COMP4161
Advanced Topics in Software Verification

$\longrightarrow$

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T3/2024

Content

- **Foundations & Principles**
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]
- **Proof & Specification Techniques**
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time on HOL

- ➜ Defining HOL

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- ➜ Higher Order Abstract Syntax

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- ➜ More automation

Term Rewriting

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Given a set of equations

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\begin{aligned}
l_1 &= r_1 \\
l_2 &= r_2 \\
&\vdots \\
l_n &= r_n
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Applications in:

- ➜ **Mathematics** (algebra, group theory, etc)
- ➜ **Functional Programming** (model of execution)
- ➜ **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$l_1 \longrightarrow r_1$$
$$l_2 \longrightarrow r_2$$
$$\vdots$$
$$l_n \longrightarrow r_n$$

decide $l = r$ **by deciding** $l \stackrel{*}{\longleftrightarrow} r$

Arrow Cheat Sheet

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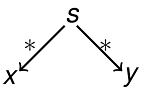
Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.

Confluence



Problem:
is a given set of reduction rules confluent?

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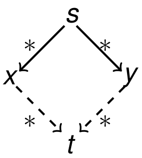

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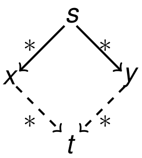

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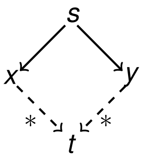

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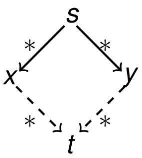

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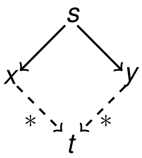

Fact: local confluence and termination $\Longrightarrow$ confluence

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$\longrightarrow$ is **terminating** if there are no infinite reduction chains
$\longrightarrow$ is **normalizing** if each element has a normal form
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② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

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True for most orders that don't treat certain parts of terms as special cases.

Example Termination Proof

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We show that the rewrite system defined by these rules is terminating.

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- $s <_i t \equiv$ num_imps $s <$ num_imps t and
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Then $<_i$ and $<_n$ are both well-founded orders (since both return nats). $<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

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\text{osize}'\ (P \vee Q) & x = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}'\ (P \longrightarrow Q) & x = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
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The other rules decrease the depth of the things osize counts, so decrease osize.

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- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
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termination:	not guaranteed (may loop)
confluence:	not guaranteed (result may depend on which rule is used first)

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  apply (simp add: <rules>) and **apply** (simp del: <rules>)

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- Adding/deleting equations locally:

  apply (simp add: $<$rules$>$) and **apply** (simp del: $<$rules$>$)
- Using only the specified set of equations:

  apply (simp only: $<$rules$>$)

DEMO

We have seen today...

➜ Equations and Term Rewriting

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- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle

Exercises

➜ Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the structure of terms from that example.