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COMP 4161

NICTA Advanced Course

Advanced Topics in Software Verification

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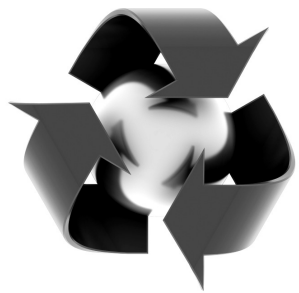

Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

[a] a1 due; [b] a2 due; [c] a3 due

Datatypes

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Example:

datatype ’a list = Nil | Cons ’a ”’a list”

Properties:

- Constructors:

$$\begin{array}{lll} \text{Nil} & :: & \text{'a list} \\ \text{Cons} & :: & \text{'a} \Rightarrow \text{'a list} \Rightarrow \text{'a list} \end{array}$$

- Distinctness: $\text{Nil} \neq \text{Cons x xs}$
- Injectivity: $(\text{Cons x xs} = \text{Cons y ys}) = (\text{x} = \text{y} \wedge \text{xs} = \text{ys})$

More Examples

Enumeration:

datatype answer = Yes | No | Maybe

Polymorphic:

datatype 'a option = None | Some 'a

datatype ('a,'b,'c) triple = Triple 'a 'b 'c

Recursion:

datatype 'a list = Nil | Cons 'a "'a list"

datatype 'a tree = Tip | Node 'a "'a tree" "'a tree"

Mutual Recursion:

datatype even = EvenZero | EvenSucc odd"

and odd = OddSucc even"

NEW

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Nested recursion:

datatype 'a tree = Tip | Node 'a "'a tree list"

datatype 'a tree = Tip | Node 'a "'a tree option" "'a tree option"

➜ Recursive call is under a type constructor.

The General Case

datatype $(\alpha_1, \ldots, \alpha_n)\ \tau \quad = \quad \mathsf{C}_1\ \tau_{1,1}\ \ldots\ \tau_{1,n_1} \mid \ldots \mid \mathsf{C}_k\ \tau_{k,1}\ \ldots\ \tau_{k,n_k}$

- Constructors: $\mathsf{C}_i :: \tau_{i,1} \Rightarrow \ldots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)\ \tau$
- Distinctness: $\mathsf{C}_i\ \ldots \neq \mathsf{C}_j\ \ldots$ if $i \neq j$
- Injectivity: $(\mathsf{C}_i\ x_1 \ldots x_{n_i} = \mathsf{C}_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

Distinctness and Injectivity applied automatically

How is this Type Defined?

datatype 'a list = Nil | Cons 'a "'a list"

- ➜ internally defined using typedef
- ➜ hence: describes a set
- ➜ set = trees with constructors as nodes
- ➜ inductive definition to characterise which trees belong to datatype

Datatype Limitations

Must be definable as set.

- ➜ Infinitely branching ok.
- ➜ Mutually recursive ok.
- ➜ Strictly positive (right of function arrow) occurrence ok.

Not ok:

$$\begin{array}{lcl} \textbf{datatype } t & = & C\ (t \Rightarrow bool) \\ & | & D\ ((bool \Rightarrow t) \Rightarrow bool) \\ & | & E\ ((t \Rightarrow bool) \Rightarrow bool) \end{array}$$

Because: Cantor's theorem (α set is larger than α)

Datatype Limitations

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Not ok (nested recursion):

datatype ('a, 'b) fun_copy = Fun "'a ⇒ 'b"

datatype 'a t = F "('a t, 'a) fun_copy

- ➜ recursion only allowed on *live* arguments
- ➜ in "'a ⇒ 'b", 'a is dead and 'b is live
- ➜ in ('a, 'b) fun_copy, 'a is dead and 'b is live
- ➜ type constructors must be registered as *BNFs** to have live arguments
- ➜ datatypes are automatically registered as BNF
- ➜ (register arbitrary type constructors as BNFs –not convered here)**

* BNF = Bounded Natural Functors.

** *Defining (Co)datatypes and Primitively (Co)recursive Functions in Isabelle/HOL*

Case

Every datatype introduces a **case** construct, e.g.

$$(\text{case } xs \text{ of } [] \Rightarrow \ldots \mid y \#ys \Rightarrow \ldots y \ldots ys \ldots)$$

In general: one case per constructor

- ➜ Nested patterns allowed: $x\#y\#zs$
- ➜ Dummy and default patterns with _
- ➜ Binds weakly, needs () in context

Cases

apply (case_tac t)

creates k subgoals

$$[\![t = C_i\ x_1 \ldots x_p; \ldots]\!] \Longrightarrow \ldots$$

one for each constructor C_i

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DEMO

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RECURSION

Why nontermination can be harmful

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How about $f\ x = f\ x + 1$?

