COMP4161 Advanced Topics in Software Verification





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Last time...

- → λ calculus syntax
- ➔ free variables, substitution
- → β reduction
- → α and η conversion
- → β reduction is confluent
- → λ calculus is expressive (Turing complete)
- → λ calculus is inconsistent (as a logic)



Content

→	Foundations & Principles	
	 Intro, Lambda calculus, natural deduction 	[1,2]
	 Higher Order Logic, Isar (part 1) 	[2,3 ^a]
	Term rewriting	[3,4]
→	Proof & Specification Techniques	
	 Inductively defined sets, rule induction 	[4,5]
	 Datatype induction, primitive recursion 	[5,7]
	 General recursive functions, termination proofs 	[7]
	 Proof automation, Isar (part 2) 	[8 ^b]
	 Hoare logic, proofs about programs, invariants 	[8,9]
	C verification	[9,10]
	 Practice, questions, exam prep 	[10 ^c]

^aa1 due; ^ba2 due; ^ca3 due

λ calculus is inconsistent

Can find term R such that $R R =_{\beta} \operatorname{not}(R R)$

There are more terms that do not make sense: 1 2, true false, etc.



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There are more terms that do not make sense: 1 2, true false, etc.

Solution: rule out ill-formed terms by using types. (Church 1940)



Idea: assign a type to each "sensible" λ term. Examples:



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- → if x has type α then $\lambda x. x$ is a function from α to α Write: $(\lambda x. x) :: \alpha \Rightarrow \alpha$



Idea: assign a type to each "sensible" λ term.

Examples:

- \rightarrow for term t has type α write $t :: \alpha$
- → if x has type α then $\lambda x. x$ is a function from α to α Write: $(\lambda x. x) :: \alpha \Rightarrow \alpha$
- → for s t to be sensible: s must be a function t must be right type for parameter

```
If s :: \alpha \Rightarrow \beta and t :: \alpha then (s t) :: \beta
```



THAT'S ABOUT IT

NOW FORMALLY AGAIN

Syntax for λ^{\rightarrow}

Terms:
$$t ::= v \mid c \mid (t \ t) \mid (\lambda x. \ t)$$

 $v, x \in V, c \in C, V, C$ sets of names

Types: $\tau ::= \mathbf{b} \mid \nu \mid \tau \Rightarrow \tau$ $\mathbf{b} \in \{\texttt{bool}, \texttt{int}, \ldots\} \text{ base types}$ $\nu \in \{\alpha, \beta, \ldots\} \text{ type variables}$

$$\alpha \Rightarrow \beta \Rightarrow \gamma \quad = \quad \alpha \Rightarrow (\beta \Rightarrow \gamma)$$



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 Γ : function from variable and constant names to types.



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Context Γ:

 Γ : function from variable and constant names to types.

Term *t* has type τ in context Γ : $\Gamma \vdash t :: \tau$



 $\Gamma \vdash (\lambda x. x) ::$



SW

$$\mathsf{F} \vdash (\lambda \mathbf{x}. \ \mathbf{x}) :: \alpha \Rightarrow \alpha$$

$$[y \leftarrow \texttt{int}] \vdash y ::$$



SW

$$\Gamma \vdash (\lambda x. x) :: \alpha \Rightarrow \alpha
 [y \leftarrow int] \vdash y :: int$$

 $[z \leftarrow \texttt{bool}] \vdash (\lambda y. \ y) \ z ::$

$$\begin{array}{l} \vdash (\lambda x. \ x) :: \alpha \Rightarrow \alpha \\ [y \leftarrow \operatorname{int}] \vdash y :: \operatorname{int} \\ [z \leftarrow \operatorname{bool}] \vdash (\lambda y. \ y) \ z :: \operatorname{bool} \\ [] \vdash \lambda f \ x. \ f \ x :: \end{array}$$



