COMP4161
Advanced Topics in Software Verification

HOL

Thomas Sewell, Miki Tanaka, Rob Sison

T3/2024

Last time...

- natural deduction rules for $\wedge$, $\vee$, $\longrightarrow$, $\neg$, iff...
- proof by assumption, by intro rule, elim rule
- safe and unsafe rules

- indent your proofs! (one space per subgoal)
- prefer implicit backtracking (chaining) or *rule_tac*, instead of *back*
- *prefer* and *defer*
- *oops* and *sorry*

Content

- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]
- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

QUANTIFIERS

Scope

- Scope of parameters: whole subgoal
- Scope of $\forall, \exists, \ldots$: ends with ; or $\Longrightarrow$

Example:

Scope

- Scope of parameters: whole subgoal
- Scope of $\forall, \exists, \ldots$: ends with ; or $\Longrightarrow$

Example:

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\quad Q\ x\ y\ ]\!] \Longrightarrow \exists x.\ Q\ x\ y$$

means

Scope

- Scope of parameters: whole subgoal
- Scope of $\forall, \exists, \ldots$: ends with ; or $\Longrightarrow$

Example:

$$\bigwedge x\ y.\ \llbracket\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\ \ Q\ x\ y\ \rrbracket \implies \exists x.\ Q\ x\ y$$

means

$$\bigwedge x\ y.\ \llbracket\ (\forall y_1.\ P\ y_1 \longrightarrow Q\ z\ y_1);\ \ Q\ x\ y\ \rrbracket \implies (\exists x_1.\ Q\ x_1\ y)$$

Natural deduction for quantifiers

$$\frac{}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x}{R}\ \text{allE}$$

$$\frac{}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x}{R}\ \text{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x}{R}\ \text{allE}$$

$$\frac{}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x}{R}\ \text{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

$$\frac{}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x}{R}\ \text{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x}{R}\ \text{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

- **allI** and **exE** introduce new parameters ($\bigwedge x$).
- **allE** and **exI** introduce new unknowns ($?x$).

Instantiating Rules

apply (rule_tac x = "*term*" in *rule*)

Like **rule**, but ?*x* in *rule* is instantiated by *term* before application.

Similar: **erule_tac**

! *x* is in *rule*, not in goal !

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

Two Successful Proofs

$$1.\ \forall x.\ \exists y.\ x = y$$

apply (rule allI)

$$1.\ \bigwedge x.\ \exists y.\ x = y$$

best practice

apply (rule_tac x = "x" in exI)

$$1.\ \bigwedge x.\ x = x$$

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

best practice

apply (rule_tac x = "x" in exI)

1. $\bigwedge x.\ x = x$

apply (rule refl)

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

best practice

apply (rule_tac x = "x" in exI)

1. $\bigwedge x.\ x = x$

apply (rule refl)

exploration

apply (rule exI)

1. $\bigwedge x.\ x = ?y\ x$

Two Successful Proofs

$$1.\ \forall x.\ \exists y.\ x = y$$

apply (rule allI)

$$1.\ \bigwedge x.\ \exists y.\ x = y$$

best practice

apply (rule_tac x = "x" in exI)

$1.\ \bigwedge x.\ x = x$

apply (rule refl)

exploration

apply (rule exI)

$1.\ \bigwedge x.\ x = ?y\ x$

apply (rule refl)

$?y \mapsto \lambda u.u$

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

best practice

apply (rule_tac x = "x" in exI)

1. $\bigwedge x.\ x = x$

apply (rule refl)

simpler & clearer

exploration

apply (rule exI)

1. $\bigwedge x.\ x = ?y\ x$

apply (rule refl)

$?y \mapsto \lambda u.u$

shorter & trickier

Two Unsuccessful Proofs

1. $\exists y.\ \forall x.\ x = y$

Two Unsuccessful Proofs

1. $\exists y.\ \forall x.\ x = y$

apply (rule_tac x = ??? in exI)

