow \mathscr{E}$.

$$(G \bullet F)(X, Z) := \int^{Y : \mathscr{D}} F(X, Y) \times G(Y, Z)$$

The identity profunctor on $\mathscr{C}$ is $\mathrm{Hom}_{\mathscr{C}}$. We now fix a monoidal category $\mathscr{C}$ and focus on the category **Prof**$(\mathscr{C})$ of profunctors from $\mathscr{C}$ to $\mathscr{C}$ and natural transformations between them.

Definition 3 (Strength)**.** A *(right) strength* for a profunctor $F : \mathbf{Prof}(\mathscr{C})$ is a family of morphisms $\mathrm{st}_{X,Y,Z} : F(X, Y) \to F(X \otimes Z, Y \otimes Z)$ (natural in X, Y, and dinatural in Z) making the following two diagrams commute. We often write st_Z or just st when the subscripts can be inferred from context.

$$F(X,Y) \xrightarrow{\text{st}_{Z\otimes Z'}} F(X\otimes(Z\otimes Z'),Y\otimes(Z\otimes Z'))$$

with the vertical map st_Z on the left, $F(\alpha^{-1},\alpha)$ on the right, to

$$F(X\otimes Z,Y\otimes Z) \xrightarrow{\text{st}_{Z'}} F((X\otimes Z)\otimes Z',(Y\otimes Z)\otimes Z')$$

and

$$F(X,Y)$$

with maps $F(\rho,\rho^{-1})$ and st_I to

$$F(X\otimes I,Y\otimes I)$$

Such a pair (F,st) is called a *strong profunctor*[1]. Note that several strengths may exist for the same profunctor. When several strengths for different profunctors (say F and G) are involved, we distinguish them using superscripts, st^F and st^G. A *strong natural transformation* from (F,st^F) to (G,st^G) is a natural transformation $\tau : F \to G$ that commutes with the strengths:

$$
\begin{array}{ccc}
F(X,Y) & \xrightarrow{\text{st}^F} & F(X\otimes Z,Y\otimes Z) \\
{\scriptstyle \tau_{X,Y}}\downarrow & & \downarrow{\scriptstyle \tau_{X\otimes Z,Y\otimes Z}} \\
G(X,Y) & \xrightarrow{\text{st}^G} & G(X\otimes Z,Y\otimes Z)
\end{array}
$$

The category **StrProf**($\mathscr{C}$) of strong profunctors and strong natural transformations is monoidal, where the tensor product is the composition operator $\bullet$, and the unit is the functor $\text{Hom}_{\mathscr{C}}$. This allows us to talk about monoids inside **StrProf**($\mathscr{C}$).

Definition 4 (Arrow). An *arrow* is a monoid in $(\textbf{StrProf}(\mathscr{C}),\bullet,\text{Hom}_{\mathscr{C}})$, i.e. a strong profunctor (A,st) together with strong natural transformations *pure* : $\text{Hom}_{\mathscr{C}} \to A$ and $\ggg : A\bullet A \to A$, satisfying the expected monoid laws.

If we unpack Definition 4 (and use the universal property of coends), an arrow on $\mathscr{C}$ can be described more concretely by the following data:

- A family of sets $A : |\mathscr{C}| \times |\mathscr{C}| \to \textbf{Set}$;
- A family of functions $pure_{XY} : \text{Hom}_{\mathscr{C}}(X,Y) \to A(X,Y)$;
- A family of functions $\ggg_{XYZ}: A(X,Y) \times A(Y,Z) \to A(X,Z)$, called *arrow composition*;
- A family of functions $\text{st}_{XYZ} : A(X,Y) \to A(X\otimes Z,Y\otimes Z)$;

subject to a number of *arrow equations* expressing functoriality, naturality, and so on (see e.g. Atkey [2]). We often use this presentation in the rest of the paper. We sometimes omit the subscripts to reduce notation when they can be inferred; and sometimes add a superscript, e.g. $\ggg^A$, to specify which arrow they belong to.

There is a notion of a *left* strength $F(X,Y) \to F(Z\otimes X,Z\otimes Y)$, analogous to Definition 3, that satisfies similar properties. When the category $\mathscr{C}$ is symmetric monoidal, left and right strengths are interdefinable:

Lemma 5. *Let $\mathscr{C}$ be symmetric monoidal. A right strength st for a profunctor $F : \textbf{Prof}(\mathscr{C})$, induces a left strength $\text{st}' : F(X,Y) \to F(Z\otimes X,Z\otimes Y)$ as $\text{st}'_{XYZ} = F(\sigma_{ZX},\sigma_{YZ}) \circ \text{st}$. The left strength satisfies the (di)-naturality conditions and analogues of the diagrams in Definition 3.*

In general, arrows give rise to *premonoidal categories* [2]. The reason we do not get a monoidal category is that there are two ways to define a map $A(X,Y) \times A(X',Y') \to A(X\otimes X',Y\otimes Y')$, and they need not coincide. Since we are interested in (symmetric) monoidal categories, we need to restrict to *commutative* arrows[2].

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[1]See also Rivas and Jaskelioff [24] for a similar definition in the case when $\mathscr{C}$ is cartesian. This definition is also the same as Pastro and Street's definition of the additional structure required for a profunctor to be a *Tambara module* [22].

[2]There is a similar situation with *strong monads*, whose Kleisli category is in general only premonoidal. To get a monoidal category, we need the stronger notion of *commutative monad* (a.k.a. *monoidal monad*). See Example 9.

Definition 6. Let $\mathscr{C}$ be symmetric monoidal. An arrow $(A, pure, \ggg, \mathrm{st})$ is *commutative* if the following diagram mixing left strength, right strength, and composition commutes:

$$
\begin{array}{ccc}
A(X,Y) \times A(X',Y') & \xrightarrow{\mathrm{st}_{X'} \times \mathrm{st}'_Y} & A(X \otimes X', Y \otimes X') \times A(Y \otimes X', Y \otimes Y') \\
\bigg\downarrow{\cong} & & \bigg\downarrow{\ggg} \\
& & A(X \otimes X', Y \otimes Y') \\
& & \bigg\uparrow{\ggg} \\
A(X',Y') \times A(X,Y) & \xrightarrow{\mathrm{st}'_X \times \mathrm{st}_{Y'}} & A(X \otimes X', X \otimes Y') \times A(X \otimes Y', Y \otimes Y')
\end{array}
$$

The following is a mild extension of results in the literature [16, 24].

Proposition 7. *Let $\mathscr{C}$ be symmetric monoidal. Every commutative arrow $A : \mathbf{Prof}(\mathscr{C})$ gives rise to a symmetric monoidal category whose objects are the objects of $\mathscr{C}$, and whose morphisms from X to Y are the elements of $A(X,Y)$. Moreover, there is a strict monoidal functor $J : \mathscr{C} \to A$ where we abuse notation and write A for both the arrow and the monoidal category it induces.*

Example 8. For any symmetric monoidal category $\mathscr{C}$, the functor $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ is an arrow, whose associated monoidal category is $\mathscr{C}$ itself.