SW

$$\begin{split} & \Gamma \vdash (\lambda x. \ x) ::: \alpha \Rightarrow \alpha \\ & [y \leftarrow \text{int}] \vdash y :: \text{ int} \\ & [z \leftarrow \text{bool}] \vdash (\lambda y. \ y) \ z :: \text{ bool} \\ & [] \vdash \lambda f \ x. \ f \ x :: (\alpha \Rightarrow \beta) \Rightarrow \alpha \Rightarrow \beta \end{split}$$



SW

$$\begin{array}{l} \mathsf{\Gamma} \vdash (\lambda x. \ x) ::: \alpha \Rightarrow \alpha \\ [y \leftarrow \mathrm{int}] \vdash y ::: \mathrm{int} \\ [z \leftarrow \mathrm{bool}] \vdash (\lambda y. \ y) \ z :: \mathrm{bool} \\ [] \vdash \lambda f \ x. \ f \ x :: (\alpha \Rightarrow \beta) \Rightarrow \alpha \Rightarrow \beta \end{array}$$

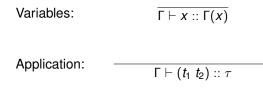
A term *t* is **well typed** or **type correct** if there are Γ and τ such that $\Gamma \vdash t :: \tau$



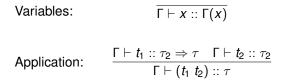
Variables:

 $\overline{\Gamma \vdash x :: \Gamma(x)}$

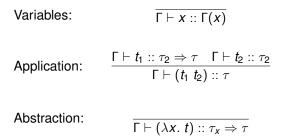




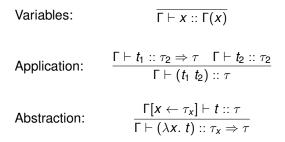














$$\frac{1}{[] \vdash \lambda x \ y. \ x ::} \qquad Abs$$

Remember:

$$\frac{\Gamma \vdash x :: \Gamma(x)}{\Gamma \vdash x :: \Gamma(x)} \quad Var \quad \frac{\Gamma \vdash t_1 :: \tau_2 \Rightarrow \tau \quad \Gamma \vdash t_2 :: \tau_2}{\Gamma \vdash (t_1 \ t_2) :: \tau} \quad App \quad \frac{\Gamma[x \leftarrow \tau_x] \vdash t :: \tau}{\Gamma \vdash (\lambda x. \ t) :: \tau_x \Rightarrow \tau} \quad Abs$$



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$$\frac{\overline{[\mathbf{x}\leftarrow\alpha]\vdash\lambda\mathbf{y}.\,\mathbf{x}::\beta\Rightarrow\alpha}}{[]\vdash\lambda\mathbf{x}\,\mathbf{y}.\,\mathbf{x}::\alpha\Rightarrow\beta\Rightarrow\alpha} \quad Abs$$

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$$\frac{\overline{[x \leftarrow \alpha, y \leftarrow \beta] \vdash x :: \alpha}}{[x \leftarrow \alpha] \vdash \lambda y. x :: \beta \Rightarrow \alpha} Abs$$

$$\overline{[] \vdash \lambda x y. x :: \alpha \Rightarrow \beta \Rightarrow \alpha} Abs$$

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$$\frac{\overline{[f \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta]} \vdash \lambda x. f x x :: \alpha \Rightarrow \beta}{[] \vdash \lambda f x. f x x :: (\alpha \Rightarrow \alpha \Rightarrow \beta) \Rightarrow \alpha \Rightarrow \beta} Abs$$

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$$\frac{ \Gamma \vdash f \times x :: \beta}{[f \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta] \vdash \lambda x. f \times x :: \alpha \Rightarrow \beta} Abs$$

$$\frac{[f \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta] \vdash \lambda x. f \times x :: \alpha \Rightarrow \beta}{[] \vdash \lambda f \times . f \times x :: (\alpha \Rightarrow \alpha \Rightarrow \beta) \Rightarrow \alpha \Rightarrow \beta} Abs$$

$$\mathsf{\Gamma} = [\mathsf{f} \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta, \mathsf{X} \leftarrow \alpha]$$

