COMP4161
Advanced Topics in Software Verification

$$\lambda^{\rightarrow}$$

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T3/2024

Last time...

- λ calculus syntax
- free variables, substitution
- β reduction
- α and η conversion
- β reduction is confluent
- λ calculus is expressive (Turing complete)
- λ calculus is inconsistent (as a logic)

Content

- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

λ calculus is inconsistent

Can find term R such that $R\ R =_\beta \texttt{not}(R\ R)$

There are more terms that do not make sense:

1 2, `true false`, etc.

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Solution: rule out ill-formed terms by using types.
(Church 1940)

Introducing types

Idea: assign a type to each "sensible" λ term.

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  Write: $(\lambda x.\ x) :: \alpha \Rightarrow \alpha$

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- if x has type α then $\lambda x.\ x$ is a function from α to α
  Write: $(\lambda x.\ x) :: \alpha \Rightarrow \alpha$
- for $s\ t$ to be sensible:
  s must be a function
  t must be right type for parameter

  If $s :: \alpha \Rightarrow \beta$ and $t :: \alpha$ then $(s\ t) :: \beta$

That's about it

Now formally again

Syntax for $\lambda^{\rightarrow}$

Terms: $t \quad ::= \quad v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

Types: $\tau \quad ::= \quad \mathtt{b} \mid \nu \mid \tau \Rightarrow \tau$

$\mathtt{b} \in \{\mathtt{bool}, \mathtt{int}, \ldots\}$ base types

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Term t has type τ in context Γ: $\quad \Gamma \vdash t :: \tau$

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A term t is **well typed** or **type correct**
if there are Γ and τ such that $\Gamma \vdash t :: \tau$

Type Checking Rules

Variables: $$\frac{}{\Gamma \vdash x :: \Gamma(x)}$$

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Example Type Derivation:

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Remember:

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Type checking and type inference on $\lambda^{\rightarrow}$ are decidable.

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This property is called **subject reduction**

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To decide if $s =_\beta t$, reduce s and t to normal form (always exists, because $\longrightarrow_\beta$ terminates), and compare result.

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  To decide if *s* $=_\beta$ *t*, reduce *s* and *t* to normal form (always exists, because $\longrightarrow_\beta$ terminates), and compare result.
- ➜ **$=_{\alpha\beta\eta}$ is decidable**
  This is why Isabelle can automatically reduce each term to $\beta\eta$ normal form.

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Checkpoint:

- ➜ untyped lambda calculus is turing complete
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- ➜ but it is inconsistent
- ➜ $\lambda^{\rightarrow}$ ”fixes” the inconsistency problem by adding types
- ➜ Problem: it is not turing complete anymore!

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But wait... typed functional languages are turing complete!

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So...

- typed functional languages are turing complete
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- How does this work?
- By adding one single constant, the Y operator (fix point operator), to $\lambda^{\rightarrow}$
- This introduces the non-termination that the types removed.

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$$Y\ t \longrightarrow_{\beta} t\ (Y\ t)$$

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$$Y :: (\tau \Rightarrow \tau) \Rightarrow \tau$$
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Fact: If we add Y to $\lambda^{\rightarrow}$ as the only constant, then each computable function can be encoded as closed, type correct $\lambda^{\rightarrow}$ term.

- ➜ Y is used for recursion
- ➜ lose decidability (what does $Y\ (\lambda x.\ x)$ reduce to?)
- ➜ (Isabelle/HOL doesn't have Y; recursion is more restricted)

Types and Terms in Isabelle

Types: $\tau \;::=\; \mathtt{b} \mid {'\nu} \mid {'\nu} :: C \mid \tau \Rightarrow \tau \mid (\tau, \ldots, \tau)\, K$

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$\nu \in \{\alpha, \beta, \ldots\}$ type variables

$K \in \{\mathtt{set}, \mathtt{list}, \ldots\}$ type constructors

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Terms: $t \;::=\; v \mid c \mid ?v \mid (t\ t) \mid (\lambda x.\ t)$

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➜ **type constructors**: construct a new type out of a parameter type.
Example: `int list`

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Terms: $t \;::=\; v \mid c \mid ?v \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- ➜ **type constructors**: construct a new type out of a parameter type.
  Example: `int list`
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  Example: α :: *order*

Types and Terms in Isabelle

Types: $\tau \ ::= \ \mathtt{b} \mid {}'\nu \mid {}'\nu :: C \mid \tau \Rightarrow \tau \mid (\tau, \ldots, \tau)\ K$

$\mathtt{b} \in \{\mathtt{bool}, \mathtt{int}, \ldots\}$ base types
$\nu \in \{\alpha, \beta, \ldots\}$ type variables
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  Example: $\alpha :: \mathit{order}$
- ➜ **schematic variables**: variables that can be instantiated.

Type Classes

➜ similar to Haskell's type classes, but with semantic properties

class order $=$

assumes order_refl: "$x \le x$"

assumes order_trans: "$\llbracket x \le y; y \le z \rrbracket \Longrightarrow x \le z$"

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- ➜ can be instantiated

  instance nat :: $''\{$order, linorder$\}''$ **by** ...

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Solution:
Isabelle has **free** (x), **bound** (x), and **schematic** (?X) variables.

Only schematic variables can be instantiated.

Free converted into schematic after proof is finished.

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Examples:

$$
\begin{array}{lll}
?X \wedge ?Y & =_{\alpha\beta\eta} & x \wedge x \\
?P\ x & =_{\alpha\beta\eta} & x \wedge x \\
P\ (?f\ x) & =_{\alpha\beta\eta} & ?Y\ x
\end{array}
$$

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Examples:

$?X \wedge ?Y$	$=_{\alpha\beta\eta}$	$x \wedge x$	$[?X \leftarrow x, ?Y \leftarrow x]$
$?P\ x$	$=_{\alpha\beta\eta}$	$x \wedge x$	$[?P \leftarrow \lambda x.\ x \wedge x]$
$P\ (?f\ x)$	$=_{\alpha\beta\eta}$	$?Y\ x$	$[?f \leftarrow \lambda x.\ x, ?Y \leftarrow P]$

Higher Order: schematic variables can be functions.

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Higher Order Pattern:

- ➜ is a term in β normal form where
- ➜ each occurrence of a schematic variable is of the form $?f\ t_1\ \dots\ t_n$
- ➜ and the $t_1\ \dots\ t_n$ are η-convertible into n distinct bound variables

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