COMP4161 Advanced Topics in Software Verification

{}

Thomas Sewell, Miki Tanaka, Rob Sison

T3/2024

Content

- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

- Conditional term rewriting

Last Time

- ➜ Conditional term rewriting
- ➜ Case Splitting with the simplifier

Last Time

➜ Conditional term rewriting
➜ Case Splitting with the simplifier
➜ Congruence rules

Last Time

- ➜ Conditional term rewriting
- ➜ Case Splitting with the simplifier
- ➜ Congruence rules
- ➜ AC Rules

Last Time

- ➜ Conditional term rewriting
- ➜ Case Splitting with the simplifier
- ➜ Congruence rules
- ➜ AC Rules
- ➜ Knuth-Bendix Completion (Waldmeister)

Last Time

- ➜ Conditional term rewriting
- ➜ Case Splitting with the simplifier
- ➜ Congruence rules
- ➜ AC Rules
- ➜ Knuth-Bendix Completion (Waldmeister)
- ➜ Orthogonal Rewrite Systems

Specification Techniques

Sets

Sets in Isabelle

Type **'a set**: sets over type 'a

Sets in Isabelle

Type **'a set**: sets over type 'a

➜ $\{\}$, $\{e_1, \ldots, e_n\}$, $\{x.\ P\ x\}$

Sets in Isabelle

Type **'a set**: sets over type 'a

- ➜ $\{\}$, $\{e_1, \ldots, e_n\}$, $\{x.\ P\ x\}$
- ➜ $e \in A$, $A \subseteq B$

Sets in Isabelle

Type **'a set**: sets over type 'a

- $\{\}$, $\{e_1, \ldots, e_n\}$, $\{x.\ P\ x\}$
- $e \in A$, $A \subseteq B$
- $A \cup B$, $A \cap B$, $A - B$, $-A$

Sets in Isabelle

Type **'a set**: sets over type 'a

- $\{\}$, $\{e_1, \ldots, e_n\}$, $\{x.\ P\ x\}$
- $e \in A$, $A \subseteq B$
- $A \cup B$, $A \cap B$, $A - B$, $-A$
- $\bigcup x \in A.\ B\ x$, $\bigcap x \in A.\ B\ x$, $\bigcap\ A$, $\bigcup A$

Sets in Isabelle

Type **'a set**: sets over type 'a

- $\{\}$, $\{e_1, \ldots, e_n\}$, $\{x.\ P\ x\}$
- $e \in A$, $A \subseteq B$
- $A \cup B$, $A \cap B$, $A - B$, $-A$
- $\bigcup x \in A.\ B\ x$, $\bigcap x \in A.\ B\ x$, $\bigcap\ A$, $\bigcup A$
- $\{i..j\}$

Sets in Isabelle

Type **'a set**: sets over type 'a

- $\{\}$, $\{e_1, \ldots, e_n\}$, $\{x.\ P\ x\}$
- $e \in A$, $A \subseteq B$
- $A \cup B$, $A \cap B$, $A - B$, $-A$
- $\bigcup x \in A.\ B\ x$, $\bigcap x \in A.\ B\ x$, $\bigcap A$, $\bigcup A$
- $\{i..j\}$
- insert :: $\alpha \Rightarrow \alpha$ set $\Rightarrow \alpha$ set

Sets in Isabelle

Type **'a set**: sets over type 'a

- $\{\}$, $\{e_1, \ldots, e_n\}$, $\{x.\ P\ x\}$
- $e \in A$, $A \subseteq B$
- $A \cup B$, $A \cap B$, $A - B$, $-A$
- $\bigcup x \in A.\ B\ x$, $\bigcap x \in A.\ B\ x$, $\bigcap A$, $\bigcup A$
- $\{i..j\}$
- insert :: $\alpha \Rightarrow \alpha$ set $\Rightarrow \alpha$ set
- $f`A \equiv \{y.\ \exists x \in A.\ y = f\ x\}$
- ...

Proofs about Sets

Natural deduction proofs:

- equalityI: $[\![A \subseteq B;\ B \subseteq A]\!] \Longrightarrow A = B$

Proofs about Sets

Natural deduction proofs:

- equalityI: $[\![A \subseteq B;\ B \subseteq A]\!] \Longrightarrow A = B$
- subsetI: $(\bigwedge x.\ x \in A \Longrightarrow x \in B) \Longrightarrow A \subseteq B$

Proofs about Sets

Natural deduction proofs:

- equalityI: $[\![A \subseteq B;\ B \subseteq A]\!] \Longrightarrow A = B$
- subsetI: $(\bigwedge x.\ x \in A \Longrightarrow x \in B) \Longrightarrow A \subseteq B$
- ... **find_theorems**

