# CakeML: bootstrapping a verified compiler

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COMP4161, 6 October 2016

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## Question

What is this function, `foo`, more often called?

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## Answer

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What about this one?

$$\texttt{bar}\ [\,] = 0$$
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## Note

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# Spot the differences

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## Example 2

$$\vdash\ (\forall f.\ \texttt{map}\ f\ [\,] = [\,]) \wedge$$
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### Answer

Example 1 is a pair of equations.
Example 2 is a theorem: it has a turnstile, a conjunction, and explicit universal quantification.
(But they mean the same thing.)

# What you learned last month

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Can you prove this?

$\forall\, l\ f.\ \texttt{length}\ (\texttt{map}\ f\ \ l) = \texttt{length}\ l$

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Yes! By induction on the list *l*, simplifying with the definitions of `map` and `length`.

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### Answer

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But we are interested in even simpler theorems...

# Simple theorems

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`map length` [[ ]; [[ ]; [ ]]; [[ ]]] = [0; 2; 1]

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Or this?

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### Answer

Simplification...

In fact, you only need the left-hand side of the equation in order to produce the theorem.

# Evaluation problems

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An *evaluation problem* is a term that does not contain any variables (only known constants and concrete data).

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A solution is a theorem $\vdash tm = tm'$, where $tm'$ cannot be simplified further.

# Example

Consider the constant `while`, which satisfies the following equation.

$\vdash$ `while` $P\ g\ x = \text{if}\ P\ x\ \text{then}$ `while` $P\ g\ (g\ x)\ \text{else}\ x$

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It is. `while` is total. We just cannot prove anything interesting about its value on the arguments above.

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Clearly this procedure can sometimes loop forever.

# Proof tools steer the kernel

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Isabelle and HOL4 support this view (“LCF-style”).

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  It can be extended with user-defined automation.

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## Call-by-value proof automation

- High-performance simplification:
  - Choose a good *evaluation strategy*.
  - Use techniques from functional programming.
- HOL4 includes such automation (called `EVAL`).
  It can be extended with user-defined automation.
- Performance is *fundamentally limited*.
  - At best, simplification is akin to interpreting a program.
  - And, every step ultimately goes through the kernel.

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  - The result theorem needs to be asserted as an axiom.
  - Much care is required to ensure this axiom is plausible.

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We will return to this later.

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Can you count the number of reductions (applications of a single rewrite rule) taken in solving an evaluation problem?

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Simplify and count: $\texttt{while}\ (\lambda\, x.\ x < 2)\ \texttt{Suc}\ 0$.

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Simplify and count: $\texttt{while}\ (\lambda x.\ x < 2)\ \texttt{Suc}\ 0$.
returns: $(\vdash \texttt{while}\ (\lambda x.\ x < 2)\ \texttt{Suc}\ 0 = 2$, 2 rewrites)
(Actually: 216 primitive inference steps.)

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### Totally different approach

Formalise simplification within the logic.

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How about *reasoning about* the number of steps?

### Problem

The simplifier is outside the logic, just using the kernel API.
Inside the logic, the number of steps is completely invisible.

### Totally different approach

Formalise simplification within the logic.
Use a deep embedding.

# Deep embeddings

### Question

What might this datatype be used for?

```
lit =
    IntLit int
  | Char char
  | StrLit string
  | Word8 byte
  | Word64 word64
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## Answer

```
exp =
    Lit lit
  | Var (string id)
  | Con (string id option) (exp list)
  | Fun string exp
  | App op (exp list)
```

# Functional semantics

## Some meanings

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## Pulling apart closures

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## Function applications tick

`evaluate` *st env* [`Call` $e_1$ $e_2$] =
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    | Some (env',e) ⇒
      if st'.clock = 0 then
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      else
        evaluate (st' with clock := st'.clock − 1) env' [e])
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The clock lets us prove termination for `evaluate`.

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- functions: higher-order, polymorphic, mutually recursive

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A real programming language.
But many similarities to HOL.

# Interlude: ML

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Nowadays a general programming language, and used in industry.

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## Characteristics

Functional, strict, impure, type safe, modular.

# Deep map

Remember this?

$\texttt{map}\ f\ [\,] = [\,]$

$\texttt{map}\ f\ (h \# t) = f\ h \# \texttt{map}\ f\ t$

# Deep map

## Remember this?

map $f$ [] = []
map $f$ $(h \# t)$ = $f$ $h \#$ map $f$ $t$

## Compare

