Since we have proved that for every primitive recursive function $f(\vec{x})$ there exists $\varphi \in \mathrm{I}\Sigma_1$ such that

$$\mathrm{I}\Sigma_1 \vdash \forall \vec{x}\ \exists! y\ \varphi(\vec{x}, y) \qquad (*)$$

we can extend the language of $\mathrm{I}\Sigma_1$ with symbols for every primitive recursive function $f(\vec{x})$, thus obtaining a conservative extension of $\mathrm{I}\Sigma_1$, which we denote by $\mathrm{I}\Sigma_1^*$. Since $(*)$ implies that $f(x) \neq y \leftrightarrow \exists z(z \neq y \wedge \varphi(\vec{x}, z))$, it is easy to see that every Σ_1 formula on the expanded language is equivalent to a Σ_1 formula on the original language of $\mathrm{I}\Sigma_1$; simply push all negations inside to atomic formulas and replace both $f(x) = y$ and $\neg(f(x) = y)$ by a Σ_1 formula. Thus we also have induction for $\mathrm{I}\Sigma_1$ formulas of the expanded language that involves extra functional symbols for all primitive recursive functions.

We now prove that if $\mathrm{I}\Sigma_1 \vdash \forall \vec{x}\ \exists y\ \varphi(\vec{x}, y)$ then there exits $f \in PR$ such that $\mathbb{N} \vDash \forall \vec{x}\ \varphi(\vec{x}, f(\vec{x}))$; in fact, we show that $\mathrm{I}\Sigma_1^* \vdash \forall \vec{x}\ \varphi(\vec{x}, f(\vec{x}))$. We first present a model theoretic proof based on the

Lemma 7. *If $\mathrm{I}\Sigma_1 \vdash \forall \vec{x}\ \exists y\ \varphi(\vec{x}, y)$ for $\varphi \in \Sigma_1$, then for some $f \in PR$,*

$$\mathrm{I}\Sigma_1^* \vdash \forall \vec{x}\ \exists y < f(\vec{x})\ \varphi(\vec{x}, f(\vec{x}))$$

Proof. Let $F(x, y)$ be the Ackermann function. Then $F(n, m) = k \leftrightarrow \exists \mathcal{C}$ ("$\mathcal{C}$ is a computation of $F(n, m) = k$"), as we showed before. Let us denote the formula

$$\exists \mathcal{C}(\text{“}\mathcal{C} \text{ is a computation of } F(x, y) = z\text{”})$$

by $F(x, y) \simeq z$, [because as we will see, $\mathrm{I}\Sigma_1 \nvdash \forall x \forall y \exists z(F(x, y) \cong z)$, i.e. $F(x, y)$ is not a total function in $\mathrm{I}\Sigma_1$.] Thus, $F(x, y)$ is not a functional symbol but just an abbreviation for $\exists \mathcal{C}$"$\mathcal{C}$ is a computation of ...". Since for each fixed n, $F(n, \vec{y})$ is primitive recursive,

$$\mathrm{I}\Sigma_1 \vdash \forall y\ \exists z F(\underline{n}, y) \cong z$$

Also the proof that for every primitive recursive function f there exist n such that for all x

$$f(x) < F(n, x)$$

can be formalized in $\mathrm{I}\Sigma_1$ because it uses only basic properties of f and induction on complexity of f. Similarly the proof that $F(x, y)$ is monotonic in both arguments can be formalized to show that

$$\mathrm{I}\Sigma_1 \vdash \forall x\ \forall y\ \forall \vec{x}\ \forall \vec{y}(x \leqslant \vec{x} \wedge y \leqslant \vec{y} \wedge \exists \vec{z}\ F(\vec{x}, \vec{y}) = \vec{z} \rightarrow \exists z\ F(x, y) = z)$$

$\square$

Lemma 8. *Let $m \vDash \mathrm{I}\Sigma_1, a, b \in m$ and assume that for all primitive recursive function f with no parameters we have $f(a) <^m b$. We denote this by $a \ll b$. Then there exists $c \in m$ such that $th(c) = a$ and such that $(c)_0 = a$ $(c)_a \leqslant b$ and $\forall i < a((c)_i \ll (c)_{i+1})$*

Proof. Since $a \ll b$, then for all n

$$m \vDash \underbrace{\forall y < \underline{n}\ \exists z < b\ F(y, 2a) \lesssim z}_{\Sigma_1 \text{ formula}}$$

