COMP4161: Advanced Topics in Software Verification

HOL

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S2/2017

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- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

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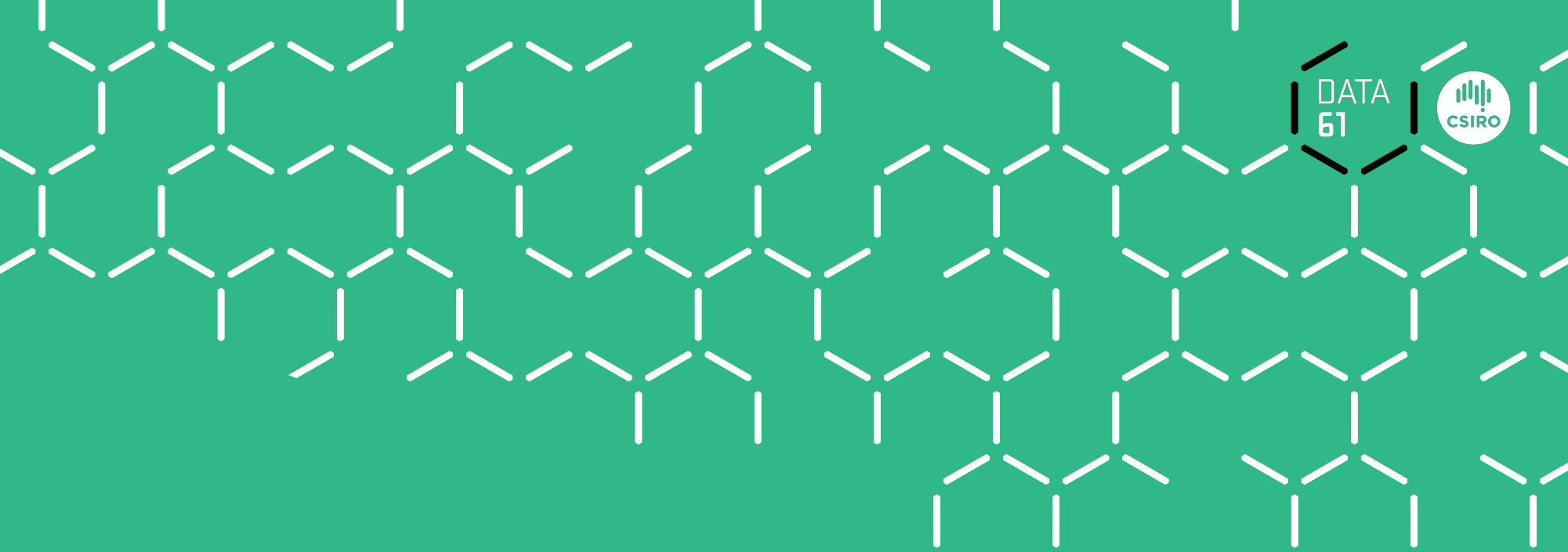
[a] a1 due; [b] a2 due; [c] a3 due



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➜ **Higher Order Logic:**

- quantification over everything, including predicates
- consistency by types
- formula $=$ term of type bool
- definition built on $\lambda^{\rightarrow}$ with certain default types and constants

Defining Higher Order Logic

Default types:

Defining Higher Order Logic

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bool

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Higher Order Abstract Syntax

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So: Use λ to encode all other binders.

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Example:

$$\text{ALL} :: (\alpha \Rightarrow \mathit{bool}) \Rightarrow \mathit{bool}$$

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Isabelle can translate usual binder syntax into HOAS.