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$$\frac{\Gamma \vdash f \, x :: \alpha \Rightarrow \beta}{\Gamma \vdash x :: \alpha \Rightarrow \beta} \frac{App}{\Gamma \vdash x :: \alpha} \frac{Var}{App} \frac{App}{\Gamma \vdash x :: \alpha} \frac{Var}{App} \frac{\Gamma \vdash f \, x \, x :: \beta}{[f \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta] \vdash \lambda x. f \, x \, x :: \alpha \Rightarrow \beta} \frac{Abs}{Abs} \frac{[f \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta] \vdash \lambda f \, x. f \, x \, x :: (\alpha \Rightarrow \alpha \Rightarrow \beta) \Rightarrow \alpha \Rightarrow \beta} \frac{Abs}{Abs}$$

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More complex Example

$$\frac{\overline{\Gamma \vdash f :: \alpha \Rightarrow (\alpha \Rightarrow \beta)} \quad Var}{\Gamma \vdash x :: \alpha \Rightarrow \beta} \quad \frac{\overline{\Gamma \vdash x :: \alpha}}{App} \quad \frac{Var}{\Gamma \vdash x :: \alpha} \quad App}{\frac{\Gamma \vdash f x :: \alpha \Rightarrow \beta}{[f \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta] \vdash \lambda x. \ f x x :: \alpha \Rightarrow \beta}} \quad Abs}{\frac{\overline{[f \leftarrow \alpha \Rightarrow \alpha \Rightarrow \beta] \vdash \lambda x. \ f x x :: \alpha \Rightarrow \beta}}{[] \vdash \lambda f x. \ f x x :: (\alpha \Rightarrow \alpha \Rightarrow \beta) \Rightarrow \alpha \Rightarrow \beta}} \quad Abs$$

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A term can have more than one type.



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Example: []
$$\vdash \lambda x. x :: \text{bool} \Rightarrow \text{bool}$$

[] $\vdash \lambda x. x :: \alpha \Rightarrow \alpha$



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 $au \lesssim \sigma$ if there is a substitution *S* such that $au = S(\sigma)$



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Fact: each type correct term has a most general type



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Formally:

 $\Gamma \vdash t :: \tau \implies \exists \sigma. \ \Gamma \vdash t :: \sigma \land (\forall \sigma'. \ \Gamma \vdash t :: \sigma' \Longrightarrow \sigma' \lesssim \sigma)$



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- → type checking: checking if $\Gamma \vdash t :: \tau$ for given Γ and τ
- → type inference: computing Γ and τ such that $\Gamma \vdash t :: \tau$

Type checking and type inference on λ^{\rightarrow} are decidable.



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Definition of β reduction stays the same.



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Definition of β reduction stays the same.

Fact: Well typed terms stay well typed during β reduction

Formally: $\Gamma \vdash s :: \tau \land s \longrightarrow_{\beta} t \Longrightarrow \Gamma \vdash t :: \tau$



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This property is called subject reduction



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β reduction in λ^{\rightarrow} always terminates.



(Alan Turing, 1942)



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$\rightarrow =_{\beta}$ is decidable

To decide if $s =_{\beta} t$, reduce *s* and *t* to normal form (always exists, because \longrightarrow_{β} terminates), and compare result.



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$\Rightarrow =_{\beta}$ is decidable

To decide if $s =_{\beta} t$, reduce *s* and *t* to normal form (always exists, because \longrightarrow_{β} terminates), and compare result.

$\Rightarrow =_{\alpha\beta\eta}$ is decidable

This is why Isabelle can automatically reduce each term to $\beta\eta$ normal form.





Checkpoint:

- → untyped lambda calculus is turing complete (all computable functions can be expressed)
- → but it is inconsistent
- → λ^{\rightarrow} "fixes" the inconsistency problem by adding types
- → Problem: it is not turing complete anymore!



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But wait...



