COMP4161
Advanced Topics in Software Verification

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Thomas Sewell, Miki Tanaka, Rob Sison

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Content

- ➜ Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

➜ Equations and Term Rewriting

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is **applicable** to term $t[s]$ with σ if

- $\sigma\ l = s$ and
- $\sigma\ P_1, \dots, \sigma\ P_n$ are provable by rewriting.

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lemma "$f\ x = g\ x \wedge g\ x = f\ x \Longrightarrow f\ x = 2$"

simp	**use and simplify** assumptions
(simp (no_asm))	**ignore** assumptions
(simp (no_asm_use))	**simplify**, but do **not use** assumptions
(simp (no_asm_simp))	**use**, but do **not simplify** assumptions

Preprocessing

Preprocessing (recursive) for maximal simplification power:

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\begin{array}{rcl}
\neg A & \mapsto & A = \mathit{False} \\
A \longrightarrow B & \mapsto & A \Longrightarrow B \\
A \wedge B & \mapsto & A,\ B \\
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$$p \Longrightarrow q = True \qquad p \Longrightarrow r = False \qquad s = True$$

DEMO

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Similar for any data type t: **t.split**

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if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
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➜ use locally with e.g. **apply** (simp cong: <rule>)

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For types nat, int etc:

- lemmas **add_ac** sort any sum ($+$)
- lemmas **mult_ac** sort any product ($*$)

Example: **apply** (simp add: add_ac) yields
$(b + c) + a \leadsto \cdots \leadsto a + (b + c)$

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If these 3 rules are present for an AC operator
Isabelle will order terms correctly

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(1)+(3)	$\{x \mapsto g\ z\}$	$a \xleftarrow{(1)} f\ (g\ z) \xrightarrow{(3)} b$
(3)+(2)	$\{z \mapsto y\}$	$b \xleftarrow{(3)} f\ (g\ y) \xrightarrow{(2)} f\ b$

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This is the main idea of the Knuth-Bendix completion algorithm.

Demo: Waldmeister

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Application: functional programming languages

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- Congruence rules
- AC rules
- More on confluence

