COMP4161
Advanced Topics in Software Verification

$$\longrightarrow$$

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Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7]
- Proof automation, Isar (part 2) [8[b]]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

Last Time on HOL

- ➜ Defining HOL
- ➜ Higher Order Abstract Syntax
- ➜ Deriving proof rules
- ➜ More automation

Term Rewriting

The Problem

Given a set of equations

$$\begin{aligned} l_1 &= r_1 \\ l_2 &= r_2 \\ &\vdots \\ l_n &= r_n \end{aligned}$$

does equation $l = r$ hold?

Applications in:

- ➜ **Mathematics** (algebra, group theory, etc)
- ➜ **Functional Programming** (model of execution)
- ➜ **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$
\begin{array}{c}
l_1 \longrightarrow r_1 \\
l_2 \longrightarrow r_2 \\
\vdots \\
l_n \longrightarrow r_n
\end{array}
$$

decide $l = r$ **by deciding** $l \stackrel{*}{\longleftrightarrow} r$

Arrow Cheat Sheet

$\xrightarrow{0}$	$=$	$\{(x,y) \mid x = y\}$	identity
$\xrightarrow{n+1}$	$=$	$\xrightarrow{n} \circ \longrightarrow$	n+1 fold composition
$\xrightarrow{+}$	$=$	$\bigcup_{i>0} \xrightarrow{i}$	transitive closure
$\xrightarrow{*}$	$=$	$\xrightarrow{+} \cup \xrightarrow{0}$	reflexive transitive closure
$\xrightarrow{=}$	$=$	$\longrightarrow \cup \xrightarrow{0}$	reflexive closure
$\xrightarrow{-1}$	$=$	$\{(y,x) \mid x \longrightarrow y\}$	inverse
$\longleftarrow$	$=$	$\xrightarrow{-1}$	inverse
$\longleftrightarrow$	$=$	$\longleftarrow \cup \longrightarrow$	symmetric closure
$\overset{+}{\longleftrightarrow}$	$=$	$\bigcup_{i>0} \overset{i}{\longleftrightarrow}$	transitive symmetric closure
$\overset{*}{\longleftrightarrow}$	$=$	$\overset{+}{\longleftrightarrow} \cup \overset{0}{\longleftrightarrow}$	reflexive transitive symmetric closure

How to Decide $l \stackrel{*}{\longleftrightarrow} r$

Same idea as for β: look for n such that $l \stackrel{*}{\longrightarrow} n$ and $r \stackrel{*}{\longrightarrow} n$

Does this always work?

If $l \stackrel{*}{\longrightarrow} n$ and $r \stackrel{*}{\longrightarrow} n$ then $l \stackrel{*}{\longleftrightarrow} r$. Ok.

If $l \stackrel{*}{\longleftrightarrow} r$, will there always be a suitable n? **No!**

Example:

Rules: $f\ x \longrightarrow a$, $g\ x \longrightarrow b$, $f\ (g\ x) \longrightarrow b$

$f\ x \stackrel{*}{\longleftrightarrow} g\ x$ because

$$f\ x \longrightarrow a \longleftarrow f\ (g\ x) \longrightarrow b \longleftarrow g\ x$$

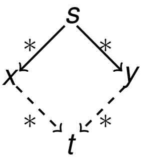
But: $f\ x \longrightarrow a$ and $g\ x \longrightarrow b$ and a, b in normal form

Works only for systems with **Church-Rosser** property:

$$l \stackrel{*}{\longleftrightarrow} r \Longrightarrow \exists n.\ l \stackrel{*}{\longrightarrow} n \wedge r \stackrel{*}{\longrightarrow} n$$

Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.