Example 9. Let $T : \mathscr{C} \to \mathscr{C}$ be a strong monad on $\mathscr{C}$. The *Kleisli arrow* of T is defined by $A_T(X,Y) := \mathrm{Hom}_{\mathscr{C}}(X, TY)$. If T is commutative, then the arrow A_T is also commutative. Its associated monoidal category is the Kleisli category $\mathscr{C}_T$. Dually, if $T : \mathscr{C} \to \mathscr{C}$ is a strong comonad on $\mathscr{C}$, its *co-Kleisli arrow* is defined by $A_T(X,Y) := \mathrm{Hom}_{\mathscr{C}}(TX, Y)$.

We use this last example to show lenses form an arrow in the next section. This example also demonstrates the interest in arrows from the functional programming community, as arrows provide a single framework within which computations with both effects and coeffects can live. To summarise: arrows provide a presentation of monoidal categories where the monoidal structure is induced by strength, and generalise both monadic and comonadic computation. We can use constructions on functors and internal monoids to build new arrows from old; this flexibility underpins their use in this paper.

3   Open Games

Rather than hit the reader with the full profunctor framework we have developed, we show first how a compositional treatment of open games is possible, which abstracts away from any specific notion of lens as the primitive building block for open games. As this is expository material we only sketch proofs, and work with **Set**-based open games.

3.1   Open games, concretely

Open games were originally presented as follows [12]:

Definition 10 (Open Game). Let X, Y, R and S be sets. A pure open game $G = (J_G, P_G, C_G, E_G) : (X,S) \to (Y,R)$ consists of:

- A set J_G, called the set of strategies of G,

- A function $P_G : J_G \times X \to Y$, called the play function of G,

- A function $C_G : J_G \times X \times R \to S$, called the coutility function of G, and

- A predicate $E_G : X \times (Y \to R) \to \mathscr{P}(J_G)$, called the equilibrium function of G[3] (where $\mathscr{P}$ is the powerset functor).

An important conceptual step was the lens-ification of open games [9]. The functor $W : \mathbf{Set} \times \mathbf{Set}^{op} \to \mathbf{Set} \times \mathbf{Set}^{op}$ defined by $W(X,S) = (X, X \to S)$ is a comonad. The category of lenses $\mathbf{Lens}$ is the co-Kleisli category of W: concretely, its objects are pairs of sets, and the morphisms from (X,S) to (Y,R) are pairs of functions (f,g) with $f : X \to Y$ and $g : X \times R \to S$. Noting that $X \cong \mathbf{Lens}(1,1)(X,S)$ and $(Y \to R) \cong \mathbf{Lens}(Y,R)(1,1)$, we have:

Theorem 11. *Let $I = (1,1)$ and A and B be objects of $\mathbf{Set} \times \mathbf{Set}^{op}$. An open game $G : A \to B$ is given by: (i) a set J_G of strategies; (ii) a function $p_G : J_G \to \mathbf{Lens}(A,B)$; and (iii) a predicate $E_G : J_G \times \mathbf{Lens}(I,A) \times \mathbf{Lens}(B,I) \to \mathsf{Bool}$.*

Apart from internalising the notion of open games into structure internal to the category of lenses, and combining the treatment of play and coplay into a single morphism, this more abstract formulation gives cleaner proofs of the following fundamental theorem:

Theorem 12. *There is a symmetric monoidal category Op whose objects are pairs of sets, and whose morphisms are open games, with games with isomorphic strategy sets identified[4]. This structure reflects the sequential and parallel composition of open games.*

3.2 A Compositional Treatment of Open Games

Rather than proving that open games and similar structures form a monoidal category in one fell swoop, i.e. non-compositionally, we adopt a more compositional, step by step approach. As mentioned in the introduction we do this by using profunctors and arrows. The first step is easy: by Example 9, the comonad $W(X,S) = (X, X \to S)$ gives rise to an arrow on $\mathbf{Set} \times \mathbf{Set}^{op}$, that we also denote by $\mathbf{Lens}$, like its induced symmetric monoidal category. Note that the strength of W is defined with respect to the monoidal structure of $\mathbf{Set} \times \mathbf{Set}^{op}$ (not its cartesian structure), which sends (X,Y) and (X',Y') to $(X \times X', Y \times Y')$. The unit of this monoidal structure is $I = (1,1)$.

Proposition 13. *The profunctor $\mathbf{Lens} : \mathbf{Prof}(\mathbf{Set} \times \mathbf{Set}^{op})$ is a commutative arrow.*

This establishes the well-known fact that the category of lenses is symmetric monoidal. Further, the unit of this arrow is the core of the lifting of functions through open games. The next part of our compositional construction requires that an arrow that satisfies a property called projection at points — a property that the key case of $\mathbf{Lens}$ has. Condition (1) below essentially expresses that the projection map p^0 is an internal natural transformation. In particular, it implies naturality of p^0 in both parameters.

Definition 14 (Projection at points). Let $A : \mathbf{Prof}(\mathbf{Set} \times \mathbf{Set}^{op})$ be an arrow. Then A has *projection at points* if there is a family of maps $p^0_{X,X'} : A(I, X \otimes X') \to A(I,X)$ such that

$$p^0_{Y,X'}(j \ggg \mathrm{st}_{X'}(a)) = p^0_{X,X'}(j) \ggg a \qquad \text{and} \qquad p^0_{X,Y'}(j \ggg \mathrm{st}'_X(a')) = p^0_{X,X'}(j) \qquad (1)$$

for all $j \in A(I, X \otimes X')$, $a \in A(X,Y)$ and $a' \in A(X',Y')$. Using symmetry, this means that there also is a projection $p^1_{X,X'} : A(I, X \otimes X') \to A(I,X')$ given by $p^1_{X,X'} := p^0_{X,X'} \circ A(\mathrm{id}, \sigma)$ that satisfies similar conditions.

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[3]Hedges [12] presents open games with best response functions rather than equilibrium functions, but we prefer the presentation using equilibria for simplicity. We will show how to also define open games with best response functions compositionally in Section 6.

[4]An alternative to this quotient would be to work with bicategories instead.

Given any arrow A (such as **Lens**) with projection at points, which we think of as representing some primitive building block, we now want to attach to it an equilibrium predicate — the key feature of open games. The type of this equilibrium predicate is $\text{Equib}(A)(X,Y) := A(I,X) \times A(Y,I) \to \text{Bool}$. Note that $\text{Equib}(A)$ itself is not an arrow: it has a structure that we call an *A-bimodule* in Section 4.1. However, by bundling together A and $\text{Equib}(A)$, we do get an arrow:

Proposition 15. *Let $A : \textbf{Prof}(\textbf{Set} \times \textbf{Set}^{op})$ be a commutative arrow with projection at points. Then the profunctor $A \times \text{Equib}(A) : \textbf{Prof}(\textbf{Set} \times \textbf{Set}^{op})$ defined by*

$$(A \times \text{Equib}(A))(X,Y) = A(X,Y) \times (A(I,X) \times A(Y,I) \to \text{Bool})$$

is a commutative arrow.