Side Track: Syntax Declarations in Isabelle

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consts drvbl :: $ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ ("_, _ ⊢ _")
Legal syntax now: $\Gamma, \Pi \vdash F$

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  $c :: (\tau_1 \Rightarrow \tau_2) \Rightarrow \tau_3$ (binder "B" $< p >$)
  $B\ x.\ P\ x$ translated into $c\ P$ (and vice versa)
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More in Isabelle/Isar Reference Manual (7.2)

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\neg P & \equiv \\
P \wedge Q & \equiv \\
P \vee Q & \equiv \\
\text{If } P\ x\ y & \equiv \\
\text{inj } f & \equiv \\
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\text{False} & \equiv & \forall P.\ P \\
\neg P & \equiv & P \longrightarrow \text{False} \\
P \wedge Q & \equiv & \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R \\
P \vee Q & \equiv & \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R \\
\text{If } P\ x\ y & \equiv & \text{SOME } z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y) \\
\text{inj } f & \equiv & \forall x\ y.\ f\ x = f\ y \longrightarrow x = y \\
\text{surj } f & \equiv & \forall y.\ \exists x.\ y = f\ x
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The Axioms of HOL

$$\frac{}{t = t}\ \text{refl} \qquad \frac{s = t \quad P\ s}{P\ t}\ \text{subst} \qquad \frac{\bigwedge x.\ f\ x = g\ x}{(\lambda x.\ f\ x) = (\lambda x.\ g\ x)}\ \text{ext}$$

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With this you can define and derive all the rest.

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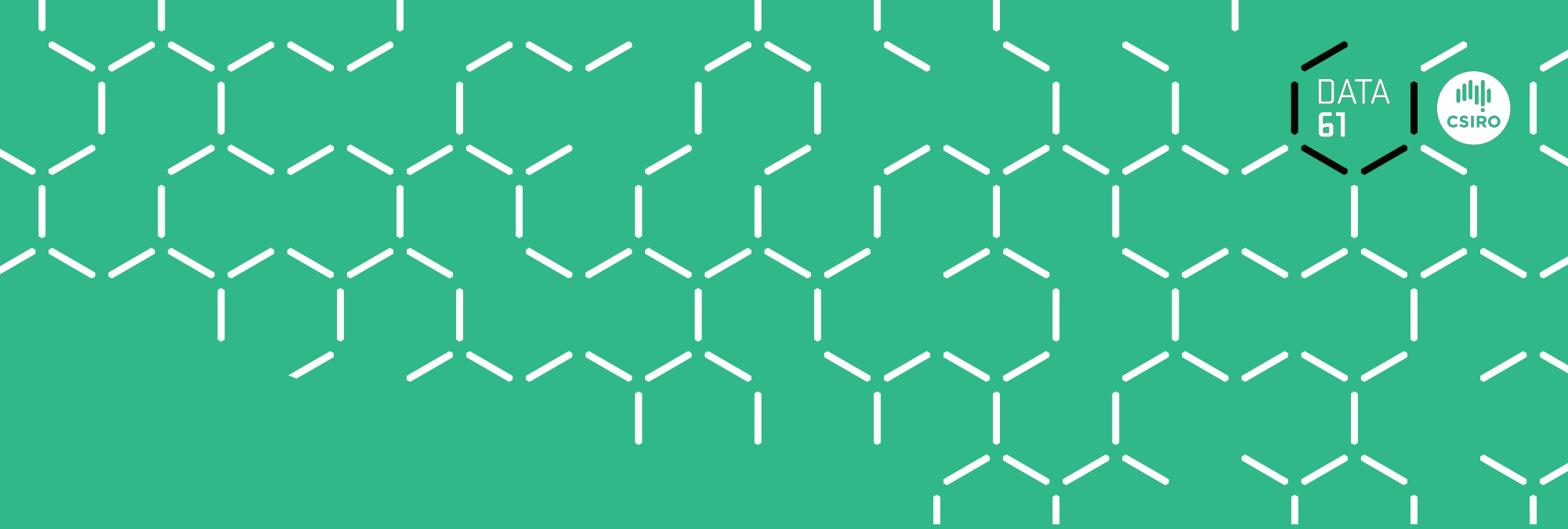
Isabelle knows 2 more axioms:

$$\frac{x = y}{x \equiv y}\ \text{eq\_reflection} \qquad \frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$



Demo:
The Definitions in Isabelle

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Convenient for deriving rules: **named assumptions in lemmas**

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assumes [$name_1$:] "$< prop >_1$"
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proves: $[\![< prop >_1;\ < prop >_2;\ \ldots]\!] \Longrightarrow < prop >$

True

consts True :: *bool*
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Intuition:
right hand side is always true

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Proof:

$$\frac{\dfrac{}{(\lambda x :: bool.\ x) = (\lambda x.\ x)}\ \text{refl}}{\text{True}}\ \text{unfold True\_def}$$

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Universal Quantifier

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Intuition:

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- ALL P should be true iff P yields true for all x, i.e. if it is equivalent to the function $\lambda x.$ True.

