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COMP4161: Advanced Topics in Software Verification



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# Exercises from last time



- Download and install Isabelle from <http://mirror.cse.unsw.edu.au/pub/isabelle/>
- Step through the demo files from the lecture web page
- Write your own theory file, look at some theorems in the library, try 'find\_theorems'
- How many theorems can help you if you need to prove something containing the term " $\text{Suc}(\text{Suc } x)$ "?
- What is the name of the theorem for associativity of addition of natural numbers in the library?

# Content



- Intro & motivation, getting started [1]
  
- Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3<sup>a</sup>]
  - Term rewriting [4]
  
- Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8<sup>b</sup>,9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11<sup>c</sup>,12]

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<sup>a</sup>a1 due; <sup>b</sup>a2 due; <sup>c</sup>a3 due

# $\lambda$ -calculus



## Alonzo Church

- lived 1903–1995
- supervised people like Alan Turing, Stephen Kleene
- famous for Church-Turing thesis, lambda calculus, first undecidability results
- invented  $\lambda$  calculus in 1930's



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## $\lambda$ -calculus

- originally meant as foundation of mathematics
- important applications in theoretical computer science
- foundation of computability and functional programming

# untyped $\lambda$ -calculus



- turing complete model of computation
- a simple way of writing down functions

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Basic intuition:

instead of  $f(x) = x + 5$   
write  $f = \lambda x. x + 5$

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$\lambda x. x + 5$

- a term



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- a nameless function

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- a simple way of writing down functions

Basic intuition:

instead of  $f(x) = x + 5$   
write  $f = \lambda x. x + 5$

$\lambda x. x + 5$

- a term
- a nameless function
- that adds 5 to its parameter

# Function Application



For applying arguments to functions

instead of  $f(a)$   
write  $f\ a$

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**Evaluating:** in  $(\lambda x. t)\ a$  replace  $x$  by  $a$  in  $t$   
(computation!)

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**Example:**  $(\lambda x. x + 5)\ a$

**Evaluating:** in  $(\lambda x. t)\ a$  replace  $x$  by  $a$  in  $t$   
(computation!)

**Example:**  $(\lambda x. x + 5)\ (a + b)$  evaluates to  $(a + b) + 5$



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**That's it!**



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# Now Formal



# Syntax



**Terms:**  $t ::= v \mid c \mid (t t) \mid (\lambda x. t)$

$v, x \in V, \quad c \in C, \quad V, C$  sets of names

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$v, x \in V, \quad c \in C, \quad V, C$  sets of names

- $v, x$  variables
- $C$  constants
- $(t t)$  application
- $(\lambda x. t)$  abstraction

# Conventions



- leave out parentheses where possible
- list variables instead of multiple  $\lambda$

**Example:** instead of  $(\lambda y. (\lambda x. (x y)))$  write  $\lambda y x. x y$

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## Rules:

- list variables:  $\lambda x. (\lambda y. t) = \lambda x y. t$
- application binds to the left:  $x y z = (x y) z \neq x (y z)$
- abstraction binds to the right:  $\lambda x. x y = \lambda x. (x y) \neq (\lambda x. x) y$
- leave out outermost parentheses

# Getting used to the Syntax



**Example:**

$\lambda x y z. x z (y z) =$

# Getting used to the Syntax



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# Getting used to the Syntax



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$\lambda x y z. ((x z) (y z)) =$

$\lambda x. \lambda y. \lambda z. ((x z) (y z)) =$



# Getting used to the Syntax



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$\lambda x y z. ((x z) (y z)) =$

$\lambda x. \lambda y. \lambda z. ((x z) (y z)) =$

$(\lambda x. (\lambda y. (\lambda z. ((x z) (y z))))))$

# Computation



**Intuition:** replace parameter by argument  
this is called  $\beta$ -reduction

## Example

$$(\lambda x y. f (y x)) 5 (\lambda x. x) \longrightarrow_{\beta}$$

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# Defining Computation



$\beta$  reduction:

$$\begin{array}{l} s \longrightarrow_{\beta} s' \implies (\lambda x. s) t \longrightarrow_{\beta} s[x \leftarrow t] \\ t \longrightarrow_{\beta} t' \implies (s t) \longrightarrow_{\beta} (s' t) \\ s \longrightarrow_{\beta} s' \implies (s t) \longrightarrow_{\beta} (s t') \\ s \longrightarrow_{\beta} s' \implies (\lambda x. s) \longrightarrow_{\beta} (\lambda x. s') \end{array}$$

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Still to do: define  $s[x \leftarrow t]$

# Defining Substitution



Easy concept. Small problem: variable capture.

