COMP4161: Advanced Topics in Software Verification

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S2/2018

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- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, invariants [8[b],9]
  - (mid-semester break)
  - C verification [10]
  - CakeML, Isar [11[c]]
  - Concurrency [12]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time on HOL

➜ Defining HOL

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- ➜ Deriving proof rules
- ➜ More automation

Term Rewriting

The Problem

Given a set of equations

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Applications in:

- ➜ **Mathematics** (algebra, group theory, etc)
- ➜ **Functional Programming** (model of execution)
- ➜ **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$\begin{array}{c} l_1 \longrightarrow r_1 \\ l_2 \longrightarrow r_2 \\ \vdots \\ l_n \longrightarrow r_n \end{array}$$

decide $l = r$ **by deciding** $l \stackrel{*}{\longleftrightarrow} r$

Arrow Cheat Sheet

$$\xrightarrow{0} \quad = \quad \{(x, y) | x = y\} \qquad \text{identity}$$

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$$\begin{array}{lcll} \xrightarrow{0} & = & \{(x,y) \mid x = y\} & \text{identity} \\ \xrightarrow{n+1} & = & \xrightarrow{n} \circ \longrightarrow & \text{n+1 fold composition} \end{array}$$

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\xrightarrow{-1} & = & \{(y,x) | x \longrightarrow y\} & \text{inverse} \\
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\longleftrightarrow & = & \longleftarrow \cup \longrightarrow & \text{symmetric closure}
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Same idea as for β:

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Does this always work?

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Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.

Confluence

$$\begin{tikzcd} & s \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x & & y \end{tikzcd}$$

Problem:
is a given set of reduction rules confluent?

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Fact: local confluence and termination $\Longrightarrow$ confluence

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② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

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True for most orders that don't treat certain parts of terms as special cases.

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We show that the rewrite system defined by these rules is terminating.

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The other rules decrease the depth of the things osize counts, so decrease osize.

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termination: not guaranteed
(may loop)

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- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

termination:	not guaranteed (may loop)
confluence:	not guaranteed (result may depend on which rule is used first)

Control

➜ Equations turned into simplification rules with **[simp]** attribute

Control

- Equations turned into simplification rules with **[simp]** attribute
- Adding/deleting equations locally:

  apply (simp add: <rules>)   and   **apply** (simp del: <rules>)

Control

- Equations turned into simplification rules with **[simp]** attribute
- Adding/deleting equations locally:

  apply (simp add: <rules>) and **apply** (simp del: <rules>)
- Using only the specified set of equations:

  apply (simp only: <rules>)

Demo

We have seen today...

➜ Equations and Term Rewriting

We have seen today...

- ➜ Equations and Term Rewriting

We have seen today...

- Equations and Term Rewriting
- Confluence and Termination of reduction systems

We have seen today...

- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle

Exercises

➜ Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the structure of terms from that example.

