COMP4161: Advanced Topics in Software Verification

Isar

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S2/2018

Content

→ Intro & motivation, getting started [1]

→ Foundations & Principles

- Lambda Calculus, natural deduction [1,2]
- Higher Order Logic [3[a]]
- Term rewriting [4]

→ Proof & Specification Techniques

- Inductively defined sets, rule induction [5]
- Datatypes, recursion, induction [6, 7]
- Hoare logic, proofs about programs, invariants [8[b],9]
- (mid-semester break)
- C verification [10]
- CakeML, Isar [11[c]]
- Concurrency [12]

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[a] a1 due; [b] a2 due; [c] a3 due

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A Language for Structured Proofs

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

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YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```


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or `by blast`

OK it's true. But WHY?

Motivation

WHY is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

Demo

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apply scripts

➜ unreadable

Isar

apply scripts

➜ unreadable
➜ hard to maintain

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What about..

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Isar

apply scripts	**What about..**
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- ➜ Large developments?

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➜ do not scale	➜ Large developments?
No structure.	**Isar!**

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** simp
  ⋮
  have $formula_n$ **by** blast
  show $formula_{n+1}$ **by** ...
qed

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(analogous to **assumes**/**shows** in lemma statements)

Isar core syntax

proof = **proof** [method] statement* **qed**
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method = (simp . . .) | (blast . . .) | (rule . . .) | . . .

Isar core syntax

$$\begin{aligned}
\text{proof} &= \textbf{proof}\ [\text{method}]\ \text{statement}^*\ \textbf{qed} \\
&\mid\ \textbf{by}\ \text{method}
\end{aligned}$$

$$\text{method} = (\text{simp} \ldots) \mid (\text{blast} \ldots) \mid (\text{rule} \ldots) \mid \ldots$$

$$\begin{aligned}
\text{statement} &= \textbf{fix}\ \text{variables} && (\bigwedge) \\
&\mid\ \textbf{assume}\ \text{proposition} && (\Longrightarrow) \\
&\mid\ [\textbf{from}\ \text{name}^+]\ (\textbf{have} \mid \textbf{show})\ \text{proposition proof} \\
&\mid\ \textbf{next} && (\text{separates subgoals})
\end{aligned}$$

Isar core syntax

proof = **proof** [method] statement* **qed**
 | **by** method

method = (simp ...) | (blast ...) | (rule ...) | ...

statement = **fix** variables ($\bigwedge$)
 | **assume** proposition ($\Longrightarrow$)
 | [**from** name$^+$] (**have** | **show**) proposition proof
 | **next** (separates subgoals)

proposition = [name:] formula

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)

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➜ **proof** (<method>) applies method to the stated goal

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- ➜ **proof** (<method>) applies method to the stated goal
- ➜ **proof** applies a single rule that fits
- ➜ **proof -** does nothing to the goal

How do I know what to Assume and Show?

Look at the proof state!

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➜ **proof** (rule conjI) changes proof state to
1. $[\![A; B]\!] \Longrightarrow A$
2. $[\![A; B]\!] \Longrightarrow B$

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- ➜ so we need 2 shows: **show** "A" and **show** "B"
- ➜ We are allowed to **assume** A,
  because A is in the assumptions of the proof state.

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goal has been stated, proof needs to follow.

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  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** **. . .**

Have

Can be used to make intermediate steps.

Example:

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lemma "$(x :: \text{nat}) + 1 = 1 + x$"

Have

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Example:

lemma "$(x :: \text{nat}) + 1 = 1 + x$"
proof -
  have A: "$x + 1 = \text{Suc } x$" **by** simp
  have B: "$1 + x = \text{Suc } x$" **by** simp
  show "$x + 1 = 1 + x$" **by** (simp only: A B)
qed

Demo

Backward and Forward

Backward reasoning: ... **have** $''A \wedge B''$ **proof**

Backward and Forward

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Forward reasoning: . . .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

Backward and Forward

Backward reasoning: ... **have** "$A \wedge B$" **proof**

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Forward reasoning: ...

assume AB: "$A \wedge B$"
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➜ first assumption of rule must unify with AB

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Forward reasoning: ...