Bounded Quantifiers

➜ $\forall x \in A.\ P\ x$

Bounded Quantifiers

- $\forall x \in A.\ P\ x \equiv \forall x.\ x \in A \longrightarrow P\ x$

Bounded Quantifiers

- $\forall x \in A.\ P\ x \equiv \forall x.\ x \in A \longrightarrow P\ x$
- $\exists x \in A.\ P\ x$

Bounded Quantifiers

- $\forall x \in A.\ P\ x \equiv \forall x.\ x \in A \longrightarrow P\ x$
- $\exists x \in A.\ P\ x \equiv \exists x.\ x \in A \wedge P\ x$

Bounded Quantifiers

- $\forall x \in A.\ P\ x \equiv \forall x.\ x \in A \longrightarrow P\ x$
- $\exists x \in A.\ P\ x \equiv \exists x.\ x \in A \wedge P\ x$
- ballI: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \Longrightarrow \forall x \in A.\ P\ x$
- bspec: $[\![\forall x \in A.\ P\ x; x \in A]\!] \Longrightarrow P\ x$

Bounded Quantifiers

- $\forall x \in A.\ P\ x \equiv \forall x.\ x \in A \longrightarrow P\ x$
- $\exists x \in A.\ P\ x \equiv \exists x.\ x \in A \wedge P\ x$
- ballI: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \Longrightarrow \forall x \in A.\ P\ x$
- bspec: $[\![\forall x \in A.\ P\ x; x \in A]\!] \Longrightarrow P\ x$
- bexI: $[\![P\ x; x \in A]\!] \Longrightarrow \exists x \in A.\ P\ x$
- bexE: $[\![\exists x \in A.\ P\ x; \bigwedge x.\ [\![x \in A; P\ x]\!] \Longrightarrow Q]\!] \Longrightarrow Q$

DEMO

SETS

The Three Basic Ways of Introducing Theorems

➜ **Axioms**:

Example: **axiomatization where** refl: "$t = t$"

The Three Basic Ways of Introducing Theorems

➜ **Axioms**:

Example: **axiomatization where** refl: "$t = t$"

Do not use. Evil. Can make your logic inconsistent.

The Three Basic Ways of Introducing Theorems

- **Axioms**:

  Example: **axiomatization where** refl: "$t = t$"

  Do not use. Evil. Can make your logic inconsistent.

- **Definitions:**

  Example: **definition** inj **where** "inj $f \equiv \forall x\ y.\ f\ x = f\ y \longrightarrow x = y$"

The Three Basic Ways of Introducing Theorems

➜ **Axioms**:

Example: **axiomatization where** refl: "$t = t$"

Do not use. Evil. Can make your logic inconsistent.

➜ **Definitions:**

Example: **definition** inj **where** "inj $f \equiv \forall x\ y.\ f\ x = f\ y \longrightarrow x = y$"
Introduces a new lemma called inj_def.

The Three Basic Ways of Introducing Theorems

➜ **Axioms**:

Example: **axiomatization where** refl: "$t = t$"

Do not use. Evil. Can make your logic inconsistent.

➜ **Definitions:**

Example: **definition** inj **where** "inj $f \equiv \forall x\ y.\ f\ x = f\ y \longrightarrow x = y$"
Introduces a new lemma called inj_def.

➜ **Proofs:**

Example: **lemma** "inj $(\lambda x.\ x + 1)$"

The Three Basic Ways of Introducing Theorems

- **Axioms**:

  Example: **axiomatization where** refl: "$t = t$"

  Do not use. Evil. Can make your logic inconsistent.

- **Definitions:**

  Example: **definition** inj **where** "inj $f \equiv \forall x\ y.\ f\ x = f\ y \longrightarrow x = y$"
  Introduces a new lemma called inj_def.

- **Proofs:**

  Example: **lemma** "inj $(\lambda x.\ x + 1)$"

  The harder, but safe choice.

The Three Basic Ways of Introducing Types

- **typedecl**: by name only

  Example: **typedecl** names
  Introduces new type *names* without any further assumptions

The Three Basic Ways of Introducing Types

- **typedecl**: by name only

  Example: **typedecl** names
  Introduces new type *names* without any further assumptions

- **type_synonym**: by abbreviation

  Example: **type_synonym** α rel = "$\alpha \Rightarrow \alpha \Rightarrow$ *bool*"
  Introduces abbreviation *rel* for existing type $\alpha \Rightarrow \alpha \Rightarrow$ *bool*
  Type abbreviations are immediately expanded internally

The Three Basic Ways of Introducing Types

- **typedecl**: by name only

  Example: **typedecl** names
  Introduces new type *names* without any further assumptions