Two Unsuccessful Proofs

1. $\exists y.\ \forall x.\ x = y$

apply (rule_tac x = ??? in exI)

apply (rule exI)

1. $\forall x.\ x = ?y$

Two Unsuccessful Proofs

1. $\exists y.\ \forall x.\ x = y$

apply (rule_tac x = ??? in exI)

apply (rule exI)

1. $\forall x.\ x = ?y$

apply (rule allI)

1. $\bigwedge x.\ x = ?y$

Two Unsuccessful Proofs

$$1.\ \exists y.\ \forall x.\ x = y$$

apply (rule_tac x = ??? in exI)

apply (rule exI)

$1.\ \forall x.\ x = ?y$

apply (rule allI)

$1.\ \bigwedge x.\ x = ?y$

apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.\ x' = x$

Two Unsuccessful Proofs

$1.\ \exists y.\ \forall x.\ x = y$

apply (rule_tac x = ??? in exI)

apply (rule exI)

$1.\ \forall x.\ x = ?y$

apply (rule allI)

$1.\ \bigwedge x.\ x = ?y$

apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.\ x' = x$

Principle:

$?f\ x_1 \ldots x_n$ ***can only be replaced by term*** t

if $params(t) \subseteq x_1, \ldots, x_n$

Safe and Unsafe Rules

Safe and Unsafe Rules

Safe allI, exE

Unsafe allE, exI

Safe and Unsafe Rules

Safe allI, exE

Unsafe allE, exI

Create parameters first, unknowns later

Demo: Quantifier Proofs

Parameter names

Parameter names are chosen by Isabelle

1. $\forall\, x.\ \exists y.\ x = y$

Parameter names

Parameter names are chosen by Isabelle

1. $\forall\, x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge\, x.\ \exists y.\ x = y$

Parameter names

Parameter names are chosen by Isabelle

1. $\forall\, x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge\, x.\ \exists y.\ x = y$

apply (rule_tac x = "x" in exI)

Brittle!

Renaming parameters

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

Renaming parameters

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

apply (rename_tac N)

1. $\bigwedge N.\ \exists y.\ N = y$

Renaming parameters

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

apply (rename_tac N)

1. $\bigwedge N.\ \exists y.\ N = y$

apply (rule_tac x = "N" in exI)

In general:
(rename_tac $x_1 \dots x_n$) renames the rightmost (inner) n parameters to $x_1 \dots x_n$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

Rule: $[\![A_1; \ldots; A_m]\!] \Longrightarrow A$

Subgoal: 1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

Rule: $[\![A_1; \ldots; A_m]\!] \Longrightarrow A$

Subgoal: 1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$

Substitution: $\sigma(B_i) \equiv \sigma(A_1)$

Forward Proof: frule and drule

apply (frule < *rule* >)

Rule:	$[\![A_1;\ldots;A_m]\!] \Longrightarrow A$
Subgoal:	1. $[\![B_1;\ldots;B_n]\!] \Longrightarrow C$
Substitution:	$\sigma(B_i) \equiv \sigma(A_1)$
New subgoals:	1. $\sigma([\![B_1;\ldots;B_n]\!] \Longrightarrow A_2)$
	$\vdots$
	m-1. $\sigma([\![B_1;\ldots;B_n]\!] \Longrightarrow A_m)$
	m. $\sigma([\![B_1;\ldots;B_n;A]\!] \Longrightarrow C)$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

Rule: $[\![A_1; \ldots; A_m]\!] \Longrightarrow A$

Subgoal: 1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$

Substitution: $\sigma(B_i) \equiv \sigma(A_1)$

New subgoals: 1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_2)$

$\vdots$

m-1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_m)$

m. $\sigma([\![B_1; \ldots; B_n; A]\!] \Longrightarrow C)$

Like **frule** but also deletes B_i: **apply** (drule $<$ *rule* $>$)

