COMP4161: Advanced Topics in Software Verification

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Content

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- Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

[a] a1 due; [b] a2 due; [c] a3 due

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Term: $a + (0 + (b + c))$

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is **applicable** to term $t[s]$ with σ if

- ➜ $\sigma\ l = s$ and
- ➜ $\sigma\ P_1, \dots, \sigma\ P_n$ are provable by rewriting.

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lemma "$f\ x = g\ x \wedge g\ x = f\ x \Longrightarrow f\ x = 2$"

simp	**use and simplify** assumptions
(simp (no_asm))	**ignore** assumptions
(simp (no_asm_use))	**simplify**, but do **not use** assumptions
(simp (no_asm_simp))	**use**, but do **not simplify** assumptions

Preprocessing

Preprocessing (recursive) for maximal simplification power:

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\begin{array}{rcl}
\neg A & \mapsto & A = \mathit{False} \\
A \longrightarrow B & \mapsto & A \Longrightarrow B \\
A \wedge B & \mapsto & A,\ B \\
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$$p \Longrightarrow q = True \qquad p \Longrightarrow r = False \qquad s = True$$

Demo

Case splitting with simp

$$P\ (\text{if } A \text{ then } s \text{ else } t) = (A \longrightarrow P\ s) \wedge (\neg A \longrightarrow P\ t)$$

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Similar for any data type t: **t.split**

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if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
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- ➜ use locally with e.g. **apply** (simp cong: <rule>)

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For types nat, int etc:

- lemmas **add_ac** sort any sum $(+)$
- lemmas **mult_ac** sort any product $(*)$

Example: **apply** (simp add: add_ac) yields
$(b + c) + a \rightsquigarrow \cdots \rightsquigarrow a + (b + c)$

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If these 3 rules are present for an AC operator
Isabelle will order terms correctly

Demo

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Rules: (1) $f\ x \longrightarrow a$ (2) $g\ y \longrightarrow b$ (3) $f\ (g\ z) \longrightarrow b$
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\begin{array}{lll}
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(3)+(2) & \{z \mapsto y\} & b \xleftarrow{(3)} f\ (g\ y) \xrightarrow{(2)} f\ b
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This is the main idea of the Knuth-Bendix completion algorithm.

Demo: Waldmeister

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Application: functional programming languages

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