```
Dletrec
 [("map","v3",
   Fun "v4"
    (Mat (VarS "v4")
      [(PconS "nil" [],ConS "nil" []);
       (PconS "::" [Pvar "v2"; Pvar "v1"],
        ConS "::"
         [Call (VarS "v3") (VarS "v2");
          Call (Call (VarS "map") (VarS "v3")) (VarS "v1")])]))]
```

# Deep map, pretty-printed

Easier to read in concrete syntax

```
fun map v3 v4 =
  case v4
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  |  v2::v1 => (v3 v2::(map v3 v1));
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Let us name this deeply-embedded declaration `map_dec`.

# Proofs about deep embeddings

## Another declaration

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val it = map (fn x => (x + 1)) [1,2,0];
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$\vdash$ `evaluate_decs` $st$ $env$ [`map_dec`; `map_suc_dec`] =
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How hard was this to prove?

Using `EVAL` the proof is short, but takes many seconds to run.

# More general proofs

Deep embeddings let us reason about the semantics in general.

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## Type safety

- We can define a type system over deeply-embedded syntax.
- We can prove that well-typed programs never crash (they only diverge or terminate with a value or un-handled exception).

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## Type safety

- We can define a type system over deeply-embedded syntax.
- We can prove that well-typed programs never crash (they only diverge or terminate with a value or un-handled exception).

## Alternative semantics

# More general proofs

Deep embeddings let us reason about the semantics in general.

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- You may have seen relational big-step semantics, as well as small-step operational semantics.

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## Alternative semantics

- You may have seen relational big-step semantics, as well as small-step operational semantics.
- We can prove equivalences between different versions of the semantics, and obtain a solid understanding of our language.

# Deep proofs are hard

Remember this?

$\vdash \forall l\ f.\ \texttt{length}\ (\texttt{map}\ f\ l) = \texttt{length}\ l$

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But can we get it automatically from the shallow proof?

(You may have seen a similar thing before, e.g., Autocorres.)

# Connecting shallow to deep

Question

What is the deep counterpart of this term?

`Suc (Suc (Suc 0))`

# Connecting shallow to deep

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(of type `v`, rather than `nat`)

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`LIST` *A ls v* means *v* relates to *ls*, if *A* relates the elements.

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$\texttt{LIST}\ A\ (h \# t)\ v \iff$

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## Definition

$$
\begin{array}{l}
(A \rightarrow B)\ f\ v \iff \\
\quad \forall\, x\ v_1 . \\
\quad\quad A\ x\ v_1 \Rightarrow \\
\quad\quad\quad \exists\, v_2\ env\ exp\ k . \\
\quad\quad\quad\quad (\mathtt{do\_call}\ [v;\ v_1] = \mathtt{Some}\ (env, exp) \wedge \\
\quad\quad\quad\quad\ \mathtt{evaluate}\ (\mathtt{st}_0\ \textit{with}\ \mathrm{clock} := k)\ env\ [exp] = \\
\quad\quad\quad\quad\quad (\mathtt{st}_0, \mathtt{Rval}\ [v_2])) \wedge B\ (f\ x)\ v_2
\end{array}
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Any closure, $\texttt{map}_\mathsf{v}$, satisfying this refinement invariant:

$$((A \to B) \to \texttt{LIST}\ A \to \texttt{LIST}\ B)\ \texttt{map}\ \texttt{map}_\mathsf{v}$$

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Yes, only closures that behave like `map` satisfy this invariant.

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Call (Fun "x" (App (Opn Plus) [VarS "x"; VarS "x"]))
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The semantics justifies the connection.

$$\vdash\ \exists k\ res.\ \texttt{evaluate}\ (\texttt{st}_0\ \textit{with}\ \text{clock} := k)\ env\ [\texttt{Call (Fun "x" (App (Opn Plus) [VarS "x"; VarS "x"])) (Lit (IntLit 3))}] = (\texttt{st}_0,\texttt{Rval}\ [res]) \wedge \texttt{NAT}\ ((\lambda x.\ x + x)\ 3)\ res$$

# Certificate theorems

### Definition

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We abbreviate this by `Cert` *env exp A*.