Proof. The unit of $A \times \text{Equib}(A)$ is built from the unit of A and an equilibrium predicate which always returns true. The composition

$$\ggg_{X,Y,Z} : (A \times \text{Equib}(A))(X,Y) \times (A \times \text{Equib}(A))(Y,Z) \to (A \times \text{Equib}(A))(X,Z)$$

is obtained as follows. We are given four arguments $s : A(X,Y), t : A(Y,Z), f : A(I,X) \times A(Y,I) \to \text{Bool}$, and $g : A(I,Y) \times A(Z,I) \to \text{Bool}$, and we define

$$(s,f) \ggg (t,g) := \left(s \ggg^A t, \lambda x \lambda z.\, f(x, t \ggg^A z) \wedge g(x \ggg^A s, z) \right)$$

where $\ggg^A$ is the multiplication of A.

To define the strength $\text{st}_{X,Y,Z} : (A \times \text{Equib}(A))(X,Y) \to (A \times \text{Equib}(A))(X \otimes Z, Y \otimes Z)$, we are given an element $s : A(X,Y)$ and a predicate $f : A(I,X) \times A(Y,I) \to \text{Bool}$. We put

$$\text{st}_Z(s,f) := \left(\text{st}_Z^A(s), \lambda x \lambda y.\, f\left(p^0(x), \text{st}'^A_Y(p^1(x)) \ggg^A y \right) \right)$$

where st^A denotes the strength of A, st'^A left strength and p^0 and p^1 are the projections at points. We omit the proof of commutativity of the arrow. □

Those familiar with the intricacies of the monoidal and sequential composition of open games will notice their essence above. What is really surprising is that this can be carried out for any set-based arrow satisfying projection at points and not just for lenses. This generality was not expected. Finally, we turn to indexing by sets of strategies. Again, we work solely with arrows and introduce an arrow constructor to deal with strategies. The simplest, used here, is a variant of the fundamental families construction from categorical logic [19, Def. 1.2.1] where we refer to a function $J \to X$ as a J-indexed collection of X's.

Proposition 16. *Let $A : \textbf{Prof}(\textbf{Set} \times \textbf{Set}^{op})$ be a commutative arrow. Then the profunctor $\text{Fam}(A) : \textbf{Prof}(\textbf{Set} \times \textbf{Set}^{op})$ defined by*

$$\text{Fam}(A)(X,Y) = \text{colimit}_{J : \textbf{Set}_{iso}}(J \to A(X,Y))$$

is a commutative arrow.

Proof. The unit of $\text{Fam}(A)$ is the singleton family built from the unit of A. The composition of $\text{Fam}(A)$ sends $(J, F : J \to A(X,Y))$ and $(K, G : K \to A(Y,Z))$ to $(J \times K, (j,k) \mapsto F(j) \ggg^A G(k))$ (see also Hancock's tensor on containers [18, Def. 2.3]). Finally, strength for $\text{Fam}(A)$ takes a family and post-composes it with the strength of A. □

Note that in the definition of $\mathsf{Fam}(A)$, taking the colimit over the category $\mathbf{Set}_{\mathrm{iso}}$ of sets and bijective functions is crucial. Indeed, it ensures that families indexed by isomorphic strategy sets are identified. This quotient is important to prove the associativity of arrow composition; the same issue comes up in the usual treatment of open games [9]. Combining Propositions 13, 15, 16 and 7, we get a symmetric monoidal category $\mathsf{Op} = \mathsf{Fam}(\mathbf{Lens} \times \mathsf{Equib}(\mathbf{Lens}))$ and an embedding $\mathbf{Set} \times \mathbf{Set}^{\mathrm{op}} \to \mathsf{Op}$, in a compositional manner.

4 Bimodules and Graded Arrows

We now upgrade the expository illustration of the previous section. The simplest observation is that there is nothing special about the category $\mathbf{Set}$ — any monoidal category can be used. We then generalise the $(- \times \mathsf{Equib}(-))$ construction using a general notion of bimodule for an arrow in Section 4.1. We decompose $\mathsf{Fam}(A)$ using a notion of *graded* arrow in Section 4.2. In Sections 5 and 6 we will use the work in this section to reconstruct open games on optics [4] and other known and new variations of open games, respectively.

4.1 Bimodules

The cumbersome proof of Proposition 15 can be further decomposed. Notice that the arrow

$$(A \times \mathsf{Equib}(A))(X,Y) = A(X,Y) \times (A(I,X) \times A(Y,I) \to \mathsf{Bool})$$

is obtained by combining the arrow A with the predicate type $\mathsf{Equib}(A)(X,Y) := A(I,X) \times A(Y,I) \to \mathsf{Bool}$. As it turns out, $\mathsf{Equib}(A)$ has a nice structure that we call an A-bimodule[5]. We will use this technology to prove Proposition 33, which generalizes Proposition 15.

We also take the opportunity to notice that there is nothing special about the Booleans in the definition of concrete open games, and replace them with an arbitrary monoid with suggestively named operations $(M, \wedge, \mathsf{tt})$ — Hedges' software implementation [15] uses this insight to collect more in-depth diagnostics when checking equilibria.

Definition 17 (Bimodule). Let $A : \mathscr{C} \nrightarrow \mathscr{C}$ be an arrow. An *A-bimodule* is a profunctor $B : \mathscr{C} \nrightarrow \mathscr{C}$ equipped with natural transformations $\ell : A \bullet B \to B$ and $r : B \bullet A \to B$, such that:

1. ℓ is a left-action, i.e., the two diagrams below commute;

2. r is a right-action (similar diagrams);

3. ℓ commutes with r, i.e.,

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[5]Note that our notion of an *A*-bimodule is equivalent to a profunctor for the (pre)monoidal category induced by *A* (Proposition 7), matching the alternative name "bimodule" for profunctors.

When B is a strong profunctor, ℓ and r are also required to be strong natural transformations.

