



COMP4161: Advanced Topics in Software Verification



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Content



- Intro & motivation, getting started [1]

- Foundations & Principles
 - Lambda Calculus, natural deduction [1,2]
 - Higher Order Logic [3^a]
 - Term rewriting [4]

- Proof & Specification Techniques
 - Inductively defined sets, rule induction [5]
 - Datatypes, recursion, induction [6, 7]
 - Hoare logic, proofs about programs, invariants [8^b, 9]
 - (mid-semester break)
 - C verification [10]
 - CakeML, Isar [11^c]
 - Concurrency [12]

^aa1 due; ^ba2 due; ^ca3 due

Last Time on HOL



→ Defining HOL

Last Time on HOL



- Defining HOL
- Higher Order Abstract Syntax

Last Time on HOL



- Defining HOL
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- Deriving proof rules

Last Time on HOL



- Defining HOL
- Higher Order Abstract Syntax
- Deriving proof rules
- More automation

Term Rewriting

The Problem



Given a set of equations

$$l_1 = r_1$$

$$l_2 = r_2$$

$$\vdots$$

$$l_n = r_n$$

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Applications in:

- **Mathematics** (algebra, group theory, etc)
- **Functional Programming** (model of execution)
- **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea



use equations as reduction rules

$$l_1 \longrightarrow r_1$$

$$l_2 \longrightarrow r_2$$

\vdots

$$l_n \longrightarrow r_n$$

decide $l = r$ by deciding $l \overset{*}{\longleftrightarrow} r$

Arrow Cheat Sheet



$$\xrightarrow{0} = \{(x, y) | x = y\} \quad \text{identity}$$

Arrow Cheat Sheet



$$\begin{aligned}\xrightarrow{0} &= \{(x, y) \mid x = y\} && \text{identity} \\ \xrightarrow{n+1} &= \xrightarrow{n} \circ \longrightarrow && \text{n+1 fold composition}\end{aligned}$$

Arrow Cheat Sheet



$\xrightarrow{0}$	$=$	$\{(x, y) x = y\}$	identity
$\xrightarrow{n+1}$	$=$	$\xrightarrow{n} \circ \longrightarrow$	n+1 fold composition
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$\xrightarrow{-1}$	$=$	$\{(y, x) x \longrightarrow y\}$	inverse

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\longleftrightarrow	$= \longleftarrow \cup \longrightarrow$	symmetric closure

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How to Decide $l \overset{*}{\longleftrightarrow} r$

Same idea as for β :



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Same idea as for β : look for n such that $l \overset{*}{\longrightarrow} n$ and $r \overset{*}{\longrightarrow} n$

Does this always work?

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Rules: $f x \longrightarrow a$, $g x \longrightarrow b$, $f (g x) \longrightarrow b$

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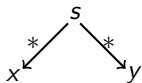
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Fact: \longrightarrow is Church-Rosser iff it is confluent.

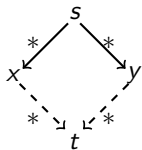
Confluence



Problem:

is a given set of reduction rules confluent?

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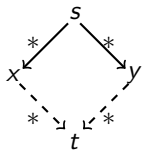


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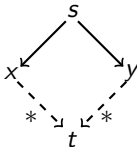


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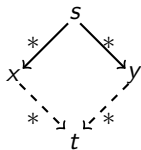
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Confluence

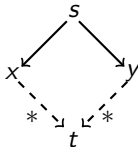


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Local Confluence



Fact: local confluence and termination \implies confluence

Termination



- is **terminating** if there are no infinite reduction chains
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- ② $<_r$ is well founded, because $<$ is well founded on \mathbb{N}

Termination in Practice



In practice: often easier to consider just the rewrite rules by themselves, rather than their application to an arbitrary term t .

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Requires $<$ to be **monotonic** with respect to the structure of terms:

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True for most orders that don't treat certain parts of terms as special cases.

Example Termination Proof



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We show that the rewrite system defined by these rules is terminating.

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This suggests a 2-part order, $<_r$: $s <_r t$ iff:

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- $s <_i t \equiv \text{num_imps } s < \text{num_imps } t$ and
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Then $<_i$ and $<_n$ are both well-founded orders (since both return nats).

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 $<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

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$$\text{osize}' c \quad x = 2^x$$

$$\text{osize}' (\neg P) \quad x = \text{osize}' P (x + 1)$$

$$\text{osize}' (P \wedge Q) \quad x = 2^x + (\text{osize}' P (x + 1)) + (\text{osize}' Q (x + 1))$$

$$\text{osize}' (P \vee Q) \quad x = 2^x + (\text{osize}' P (x + 1)) + (\text{osize}' Q (x + 1))$$

$$\text{osize}' (P \longrightarrow Q) \quad x = 2^x + (\text{osize}' P (x + 1)) + (\text{osize}' Q (x + 1))$$

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The other rules decrease the depth of the things osize counts, so decrease osize.

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apply simp

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- (almost) blindly from left to right
- until no rule is applicable.

termination: not guaranteed
(may loop)

confluence: not guaranteed
(result may depend on which rule is used first)

Control



→ Equations turned into simplification rules with **[simp]** attribute

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- Adding/deleting equations locally:
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- Adding/deleting equations locally:
apply (simp add: <rules>) and **apply** (simp del: <rules>)
- Using only the specified set of equations:
apply (simp only: <rules>)

A background pattern of white hexagons on a dark teal background, arranged in a staggered grid.

DATA
61



Demo

We have seen today...



→ Equations and Term Rewriting

We have seen today...



→ Equations and Term Rewriting

We have seen today...



- Equations and Term Rewriting
- Confluence and Termination of reduction systems

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- Equations and Term Rewriting
- Confluence and Termination of reduction systems
- Term Rewriting in Isabelle

Exercises



- Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the structure of terms from that example.