COMP4161 Advanced Topics in Software Verification

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- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- Proof & Specification Techniques
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[a] a1 due; [b] a2 due; [c] a3 due

Last Time

- Sets
- Type Definitions
- Inductive Definitions

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Inductive Definitions

How They Work

The Nat Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

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- ➜ Objective: **no junk**. Only what must be in X shall be in X.

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Why the smallest set?

- Objective: **no junk**. Only what must be in X shall be in X.
- Gives rise to a nice proof principle (rule induction)

Formally

$$\text{Rules } \frac{a_1 \in X \quad \dots \quad a_n \in X}{a \in X} \text{ with } a_1, \dots, a_n, a \in A$$

define set $X \subseteq A$

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Example:

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Example:

$$\begin{array}{lcl} R & \equiv & \{(\{\}, 0)\} \cup \{(\{n\}, n+1).\ n \in \mathbb{R}\} \\ \hat{R}\; \{3, 6, 10\} & = & \end{array}$$

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Definition: B is R-closed iff $\hat{R}\ B \subseteq B$

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This does always exist:

Fact: $X = \bigcap\{B \subseteq A.\ B\ R{-}\text{closed}\}$

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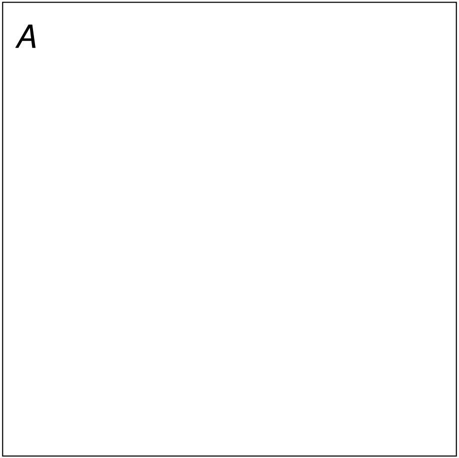

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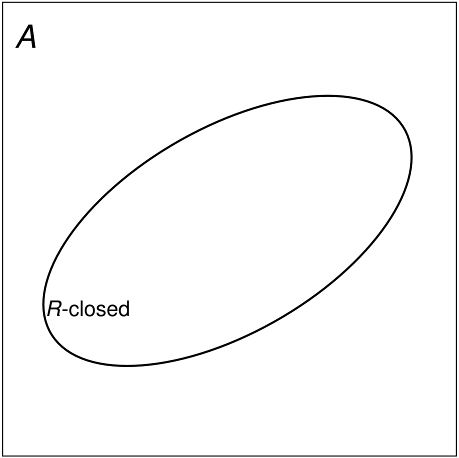

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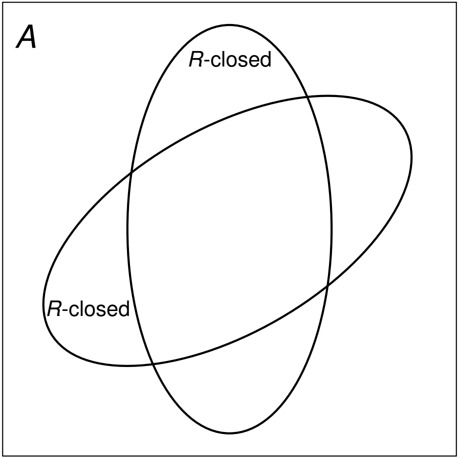

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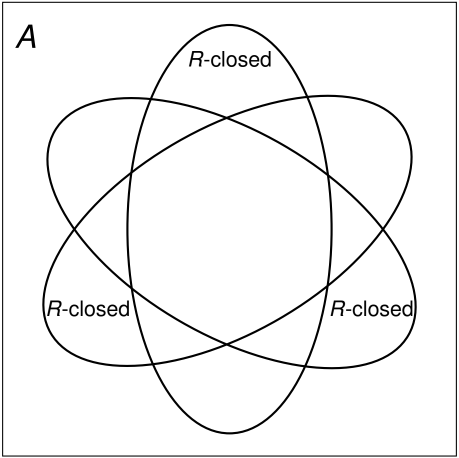

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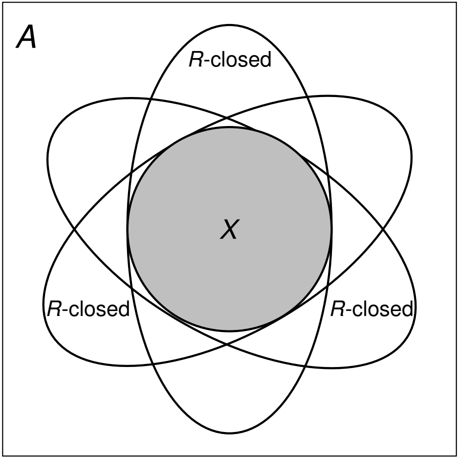

Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

induces induction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n+1)]\!] \Longrightarrow \forall x \in N.\ P\ x$$

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In general:

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

Why does this work?