Definition 18. Let A be an arrow and $B : \mathbf{StrProf}(\mathscr{C})$ an A-bimodule. We say that B is *commutative* when the following diagram (akin to the one of Definition 6) commutes.

$$
\begin{array}{ccc}
A(X,Y) \times B(X',Y') & \xrightarrow{\ \mathrm{st}_{X'} \times \mathrm{st}'_Y\ } & A(X \otimes X', Y \otimes X') \times B(Y \otimes X', Y \otimes Y') \\
\Big\downarrow{\cong} & & \Big\downarrow{\ell} \\
& & B(X \otimes X', Y \otimes Y') \\
& & \Big\uparrow{r} \\
B(X',Y') \times A(X,Y) & \xrightarrow{\ \mathrm{st}'_X \times \mathrm{st}_{Y'}\ } & B(X \otimes X', X \otimes Y') \times A(X \otimes Y', Y \otimes Y')
\end{array}
$$

Recall that profunctors inherit the cartesian structure of **Set**. Given $F, G : \mathscr{C} \nrightarrow \mathscr{D}$, their product is defined by $(F \times G)(c,d) := F(c,d) \times G(c,d)$; and the terminal profunctor $1 : \mathscr{C} \nrightarrow \mathscr{D}$ is the constant functor that picks out a singleton set. It is easy to check that this is also a cartesian structure on $\mathbf{StrProf}(\mathscr{C})$. Finally, note that $\bullet$ distributes over $\times$, in the sense that there is a strong natural transformation of type $F \bullet (G \times H) \to (F \bullet G) \times (F \bullet H)$, and similarly with F on the right.

Definition 19 (Bimodule with monoid)**.** When B additionally has a monoid structure (B,m,e) w.r.t. the cartesian product $(\times,1)$ in $\mathbf{StrProf}(\mathscr{C})$, we also require that ℓ and r preserve the monoid structure of B:

$$
\begin{array}{ccc}
A \bullet (B \times B) & \xrightarrow{\ \text{distributivity}\ } & (A \bullet B) \times (A \bullet B) \\
{\scriptstyle A \bullet m}\Big\downarrow & & \Big\downarrow{\scriptstyle \ell \bullet \ell} \\
A \bullet B \xrightarrow[\ell]{} B \xleftarrow[m]{} & & B \times B
\end{array}
\qquad
\begin{array}{ccc}
A \bullet 1 & \xrightarrow{\ \mathrm{id} \bullet e\ } & A \bullet B \\
{\scriptstyle !}\Big\downarrow & & \Big\downarrow{\scriptstyle \ell} \\
1 & \xrightarrow[\ e\ ]{} & B
\end{array}
$$

Theorem 20. *Assume $A : \mathscr{C} \nrightarrow \mathscr{C}$ is an arrow, and $B : \mathbf{StrProf}(\mathscr{C})$ is a monoid w.r.t. cartesian product $(\times,1)$ and an A-bimodule, then $A \times B$ is an arrow. Moreover, if both A and B are commutative (as an arrow, as a monoid and as a bimodule), then so is $A \times B$.*

Proof. Since A and B are strong profunctors, then so is $A \times B$. Moreover, our hypothesis gives us the following strong natural transformations:

$$
\begin{array}{lll}
pure^A : \mathrm{Hom}_\mathscr{C} \to A & \ell : A \bullet B \to B & e : 1 \to B \\
\ggg^A : A \bullet A \to A & r : B \bullet A \to B & m : B \times B \to B
\end{array}
$$

Then we define $pure^{A \times B} := pure^A \times e$, and $\ggg^{A \times B}$ is given by

$$
(A \times B) \bullet (A \times B) \xrightarrow{\ \text{distributivity}\ } \begin{array}{c}(A \bullet A) \times (A \bullet B) \\ \times (B \bullet A) \times (B \bullet B)\end{array} \xrightarrow{\ \ggg^A \times \ell \times r \times !\ } A \times B \times B \xrightarrow{\ \mathrm{id} \times m\ } A \times B
$$

One can then check that the arrow laws are satisfied.   □

In virtue of Theorem 20, the proof of Proposition 15 now amounts to showing that $\mathrm{Equib}(A)$ is a commutative A-bimodule with a $(\times,1)$-monoid structure. In fact, we can further decompose this proof by introducing the following notion of *context*. A similar notion of context appears in Hedges [14], where it is axiomatized at the level of the monoidal category induced by the arrow A.

Definition 21 (Context). Let $A : \mathscr{C} \nrightarrow \mathscr{C}$ be an arrow. A profunctor $B : \mathscr{C} \nrightarrow \mathscr{C}$ is called a *context* when it is a commutative A-bimodule equipped with a *costrength* $\mathrm{cst}_{XYZ} : B(X \otimes Z, Y \otimes Z) \to B(X,Y)$, satisfying some coherence diagrams similar to the ones for strength (full details in Appendix A).

The notion of costrength is a weakening of the notion of a trace for a symmetric monoidal category to arbitrary profunctors — in particular if the profunctor B is the Hom-functor of a traced monoidal category, then B has a canonical costrength. The meat of Proposition 15 lies in the next lemma. Notice that the position of X and Y is reversed; hence the pre-composition with *flip* : $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}^{\mathrm{op}}$ in Theorem 23. Another example of context, which does not require projection at points, will be given later in Definition 31.

Lemma 22. *If A has projection at points, then* $\mathsf{Ctxt}_A(X,Y) = A(I,Y) \times A(X,I)$ *is a context.*

Proof. The left action is given by the family $\ell_{XYZ} : A(X,Y) \times \mathsf{Ctxt}_A(Y,Z) \to \mathsf{Ctxt}_A(X,Z)$, defined by:

$$\ell_{XYZ} := (s,z,y) \mapsto (z, s \ggg^A y) : A(X,Y) \times A(I,Z) \times A(Y,I) \longrightarrow A(I,Z) \times A(X,I)$$

The right action is similar. To define the costrength, we use the strength of A and the projections p^0, p^1:

$$\mathrm{cst}_{XYZ} := (y,x) \mapsto (p^0(y), \mathrm{st}_X'^A(p^1(y)) \ggg^A x) : A(I, Y \otimes Z) \times A(X \otimes Z, I) \longrightarrow A(I,Y) \times A(X,I) \qquad \square$$

Theorem 23. *Let A be a commutative arrow, $B : \mathscr{C} \nrightarrow \mathscr{C}$ be a context for A, and $(M, \wedge, \mathsf{tt})$ be a commutative monoid (in **Set**). Then $M^{B \circ flip}$ is a commutative A-bimodule and $(\times, 1)$-monoid.*

Proof. The left and right actions are given by the families of functions

$$\ell_{XYZ} : A(X,Y) \times M^{B(Z,Y)} \longrightarrow M^{B(Z,X)} \qquad\qquad r_{XYZ} : M^{B(Y,X)} \times A(Y,Z) \longrightarrow M^{B(Z,X)}$$
$$(a,h) \longmapsto \lambda b.\, h(r^B\, b\, a) \qquad\qquad\qquad (h,a) \longmapsto \lambda b.\, h(\ell^B\, a\, b)$$

The $(\times, 1)$-monoid structure is inherited from the one of M:

$$e_{XY} : 1 \longrightarrow M^{B(Y,X)} \qquad\qquad m_{XY} : M^{B(Y,X)} \times M^{B(Y,X)} \longrightarrow M^{B(Y,X)}$$
$$* \longmapsto \lambda b.\, \mathsf{tt} \qquad\qquad\qquad (h,h') \longmapsto \lambda b.\, h\,b \wedge h'\,b$$