Hence $F(y, 2a) \lesssim z$ means: there exists w s.t. $F(y, 2a) \simeq w < z$.
Thus, by overspill we have for some nonstandard d,

$$m \vDash \forall y \leqslant d\ \exists z < b\ (F(y, 2a) \cong z)$$

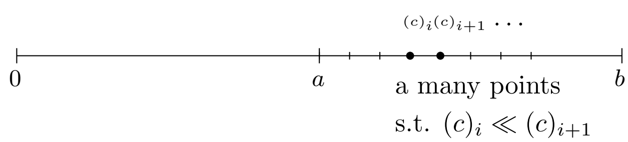
Since $F(x, y)$ is provably monotonic (in $\mathrm{I}\Sigma_1$) we also have

$$m \vDash \forall i < a\ \forall y \leqslant d\ \exists z < b\ (F(y, a + i) \cong z)$$

We let $(c)_i = F(d, a + i)$. Then

$$\begin{aligned} (c)_{i+1} &= F(d, a + i + 1) \\ &= F(d - 1, F(d, a + i)) \\ &= F(d - 1, (c)_i) \\ &> F(n, (c)_i) \end{aligned}$$

for all n, and thus $c_i \ll c_{i+1}$, and $(c)_a = F(d, a + a) < b$.



□

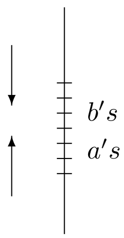
<u>Proof of the main lemma:</u> we now show that $\mathrm{I}\Sigma_1 \vdash \forall x\ \exists y\ \varphi(x, y) \Rightarrow \mathrm{I}\Sigma_1^* \vdash \forall \vec{x}\ \exists y < f(x)\ \varphi(x, f(x))$ for some PR function f.
Let $f_1 \dots f_n(x) \dots$ be an enumeration of all PR functions of x, and assume opposite: for all n, $\mathrm{I}\Sigma \nvdash \forall x\ \exists y < f_n(x)\ \varphi(x, f_n(x))$
<u>Claim:</u> Let c be a new constant. Then the above assumption implies that the theory $\mathbf{T}^* = \mathrm{I}\Sigma_1 + \{\neg\exists z < f_n(c)\ \varphi(c, f(c));\ n \in \mathbb{N}\}$ is consistent.

Proof. If $\mathrm{I}\Sigma_1 \vdash \exists z((z < f_n(c) \vee \cdots \vee z < f_n(c)) \wedge \varphi(c, f(c))$ then

$$\mathrm{I}\Sigma_1 \vdash \exists z < \max_{i<n} f_i(c)\ \varphi(c, f(c)) \quad \text{i.e}$$

$$\mathrm{I}\Sigma_1 \vdash \forall x \exists z < \max_{i<n} f_i(x)\ \varphi(x, f(x))$$

because c is a new symbol. But $\max_{i<n} f_i(x)$ is a PR function $\rightarrow\leftarrow$.
Let $m \vDash \mathbf{T}^*$; then $m \vDash \forall x\ \exists y\ \varphi(x, y)$; thus $m \vDash \exists y\ \varphi(a, y)$. Let b be s.t. $m \vDash \varphi(a, b)$. Then $a \ll b$. By our lemma, there exists a sequence $(c)_0 = a, (c)_a < b$.
We now build a submodel n of m s.t. $n \subset_e m$ (i.e n is an initial segment of m, i.e. $a \in n, b < a \wedge b \in m \Rightarrow b \in n$).



n will satisfy IΣ_1. Since m is assumed countable, we can enumerate its elements. □

Our construction will also produce a cut in m i.e. if $x \in m$ is s.t. for all i $x > a_i$ then $x \geqslant b_j$ for some j. Thus "$sup(a_s) = inf(b_s)$". We will ensure:

$$(1) \quad N = \{x | x < a_s \text{ for some } s \in w\}$$
$$(2) \quad (N, +, \cdot, 0, <) = n \vDash \mathrm{I}\Sigma_1$$

We make mod s construction picking in stages elements to ensure various parts of our requirements; let x_s be a listing of all elements of m. Let $a_0 = a, b_0 = b$.
<u>$n = 3s$:</u> Consider x_s if $x_s \leqslant a_{3s-1}$ or $x \geqslant b_{3s-1}$ then let $a_{3s} = a_{3s-1}$, $b_{3s} = b_{3s-1}$. If not then if $a_{3s-1} \ll x_s$ put $b_{3s} = x_s$, $a_{3s} = a_{3s-1}$ if not, then let $a_{3s+1} = x_s$, $b_{3s} = b_{3s-1}$.
<u>Claim:</u> If it is not $a_{3s-1} \ll x_s$, then $x_s \ll b_{3s}$, providing $a_{3s-1} \ll b_{3s-1}$.

