DATA 61

COMP4161: Advanced Topics in Software Verification

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S2/2016

Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

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[a] a1 due; [b] a2 due; [c] a3 due

Last Time

➜ Sets

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- ➜ Inductive Definitions

Inductive Definitions

How They Work

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- Objective: **no junk**. Only what must be in X shall be in X.
- Gives rise to a nice proof principle (rule induction)

Formally

$$\text{Rules } \frac{a_1 \in X \quad \ldots \quad a_n \in X}{a \in X} \text{ with } a_1, \ldots, a_n, a \in A$$

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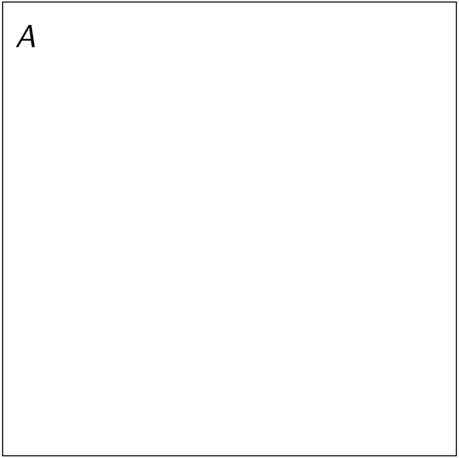
Fact: $X = \bigcap\{B \subseteq A.\ B\ R\text{−closed}\}$

Generation from Above



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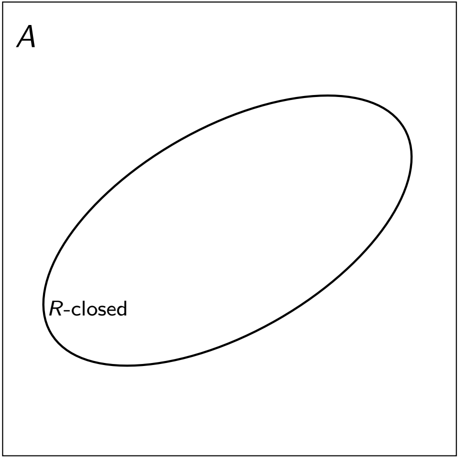

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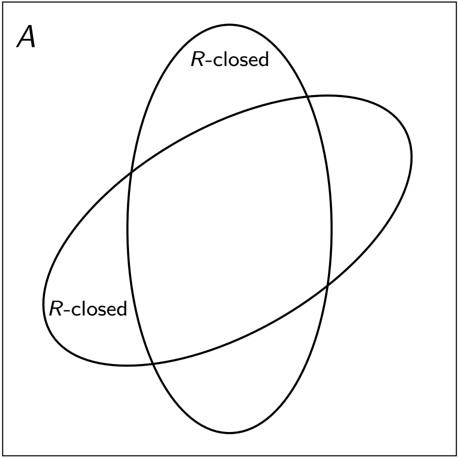

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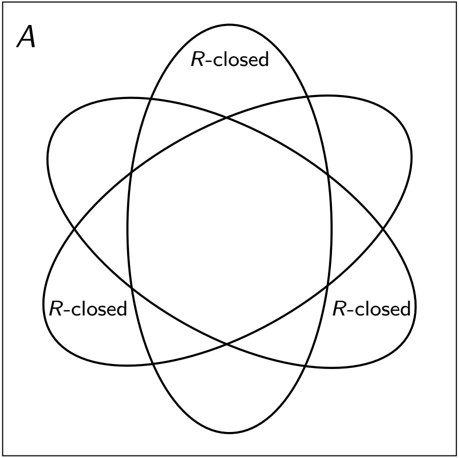

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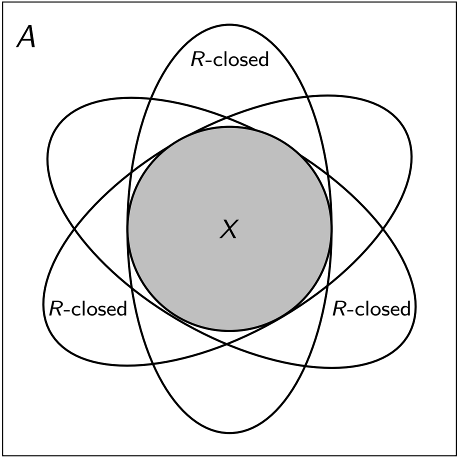

Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

induces induction principle

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In general:

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

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Rules with side conditions

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induction scheme:

$$(\forall (\{a_1, \dots a_n\}, a) \in R.\ P\ a_1 \wedge \dots \wedge P\ a_n \wedge$$
$$C_1 \wedge \dots \wedge C_m \wedge$$
$$\{a_1, \dots, a_n\} \subseteq X \Longrightarrow P\ a)$$
$$\Longrightarrow$$
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X as Fixpoint

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$$X_\omega = \bigcup_{n \in \mathbb{N}} (R^n\ \{\}) = X$$

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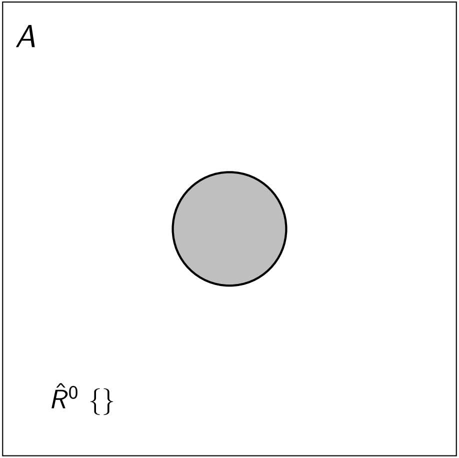

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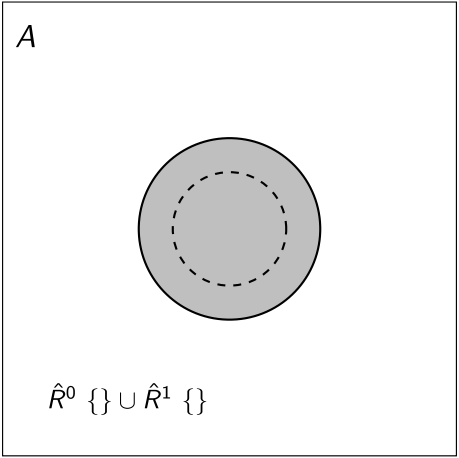

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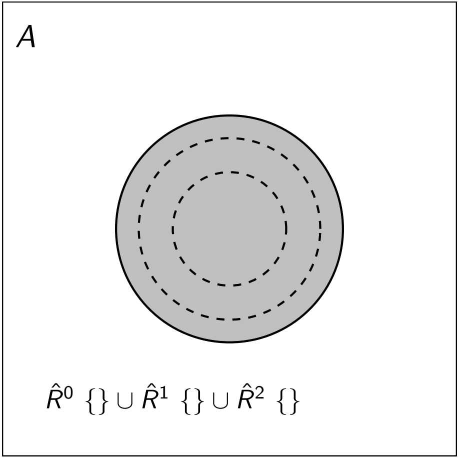

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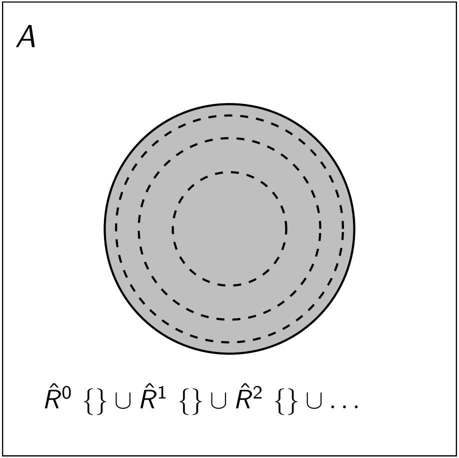

Does this always work?

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Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

Exercise

Formalize the this lecture in Isabelle:

- ➜ Define **closed** $f\ A :: (\alpha \text{ set} \Rightarrow \alpha \text{ set}) \Rightarrow \alpha \text{ set} \Rightarrow \text{bool}$
- ➜ Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)
- ➜ Define **lfpt** f as the intersection of all f-closed sets
- ➜ Show that lfpt f is a fixpoint of f if f is monotone
- ➜ Show that lfpt f is the least fixpoint of f
- ➜ Declare a constant $R :: (\alpha \text{ set} \times \alpha) \text{ set}$
- ➜ Define $\hat{R} :: \alpha \text{ set} \Rightarrow \alpha \text{ set}$ in terms of R
- ➜ Show soundness of rule induction using R and lfpt $\hat{R}$

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