Finally, the strength st : $M^{B(Y,X)} \longrightarrow M^{B(Y \otimes Z, X \otimes Z)}$ is obtained by pre-composing with cst^B. All the bimodule axioms of $M^{B \circ flip}$ can then be easily derived from those of B. $\qquad \square$

4.2 Graded Arrows and graded bimodules

The $\mathsf{Fam}(A)$ construction is really a two step construction: it takes an arrow A and parameterises by elements of sets J, then the parameter sets are hidden using a colimit. The intermediate step in this process is not an arrow because its composition composes morphisms with different parameter sets. However, it is a *graded* arrow, which we define by adapting the definition of graded monads [20]. Graded monads have been used to model programming languages with variable sets of possible side effects. The multiplication of a graded monad is a natural transformation $T_p T_q \Rightarrow T_{pq}$, where the p, q are from a symmetric monoidal category $(\mathscr{P}, i, \cdot)$ of grades. Graded monads are lax monoidal functors $\mathscr{P} \to \mathrm{Endo}(\mathscr{C})$. Swapping endofunctors for strong profunctors yields:

Definition 24. A $\mathscr{P}$-*graded arrow* is a lax monoidal functor $A : \mathscr{P} \to \mathbf{StrProf}(\mathscr{C})$.

Unfolding this definition shows that a graded arrow comes equipped with graded composition $A_p(X,Y) \times A_q(Y,Z) \to A_{pq}(X,Z)$ and unit $\text{Hom}_{\mathscr{C}}(X,Y) \to A_i(X,Y)$ that use the monoidal structure of $\mathscr{P}$. Strength $A_p(X,Y) \to A_p(X \otimes Z, Y \otimes Z)$ is provided for every grade p, but when we use strength to define parallel composition we will use the monoidal structure of grades. To make sure that parallel composition makes sense, graded arrows must be commutative. Note that the symmetry of $\mathscr{P}$'s monoidal structure is required for this definition to make sense.

Definition 25 (Commutative Graded Arrow). A $\mathscr{P}$-graded arrow A is *commutative* if the two functions $A_p(X,Y) \times A_q(X',Y') \to A_{pq}(X \otimes X', Y \otimes Y')$ defined using left and right strength are equal.

When a graded arrow is commutative we obtain a canonical parallel composition of morphisms that commutes with sequential composition. Note that the composition of parameters is the same as in the sequential case, forced by an Eckmann-Hilton style argument.

We now use graded arrows to reconstuct $\text{Fam}(A)$. The first step is this proposition, which follows because the graded arrow is constructed pointwise.

Proposition 26. *Let $(\mathscr{P}, i, \cdot)$ be a symmetric monoidal subcategory of $(\mathbf{Set}, 1, \times)$, and let $A : \mathbf{StrProf}(\mathscr{C})$ be an arrow. Then $(- \to A)_p(X,Y) = p \to A(X,Y)$ is a $\mathscr{P}^{op}$-graded arrow. If A is commutative, then so is $- \to A$.*

The second step in the $\text{Fam}(A)$ construction is the following result, which shows that there is a general way of "summing out" the grading of a graded arrow to recover an (ungraded) arrow.

Proposition 27. *For every graded arrow $A : \mathscr{P} \to \mathbf{StrProf}(\mathscr{C})$, $(\text{colim}\, A)(X,Y) = \text{colim}_p A_p(X,Y)$ is an arrow. If A is commutative, then so is $\text{colim}\, A$.*

Proof. $\text{colim}\, A$ is certainly a profunctor, so it remains to construct the monoid operations. The unit is defined by the composite $\text{Hom}_{\mathscr{C}}(X,Y) \overset{pure_i}{\to} A_i(X,Y) \overset{\iota}{\to} \text{colim}_p A_p(X,Y)$, where ι is the injection into the colimit. Composition is defined using colimits commuting with products in $\mathbf{Set}$: $(\text{colim}_p F_p(X,Y)) \times (\text{colim}_q F_q(Y,Z)) \cong \text{colim}_{p,q} (F_p(X,Y) \times F_q(Y,Z)) \overset{\text{colim} \gg_{p,q}}{\longrightarrow} \text{colim}_{p,q} F_{pq}(X,Z) \to \text{colim}_r F_r(X,Z)$. Strength is defined as $\text{colim}_p F_p(X,Y) \overset{\text{colim st}}{\longrightarrow} \text{colim}_p F_p(X \otimes Z, Y \otimes Z)$. Each of the axioms follows from the corresponding axiom for the graded arrow A. $\square$

Thus we have decomposed the $\text{Fam}(A)$ construction into $\text{colim}\,(- \to A)$, using the intermediate concept of graded arrow. We will also use this technology in Proposition 30 to prove that optics form an arrow, as well as in Section 6 to introduce other indexing operators besides Fam. For variations such as best-response games and probabilistic games, we need to introduce a graded version of bimodules:

Definition 28 (Graded bimodule). Let $A : \mathscr{P} \to \mathbf{StrProf}(\mathscr{C})$ be a graded arrow. A *graded A-bimodule* is a functor $B : \mathscr{P} \to \mathbf{Prof}(\mathscr{C})$ equipped with natural transformations $\ell_{pq} : A_p \bullet B_q \to B_{pq}$ and $r_{pq} : B_p \bullet A_q \to B_{pq}$ natural in p and q, satisfying the bimodule axioms:

1. ℓ is a left action,

2. r is a right action,

3. ℓ commutes with r.

When $B : \mathscr{P} \to \mathbf{StrProf}(\mathscr{C})$, we moreover require ℓ_{pq} and r_{pq} to be strong natural transformations. As in Section 4.1, we can define the notion of a *commutative* graded bimodule; and when each B_p has a $(\times, 1)$-monoid structure, we require that ℓ and r should preserve it.

The following theorem is the graded generalisation of Theorem 20.

Theorem 29. *Assume $A : \mathscr{P} \to \mathbf{StrProf}(\mathscr{C})$ is a commutative graded arrow, and $B : \mathscr{P} \to \mathbf{StrProf}(\mathscr{C})$ is a commutative graded A-bimodule, such that B_p is a commutative $(\times, 1)$-monoid for every p. Then $A \times B : \mathscr{P} \to \mathbf{StrProf}(\mathscr{C})$ is a commutative graded arrow.*

Proof. The proof is very similar to the proof of Theorem 20. □

5   Open games based on optics

In this section, we use graded arrows and bimodules compositionally define open games starting from a monoidal category, so as to treat open games containing computational effects via the Kleisli category. We fundamentally achieve the same result as [4], but we do so in a compositonal manner. Lenses only define a monoidal category when the underlying category is cartesian. In a monoidal category we work with optics [22, 5, 23, 3] instead:

Proposition 30. *Let $\mathscr{C}$ be symmetric monoidal, and $A : \mathbf{Prof}(\mathscr{C})$ a commutative arrow. Then the profunctor $\mathsf{Optic}_A : \mathbf{Prof}(\mathscr{C} \times \mathscr{C}^{op})$ defined by*

$$\mathsf{Optic}_A((X,S),(Y,R)) = \int^{P \in \mathscr{C}} A(X, P \otimes Y) \times A(P \otimes R, S)$$

is also a commutative arrow.

