COMP4161
Advanced Topics in Software Verification

fun

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Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7]
- Proof automation, Isar (part 2) [8[b]]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

General Recursion

The Choice

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➜ Limited expressiveness, automatic termination
  - primrec

➜ High expressiveness, termination proof may fail
  - fun

➜ High expressiveness, tweakable, termination proof manual
  - function

fun — examples

fun sep :: "'a ⇒ 'a list ⇒ 'a list"
where

  "sep a (x # y # zs) = x # a # sep a (y # zs)" |
  "sep a xs = xs"

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fun ack :: "nat ⇒ nat ⇒ nat"
where
"ack 0 n = Suc n" |
"ack (Suc m) 0 = ack m 1" |
"ack (Suc m) (Suc n) = ack m (ack (Suc m) n)"

fun

➜ Much more permissive than **primrec**:
  - pattern matching in all parameters
  - nested, linear constructor patterns
  - reads equations sequentially like in Haskell (top to bottom)
  - proves termination automatically in many cases
    (tries lexicographic order and datatype size)

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- ➜ May fail to prove termination:
  - use **function** instead
    - **function(sequential)** preserves sequential behaviour
  - allows you to prove termination manually

DEMO

Why Termination?

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MINI-DEMO

Conservative Extensions

These are some **definitional mechanisms** of Isabelle/HOL:

- **definition**
- **primrec**
- **inductive**
- **datatype** (sort of)
- **fun**
- **function**

They all add a new constant (or constants) and their defining facts.

They all try to make a **conservative extension** of the logic:

- new symbols, thus new type-correct statements
- some of these new statements are provable
- previously type-correct statements **should not change meaning**

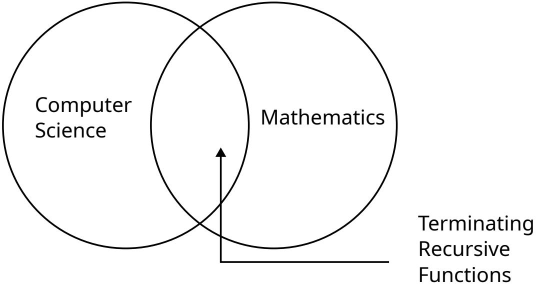
A Dramatic Aside

Ondřej Kunčar and Andrei Popescu
A Consistent Foundation for Isabelle/HOL
In ITP 2015, Nanjing

`https://andreipopescu.uk/pdf/ITP2015.pdf`

- discusses a (debatable) proof of `False` in Isabelle 2014.

Terminating Functions in the Intersection



If a recursive computational function *f* $:: \alpha \Rightarrow \beta$ terminates, then its type in the logic can be *f* $:: \alpha \Rightarrow \beta$.

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- Example **sep.induct**:

$$
\begin{array}{l}
[\![\bigwedge a.\ P\ a\ [];\\
\quad \bigwedge a\ w.\ P\ a\ [w]\\
\quad \bigwedge a\ x\ y\ zs.\ P\ a\ (y\#zs) \Longrightarrow P\ a\ (x\#y\#zs);\\
]\!] \Longrightarrow P\ a\ xs
\end{array}
$$

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- ➜ For most functions this works with a lexicographic termination relation.
- ➜ Sometimes not $\Rightarrow$ error message with unsolved subgoal
- ➜ You can prove termination separately.

function (sequential) quicksort **where**
quicksort [] = [] |
quicksort ($x \# xs$) = quicksort [$y \leftarrow xs. y \leq x$]@[$x$]@ quicksort [$y \leftarrow xs. x < y$]
by pat_completeness auto

termination
by (relation "measure length") (auto simp: less_Suc_eq_le)

DEMO

How does fun/function work? Option 1.

You may remember the previous explanation of how the **rec_list** constant (used by **primrec**) is defined via a relation.

For **fun** $f :: \alpha \Rightarrow \beta$, first define $f_{rel} :: (\alpha \times \beta)$ set.