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### Example

$\vdash\ \texttt{Cert}\ env\ (\texttt{ConS}\ \texttt{"::"}\ [\texttt{Con None}\ [\,];\ \texttt{ConS}\ \texttt{"nil"}\ [\,]])$
$\quad (\text{LIST UNIT}\ [()])$

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Just a variable: `VarS "map"`.

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But it is only correct in the right environment:

$\vdash$ `lookup_var "map"` *env* = Some $\mathtt{map}_\mathsf{v}$ $\Rightarrow$

Cert *env* (`VarS "map"`) $(((a \to b) \to \mathtt{LIST}\ a \to \mathtt{LIST}\ b)\ \mathtt{map})$

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What is the deep counterpart of `map`, considered as an expression?

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Just a variable: `VarS "map"`.

But it is only correct in the right environment:

$$\vdash \texttt{lookup\_var "map"}\ \mathit{env} = \texttt{Some}\ \texttt{map}_{\mathsf{v}} \Rightarrow$$
$$\texttt{Cert}\ \mathit{env}\ (\texttt{VarS "map"})\ (((a \to b) \to \texttt{LIST}\ a \to \texttt{LIST}\ b)\ \texttt{map})$$

Now, how can we use this certificate theorem?

# Deep results for shallow proofs

Remember this?

$\vdash$ `length` (`map` $f$ $l$) = `length` $l$

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`lookup_var "l"` $env$ = `Some` $l_v$ $\wedge$ LIST $a$ $l$ $l_v$ $\Rightarrow$

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`lookup_var "f"` $env$ = `Some` $f_\mathsf{v}$ $\wedge$ $(a \rightarrow b)$ $f$ $f_\mathsf{v}$ $\Rightarrow$
`Cert` $env$ (`Call` (`VarS "length"`) (`VarS "l"`))
(`NAT` (`length` $l$)) $\wedge$
`Cert` $env$
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# Deep results for shallow proofs

## Remember this?

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      (`NAT` (`length` $l$)) $\wedge$
     `Cert` $env$
      (`Call` (`VarS "length"`)
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Follows directly from the certificate theorems for `map` and `length`.

# Certificate theorems compose

Derived rules

$\vdash$ `Cert` *env* (`Lit` (`IntLit` (`toInt` *n*))) (`NAT` *n*)

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## Derived rules

$\vdash \texttt{Cert}\ \mathit{env}\ (\texttt{Lit}\ (\texttt{IntLit}\ (\texttt{toInt}\ n)))\ (\texttt{NAT}\ n)$

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$\quad \texttt{Cert}\ \mathit{env}\ e_2\ (A\ x) \Rightarrow \texttt{Cert}\ \mathit{env}\ (\texttt{Call}\ e_1\ e_2)\ (B\ (f\ x))$

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  `Cert` *env* $e_2$ (`BOOL` $b_2$) $\Rightarrow$
   `Cert` *env*
    (`If` $e_1$ $e_2$
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    (`BOOL` $(b_1 \Rightarrow b_2)$)

# Certificate theorems compose

## Derived rules

$\vdash$ Cert *env* (Lit (IntLit (toInt $n$))) (NAT $n$)

$\vdash$ Cert *env* $e_1$ (($A \to B$) $f$) $\Rightarrow$
  Cert *env* $e_2$ ($A$ $x$) $\Rightarrow$ Cert *env* (Call $e_1$ $e_2$) ($B$ ($f$ $x$))

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$\vdash$ $A\ x\ v \Rightarrow$
`lookup_var` *n* *env* = `Some` *v* $\Rightarrow$ `Cert` *env* (`VarS` *n*) $(A\ x)$

By composing certificates, we can generate a certified deep embedding by traversing a shallow term.

# Proof-producing code generation

## That is the idea

From shallow embeddings we can *automatically* generate *certified* deep embeddings.

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“Certified implementations from verified algorithms”

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Fast simplification within the logic using `EVAL`.

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Fast simplification within the logic using `EVAL`.
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Automatic generation of a *real* functional program from a HOL term, *plus*... a certificate theorem stating that the CakeML code correctly implements the HOL term.

