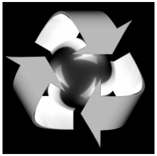
COMP4161 Advanced Topics in Software Verification



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Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7]
- Proof automation, Isar (part 2) [8[b]]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Datatypes

Example:

datatype 'a list = Nil | Cons 'a "'a list"

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Properties:

- ➜ Constructors:

Nil	::	'a list
Cons	::	'a $\Rightarrow$ 'a list $\Rightarrow$ 'a list

- ➜ Distinctness: Nil $\neq$ Cons x xs
- ➜ Injectivity: (Cons x xs $=$ Cons y ys) $=$ (x $=$ y $\wedge$ xs $=$ ys)

More Examples

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Mutual Recursion:

datatype even = EvenZero | EvenSucc odd
and odd = OddSucc even

Nested

Nested recursion:

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➜ Recursive call is under a type constructor.

The General Case

$$\textbf{datatype}\ (\alpha_1, \ldots, \alpha_n)\ \tau \quad = \quad \mathsf{C}_1\ \tau_{1,1}\ \ldots\ \tau_{1,n_1}$$
$$\mid \quad \ldots$$
$$\mid \quad \mathsf{C}_k\ \tau_{k,1}\ \ldots\ \tau_{k,n_k}$$

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Distinctness and Injectivity applied automatically

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More detail: Tutorial on (Co-)datatypes Definitions at isabelle.in.tum.de

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Because: Cantor's theorem (α set is larger than α)

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[*] BNF = Bounded Natural Functors.

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- ➜ BNF defines well-behaved type constructors, ie where recursion is allowed
- ➜ datatypes automatically are BNFs (that's how they are constructed)
- ➜ can register other type constructors as BNFs — not covered here**

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** *Defining (Co)datatypes and Primitively (Co)recursive Functions in Isabelle/HOL*

Case

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- ➜ Binds weakly, needs () in context

Cases

apply (case_tac *t*)

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apply (case_tac t)

creates k subgoals

$$\llbracket t = C_i\ x_1 \ldots x_p; \ldots \rrbracket \Longrightarrow \ldots$$

one for each constructor C_i

Demo

Recursion

Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

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! All functions in HOL must be total !

Primitive Recursion

primrec guarantees termination structurally

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where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

The General Case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \tau \Rightarrow \tau'$ can be defined by **primitive recursion**:

$$\begin{array}{lcl} f\ (C_1\ y_{1,1}\ \ldots\ y_{1,n_1}) & = & r_1 \\ \vdots & & \\ f\ (C_k\ y_{k,1}\ \ldots\ y_{k,n_k}) & = & r_k \end{array}$$

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The recursive calls in r_i must be **structurally smaller**
(of the form $f\ a_1\ \ldots\ y_{i,j}\ \ldots\ a_p$)

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Example: rec_list :: "'a ⇒ ('b ⇒ 'b list ⇒ 'a ⇒ 'a) ⇒ 'b list ⇒ 'a"

$$\begin{array}{lcl} \text{rec\_list } f_1 \; f_2 \text{ Nil} & = & f_1 \\ \text{rec\_list } f_1 \; f_2 \text{ (Cons } x \; xs\text{)} & = & f_2 \; x \; xs \text{ (rec\_list } f_1 \; f_2 \; xs\text{)} \end{array}$$

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$\text{app} \equiv \text{rec\_list } (\lambda ys.\ ys)\ (\lambda x\ xs\ xs'.\ \lambda ys.\ \text{Cons } x\ (xs'\ ys))$

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
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$$\frac{}{(\mathrm{Nil}, f_1) \in \mathrm{list\_rel}\ f_1\ f_2} \qquad \frac{(xs, xs') \in \mathrm{list\_rel}\ f_1\ f_2}{(\mathrm{Cons}\ x\ xs, f_2\ x\ xs\ xs') \in \mathrm{list\_rel}\ f_1\ f_2}$$

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$\mathrm{rec\_list}\ f_1\ f_2\ xs \equiv \mathrm{THE}\ y.\ (xs, y) \in \mathrm{list\_rel}\ f_1\ f_2$
Automatic proof that set def indeed is total function
(the equations for rec_list are lemmas!)

Predefined Datatypes

nat is a datatype

datatype nat $=$ 0 $|$ Suc nat

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Functions on nat definable by primrec!

primrec

$$\begin{array}{lcl} f\ 0 & = & \ldots \\ f\ (\mathrm{Suc}\ n) & = & \ldots\ f\ n\ \ldots \end{array}$$

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'b ⇒ 'a option $\sim$ partial function:

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Example:
primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option
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primrec lookup :: 'k $\Rightarrow$ ('k $\times$ 'v) list $\Rightarrow$ 'v option
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lookup k [] = None |
lookup k (x #xs) = (if fst x = k then Some (snd x) else lookup k xs)

Demo

PRIMREC

INDUCTION

Structural induction

P xs holds for all lists *xs* if

- ➜ *P* Nil
- ➜ and for arbitrary *x* and *xs*, $P\ xs \Longrightarrow P\ (x \# xs)$

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  $[\![P\ [\,];\ \bigwedge a\ list.\ P\ list \Longrightarrow P\ (a\#list)]\!] \Longrightarrow P\ list$
- ➜ General proof method for induction: **(induct x)**
  - x must be a free variable in the first subgoal.
  - type of x must be a datatype.

Basic heuristics

Theorems about recursive functions are proved by induction

Induction on argument number *i* of *f*
if *f* is defined by recursion on argument number *i*

Example

A tail recursive list reverse:

primrec itrev :: 'a list ⇒ 'a list ⇒ 'a list
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itrev [] $\qquad ys =$ |

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primrec itrev :: 'a list $\Rightarrow$ 'a list $\Rightarrow$ 'a list
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lemma itrev xs [] $=$ rev xs

Demo

Proof Attempt

Generalisation

Replace constants by variables

lemma itrev $xs\ ys$ = rev $xs @ ys$

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Quantify free variables by $\forall$
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Quantify free variables by ∀
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lemma ∀*ys*. itrev *xs ys* = rev *xs*@*ys*

Or: **apply (induct xs arbitrary: ys)**

We have seen today ...

- ➜ Datatypes
- ➜ Primitive recursion
- ➜ Case distinction
- ➜ Structural Induction

Exercises

- ➜ define a primitive recursive function **lsum** :: nat list ⇒ nat that returns the sum of the elements in a list.
- ➜ show "$2 * \text{lsum}\ [0..< \mathit{Suc}\ n] = n * (n + 1)$"
- ➜ show "$\text{lsum (replicate}\ n\ a) = n * a$"
- ➜ define a function **lsumT** using a tail recursive version of listsum.
- ➜ show that the two functions are equivalent: lsum *xs* = lsumT *xs*