COMP4161: Advanced Topics in Software Verification

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Content

➜ Intro & motivation, getting started [1]

➜ Foundations & Principles

- Lambda Calculus, natural deduction [1,2]
- Higher Order Logic [3[a]]
- Term rewriting [4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [5]
- Datatypes, recursion, induction [6, 7]
- Hoare logic, proofs about programs, C verification [8[b],9]
- (mid-semester break)
- Writing Automated Proof Methods [10]
- Isar, codegen, typeclasses, locales [11[c],12]

[a] a1 due; [b] a2 due; [c] a3 due

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- ➜ Orthogonal Rewrite Systems

Specification Techniques

Sets

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- $f`A \equiv \{y.\ \exists x \in A.\ y = f\ x\}$
- ...

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- ... (see Tutorial)

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- bspec: $[\![\forall x \in A.\ P\ x; x \in A]\!] \Longrightarrow P\ x$

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- bexI: $[\![P\ x; x \in A]\!] \Longrightarrow \exists x \in A.\ P\ x$
- bexE: $[\![\exists x \in A.\ P\ x; \bigwedge x.\ [\![x \in A; P\ x]\!] \Longrightarrow Q]\!] \Longrightarrow Q$

Demo

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  The harder, but safe choice.

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- **typedef**: by definiton as a set

  Example: **typedef** new_type = "{some set}" <proof>
  Introduces a new type as a subset of an existing type.
  The proof shows that the set on the rhs in non-empty.

How typedef works

new type

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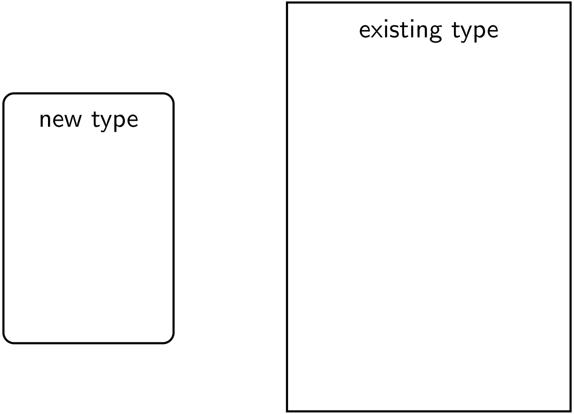

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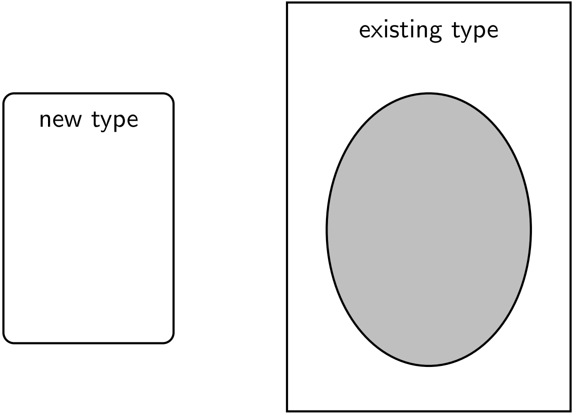

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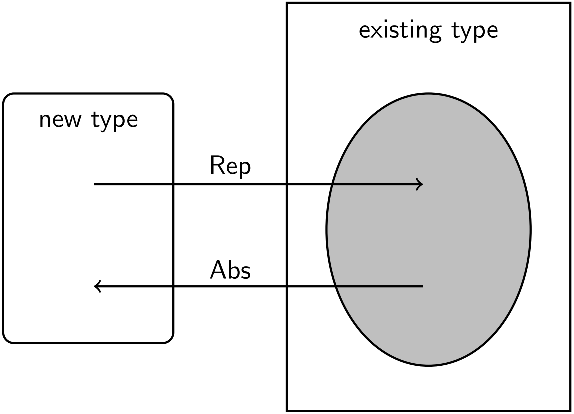

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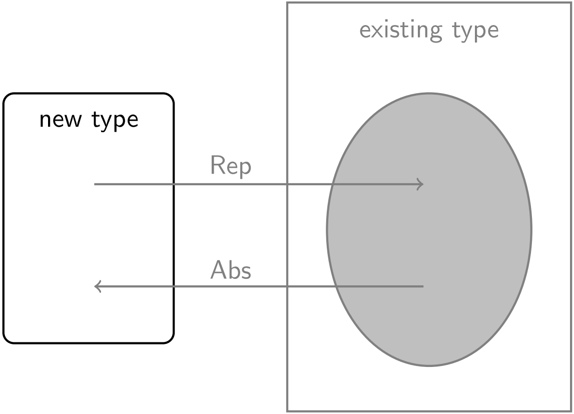

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④ We now can:

- define constants Pair, fst, snd in terms of Abs_Prod and Rep_Prod
- derive all characteristic theorems
- forget about Rep/Abs, use characteristic theorems instead

Demo

Introducing new Types

Inductive Definitions

Example

$$\frac{}{\langle \mathsf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{[\![e]\!]\sigma = v}{\langle \mathsf{x} := \mathsf{e}, \sigma\rangle \longrightarrow \sigma[x \mapsto v]}$$

$$\frac{\langle c_1, \sigma\rangle \longrightarrow \sigma' \quad \langle c_2, \sigma'\rangle \longrightarrow \sigma''}{\langle c_1; c_2, \sigma\rangle \longrightarrow \sigma''}$$

$$\frac{[\![b]\!]\sigma = \mathsf{False}}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma\rangle \longrightarrow \sigma}$$

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But which set?

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- ➜ Gives rise to a nice proof principle (rule induction)
- ➜ Alternative (greatest set) occasionally also useful: coinduction

Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

induces induction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n+1)]\!] \Longrightarrow \forall x \in N.\ P\ x$$

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