COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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Slide 1

Content

[a] a1 due; [b] a2 due; [c] a3 due

Slide 2

Datatypes

Example:

datatype 'a list = Nil | Cons 'a "'a list"

Properties:

- Constructors:

  Nil :: 'a list
  Cons :: 'a ⇒ 'a list ⇒ 'a list

- Distinctness: Nil $\neq$ Cons x xs
- Injectivity: (Cons x xs $=$ Cons y ys) $=$ (x $=$ y $\wedge$ xs $=$ ys)

Slide 3

The General Case

$$\textbf{datatype}\ (\alpha_1,\ldots,\alpha_n)\,\tau \;=\; \mathsf{C}_1\ \tau_{1,1}\ \ldots\ \tau_{1,n_1} \;\mid\; \ldots \;\mid\; \mathsf{C}_k\ \tau_{k,1}\ \ldots\ \tau_{k,n_k}$$

- Constructors: $\mathsf{C}_i :: \tau_{i,1} \Rightarrow \ldots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1,\ldots,\alpha_n)\,\tau$
- Distinctness: $\mathsf{C}_i\ \ldots \neq \mathsf{C}_j\ \ldots$ if $i \neq j$
- Injectivity: $(\mathsf{C}_i\ x_1 \ldots x_{n_i} = \mathsf{C}_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

Distinctness and Injectivity applied automatically

Slide 4

How is this Type Defined?

datatype 'a list = Nil | Cons 'a "'a list"

- ➜ internally defined using typedef
- ➜ hence: describes a set
- ➜ set = trees with constructors as nodes
- ➜ inductive definition to characterise which trees belong to datatype

More detail: HOL/Datatype.thy

Slide 5

Datatype Limitations

Must be definable as set.

- ➜ Infinitely branching ok.
- ➜ Mutually recursive ok.
- ➜ Strictly positive (right of function arrow) occurrence ok.

Not ok:

datatype t	=	C (t ⇒ bool)
	|	D ((bool ⇒ t) ⇒ bool)
	|	E ((t ⇒ bool) ⇒ bool)

Because: Cantor's theorem (α set is larger than α)

Slide 6

Case

Every datatype introduces a **case** construct, e.g.

(case xs of [] ⇒ ... | $y\ \#ys$ ⇒ ... y ... ys ...)

In general: one case per constructor

- ➜ Nested patterns allowed: $x\#y\#zs$
- ➜ Dummy and default patterns with _
- ➜ Binds weakly, needs () in context

Slide 7

Cases

apply (case_tac t)

creates k subgoals

$$[\![t = C_i\ x_1 \dots x_p; \dots]\!] \Longrightarrow \dots$$

one for each constructor C_i

Slide 8

DEMO

Slide 9

RECURSION

Slide 10

Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

Subtract $f\ x$ on both sides.

$$\Longrightarrow$$
$$0 = 1$$

! **All functions in HOL must be total** **!**

Slide 11

Primitive Recursion

primrec guarantees termination structurally

Example primrec def:

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

Slide 12

The General Case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \tau \Rightarrow \tau'$ can be defined by **primitive recursion**:

$$\begin{array}{lcl} f\ (C_1\ y_{1,1}\ \ldots\ y_{1,n_1}) & = & r_1 \\ \vdots & & \\ f\ (C_k\ y_{k,1}\ \ldots\ y_{k,n_k}) & = & r_k \end{array}$$

The recursive calls in r_i must be **structurally smaller**
(of the form $f\ a_1\ \ldots\ y_{i,j}\ \ldots\ a_p$)

Slide 13

How does this Work?

primrec just fancy syntax for a **recursion operator**

Example: list_rec :: "'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"

list_rec f_1 f_2 Nil $= f_1$

list_rec f_1 f_2 (Cons x xs) $= f_2\ x\ xs$ (list_rec f_1 f_2 xs)

app ≡ list_rec $(\lambda ys.\ ys)$ $(\lambda x\ xs\ xs'.\ \lambda ys.$ Cons $x\ (xs'\ ys))$

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

Slide 14

list_rec

Defined: automatically, first inductively (set), then by epsilon

$$\frac{}{(\text{Nil}, f_1) \in \text{list\_rel}\ f_1\ f_2} \qquad \frac{(xs, xs') \in \text{list\_rel}\ f_1\ f_2}{(\text{Cons}\ x\ xs, f_2\ x\ xs\ xs') \in \text{list\_rel}\ f_1\ f_2}$$

list_rec f_1 f_2 $xs \equiv$ THE $y.\ (xs, y) \in$ list_rel f_1 f_2

Automatic proof that set def indeed is total function
(the equations for list_rec are lemmas!)

Slide 15

PREDEFINED DATATYPES

Slide 16

nat is a datatype

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datatype nat $=$ 0 | Suc nat

Functions on nat definable by primrec!

primrec

$f\ 0 \quad = \quad ...$

$f\ (\text{Suc}\ n) \quad = \quad ...\ f\ n\ ...$

Slide 17

Option

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datatype 'a option = None | Some 'a

Important application:

'b ⇒ 'a option $\sim$ partial function:

None $\sim$ no result

Some a $\sim$ result a

Example:

primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option

where

lookup k [] = None |

lookup k (x #xs) = (if fst x = k then Some (snd x) else lookup k xs)

Slide 18

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Demo: primrec

Slide 19

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Induction

Slide 20

Structural induction

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$P\ xs$ holds for all lists xs if

- ➜ P Nil
- ➜ and for arbitrary x and xs, $P\ xs \Longrightarrow P\ (x\#xs)$

Induction theorem **list.induct:**
$[\![P\ [\,];\ \bigwedge a\ list.\ P\ list \Longrightarrow P\ (a\#list)]\!] \Longrightarrow P\ list$

- ➜ General proof method for induction: **(induct x)**
  - x must be a free variable in the first subgoal.
  - type of x must be a datatype.

Slide 21

Basic heuristics

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Theorems about recursive functions are proved by induction

Induction on argument number i of f
if f is defined by recursion on argument number i

Slide 22

Example

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A tail recursive list reverse:

primrec itrev :: 'a list ⇒ 'a list ⇒ 'a list
where
itrev [] $ys = ys$ |
itrev $(x\#xs)$ $ys =$ itrev xs $(x\#ys)$

lemma itrev xs [] = rev xs

Slide 23

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DEMO: PROOF ATTEMPT

Slide 24

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Generalisation

Replace constants by variables

lemma itrev $xs\ ys =$ rev $xs@ys$

Quantify free variables by $\forall$
(except the induction variable)

lemma $\forall ys.$ itrev $xs\ ys =$ rev $xs@ys$

Or: **apply (induct xs arbitrary: ys)**

Slide 25

We have seen today ...

- ➜ Datatypes
- ➜ Primitive recursion
- ➜ Case distinction
- ➜ Structural Induction

Slide 26

Exercises

- ➜ define a primitive recursive function **lsum** :: nat list ⇒ nat that returns the sum of the elements in a list.
- ➜ show "$2 *$ lsum $[0..< Suc\ n] = n * (n + 1)$"
- ➜ show "lsum (replicate $n\ a) = n * a$"
- ➜ define a function **lsumT** using a tail recursive version of listsum.
- ➜ show that the two functions are equivalent: lsum xs = lsumT xs

Slide 27