- ➜ extract *recursion scheme* for equations in f
- ➜ define graph f_{rel} inductively, encoding recursion scheme
  - $f\ (\mathrm{Suc}\ x) = f\ x * 2 \mapsto (f_{rel} x v \longrightarrow f_{rel} (\mathrm{Suc}\ x)(v * 2))$
- ➜ prove totality (= termination)

- ➜ prove uniqueness (automatic)
- ➜ derive f and original equations from ϵ choice and f_{rel}
- ➜ export induction scheme from f_{rel}

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- ➜ prove totality (= termination)
  - recall that inductive relations are the least fixpoint
  - nonterminating recursion chains are not in the set
- ➜ prove uniqueness (automatic)
- ➜ derive f and original equations from ϵ choice and f_{rel}
- ➜ export induction scheme from f_{rel}

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- ➜ still have conditional equations for partial functions
- ➜ note that for $f :: \alpha \Rightarrow \beta$, this $f\_rel :: (\alpha \times \alpha)$ set.

DEMO

Proving Termination

termination fun_name sets up termination goal
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➜ **lexicographic_order** (default tried by **fun**)
➜ **size_change** (automated translation to simpler size-change graph[1])
➜ **relation R** (manual proof via well-founded relation)

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Well Founded Orders

Definition

$<_r$ is well founded if well founded induction holds

$\text{wf}(<_r) \equiv \forall P.\ (\forall x.\ (\forall y <_r x. P\ y) \longrightarrow P\ x) \longrightarrow (\forall x.\ P\ x)$

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Alternative definition (equivalent):

there are no infinite descending chains, or (equivalent):
the accessible part is everything, or (equivalent):
every nonempty set has a minimal element wrt $<_r$

$$\begin{array}{lcl} \text{min}\ (<_r)\ Q\ x & \equiv & \forall y \in Q.\ y \not<_r x \\ \text{wf}\ (<_r) & = & (\forall Q \neq \{\}.\ \exists m \in Q.\ \text{min}\ r\ Q\ m) \end{array}$$

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- $A <_r B = A \subset B \wedge \text{finite } B$ is well founded
- $\subseteq$ and $\subset$ in general are **not** well founded

More about well founded relations: *Term Rewriting and All That*

Extracting the Recursion Scheme

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Examples:

➜ **fun** fib **where**
 fib 0 = 1 |
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- **fun** f **where** f x = (if x = 0 then 0 else f (x - 1) * 2)

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- **fun** f **where** f x = (if x = 0 then 0 else f (x - 1) * 2)

  Recursion: $x \neq 0 \Longrightarrow x \leadsto x - 1$

Extracting the Recursion Scheme

Higher Order:

- **datatype** 'a tree = Leaf 'a | Branch 'a tree list

  fun treemap :: ('a ⇒ 'a) ⇒ 'a tree ⇒ 'a tree **where**
  treemap fn (Leaf n) = Leaf (fn n) |
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How does Isabelle extract context information for the call?

Extracting the Recursion Scheme

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Congruence Rules!

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Congruence Rules!

Recall rule **if_cong**:

$$\llbracket\ b = c;\ c \Longrightarrow x = u;\ \neg\ c \Longrightarrow y = v\ \rrbracket \Longrightarrow$$
$$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$$

Read: for transforming *x*, use *b* as context information, for *y* use $\neg b$.

Extracting the Recursion Scheme

Extracting context for equations

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Read: for transforming *x*, use *b* as context information, for *y* use ¬*b*.
In fun_def: for recursion in x, use *b* as context, for *y* use ¬*b*.

Congruence Rules for fun_defs

The same works for function definitions.

declare my_rule[fundef_cong]

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declare my_rule[fundef_cong]
(if_cong already added by default)

Another example (higher-order):
$[\![\text{xs} = \text{ys}; \bigwedge x.\ x \in \text{set ys} \Longrightarrow f\ x = g\ x]\!] \Longrightarrow \text{map}\ f\ \text{xs} = \text{map}\ g\ \text{ys}$

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Another example (higher-order):
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Read: for recursive calls in *f*, *f* is called with elements of *xs*

DEMO

Further Reading

Alexander Krauss,
Automating Recursive Definitions and Termination Proofs in Higher-Order Logic.
PhD thesis, TU Munich, 2009.

`https://www21.in.tum.de/~krauss/papers/krauss-thesis.pdf`

Ondřej Kunčar and Andrei Popescu
A Consistent Foundation for Isabelle/HOL
In ITP 2015

`https://andreipopescu.uk/pdf/ITP2015.pdf`

Rob Arthan
HOL constant definition done right
In ITP 2014

We have seen today ...

- ➜ General recursion with **fun**/**function**
- ➜ Induction over recursive functions
- ➜ How **fun** works
- ➜ Termination, partial functions, congruence rules