Examples for Forward Rules

$$\frac{P \wedge Q}{P}\ \text{conjunct1} \qquad \frac{P \wedge Q}{Q}\ \text{conjunct2}$$

$$\frac{P \longrightarrow Q \quad P}{Q}\ \text{mp}$$

$$\frac{\forall x.\ P\ x}{P\ ?x}\ \text{spec}$$

Forward Proof: OF

$$r\ [\textbf{OF}\ r_1 \ldots r_n]$$

Prove assumption 1 *of theorem* r *with theorem* r_1*, and assumption* 2 *with theorem* r_2*, and ...*

Forward Proof: OF

$$r\ [\textbf{OF}\ r_1 \ldots r_n]$$

Prove assumption 1 *of theorem* r *with theorem* r_1*, and assumption* 2 *with theorem* r_2*, and* ...

Rule r	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots; B_n]\!] \Longrightarrow B$

Forward Proof: OF

$$r\ [\textbf{OF}\ r_1 \ldots r_n]$$

Prove assumption 1 *of theorem* r *with theorem* r_1*, and assumption* 2 *with theorem* r_2*, and* ...

Rule r	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$

Forward Proof: OF

$$r\ [\textbf{OF}\ r_1 \dots r_n]$$

Prove assumption 1 *of theorem* r *with theorem* r_1*, and assumption* 2 *with theorem* r_2*, and* ...

Rule r	$[\![A_1; \dots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \dots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \dots; B_n; A_2; \dots; A_m]\!] \Longrightarrow A)$

Forward Proof: OF

$$r\ [\textbf{OF}\ r_1 \ldots r_n]$$

Prove assumption 1 *of theorem* r *with theorem* r_1*, and assumption* 2 *with theorem* r_2*, and* ...

Rule r	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \ldots; B_n; A_2; \ldots; A_m]\!] \Longrightarrow A)$

Example:

dvd_add : $[\![\ ?a\ dvd\ ?b; ?a\ dvd\ ?c\ ]\!] \Longrightarrow ?a\ dvd\ ?b + ?c$

dvd_refl : $?a\ dvd\ ?a$

$dvd\_add[OF\ dvd\_refl]$: $[\![\ ?a\ dvd\ ?c\ ]\!] \Longrightarrow ?a\ dvd\ ?a + ?c$

Forward proofs: THEN

r_1 [THEN r_2] means r_2 [OF r_1]

Demo: Forward Proofs

Hilbert's Epsilon Operator

(David Hilbert, 1862-1943)

$\varepsilon\, x.\ Px$ is a value that satisfies P (if such a value exists)

Hilbert's Epsilon Operator

(David Hilbert, 1862-1943)

$\varepsilon\ x.\ Px$ is a value that satisfies P (if such a value exists)

ε also known as **description operator**.
In Isabelle the ε-operator is written SOME $x.\ P\ x$

Hilbert's Epsilon Operator

(David Hilbert, 1862-1943)

$\varepsilon\ x.\ Px$ is a value that satisfies P (if such a value exists)

ε also known as **description operator**.
In Isabelle the ε-operator is written SOME $x.\ P\ x$

$$\frac{P\ ?x}{P\ (\text{SOME}\ x.\ P\ x)}\ \text{someI}$$

More Epsilon

$\mathcal{E}$ implies Axiom of Choice:

$$\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

More Epsilon

$\mathcal{E}$ implies Axiom of Choice:

$$\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

Isabelle also knows the definite description operator **THE** (aka ι):

$$\frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules

Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules
apply clarify	applies all safe rules that do not split the goal

Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules
apply clarify	applies all safe rules that do not split the goal
apply safe	applies all safe rules

Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules
apply clarify	applies all safe rules that do not split the goal
apply safe	applies all safe rules
apply blast	an automatic tableaux prover (works well on predicate logic)

Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules
apply clarify	applies all safe rules that do not split the goal
apply safe	applies all safe rules
apply blast	an automatic tableaux prover (works well on predicate logic)
apply fast	another automatic search tactic

Epsilon and Automation Demo

We have learned so far...

➜ Proof rules for predicate calculus

We have learned so far...