**Example:**  $(\lambda x. x z)[z \leftarrow x]$



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We do **not** want:  $(\lambda x. x x)$  as result.

What do we want?

# Defining Substitution



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**Example:**  $(\lambda x. x z)[z \leftarrow x]$

We do **not** want:  $(\lambda x. x x)$  as result.

What do we want?

In  $(\lambda y. y z)[z \leftarrow x] = (\lambda y. y x)$  there would be no problem.

So, solution is: rename bound variables.

# Free Variables



**Bound variables:** in  $(\lambda x. t)$ ,  $x$  is a bound variable.

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**Free variables**  $FV$  of a term:

$$FV(x) = \{x\}$$

$$FV(c) = \{\}$$

$$FV(st) = FV(s) \cup FV(t)$$

$$FV(\lambda x. t) = FV(t) \setminus \{x\}$$

**Example:**  $FV(\lambda x. (\lambda y. (\lambda x. x) y) y x)$

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Term  $t$  is called **closed** if  $FV(t) = \{\}$

The substitution example,  $(\lambda x. x z)[z \leftarrow x]$ , is problematic because the bound variable  $x$  is a free variable of the replacement term “ $x$ ”.

# Substitution



$$x [x \leftarrow t] = t$$

$$y [x \leftarrow t] = y$$

$$c [x \leftarrow t] = c$$

if  $x \neq y$

$$(s_1 s_2) [x \leftarrow t] =$$



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$$(\lambda x. s) [x \leftarrow t] =$$

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$$(\lambda x. s) [x \leftarrow t] = (\lambda x. s)$$

$$(\lambda y. s) [x \leftarrow t] =$$

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$$(\lambda x. s) [x \leftarrow t] = (\lambda x. s)$$

$$(\lambda y. s) [x \leftarrow t] = (\lambda y. s[x \leftarrow t])$$

if  $x \neq y$  and  $y \notin FV(t)$

$$(\lambda y. s) [x \leftarrow t] =$$

# Substitution



$$\begin{aligned}x [x \leftarrow t] &= t \\y [x \leftarrow t] &= y \\c [x \leftarrow t] &= c\end{aligned} \quad \text{if } x \neq y$$

$$(s_1 s_2) [x \leftarrow t] = (s_1[x \leftarrow t] s_2[x \leftarrow t])$$

$$\begin{aligned}(\lambda x. s) [x \leftarrow t] &= (\lambda x. s) \\(\lambda y. s) [x \leftarrow t] &= (\lambda y. s[x \leftarrow t]) \quad \text{if } x \neq y \text{ and } y \notin FV(t) \\(\lambda y. s) [x \leftarrow t] &= (\lambda z. s[y \leftarrow z])[x \leftarrow t] \quad \text{if } x \neq y \\&\quad \text{and } z \notin FV(t) \cup FV(s)\end{aligned}$$

# Substitution Example



$$(x (\lambda x. x) (\lambda y. z x))[x \leftarrow y]$$

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$$\begin{aligned} & (x \ (\lambda x. x) \ (\lambda y. z \ x))[x \leftarrow y] \\ = & (x[x \leftarrow y]) \ ((\lambda x. x)[x \leftarrow y]) \ ((\lambda y. z \ x)[x \leftarrow y]) \end{aligned}$$

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# $\alpha$ Conversion



**Bound names are irrelevant:**

$\lambda x. x$  and  $\lambda y. y$  denote the same function.

**$\alpha$  conversion:**

$s =_{\alpha} t$  means  $s = t$  up to renaming of bound variables.



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## Formally:

$$\begin{array}{l} s \longrightarrow_{\alpha} s' \implies (\lambda x. t) \longrightarrow_{\alpha} (\lambda y. t[x \leftarrow y]) \text{ if } y \notin FV(t) \\ t \longrightarrow_{\alpha} t' \implies (s t) \longrightarrow_{\alpha} (s' t) \\ s \longrightarrow_{\alpha} s' \implies (\lambda x. s) \longrightarrow_{\alpha} (\lambda x. s') \end{array}$$

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$$s =_{\alpha} t \text{ iff } s \longrightarrow_{\alpha}^* t$$

( $\longrightarrow_{\alpha}^*$  = transitive, reflexive closure of  $\longrightarrow_{\alpha}$  = multiple steps)

# $\alpha$ Conversion



**Equality in Isabelle is equality modulo  $\alpha$  conversion:**

if  $s =_{\alpha} t$  then  $s$  and  $t$  are syntactically equal.