Proof. By Proposition 27, and using that coends can be reduced to colimits using the twisted arrow category $\mathsf{Tw}(\mathscr{C}^{op})^{op}$ (with objects arrows $f : P' \to P$ in $\mathscr{C}$) [21, Remark 1.2.3], it is enough to show that the assignment $\mathsf{Optic}_A^{\mathsf{Tw}} : \mathsf{Tw}(\mathscr{C}^{op})^{op} \to \mathbf{StrProf}(\mathscr{C} \times \mathscr{C}^{op})$ given by

$$\mathsf{Optic}_A^{\mathsf{Tw}}\ (P \xleftarrow{f} P', (X,S), (Y,R)) = A(X, P' \otimes Y) \times A(P \otimes R, S)$$

is a commutative graded arrow, which follows from A being a commutative arrow and $\mathscr{C}$ being symmetric monoidal. □

This compositional proof has factored out the coend calculations involved into a separate statement about graded arrows. We expect many other monoidal categories defined in similar ways, e.g. combs [25], can be proved monoidal using graded arrows to similarly avoid coend manipulation. Next, we follow Bolt, Hedges and Zahn [4, §3.4] in generalising the type of the equilibrium function as needed when working in a monoidal setting.

Definition 31. Let $A : \mathbf{Prof}(\mathscr{C})$ be an arrow, and M a monoid. We define the *context* and *equilibrium type* profunctors $\mathsf{Ctxt}_A^{\mathscr{C}}, \mathsf{Equib}_M^{\mathscr{C}}(A) : \mathbf{Prof}(\mathscr{C})$ by

$$\mathsf{Ctxt}_A^{\mathscr{C}}(X,Y) = \mathsf{Optic}_A((I,I),(Y,X))$$
$$\mathsf{Equib}_M^{\mathscr{C}}(A)(X,Y) = \mathsf{Ctxt}_A^{\mathscr{C}}(Y,X) \to M$$

We now prove the optics analogue of Lemma 22:

Lemma 32. *Let $A : \mathbf{Prof}(\mathscr{C})$ be a commutative arrow. Then $\mathsf{Ctxt}_A^{\mathscr{C}}$ is a context, i.e. a commutative A-bimodule coherently equipped with a costrength.*

Proof. The left costrength is given by the Θ-indexed family of maps $(\mathsf{cst}_Z^l)_\Theta = \iota_{\Theta \otimes Z}$. The actions are defined using composition and the strength of A, e.g. for the left action $A(X,Y) \times \mathsf{Ctxt}_A^{\mathscr{C}}(Y,Z) \to \mathsf{Ctxt}_A^{\mathscr{C}}(X,Z)$:

$$A(X,Y) \times \mathsf{Ctxt}_A^{\mathscr{C}}(Y,Z)$$
$$\cong \int^P A(I,P\otimes Z) \times A(X,Y) \times A(P\otimes Y,I)$$

$$\int^P (\mathsf{id}\times\mathsf{st}_P'\times\mathsf{id}) \downarrow$$

$$\int^P A(I,P\otimes Z) \times A(P\otimes X,P\otimes Y) \times A(P\otimes Y,I) \xrightarrow{\int^P(\mathsf{id}\times\ggg)} \mathsf{Ctxt}_A^{\mathscr{C}}(X,Z)$$

One can then check that all the required laws of a bimodule with costrength are satisfied. $\qquad\square$

Applying Theorem 20 together with Theorem 23, we thus immediately get the optics analogue of Proposition 15, but this time we do not need the assumption of projection at points:

Proposition 33. *Let A : $\mathbf{Prof}(\mathscr{C})$ be a commutative arrow and M a monoid. The profunctor $A \times Equib_M^{\mathscr{C}}(A)$: $\mathbf{Prof}(\mathscr{C})$ defined by*

$$(A \times Equib_M^{\mathscr{C}}(A))(X,Y) := A(X,Y) \times Equib_M^{\mathscr{C}}(A)(X,Y)$$

is a commutative arrow.

To parameterise by strategies, we use graded arrows from Section 4.2. By Proposition 26 and Proposition 27, when A is an arrow, $\mathsf{Fam}(A) = \mathrm{colim}\,(-\to A)$ is an arrow. Composing the steps, we get the following theorem. By using the abstract framework in the previous section, the proofs are much shorter.

Theorem 34. *Let $\mathscr{C}$ be a symmetric monoidal category and M a monoid. There is a symmetric monoidal category $\mathsf{Op}_{\mathscr{C}} = \mathsf{Fam}(\mathsf{Optic}_{\mathrm{Hom}_{\mathscr{C}}} \times Equib_M^{\mathscr{C}}(\mathsf{Optic}_{\mathrm{Hom}_{\mathscr{C}}}))$ of open games in $\mathscr{C}$, and an embedding $\mathscr{C} \times \mathscr{C}^{op} \to \mathsf{Op}_{\mathscr{C}}$.*

6 Other variants of open games

We demonstrate the power of our framework by showing how a host of variations of open games can be easily proven to form monoidal categories.

Example 35 (Fibred Open games)**.** Lets say we want to add a little bit of type dependency so coutility can depend on state and utility on the moved played. Type dependency is modelled fibrationally [19], e.g. via the families fibration, which is split. Given a (split) fibration $p : \mathscr{E} \to \mathscr{B}$, define the monoidal category of fibrational lenses $\mathbf{Lens}_p$ to have the same objects as $\mathscr{E}$, and with morphisms from E to E' given by pairs (f,g), where $f : pE \to pE'$ is a morphism in $\mathscr{B}$ and $g : f^*E' \to E$ is a morphism in $\mathscr{E}$. If p is a symmetric monoidal fibration [26] then $\mathbf{Lens}_p$ is a commutative arrow. Defining Fam and $(-\times \mathrm{Equib}_M^{\mathscr{C}}(-))$ as in Section 5, we hence get an arrow $\mathsf{Fam}(\mathbf{Lens}_p \times \mathrm{Equib}_M^{\mathscr{C}}(\mathbf{Lens}_p))$ of fibrational open games. This subsumes the use of container morphisms [1] as dependently typed open games. The point of this example is not that this is necessarily the right way to do dependent games — clearly a lot more dependency is possible — but rather to show how new variants of open games can easily be shown monoidal in our compositional framework.