- ➜ Proof rules for predicate calculus
- ➜ Safe and unsafe rules

We have learned so far...

- ➜ Proof rules for predicate calculus
- ➜ Safe and unsafe rules
- ➜ Forward Proof

We have learned so far...

- ➜ Proof rules for predicate calculus
- ➜ Safe and unsafe rules
- ➜ Forward Proof
- ➜ The Epsilon Operator

We have learned so far...

- ➜ Proof rules for predicate calculus
- ➜ Safe and unsafe rules
- ➜ Forward Proof
- ➜ The Epsilon Operator
- ➜ Some automation

Isar (Part 1)

A Language for Structured Proofs

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```

or `by blast`

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```

or `by blast`

OK it's true. But WHY?

Motivation

WHY is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

Demo

Isar

apply scripts

➜ hard to read

Isar

apply scripts

- ➜ hard to read
- ➜ hard to maintain

Isar

apply scripts

- ➜ hard to read
- ➜ hard to maintain

No explicit structure.

Isar

apply scripts

- ➜ hard to read
- ➜ hard to maintain

➜ Elegance?

No explicit structure.

Isar

apply scripts

- ➜ hard to read
- ➜ hard to maintain

What about..

- ➜ Elegance?
- ➜ Explaining deeper insights?

No explicit structure.

Isar

apply scripts

- ➜ hard to read
- ➜ hard to maintain

No explicit structure.

What about..

- ➜ Elegance?
- ➜ Explaining deeper insights?

Isar!

A typical Isar proof

proof
  assume $\mathit{formula}_0$
  have $\mathit{formula}_1$ **by** simp
  $\vdots$
  have $\mathit{formula}_n$ **by** blast
  show $\mathit{formula}_{n+1}$ **by** ...
qed

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** simp
  ⋮
  have $formula_n$ **by** blast
  show $formula_{n+1}$ **by** ...
qed

proves $formula_0 \Longrightarrow formula_{n+1}$

A typical Isar proof

proof
  assume *formula*$_0$
  have *formula*$_1$ **by** simp
  ⋮
  have *formula*$_n$ **by** blast
  show *formula*$_{n+1}$ **by** . . .
qed

proves $\mathit{formula}_0 \Longrightarrow \mathit{formula}_{n+1}$

(analogous to **assumes**/**shows** in lemma statements)

Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

method = (simp . . .) | (blast . . .) | (rule . . .) | . . .

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

method = (simp ...) | (blast ...) | (rule ...) | ...

statement = **fix** variables $(\bigwedge)$
| **assume** proposition $(\Longrightarrow)$
| [**from** name$^+$] (**have** | **show**) proposition proof
| **next** (separates subgoals)

Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

method = (simp . . .) | (blast . . .) | (rule . . .) | . . .

statement = **fix** variables ($\bigwedge$)
  | **assume** proposition ($\Longrightarrow$)
  | [**from** name$^+$] (**have** | **show**) proposition proof
  | **next** (separates subgoals)

proposition = [name:] formula

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)
    assume A: "A"
    from A **show** "A" **by** assumption

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)
    assume A: "A"
    from A **show** "A" **by** assumption
next

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)
    assume A: "A"
    from A **show** "A" **by** assumption
next
    assume B: "B"
    from B **show** "B" **by** assumption

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)
  assume A: "A"
  from A **show** "A" **by** assumption
next
  assume B: "B"
  from B **show** "B" **by** assumption
qed

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)
    assume A: "A"
    from A **show** "A" **by** assumption
next
    assume B: "B"
    from B **show** "B" **by** assumption
qed

➜ **proof** (<method>) applies method to the stated goal

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)
  assume A: "A"
  from A **show** "A" **by** assumption
next
  assume B: "B"
  from B **show** "B" **by** assumption
qed

➜	**proof** (<method>)	applies method to the stated goal
➜	**proof**	applies a single rule that fits

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)
    assume A: "A"
    from A **show** "A" **by** assumption
next
    assume B: "B"
    from B **show** "B" **by** assumption
qed

➜	**proof** (<method>)	applies method to the stated goal
➜	**proof**	applies a single rule that fits
➜	**proof -**	does nothing to the goal

How do I know what to Assume and Show?