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Proof Rules:

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

Proof: Isabelle Demo

False

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$$\frac{\text{False}}{P}\ \text{FalseE} \qquad \frac{}{\text{True} \neq \text{False}}$$

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Proof Rules:

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Proof: Isabelle Demo

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Proof Rules:

$$\frac{A \quad B}{A \wedge B}\ \text{conjI} \qquad \frac{A \wedge B \quad [\![A; B]\!] \Longrightarrow C}{C}\ \text{conjE}$$

Proof: Isabelle Demo

Disjunction

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Proof Rules:

$$\frac{A}{A \vee B}\ \frac{B}{A \vee B}\ \text{disjI1/2} \qquad \frac{A \vee B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C}\ \text{disjE}$$

Proof: Isabelle Demo

If-Then-Else

consts If :: $bool \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha$ (if_ then _ else _)
If $P\ x\ y \equiv$ SOME z. $(P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

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Intuition:

➜ for $P = \text{True}$, right hand side collapses to SOME z. $z = x$

If-Then-Else

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If $P\ x\ y \equiv$ SOME z. $(P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

Intuition:

- ➜ for $P = \text{True}$, right hand side collapses to SOME z. $z = x$
- ➜ for $P = \text{False}$, right hand side collapses to SOME z. $z = y$

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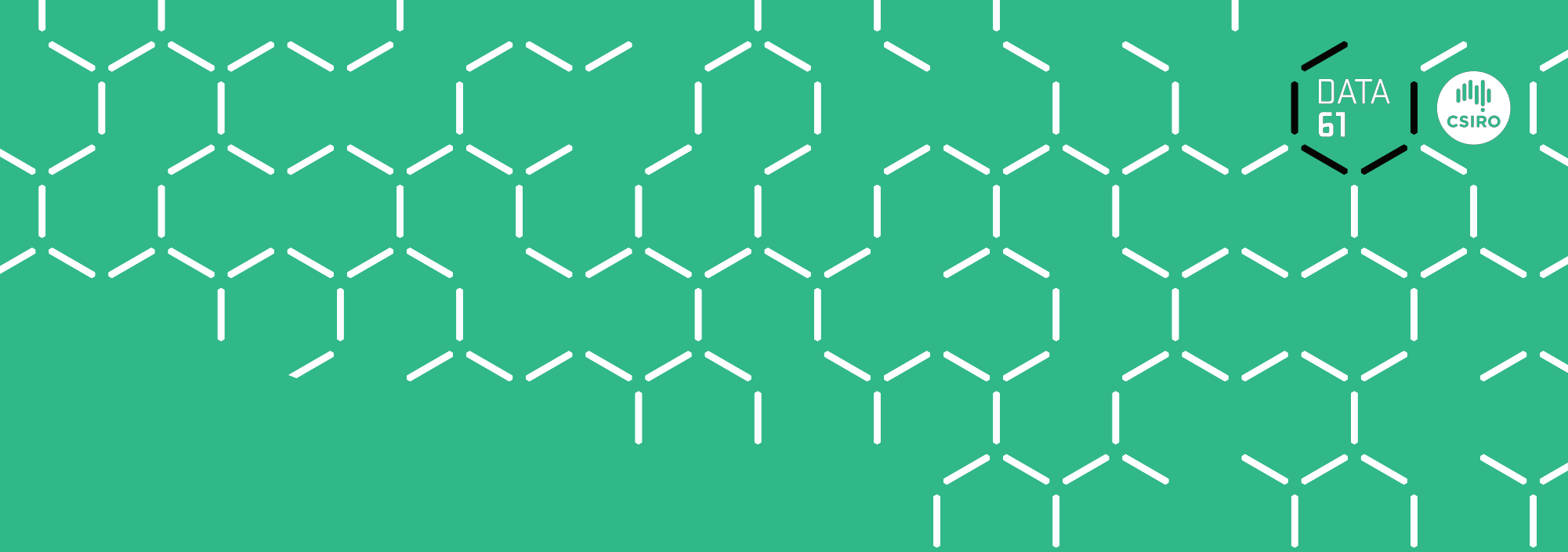
Proof Rules:

$$\frac{}{\text{if True then } s \text{ else } t = s}\ \text{ifTrue} \qquad \frac{}{\text{if False then } s \text{ else } t = t}\ \text{ifFalse}$$

Proof: Isabelle Demo



That was HOL

More on Automation

Last time: safe and unsafe, heuristics: use safe before unsafe

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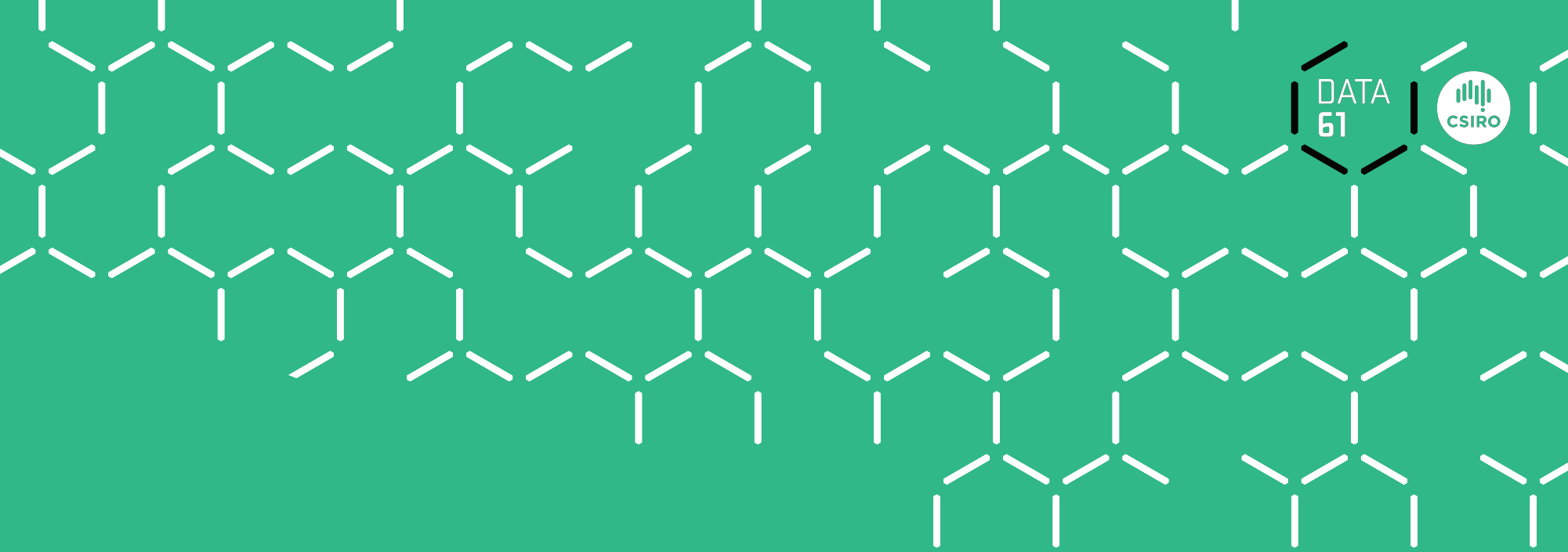
Example:

declare attribute globally	**declare** conjI [intro!] allE [elim]
remove attribute gloablly	**declare** allE [rule del]
use locally	**apply** (blast intro: someI)
delete locally	**apply** (blast del: conjI)



Demo: Automation

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