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from AB **have** "..." **proof**

- ➜ now **proof** picks an **elim** rule automatically
- ➜ triggered by **from**
- ➜ first assumption of rule must unify with AB

General case: **from** $A_1 \ldots A_n$ **have** R **proof**

- ➜ first n assumptions of rule must unify with $A_1 \ldots A_n$
- ➜ conclusion of rule must unify with R

Fix and Obtain

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Introduces new variables together with property

Demo

Fancy Abbreviations

this = the previous fact proved or assumed

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this	=	the previous fact proved or assumed
then	=	**from** this

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Fancy Abbreviations

this	=	the previous fact proved or assumed
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hence	=	**then have**
with $A_1 \ldots A_n$	=	**from** $A_1 \ldots A_n$ this
?thesis	=	the last enclosing goal statement

Moreover and Ultimately

have X_1: P_1 ...
have X_2: P_2 ...
$\vdots$
have X_n: P_n ...
from $X_1 \ldots X_n$ **show** ...

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have X_n: P_n ...
from $X_1 \dots X_n$ **show** ...

wastes lots of brain power
on names $X_1 \dots X_n$

Moreover and Ultimately

have X_1: P_1 ...	**have** P_1 ...
have X_2: P_2 ...	**moreover have** P_2 ...
$\vdots$	$\vdots$
have X_n: P_n ...	**moreover have** P_n ...
from $X_1 \dots X_n$ **show** ...	**ultimately show** ...

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General Case Distinctions

show *formula*
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  have $P_1 \vee P_2 \vee P_3$ <proof>
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  moreover { **assume** P_3 ... **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

General Case Distinctions

show *formula*
proof -
  have $P_1 \lor P_2 \lor P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
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{ ... } is a proof block similar to **proof** ... **qed**

General Case Distinctions

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{ ... } is a proof block similar to **proof** ... **qed**

{ **assume** P_1 ... **have** P <proof> }
stands for $P_1 \Longrightarrow P$

Mixing proof styles

```
from ...
have ...
  apply -        make incoming facts assumptions
  apply (...)
  ⋮
  apply (...)
  done
```


Datatypes in Isar

Datatype case distinction

proof (cases *term*)
  case Constructor_1
  ⋮
next
⋮
next
  case $(\text{Constructor}_k\ \vec{x})$
  $\cdots\ \vec{x}\ \cdots$
qed

Datatype case distinction

proof (cases *term*)
  case Constructor_1
  ⋮
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next
  case $(\text{Constructor}_k\ \vec{x})$
  $\cdots\ \vec{x}\ \cdots$
qed

$$\textbf{case}\ (\text{Constructor}_i\ \vec{x}) \quad \equiv$$
$$\textbf{fix}\ \vec{x}\ \textbf{assume}\ \text{Constructor}_i : \text{"}\mathit{term} = \text{Constructor}_i\ \vec{x}\text{"}$$

Structural induction for nat

show $P\ n$
proof (induct n)
  case 0 $\equiv$ **let** $?case = P\ 0$
  ...
  show $?case$
next
  case (Suc n) $\equiv$ **fix** n **assume** Suc: $P\ n$
    let $?case = P\ (\text{Suc}\ n)$
  ...
  ... n ...
  show $?case$
qed

Structural induction: $\Longrightarrow$ and $\bigwedge$

show "$\bigwedge x.\ A\ n \Longrightarrow P\ n$"
proof (induct n)
  case 0 $\equiv$ **fix** x **assume** 0: "$A\ 0$"
  let ?$case$ = "$P\ 0$"
  ...
  show ?$case$
next
  case (Suc n) $\equiv$ **fix** n and x
  assume Suc: "$\bigwedge x.\ A\ n \Longrightarrow P\ n$"
  "A (Suc n)"
  let ?$case$ = "P (Suc n)"
  ...
  ... n ...
  ...
  show ?$case$
qed