Look at the proof state!

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)

How do I know what to Assume and Show?

Look at the proof state!

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)

➜ **proof** (rule conjI) changes proof state to
1. $[\![A; B]\!] \Longrightarrow A$
2. $[\![A; B]\!] \Longrightarrow B$

How do I know what to Assume and Show?

Look at the proof state!

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)

- ➜ **proof** (rule conjI) changes proof state to
  1. $\llbracket A; B \rrbracket \Longrightarrow A$
  2. $\llbracket A; B \rrbracket \Longrightarrow B$
- ➜ so we need 2 shows: **show** "A" and **show** "B"

How do I know what to Assume and Show?

Look at the proof state!

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)

- ➜ **proof** (rule conjI) changes proof state to
  1. $[\![A; B]\!] \Longrightarrow A$
  2. $[\![A; B]\!] \Longrightarrow B$
- ➜ so we need 2 shows: **show** "A" and **show** "B"
- ➜ We are allowed to **assume** A,
  because A is in the assumptions of the proof state.

The Three Modes of Isar

➜ **[prove]**:
goal has been stated, proof needs to follow.

The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.

The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"

The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**

The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**

The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**
  assume A: "A" **[state]**

The Three Modes of Isar

- ➜ **[prove]**:
  goal has been stated, proof needs to follow.
- ➜ **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- ➜ **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**
  assume A: "A" **[state]**
  from A **[chain]**

The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has opened or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**
  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** . . .

Have

Can be used to make intermediate steps.

Example:

Have

Can be used to make intermediate steps.

Example:

lemma "(x :: nat) $+ 1 = 1 + x$"

Have

Can be used to make intermediate steps.

Example:

lemma "(x :: nat) $+ 1 = 1 + x$"
proof -
  have A: "$x + 1 =$ Suc x" **by** simp
  have B: "$1 + x =$ Suc x" **by** simp
  show "$x + 1 = 1 + x$" **by** (simp only: A B)
qed

Demo

Backward and Forward

Backward reasoning: ... **have** "$A \wedge B$" **proof**

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

➜ **proof** picks an **intro** rule automatically

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

➜ **proof** picks an **intro** rule automatically

➜ conclusion of rule must unify with $A \wedge B$

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

➜ **proof** picks an **intro** rule automatically

➜ conclusion of rule must unify with $A \wedge B$

Forward reasoning: . . .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

Backward and Forward

Backward reasoning: ... **have** "$A \wedge B$" **proof**

➜ **proof** picks an **intro** rule automatically

➜ conclusion of rule must unify with $A \wedge B$

Forward reasoning: ...

assume AB: "$A \wedge B$"
from AB **have** "..." **proof**

➜ now **proof** picks an **elim** rule automatically

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

➜ **proof** picks an **intro** rule automatically
➜ conclusion of rule must unify with $A \wedge B$

Forward reasoning: . . .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

➜ now **proof** picks an **elim** rule automatically
➜ triggered by **from**

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

- ➜ **proof** picks an **intro** rule automatically
- ➜ conclusion of rule must unify with $A \wedge B$

Forward reasoning: . . .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

- ➜ now **proof** picks an **elim** rule automatically
- ➜ triggered by **from**
- ➜ first assumption of rule must unify with AB

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

- ➜ **proof** picks an **intro** rule automatically
- ➜ conclusion of rule must unify with $A \wedge B$

Forward reasoning: . . .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

- ➜ now **proof** picks an **elim** rule automatically
- ➜ triggered by **from**
- ➜ first assumption of rule must unify with AB

General case: **from** A_1 . . . A_n **have** R **proof**

- ➜ first n assumptions of rule must unify with A_1 . . . A_n
- ➜ conclusion of rule must unify with R