Example 36 (Best Response)**.** Using graded bimodules (Definition 28) and Theorem 29, we can recover *best response* games [12, §2.1.5]. In these games, the equilibrium predicate is a relation on strategies in context, expressed as a graded bimodule w.r.t. the graded arrow $A_J := J \to A$, for a given arrow A.

$$\mathrm{BestResp}_J(X,Y) = \mathsf{Ctxt}_A^{\mathscr{C}}(Y,X) \to J \times J \to \mathsf{Bool}$$

The left action, of type $A_J(X,Y) \times \text{BestResp}_{J'}(Y,Z) \to \text{BestResp}_{J \times J'}(X,Z)$, can be defined as follows. Given a context $c \in \text{Ctxt}_A^\mathscr{C}(Z,X)$ and a strategy $j \in J$, we can use the right action of the context (which is an A-bimodule) to get a context $c' \in \text{Ctxt}_A^\mathscr{C}(Z,Y)$. Thus, given four parameters (j_1, j_1', j_2, j_2'), we use j_1 to create a new context and feed it to $\text{BestResp}_{J'}(Y,Z)$ together with (j_1', j_2') to obtain a boolean. Note that j_2 is discarded, which is also what happens when we define the sequential composition of best response games. Best response games are then defined as $\text{colim}_J \big((J \to \textbf{Lens}(X,Y)) \times \text{BestResp}_J(X,Y) \big)$, which automatically gives a symmetric monoidal category by Theorem 29 and Proposition 7.

Example 37 (Probabilistic games). Surprisingly, Probabilistic Open Games [11] can be recovered using a construction similar to the one of Example 36. The main difference is that the graded bimodule is now indexed by $\mathscr{D}(J)$, where $\mathscr{D}$ is the finite distribution monad. However, for it to work properly, we need to make some assumptions on $\mathscr{C}$ and A.

For base category, we take $\mathscr{C} = \textbf{Set} \times (\mathscr{D}\text{-}\textbf{Alg})^{\text{op}}$, where $\mathscr{D}\text{-}\textbf{Alg}$ is the category of algebras for the monad $\mathscr{D}$. There is a forgetful functor $U : \textbf{Set} \times (\mathscr{D}\text{-}\textbf{Alg})^{\text{op}} \to \textbf{Set} \times \textbf{Set}^{\text{op}}$ to the "usual" base category $\textbf{Set} \times \textbf{Set}^{\text{op}}$, obtained by sending an algebra to its carrier set. Over this category, the profunctor $\textbf{Lens}(U-,U-) = \textbf{Lens} \circ (U^{\text{op}} \times U)$ can easily be seen to be a commutative arrow (because $\textbf{Lens}$ is), and thus by Proposition 26, $A_J(X,Y) = J \to \textbf{Lens}(U(X),U(Y))$ is a $\textbf{Set}_{\text{iso}}$-graded arrow. With these choices of $\mathscr{C}$ and A, the graded A-bimodule for probabilistic equilibria selection is:

$$\text{ProbEquib}_J(X,Y) = \text{Ctxt}_{\textbf{Lens}}^\mathscr{C}(U(Y),U(X)) \to \mathscr{D}(J) \to \text{Bool}$$

To match Definition 12 in [11], the left and right actions are defined quite differently: while the left action relies on the so-called *Kleisli predicate lifting* $\overline{\mathscr{D}}^\#$ [11, Def. 11], the right action uses the $\mathscr{D}$-algebra structure to compute the expectation of the payoff.

Finally, we define $\text{Op}_{\textbf{Prob}} = \text{colim}_J \big(A_J(X,Y) \times \text{ProbEquib}_J(X,Y) \big)$, which again automatically yields a symmetric monoidal category by our results. This links nicely to the definition of the equilibrium for sequential composition of probabilistic open games [11, Def. 12]: the definition in *loc.cit.* uses a conjunction whose conjuncts correspond to the left and right bimodule actions, respectively. This matches the arrow composition of $\text{Op}_{\textbf{Prob}}$ given by Theorem 29 (cf. proof of Theorem 20).

So far all examples have had sets of strategies. Its not surprising this does not have to be the case and there are other options. Our framework means this can be handled compositionally.

Example 38 (Strategies need not be Sets). Consider the parametrisation functor $\text{Para} : \textbf{Prof}(\mathscr{C}) \to \textbf{Prof}(\mathscr{C})$ defined by $(\text{Para}\, A)(X,Y) = \text{colim}_{J:\mathscr{C}_{\text{iso}}} A(J \otimes X, Y)$ [7, 8]. By our theorem on colimits of graded arrows, Para maps commutative arrows to commutative arrows and hence could replace Fam. More generally, given $F : \mathscr{B} \times \mathscr{C} \to \mathscr{C}$, we define $(\text{Para}_F\, A)(X,Y) = \text{colim}_{J:\mathscr{B}_{\text{iso}}} A(F(J,X),Y)$. Again Para_F maps commutative arrows to commutative arrows, and can hence be used to construct open games $\text{Para}_F(\textbf{Lens} \times \text{Equib}_M^\mathscr{C}(\textbf{Lens}))$ with strategies living in a different category than the play/coplay function.

Example 39 (Learners). If we take the category of lenses, omit the equilibria (or, equivalently, take equilibria valued in the trivial monoid), and use the Para construction above, we get Fong, Spivak and Tuyéras' monoidal category of learners [7]. That is, defining $\text{Learn} = \text{Para}(\textbf{Lens})$ makes learners an instance of our framework.

Example 40 (Effectful coplay and utility functions). Let $\mathscr{C}$ be a symmetric monoidal category. In Section 5, we used optics to define an arrow on $\mathscr{C} \times \mathscr{C}^{\text{op}}$, but the coend machinery used in its definition is quite heavy, e.g. one might worry about the existence of the coends. An alternative construction (folklore according to [27, §2.2]), instead defines an arrow on $\text{Comon}(\mathscr{C}) \times \mathscr{C}^{\text{op}}$, where $\text{Comon}(\mathscr{C})$ is the category of *commutative comonoids* in $\mathscr{C}$. The underlying profunctor is defined

by $\mathbf{Lens}_{\mathscr{C}}((X,S),(Y,R)) = \mathrm{Hom}_{\mathsf{Comon}(\mathscr{C})}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(X \otimes R, S)$, i.e. the same as for ordinary lenses, except that the play functions must respect the comonoid structure. In particular, If $\mathscr{C}$ is a Kleisli category for a commutative monad on a cartesian category, then every object has a natural comonoid structure. Further, in this case $\mathbf{Lens}_{\mathscr{C}}$ has projections at points, and so we can use Lemma 22 and Theorems 20 and 23 to equip it with a notion of equilibrium function $\mathbf{Lens}_{\mathscr{C}} \times E(\mathbf{Lens}_{\mathscr{C}})$. In the resulting notion of games $\mathsf{Fam}(\mathbf{Lens}_{\mathscr{C}} \times E(\mathbf{Lens}_{\mathscr{C}}))$, comonoid homomorphisms are intuitively 'pure' maps, so this notion of game allows 'effectful' coplay and utility functions, while the play function and the state in the equilibrium predicate remain pure.