Proof. If $a_{3s-1} \not\ll x_s$, then for some primitive recursive f, $f(a_{3s-1}) > x_s$. Similarly if $x_s \not\ll b_{3s-1}$, then $g(x_s) < b_{3s-1}$ for some primitive recursive g. Consider $G(x) = \max\limits_{y < f(x)} g(y)$, then obviously $G(x)$ is primitive recursive and $G(a_{3s-1}) > b_{3s-1} \rightarrow\leftarrow$. □

<u>$n = 3s + 1$:</u> Let $a_{3s+1} = a_{3s}^2$ $b_{3s+1} = b_{3s}$; obviously $a_{3s+1} \ll b_{3s+1}$ since $f(a_{3s}^2) > b_{3s} \rightarrow\leftarrow$ with $a_{3s} \ll b_{3s}$ with $F(x) = f(x^2)$.
<u>$n = 3s + 2$:</u> We assume that during the whole construction we have a listing of all finite sequences of the form $h_s = (\psi, e_i, \ldots, e_n, d)$ s.t. ψ is a Σ_1 formula of $L, e_i, \ldots, e_n, d \in m$, and listing is with <u>infinitely many repetitions</u>. Now, at stages $3s + 2$ we look at $h_s = (\psi, e_i, \ldots, e_n, d)$ and if for all $i \leqslant n$ $e_i < a_{3s+1}$ and $d < a_{3s+1}$, using our lemma and putting $\tilde{a} = a_{3s+1}, \tilde{b} = b_{3s+1}$ we can divide $(\tilde{a}, \tilde{b})$ in $\tilde{a}$ many parts s.t. $\tilde{a} = \alpha_0 \ll \alpha_1 \ll \ldots \ll \alpha_{\tilde{a}} < b$, let $\psi(\vec{x}, y) \equiv \exists t\ \psi^*(t, \vec{x}, y)$.
<u>Claim:</u> There is an $i < \tilde{a}$ s.t. for any $\vec{y} < d$ $\exists t < c_{i+1}$ $\psi^*(t, e_1 \ldots, e_n, \vec{y}) \rightarrow \exists t < c_i$ $\psi^*(t, e_1 \ldots, e_n, \vec{y})$ i.e. for no $\vec{y} < d$ interval $[c_i, c_{i+1})$ contains the least witness for $\exists t\ \psi^*(t, \vec{x}, \vec{y})[e_1, \ldots, e_n, \vec{y}]$

Proof. Assume opposite, define a mapping $\phi : d \rightarrow \tilde{a}$ s.t. $\phi(\vec{y}) = i \leftrightarrow \exists t < c_{i+1}$ $\psi^*(t, e_1 \ldots, e_n, \vec{y}) \wedge \neg\exists t < c_i$ ψ^*. Obviously ϕ is Σ_1 and is an <u>onto</u> mapping

of $d \to \tilde{a}$ for $d < \tilde{a}$ which is $\to\leftarrow$ by simple Σ_1 induction.
Let i be as in the claim, let $a_{3s+2} = c_i$ $b_{3s+2} = c_{i+1}$. $\square$

Claim: $N = \{x | x < a_i$ for some $i \in w\}$ with $+, \cdot, s, 0$ is a model of $\mathbf{T}^*$ containing a and not containing b.

Proof. $a \in N, b \notin N$. Steps $3s+1$ make sure that N is closed for $+, \cdot, T_\sigma$ see that $N \vDash \mathrm{I}\Sigma_1$, let $\psi \in \Sigma_1, \psi \equiv \exists z\ \psi^*(z, \vec{x}, y)$, and let $\vec{a} \in N, d \in N$. Then, since our list $\{h_s\}_{s \in W}$ has infinitely many repetitions of each member, if s is s.t. $\vec{a}, d < a_{3s+1}$ there is a $\hat{s} > s$ s.t. $h_{\hat{s}} \equiv (\psi, \vec{a}, d)$. By our construction, let c_i, c_{i+1} be s.t. $m \vDash \forall t < d\ (\exists z < c_{i+1}\ \psi^*(\vec{a}, z, t) \to \exists z < c_i\ \psi^*(\vec{a}, z, t))$. Since $c_i \in N, c_{i+1} \notin N$ we have: for all $t < d$ $n \vDash \exists z\ \psi^*(\vec{a}, z, t) \Rightarrow m \vDash \exists z < c_{i+1}\ \psi^*(\vec{a}, z, t)$. If $m \vDash \exists z < c_{i+1}\ \psi^*(\vec{a}, z, t) \to m \vDash \exists z < c_i\ \psi^*(\vec{a}, z, t) \Rightarrow n \vDash \exists z\ \psi^*(\vec{a}, z, t)$. Thus for all $\underline{t < d}$ $m \vDash \exists z < c_{i+1}\ \psi^*(\vec{a}, z, t) \leftrightarrow n \vDash \exists z\ \psi^*(\vec{a}, z, t)$.
Now assume $n \vDash \exists z\ \psi^*(\vec{a}, z, \vec{0}) \wedge \forall x(\exists z\ \psi^*(\vec{a}, z, t) \to \exists z\ \psi^*(\vec{a}, z, s(x)))$. We want to show $n \vDash \forall x(\exists z\ \psi^*(\vec{a}, z, x))$. Let b be arbitrary and let d be such that $a_i, b < d$. Then by the above