Fix and Obtain

fix $v_1 \ldots v_n$

Fix and Obtain

fix $v_1 \ldots v_n$

Introduces new arbitrary but fixed variables
($\sim$ parameters, $\bigwedge$)

Fix and Obtain

fix $v_1 \ldots v_n$

Introduces new arbitrary but fixed variables
($\sim$ parameters, $\bigwedge$)

obtain $v_1 \ldots v_n$ **where** <prop> <proof>

Fix and Obtain

fix $v_1 \ldots v_n$

Introduces new arbitrary but fixed variables
($\sim$ parameters, $\bigwedge$)

obtain $v_1 \ldots v_n$ **where** <prop> <proof>

Introduces new variables together with property

Fancy Abbreviations

this = the previous fact proved or assumed

Fancy Abbreviations

this	=	the previous fact proved or assumed
then	=	**from** this

Fancy Abbreviations

this	=	the previous fact proved or assumed
then	=	**from** this
thus	=	**then show**

Fancy Abbreviations

this	=	the previous fact proved or assumed
then	=	**from** this
thus	=	**then show**
hence	=	**then have**

Fancy Abbreviations

this	=	the previous fact proved or assumed
then	=	**from** this
thus	=	**then show**
hence	=	**then have**
with $A_1 \ldots A_n$	=	**from** $A_1 \ldots A_n$ this

Fancy Abbreviations

this	=	the previous fact proved or assumed
then	=	**from** this
thus	=	**then show**
hence	=	**then have**
with $A_1 \ldots A_n$	=	**from** $A_1 \ldots A_n$ this
?thesis	=	the last enclosing goal statement

Demo

Moreover and Ultimately

have X_1: P_1 ...
have X_2: P_2 ...
$\vdots$
have X_n: P_n ...
from $X_1 \ldots X_n$ **show** ...

Moreover and Ultimately

have X_1: P_1 ...
have X_2: P_2 ...
⋮
have X_n: P_n ...
from $X_1 \ldots X_n$ **show** ...

wastes lots of brain power
on names $X_1 \ldots X_n$

Moreover and Ultimately

have X_1: P_1 ...
have X_2: P_2 ...
⋮
have X_n: P_n ...
from $X_1 \dots X_n$ **show** ...

have P_1 ...
moreover have P_2 ...
⋮
moreover have P_n ...
ultimately show ...

wastes lots of brain power
on names $X_1 \dots X_n$

General Case Distinctions

show *formula*
proof -

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
  moreover { **assume** P_3 ... **have** ?thesis <proof> }

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
  moreover { **assume** P_3 ... **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
  moreover { **assume** P_3 ... **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

{ ... } is a proof block similar to **proof** ... **qed**

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
  moreover { **assume** P_3 ... **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

{ ... } is a proof block similar to **proof** ... **qed**

{ **assume** P_1 ... **have** P <proof> }
stands for $P_1 \Longrightarrow P$

Mixing proof styles

from . . .
have . . .
  apply -   make incoming facts assumptions
  apply (. . .)
  ⋮
  apply (. . .)
  done

We have learned so far...

➜ Isar style proofs

We have learned so far...

- ➜ Isar style proofs
- ➜ proof, qed

We have learned so far...

- ➜ Isar style proofs
- ➜ proof, qed
- ➜ assumes, shows

We have learned so far...

- Isar style proofs
- proof, qed
- assumes, shows
- fix, obtain

We have learned so far...

- ➜ Isar style proofs
- ➜ proof, qed
- ➜ assumes, shows
- ➜ fix, obtain
- ➜ moreover, ultimately

We have learned so far...

- ➜ Isar style proofs
- ➜ proof, qed
- ➜ assumes, shows
- ➜ fix, obtain
- ➜ moreover, ultimately
- ➜ forward, backward

We have learned so far...

- ➜ Isar style proofs
- ➜ proof, qed
- ➜ assumes, shows
- ➜ fix, obtain
- ➜ moreover, ultimately
- ➜ forward, backward
- ➜ mixing proof styles