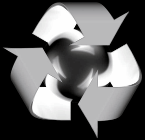
COMP4161: Advanced Topics in Software Verification



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Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

[a] a1 due; [b] a2 due; [c] a3 due

Datatypes

Example:

$$\textbf{datatype}\ \text{'a list} = \text{Nil} \mid \text{Cons 'a ''a list''}$$

Properties:

- ➜ Constructors:

$$\begin{array}{lll} \text{Nil} & :: & \text{'a list} \\ \text{Cons} & :: & \text{'a} \Rightarrow \text{'a list} \Rightarrow \text{'a list} \end{array}$$

- ➜ Distinctness: $\text{Nil} \neq \text{Cons x xs}$
- ➜ Injectivity: $(\text{Cons x xs} = \text{Cons y ys}) = (\text{x} = \text{y} \wedge \text{xs} = \text{ys})$

More Examples

Enumeration:

datatype answer = Yes | No | Maybe

Polymorphic:

datatype 'a option = None | Some 'a
datatype ('a,'b,'c) triple = Triple 'a 'b 'c

Recursion:

datatype 'a list = Nil | Cons 'a "'a list"
datatype 'a tree = Tip | Node 'a "'a tree" "'a tree"

Mutual Recursion:

datatype even = EvenZero | EvenSucc odd

Nested

Nested recursion:

datatype 'a tree = Tip | Node 'a "'a tree list"

datatype 'a tree = Tip | Node 'a "'a tree option" "'a tree option"

➜ Recursive call is under a type constructor.

The General Case

$$
\textbf{datatype}\ (\alpha_1, \ldots, \alpha_n)\ \tau \quad = \quad C_1\ \tau_{1,1}\ \ldots\ \tau_{1,n_1} \\
\quad | \quad \ldots \\
\quad | \quad C_k\ \tau_{k,1}\ \ldots\ \tau_{k,n_k}
$$

- ➜ Constructors: $C_i :: \tau_{i,1} \Rightarrow \ldots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)\ \tau$
- ➜ Distinctness: $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$
- ➜ Injectivity: $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

Distinctness and Injectivity applied automatically

How is this Type Defined?

datatype 'a list = Nil | Cons 'a "'a list"

- ➜ internally defined using typedef
- ➜ hence: describes a set
- ➜ set = trees with constructors as nodes
- ➜ inductive definition to characterise which trees belong to datatype

Datatype Limitations

Must be definable as set.

- Infinitely branching ok.
- Mutually recursive ok.
- Strictly positive (right of function arrow) occurrence ok.

Not ok:

$$
\begin{array}{lcl}
\textbf{datatype}\ \mathsf{t} & = & \mathsf{C}\ (\mathsf{t} \Rightarrow \mathsf{bool}) \\
 & | & \mathsf{D}\ ((\mathsf{bool} \Rightarrow \mathsf{t}) \Rightarrow \mathsf{bool}) \\
 & | & \mathsf{E}\ ((\mathsf{t} \Rightarrow \mathsf{bool}) \Rightarrow \mathsf{bool})
\end{array}
$$

Because: Cantor's theorem (α set is larger than α)

Datatype Limitations

Not ok (nested recursion):

datatype ('a, 'b) fun_copy = Fun "'a ⇒ 'b"

datatype 'a t = F "('a t, 'a) fun_copy"

- ➜ recursion only allowed on *live* arguments
- ➜ in "'a ⇒ 'b", 'a is dead and 'b is live
- ➜ in ('a, 'b) fun_copy, 'a is dead and 'b is live
- ➜ type constructors must be registered as *BNFs** to have live arguments
- ➜ datatypes are automatically registered as BNF
- ➜ can register other type constructors as BNFs — not covered here**

* BNF = Bounded Natural Functors.
** *Defining (Co)datatypes and Primitively (Co)recursive Functions in Isabelle/HOL*

Case

Every datatype introduces a **case** construct, e.g.

$$(\text{case } xs \text{ of } [] \Rightarrow \ldots \mid y \,\#ys \Rightarrow \ldots\, y \,\ldots\, ys \,\ldots)$$

In general: one case per constructor

- Nested patterns allowed: $x\#y\#zs$
- Dummy and default patterns with _
- Binds weakly, needs () in context

Cases

apply (case_tac t)

creates k subgoals

$$[\![t = C_i\ x_1 \ldots x_p; \ldots]\!] \Longrightarrow \ldots$$

one for each constructor C_i

Demo

Recursion

Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

Subtract $f\ x$ on both sides.

$$\Longrightarrow$$
$$0 = 1$$

! All functions in HOL must be total !