**Examples:**

$x (\lambda x y. x y)$

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# Back to $\beta$



We have defined  $\beta$  reduction:  $\longrightarrow_{\beta}$

Some notation and concepts:

$\rightarrow_{\beta}$  **conversion**:  $s =_{\beta} t$  iff  $\exists n. s \longrightarrow_{\beta}^* n \wedge t \longrightarrow_{\beta}^* n$



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- $\beta$  **conversion**:  $s =_{\beta} t$  iff  $\exists n. s \longrightarrow_{\beta}^* n \wedge t \longrightarrow_{\beta}^* n$
- $t$  is **reducible** if there is an  $s$  such that  $t \longrightarrow_{\beta} s$

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- $(\lambda x. s) t$  is called a **redex** (reducible expression)
- $t$  is reducible iff it contains a redex
- if it is not reducible,  $t$  is in **normal form**

# Does every $\lambda$ term have a normal form?



**Example:**

$$(\lambda x. x x) (\lambda x. x x) \longrightarrow_{\beta}$$

# Does every $\lambda$ term have a normal form?



**Example:**

$$\begin{aligned}(\lambda x. x x) (\lambda x. x x) &\longrightarrow_{\beta} \\(\lambda x. x x) (\lambda x. x x) &\longrightarrow_{\beta}\end{aligned}$$

# Does every $\lambda$ term have a normal form?



**No!**

**Example:**

$$\begin{aligned}(\lambda x. x x) (\lambda x. x x) &\longrightarrow_{\beta} \\(\lambda x. x x) (\lambda x. x x) &\longrightarrow_{\beta} \\(\lambda x. x x) (\lambda x. x x) &\longrightarrow_{\beta} \dots\end{aligned}$$

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$$\text{(but: } (\lambda x y. y) ((\lambda x. x x) (\lambda x. x x)) \longrightarrow_{\beta} \lambda y. y \text{)}$$



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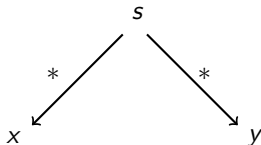
$$\text{(but: } (\lambda x y. y) ((\lambda x. x x) (\lambda x. x x)) \longrightarrow_{\beta} \lambda y. y \text{)}$$

**$\lambda$  calculus is not terminating**

# $\beta$ reduction is confluent



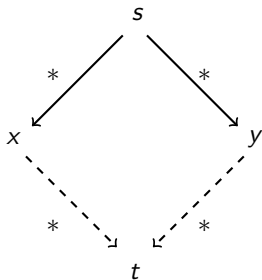
**Confluence:**  $s \rightarrow_{\beta}^* x \wedge s \rightarrow_{\beta}^* y \implies \exists t. x \rightarrow_{\beta}^* t \wedge y \rightarrow_{\beta}^* t$



# $\beta$ reduction is confluent



**Confluence:**  $s \rightarrow_{\beta}^* x \wedge s \rightarrow_{\beta}^* y \implies \exists t. x \rightarrow_{\beta}^* t \wedge y \rightarrow_{\beta}^* t$



**Order of reduction does not matter for result**  
**Normal forms in  $\lambda$  calculus are unique**

# $\beta$ reduction is confluent



**Example:**

$(\lambda x y. y) ((\lambda x. x x) a)$

$(\lambda x y. y) ((\lambda x. x x) a)$

# $\beta$ reduction is confluent



**Example:**

$$(\lambda x y. y) ((\lambda x. x x) a) \longrightarrow_{\beta} (\lambda x y. y) (a a)$$

$$(\lambda x y. y) ((\lambda x. x x) a) \longrightarrow_{\beta} \lambda y. y$$

# $\beta$ reduction is confluent



**Example:**

$$(\lambda x y. y) ((\lambda x. x x) a) \longrightarrow_{\beta} (\lambda x y. y) (a a) \longrightarrow_{\beta} \lambda y. y$$

$$(\lambda x y. y) ((\lambda x. x x) a) \longrightarrow_{\beta} \lambda y. y$$

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Another case of trivially equal functions:  $t = (\lambda x. t x)$

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**Another case of trivially equal functions:**  $t = (\lambda x. t x)$