$$m \vDash \exists z < c_{i+1}\ \psi^*(\vec{a}, z, \vec{0}) \wedge \forall x < d (\exists z < c_{i+1}\ \psi^*(\vec{a}, z, x)) \to \exists z < c_{i+1}\ \psi^*(\vec{a}, z, s(x))$$

Namely, if for some $t < d-1$ $m \vDash \exists z < c_{i+1}\ \psi^*(a, z, t)$ then $n \vDash \exists z\ \psi^*(a, z, t) \to n \vDash \exists z\ \psi^*(a, z, s(t)) \to m \vDash \exists z < c_{i+1}\ \psi^*(a, z, s(t))$. Now we use induction in m applied on $x < d \to \exists z < c_{i+1}\ \psi^*(\vec{a}, z, x)$.
Thus at this step we use only $\triangle_0$ induction, but $\mathrm{I}\Sigma_1$ was needed to get the division of $\tilde{a} \ll \tilde{b}$.
Thus $m \vDash \forall x\ (x < d \to \exists z < c_{i+1}\ \psi^*(\vec{a}, z, x))$ and
so $m \vDash \exists z < c_{i+1}\ \psi^*(\vec{a}, z, b) \Rightarrow n \vDash \exists z\ \psi^*(\vec{a}, z, b)$.
Thus we have shown $n \vDash$ but then $n \vDash \forall x\ \exists y\ \varphi(x, y)$ and so $n \vDash \exists y\ \varphi(a, y)$ i.e for some $\tilde{b} \in |n|$ $n \vDash \varphi(a, \tilde{b}) \Rightarrow m \vDash \varphi(a, \tilde{b})$ because Σ_1-formulas are preserved upwards. But then $\to\leftarrow$ with $\tilde{b} < b$ and b was chosen least s.t. $m \vDash \varphi(a, b)$. $\square$

Corollary. *If* $\mathrm{I}\Sigma_1 \vdash \forall x\ \exists! y\ \varphi(x, y), \varphi \in \Sigma_1$ *then there is a primitive recursive functions* g *s.t.* $\mathrm{I}\Sigma_1 \vdash \forall x\ \varphi(x, g(x))$.

Proof. Assume $\mathrm{I}\Sigma_1 \vdash \forall x\ \exists! y\ \varphi(x, y)$, let $\varphi(x, y) = \exists z \varphi^*(x, y, z)$. Then φ^* is $\triangle_0$. Then $\mathrm{I}\Sigma_1 \vdash \forall x\ \exists y\ \exists z\ \varphi^*(x, y, z)$ and so $\mathrm{I}\Sigma_1 \vdash \forall x\ \exists w (\exists y < w\ \exists z < w\ \varphi^*(x, y, z))$. By the previous theorem, for some primitive recursive f,
$\mathrm{I}\Sigma_1 \vdash \forall x\ \exists w < f(x) (\exists y < w\ \exists z < w\ \varphi^*(x, y, z))$ and so
$\mathrm{I}\Sigma_1 \vdash \forall x\ \exists y < f(x)\ \exists z < f(x)\ \varphi^*(x, y, z)$.
Define

$$g(x) = y(\ \exists z < f(x) \varphi^*(x, y, z))$$

g is obviously primitive recursive and also $\mathrm{I}\Sigma_1 \vdash \forall x \varphi(x, g(x))$. Since $\mathbf{T} \vdash \forall x\ \exists! y \varphi(x, y)$ g is uniquely determined. $\square$

Corollary. *Ackermann's function is not provably total in* $\mathbf{T}_0$