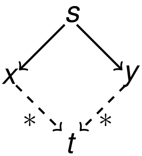
Confluence



Problem:
is a given set of reduction rules confluent?

undecidable

Local Confluence



Fact: local confluence and termination $\Longrightarrow$ confluence

Termination

$\longrightarrow$ is **terminating** if there are no infinite reduction chains
$\longrightarrow$ is **normalizing** if each element has a normal form
$\longrightarrow$ is **convergent** if it is terminating and confluent

Example:

$\longrightarrow_\beta$ in λ is not terminating, but confluent
$\longrightarrow_\beta$ in $\lambda^\rightarrow$ is terminating and confluent, i.e. convergent

Problem: is a given set of reduction rules terminating?

undecidable

When is $\longrightarrow$ Terminating?

Basic idea: when each rule application makes terms simpler in some way.

More formally: $\longrightarrow$ is terminating when there is a well founded order $<$ on terms for which $s < t$ whenever $t \longrightarrow s$
(well founded = no infinite decreasing chains $a_1 > a_2 > \ldots$)

Example: $f\ (g\ x) \longrightarrow g\ x$, $g\ (f\ x) \longrightarrow f\ x$

This system always terminates. Reduction order:

$s <_r t$ iff $size(s) < size(t)$ with
$size(s) =$ number of function symbols in s

① Both rules always decrease *size* by 1 when applied to any term t

② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

Termination in Practice

In practice: often easier to consider just the rewrite rules by themselves,

rather than their application to an arbitrary term t.

Show for each rule $l_i = r_i$, that $r_i < l_i$.

Example:

$g\ x < f\ (g\ x)$ and $f\ x < g\ (f\ x)$

Requires

u to become smaller whenever any subterm of u is made smaller.

Formally:

Requires $<$ to be **monotonic** with respect to the structure of terms:

$$s < t \longrightarrow u[s] < u[t].$$

True for most orders that don't treat certain parts of terms as

Example Termination Proof

Problem: Rewrite formulae containing $\neg$, $\wedge$, $\vee$ and $\longrightarrow$, so that they don't contain any implications and $\neg$ is applied only to variables and constants.

Rewrite Rules:

- ➜ Remove implications:

  imp: $(A \longrightarrow B) = (\neg A \vee B)$

- ➜ Push $\neg$s down past other operators:

  notnot: $(\neg\neg P) = P$

  notand: $(\neg(A \wedge B)) = (\neg A \vee \neg B)$

  notor: $(\neg(A \vee B)) = (\neg A \wedge \neg B)$

We show that the rewrite system defined by these rules is terminating.

Order on Terms

Each time one of our rules is applied, either:

- ➜ an implication is removed, or
- ➜ something that is not a $\neg$ is hoisted upwards in the term.

This suggests a 2-part order, $<_r$: $s <_r t$ iff:

- ➜ num_imps $s <$ num_imps t, or
- ➜ num_imps $s =$ num_imps $t \wedge$ osize $s <$ osize t.

Let:

- ➜ $s <_i t \equiv$ num_imps $s <$ num_imps t and
- ➜ $s <_n t \equiv$ osize $s <$ osize t

Then $<_i$ and $<_n$ are both well-founded orders (since both return nats).

$<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

Order Decreasing

imp clearly decreases num_imps.
osize adds up all non-¬ operators and variables/constants, weights each one according to its depth within the term.

$$
\begin{array}{ll}
\text{osize}'\ c & x = 2^x \\
\text{osize}'\ (\neg P) & x = \text{osize}'\ P\ (x+1) \\
\text{osize}'\ (P \wedge Q) & x = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}'\ (P \vee Q) & x = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}'\ (P \longrightarrow Q) & x = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}\ \ P & \quad = \text{osize}'\ P\ 0
\end{array}
$$

The other rules decrease the depth of the things osize counts, so decrease osize.

Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

termination: not guaranteed
(may loop)

confluence: not guaranteed
(result may depend on which rule is used first)

Control

- Equations turned into simplification rules with **[simp]** attribute
- Adding/deleting equations locally:

  apply (simp add: $<$rules$>$)   and   **apply** (simp del: $<$rules$>$)
- Using only the specified set of equations:

  apply (simp only: $<$rules$>$)

Demo

We have seen today...

- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle

Exercises

➜ Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the structure of terms from that example.