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qed

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Rules with side conditions

$$\frac{a_1 \in X \quad \dots \quad a_n \in X \qquad C_1 \quad \dots \quad C_m}{a \in X}$$

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Rules with side conditions

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induction scheme:

$$\begin{aligned}(\forall(\{a_1, \dots a_n\}, a) \in R.\ & P\ a_1 \wedge \dots \wedge P\ a_n \wedge \\ & C_1 \wedge \dots \wedge C_m \wedge \\ & \{a_1, \dots, a_n\} \subseteq X \Longrightarrow P\ a)\end{aligned}$$

$$\Longrightarrow$$

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X as Fixpoint

How to compute X?

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$X = \bigcap\{B \subseteq A.\ B\ R-\text{closed}\}$ hard to work with.

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$$\vdots$$
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$$X_\omega = \bigcup_{n \in \mathbb{N}} (\hat{R}^n\ \{\}) = X$$

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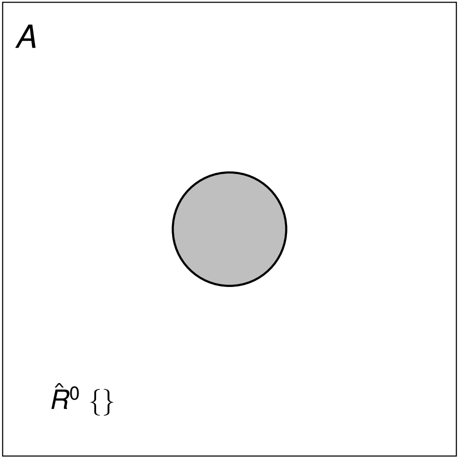

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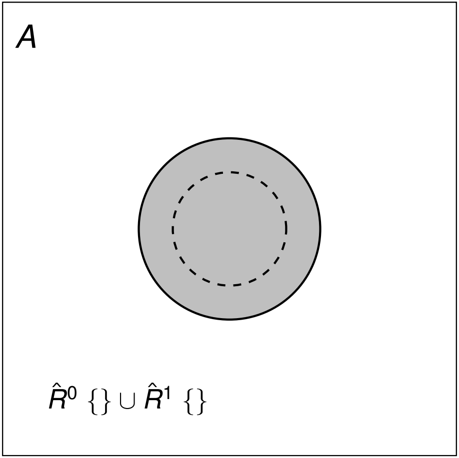

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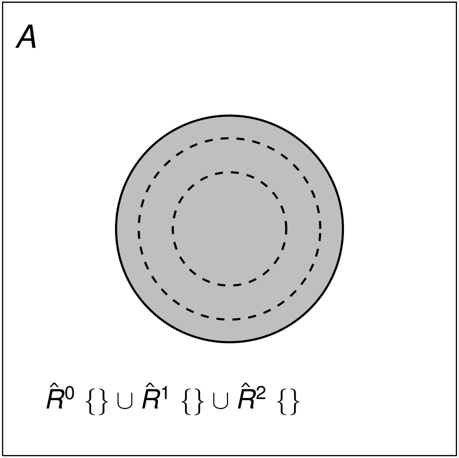

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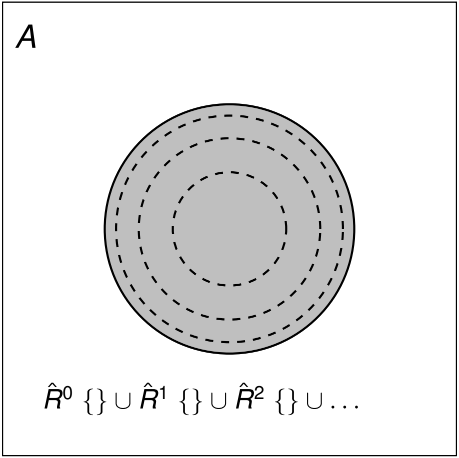

Does this always work?

Knaster-Tarski Fixpoint Theorem:
Let $(A, \leq)$ be a complete lattice, and $f :: A \Rightarrow A$ a monotone function.
Then the fixpoints of f again form a complete lattice.

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Implications:

➜ least and greatest fixpoints exist (complete lattice always non-empty).

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Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

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Exercise

Formalize this lecture in Isabelle:

- Define **closed** $f\ A :: (\alpha \text{ set} \Rightarrow \alpha \text{ set}) \Rightarrow \alpha \text{ set} \Rightarrow \text{bool}$
- Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)
- Define **lfpt** f as the intersection of all f-closed sets
- Show that lfpt f is a fixpoint of f if f is monotone
- Show that lfpt f is the least fixpoint of f
- Declare a constant $R :: (\alpha \text{ set} \times \alpha) \text{ set}$
- Define $\hat{R} :: \alpha \text{ set} \Rightarrow \alpha \text{ set}$ in terms of R
- Show soundness of rule induction using R and lfpt $\hat{R}$

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We have learned today ...

- ➜ Formal background of inductive definitions
- ➜ Definition by intersection
- ➜ Computation by iteration
- ➜ Formalisation in Isabelle