Subtract $f\ x$ on both sides.

$$\Longrightarrow$$

$$0 = 1$$

! All functions in HOL must be total !

Primitive Recursion

primrec guarantees termination structurally

Example primrec def:

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

The General Case

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If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \tau \Rightarrow \tau'$ can be defined by **primitive recursion**:

$$\begin{array}{lcl} f\ (C_1\ y_{1,1}\ \ldots\ y_{1,n_1}) & = & r_1 \\ \vdots & & \\ f\ (C_k\ y_{k,1}\ \ldots\ y_{k,n_k}) & = & r_k \end{array}$$

The recursive calls in r_i must be **structurally smaller**
(of the form $f\ a_1\ \ldots\ y_{i,j}\ \ldots\ a_p$)

How does this Work?

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primrec just fancy syntax for a **recursion operator**

Example:

list_rec :: "'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"

list_rec f_1 f_2 Nil = f_1

list_rec f_1 f_2 (Cons x xs) = f_2 x xs (list_rec f_1 f_2 xs)

app ≡ list_rec ($\lambda ys.\ ys$) ($\lambda x\ xs\ xs'.\ \lambda ys.$ Cons x (xs' ys))

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

list_rec

Defined: automatically, first inductively (set), then by epsilon

$$\frac{}{(\mathsf{Nil}, f_1) \in \mathsf{list\_rel}\ f_1\ f_2} \qquad \frac{(xs, xs') \in \mathsf{list\_rel}\ f_1\ f_2}{(\mathsf{Cons}\ x\ xs, f_2\ x\ xs\ xs') \in \mathsf{list\_rel}\ f_1\ f_2}$$

$$\mathsf{list\_rec}\ f_1\ f_2\ xs \equiv \mathsf{THE}\ y.\ (xs, y) \in \mathsf{list\_rel}\ f_1\ f_2$$

Automatic proof that set def indeed is total function
(the equations for list_rec are lemmas!)

PREDEFINED DATATYPES

nat is a datatype

datatype nat $= 0 \mid$ Suc nat

Functions on nat definable by primrec!

primrec

$$\begin{array}{lcl} f\ 0 & = & \ldots \\ f\ (\text{Suc}\ n) & = & \ldots\ f\ n\ \ldots \end{array}$$

Option

datatype 'a option = None | Some 'a

Important application:

'b ⇒ 'a option	∼	partial function:
None	∼	no result
Some *a*	∼	result *a*

Example:

primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option
where
lookup k [] = None |
lookup k (x #xs) = (if fst x = k then Some (snd x) else lookup k xs)

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DEMO: PRIMREC

INDUCTION

Structural induction

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$P\ xs$ holds for all lists xs if

- ➜ P Nil
- ➜ and for arbitrary x and xs, $P\ xs \Longrightarrow P\ (x\#xs)$
  Induction theorem **list.induct:**
  $[\![P\ []; \bigwedge a\ list.\ P\ list \Longrightarrow P\ (a\#list)]\!] \Longrightarrow P\ list$
- ➜ General proof method for induction: **(induct x)**
  - x must be a free variable in the first subgoal.
  - type of x must be a datatype.

Basic heuristics

Theorems about recursive functions are proved by induction

Induction on argument number *i* of *f*
if *f* is defined by recursion on argument number *i*

Example

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A tail recursive list reverse:

primrec itrev :: 'a list ⇒ 'a list ⇒ 'a list
where
itrev [] $ys = ys$ |
itrev $(x\#xs)$ $ys =$ itrev xs $(x\#ys)$

lemma itrev xs [] = rev xs

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DEMO: PROOF ATTEMPT

Generalisation

Replace constants by variables

lemma itrev xs ys = rev xs@ys

Quantify free variables by $\forall$
(except the induction variable)

lemma $\forall ys$. itrev xs ys = rev xs@ys

Or: **apply (induct xs arbitrary: ys)**

We have seen today ...

- Datatypes
- Primitive recursion
- Case distinction
- Structural Induction

Exercises

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- define a primitive recursive function **lsum** :: nat list $\Rightarrow$ nat that returns the sum of the elements in a list.
- show "$2 * \mathsf{lsum}\ [0..<\mathit{Suc}\ n] = n * (n + 1)$"
- show "$\mathsf{lsum}\ (\mathsf{replicate}\ n\ a) = n * a$"
- define a function **lsumT** using a tail recursive version of listsum.
- show that the two functions are equivalent: lsum *xs* = lsumT *xs*