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Fast simplification within the logic using `EVAL`.
"Evaluate" HOL terms as if with a functional-program interpreter.

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Automatic generation of a *real* functional program from a HOL term, *plus*... a certificate theorem stating that the CakeML code correctly implements the HOL term.

## Next up

Verified compilation

# What else can we do with syntax?

## Functions on syntax

Within the logic, we have defined semantic functions.

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(You saw something like this in Assignment 2)

# Anatomy of a compiler

## Compiler definition

Would something like this work

$\texttt{compile\_exp}\ cs\ (\texttt{Lit}\ (\texttt{IntLit}\ 2)) = (cs, [184w;\ 2w;\ 0w;\ 0w;\ 0w])$

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Compilation is rather more involved than the semantics.

# Intermediate languages

Many phases

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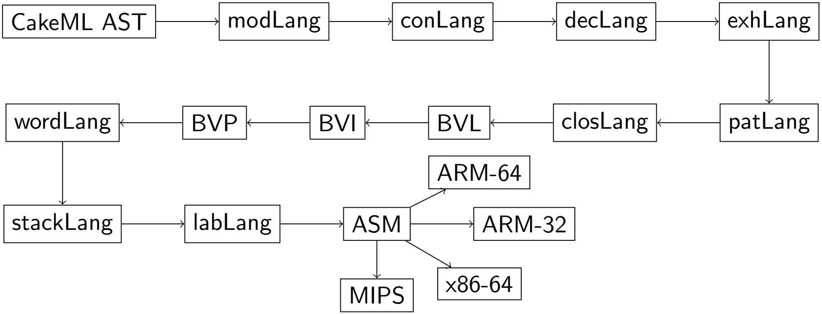
## Many phases

CakeML compiler backend:



# Compiling pattern matching

## A small peek

exhLang ⟶ patLang compiles `case` to nested `if`.

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## Or, in the deep embedding

```
⊢ compile []
    (Mat (Con 0 [Lit (IntLit 1)])
      [(Pcon 1 [],Raise (Con 2 []));
       (Pcon 0 [Pvar "x"],Var "x")]) =
    Let (Con 0 [Lit (IntLit 1)])
     (If (App (Op Eq) [Vardb 0; Con 1 []])
       (Raise (Con 2 [])) (App (El 0) [Vardb 0]))
```

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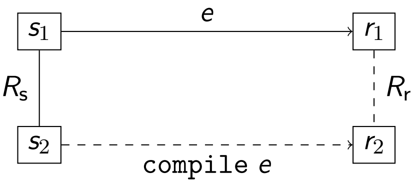

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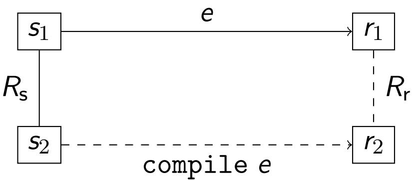


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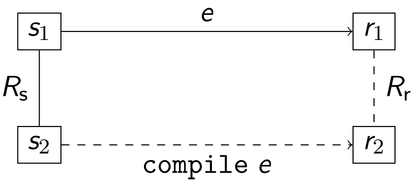


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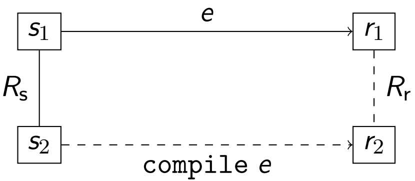


Proof idea: induction on source semantics. A natural fit.

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- Programs that crash...
- Programs that diverge (loop forever)...
- It depends on what style of semantics you use.

# Functional big-step

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## The clock enables divergence preservation

- If we prove the compiler preserves timeouts,
- then the compiled code diverges if and only if the source code diverges.

# Divergence preservation

## Definition (or theorem)

*exp diverges* iff:

$$\forall k.\ \exists s'.\ \texttt{evaluate}\ (s\ \textit{with}\ \mathrm{clock}\ := k)\ \mathit{env}\ [\mathit{exp}] = (s', \texttt{Rerr}\ (\texttt{Rabort}\ \mathit{Rtimeout}))$$

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(As long as both source and target semantics are *deterministic*.)