Definition:

$$\begin{array}{l} s \longrightarrow_{\eta} s' \implies (\lambda x. t x) \longrightarrow_{\eta} t \quad \text{if } x \notin FV(t) \\ t \longrightarrow_{\eta} t' \implies (s t) \longrightarrow_{\eta} (s' t) \\ s \longrightarrow_{\eta} s' \implies (s t) \longrightarrow_{\eta} (s t') \\ s \longrightarrow_{\eta} s' \implies (\lambda x. s) \longrightarrow_{\eta} (\lambda x. s') \end{array}$$

$$s =_{\eta} t \quad \text{iff} \quad \exists n. s \longrightarrow_{\eta}^* n \wedge t \longrightarrow_{\eta}^* n$$

**Example:**  $(\lambda x. f x) (\lambda y. g y) \longrightarrow_{\eta}$



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**Example:**  $(\lambda x. f x) (\lambda y. g y) \longrightarrow_{\eta} (\lambda x. f x) g \longrightarrow_{\eta} f g$

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**Another case of trivially equal functions:**  $t = (\lambda x. t x)$

Definition:

$$\begin{array}{l} s \longrightarrow_{\eta} s' \implies (\lambda x. t x) \longrightarrow_{\eta} t \quad \text{if } x \notin FV(t) \\ t \longrightarrow_{\eta} t' \implies (s t) \longrightarrow_{\eta} (s' t) \\ s \longrightarrow_{\eta} s' \implies (s t) \longrightarrow_{\eta} (s t') \\ s \longrightarrow_{\eta} s' \implies (\lambda x. s) \longrightarrow_{\eta} (\lambda x. s') \end{array}$$

$$s =_{\eta} t \quad \text{iff} \quad \exists n. s \longrightarrow_{\eta}^* n \wedge t \longrightarrow_{\eta}^* n$$

**Example:**  $(\lambda x. f x) (\lambda y. g y) \longrightarrow_{\eta} (\lambda x. f x) g \longrightarrow_{\eta} f g$

- $\eta$  reduction is confluent and terminating.
- $\longrightarrow_{\beta\eta}$  is confluent.  
 $\longrightarrow_{\beta\eta}$  means  $\longrightarrow_{\beta}$  and  $\longrightarrow_{\eta}$  steps are both allowed.
- Equality in Isabelle is also modulo  $\eta$  conversion.

# In fact ...



**Equality in Isabelle is modulo  $\alpha$ ,  $\beta$ , and  $\eta$  conversion.**

We will see later why that is possible.

# So, what can you do with $\lambda$ calculus?



$\lambda$  calculus is very expressive, you can encode:

- logic, set theory
- turing machines, functional programs, etc.

**Examples:**

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`not`  $\equiv \lambda x. \text{if } x \text{ false true}$

`and`  $\equiv \lambda x y. \text{if } x y \text{ false}$

`or`  $\equiv \lambda x y. \text{if } x \text{ true } y$

# More Examples



## Encoding natural numbers (Church Numerals)

$$0 \equiv \lambda f x. x$$

$$1 \equiv \lambda f x. f x$$

$$2 \equiv \lambda f x. f (f x)$$

$$3 \equiv \lambda f x. f (f (f x))$$

...

Numeral  $n$  takes arguments  $f$  and  $x$ , applies  $f$   $n$ -times to  $x$ .

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$$\text{add} \equiv \lambda m n. \lambda f x. m f (n f x)$$

# Fix Points



$$(\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t \longrightarrow_{\beta}$$

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$(\lambda x f. f (x x f)) (\lambda x f. f (x x f))$  is Turing's fix point operator

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allows arbitrary quantification over predicates
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- **Whitehead & Russell** (Principia Mathematica, 1910-1913):  
Fix the problem
- **Church** (1930):  $\lambda$  calculus as logic, true, false,  $\wedge$ , ... as  $\lambda$  terms

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you can write  $R \equiv \lambda x. \text{not } (x x)$   
and get  $(R R) =_{\beta} \text{not } (R R)$   
because  $(R R) = (\lambda x. \text{not } (x x)) R \longrightarrow_{\beta} \text{not } (R R)$





DATA  
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# Isabelle Demo

# We have learned so far...



- $\lambda$  calculus syntax
- free variables, substitution
- $\beta$  reduction
- $\alpha$  and  $\eta$  conversion
- $\beta$  reduction is confluent
- $\lambda$  calculus is very expressive (turing complete)
- $\lambda$  calculus is inconsistent