Demo: Datatypes in Isar

Calculational Reasoning

The Goal

Prove:
$x \cdot x^{-1} = 1$

using:

assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
left_inv:	$x^{-1} \cdot x = 1$
left_one:	$1 \cdot x = x$

The Goal

Prove:

$$
\begin{aligned}
x \cdot x^{-1} &= 1 \cdot (x \cdot x^{-1}) \\
\ldots &= 1 \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (x^{-1} \cdot x) \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot 1 \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (1 \cdot x^{-1}) \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \\
\ldots &= 1
\end{aligned}
$$

assoc: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

left_inv: $x^{-1} \cdot x = 1$

left_one: $1 \cdot x = x$

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\end{aligned}
$$

assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
left_inv:	$x^{-1} \cdot x = 1$
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Can we do this in Isabelle?

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\end{aligned}
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assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
left_inv:	$x^{-1} \cdot x = 1$
left_one:	$1 \cdot x = x$

Can we do this in Isabelle?

➜ Simplifier: too eager

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\ldots &= 1 \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (x^{-1} \cdot x) \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot 1 \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (1 \cdot x^{-1}) \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \\
\ldots &= 1
\end{aligned}
$$

assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
left_inv:	$x^{-1} \cdot x = 1$
left_one:	$1 \cdot x = x$

Can we do this in Isabelle?

- ➜ Simplifier: too eager
- ➜ Manual: difficult in apply style

The Goal

Prove:

$$
\begin{aligned}
x \cdot x^{-1} &= 1 \cdot (x \cdot x^{-1}) \\
\ldots &= 1 \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \cdot x \cdot x^{-1} \\
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Can we do this in Isabelle?

- ➜ Simplifier: too eager
- ➜ Manual: difficult in apply style
- ➜ Isar: with the methods we know, too verbose

Chains of equations

The Problem

$$\begin{array}{rcl} a & = & b \\ \ldots & = & c \\ \ldots & = & d \end{array}$$

shows $a = d$ by transitivity of $=$

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Solution in Isar:

- ➜ Keywords **also** and **finally** to delimit steps
- ➜ **...**: predefined schematic term variable, refers to right hand side of last expression
- ➜ Automatic use of transitivity rules to connect steps

also/finally

have "$t_0 = t_1$" [proof]
also

also/finally

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also

calculation register
"$t_0 = t_1$"

also/finally

have "$t_0 = t_1$" [proof]
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also/finally

have "$t_0 = t_1$" [proof]
also
have "$\ldots = t_2$" [proof]
also
$\vdots$
also

calculation register
"$t_0 = t_1$"

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$\vdots$
"$t_0 = t_{n-1}$"

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also/finally

have "$t_0 = t_1$" [proof]	calculation register
also	"$t_0 = t_1$"
have "$\ldots = t_2$" [proof]	
also	"$t_0 = t_2$"
$\vdots$	$\vdots$
also	"$t_0 = t_{n-1}$"
have "$\cdots = t_n$" [proof]	
finally	$t_0 = t_n$
show P	

— 'finally' pipes fact "$t_0 = t_n$" into the proof

More about also

➜ Works for all combinations of $=$, $\leq$ and $<$.

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- Uses all rules declared as `[trans]`.
- To view all combinations: `print_trans_rules`

Designing [trans] Rules

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Examples:

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Examples:

- pure transitivity: $[\![a = b; b = c]\!] \Longrightarrow a = c$
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- substitution: $[\![P\ a; a = b]\!] \Longrightarrow P\ b$

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- ➜ substitution: $[\![P\ \ a; a = b]\!] \Longrightarrow P\ \ b$
- ➜ antisymmetry: $[\![a < b; b < a]\!] \Longrightarrow False$
- ➜ monotonicity: $[\![a = f\ \ b; b < c; \bigwedge x\ y.\ x < y \Longrightarrow f\ \ x < f\ \ y]\!] \Longrightarrow a < f\ \ c$

Demo