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<sup>a</sup>a1 due; <sup>b</sup>a2 due; <sup>c</sup>a3 due



➔ Equations and Term Rewriting



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- → Confluence and Termination of reduction systems



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- → Term Rewriting in Isabelle



 $\rightarrow$  *l*  $\rightarrow$  *r* **applicable** to term *t*[*s*]



→ I → r applicable to term t[s] if there is substitution σ such that σ I = s



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Rewrite rules can be conditional:

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is **applicable** to term t[s] with  $\sigma$  if

 $\rightarrow \sigma l = s$  and

→  $\sigma P_1, \ldots, \sigma P_n$  are provable by rewriting.

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simp (simp (no\_asm)) (simp (no\_asm\_use)) (simp (no\_asm\_simp))

use and simplify assumptions ignore assumptions simplify, but do not use assumptions use, but do not simplify assumptions

## Preprocessing



Preprocessing (recursive) for maximal simplification power:

$$\begin{array}{cccc} \neg A & \mapsto & A = False \\ A \longrightarrow B & \mapsto & A \Longrightarrow B \\ A \land B & \mapsto & A, B \\ \forall x. \ A \ x & \mapsto & A \ ?x \\ A & \mapsto & A = True \end{array}$$

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Example:

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 $\mapsto$ 

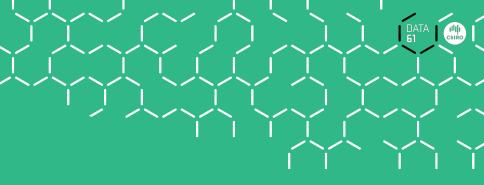
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Example:  $(p \longrightarrow q \land \neg r) \land s$  $\mapsto$  $p \Longrightarrow q = True$   $p \Longrightarrow r = False$  s = True







P (if A then s else t) $= (A \longrightarrow P s) \land (\neg A \longrightarrow P t)$ 



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$$\begin{array}{l} P \ (\text{case } e \ \text{of } 0 \ \Rightarrow \ a \mid \text{Suc } n \ \Rightarrow \ b) \\ = \\ (e = 0 \longrightarrow P \ a) \land (\forall n. \ e = \text{Suc } n \longrightarrow P \ b) \end{array}$$



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Similar for any data type t: t.split





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  - $\Rightarrow$  the result is  $P' \longrightarrow Q'$

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**if\_cong**: 
$$\llbracket b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v \rrbracket \Longrightarrow$$
  
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- → use locally with e.g. apply (simp cong: <rule>)



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For types nat, int etc:

- lemmas add\_ac sort any sum (+)
- lemmas mult\_ac sort any product (\*)
- **Example:** apply (simp add: add\_ac) yields  $(b+c) + a \rightsquigarrow \cdots \rightsquigarrow a + (b+c)$



Example for associative-commutative rules: Associative:  $(x \odot y) \odot z = x \odot (y \odot z)$ Commutative:  $x \odot y = y \odot x$ 



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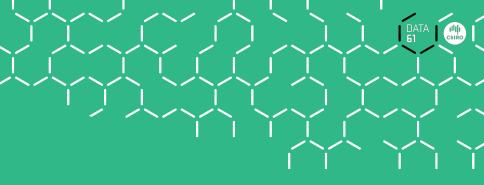
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If these 3 rules are present for an AC operator Isabelle will order terms correctly





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#### Definition:

Let  $l_1 \longrightarrow r_1$  and  $l_2 \longrightarrow r_2$  be two rules with disjoint variables.

They form a **critical pair** if a non-variable subterm of  $l_1$  unifies with  $l_2$ .



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#### (1) $f x \longrightarrow a$ (2) $g y \longrightarrow b$ (3) $f (g z) \longrightarrow b$ is not confluent

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#### (1) $f \times \longrightarrow a$ (2) $g \times \longrightarrow b$ (3) $f (g \times z) \longrightarrow b$ is not confluent

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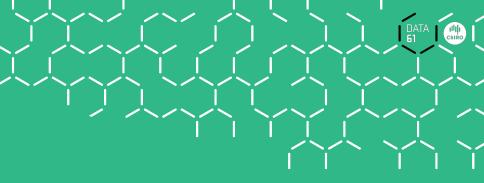
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shows that  $a = b$  (because  $a \stackrel{*}{\longleftrightarrow} b$ ), so we add  $a \longrightarrow b$  as a rule

This is the main idea of the Knuth-Bendix completion algorithm.



# **Demo: Waldmeister**



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#### Orthogonal rewrite systems are confluent

Application: functional programming languages



➔ Conditional term rewriting



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- ➔ Congruence rules



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- → AC rules



- ➔ Conditional term rewriting
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- ➔ More on confluence