7   Conclusions and future work

We have developed a new framework for open games. Conceptually, our framework is not just an axiomatisation of open games, but a compositional framework allowing us to vary and reorder components without having to redevelop the whole theory of the resulting open games. This is achieved using standard mathematical structures: arrows, bimodules and gradings. These structures allow us to define new arrows from old which is the hallmark of compositionality. Collectively, we demonstrate the utility of this framework by showing how a number of existing and new variants of open games can be shown to form monoidal categories with relative ease. This is important as the monoidal structure reflects the parallel and sequential composition of open games.

As is becoming clear, there is a huge variety of open games each with their pros and cons and each of which makes sense to use in some circumstances. We must now work to determine which of the categories of Open Games we have constructed are useful for Compositional Game Theory. To do this, we will need to gain experience in modelling game theoretic concepts in our categories, and to observe how modifying the constructions using the building blocks we have developed here affects the game theoretic models. We will be aided in this investigation by studying embeddings between the various categories of Open Games.

We have built tools to prove that categories are symmetric monoidal. However, there are other operators on games beyond parallel and sequential composition, e.g. an operator for repeated games is needed. A start is given by [10], but we want to integrate such operators into our general compositional framework.

All our profunctors have been of the form $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ but we could replace $\mathbf{Set}$ by some other cocomplete monoidal category with (we conjecture) minimal disruption. This would give us an enriched theory of open games which will be important for the investigation of topological games, with notions of limit games.

Finally, it will be interesting to see if this framework has ramifications for software for open games by providing all the benefits compositionality does. That is, rather than implementing a specific notion of open games and expecting users to use that notion, our compositional framework would allow users to build their own notion of open games suitable for their own needs (mixing both predefined components such as for example Fam or Para with their own specific definitions). The fact that we have used canonical categorical structures such as profunctors, modules and gradings means the road to a Haskell implementation should be a pleasant road to engage with. We hope to make it so as a matter of priority.

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A   Definition of a bimodule with costrength

For the sake of completeness, we spell in full details the definition of a costrength, and of a bimodule equipped with a costrength (used to define the notion of context, Definition 21).

Definition 41 (Costrength). A *(right) costrength* for a profunctor $F : \mathscr{C} \nrightarrow \mathscr{C}$ is a family of morphisms $\mathrm{cst}_{X,Y,Z} : F(X \otimes Z, Y \otimes Z) \to F(X,Y)$ (natural in X, Y and dinatural in Z) making two diagrams commute:

$$
\begin{array}{ccc}
F((X \otimes Z) \otimes Z', (Y \otimes Z) \otimes Z') & \xrightarrow{\ \mathrm{cst}_{Z'}\ } & F(X \otimes Z, Y \otimes Z) \\
{\scriptstyle F(\alpha^{-1}, \alpha)} \downarrow & & \downarrow {\scriptstyle \mathrm{cst}_Z} \\
F(X \otimes (Z \otimes Z'), Y \otimes (Z \otimes Z')) & \xrightarrow[\ \mathrm{cst}_{Z \otimes Z'}\ ]{} & F(X,Y)
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
F(X \otimes I, Y \otimes I) \\
{\scriptstyle F(\rho^{-1}, \rho)} \Big\downarrow\ \Big\downarrow {\scriptstyle \mathrm{cst}_I} \\
F(X,Y)
\end{array}
$$

Given a profunctor $F : \mathscr{C} \nrightarrow \mathscr{C}$ with a right costrength, assuming $\mathscr{C}$ is symmetric, we can define the left costrength $\mathrm{cst}'_{X,Y,Z} : F(Z \otimes X, Z \otimes Y) \to F(X,Y)$ by $\mathrm{cst}' := \mathrm{cst} \circ F(\sigma, \sigma)$.

When an A-bimodule has a strength, the two actions ℓ and r are required to be strong natural transformations. When it has a costrength, we require similar conditions:

Definition 42 (Bimodule with costrength). Let $A : \mathscr{C} \nrightarrow \mathscr{C}$ be an arrow. When an A-bimodule $B : \mathscr{C} \nrightarrow \mathscr{C}$ is equipped with a costrength, we also require the two commutative diagrams below, relating the strength of A, costrength of B and the left and right actions ℓ and r:

$$
\begin{array}{ccc}
A(X \otimes W, Y \otimes W) \times B(Y \otimes W, Z \otimes W) & \xrightarrow{\quad \ell \quad} & B(X \otimes W, Z \otimes W) \\
{\scriptstyle \mathrm{st}_W \times \mathrm{id}} \uparrow & & \downarrow \\
A(X,Y) \times B(Y \otimes W, Z \otimes W) & & {\scriptstyle \mathrm{cst}_W} \\
{\scriptstyle \mathrm{id} \times \mathrm{cst}_W} \downarrow & & \downarrow \\
A(X,Y) \times B(Y,Z) & \xrightarrow{\quad \ell \quad} & B(X,Z)
\end{array}
$$

and

$$
\begin{array}{ccc}
B(X \otimes W, Y \otimes W) \times A(Y \otimes W, Z \otimes W) & \xrightarrow{\quad r \quad} & B(X \otimes W, Z \otimes W) \\
{\scriptstyle \mathrm{id} \times \mathrm{st}_W} \uparrow & & \downarrow \\
B(X \otimes W, Y \otimes W) \times A(Y,Z) & & {\scriptstyle \mathrm{cst}_W} \\
{\scriptstyle \mathrm{cst}_W \times \mathrm{id}} \downarrow & & \downarrow \\
B(X,Y) \times A(Y,Z) & \xrightarrow{\quad r \quad} & B(X,Z)
\end{array}
$$

Finally, we also have a counterpart of the commutativity condition, mixing strength and costrength.

Definition 43 (Commutative bimodule with costrength). An A-bimodule B with a costrength is commutative when the following diagram commutes:

$$
\begin{array}{ccc}
A(X,Y) \times B(Y \otimes X', X \otimes Y') & \xrightarrow{\quad \cong \quad} & B(Y \otimes X', X \otimes Y') \times A(X,Y) \\
{\scriptstyle \mathrm{st}_{X'} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \mathrm{st}_{Y'}} \\
A(X \otimes X', Y \otimes X') \times B(Y \otimes X', X \otimes Y') & & B(Y \otimes X', X \otimes Y') \times A(X \otimes Y', Y \otimes Y') \\
{\scriptstyle \ell} \downarrow & & \downarrow {\scriptstyle r} \\
B(X \otimes X', X \otimes Y') & & B(Y \otimes X', Y \otimes Y') \\
& {\scriptstyle \mathrm{cst}'_X} \searrow \quad \swarrow {\scriptstyle \mathrm{cst}'_Y} & \\
& B(X', Y') &
\end{array}
$$