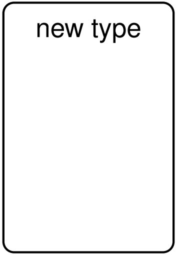
- **type_synonym**: by abbreviation

  Example: **type_synonym** α rel = "$\alpha \Rightarrow \alpha \Rightarrow$ *bool*"
  Introduces abbreviation *rel* for existing type $\alpha \Rightarrow \alpha \Rightarrow$ *bool*
  Type abbreviations are immediately expanded internally

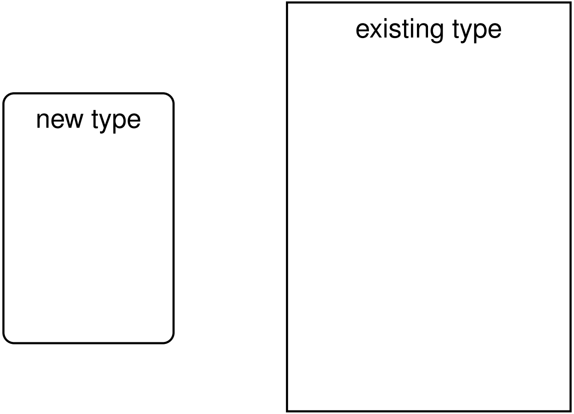
- **typedef**: by definition as a set

  Example: **typedef** new_type = "{some set}" <proof>
  Introduces a new type as a subset of an existing type.
  The proof shows that the set on the rhs in non-empty.

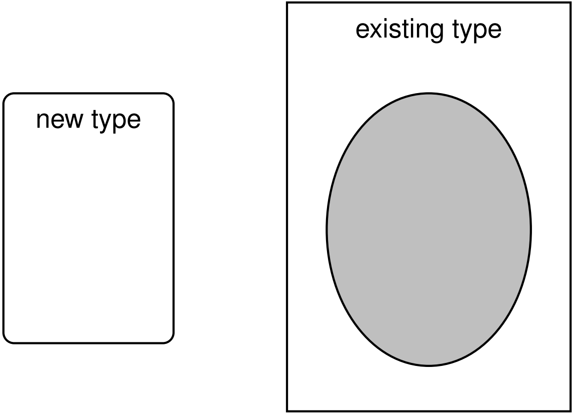
How typedef works



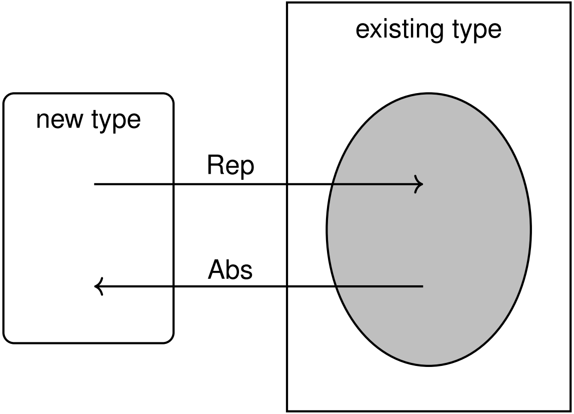
How typedef works



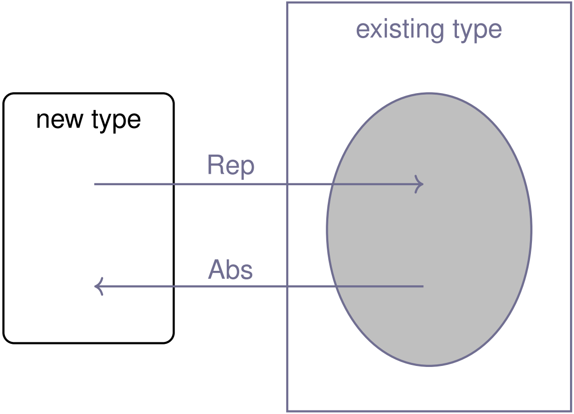
How typedef works



How typedef works



How typedef works



Example: Pairs

$$(\alpha, \beta)\ \text{Prod}$$

① Pick existing type:

Example: Pairs

$$(\alpha, \beta)\ \text{Prod}$$

① Pick existing type: $\alpha \Rightarrow \beta \Rightarrow \text{bool}$

② Identify subset:

Example: Pairs

$$(\alpha, \beta) \text{ Prod}$$

① Pick existing type: $\alpha \Rightarrow \beta \Rightarrow \text{bool}$

② Identify subset:
$(\alpha, \beta) \text{ Prod} = \{f.\ \exists a\ b.\ f = \lambda(x :: \alpha)\ (y :: \beta).\ x = a \wedge y = b\}$

③ We get from Isabelle:

Example: Pairs

$$(\alpha, \beta)\ \text{Prod}$$

① Pick existing type: $\alpha \Rightarrow \beta \Rightarrow \text{bool}$

② Identify subset:
$(\alpha, \beta)\ \text{Prod} = \{f.\ \exists a\ b.\ f = \lambda(x :: \alpha)\ (y :: \beta).\ x = a \wedge y = b\}$

③ We get from Isabelle:

- functions Abs_Prod, Rep_Prod
- both injective
- Abs_Prod (Rep_Prod x) $= x$

④ We now can:

Example: Pairs

$$(\alpha, \beta)\ \mathsf{Prod}$$

① Pick existing type: $\alpha \Rightarrow \beta \Rightarrow$ bool

② Identify subset:
(α, β) Prod $= \{f.\ \exists a\ b.\ f = \lambda(x :: \alpha)\ (y :: \beta).\ x = a \wedge y = b\}$

③ We get from Isabelle:

- functions Abs_Prod, Rep_Prod
- both injective
- Abs_Prod (Rep_Prod x) $= x$

④ We now can:

- define constants Pair, fst, snd in terms of Abs_Prod and Rep_Prod
- derive all characteristic theorems
- forget about Rep/Abs, use characteristic theorems instead

Demo

Introducing new Types

Inductive Definitions

Example

$$\frac{}{\langle \mathsf{skip}, \sigma \rangle \longrightarrow \sigma} \qquad \frac{[\![e]\!]\sigma = v}{\langle \mathsf{x} := \mathsf{e}, \sigma \rangle \longrightarrow \sigma[x \mapsto v]}$$

$$\frac{\langle c_1, \sigma \rangle \longrightarrow \sigma' \quad \langle c_2, \sigma' \rangle \longrightarrow \sigma''}{\langle c_1 ; c_2, \sigma \rangle \longrightarrow \sigma''}$$

$$\frac{[\![b]\!]\sigma = \mathsf{False}}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma \rangle \longrightarrow \sigma}$$

$$\frac{[\![b]\!]\sigma = \mathsf{True} \quad \langle c, \sigma \rangle \longrightarrow \sigma' \quad \langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma' \rangle \longrightarrow \sigma''}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma \rangle \longrightarrow \sigma''}$$

What does this mean?

What does this mean?

- ➜ $\langle c, \sigma \rangle \longrightarrow \sigma'$ fancy syntax for a relation $(c, \sigma, \sigma') \in E$

What does this mean?

- $\langle c, \sigma \rangle \longrightarrow \sigma'$ fancy syntax for a relation $(c, \sigma, \sigma') \in E$
- relations are sets: E :: (com $\times$ state $\times$ state) set

What does this mean?

- $\langle c, \sigma \rangle \longrightarrow \sigma'$ fancy syntax for a relation $(c, \sigma, \sigma') \in E$
- relations are sets: E :: (com $\times$ state $\times$ state) set
- the rules define a set inductively

What does this mean?

- $\langle c, \sigma \rangle \longrightarrow \sigma'$ fancy syntax for a relation $(c, \sigma, \sigma') \in E$
- relations are sets: E :: (com $\times$ state $\times$ state) set
- the rules define a set inductively

But which set?

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

➜ N is the set of natural numbers $\mathbb{N}$

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

- ➜ N is the set of natural numbers $\mathbb{N}$
- ➜ But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n + 1 \in \mathbb{R}$

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

- N is the set of natural numbers $\mathbb{N}$
- But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n + 1 \in \mathbb{R}$
- $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

- ➜ N is the set of natural numbers $\mathbb{N}$
- ➜ But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n + 1 \in \mathbb{R}$
- ➜ $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

- N is the set of natural numbers $\mathbb{N}$
- But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n + 1 \in \mathbb{R}$
- $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

- Objective: **no junk**. Only what must be in X shall be in X.

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

- ➜ N is the set of natural numbers $\mathbb{N}$
- ➜ But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n+1 \in \mathbb{R}$
- ➜ $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

- ➜ Objective: **no junk**. Only what must be in X shall be in X.
- ➜ Gives rise to a nice proof principle (rule induction)

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

- N is the set of natural numbers $\mathbb{N}$
- But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n + 1 \in \mathbb{R}$
- $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

- Objective: **no junk**. Only what must be in X shall be in X.
- Gives rise to a nice proof principle (rule induction)
- Alternative (greatest set) occasionally also useful: coinduction

Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

induces induction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n + 1)]\!] \Longrightarrow \forall x \in N.\ P\ x$$

DEMO

INDUCTIVE DEFINITIONS

We have learned today ...

- ➜ Sets

We have learned today ...

- ➜ Sets
- ➜ Type Definitions

We have learned today ...

- ➜ Sets
- ➜ Type Definitions
- ➜ Inductive Definitions