Primitive Recursion

primrec guarantees termination structurally

Example primrec def:

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

The General Case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \tau \Rightarrow \tau'$ can be defined by **primitive recursion**:

$$
\begin{array}{lcl}
f\ (C_1\ y_{1,1}\ \ldots\ y_{1,n_1}) & = & r_1 \\
\vdots & & \\
f\ (C_k\ y_{k,1}\ \ldots\ y_{k,n_k}) & = & r_k
\end{array}
$$

The recursive calls in r_i must be **structurally smaller**
(of the form $f\ a_1\ \ldots\ y_{i,j}\ \ldots\ a_p$)

How does this Work?

primrec just fancy syntax for a **recursion operator**

Example: list_rec :: "'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"

$$\text{list\_rec } f_1\ f_2\ \text{Nil} = f_1$$
$$\text{list\_rec } f_1\ f_2\ (\text{Cons } x\ xs) = f_2\ x\ xs\ (\text{list\_rec } f_1\ f_2\ xs)$$

$$\text{app} \equiv \text{list\_rec } (\lambda ys.\ ys)\ (\lambda x\ xs\ xs'.\ \lambda ys.\ \text{Cons } x\ (xs'\ ys))$$

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

list_rec

Defined: automatically, first inductively (set), then by epsilon

$$\frac{}{(\text{Nil}, f_1) \in \text{list\_rel}\ f_1\ f_2} \qquad \frac{(xs, xs') \in \text{list\_rel}\ f_1\ f_2}{(\text{Cons}\ x\ xs, f_2\ x\ xs\ xs') \in \text{list\_rel}\ f_1\ f_2}$$

$$\text{list\_rec}\ f_1\ f_2\ xs \equiv \text{THE}\ y.\ (xs, y) \in \text{list\_rel}\ f_1\ f_2$$

Automatic proof that set def indeed is total function

(the equations for list_rec are lemmas!)

Predefined Datatypes

nat is a datatype

datatype nat = 0 | Suc nat

Functions on nat definable by primrec!

primrec

$$\begin{array}{lcl} f\ 0 & = & \ldots \\ f\ (\text{Suc}\ n) & = & \ldots\ f\ n\ \ldots \end{array}$$

Option

$$\textbf{datatype}\ \text{'a option} = \text{None} \mid \text{Some 'a}$$

Important application:

$$\begin{array}{rcl} \text{'b} \Rightarrow \text{'a option} & \sim & \text{partial function:} \\ \text{None} & \sim & \text{no result} \\ \text{Some } a & \sim & \text{result } a \end{array}$$

Example:

primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option
where
lookup k [] = None |
lookup k (x #xs) = (if fst x = k then Some (snd x) else lookup k xs)

Demo

primrec

Induction

Structural induction

$P\ xs$ holds for all lists xs if

- ➜ P Nil
- ➜ and for arbitrary x and xs, $P\ xs \Longrightarrow P\ (x\#xs)$
  Induction theorem **list.induct:**
  $[\![P\ [];\ \bigwedge a\ list.\ P\ list \Longrightarrow P\ (a\#list)]\!] \Longrightarrow P\ list$
- ➜ General proof method for induction: **(induct x)**
  - x must be a free variable in the first subgoal.
  - type of x must be a datatype.

Basic heuristics

Theorems about recursive functions are proved by induction

Induction on argument number i of f
if f is defined by recursion on argument number i

Example

A tail recursive list reverse:

primrec itrev :: 'a list ⇒ 'a list ⇒ 'a list
where
itrev [] $ys = ys$ |
itrev $(x\#xs)$ $ys =$ itrev xs $(x\#ys)$

lemma itrev xs [] = rev xs

Demo

Proof Attempt

Generalisation

Replace constants by variables

lemma itrev xs ys = rev xs@ys

Quantify free variables by $\forall$

(except the induction variable)

lemma $\forall ys$. itrev xs ys = rev xs@ys

Or: **apply (induct xs arbitrary: ys)**

We have seen today ...

- Datatypes
- Primitive recursion
- Case distinction
- Structural Induction

Exercises

- define a primitive recursive function **lsum** :: nat list ⇒ nat that returns the sum of the elements in a list.
- show "$2 * \text{lsum}\ [0.. < Suc\ n] = n * (n + 1)$"
- show "$\text{lsum}\ (\text{replicate}\ n\ a) = n * a$"
- define a function **lsumT** using a tail recursive version of listsum.
- show that the two functions are equivalent: $\text{lsum}\ xs = \text{lsumT}\ xs$