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cs → string → compiler_result
where compiler_result =
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      | Success cs (byte list)
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- If a program parses and type checks, then its semantics is one of:
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- The latest version of CakeML adds a trace of I/O events to each of these options.

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- Specifies a machine state (memory, registers, etc.), and a "next state" relation for each instruction.
- Validated (in some cases) by evaluation of the model compared with execution of real hardware.

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- Furthermore the I/O events match up with the semantics.

Shorthand: "`compile` *cs prog* implements *prog*"

# What we have seen so far

## Evaluation problems

Fast simplification within the logic using `EVAL`.
"Evaluate" HOL terms as if with a functional-program interpreter.

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Automatic generation of a *real* functional program from a HOL term, *plus*... a certificate theorem stating that the CakeML code correctly implements the HOL term.

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An algorithm turning CakeML code into machine code, *plus*... a theorem stating that the semantics of the compiled code is permitted by the semantics of the source code.

# Compiler as shallow embedding

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How can we run the verified compiler?

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How can we run the verified compiler?

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Recall, the compiler is a function in HOL...

Answer

Running the compiler is an *evaluation problem*.
We can use `EVAL` to run the compiler in the logic.

# Evaluating the compiler

## Remember this?

`map_suc_dec`, pretty-printed:

```
val it = map (fn x => (x + 1)) [1,2,0];
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## Example

Input term:

`compile_ast` $\texttt{cs}_0$ `[Tdec map_dec; Tdec map_suc_dec]`.

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`map_suc_dec`, pretty-printed:

```
val it = map (fn x => (x + 1)) [1,2,0];
```

### Example

Input term:

`compile_ast` $cs_0$ `[Tdec map_dec; Tdec map_suc_dec]`.

Produces:

$\vdash$ `compile_ast` $cs_0$ `[Tdec map_dec; Tdec map_suc_dec]` =
`Success` $cs_1$ `map_suc_code`

for some $cs_1$ and `map_suc_code`.

# Compiler as deep embedding

**Question**

But how can we run the compiler quicker?

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### Possible answer

Can we use proof-producing code generation?

Yes, to produce CakeML code implementing the compiler.

# Compiler as deep embedding

Generating code implementing the compiler

What is the deep counterpart of `compile`?

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`compile_dec = Dletrec [("compile","cs",Fun "prog" ...)]`

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$\vdash$ `EnvContains` *env* `compiler_decs` $\Rightarrow$
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compiler_decs includes all the preliminary declarations required to define the compiler.

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## But how do we run it?

Now we have the compiler as CakeML code...

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## To run CakeML code, first compile it

- Evaluation problem:
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- We assume our machine model (and loader etc.) is correct.

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## Put them together

- 1 and 3: $\vdash$ `compiler_code` implements `compiler_decs`.

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## Put them together

- 1 and 3: $\vdash$ compiler_code implements compiler_decs.
- plus 2: $\vdash$ compiler_code implements compile.

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We have verified machine code implementing the compiler.

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  string $\rightarrow$ AST $\rightarrow$ ILs $\rightarrow \cdots \rightarrow$ asm $\rightarrow$ bytes
- Which level of the compiler is verified:
  algorithm (shallow), high-level code (deep), machine code
- CakeML covers the full spectrum of both dimensions.

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## Alternative to simplification outside the logic

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## Yes, but not yet

- Still need to run the machine code outside the logic, and lose the connection.
- Work in progress: building a verified theorem prover that includes evaluation by compilation...

# What we have seen

## Evaluation problems

Fast simplification within the logic using `EVAL`.
"Evaluate" HOL terms as if with a functional-program interpreter.

## Certified deep embeddings

Automatic generation of a *real* functional program from a HOL term, *plus*... a certificate theorem stating that the CakeML code correctly implements the HOL term.

## Verified compilation

An algorithm turning CakeML code into machine code, *plus*... a theorem stating that the semantics of the compiled code is permitted by the semantics of the source code.

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An algorithm turning CakeML code into machine code, *plus*... a theorem stating that the semantics of the compiled code is permitted by the semantics of the source code.

## Bootstrapping

Combining the above to get a verified compiler in machine code.

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## People involved

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You can be involved!
`https://cakeml.org`