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But wait... typed functional languages are turing complete!



So...

- → typed functional languages are turing complete
- → but λ^{\rightarrow} is not...
- ➔ How does this work?



So...

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- → but λ^{\rightarrow} is not...
- → How does this work?
- → By adding one single constant, the Y operator (fix point operator), to λ^{\rightarrow}
- → This introduces the non-termination that the types removed.

$$\begin{array}{l} \mathbf{Y} :: (\tau \Rightarrow \tau) \Rightarrow \tau \\ \mathbf{Y} \ t \longrightarrow_{\beta} t \ (\mathbf{Y} \ t) \end{array}$$

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Fact: If we add *Y* to λ^{\rightarrow} as the only constant, then each computable function can be encoded as closed, type correct λ^{\rightarrow} term.

- →

 Y is used for recursion
- → lose decidability (what does $Y(\lambda x. x)$ reduce to?)
- → (Isabelle/HOL doesn't have Y; recursion is more restricted)



Types:
$$\tau ::= \mathbf{b} \mid '\nu \mid '\nu :: \mathbf{C} \mid \tau \Rightarrow \tau \mid (\tau, ..., \tau) \mathsf{K}$$

 $\mathbf{b} \in \{ \text{bool, int, ...} \}$ base types
 $\nu \in \{\alpha, \beta, ...\}$ type variables
 $\mathsf{K} \in \{ \text{set, list, ...} \}$ type constructors
 $\mathsf{C} \in \{ \text{order, linord, ...} \}$ type classes

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- → type constructors: construct a new type out of a parameter type. Example: int list
- → type classes: restrict type variables to a class defined by axioms. Example: α :: order



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$$t ::= v | c | ?v | (t t) | (\lambda x. t)$$

 $v, x \in V, c \in C, V, C$ sets of names

- → type constructors: construct a new type out of a parameter type. Example: int list
- → type classes: restrict type variables to a class defined by axioms. Example: α :: order
- → schematic variables: variables that can be instantiated.



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→ similar to Haskell's type classes, but with semantic properties class order = assumes order_refl: "x ≤ x" assumes order_trans: " [[x ≤ y; y ≤ z]] ⇒ x ≤ z" ...

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Type Classes

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instance nat :: " {order, linorder}" by ...



Schematic Variables

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Solution:

Isabelle has free (x), bound (x), and schematic (?X) variables.

Only schematic variables can be instantiated.

Free converted into schematic after proof is finished.



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Find substitution σ on variables for terms s, t such that $\sigma(s) = \sigma(t)$



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Examples:

?X∧?Y	$=_{\alpha\beta\eta}$	$X \wedge X$
?P x	$=_{\alpha\beta\eta}$	$X \wedge X$
<i>P</i> (? <i>f x</i>)	$=_{\alpha\beta\eta}$?Y x



Unification:

Find substitution σ on variables for terms s, t such that $\sigma(s) = \sigma(t)$

In Isabelle:

Find substitution σ on schematic variables such that $\sigma(s) =_{\alpha\beta\eta} \sigma(t)$

Examples:

$$\begin{array}{ll} ?X \land ?Y &=_{\alpha\beta\eta} & x \land x & \qquad [?X \leftarrow x, ?Y \leftarrow x] \\ ?P x &=_{\alpha\beta\eta} & x \land x & \qquad [?P \leftarrow \lambda x. \ x \land x] \\ P (?f x) &=_{\alpha\beta\eta} & ?Y x & \qquad [?f \leftarrow \lambda x. \ x, ?Y \leftarrow P] \end{array}$$

Higher Order: schematic variables can be functions.



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Higher Order Pattern:

- \rightarrow is a term in β normal form where
- → each occurrence of a schematic variable is of the form ?f $t_1 \ldots t_n$
- → and the $t_1 \ldots t_n$ are η -convertible into *n* distinct bound variables



→ Simply typed lambda calculus: λ^{\rightarrow}



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- ➔ Types and terms in Isabelle