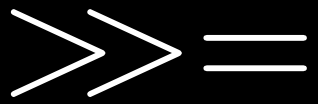
COMP4161: Advanced Topics in Software Verification



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data61.csiro.au

Last Time

- ➜ Weakest preconditions
- ➜ Verification conditions
- ➜ Example program proofs
- ➜ Arrays, pointers

Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

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[a] a1 due; [b] a2 due; [c] a3 due

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Disadvantages:

➜ Semantically equivalent programs are not obviously equal.

➜ e.g. "IF True THEN SKIP ELSE SKIP = SKIP" is not a true theorem.

➜ Many concepts that we already have in the logic are reinvented in the language.

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Today we learn about a formalism suitable for shallowly embedding C semantics.

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Demo

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AutoCorres: verified translation of C to monadic representation

- ➜ Specifically designed for humans to do proofs over.

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modify – applies its argument to modify the state; returns ():

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Monads, Laws

Formally: a monad **M** is a type constructor with two associated operations.

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A fragment of C:

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void f(int *p) {
    int x = *p;
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record state =
    hp :: int ptr ⇒ int

f :: "int ptr ⇒ (state ⇒ (unit,state))"

f *p* ≡
do
  x ← gets (λs. hp s p);
  if x < 10 **then**
    modify (hp_update (λh. (h(p := x + 1))))
  else
    return ()
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guard – fails when given condition applied to the state is False:

guard P $\equiv$ get $\gg\!=$ (λs. assert (P s))

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f p ≡
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    y ←guard (λs. valid s p);
    x ← gets (λs. hp s p);
    if x < 10 then
      modify (hp_update (λh. (h(p := x + 1))))
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select – nondeterministic selection from a set:

$$\text{select}\ A \equiv \lambda s.\ \ ((A \times \{s\}),\text{False})$$

Demo

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Example: whileLoop $(\lambda p\ s.\ \text{hp}\ s\ p = 0)$ $(\lambda \text{p}.\ \text{return}\ (\text{ptrAdd}\ p\ 1))$ p

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Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow \forall (r,s') \in \text{fst}\ (m\ s).\ Q\ r\ s'$$

➜ Post-condition Q is a predicate of return-value and result state.

Weakest Precondition Rules

$$\{\!|\ \ |\!\}\ \text{return}\ x\ \{\!|\lambda r\ s.\ P\ r\ s|\!\} \qquad \{\!|\ \ |\!\}\ \text{get}\ \{\!|P|\!\} \qquad \{\!|\ \ |\!\}\ \text{put}\ x\ \{\!|P|\!\}$$

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More Hoare Logic Rules

$$\frac{}{\{\!|\quad\quad|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

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Demo

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